AFFINE REPRESENTATIONS OF FRACTIONAL PROCESSES WITH APPLICATIONS IN MATHEMATICAL FINANCE

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Abstract. Fractional processes have gained popularity in financial modeling due to the dependence structure of their increments and the roughness of their sample paths. The non-Markovianity of these processes gives, however, rise to conceptual and practical difficulties in computation and calibration. To address these issues, we show that a certain class of fractional processes can be represented as linear functionals of an infinite dimensional affine process. We demonstrate by means of several examples that the affine structure allows one to construct tractable financial models with fractional features. In future work, the Markovian representation could be useful for option pricing by PDE methods in fractional volatility models.

1. Introduction

Empirical evidence suggests that certain financial time series may not be captured well by low-dimensional Markovian models. In particular, this applies to short-term interest rates, which tend to have long-range dependence [1], and to volatilities of stock prices, which have rough sample paths and behave essentially as fractional Brownian motion with small Hurst index [6]. Dependent increments and rough sample paths are, however, characteristic features of fractional processes.

The wide-spread adoption of fractional processes in financial modeling was impeded by several difficulties. Conceptually, one of the major challenges is the lack of the Markov property. In the absence of the Markov property, it is unclear what the states of the model are. This makes it difficult to talk about calibration in a sensible way and to compare the model across time. Moreover, PDE methods for option pricing cannot be used.

In this paper we introduce a class of fractional processes which can be represented as linear functionals of an infinite-dimensional affine process. The key idea, which goes back to Carmona and Coutin [4], is to express the fractional integral in the Mandelbrot-Van Ness representation of fractional Brownian motion by a Laplace transform. This allows one to express this fractional integral as a superposition of infinitely many Ornstein-Uhlenbeck (OU) processes with varying speed of mean reversion. We show that the collection of OU processes, indexed by the speed of mean reversion, is a Banach-space valued affine process. Linear functionals of this process are in general not semimartingales. Instead, they are fractional

Date: September 20, 2015.
2010 *Mathematics Subject Classification.* 91G30, 60J25, 60J60,
Supported in part by SNF grant 149879.
We gratefully acknowledge support by ETH Foundation.

processes with positively or negatively correlated increments and are closely related to fractional Brownian motion. More precisely, fractional Brownian motion is obtained by randomizing the initial condition of the OU processes according to their stationary distribution.

Our result is important for many reasons. First, it is within reach to solve some simple fractional models where the affine structure is preserved. We demonstrate this by means of several examples in this paper. In particular, we construct interest rate models where either the short rate or the bank account process is modeled by a fractional process. In contrast to [14] and [2], we build the model such that discounted zero-coupon bond prices are martingales. This implies absence of arbitrage by construction, while certain quantities of the model such as the short rate may very well be non-semimartingales. We also build a fractional version of the stochastic volatility model by Stein and Stein [15].

Second, there is currently a high interest in non-affine fractional volatility models such as the fractional Bergomi and SABR models [12, 7]. It is a major challenge to derive short-time, large-time, and wing asymptotics for these models, as well as to develop numerical schemes for pricing and calibration. Hopefully, the Markovian point of view and the affine structure will be helpful for achieving these goals.

The paper is structured as follows. In Section 2 we prove that the collection of OU processes is indeed a Banach-space valued affine process. In Section 3 we deduce the affine representation of fractional Brownian motion. Section 4 is dedicated to applications in interest rate modeling and Section 5 to a fractional version of the stochastic volatility model of Stein and Stein [15].

2. Infinite-dimensional Ornstein-Uhlenbeck process

2.1. **Setup and notation.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{Q})$ be a filtered probability space satisfying the usual conditions and let W be two-sided $(\mathcal{F}_t)$-Brownian motion on Ω. The measure $\mathbb{Q}$ plays the role of a risk-neutral measure.

Definition 2.1 (OU processes)**.** Given a collection of $\mathcal{F}_0$-measurable $\mathbb{R}$-valued random variables Y_0^x, Z_0^x indexed by $x \in (0,\infty)$, let for each $t \geq 0$

$$\begin{aligned} Y_t^x &= Y_0^x e^{-tx} + \int_0^t e^{-(t-s)x} dW_s, \\ Z_t^x &= Z_0^x e^{-tx} + \int_0^t e^{-(t-s)x} Y_s^x ds. \end{aligned} \tag{2.1}$$

Remark 2.2 (SDE representation)**.** *For each $x > 0$, the process $(Y_t^x, Z_t^x)_{t\geq 0}$ solves the SDE*

$$dY_t^x = -xY_t^x dt + dW_t, \quad dZ_t^x = (-xZ_t^x + Y_t^x)dt. \tag{2.2}$$

Therefore, it is a bi-variate OU process, and the variable x is related to the speed of mean reversion of the process (see Lemma D.1 in the appendix for details).

2.2. **Banach-space valued OU process.** Let $Y_t := (Y_t^x)_{x>0}$ and $Z_t := (Z_t^x)_{x>0}$ denote the collection of OU processes indexed by the speed of mean reversion x. We show in this section that the process $(Y_t, Z_t)_{t\geq 0}$ takes values in $L^1(\mu) \times L^1(\nu)$, where the measures μ and ν are subject to the following conditions.

Assumption 2.3. *μ and ν are sigma-finite measures on $(0,\infty)$ such that ν has a density p with respect to μ and for each $t>0$,*

$$\int_0^\infty (1\wedge x^{-\frac{1}{2}})\mu(dx)<\infty, \qquad \int_0^\infty (1\wedge x^{-\frac{3}{2}})\nu(dx)<\infty, \qquad \sup_{x\in(0,\infty)} p(x)e^{-tx}<\infty.$$

Theorem 2.4. *Let μ,ν satisfy Assumption 2.3 and let $(Y_0,Z_0)\in L^1(\mu)\times L^1(\nu)$. Then the process $(Y_t,Z_t)_{t\geq 0}$ has a predictable $L^1(\mu)\times L^1(\nu)$-valued version.*

Proof. It is shown in Lemma D.1 that for each $x\in(0,\infty)$ the process $(Y_t^x,Z_t^x)_{t\geq 0}$ can be represented as

$$\begin{aligned} Y_t^x &= Y_0^x e^{-tx} + \int_0^t e^{-(t-s)x}dW_s, \\ Z_t^x &= Z_0^x e^{-tx} + Y_0^x t e^{-tx} + \int_0^t (t-s)e^{-(t-s)x}dW_s. \end{aligned} \tag{2.3}$$

By Assumption 2.3, the deterministic parts in the above representation are $L^1(\mu)$- and $L^1(\nu)$-valued functions, respectively. Therefore, we can assume without loss of generality that Y_0 and Z_0 are zero.

In Lemma D.2, the stochastic Fubini theorem is used to show that for each fixed $t\geq 0$, $(Y_t,Z_t)\in L^1(\mu)\times L^1(\nu)$ holds almost surely. Moreover, for any $(u,v)\in L^\infty(\mu)\times L^\infty(\nu)$, the random variables $\langle Y_t,u\rangle_\mu$ and $\langle Z_t,v\rangle_\nu$ are centered Gaussian as shown in Lemma D.3. Let $P_t\colon L^\infty(\mu)\to L^1(\mu)$ and $Q_t\colon L^\infty(\mu)\to L^1(\nu)$ be the associated covariance operators, which are calculated explicitly in Lemma D.4.

We now show that Y_t is a version of an $L^1(\mu)$-valued stochastic convolution. To this aim, let $\mathsf{H}_t\subseteq L^1(\mu)$ be the reproducing kernel Hilbert space of P_t (see Appendix B). The inclusion of H_t in $L^1(\mu)$ is γ-radonifying because Y_t provides an instance of a Gaussian random variable with covariance operator P_t [13, Theorem 7.4]. For each $s>0$ define $\Theta_1(s)\in L^1(\mu)$ and $\Theta_1^*(s):L^\infty(\mu;\mathbb{C})\to\mathbb{C}$ by

$$\Theta_1(s)(x)=e^{-sx}, \qquad \Theta_1^*(s)(u)=\langle\Theta_1(s),u\rangle_\mu .$$

Then Θ_1^* satisfies for each $t\geq 0$

$$\int_0^t \left(\Theta_1^*(t-s)u\right)^2 ds = \int_0^t \left(\int_0^\infty e^{-x(t-s)}u(x)\mu(dx)\right)^2 ds = \langle P_t u,u\rangle_\mu<\infty,$$

where the order of integration can be exchanged because condition (A.1) is satisfied by Equation (C.21). By [3, Theorem 3.3], the bound on Θ_1^* and the γ-radonifying property of the inclusion of H_t in $L^1(\mu)$ imply that the stochastic convolution of Θ_1 with W exists as an $L^1(\mu)$-valued $(\mathcal{F}_t)_{t\geq 0}$-predictable process $\widetilde{Y}$ such that for each $t\geq 0$ and any $u\in L^\infty(\mu;\mathbb{C})$,

$$\langle\widetilde{Y}_t,u\rangle_\mu=\int_0^t\Theta_1^*(t-s)(u)dW_s$$

holds almost surely. The same equation is also satisfied by Y_t. As stochastic convolutions are unique up to modifications [3, Theorem 3.3], $\mathbb{P}(Y_t=\widetilde{Y}_t)=1$ holds for each $t\geq 0$. This proves that Y has a predictable, $L^1(\mu)$-valued version.

We use the same argument to show that Z has a predictable, $L^1(\nu)$-valued version. For each $s>0$ define $\Theta_2(s)\in L^1(\nu)$ and $\Theta_2^*(s):L^\infty(\nu;\mathbb{C})\to\mathbb{C}$ by

$$\Theta_2(s)(x)=se^{-sx}, \qquad \Theta_2^*(s)(v)=\langle\Theta_2(s),v\rangle_\nu .$$

Then, Θ_2^* satisfies for each $t \geq 0$

$$\int_0^t \big(\Theta_2^*(t-s)v\big)^2 ds = \int_0^t \left(\int_0^\infty (t-s)e^{-x(t-s)}v(x)\nu(dx)\right)^2 ds = \langle Q_t v, v\rangle_\nu < \infty,$$

where the order of integration can be exchanged because condition (A.1) is satisfied by Equation (C.22). By the same argument as above there exists an $L^1(\nu)$-valued $(\mathcal{F}_t)_{t\geq 0}$-predictable process $\widetilde{Z}$ such that for each $t \geq 0$ and any $v \in L^\infty(\nu)$,

$$\langle \widetilde{Z}_t, v\rangle_\nu = \int_0^t \Theta_2^*(t-s)(u)dW_s,$$

holds almost surely. As Z satisfies the same equation and stochastic convolutions are unique up to modifications [3, Theorem 3.3], $\widetilde{Z}$ is a version of Z. □

2.3. **Affine structure.** We derive an infinite dimensional affine transformation formula for the conditional exponential moments of $\langle Y, u\rangle_\mu$ and $\langle Z, v\rangle_\nu$.

Theorem 2.5 (Affine structure)**.** *Under Assumption 2.3, the process (Y, Z) is affine in the sense that for each $0 \leq t \leq T$ and $(u, v) \in L^\infty(\mu) \times L^\infty(\nu)$, the relation*

$$\mathbb{E}\left[e^{\langle Y_T,u\rangle_\mu + \langle Z_T,v\rangle_\nu}\middle|\mathcal{F}_t\right] = e^{\phi_0(T-t,u,v)+\langle Y_t,\phi_1(T-t,u,v)\rangle_\mu + \langle Z_t,\phi_2(T-t,u,v)\rangle_\nu}$$

holds with probability one, where the functions

$$\begin{aligned}
\phi_0 &\colon [0,\infty) \times L^\infty(\mu;\mathbb{C}) \times L^\infty(\nu;\mathbb{C}) \to \mathbb{C},\\
\phi_1 &\colon [0,\infty) \times L^\infty(\mu;\mathbb{C}) \times L^\infty(\nu;\mathbb{C}) \to L^\infty(\mu;\mathbb{C}),\\
\phi_2 &\colon [0,\infty) \times L^\infty(\mu;\mathbb{C}) \times L^\infty(\nu;\mathbb{C}) \to L^\infty(\nu;\mathbb{C})
\end{aligned}$$

are given by

$$\begin{aligned}
\phi_0(\tau,u,v) &= \frac{1}{2}\int_0^\tau \left(\int_0^\infty \phi_1(s,u,v)(x)\mu(dx)\right)^2 ds,\\
\phi_1(\tau,u,v)(x) &= e^{-\tau x}\big(u(x) + \tau v(x)p(x)\big),\\
\phi_2(\tau,u,v)(x) &= e^{-\tau x}v(x).
\end{aligned} \tag{2.4}$$

Proof. Lemma D.3 states that for each $0 \leq t \leq T$, the random variable $\langle Y_T, u\rangle_\mu + \langle Z_T, v\rangle_\nu$ is Gaussian, given $\mathcal{F}_t$, with mean

$$\begin{aligned}
&\int_0^\infty Y_t^x e^{-(T-t)x}u(x)\mu(dx) + \int_0^\infty \left(Z_t^x e^{-x(T-t)} + Y_t^x(T-t)e^{-x(T-t)}\right)v(x)\nu(dx)\\
&\quad = \langle Y_t, \phi_1(T-t,u,v)\rangle_\mu + \langle Z_t, \phi_2(T-t,u,v)\rangle_\nu .
\end{aligned}$$

By Itō's isometry, the conditional variance of $\langle Y_T, u\rangle_\mu + \langle Z_T, v\rangle_\nu$ given $\mathcal{F}_t$ is

$$\int_t^T \left(\int_0^\infty e^{-(T-s)x}u(x)\mu(dx) + \int_0^\infty (T-s)e^{-x(T-s)}v(x)\nu(dx)\right)^2 ds,$$

which equals $2\phi_0(T-t,u,v)$. Thus,

$$\begin{aligned}
\mathbb{E}\left[e^{\langle Y_T,u\rangle_\mu + \langle Z_T,v\rangle_\nu}\middle|\mathcal{F}_t\right] &= e^{\frac{1}{2}\operatorname{Var}\left(\langle Y_T,u\rangle_\mu + \langle Z_T,v\rangle_\nu|\mathcal{F}_t\right) + \mathbb{E}\left[\langle Y_T,u\rangle_\mu + \langle Z_T,v\rangle_\nu|\mathcal{F}_t\right]}\\
&= e^{\phi_0(T-t,u,v)+\langle Y_t,\phi_1(T-t,u,v)\rangle_\mu + \langle Z_t,\phi_2(T-t,u,v)\rangle_\nu}.
\end{aligned}$$ □

The coefficient functions (ϕ_0, ϕ_1, ϕ_2) are solutions of an infinite-dimensional system of Riccati equations.

Lemma 2.6. *The functions ϕ_0, ϕ_1, ϕ_2 defined in Equation (2.4) are the unique solution of the Riccati equations*

$$\begin{aligned} &\partial_\tau \phi_0(\tau,u,v) = R_0\big(\phi_1(\tau,u,v),\phi_2(\tau,u,v)\big), \quad \phi_0(0,u,v)=0,\\ &\partial_\tau \phi_1(\tau,u,v) = R_1\big(\phi_1(\tau,u,v),\phi_2(\tau,u,v)\big), \quad \phi_1(0,u,v)=u,\\ &\partial_\tau \phi_2(\tau,u,v) = R_2\big(\phi_1(\tau,u,v),\phi_2(\tau,u,v)\big), \quad \phi_2(0,u,v)=v, \end{aligned} \tag{2.5}$$

where $R_0(u,v) \in \mathbb{C}$ is given by

$$R_0(u,v) = \frac{1}{2}\left(\int_0^\infty u(x)\mu(dx)\right)^2,$$

and $R_1(u,v), R_2(u,v)$ are measurable functions on $(0,\infty)$ given by

$$\begin{aligned} R_1(u,v)(x) &= -xu(x) + p(x)v(x),\\ R_2(u,v)(x) &= -xv(x). \end{aligned}$$

Proof. It is straightforward to verify that the functions (ϕ_0,ϕ_1,ϕ_2) given by Equation (2.4) solve the Riccati equations. Let $(\overline{\phi}_0,\overline{\phi}_1,\overline{\phi}_2)$ be any other solution. Then $e^{xt}(\phi_2-\overline{\phi}_2)$ has vanishing derivative and initial condition, implying that it is constant and $\phi_2=\overline{\phi}_2$. The same applies to $e^{xt}(\phi_1-\overline{\phi}_1)$, showing that $\phi_1=\overline{\phi}_1$, and to $\phi_0-\overline{\phi}_0$, showing that $\phi_0=\overline{\phi}_0$. $\square$

2.4. **Continuity of sample paths.** Under some more restrictive conditions on the measures μ and ν, the process (Y,Z) has continuous sample paths in $L^1(\mu)\times L^1(\nu)$.

Assumption 2.7. *μ and ν are sigma-finite measures on $(0,\infty)$ satisfying*

$$\int_0^\infty \log(1+tx)x^{-\frac{1}{2}}\mu(dx)<\infty, \qquad \int_0^\infty \log(1+tx)x^{-\frac{3}{2}}\nu(dx)<\infty.$$

Moreover, ν has a density p with respect to μ, such that for each $t>0$

$$\sup_{x\in(0,\infty)} p(x)e^{-tx}<\infty.$$

Remark 2.8. *Note that Assumption 2.7 is stronger than Assumption 2.3 since*

$$\forall t>0\colon \quad \frac{(1+tx)x^{-\frac{1}{2}}}{1\wedge x^{-\frac{1}{2}}}\to\infty, \quad \textit{and} \quad \frac{(1+tx)x^{-\frac{3}{2}}}{1\wedge x^{-\frac{3}{2}}}\to\infty$$

as $x\to 0^+$ or $x\to\infty$.

Theorem 2.9 (Continuous sample paths)**.** *Under Assumption 2.7, the process (Y,Z) has continuous sample paths in $L^1(\mu)\times L^1(\nu)$ if its initial condition (Y_0,Z_0) lies in this space.*

Proof. The expressions $Y_0^x e^{-tx}$ and $Z_0^x e^{-tx}+Y_0^x t e^{-tx}$ define continuous $L^1(\mu)$- and $L^1(\nu)$-valued functions, respectively. Thus, it follows from the representation of (Y,Z) in Equation (2.3) that we may assume $(Y_0,Z_0)=0$ without loss of generality. By Lemma D.5, and Assumption 2.7 on μ, integration with respect to μ yields

$$\mathbb{E}\left[\int_0^\infty \sup_{s\in[0,t]}|Y_s^x|\mu(dx)\right]\le C\int_0^\infty \log(1+tx)x^{-\frac{1}{2}}\mu(dx)<\infty,$$

where we are allowed to exchange the order of integration since the integrand is positive. This implies that $\mathbb{P}[\forall t\colon Y_t\in L^1(\mu)]=1$. Moreover, by the dominated convergence theorem with the sup process of Y as majorant, $\mathbb{P}[Y\in C([0,\infty);L^1(\mu))]=1$.

For the process Z, the estimate of Lemma D.5 and Assumption 2.7 on ν show that $\mathbb{P}[\forall t\colon Z_t^x \in L^1(\nu)] = 1$. As before, the dominated convergence theorem with the sup process of Z as majorant implies $\mathbb{P}[Z \in C([0,\infty); L^1(\nu))] = 1$. □

2.5. **Semimartingale property.** We investigate in this section under what conditions linear functionals of the process (Y, Z) are semimartingales. As this will be needed later in applications, we consider time-dependent linear functionals.

Theorem 2.10 (Semimartingale property)**.** *Let Assumption 2.3 be in place. Let f_t^x and g_t^x be real-valued, deterministic, jointly measurable in $(x,t) \in (0,\infty)\times[0,\infty)$, differentiable in t and satisfy*

$$\forall t \geq 0\colon (f_t, g_t) \in L^\infty(\mu) \times L^\infty(\nu).$$

Assume $(Y_0, Z_0) \in L^1(\mu)\times L^1(\nu)$, a.s., and for each $t \geq 0$

$$\int_0^\infty \int_0^t |\partial_s f_s^x - x f_s^x|(1\wedge x^{-\frac{1}{2}}) ds\mu(dx) < \infty, \tag{2.6}$$

$$\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds}\,\mu(dx) < \infty, \tag{2.7}$$

$$\int_0^\infty \int_0^t |\partial_s g_s^x - x g_s^x|(1\wedge x^{-\frac{3}{2}}) ds\nu(dx) < \infty, \tag{2.8}$$

$$\int_0^\infty \int_0^t |g_s^x|(1\wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty. \tag{2.9}$$

Then $\langle Y_t, f_t\rangle_\mu$ and $\langle Z_t, g_t\rangle_\nu$ are semimartingales with decompositions

$$\begin{aligned}
\langle Y_t, f_t\rangle_\mu &= \langle Y_0, f_0\rangle_\mu + \int_0^t \int_0^\infty \left(\partial_s f_s^x - x f_s^x\right) Y_s^x \mu(dx) ds \\
&\quad + \int_0^t \int_0^\infty f_s^x \mu(dx) dW_s, \\
\langle Z_t, g_t\rangle_\nu &= \langle Z_0, g_0\rangle_\nu + \int_0^t \int_0^\infty \left(\partial_s g_s^x - x g_s^x\right) Z_s^x \nu(dx) ds \\
&\quad + \int_0^t \int_0^\infty g_s^x Y_s^x \nu(dx) ds.
\end{aligned} \tag{2.10}$$

Proof. First observe that

$$\begin{aligned}
\langle Y_t, f_t\rangle_\mu &= \left\langle Y_t - Y_0^x e^{-xt}, f_t\right\rangle_\mu + \left\langle Y_0^x e^{-xt}, f_t\right\rangle_\mu, \\
\langle Z_t, g_t\rangle_\nu &= \left\langle Z_t - Z_0^x e^{-xt} - Y_0^x t e^{-xt}, g_t\right\rangle_\nu + \left\langle Z_0^x e^{-xt}, g_t\right\rangle_\nu + \left\langle Y_0^x t e^{-xt}, g_t\right\rangle_\nu.
\end{aligned}$$

Since $\langle Y_0^x e^{-xt}, f_t\rangle_\mu$, $\langle Z_0^x e^{-xt}, g_t\rangle_\nu$ and $\langle Y_0^x t e^{-xt}, g_t\rangle_\nu$ are finite variation processes we assume without loss of generality that $Y_0 = Z_0 = 0$. By SDE (2.2) for (Y, Z) and Itō's formula, the semimartingale decomposition of the process $(f_t^x Y_t^x, g_t^x Z_t^x)$ is given by

$$\begin{aligned}
f_t^x Y_t^x &= \int_0^t \left(\partial_s f_s^x - x f_s^x\right) Y_s^x ds + \int_0^t f_s^x dW_s, \\
g_t^x Z_t^x &= \int_0^t \left(\partial_s g_s^x - x g_s^x\right) Z_s^x ds + \int_0^t g_s^x Y_s^x ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle Y_t, f_t\rangle_\mu &= \int_0^\infty \int_0^t \left(\partial_s f_s^x - x f_s^x\right) Y_s^x ds\mu(dx) + \int_0^\infty \int_0^t f_s^x dW_s \mu(dx),\\
\langle Z_t, g_t\rangle_\nu &= \int_0^\infty \int_0^t \left(\partial_s g_s^x - x g_s^x\right) Z_s^x ds\nu(dx) + \int_0^\infty \int_0^t g_s^x Y_s^x ds\nu(dx).
\end{aligned}$$

By Theorem A.1, one obtains the semimartingale decompositions of $\langle Y_t, f_t\rangle_\mu$ and $\langle Z_t, g_t\rangle_\nu$. By Lemma D.6 and Equations (2.6)-(2.9) conditions (A.1) and (A.2) are satisfied. □

2.6. **Stationary distribution.** We show that the stationary distribution of (Y, Z) is in general not a Gaussian distribution on $L^1(\mu) \times L^1(\nu)$, but only on a larger space $L^1(\mu_\infty) \times L^1(\nu_\infty)$ corresponding to stronger integrability conditions on the measures μ_∞ and ν_∞.

Assumption 2.11 (Conditions for stationary distribution)**.** *μ_∞, ν_∞ are sigma-finite measures on $(0,\infty)$ such that $\nu_\infty(dx) = p_\infty(x)\mu_\infty(dx)$ for some non-negative measurable function p_∞ and*

$$\int_0^\infty x^{1/2}\mu_\infty(dx) < \infty, \qquad \int_0^\infty x^{3/2}\nu_\infty(dx) < \infty, \qquad \sup_{x\in(0,\infty)} p_\infty(x)e^{-tx} < \infty.$$

Remark 2.12. *Assumption 2.11 is more stringent than Assumption 2.3. The difference is the decay of the measures near zero: μ, ν satisfy Assumption 2.3 if and only if the measures*

$$\mu_\infty(dx) := (1 \wedge x^{1/2})\mu(dx), \qquad \nu_\infty(dx) := (1 \wedge x^{1/2})\nu(dx) \tag{2.11}$$

satisfy Assumption 2.11. In this case, $L^1(\mu) \times L^1(\nu) \subset L^1(\mu_\infty) \times L^1(\nu_\infty)$.

Theorem 2.13 (Stationary distribution)**.** *The random variables $Y_\infty = (Y_\infty^x)_{x>0}$ and $Z_\infty = (Z_\infty^x)_{x>0}$ defined by*

$$Y_\infty^x = \int_{-\infty}^0 e^{sx} dW_s, \qquad Z_\infty^x = -\int_{-\infty}^0 s e^{xs} dW_s \tag{2.12}$$

are normally distributed on $L^1(\mu_\infty) \times L^1(\nu_\infty)$. Their distribution is stationary in the sense that (Y_t, Z_t) is equal in distribution to (Y_∞, Z_∞) if (Y_0, Z_0) is equal in distribution to (Y_∞, Z_∞). Moreover, for any initial condition $(Y_0, Z_0) \in L^1(\mu_\infty) \times L^1(\nu_\infty)$, the random variables (Y_t, Z_t) converge in distribution on $L^1(\mu_\infty) \times L^1(\nu_\infty)$ to (Y_∞, Z_∞) as t tends to infinity.

Proof. $(Y_\infty, Z_\infty) \in L^1(\mu_\infty) \times L^1(\nu_\infty)$ holds almost surely because

$$\begin{aligned}
\mathbb{E}\left[\|Y_\infty\|_{L^1(\mu_\infty)}\right] &= \int_0^\infty \mathbb{E}\left[\left|\int_{-\infty}^0 e^{sx} dW_s\right|\right]\mu_\infty(dx) = \int_0^\infty \sqrt{\frac{1}{\pi x}}\mu_\infty(dx) < \infty,\\
\mathbb{E}\left[\|Z_\infty\|_{L^1(\nu_\infty)}\right] &= \int_0^\infty \mathbb{E}\left[\left|\int_{-\infty}^0 s e^{sx} dW_s\right|\right]\nu_\infty(dx) = \int_0^\infty \sqrt{\frac{1}{2\pi x^3}}\nu_\infty(dx) < \infty.
\end{aligned}$$

For each $u, v \in L^\infty(\mu_\infty) \times L^\infty(\nu_\infty)$, the random variable $\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty}$ can be expressed by Fubini (Theorem A.1) as

$$\begin{aligned}\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty} &= \int_{-\infty}^0 \int_0^\infty e^{sx} u(x)\mu_\infty(dx)dW_s \\ &\quad + \int_{-\infty}^0 \int_0^\infty s e^{sx} v(x)\nu_\infty(dx)dW_s.\end{aligned}$$

Condition (A.2) of Fubini's theorem is satisfied because

$$\begin{aligned}&\int_0^\infty \sqrt{\int_{-\infty}^0 e^{2sx}u(x)^2 ds}\mu_\infty(dx) \le \|u\|_{L^\infty(\mu_\infty)} \int_0^\infty \sqrt{\frac{1}{2x}}\mu_\infty(dx) < \infty, \\ &\int_0^\infty \sqrt{\int_{-\infty}^0 s^2 e^{2sx}v(x)^2 ds}\nu_\infty(dx) \le \|v\|_{L^\infty(\mu_\infty)} \int_0^\infty \sqrt{\frac{1}{4x^3}}\nu_\infty(dx) < \infty.\end{aligned}$$

Therefore, $\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty}$ is a centered Gaussian real-valued random variable. By Itō's isometry its variance is given by

$$\mathbb{E}\left[(\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty})^2\right] = \int_{-\infty}^0 \left(\int_0^\infty e^{sx}\big(u(x) + sv(x)p_\infty(x)\big)\mu_\infty(dx)\right)^2 ds.$$

As $\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty}$ is Gaussian for all $(u, v) \in L^\infty(\mu_\infty) \times L^\infty(\nu_\infty)$, it follows that (Y_∞, Z_∞) is a Gaussian random variable on $L^1(\mu_\infty) \times L^1(\nu_\infty)$. As the measures μ_∞ and ν_∞ satisfy the conditions of Theorem 2.5,

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left[e^{\langle Y_t, u\rangle_{\mu_\infty} + \langle Z_t, v\rangle_{\nu_\infty}}\right] &= \lim_{t\to\infty} e^{\phi_0(t,u,v) + \langle Y_0, \phi_1(t,u,v)\rangle_{\mu_\infty} + \langle Z_0, \phi_2(t,u,v)\rangle_{\nu_\infty}} \\ &= e^{\frac{1}{2}\int_0^\infty \left(\int_0^\infty e^{-tx}(u(x) + tv(x)p_\infty(x))\mu(dx)\right)^2 ds} \\ &= e^{\frac{1}{2}\operatorname{Var}(\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty})} \\ &= \mathbb{E}\left[e^{\langle Y_\infty, u\rangle_{\mu_\infty} + \langle Z_\infty, v\rangle_{\nu_\infty}}\right],\end{aligned}$$

which shows that (Y_t, Z_t) converges in distribution on $L^1(\mu_\infty) \times L^1(\nu_\infty)$ to (Y_∞, Z_∞). □

3. Fractional Brownian motion as a functional of a Markov process

The goal in this section is to obtain a Markovian representation of fractional Brownian motion (fBM) in terms of (Y, Z). We use the representation of Mandelbrot and Van Ness [11] to define fBM.

Definition 3.1. Fractional Brownian motion W^H with initial value $w_0^H \in \mathbb{R}$ and Hurst index $H \in (0, 1)$ is defined for each $t \ge 0$ as

$$W_t^H = w_0^H + \frac{1}{\Gamma(H + \frac{1}{2})} \int_{-\infty}^t \left((t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right) dW_s,$$

where $W = (W_t)_{t\in\mathbb{R}}$ is two-sided Brownian motion as defined in Section 2.1.

Remark 3.2. *Definition 3.1 can be written as*

$$W_t^{\mathrm{H}} = w_0^H + \int_{-\infty}^0 \left((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\right) dW_s + \int_0^t (t-s)^{H-\frac{1}{2}} dW_s.$$

Under the condition $H < \frac{1}{2}$, Carmona and Coutin [4] found a Markovian representation of the fractional integral $\int_0^{\cdot}(\cdot - s)^{H-\frac{1}{2}} dW_s$ in terms of the process Y. We extend their result by incorporating the integrals $\int_{-\infty}^0$ in our representation and by treating also the case $H > \frac{1}{2}$. This allows us to derive a representation of fBM with general Hurst index.

Theorem 3.3. *fBM has the representation*

$$W_t^H = \begin{cases} w_0^H + \displaystyle\int_0^\infty (Y_t^x - Y_0^x)\mu(dx), & \text{if } H < \dfrac{1}{2}, \\ w_0^H + \displaystyle\int_0^\infty (Z_t^x - Z_0^x)\nu(dx), & \text{if } H > \dfrac{1}{2}, \end{cases}$$

where μ, ν are measures on $(0, \infty)$ given by

$$\mu(dx) := \frac{dx}{x^{\frac{1}{2}+H}\Gamma(H+\frac{1}{2})\Gamma(\frac{1}{2}-H)}, \qquad \nu(dx) := \frac{dx}{x^{H-\frac{1}{2}}\Gamma(\frac{1}{2}+H)\Gamma(\frac{3}{2}-H)},$$

$(Y_0, Z_0) := (Y_\infty, Z_\infty)$ are random variables with values in $L^1(\mu_\infty) \times L^1(\nu_\infty)$, and $(Y_\infty, Z_\infty), (\mu_\infty, \nu_\infty)$ are given by Equations (2.11) and (2.12).

Remark 3.4. *The measures μ, ν in the above representation of fBM satisfy Assumptions 2.3 and 2.7 with $p(x) = x$, but not Assumption 2.11. It follows by Theorem 2.9 that (Y, Z) has continuous sample paths in $L^1(\mu_\infty) \times L^1(\nu_\infty)$, but not necessarily in $L^1(\mu) \times L^1(\nu)$. Nevertheless, $(Y - Y_0, Z - Z_0)$ has continuous sample paths in $L^1(\mu) \times L^1(\nu)$, as shown in the proof of Theorem 3.3.*

Proof of Theorem 3.3 for $H < \frac{1}{2}$. The function $\tau \mapsto \tau^{H-\frac{1}{2}}/\Gamma(H + \frac{1}{2})$ on $(0, \infty)$ appearing in the definition of W^H is the Laplace transform of μ, i.e., for each $\tau > 0$ and $H < \frac{1}{2}$

$$\mathcal{L}(\mu)(\tau) = \int_0^\infty e^{-\tau x}\mu(dx) = \frac{\tau^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}.$$

Therefore,

$$\begin{aligned} W_t^H = w_0^H &+ \int_{-\infty}^0 \int_0^\infty \left(e^{-x(t-s)} - e^{-x(-s)}\right) \mu(dx) dW_s \\ &+ \int_0^t \int_0^\infty e^{-x(t-s)}\mu(dx) dW_s. \end{aligned}$$

By the stochastic Fubini's theorem A.1,

$$\begin{aligned} W_t^H = w_0^H &+ \int_0^\infty \int_{-\infty}^0 \left(e^{-x(t-s)} - e^{-x(-s)}\right) dW_s \mu(dx) \\ &+ \int_0^\infty \int_0^t e^{-x(t-s)} dW_s \mu(dx). \end{aligned}$$

Condition (A.2) of Fubini's theorem is satisfied because

$$\begin{aligned}
&\int_0^\infty \sqrt{\int_{-\infty}^0 \left(e^{-x(t-s)} - e^{-x(-s)}\right)^2 ds}\mu(dx) = \int_0^\infty \frac{1-e^{-tx}}{\sqrt{2x}}\mu(dx) \\
&\quad \leq \int_0^\infty \sqrt{\frac{1-e^{-tx}}{x}}\mu(dx) < \infty, \\
&\int_0^\infty \sqrt{\int_0^t e^{-2x(t-s)}}\mu(dx) \leq \int_0^\infty \sqrt{\frac{1-e^{-2tx}}{x}}\mu(dx) < \infty,
\end{aligned}$$

where we use $1-e^{-tx} \leq \sqrt{1-e^{-tx}}$ and Equation (C.11). By the definition of Y_t^x,

$$\begin{aligned}
W_t^H &= w_0^H + \int_0^\infty \left(e^{-xt}-1\right) Y_0^x \mu(dx) + \int_0^\infty \int_0^t e^{-x(t-s)} dW_s \mu(dx) \\
&= w_0^H + \int_0^\infty (Y_t^x - Y_0^x)\mu(dx).
\end{aligned}$$

The expressions

$$\left(e^{-xt}-1\right) Y_0^x \quad \text{and} \quad \int_0^t e^{-x(t-s)} dW_s$$

define continuous $L^1(\mu)$-valued processes: the first expression has majorant $(1 \vee t)(1 \wedge x)Y_0^x$ in $L^1(\mu)$, which allows one to apply the dominated convergence theorem, and the second expression is treated in Theorem 2.9. □

Proof of Theorem 3.3 for $H > \frac{1}{2}$. As the function $\tau^{H-\frac{3}{2}}/\Gamma(H+\frac{1}{2})$ is the Laplace transform of the measure ν, the relation

$$\tau \mathcal{L}(\nu)(\tau) = \tau \int_0^\infty e^{-x\tau}\nu(dx) = \frac{\tau^{H-\frac{1}{2}}}{\Gamma(H+\frac{1}{2})},$$

holds for each $\tau > 0$ and $H \in (\frac{1}{2}, 1)$. Therefore,

$$\begin{aligned}
W_t^H = w_0^H &+ \int_{-\infty}^0 \int_0^\infty \left((t-s)e^{-x(t-s)} + se^{xs}\right) \nu(dx) dW_s \\
&+ \int_0^t \int_0^\infty (t-s)e^{-x(t-s)}\nu(dx)dW_s.
\end{aligned}$$

By the stochastic Fubini's theorem A.1,

$$\begin{aligned}
W_t^H = w_0^H &+ \int_0^\infty \int_{-\infty}^0 \left((t-s)e^{-x(t-s)} + se^{xs}\right) dW_s \nu(dx) \\
&+ \int_0^\infty \int_0^t (t-s)e^{-x(t-s)} dW_s \nu(dx).
\end{aligned} \tag{3.1}$$

Condition (A.2) of Fubini's theorem is satisfied because

$$\begin{aligned}
&\int_0^\infty \sqrt{\int_{-\infty}^0 \left((t-s)e^{-x(t-s)} + se^{xs}\right)^2 ds\nu(dx)} \\
&\quad = \int_0^\infty \sqrt{\frac{1-2e^{-tx}(tx+1)+2txe^{-2tx}(tx+1)+e^{-2tx}}{4x^3}}\nu(dx) \\
&\quad \leq \int_0^{1/t} \sqrt{\frac{t^2}{6x}}\nu(dx) + \int_{1/t}^\infty \sqrt{\frac{2}{x^3}}\nu(dx) \\
&\quad \leq \sqrt{2}(t\vee 1)\int_0^\infty (x^{-\frac{1}{2}} \wedge x^{-\frac{3}{2}})\nu(dx) < \infty, \\
&\int_0^\infty \sqrt{\int_0^t (t-s)^2 e^{-2x(t-s)}\nu(dx)} = \int_0^\infty \sqrt{\frac{1-e^{-2tx}\left(1+2tx+2t^2x^2\right)}{4x^3}}\nu(dx) < \infty,
\end{aligned}$$

where we used Equations (C.12) and (C.13). Using the definition of (Y^x, Z^x) in Equation (2.1), Equation (3.1) can be expressed as

$$\begin{aligned}
W_t^H &= w_0^H + \int_0^\infty \int_{-\infty}^0 e^{xs}\left(te^{-xt} + s(1-e^{-xt})\right) dW_s\nu(dx) \\
&\quad + \int_0^\infty \int_0^t (t-s)e^{-x(t-s)} dW_s\nu(dx) \\
&= w_0^H + \int_0^\infty \left(te^{-xt}\int_{-\infty}^0 e^{xs}dW_s + (1-e^{-xt})\int_{-\infty}^0 se^{xs}dW_s\right)\nu(dx) \\
&\quad + \int_0^\infty \left(Z_t^x - Z_0^x e^{-xt} - Y_0^x te^{-xt}\right)\nu(dx) \\
&= w_0^H + \int_0^\infty \left(Z_t^x - Z_0^x\right)\nu(dx).
\end{aligned}$$

By Lemma D.1, $Z_t^x - Z_0^x$ can be written as the sum of the following expressions:

$$Z_0^x(e^{-tx}-1), \qquad Y_0^x te^{-tx}, \qquad \int_0^t (t-s)e^{-(t-s)x}dW_s.$$

All three expressions define continuous $L^1(\nu)$-valued processes: the first and second expression have $|Z_0^x|(1\vee t)(1\wedge x)$ and $|Y_0^x|(1\vee t)(1\wedge x^{-1})$ as majorants in $L^1(\nu)$, which allows one to apply the dominated convergence theorem, and the third expression is treated in Theorem 2.9. □

4. Applications to interest rate modeling

In this section we construct two interest rate models: one with fractional short rate and another one with fractional bank account process. In both models, the affine structure gives rise to explicit formulas for zero-coupon bond (ZCB) prices, forward rates, and calls and puts on ZCB's.

4.1. **Essentials of interest rate modeling.** We refer to [5] for further reference.

Definition 4.1. The bank account is given by a positive process $B = (B_t)_{t\geq 0}$ such that B_t^{-1} is integrable for all $t \geq 0$. Zero-coupon bond (ZCB) prices are given by

$$P(t,T) = \mathbb{E}\left[\frac{B_t}{B_T}\middle|\mathcal{F}_t\right], \quad T \geq t \geq 0,$$

and the (instantaneous) forward rates are given by

$$h(t)(\tau) = -\partial_T \log P(t,T)|_{T=t+\tau}, \quad t,\tau \geq 0.$$

Remark 4.2. *Note that for each $T > 0$ the process $B^{-1}P(\cdot,T)$ is by definition a martingale. This means that $\mathbb{Q}$ is an equivalent martingale measure by construction, and that the model is free of arbitrage.*

Definition 4.3. The prices at time $t \geq 0$ of European call and put options with expiry date $T < S$ and strike K on the ZCB with maturity date $S > T$ are given by

$$\pi_t^{\mathrm{Call}}(T,S,K) = \mathbb{E}\left[\frac{B_0}{B_T}\big(P(T,S)-K\big)^+\middle|\mathcal{F}_t\right],$$
$$\pi_t^{\mathrm{Put}}(T,S,K) = \mathbb{E}\left[\frac{B_0}{B_T}\big(K-P(T,S)\big)^+\middle|\mathcal{F}_t\right].$$

Definition 4.4. Consider interest rate cap and floor with maturity T_n, strike rate κ and payment dates $0 < T_0 < T_1 < \ldots < T_n$ where $T_i - T_{i-1} = \Delta$. At time $t < T_0$ the cap and floor prices are given

$$\mathrm{Cp}_t = \sum_{i=1}^n \mathrm{Cpl}_t(T_{i-1},T_i,\kappa), \quad \text{and} \quad \mathrm{Fl}_t = \sum_{i=1}^n \mathrm{Fll}_t(T_{i-1},T_i,\kappa),$$

where

$$\mathrm{Cpl}_t(T_{i-1},T_i,\kappa) = (1+\Delta\kappa)\pi_t^{\mathrm{Put}}\left(T_{i-1},T_i,(1+\Delta\kappa)^{-1}\right),$$
$$\mathrm{Fll}_t(T_{i-1},T_i,\kappa) = (1+\Delta\kappa)\pi_t^{\mathrm{Call}}\left(T_{i-1},T_i,(1+\Delta\kappa)^{-1}\right).$$

In order to calculate prices of call and put options on ZCB's it is convenient to consider forward measure changes.

Definition 4.5. For $0 \leq t \leq T$ define the T-forward measure $\mathbb{Q}^T$ by the following Radon-Nikodým derivative

$$\xi(t,T) = \left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_{\mathcal{F}_t} = \mathbb{E}\left[\frac{B_0}{P(0,T)B_T}\middle|\mathcal{F}_t\right] = \frac{B_t^{-1}P(t,T)}{B_0^{-1}P(0,T)},$$

where the last equality follows directly from Definition 4.1.

The following property is useful for computations. The symbol $\mathcal{E}$ denotes the stochastic exponential, see e.g. [5, Section 4.1].

Theorem 4.6. *Assume there exists a process $v(\cdot,T)$ such that*

$$\xi(t,T) = \mathcal{E}\left(\int_0^\cdot v(s,T)dW_s\right)_t, \quad 0 \leq t \leq T. \tag{4.1}$$

Then, for any $S,T > 0$ the process $W^T = W - \int_0^\cdot v(s,T)ds$ is $\mathbb{Q}^T$-Brownian motion and the price process of the ZCB with maturity date S discounted by the ZCB with maturity T

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)}\mathcal{E}\left(\int_0^\cdot \big(v(s,S)-v(s,T)\big)dW_s^T\right)_t, \quad t \in [0,S\wedge T],$$

is a $\mathbb{Q}^T$-martingale. Moreover, assuming that $v(\cdot, T)$ is deterministic, call and put option prices are given by the following version of the Black-Scholes formula

$$\pi_t^{\mathrm{Call}} = P(t,S)\Phi_0^{\mathrm{Gauss}}(d_1) - KP(t,T)\Phi_0^{\mathrm{Gauss}}(d_2),$$
$$\pi_t^{\mathrm{Put}} = KP(t,T)\Phi_0^{\mathrm{Gauss}}(-d_2) - P(t,S)\Phi_0^{\mathrm{Gauss}}(-d_1),$$

where Φ_0^{Gauss} is the standard Gaussian cumulative distribution function and

$$d_{1,2} = \frac{\log\left(\frac{P(t,S)}{KP(t,T)}\right) \pm \frac{1}{2}\int_t^T \big(v(\cdot,S) - v(\cdot,T)\big)^2 ds}{\sqrt{\int_t^T \big(v(\cdot,S) - v(\cdot,T)\big)^2 ds}}.$$

Proof. The derivation of [5, Section 7] can also be used in this setting because the discounted ZCB price process $(B_t P(t,T))_{t\geq 0}$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale by construction. □

4.2. **Fractional short rate process.** In this section, we construct an interest rate model with a fractional short rate. To this aim, we fix measures μ, ν on $(0,\infty)$ satisfying the following slightly strengthened version of Assumption 2.3.

Assumption 4.7. *μ and ν are sigma-finite measures on $(0,\infty)$. The measure ν has a density p with respect to μ, and there exists $\alpha \in (0,2)$ such that for each $t > 0$,*

$$\int_0^\infty (1 \wedge x^{-\frac{1}{2}})\mu(dx) < \infty, \qquad \int_0^\infty (1 \wedge x^{-\frac{3}{2}})\nu(dx) < \infty, \qquad \sup_{x\in(0,\infty)} p(x)(1 \wedge x^{-\alpha}) < \infty.$$

Moreover, we fix $(u,v) \in L^\infty(\mu) \times L^\infty(\nu)$, $\ell \in \mathbb{R}$, and an initial value $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ for the process (Y,Z) defined in Section 2. Given these model parameters, we define the short rate and bank account as

$$r_t = \ell + \langle Y_t, u\rangle_\mu + \langle Z_t, v\rangle_\nu, \qquad B_t = \exp\left(\int_0^t r_s ds\right). \tag{4.2}$$

Example 4.8. Set $u = v = 1$ and consider measures of the form $\mu(dx) \propto x^{-\alpha}dx$ and $\nu(dx) \propto x^{\frac{1}{2}-\alpha}$ for $\alpha \in (\frac{1}{2}, 1)$. Then Assumption 4.7 is satisfied. The process (Y,Z) takes values in $L^1(\mu) \times L^1(\nu)$ and has continuous sample paths by Theorem 2.9. Therefore, it can be used to construct fractional processes as in Section 3. In particular, $\langle Y, u\rangle_\mu$ is a fractional process of the same roughness as fBM with Hurst index $H = \alpha - \frac{1}{2} \in (0, \frac{1}{2})$, and $\langle Z, v\rangle_\nu$ has the same roughness as fBM with $H = \alpha \in (\frac{1}{2}, 1)$. None of the two processes are semimartingales.

Remark 4.9. *While the short rate may take negative values, the probability of yields becoming negative can be reduced by shifting the parameter ℓ and scaling the parameters u, v. Often times, either u or v will be set to zero, unless one is interested in mixing processes with long and short range dependence.*

Theorem 4.10. *In the fractional short rate model (4.2), ZCB prices and forward rates are given by*

$$P(t,T) = e^{-\ell(T-t)+\Phi_0(T-t,u,v)+\langle Y_t, \Phi_1(T-t,u,v)\rangle_\mu + \langle Z_t, \Phi_2(T-t,u,v)\rangle_\nu}, \qquad 0 \leq t \leq T,$$
$$h(t)(\tau) = \ell - \partial_\tau\Phi_0(\tau,u,v) - \langle Y_t, \partial_\tau\Phi_1(\tau,u,v)\rangle_\mu - \langle Z_t, \partial_\tau\Phi_2(\tau,u,v)\rangle_\nu, \quad t,\tau \geq 0,$$

where for each $\tau \geq 0$ and $x \in (0,\infty)$

$$\Phi_0(\tau,u,v) = \frac{1}{2}\int_0^\tau \langle \Phi_1(s,u,v),1\rangle_\mu^2 ds,$$
$$\Phi_1(\tau,u,v)(x) = \frac{e^{-\tau x}-1}{x}u(x) + \left(\frac{e^{-\tau x}-1}{x^2} + \frac{\tau}{x}e^{-\tau x}\right)p(x)v(x),$$
$$\Phi_2(\tau,u,v)(x) = \frac{e^{-\tau x}-1}{x}v(x).$$

Proof. Lemma E.2 implies that the random variable $\int_t^T \Big(\langle Y_s,u\rangle_\mu + \langle Z_s,v\rangle_\nu\Big)\,ds$ is Gaussian, given $\mathcal{F}_t$, with mean

$$-\langle Y_t,\Phi_1(T-t,u,v)\rangle_\mu - \langle Z_t,\Phi_2(T-t,u,v)\rangle_\nu$$

and variance $2\Phi_0(T-t,u,v)$. Thus, the formula for ZCB prices follows from the formula of the moment generating function of the normal distribution. The expression for the forward rates follows by differentiation with respect to the time to maturity. □

Remark 4.11. *The functions Φ_0,Φ_1,Φ_2 are the unique solution of the Riccati equations*

$$\begin{aligned}
&\partial_\tau\Phi_0(\tau,u,v) = R_0\big(\Phi_1(\tau,u,v),\Phi_2(\tau,u,v)\big), && \Phi_0(0,u,v)=0,\\
&\partial_\tau\Phi_1(\tau,u,v) = R_1\big(\Phi_1(\tau,u,v),\Phi_2(\tau,u,v)\big) - u, && \Phi_1(0,u,v)=0,\\
&\partial_\tau\Phi_2(\tau,u,v) = R_2\big(\Phi_1(\tau,u,v),\Phi_2(\tau,u,v)\big) - v, && \Phi_2(0,u,v)=0,
\end{aligned}\tag{4.3}$$

with R_0,R_1,R_2 as in Lemma 2.6. This can be verified as in the proof of Lemma 2.6.

Theorem 4.12 (HJM equation)**.** *In the fractional short rate model* (4.2) *bond prices $(P(t,T))_{0\leq t\leq T}$ and forward rates $(h(t)(\tau))_{t\geq 0}$ are semimartingales for each fixed $T,\tau>0$. The forward rate process $h=(h(t)(\cdot))_{t\geq0}$ is a solution of the HJM equation*

$$dh(t) = \big(\mathcal{A}h(t) + \mu^{\mathrm{HJM}}\big)\,dt + \sigma^{\mathrm{HJM}}dW_t,\tag{4.4}$$

where $\mathcal{A}$ denotes differentiation with respect to time to maturity τ and $\mu^{\mathrm{HJM}},\sigma^{\mathrm{HJM}}$ are measurable functions on $(0,\infty)$ given by

$$\mu^{\mathrm{HJM}}(\tau) = \partial_\tau^2\Phi_0(\tau,u,v),\quad \sigma^{\mathrm{HJM}}(\tau) = -\langle\partial_\tau\Phi_1(\tau,u,v),1\rangle_\mu.$$

Proof. The semimartingale property of prices and forward rates follows from Lemmas E.3 and E.4, which are based on Theorem 2.10. The semimartingale decomposition of $h(\cdot)(\tau)$ is obtained by collecting the terms in Equation (2.10):

$$\begin{aligned}
dh(t)(\tau) &= -d\,\langle Y_t,\partial_\tau\Phi_1(\tau,u,v)\rangle_\mu - d\,\langle Z_t,\partial_\tau\Phi_2(\tau,u,v)\rangle_\nu\\
&= \left(\langle Y_t, x\partial_\tau\Phi_1(\tau,u,v) - \partial_\tau\Phi_2(\tau,u,v)p\rangle_\mu + \langle Z_t, x\partial_\tau\Phi_2(\tau,u,v)\rangle_\nu\right)dt\\
&\quad - \langle\partial_\tau\Phi_1(\tau,u,v),1\rangle_\mu dW_t.
\end{aligned}$$

Note that by abuse of notation, we wrote $x\partial_\tau\Psi_i(\tau,u,v)$ to designate the function $x\mapsto\partial_\tau\Psi_i(\tau,u,v)(x)$ for $i=1,2$. The second derivatives of Ψ_i are

$$\begin{aligned}
\partial_\tau^2\Phi_1(\tau,u,v) &= -x\partial_\tau\Phi_1(\tau,u,v) + \partial_\tau\Phi_2(\tau,u,v)p,\\
\partial_\tau^2\Phi_2(\tau,u,v) &= -x\partial_\tau\Phi_2(\tau,u,v).
\end{aligned}$$

Therefore, we have for all $t \geq 0$ and $\tau > 0$

$$\begin{aligned}\mathcal{A}h(t)(\tau) = -\partial_\tau^2 \Phi_0(\tau, u, v) + \langle Y_t, x\partial_\tau \Phi_1(\tau, u, v) - \partial_\tau \Phi_2(\tau, u, v) p\rangle_\mu \\ + \langle Z_t, x\partial_\tau \Phi_2(\tau, u, v)\rangle_\nu .\end{aligned}$$

It follows that

$$dh(t)(\tau) = \big(\mathcal{A}h(t)(\tau) + \partial_\tau^2 \Phi_0(\tau, u, v)\big)\, dt - \langle \partial_\tau \Phi_1(\tau, u, v), 1\rangle_\mu dW_t,$$

which allows one to identify μ^{HJM} and σ^{HJM}. □

Remark 4.13. *The HJM drift condition is satisfied because*

$$\begin{aligned}\mu^{\mathrm{HJM}} &= \partial_\tau^2 \Phi_0(\tau, u, v) = \langle \partial_\tau \Phi_1(\tau, u, v), 1\rangle_\mu \langle \Phi_1(\tau, u, v), 1\rangle_\mu \\ &= \sigma^{\mathrm{HJM}}(\tau) \int_0^\tau \sigma^{\mathrm{HJM}}(s) ds.\end{aligned}$$

To show that the Black-Scholes formula of Theorem 4.6 holds, we verify that the T-forward density process is a stochastic exponential of a deterministic function $v(\cdot, T)$.

Corollary 4.14. *For $0 \leq t \leq T$ the density process $\xi(t, T)$ takes the form* (4.1) *with deterministic $v(t, T) = \langle \Phi_1(T - t, u, v), 1\rangle_\mu$.*

Proof. In Lemma E.3 we verified that the expressions $\langle Y, \Phi_1(T - \cdot, u, v)\rangle_\mu$ and $\langle Z, \Phi_2(T - \cdot, u, v)\rangle_\nu$ are semimartingales. Their semimartingale decompositions are given by Equation (2.10):

$$\begin{aligned}d\,\langle Y_t, \Phi_1(T - t, u, v)\rangle_\mu &= \int_0^\infty \left(u(x) - \frac{e^{-(T-t)x} - 1}{x} p(x) v(x) \right) Y_t^x \mu(dx) dt \\ &\quad + \langle \Phi_1(T - t, u, v), 1\rangle_\mu dW_t, \\ d\,\langle Z_t, \Phi_2(T - t, u, v)\rangle_\nu &= \Big(\langle v, Z_t\rangle_\nu + \langle \Phi_2(T - t, u, v), Y_t\rangle_\nu \Big) dt.\end{aligned}$$

By the formula for bond prices in Theorem 4.10, $\log(\xi(t, T))$ satisfies

$$\begin{aligned}d(\log \xi(t, T)) = \Big(- \langle Y_t, u\rangle_\mu - \langle Z_t, v\rangle_\nu - \partial_\tau \Phi_0(T - t, u, v) \Big)\, dt \\ + d\,\langle Y_t, \Phi_1(T - t, u, v)\rangle_\mu + d\,\langle Z_t, \Phi_2(T - t, u, v)\rangle_\nu .\end{aligned}$$

Applying Itō's formula and canceling out terms yields

$$\begin{aligned}d\xi(t, T) &= \xi(t, T) \left(d(\log \xi(t, T)) + \frac{1}{2} d\,[\log \xi(\cdot, T)]_t \right) \\ &= \xi(t, T)\, \langle \Phi_1(T - t, u, v), 1\rangle_\mu\, dW_t,\end{aligned}$$

which implies that ξ is a stochastic exponential of the form (4.1) with $v(t, T) = \langle \Phi_1(T - t, u, v), 1\rangle_\mu$. □

We show in the following corollary that the (signed) measures $u\mu$ and $v\nu$ can be estimated from quadratic variations of forward rates. This estimation can be performed directly under real world observations since the martingale part of $h(\cdot)(\tau)$ does not change under Girsanov's change of measure.

Corollary 4.15. *For each $\tau_1, \tau_2 > 0$ the following relation holds:*

$$d[h(\cdot)(\tau_1), h(\cdot)(\tau_2)]_t = \langle \partial_\tau \Phi_1(\tau_1, u, v), 1\rangle_\mu \langle \partial_\tau \Phi_1(\tau_2, u, v), 1\rangle_\mu dt. \tag{4.5}$$

In particular, Equation (4.5) can be written for $u \geq 0$ and $v = 0$ as

$$(u\mu)(dx) = \mathcal{L}^{-1}\left(\sqrt{\frac{d}{dt}[h(\cdot, \cdot + \tau)]_t}\right)(dx),$$

and for $u = 0$ and $v \geq 0$ as

$$(v\nu)(dx) = \mathcal{L}^{-1}\left(\tau^{-1}\sqrt{\frac{d}{dt}[h(\cdot, \cdot + \tau)]_t}\right)(dx),$$

where the inverse Laplace transform $\mathcal{L}^{-1}$ is calculated with respect to the variable τ.

Proof. Equation (4.5) follows directly from Theorem 4.12. For $u \geq 0$ and $v = 0$ we have

$$|\langle \partial_\tau \Phi_1(\tau, u, 0), 1\rangle_\mu| = \int_0^\infty e^{-x\tau} u(x)\mu(dx) = \mathcal{L}(u\mu)(\tau),$$

and for $u = 0$ and $v \geq 0$ we have

$$|\langle \partial_\tau \Phi_1(\tau, u, 0), 1\rangle_\mu| = \tau \int_0^\infty e^{-x\tau} v(x)\nu(dx) = \tau\mathcal{L}(v\nu)(\tau).$$

Hence, the statements about the measures $u\mu$ and $v\nu$ follow. □

Remark 4.16. *We summarize the results of Section 4.2. We considered a model with fractional short rate process constructed as a superposition of infinitely many OU processes. We derived closed-form expressions for ZCB prices and forward rates. Bond prices and forward rates are semimartingales, and HJM equation* (4.4) *holds. It follows that prices of interest rate derivatives can be calculated as in the standard HJM framework (see e.g. [5, Section 6 and 7]), even though the short rate is not a semimartingale. The model parameters $u\mu$ and $v\nu$ can be estimated from quadratic variations of forward rates by an inverse Laplace transform.*

4.3. **Fractional bank account process.** In this section, we construct an interest rate model with a fractional bank account process. To this aim, we fix measures μ, ν on $(0, \infty)$ satisfying Assumption 2.3. Moreover, we fix $(u, v) \in L^\infty(\mu) \times L^\infty(\nu)$, $\ell \in \mathbb{R}$, and an initial value $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ for the process (Y, Z) defined in Section 2. Given these model parameters, we define the bank account process as

$$B_t = e^{\ell t + \langle Y_t, u\rangle_\mu + \langle Z_t, v\rangle_\nu}. \tag{4.6}$$

Theorem 4.17. *In the fractional bank account model* (4.6)*, ZCB prices and forward rates are given by*

$$P(t, T) = e^{-\ell(T-t) + \phi_0(T-t, -u, -v) + \langle Y_t, \phi_1(T-t, -u, -v) + u\rangle_\mu + \langle Z_t, \phi_2(T-t, -u, -v) + v\rangle_\nu},$$

$$h(t)(\tau) = \ell - \partial_\tau \phi_0(\tau, -u, -v) - \langle Y_t, \partial_\tau \phi_1(\tau, -u, -v)\rangle_\mu - \langle Z_t, \partial_\tau \phi_2(\tau, -u, -v)\rangle_\nu$$

for each $0 \leq t \leq T$ and $\tau > 0$, where ϕ_0, ϕ_1, and ϕ_2 are given by Theorem 2.5.

Proof. The formula for the ZCB prices follows directly from Theorem 2.5 and Equation (4.6), and the formula for the forward rates follows by definition. □

Theorem 4.18 (HJM equation)**.** *Discounted bond prices $(B_t^{-1}P(t,T))_{t\geq 0}$ and forward rates $(h(t)(\tau))_{t\geq 0}$ are semimartingales for each $T,\tau>0$. The forward rates solve HJM equation (4.4) with μ^{HJM} and σ^{HJM} given by*

$$\mu^{\mathrm{HJM}}(\tau)=\partial_\tau^2\phi_0(\tau,-u,-v),\quad \sigma^{\mathrm{HJM}}(\tau)=-\left\langle \partial_\tau\phi_1(\tau,-u,-v),1\right\rangle_\mu .$$

Proof. Discounted bond prices are martingales by definition. Forward rates are semimartingales because $\left\langle Y,\partial_\tau\phi_1(\tau,-u,-v)\right\rangle_\mu$ and $\left\langle Z,\partial_\tau\phi_2(\tau,-u,-v)\right\rangle_\nu$ are semimartingales by Lemma F.1. The semimartingale decomposition of the forward rate process is given by Equation (2.10) and reads as

$$\begin{aligned} dh(t)(\tau) &= -d\left\langle Y_t,\partial_\tau\phi_1(\tau,-u,-v)\right\rangle_\mu - d\left\langle Z_t,\partial_\tau\phi_2(\tau,-u,-v)\right\rangle_\nu \\ &= \langle Y_t, x\partial_\tau\phi_1(\tau,-u,-v)-\partial_\tau\phi_2(\tau,-u,-v)p\rangle_\mu dt \\ &\quad + \langle Z_t, x\partial_\tau\phi_2(\tau,-u,-v)\rangle_\nu dt - \langle\partial_\tau\phi_1(\tau,-u,-v),1\rangle_\mu dW_t, \end{aligned}$$

where by abuse of notation we wrote $x\partial_\tau\phi_1(\tau,-u,-v)$ to designate the function $x\mapsto x\partial_\tau\phi_1(\tau,-u,-v)(x)$. The second derivatives of ϕ_1,ϕ_2 are

$$\begin{aligned} \partial_\tau^2\phi_1(\tau,-u,-v) &= -x\partial_\tau\phi_1(\tau,-u,-v)+p\partial_\tau\phi_2(\tau,-u,-v), \\ \partial_\tau^2\phi_2(\tau,-u,-v) &= -x\partial_\tau\phi_2(\tau,-u,-v). \end{aligned}$$

Hence, for all $t\geq 0$ and for all $\tau>0$ we have

$$\begin{aligned} \mathcal{A}h(t)(\tau) = &-\partial_\tau^2\phi_0(\tau,-u,-v)+\left\langle Y_t,x\partial_\tau\phi_1(\tau,-u,-v)\right\rangle_\mu+\left\langle Z_t,x\partial_\tau\phi_2(\tau,-u,-v)\right\rangle_\nu \\ &-\left\langle Y_t,\partial_\tau\phi_2(\tau,-u,-v)\right\rangle_\nu . \end{aligned}$$

Therefore, the semimartingale decomposition of $h(\cdot)(\tau)$ can be written as

$$dh(t)(\tau)=\big(\mathcal{A}h(t)(\tau)+\partial_\tau^2\phi_0(\tau,-u,-v)\big)\,dt-\left\langle\partial_\tau\phi_1(\tau,-u,-v),1\right\rangle_\mu dW_t,$$

which allows one to identify μ^{HJM} and σ^{HJM}. □

Remark 4.19. *The HJM drift condition is satisfied:*

$$\begin{aligned} \mu^{\mathrm{HJM}}(\tau) &= \partial_\tau^2\phi_0(\tau,-u,-v)=\left\langle\partial_\tau\phi_1(\tau,-u,-v),1\right\rangle_\mu\left\langle\phi_1(\tau,-u,-v),1\right\rangle_\mu \\ &= \sigma^{\mathrm{HJM}}(\tau)\int_0^\tau \sigma^{\mathrm{HJM}}(s)ds. \end{aligned}$$

Remark 4.20. *In contrast to discounted bond prices and forward rates, undiscounted bond prices $(P(t,T))_{0\leq t\leq T}$ are not semimartingales, in general. For example, they are fractional processes if μ,ν are chosen as in Section 3 and $u=v=1$.*

To show that the Black-Scholes formula of Theorem 4.6 holds, we verify that the T-forward density process is a stochastic exponential of a deterministic function $v(\cdot,T)$.

Corollary 4.21. *For $0\leq t\leq T$ the density process $\xi(t,T)$ takes the form* (4.1) *with deterministic $v(t,T)=\left\langle\phi_1(\tau,-u,-v),1\right\rangle_\mu$.*

Proof. By Theorem 4.17, the density process $\xi(t,T)$ can be expressed equivalently as

$$\begin{aligned} d(\log\xi(t,T)) = &-\partial_\tau\phi_0(T-t,-u,-v)+d\left\langle Y_t,\phi_1(T-t,-u,-v)\right\rangle_\mu \\ &+d\left\langle Z_t,\phi_2(T-t,-u,-v)\right\rangle_\nu . \end{aligned}$$

The processes $(\langle Y_t, \phi_1(T-t,-u,-v)\rangle_\mu)_{t\geq 0}$ and $(\langle Z_t, \phi_2(T-t,-u,-v)\rangle_\nu)_{t\geq 0}$ are semimartingales with decompositions given by

$$\begin{aligned} d\left\langle Y_t, \phi_1(T-t,-1,-1)\right\rangle_\mu &= -\left\langle Y_t, \phi_2(T-t,-u,-v)\right\rangle_\nu dt \\ &\quad + \left\langle \phi_1(T-t,-u,-v), 1\right\rangle_\mu dW_t, \\ d\left\langle Z_t, \phi_2(T-t,-u,-v)\right\rangle_\nu &= \left\langle Y_t, \phi_2(T-t,-u,-v)\right\rangle_\nu dt. \end{aligned}$$

By Itō's formula, using ODE (2.5) for ϕ_0, one obtains

$$\begin{aligned} d\xi(t,T) &= \xi(t,T)\left(d(\log \xi(t,T)) + \frac{1}{2} d\left[\log \xi(\cdot,T)\right]_t\right) \\ &= \xi(t,T)\left\langle \phi_1(T-t,-u,-v), 1\right\rangle_\mu dW_t. \end{aligned}$$

□

We show in the following corollary that the (signed) measures $u\mu$ and $v\nu$ can be estimated from quadratic variations of forward rates. This estimation can be performed directly under real world observations since the martingale part of $h(\cdot)(\tau)$ does not change under Girsanov's change of measure.

Corollary 4.22. *For fixed $\tau_1, \tau_2 > 0$*

$$d[h(\cdot)(\tau_1), h(\cdot)(\tau_2)]_t = \left\langle \partial_\tau \phi_1(\tau_1,-u,-v), 1\right\rangle_\mu \left\langle \partial_\tau \phi_1(\tau_2,-u,-v), 1\right\rangle_\mu dt. \tag{4.7}$$

In particular, Equation (4.7) can be written for $u \geq 0$ and $v = 0$ as

$$(u\mu)(dx) = x^{-1}\mathcal{L}^{-1}\left(\sqrt{\frac{d}{dt}[h(\cdot,\cdot+\tau)]_t}\right)(dx),$$

where the inverse Laplace transform $\mathcal{L}^{-1}$ is calculated with respect to τ.
In the case $u = 0$ and $v \geq 0$ we assume that $v(x)\nu(dx) = f_v(x)dx$ for a non-negative decreasing differentiable function f_v on $(0,\infty)$ satisfying $xf_v(x) \to 0$ as $x \to 0^+$. Then Equation (4.7) is equivalent to the following ODE

$$xf_v'(x) = -\mathcal{L}^{-1}\left(\sqrt{\frac{d}{dt}[h(\cdot,\cdot+\tau)]_t}\right)(x).$$

Remark 4.23. *If $v = 1$ and $\nu(dx) \propto x^{-\alpha}$ for $\alpha \in (0, 1/2)$, then the conditions on the density f_v in Corollary 4.22 are satisfied.*

Proof. Equation (4.7) follows directly from Theorem 4.18. Let $\widetilde{\mu}(dx) = xu(x)\mu(dx)$. For $u \geq 0$ and $v = 0$ the first equation follows from

$$\sqrt{\frac{d}{dt}[h(\cdot,\cdot+\tau)]_t} = \int_0^\infty e^{-\tau x} xu(x)\mu(dx) = \mathcal{L}(\widetilde{\mu})(\tau).$$

For $u = 0$ and $v \geq 0$ the second equation follows from partial integration using the assumptions on f_v:

$$\begin{aligned} \sqrt{\frac{d}{dt}[h(\cdot,\cdot+\tau)]_t} &= \left|\int_0^\infty e^{-\tau x} v(x)(1-\tau x)\nu(dx)\right| = \left|\mathcal{L}(f_v)(\tau) - \tau\mathcal{L}(x \mapsto xf_v(x))(\tau)\right| \\ &= \left|\mathcal{L}(f_v)(\tau) - \mathcal{L}(x \mapsto \partial_x(xf_v(x))(\tau)\right| = \left|\mathcal{L}\left(x \mapsto xf_v'\right)(\tau)\right| \\ &= -\mathcal{L}\left(x \mapsto xf_v'\right)(\tau). \end{aligned}$$

□

Remark 4.24. *We summarize the results of Section 4.3. We defined an interest rate model where the logarithmic bank account is a fractional process constructed as a superposition of infinitely many OU process. We derived closed-form expressions for ZCB prices and forward rates. While ZCB prices are typically not semimartingales, discounted ZCB prices and forward rates are. The forward rates satisfy HJM equation* (4.4)*, which allows one to apply standard techniques for calculating derivatives' prices. Under some mild conditions, the model parameters $u\mu$ and $v\mu$ can be estimated from quadratic variations of forward rates.*

5. Fractional Stein & Stein model

In this section we generalize an affine asset price model by Stein and Stein [15] to fractional volatility. In the original model, the volatility process is a single OU process. In our model, it is a fractional process constructed as a superposition of infinitely many OU processes. In accordance with empirical facts about realized volatility [6] we restrict ourselves to fractional processes with roughness and dependence structure similar to fBM of Hurst index $H \in (0, 1/2)$.

5.1. **Setup and notation.** Let $\widetilde{W}$ be $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion with correlation $d\langle W, \widetilde{W}\rangle_t = \rho dt$ for some $\rho \in (-1, 1)$. We fix a measure μ on $(0, \infty)$ satisfying Assumption 2.3, a function $u \in L^\infty(\mu)$, and an initial value $Y_0 \in L^1(\mu)$ for the process Y defined in Section 2. Given these model parameters, the price process $S = (S_t)_{t\geq 0}$ is defined by the SDE

$$dS_t = S_t \langle Y_t, u\rangle_\mu d\widetilde{W}_t.$$

To bring this equation in affine form, we introduce for each $t \geq 0$ the tensor product $\Pi_t := Y_t^{\otimes 2} \in L^1(\mu)^{\otimes 2} \subset L^1(\mu^{\otimes 2})$. Similarly $u^{\otimes 2} \in L^\infty(\mu)^{\otimes 2} \subset L^\infty(\mu^{\otimes 2})$, and the relation $\langle Y_t, u\rangle_\mu^2 = \langle Y_t^{\otimes 2}, u^{\otimes 2}\rangle_{\mu\otimes\mu}$ holds. Therefore, the log-price process $X = \log(S)$ satisfies

$$dX_t = -\frac{1}{2}\left\langle \Pi_t, u^{\otimes 2}\right\rangle_{\mu^{\otimes 2}} dt + \sqrt{\left\langle \Pi_t, u^{\otimes 2}\right\rangle_{\mu^{\otimes 2}}}\, d\widetilde{W}_t. \tag{5.1}$$

Our aim is to show that (X, Π) is an affine process.

5.2. **Affine structure.** For each $(x, y) \in (0, 1)^2$, the tuple $(\Pi^{x,x}, \Pi^{x,y}, \Pi^{y,y})$ is an affine process, as shown by the following lemma.

Theorem 5.1 (SDE for Π)**.** *For each $(x, y) \in (0, 1)^2$ the process $\Pi_t^{x,y} := Y_t^x Y_t^y$ satisfies the affine equation*

$$d\Pi_t^{x,y} = (1 - (x + y)\Pi_t^{x,y})\, dt + \sqrt{\Pi_t^{x,x} + 2\Pi_t^{x,y} + \Pi_t^{y,y}}\, dW_t.$$

Proof. By Itō's formula we obtain for each $x, y \in (0, \infty)$

$$d\Pi_t^{x,y} = (1 - (x + y)\Pi_t^{x,y})\, dt + (Y_t^x + Y_t^y)\, dW_t.$$

The statement follows from $(Y_t^x + Y_t^y)^2 = \Pi_t^{x,x} + 2\Pi_t^{x,y} + \Pi_t^{y,y}$. □

The following theorem characterized Π as an affine process with values in $L^1(\mu)^{\otimes 2}$.

Theorem 5.2 (Affine structure)**.** *Let Assumption 2.3 be in place and $Y_0 \in L^1(\mu)$. For all symmetric tensors $w \in L^\infty(\mu; \mathbb{C})^{\otimes 2}$ and $0 \leq t \leq T$,*

$$\mathbb{E}\left[e^{\langle \Pi_T, w\rangle_{\mu\otimes\mu}} \middle| \mathcal{F}_t\right] = e^{\psi_0(T-t,w) + \langle \Pi_t, \psi_1(T-t,w)\rangle_{\mu\otimes\mu}},$$

where $(\psi_0,\psi_1)\colon [0,\infty)\times L^\infty(\mu;\mathbb{C})^{\otimes 2}\to\mathbb{C}\times L^\infty(\mu;\mathbb{C})^{\otimes 2}$ *are given by*

$$\psi_0(\tau,w)=-\frac{1}{2}\operatorname{Tr}\left(\log\left(\mathbb{1}-2\left(P_\tau\otimes P_\tau\right)w\right)\right),$$

$$\psi_1(\tau,w)(x,y)=e^{-\tau(x+y)}\left(\frac{\left(P_\tau\otimes P_\tau\right)w}{1-2\left(P_\tau\otimes P_\tau\right)w}\right)(x,y).$$

Here P_τ is the covariance operator of Y_τ, seen as a bounded linear operator $P_\tau\colon L^\infty(\mu)\to\mathsf{H}_\tau$ taking values in the reproducing kernel Hilbert space H_t of Y_τ. Then $(P_\tau\otimes P_\tau)w$ can be interpreted as a symmetric linear operator on H_τ, and non-linear functions of $(P_\tau\otimes P_\tau)w$ are to be understood in the sense of functional calculus.

Proof. Let $0\le t\le T$ be fixed and let $w=\sum_{k=1}^n\vartheta_k u_k^{\otimes 2}$ be a decomposition of w into sums of squares in the sense of Lemma G.1. By Lemmas D.3, D.4, and G.1 the random variables $\langle Y_T,u_1\rangle_\mu,\dots,\langle Y_T,u_n\rangle_\mu$ are independent Gaussian, given $\mathcal{F}_t$, with conditional means

$$\mathbb{E}\left[\langle Y_T,u_k\rangle_\mu\middle|\mathcal{F}_t\right]=\langle Y_t,\phi_1(T-t,u_k,0)\rangle_\mu,\quad k\in\{1,\dots,n\},$$

and unit variances. Hence, the random variables $\langle Y,u_1\rangle_\mu^2,\dots,\langle Y,u_n\rangle_\mu^2$ are independent non-central χ^2, given $\mathcal{F}_t$, with non centrality parameters

$$\langle Y_t,\phi_1(T-t,u_k,0)\rangle_\mu^2=\left\langle\Pi_t,\phi_1(T-t,u_k,0)^{\otimes 2}\right\rangle_{\mu\otimes\mu},\quad k\in\{1,\dots,n\}.$$

We obtain the affine transformation formula using independence and the characteristic function of the non-central χ^2 distribution

$$\begin{aligned}\mathbb{E}\left[e^{\langle\Pi_T,w\rangle_{\mu\otimes\mu}}\middle|\mathcal{F}_t\right]=\prod_{k=1}^n\mathbb{E}\left[e^{\vartheta_k\langle Y_T,u_k\rangle_\mu^2}\middle|\mathcal{F}_t\right]&=\exp\left(-\frac{1}{2}\sum_{k=1}^n\log\left(1-2\vartheta_k\right)\right)\times\\&\exp\left(\sum_{k=1}^n\frac{\vartheta_k}{1-2\vartheta_k}\left\langle\Pi_t,\phi_1(T-t,u_k,0)^{\otimes 2}\right\rangle_{\mu\otimes\mu}\right).\end{aligned}$$

To rewrite this in terms of the tensor w, we interpret $(P_{T-t}\otimes P_{T-t})w$ as a symmetric bounded linear operator on the Hilbert space H_{T-t} and apply the functional calculus:

$$\psi_0(T-t,w)=-\frac{1}{2}\sum_{k=1}^n\log\left(1-2\vartheta_k\right)=-\frac{1}{2}\operatorname{Tr}\left(\log(1-2(P_{T-t}\otimes P_{T-t})w)\right),$$

$$\begin{aligned}\psi_1(T-t,w)(x,y)&=\sum_{k=1}^n\frac{\vartheta_k}{1-2\vartheta_k}\phi_1(T-t,u_k,0)^{\otimes 2}(x,y)\\&=\sum_{k=1}^n\frac{\vartheta_k}{1-2\vartheta_k}u_k(x)u_k(y)e^{-(T-t)(x+y)}\\&=e^{-(T-t)(x+y)}\left(\frac{\left(P_{T-t}\otimes P_{T-t}\right)w}{1-2\left(P_{T-t}\otimes P_{T-t}\right)w}\right)(x,y).\end{aligned}$$

□

It is useful to consider the following special case for calculations in our model.

Corollary 5.3. *Let $u\in L^\infty(\mu;\mathbb{C})$. Then, with probability one*

$$\mathbb{E}\left[e^{\left\langle\Pi_T,u^{\otimes 2}\right\rangle_{\mu\otimes\mu}}\middle|\mathcal{F}_t\right]=e^{\psi_0\left(T-t,u^{\otimes 2}\right)+\left\langle\Pi_t,\psi_1\left(T-t,u^{\otimes 2}\right)\right\rangle_{\mu\otimes\mu}},\quad 0\le t\le T,$$

where $\psi_0\left(\tau, u^{\otimes 2}\right) \in \mathbb{C}$ *and* $\psi_1\left(\tau, u^{\otimes 2}\right) \in L^\infty(\mu;\mathbb{C})^{\otimes 2}$ *are given by*

$$\psi_0\left(\tau, u^{\otimes 2}\right) = -\frac{1}{2}\log\left(1-4\phi_0(\tau,u,0)\right),$$
$$\psi_1\left(\tau, u^{\otimes 2}\right) = \frac{\phi_1(\tau,u,0)^{\otimes 2}}{1-4\phi_0(\tau,u,0)}.$$

Proof. This follows from the general result of Theorem 5.2, but we provide a manual proof for illustration. By Lemma D.3 the random variable $\frac{1}{\sqrt{2\phi_0(T-t,u,0)}}\left\langle Y_T, u\right\rangle_\mu$ is Gaussian, given $\mathcal{F}_t$, with mean

$$\frac{\left\langle Y_t, \phi_1(T-t,u,0)\right\rangle_\mu}{\sqrt{2\phi_0(T-t,u,0)}},$$

and unit variance. Hence, the random variable

$$\frac{\left\langle \Pi_T, u^{\otimes 2}\right\rangle_{\mu\otimes\mu}}{2\phi_0(T-t,u,0)} = \left(\frac{\left\langle Y_T, u\right\rangle_\mu}{\sqrt{2\phi_0(T-t,u,0)}}\right)^2,$$

is non central χ^2-distributed, given $\mathcal{F}_t$, with one degree of freedom and non centrality parameter

$$\frac{\left\langle Y_t, \phi_1(T-t,u,0)\right\rangle_\mu^2}{2\phi_0(T-t,u,0)} = \frac{\left\langle \Pi_t, \phi_1(T-t,u,0)^{\otimes 2}\right\rangle_{\mu\otimes\mu}}{2\phi_0(T-t,u,0)}.$$

The statement follows from the formula for the characteristic function of the non central χ^2 distribution. □

The coefficient functions (ψ_0,ψ_1) of Corollary 5.3 are solutions of an infinite dimensional version of the Riccati ODE's.

Lemma 5.4. *For any* $u \in L^\infty(\mu;\mathbb{C})$*, the functions* $\psi_0\left(\cdot, u^{\otimes 2}\right)$ *and* $\psi_1\left(\cdot, u^{\otimes 2}\right)$ *given by Corollary 5.3 solve the following system of differential equations*

$$\partial_\tau\psi_0\left(\tau,u^{\otimes 2}\right) = F_0\big(\psi_1\left(\tau,u^{\otimes 2}\right)\big), \quad \psi_0\left(0,u^{\otimes 2}\right)=0,$$
$$\partial_\tau\psi_1\left(\tau,u^{\otimes 2}\right) = F_1\big(\psi_1\left(\tau,u^{\otimes 2}\right)\big), \quad \psi_1\left(0,u^{\otimes 2}\right)=u^{\otimes 2},$$

where for any $w \in L^\infty(\mu;\mathbb{C})^{\otimes 2}$*,* $F_0(w)$ *is a complex number given by*

$$F_0(w) = \int_0^\infty\int_0^\infty w(x,y)\mu(dx)\mu(dy),$$

and $F_1(w)$ *is a measurable function on* $(0,\infty)^2$ *given by*

$$F_1(w)(x,y) = -(x+y)w(x,y) + 2\int_0^\infty\int_0^\infty w(x,x')w(y,y')\mu(dx')\mu(dy').$$

Proof. The initial conditions are satisfied by Lemma 2.6. We differentiate with respect to τ and use Lemma 2.6:

$$\partial_\tau\psi_0\left(\tau,u^{\otimes 2}\right) = \frac{2}{1-4\psi_0(\tau,u,0)}\partial_\tau\phi_0(\tau,u,0) = F_0\left(\psi_1\left(\tau,u^{\otimes 2}\right)\right),$$

$$\begin{aligned}\partial_\tau\psi_1\left(\tau,u^{\otimes 2}\right)(x,y) &= \frac{-x\phi_1(\tau,u,0)(x)\phi_1(\tau,u,0)(y) - y\phi_1(\tau,u,0)(x)\phi_1(\tau,u,0)(y)}{1-4\psi_0(\tau,u,0)}\\ &\quad + \frac{2\phi_1(\tau,u,0)(x)\phi_1(\tau,u,0)(y)}{\left(1-4\phi_0(\tau,u,0)\right)^2}\left(\int_0^\infty \phi_1(\tau,u,0)(z)\mu(dz)\right)^2\\ &= F_1\left(\psi_1\left(\tau,u^{\otimes 2}\right)\right)(x,y).\end{aligned}$$

□

Corollary 5.5. *For each real-valued $u \in L^\infty(\mu)^{\otimes 2}$, the conditional mean of $\left\langle \Pi, u^{\otimes 2}\right\rangle_{\mu^{\otimes 2}}$ is given by*

$$\mathbb{E}\left[\left\langle \Pi_T, u^{\otimes 2}\right\rangle_{\mu\otimes\mu}\middle|\mathcal{F}_t\right] = 2\phi_0(\tau, u, 0) + \left\langle \Pi_t, \phi_1(\tau, u, 0)^{\otimes 2}\right\rangle_{\mu\otimes\mu}.$$

Proof. By Corollary 5.3,

$$\begin{aligned}
\mathbb{E}\left[\left\langle \Pi_T, u^{\otimes 2}\right\rangle_{\mu\otimes\mu}\middle|\mathcal{F}_t\right] &= \frac{1}{i}\partial_q|_{q=0}\mathbb{E}\left[e^{iq\left\langle \Pi_T, u^{\otimes 2}\right\rangle_{\mu\otimes\mu}}\middle|\mathcal{F}_t\right] \\
&= \frac{1}{i}\partial_q|_{q=0} e^{\psi_0\left(T-t, u^{\otimes 2} iq\right)+\left\langle \Pi_t, \psi_1\left(T-t, u^{\otimes 2} iq\right)\right\rangle_{\mu\otimes\mu}} \\
&= \frac{1}{i}\partial_q|_{q=0} e^{-\frac{1}{2}\log\left(1-4\phi_0(\tau, u\sqrt{iq}, 0)\right)+\left\langle \Pi_t, \frac{\phi_1(\tau, u\sqrt{iq}, 0)^{\otimes 2}}{1-4\phi_0(\tau, u\sqrt{iq}, 0)}\right\rangle_{\mu\otimes\mu}} \\
&= \frac{1}{i}\partial_q|_{q=0} e^{-\frac{1}{2}\log(1-4iq\phi_0(\tau,u,0))+iq\left\langle \Pi_t, \frac{\phi_1(\tau,u,0)^{\otimes 2}}{1-4iq\phi_0(\tau,u,0)}\right\rangle_{\mu\otimes\mu}} \\
&= 2\phi_0(\tau, u, 0) + \left\langle \Pi_t, \phi_1(\tau, u, 0)^{\otimes 2}\right\rangle_{\mu\otimes\mu}. \qquad \square
\end{aligned}$$

5.3. **The uncorrelated case.** By "uncorrelated" we mean $\langle W, \widetilde{W}\rangle_t = \rho dt = 0$.

Theorem 5.6. *If $\rho = 0$, then (X, Π) is an affine process on $\mathbb{R} \times L^1(\mu)^{\otimes 2}$.*

Proof. We fix $u \in i\mathbb{R}$ and $v \in iL^\infty(\mu)^{\otimes 2}$. Let $\mathcal{G}_t$ be the sigma algebra generated by $(\langle \Pi_s, v\rangle_{\mu^{\otimes 2}}$. Conditionally on $\mathcal{F}_t \vee \mathcal{G}_t$, the random variable $X_T - X_t$ is Gaussian with mean $-\frac{1}{2}\int_t^T \langle \Pi_s, v\rangle_{\mu^{\otimes 2}} ds$ and variance $\int_t^T \langle \Pi_s, v\rangle_{\mu^{\otimes 2}} ds$. Therefore,

$$\begin{aligned}
\mathbb{E}\left[e^{X_T u + \langle \Pi_t, v\rangle_{\mu^{\otimes 2}}}\middle|\mathcal{F}_t\right] &= \mathbb{E}\left[e^{X_t u}\mathbb{E}\left[e^{(X_T - X_t)u}\middle|\mathcal{F}_t \vee \mathcal{G}_t\right] e^{\langle \Pi_t, v\rangle_{\mu^{\otimes 2}}}\middle|\mathcal{F}_t\right] \\
&= \mathbb{E}\left[e^{X_t u + \frac{1}{2}(u^2-u)\langle \int_t^T \Pi_s ds, v\rangle_{\mu^{\otimes 2}} + \langle \Pi_t, v\rangle_{\mu^{\otimes 2}}}\middle|\mathcal{F}_t\right].
\end{aligned}$$

As Π is affine by Theorem 5.2, this evaluates to an exponentially affine function in (X_t, Π_t) (see [10, Section 4.3]). $\square$

The proof of Theorem 5.6 shows that the distribution of X_T depends immediately on the distribution of the integrated variance, which is defined for each $0 \le t \le T$ as

$$\mathrm{IV}(t, T) = \frac{1}{T-t}\int_t^T \langle Y_s, u\rangle_\mu^2 ds = \frac{1}{T-t}\int_t^T \langle \Pi_s, u^{\otimes 2}\rangle_{\mu^{\otimes 2}} ds.$$

In the following lemma, we make this relation precise.

Lemma 5.7. *If $\rho = 0$ holds, then we have for each $0 \le t \le T$*

$$\mathbb{Q}\left[X_T \le x | \mathcal{F}_t\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \mathbb{E}\left[\exp\left(-\frac{\left(y - X_t + \frac{T-t}{2}\mathrm{IV}(t,T)\right)^2}{2(T-t)\mathrm{IV}(t,T)}\right)\middle|\mathcal{F}_t\right] dy.$$

Proof. This can be seen as in [15] by conditioning on the sigma algebra generated by $(\langle Y_t, u\rangle)_{0 \le t \le T}$ and by using the independence of W and $\widetilde{W}$. $\square$

The Fourier transform of the integrate variance process can be calculated explicitly using the affine structure of the process Π. Thus, in theory, it is possible to characterize the conditional distribution of the integrated variance. An example is given in the next lemma.

Corollary 5.8. *The mean of the integrated variance is given by*

$$\mathbb{E}\left[\mathrm{IV}(t,T)|\mathcal{F}_t\right] = \int_t^T \left(2\phi_0(s-t,u,0) + \left\langle \Pi_t, \phi_1(s-t,u,0)^{\otimes 2}\right\rangle_{\mu\otimes\mu}\right) ds.$$

Proof. We obtain the formula for the conditional mean from using Corollary 5.5. Note that we are allowed to exchange the conditional expectation and integration because the integrand is positive. □

Remark 5.9. *We summarize the results of Section 5. We generalized the stochastic volatility model by Stein and Stein [15] to fractional volatility. We introduced an affine framework for formulating the model. Namely, we conjecture that (X,Π) is affine, where X is the log-price and $\Pi = Y \otimes Y$. A preliminary result in this direction is that Π is affine and that the dynamics of X are affine in Π. Moreover, in the uncorrelated case $\rho = 0$, we could show that the process (X,Π) is affine. We are currently working on a proof of the general case.*

Appendix A. Stochastic Fubini's theorem

We refer to the version of the theorem proved in [16]. Let μ be a σ-finite measure on $(0,\infty)$. Fix $T \geq 0$ and denote by $\mathcal{P}$ the σ-algebra on $[0,T]\times\Omega$ generated by all progressive measurable processes.

Theorem A.1. *Let $G : (0,\infty)\times[0,T]\times\Omega \to \mathbb{R}$ be measurable with respect to the product σ-algebra $\mathcal{B}(0,\infty)\otimes\mathcal{P}$. Define $\zeta_{1,2} : (0,\infty)\times[0,T]\times\Omega \to \mathbb{R}$ and $\eta : [0,T]\times\Omega\to\mathbb{R}$ by*

$$\begin{aligned}
\zeta_1(x,t,\omega) &= \int_0^t G(x,s,\omega)ds,\\
\zeta_2(x,t,\omega) &= \left(\int_0^t G(x,s,\cdot)dW_s\right)(\omega), \quad \text{and}\\
\eta(t,\omega) &= \int_0^\infty G(x,t,\omega)\mu(dx).
\end{aligned}$$

(i) Assume G satisfies for almost all $\omega\in\Omega$

$$\int_0^\infty \int_0^T |G(x,s,\omega)|\, ds\mu(dx) < \infty. \tag{A.1}$$

Then, for almost all $\omega\in\Omega$ and for all $t\in[0,T]$ we have $\zeta_1(\cdot,t,\omega)\in L^1(\mu)$ and

$$\int_0^\infty \zeta_1(x,t,\omega)\mu(dx) = \int_0^t \eta(s,\omega)ds.$$

(ii) Assume G satisfies for almost all $\omega\in\Omega$

$$\int_0^\infty \sqrt{\int_0^T G(x,s,\omega)^2 ds}\,\mu(dx) < \infty. \tag{A.2}$$

Then, for almost all $\omega\in\Omega$ and for all $t\in[0,T]$ we have $\zeta_2(\cdot,t,\omega)\in L^1(\mu)$ and

$$\int_0^\infty \zeta_2(x,t,\omega)\mu(dx) = \left(\int_0^t \eta(s,\cdot)dW_s\right)(\omega).$$

Remark A.2. *Note that*

$$\int_0^\infty \int_0^T \mathbb{E}\left[|G(x,s)|\right] ds\mu(dx) < \infty \quad \textit{and} \quad \int_0^\infty \mathbb{E}\left[\sqrt{\int_0^T G(x,s)^2 ds}\right]\mu(dx) < \infty$$

imply that conditions (A.1) *and* (A.2) *hold with probability one.*

Appendix B. Reproducing kernel Hilbert spaces

We adapt the exposition of [13, Section 8] to our setting and refer to this reference for further details. Let $P\colon L^\infty(\mu;\mathbb{C}) \to L^1(\mu;\mathbb{C})$ be a positive and symmetric bounded linear operator, i.e., $\langle Pu, u\rangle_\mu \geq 0$ and $\langle Pu, v\rangle_\mu = \langle Pv, u\rangle_\mu$ for all $u, v \in L^\infty(\mu;\mathbb{C})$. The bilinear form $(Pu, Pv) \mapsto \langle Pu, v\rangle_\mu$ defines an inner product on the image of P. The completion of the image of P with respect to this inner product is a Hilbert space, which we denote by $\overline{\operatorname{im}(P)}$. The inclusion of the image of P in $L^1(\mu;\mathbb{C})$ extends to a bounded injective operator $i\colon \overline{\operatorname{im}(P)} \to L^1(\mu;\mathbb{C})$. The space $\mathsf{H} := \operatorname{im}(i) \subseteq L^1(\mu;\mathbb{C})$ with the Hilbert structure induced by the bijection $i\colon \overline{\operatorname{im}(P)} \to \mathsf{H}$ is called the reproducing kernel Hilbert space of P. If $u, v \in L^\infty(\mu;\mathbb{C})$, then $Pu, Pv \in \mathsf{H}$ and $\langle Pu, Pv\rangle_{\mathsf{H}} = \langle Pu, v\rangle_\mu$.[1]

Appendix C. Basic estimates

Lemma C.1 (Elementary inequalities)**.** *The following inequalities hold true for all* $x, y > 0$

$$1 \wedge xy \leq (1 \vee x)(1 \wedge y), \tag{C.1}$$

$$y \wedge x^{-1} \leq (1 \vee y)\left(1 \wedge x^{-1}\right), \tag{C.2}$$

and for all $\alpha, \tau > 0$

$$e^{-x\tau} \leq \left(1 \vee \left(\frac{\tau}{\alpha}\right)^{-\alpha}\right)\left(1 \wedge x^{-\alpha}\right), \tag{C.3}$$

$$\frac{1 - e^{-\tau x}}{x} \leq (1 \vee \tau)\left(1 \wedge x^{-1}\right), \tag{C.4}$$

$$\frac{1 - e^{-\tau x}(1 + \tau x)}{x^2} \leq \left(1 \vee \tau^2\right)\left(1 \wedge x^{-2}\right), \tag{C.5}$$

$$\frac{1 - e^{-\tau x}\left(1 + \tau x + \frac{1}{2}\tau^2 x^2\right)}{x^3} \leq \left(1 \vee \tau^3\right)\left(1 \wedge x^{-3}\right). \tag{C.6}$$

Proof. For the inequalities (C.1)-(C.2) consider the following four cases separately.

1. If $0 < x, y \leq 1$. Then, $1 \wedge xy = xy \leq y = (1 \vee x)(1 \wedge y)$ and $y \wedge x^{-1} = y \leq 1 = (1 \vee y)\left(1 \wedge x^{-1}\right)$.
2. If $0 < x \leq 1 \leq y$. Then, $1 \wedge xy \leq 1 = (1 \vee x)(1 \wedge y)$ and $y \wedge x^{-1} \leq y = (1 \vee y)\left(1 \wedge x^{-1}\right)$.
3. If $0 < y \leq 1 \leq x$. Then, $1 \wedge xy \leq xy = (1 \vee x)(1 \wedge y)$ and $y \wedge x^{-1} \leq x^{-1} = (1 \vee y)\left(1 \wedge x^{-1}\right)$.
4. If $1 \leq x, y$. Then, $1 \wedge xy = 1 \leq x = (1 \vee x)(1 \wedge y)$ and $y \wedge x^{-1} = x^{-1} \leq yx^{-1} = (1 \vee y)\left(1 \wedge x^{-1}\right)$.

[1] In [13] the space $\overline{\operatorname{im}(P)}$ is called reproducing kernel Hilbert space of P.

Consider the functions $f(x,\tau) = e^{-x\tau}$ and $g(x,\tau,\alpha) = x^\alpha f(x,\tau)$. Obviously, $f(x,\tau) \le 1$ for all $x,\tau > 0$. Note that $\partial_x g(x,\tau,\alpha) = x^{\alpha-1}e^{-x\tau}(\alpha - \tau x)$ and g attains its maximum in x at $\frac{\alpha}{\tau}$. Hence, Equation (C.3) follows from

$$f(x,\tau) = \frac{g(x,\tau,\alpha)}{x^\alpha} \le \frac{g\left(\frac{\alpha}{\tau},\tau,\alpha\right)}{x^\alpha} = \left(\frac{\tau}{\alpha}x\right)^{-\alpha} e^{-\alpha} \le \left(\frac{\tau}{\alpha}x\right)^{-\alpha},$$

and Equation (C.1).

Define $k_1(x,\tau) = \frac{1-e^{-\tau x}}{x}$, $k_2(x,\tau) = \frac{1-e^{-\tau x}(1+\tau x)}{x^2}$ and $k_3(x,\tau) = \frac{1-e^{-\tau x}\left(1+\tau x+\frac{1}{2}\tau^2x^2\right)}{x^3}$. Computing the derivatives with respect to x shows that $k_{1,2,3}(\cdot,\tau)$ are decreasing functions in x for all $\tau > 0$. The inequalities (C.4)-(C.6) follow from

$$\lim_{x\to\infty} k_{1,2,3}(x,\tau) = 0, \quad \lim_{x\to 0^+} k_i(x,\tau) = \begin{cases} \tau, & i = 1, \\ \frac{\tau^2}{2}, & i = 2, \\ \frac{\tau^3}{6}, & i = 3, \end{cases}$$

and Equation (C.2). $\square$

Lemma C.2 (Integrability of elementary expressions). *Let Assumption 2.3 be in place and let $\tau,\alpha > 0$. Then*

$$\int_0^\infty e^{-x\tau}\mu(dx) < \infty, \tag{C.7}$$

$$\int_0^\infty e^{-x\tau}\nu(dx) < \infty, \tag{C.8}$$

$$\int_0^\infty x^\alpha e^{-x\tau}\mu(dx) < \infty, \tag{C.9}$$

$$\int_0^\infty x^\alpha e^{-x\tau}\nu(dx) < \infty, \tag{C.10}$$

$$\int_0^\infty \sqrt{\frac{1-e^{-2\tau x}}{x}}\mu(dx) < \infty, \tag{C.11}$$

$$\int_0^\infty \sqrt{\frac{1-e^{-2\tau x}\left(1+2\tau x+2\tau^2x^2\right)}{x^3}}\nu(dx) < \infty, \tag{C.12}$$

$$\int_0^\infty \sqrt{\frac{1-2e^{-\tau x}(\tau x+1)(1-\tau x e^{-\tau x})+e^{-2\tau x}}{x^3}}\mu(dx) < \infty. \tag{C.13}$$

Furthermore, for each $0 \le t < T$ we have

$$\int_0^\infty \sqrt{\int_t^T e^{-2x(T-s)}ds}\,\mu(dx) < \infty, \tag{C.14}$$

$$\int_0^\infty \sqrt{\int_t^T (T-s)^2 e^{-2x(T-s)}ds}\,\nu(dx) < \infty, \tag{C.15}$$

$$\int_0^\infty \int_t^T e^{-x(T-s)}ds\mu(dx) < \infty, \tag{C.16}$$

$$\int_0^\infty \int_t^T (T-s)e^{-x(T-s)}ds\nu(dx) < \infty, \tag{C.17}$$

$$\int_0^\infty \int_t^T \frac{1-e^{-x(T-s)}}{x}(1\wedge x^{-\frac{1}{2}})ds\nu(dx) < \infty, \tag{C.18}$$

$$\int_0^\infty \sqrt{\int_t^T \left(\frac{1-e^{-x(T-s)}}{x}\right)^2 ds}\,\mu(dx) < \infty, \tag{C.19}$$

$$\int_0^\infty \sqrt{\int_t^T \left(1-\frac{e^{-x(T-s)}\left(1+x(T-s)\right)}{x^2}\right)^2 ds}\,\mu(dx) < \infty, \tag{C.20}$$

$$\int_0^\infty \int_0^\infty \int_t^T e^{-(x+y)(T-s)} ds\mu(dx)\mu(dy) < \infty, \tag{C.21}$$

$$\int_0^\infty \int_0^\infty \int_t^T (T-s)^2 e^{-(x+y)(T-s)} ds\nu(dx)\nu(dy) < \infty. \tag{C.22}$$

Proof. Equations (C.7) and (C.8) follow directly from (C.3) for $\alpha = \frac{1}{2}$ and $\alpha = \frac{3}{2}$, respectively. Applying Equation (C.3) for $\beta > \alpha$ we obtain

$$\begin{aligned}
\int_0^\infty x^\alpha e^{-x\tau}\mu(dx) &\le \int_0^1 e^{-x\tau}\mu(dx) + \int_1^\infty x^\alpha e^{-x\tau}\mu(dx)\\
&\le \int_0^1 \left(1 \vee \left(\frac{\tau}{\beta}\right)^{-\beta}\right)\mu(dx) + \int_1^\infty x^{\alpha-\beta}\left(1 \vee \left(\frac{\tau}{\beta}\right)^{-\beta}\right)\mu(dx)\\
&= \left(1 \vee \left(\frac{\tau}{\beta}\right)^{-\beta}\right)\int_0^\infty \left(1 \wedge x^{\alpha-\beta}\right)\mu(dx),
\end{aligned}$$

and in the same way $\int_0^\infty x^\alpha e^{-x\tau}\nu(dx) \le (1 \vee (\frac{\tau}{\beta})^{-\beta})\int_0^\infty \left(1 \wedge x^{\alpha-\beta}\right)\nu(dx)$. Setting $\beta = \alpha + \frac{1}{2}$ and $\beta = \alpha + \frac{3}{2}$ one proves (C.9) and (C.10), respectively. By Equation (C.4) we obtain Equation (C.11)

$$\int_0^\infty \sqrt{\frac{1-e^{-2\tau x}}{x}}\mu(dx) \le \left(1 \vee (2\tau)^{\frac{1}{2}}\right)\int_0^\infty \left(1 \wedge x^{-\frac{1}{2}}\right)\mu(dx) < \infty. \tag{C.23}$$

By Equation (C.11) we obtain Equation (C.14)

$$\int_0^\infty \sqrt{\int_t^T e^{-2x(T-s)} ds}\,\mu(dx) = \int_0^\infty \sqrt{\frac{1-e^{-2(T-t)x}}{2x}}\mu(dx) < \infty.$$

By Equation (C.6) we obtain Equation (C.12)

$$\begin{aligned}
&\int_0^\infty \sqrt{\frac{1-e^{-2\tau x}\left(1+2\tau x+2\tau^2x^2\right)}{x^3}}\nu(dx)\\
&\qquad \le (1 \vee (2\tau)^{\frac{3}{2}})\int_0^\infty (1 \wedge x^{-\frac{3}{2}})\nu(dx) < \infty.
\end{aligned} \tag{C.24}$$

Equation (C.13) follows from

$$\begin{aligned}
&\int_0^\infty \sqrt{\frac{1-2e^{-\tau x}(\tau x+1)+2\tau x e^{-2\tau x}(\tau x+1)+e^{-2\tau x}}{4x^3}}\nu(dx)\\
&\le \int_0^{1/\tau} \sqrt{\frac{\tau^2}{6x}}\nu(dx) + \int_{1/\tau}^\infty \sqrt{\frac{2}{x^3}}\nu(dx)\\
&\le \sqrt{2}(\tau \vee 1)\int_0^\infty (x^{-\frac{1}{2}} \wedge x^{-\frac{3}{2}})\nu(dx) < \infty.
\end{aligned}$$

Equation (C.12) implies Equation (C.15)

$$\begin{aligned}&\int_0^\infty \sqrt{\int_t^T (T-s)^2 e^{-2x(T-s)} ds \nu(dx)}\\&\quad = \int_0^\infty \sqrt{\frac{1-e^{-2(T-t)x}\left(1+2(T-t)x+2(T-t)^2x^2\right)}{4x^3}}\nu(dx) < \infty.\end{aligned}$$

Equation (C.16) is obtained using (C.3) for $\alpha = \frac{1}{2}$

$$\begin{aligned}&\int_0^\infty \int_t^T e^{-x(T-s)} ds \mu(dx) \le \int_t^T (1 \vee (T-s)^{-\frac{1}{2}}) ds \int_0^\infty (1 \wedge x^{-\frac{1}{2}}) \mu(dx)\\&\quad = \left(t \vee (T-1) - t + 2\sqrt{T-(t\vee(T-1))}\right) \int_0^\infty (1 \wedge x^{-\frac{1}{2}}) \mu(dx) < \infty.\end{aligned}$$

Equation (C.17) is obtained using (C.3) for $\alpha = \frac{3}{2}$

$$\begin{aligned}&\int_0^\infty \int_t^T (T-s) e^{-x(T-s)} ds \nu(dx)\\&\quad \le \int_t^T (T-s)\left(1 \vee (T-s)^{-\frac{3}{2}}\right) ds \int_0^\infty (1 \wedge x^{-\frac{3}{2}}) \mu(dx)\\&\quad \le \left(\int_t^T (T-s) ds \vee \int_t^T (T-s)^{-\frac{1}{2}} ds\right) \int_0^\infty (1 \wedge x^{-\frac{3}{2}}) \mu(dx)\\&\quad = \left(\frac{(T-t)^2}{2} \vee 2\sqrt{T-t}\right) \int_0^\infty (1 \wedge x^{-\frac{3}{2}}) \mu(dx) < \infty.\end{aligned}$$

Equation (C.4) immediately implies Equation (C.18)

$$\begin{aligned}&\int_0^\infty \int_t^T \frac{1-e^{-x(T-s)}}{x} (1 \wedge x^{-\frac{1}{2}}) ds \nu(dx)\\&\quad \le \int_0^\infty \left(1 \wedge x^{-\frac{3}{2}}\right) \nu(dx) \int_t^T \left(1 \vee (T-s)\right) ds < \infty,\end{aligned}$$

and Equation (C.19)

$$\begin{aligned}&\int_0^\infty \sqrt{\int_t^T \left(\frac{1-e^{-x(T-s)}}{x}\right)^2 ds \mu(dx)}\\&\quad \le \sqrt{\int_t^T \left(1 \vee (T-s)^2\right) ds} \int_0^\infty \left(1 \wedge \frac{1}{x}\right) \mu(dx) < \infty.\end{aligned}$$

Equation (C.5) immediately implies Equation (C.20)

$$\begin{aligned}&\int_0^\infty \sqrt{\int_t^T \left(\frac{e^{-x(T-s)}(1+x(T-s))-1}{x^2}\right)^2 ds \mu(dx)}\\&\quad \le \sqrt{\int_t^T \left(1 \vee (T-s)^4\right) ds} \int_0^\infty \left(1 \wedge x^{-2}\right) \mu(dx) < \infty.\end{aligned}$$

Equation (C.21) follows from Equation (C.14) applying Cauchy-Schwarz inequality

$$\begin{aligned}&\int_0^\infty \int_0^\infty \int_t^T e^{-(x+y)(T-s)} ds\mu(dx)\mu(dy)\\&\quad\leq \int_0^\infty \int_0^\infty \sqrt{\int_t^T e^{-2x(T-s)}ds}\sqrt{\int_t^T e^{-2y(T-s)}ds}\mu(dx)\mu(dy)\\&\quad= \left(\int_0^\infty \sqrt{\int_t^T e^{-2x(T-s)}ds}\mu(dx)\right)^2 < \infty.\end{aligned}$$

In the same way Equation (C.22) follows from Equation (C.15)

$$\begin{aligned}&\int_0^\infty \int_0^\infty \int_t^T (T-s)^2 e^{-(x+y)(T-s)} ds\nu(dx)\nu(dy)\\&\quad\leq \left(\int_0^\infty \sqrt{\int_t^T (T-s)^2 e^{-2y(T-s)}ds}\right)^2 < \infty.\end{aligned}$$

□

Appendix D. Auxiliary results for Section 2

Lemma D.1. *For each $x \in (0,\infty)$ and $0 \leq t \leq T$, the process (Y^x, Z^x) can be represented as*

$$\begin{aligned}Y_T^x &= Y_t^x e^{-(T-t)x} + \int_t^T e^{-(T-s)x} dW_s,\\Z_T^x &= Z_t^x e^{-(T-t)x} + Y_t^x (T-t) e^{-(T-t)x} + \int_t^T (T-s) e^{-(T-s)x} dW_s.\end{aligned}$$

Moreover, for $x_1, \ldots, x_n \in (0,\infty)$ the process $(Y^{x_i}, \ldots, Y^{x_n}, Z^{x_1}, \ldots, Z^{x_n})$ is a multivariate OU process. In particular, the random vector $(Y_T^{x_1}, \ldots, Y_T^{x_n}, Z_T^{x_1}, \ldots, Z_T^{x_n})$ is multivariate Gaussian, given $\mathcal{F}_t$, with conditional means

$$\begin{aligned}\mathbb{E}\left[Y_T^{x_i} | \mathcal{F}_t\right] &= Y_t^{x_i} e^{-(T-t)x_i},\\\mathbb{E}\left[Z_T^{x_i} | \mathcal{F}_t\right] &= Z_t^{x_i} e^{-(T-t)x_i} + Y_t^{x_i}(T-t)e^{-(T-t)x_i},\end{aligned}$$

and conditional covariances

$$\begin{aligned}\operatorname{Cov}\left(Y_T^{x_i}, Y_T^{x_j} \middle| \mathcal{F}_t\right) &= \frac{1 - e^{-(T-t)(x_i+x_j)}}{x_i + x_j},\\\operatorname{Cov}\left(Z_T^{x_i}, Z_T^{x_j} \middle| \mathcal{F}_t\right) &= \frac{2 - e^{-(T-t)(x_i+x_j)}\left(2 + 2(T-t)(x_i+x_j) + (T-t)^2(x_i+x_j)^2\right)}{(x_i+x_j)^3},\\\operatorname{Cov}\left(Y_T^{x_i}, Z_T^{x_j} \middle| \mathcal{F}_t\right) &= \frac{1 - e^{-(T-t)(x_i+x_j)}\left(1 + (T-t)(x_i+x_j)\right)}{(x_i+x_j)^2},\end{aligned}$$

for $i, j = 1, \ldots, n$.

Proof. The representation can be deduced from the SDE (2.2) for (Y^x, Z^x) using Theorem A.1(ii)

$$\begin{aligned}
Z_T^x &= Z_t^x e^{-(T-t)x} + \int_t^T e^{-(T-s)x}\left(Y_t^x e^{-(s-t)x} + \int_t^s e^{-(s-u)x} dW_u\right) ds \\
&= Z_t^x e^{-(T-t)x} + Y_t^x (T-t) e^{-(T-t)x} + \int_t^T \int_t^s e^{-(T-u)x} dW_u ds \\
&= Z_t^x e^{-(T-t)x} + Y_t^x (T-t) e^{-(T-t)x} + \int_t^T \int_u^T e^{-(t-u)x} ds dW_u \\
&= Z_t^x e^{-(T-t)x} + Y_t^x (T-t) e^{-(T-t)x} + \int_t^T (T-u) e^{-(T-u)x} dW_u.
\end{aligned}$$

The condition (A.2) is satisfied because $\int_t^T \sqrt{\int_t^s e^{-2(t-u)x} du} ds < \infty$. The $\mathbb{R}^{2n}$-valued process $U = (Y^{x_1}, \dots, Y^{x_n}, Z^{x_1}, \dots, Z^{x_n})^\top$ satisfies

$$dU_t = MU_t dt + dW_t,$$

where $M = \left[\begin{array}{c|c} -\mathrm{diag}(x_1, \dots, x_n) & 0 \\ \hline \mathrm{diag}(1, \dots, 1) & -\mathrm{diag}(x_1, \dots, x_n) \end{array}\right] \in \mathbb{R}^{2n \times 2n}$ for diagonal matrices $\mathrm{diag}(1, \dots, 1) \in \mathbb{R}^{n \times n}$ and $\mathrm{diag}(x_1, \dots, x_n) \in \mathbb{R}^{n \times n}$. Therefore, U ia a multivariate OU process and we have

$$U_T = e^{M(T-t)} U_t + \int_t^T e^{M(T-s)} e dW_s,$$

where $e = (1, \dots, 1)^\top \in \mathbb{R}^n$. This implies that U_T is multivariate Gaussian, given $\mathcal{F}_t$, with mean $e^{M(T-t)} U_t$ and covariance matrix

$$\int_t^T e^{M(T-s)} e e^\top e^{M^\top (T-s)} ds.$$

The formulas for the means and the covariances follow from

$$e^{M(T-t)} = \left[\begin{array}{c|c} \mathrm{diag}(e^{-x_1(T-t)}, \dots, e^{-x_n(T-t)}) & 0 \\ \hline (T-t)\mathrm{diag}(e^{-x_1(T-t)}, \dots, e^{-x_n(T-t)}) & \mathrm{diag}(e^{-x_1(T-t)}, \dots, e^{-x_n(T-t)}) \end{array}\right].$$

□

Lemma D.2. *Let Assumption 2.3 be in place and assume* $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ *a.s. Then, for all* $(u, v) \in L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C})$ *and each* $t \geq 0$*, with probability one,* $Y_t u \in L^1(\mu)$ *and* $Z_t v \in L^1(\nu)$*.*

Proof. By Lemma D.1 we have for (Y^x, Z^x)

$$\begin{aligned}
Y_t^x u(x) &= Y_0^x e^{-tx} u(x) + \int_0^t e^{-(t-s)x} u(x) dW_s, \\
Z_t^x v(x) &= Z_0^x e^{-tx} v(x) + Y_0^x t e^{-tx} v(x) + \int_0^t (t-s) e^{-(t-s)x} v(x) dW_s.
\end{aligned}$$

We verify integrability of $(Y_t u, Z_t v)$ with respect to μ and ν, respectively. Note that

$$\begin{aligned}
&\int_0^\infty |Y_0^x u(x)| e^{-tx}\mu(dx) \leq \|u\|_{L^\infty(\mu)} \int_0^\infty |Y_0^x|\mu(dx) < \infty,\\
&\int_0^\infty |Z_0^x v(x)| e^{-tx}\nu(dx) \leq \|v\|_{L^\infty(\nu)} \int_0^\infty |Z_0^x|\nu(dx) < \infty,\\
&\int_0^\infty |Y_0^x v(x)| t e^{-tx}\nu(dx) \leq \|v\|_{L^\infty(\nu)} \sup_{x\in(0,\infty)} \left(p(x)e^{-xt}\right) t \int_0^\infty |Y_0^x|\mu(dx) < \infty,
\end{aligned}$$

where in the latter we use Assumption 2.3. By Theorem A.1 the functions $x \mapsto \int_0^t e^{-(t-s)x} u(x) dW_s$ and $x \mapsto \int_0^t (t-s)e^{-(t-s)x} v(x) dW_s$ are in $L^1(\mu)$ and $L^1(\nu)$, respectively. The condition (A.2) is satisfied by Equations (C.14) and (C.15). $\square$

Lemma D.3. *Let Assumption 2.3 be in place and assume $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ a.s. Then, for each $0 \leq t \leq T$ the process (Y, Z) satisfies for all $(u, v) \in L^\infty(\mu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C})$*

$$\begin{aligned}
\langle Y_T, u\rangle_\mu &= \int_0^\infty Y_t^x e^{-(T-t)x} u(x)\mu(dx) + \int_t^T \int_0^\infty e^{-x(T-s)} u(x)\mu(dx) dW_s,\\
\langle Z_T, v\rangle_\nu &= \int_0^\infty \left(Z_t^x e^{-(T-t)x} + Y_t^x (T-t) e^{-(T-t)x}\right) v(x)\nu(dx)\\
&\quad + \int_t^T \int_0^\infty (T-s) e^{-x(T-s)} v(x)\nu(dx) dW_s.
\end{aligned}$$

In particular, the random variable $\langle Y_T, u\rangle_\mu + \langle Z_T, v\rangle_\nu$ is Gaussian, given $\mathcal{F}_t$.

Proof. The statement follows from Lemmas D.1 and D.2 and Theorem A.1. The condition (A.2) for the application of the stochastic Fubini's theorem are satisfied by Equations (C.14) and (C.15). $\square$

Lemma D.4 (Covariance operators)**.** *Let Assumption 2.3 be in place and $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ a.s. Then, for all $(u_1, u_2) \in L^\infty(\mu; \mathbb{C}) \times L^\infty(\mu; \mathbb{C})$, $(v_1, v_2) \in L^\infty(\nu; \mathbb{C}) \times L^\infty(\nu; \mathbb{C})$ and for all $0 \leq t \leq T$*

$$\begin{aligned}
\operatorname{Cov}\left(\langle Y_T, u_1\rangle_\mu, \langle Y_T, u_2\rangle_\mu \middle| \mathcal{F}_t\right) &= \langle P_{T-t} u_1, u_2\rangle_\mu,\\
\operatorname{Cov}\left(\langle Z_T, v_1\rangle_\mu, \langle Z_T, v_2\rangle_\mu \middle| \mathcal{F}_t\right) &= \langle Q_{T-t} v_1, v_2\rangle_\nu,
\end{aligned}$$

where $P_\tau\colon L^\infty(\mu; \mathbb{C}) \to L^1(\mu; \mathbb{C})$ and $Q_\tau\colon L^\infty(\nu; \mathbb{C}) \to L^1(\nu; \mathbb{C})$ are bounded linear operators given by

$$\begin{aligned}
P_\tau u(x) &= \int_0^\infty \frac{1 - e^{-\tau(x+y)}}{x+y} u(y)\mu(dy),\\
Q_\tau v(x) &= \int_0^\infty \frac{1 - e^{-\tau(x+y)} - \tau(x+y)e^{-\tau(x+y)}}{(x+y)^2} v(y)\nu(dy),
\end{aligned}$$

for $u \in L^\infty(\mu; \mathbb{C})$, $v \in L^\infty(\nu; \mathbb{C})$ and $\tau \geq 0$. In particular, Y_T and Z_T are Gaussian random variables, given $\mathcal{F}_t$, with covariance operators P_{T-t} and Q_{T-t}, respectively.

Proof. For each $t \geq 0$ and any $u_{1,2} \in L^\infty(\mu)$ and $v_{1,2} \in L^\infty(\nu)$ we have using the representation of Lemma D.3

$$\begin{aligned}
&\operatorname{Cov}\left(\langle Y_T, u_1\rangle_\mu, \langle Y_T, u_2\rangle_\mu \middle| \mathcal{F}_t\right) = \int_0^\infty \int_0^\infty \operatorname{Cov}\left(Y_T^x, Y_T^y | \mathcal{F}_t\right) u_1(x) u_2(y) \mu(dy)\mu(dx) \\
&= \int_0^\infty \int_0^\infty \int_t^T e^{-(T-s)x} e^{-(T-s)y} ds u_1(x) u_2(y) \mu(dy)\mu(dx) \\
&= \int_0^\infty \int_0^\infty \frac{1 - e^{-(T-t)(x+y)}}{x+y} u_1(x) u_2(y) \mu(dy)\mu(dx) \\
&= \langle P_{T-t} u_1, u_2\rangle_\mu, \\
&\operatorname{Cov}\left(\langle Z_T, v_1\rangle_\mu, \langle Z_T, v_2\rangle_\mu \middle| \mathcal{F}_t\right) = \int_0^\infty \int_0^\infty \operatorname{Cov}\left(Z_T^x, Z_T^y | \mathcal{F}_t\right) v_1(x) v_2(y) \mu(dy)\mu(dx) \\
&= \int_0^\infty \int_0^\infty \int_t^T (T-s)^2 e^{-(T-s)x} e^{-(T-s)y} ds v_1(x) v_2(y) \nu(dy)\nu(dx) \\
&= \int_0^\infty \int_0^\infty \frac{1 - e^{-(x+y)(T-t)}\left(1 + (T-t)(x+y)\right)}{(x+y)^2} v_1(x) v_2(y) \nu(dy)\nu(dx)) \\
&= \langle Q_{T-t} v_1, v_2\rangle_\nu
\end{aligned}$$

where P_τ and Q_τ are given by

$$\begin{aligned}
P_\tau u(x) &= \int_0^\infty \int_0^\tau e^{-sx} e^{-sy} ds u(y) \mu(dy) = \int_0^\infty \frac{1 - e^{-\tau(x+y)}}{x+y} \mu(dy), \\
Q_\tau v(x) &= \int_0^\infty \int_0^\tau s^2 e^{-sx} e^{-sy} ds v(y) \nu(dy) \\
&= \int_0^\infty \frac{1 - e^{-\tau(x+y)}\left(1 + \tau(x+y)\right)}{(x+y)^2} v(y) \nu(dy).
\end{aligned}$$

By Equations (C.21) and (C.22) we have

$$\begin{aligned}
\|P_\tau u\|_{L^1(\mu)} &= \int_0^\infty \int_0^\infty \int_0^\tau e^{-s(x+y)} |u(x)| ds \mu(dx)\mu(dy) \leq C \|u\|_{L^\infty(\mu)} < \infty, \\
\|Q_\tau v\|_{L^1(\nu)} &= \int_0^\infty \int_0^\infty \int_0^\tau s^2 e^{-s(x+y)} |v(x)| ds \nu(dx)\nu(dy) \leq C \|u\|_{L^\infty(\mu)} < \infty,
\end{aligned}$$

for some constant C. The last two inequalities imply that $P_\tau \colon L^\infty(\mu; \mathbb{C}) \to L^1(\mu)$ and $Q_\tau \colon L^\infty(\nu; \mathbb{C}) \to L^1(\nu)$ are bounded linear operators. □

Lemma D.5 (Maximum inequality for OU processes)**.** *There exists a constant $C > 0$ such that for each $t \geq 0$ and $x > 0$*

$$\begin{aligned}
\mathbb{E}\left[\sup_{s \in [0,t]} |Y_s^x|\right] &\leq C \log(1 + tx) x^{-\frac{1}{2}}, \\
\mathbb{E}\left[\sup_{s \in [0,t]} |Z_s^x|\right] &\leq C \log(1 + tx) x^{-\frac{3}{2}}.
\end{aligned}$$

Proof. The inequality for Y^x follows from the maximal inequalities for OU processes developed by Graversen and Peskir [8]. For the process Z^x, we estimate for each

$t \geq 0$ and $x > 0$

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,t]}|Z_s^x|\right] &\leq \mathbb{E}\left[\int_0^t e^{-(t-s)x}|Y_s^x|ds\right] \leq C\int_0^t e^{-(t-s)x}\log(1+sx)x^{-\frac{1}{2}}ds \\ &= C\left[e^{-(t-s)x}\log(1+sx)x^{-\frac{3}{2}}\right]_0^t - C\int_0^t e^{-(t-s)x}(1+sx)^{-1}x^{-\frac{1}{2}}dt \\ &= C\log(1+tx)x^{-\frac{3}{2}} - C\int_0^t e^{-(t-s)x}(1+sx)^{-1}x^{-\frac{1}{2}}ds \\ &\leq C\log(1+tx)x^{-\frac{3}{2}}.\end{aligned}$$

□

Lemma D.6. *Let $G(x,t)$ be deterministic and jointly measurable in $(x,t) \in (0,\infty)\times[0,\infty)$. Assume $Y_0 = Z_0 = 0$. Then, with probability one,*

$$\int_0^\infty\int_0^t |G(x,t)Y_s^x(\omega)|ds\mu(dx) \leq (1\vee t^{\frac{1}{2}})\int_0^\infty\int_0^t |G(x,t)|(1\wedge x^{-\frac{1}{2}})ds\mu(dx),$$
$$\int_0^\infty\int_0^t |G(x,s)Z_s^x(\omega)|ds\nu(dx) \leq (1\vee t^{\frac{3}{2}})\int_0^\infty\int_0^t |G(x,s)|\,(1\wedge x^{-\frac{3}{2}})ds\nu(dx),$$

Proof. Note that for each $s \geq 0$ the random variables $|Y_s^x|$ and $|Z_s^x|$ are half-normal distributed with mean

$$\mathbb{E}\left[|Y_s^x|\right] = \sqrt{\frac{1-e^{-2sx}}{\pi x}}, \quad \text{and} \quad \mathbb{E}\left[|Z_s^x|\right] = \sqrt{\frac{1-e^{-2sx}\left(1+2sx+2s^2x^2\right)}{2\pi x^3}}.$$

By (C.4) we have

$$\begin{aligned}&\int_0^\infty\int_0^t \mathbb{E}\left[|G(x,s)Y_s^x|\right]ds\mu(dx) = \int_0^\infty\int_0^t |G(x,s)|\sqrt{\frac{1-e^{-2sx}}{\pi x}}ds\mu(dx) \\ &\leq (1\vee t^{\frac{1}{2}})\int_0^\infty\int_0^t |G(x,s)|(1\wedge x^{-\frac{1}{2}})ds\mu(dx).\end{aligned}$$

By (C.6) we have

$$\begin{aligned}&\int_0^\infty\int_0^t \mathbb{E}\left[|G(x,s)Z_s^x|\right]ds\nu(dx) \\ &= \int_0^\infty\int_0^t |G(x,s)|\sqrt{\frac{1-e^{-2sx}\left(1+2sx+2s^2x^2\right)}{2\pi x^3}}ds\mu(dx) \\ &\leq (1\vee t^{\frac{3}{2}})\int_0^\infty\int_0^t |G(x,s)|(1\wedge x^{-\frac{3}{2}})ds\nu(dx).\end{aligned}$$

Then, the inequalities hold true with probability one. □

Appendix E. Auxiliary results for Section 4.2

Lemma E.1. *Under Assumption 4.7, the following condition is satisfied:*

$$\sup_{x\in(0,\infty)} p(x)\int_0^t se^{-sx}ds < \infty.$$

Proof. By assumption, there is $\alpha \in (0,2)$ such that $p(x)(1 \wedge x^{-\alpha})$ is bounded in x. Then the lemma follows from the estimate

$$\int_0^t se^{-sx}ds \le \int_0^t s\left(1 \vee \left(\frac{s}{\alpha}\right)^{-\alpha}\right)\left(1 \wedge x^{-\alpha}\right)ds \le \int_0^t \left(s + \left(\frac{s}{\alpha}\right)^{1-\alpha}\right)ds\left(1 \wedge x^{-\alpha}\right)$$
$$= \left(\frac{t^2}{2} + \frac{1}{2-\alpha}\left(\frac{t}{\alpha}\right)^{2-\alpha}\right)\left(1 \wedge x^{-\alpha}\right).$$

□

Lemma E.2. *Let Assumption 4.7 be in place and assume $(Y_0, Z_0) \in L^1(\mu) \times L^1(\nu)$ a.s. Then, for each $0 \le t \le T$ and for all $(u,v) \in L^\infty(\mu;\mathbb{C}) \times L^\infty(\nu;\mathbb{C})$ one has*

$$\int_t^T \left(\langle Y_s, u\rangle_\mu + \langle Z_s, v\rangle_\nu\right)ds = -\langle Y_t, \Phi_1(T-t,u,v)\rangle_\mu - \langle Z_t, \Phi_2(T-t,u,v)\rangle_\nu$$
$$- \int_t^T \langle \Phi_1(T-s,u,v), 1\rangle_\mu \, dW_s$$

with Φ_1, Φ_2 as in Theorem 4.10. In particular, the random variable $\int_t^T(\langle Y_s,u\rangle_\mu + \langle Z_s, v\rangle_\nu)ds$ is Gaussian, given $\mathcal{F}_t$.

$$\Phi_0(\tau,u,v) = \frac{1}{2}\int_0^\tau \langle \Phi_1(s), 1\rangle_\mu^2 ds,$$
$$\Phi_1(\tau,u,v)(x) = \frac{e^{-\tau x}-1}{x}u(x) + \left(\frac{e^{-\tau x}-1}{x^2} + \frac{\tau}{x}e^{-\tau x}\right)p(x)v(x),$$
$$\Phi_2(\tau,u,v)(x) = \frac{e^{-\tau x}-1}{x}v(x).$$

Proof. The time-derivatives of Φ_1, Φ_2 are given by

$$\partial_\tau \Phi_1(\tau,u,v)(x) = -e^{-\tau x}\big(u(x) + \tau p(x)v(x)\big), \quad \partial_\tau\Phi_2(\tau,u,v)(x) = -e^{-\tau x}v(x).$$

It follows from Lemma D.3 that for any $0 \le t \le s$,

$$\langle Y_s, u\rangle + \langle Z_s, v\rangle = -\langle Y_t, \partial_\tau\Phi_1(s-t,u,v)\rangle_\mu - \langle Z_t, \partial_\tau\Phi_2(s-t,u,v)\rangle_\mu$$
$$- \int_t^s \langle \partial_\tau\Phi_1(s-r,u,v), 1\rangle_\mu dW_r.$$

The result follows by integrating over $s \in [t,T]$ and applying Fubini's theorem (Theorem A.1) to each of the three summands above. For the first summand, Condition (A.1) of Theorem A.1 is satisfied by Lemma E.1 and the estimate

$$\int_0^\infty \int_t^T |Y_t^x \partial_\tau \Phi_1(s-t,u,v)| ds\mu(dx)$$
$$\le \|u\|_{L^\infty(\mu)}\|Y_t\|_{L^1(\mu)} + \|v\|_{L^\infty(\nu)}\int_0^\infty |Y_t^x| \int_t^T (s-t)e^{-(s-t)x}dsp(x)\mu(dx)$$
$$= \|u\|_{L^\infty(\mu)}\|Y_t\|_{L^1(\mu)} + \|v\|_{L^\infty(\nu)}\int_0^\infty |Y_t^x| \int_0^{T-t} se^{-sx}dsp(x)\mu(dx) < \infty.$$

For the second summand, Condition (A.1) reads as

$$\int_0^\infty \int_t^T |Z_t^x e^{-(s-t)x}v(x)| ds\nu(dx) \le (T-t)\|v\|_{L^\infty(\nu)}\|Z_t\|_{L^1(\nu)} < \infty.$$

For the third summand, we first use Fubini's theorem to exchange the order of integration with respect to $\mu(dx)$ and dW_r:

$$\int_t^s \langle \partial_\tau \Phi_1(s-r,u,v),1\rangle_\mu dW_r = -\int_0^\infty \int_t^s e^{-(s-r)x}\big(u(x)+(s-r)p(x)v(x)\big)dW_r\mu(dx).$$

This is allowed because Equation (A.2) is satisfied by Equations (C.11) and (C.15):

$$\int_0^\infty \sqrt{\int_t^s e^{-2(s-r)x}}|u(x)|\mu(dx) < \infty,$$
$$\int_0^\infty \sqrt{\int_t^s (s-r)^2 e^{-2(s-r)x}}|v(x)|\nu(dx) < \infty,$$

Then we interchange the order of integration with respect to dW_r and the product measure $\mu(dx)ds$. Then the third summand reads as:

$$\begin{aligned}-\int_t^T \int_0^\infty \int_t^s & e^{-(s-r)x}\big(u(x)+(s-r)p(x)v(x)\big)dW_r\mu(dx)ds \\ &= -\int_t^T \int_r^T \int_0^\infty e^{-(s-r)x}\big(u(x)+(s-r)p(x)v(x)\big)\mu(dx)dsdW_r.\end{aligned}$$

This is allowed because Condition (A.2) is satisfied by Equations (C.23) and (C.24):

$$\int_t^T \int_0^\infty \sqrt{\int_t^s e^{-2(s-r)}dr}|u(x)|\mu(dx)ds < \infty,$$
$$\int_t^T \int_0^\infty \sqrt{\int_t^s (s-r)^2 e^{-2(s-r)}dr}|v(x)|\nu(dx)ds < \infty.$$

Finally, we exchange the innermost integrals $\mu(dx)$ and ds, which is justified by Condition (A.1) and Equations (C.16) and (C.17). Then the third summand is given by

$$\begin{aligned}-\int_t^T \int_0^\infty \int_r^T & e^{-(s-r)x}\big(u(x)+(s-r)p(x)v(x)\big)ds\mu(dx)dW_r \\ &= -\int_t^T \langle \Phi_1(T-r,u,v),1\rangle_\mu dW_r.\end{aligned}$$

□

Lemma E.3. *Under Assumption 4.7, the expressions* $\langle Y_t, \Phi_1(T-t,u,v)\rangle_\mu$ *and* $\langle Z_t, \Phi_2(T-t,u,v)\rangle_\nu$ *are continuous semimartingales in* $t \in [0,T]$*, for each fixed* $T > 0$ *and* $(u,v) \in L^\infty(\mu)\times L^\infty(\nu)$*.*

Proof. We verify the conditions of Theorem 2.10. In the following estimates it can be assumed without loss of generality that the functions u and v are equal to 1 because they are bounded. Conditions (2.6) and (2.7) for $f_t^x = \Phi_1(T-t,u,v)(x)$

are satisfied by Equations (C.18), (C.19) and (C.20):

$$\int_0^\infty \int_0^t |\partial_s f_s^x - x f_s^x|(1 \wedge x^{-\frac{1}{2}}) ds\mu(dx)$$
$$= \int_0^\infty \int_0^t \left(1 + \frac{1-e^{-(T-s)x}}{x} p(x)\right)(1 \wedge x^{-\frac{1}{2}}) ds\mu(dx) < \infty,$$
$$\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds}\mu(dx) \le \int_0^\infty \sqrt{\int_0^t 2\left(\frac{e^{-(T-s)x}-1}{x}\right)^2 ds}\mu(dx)$$
$$+ \int_0^\infty \sqrt{\int_0^t 2\left(\frac{e^{-(T-s)x}-1}{x^2} + \frac{\tau}{x} e^{-(T-s)x}\right)^2 ds}\nu(dx) < \infty.$$

Conditions (2.8) and (2.9) are satisfied for $g_t^x = \Phi_2(T-t,u,v)(x)$ by Equation (C.18):

$$\int_0^\infty \int_0^t |\partial_s g_s^x - x g_s^x|(1 \wedge x^{-\frac{3}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t (1 \wedge x^{-\frac{3}{2}}) ds\nu(dx) < \infty,$$
$$\int_0^\infty \int_0^t |g_s^x|(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t \frac{1-e^{-\tau x}}{x}(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty.$$

Thus, we have verified the conditions of Theorem 2.10 and the statement of the lemma follows. □

Lemma E.4. *Under Assumption 4.7, the expressions* $\langle Y_t, \partial_\tau \Phi_1(\tau,u,v)\rangle_\mu$ *and* $\langle Z_t, \partial_\tau \Phi_2(\tau,u,v)\rangle_\nu$ *are continuous semimartingales in* $t \in [0,T]$*, for each fixed* $\tau > 0$ *and* $(u,v) \in L^\infty(\mu) \times L^\infty(\nu)$.

Proof. We calculate

$$\partial_\tau \Phi_1(\tau,u,v)(x) = -e^{-\tau x}\big(u(x) + \tau p(x) v(x)\big), \quad \partial_\tau \Phi_2(\tau,u,v)(x) = -e^{-\tau x} v(x).$$

We show the semimartingale property by verifying the conditions of Theorem 2.10. In the following estimates it can be assumed without loss of generality that the functions u and v are equal to 1 because they are bounded. Conditions (2.6) and (2.7) for $f_t^x = \partial_\tau \Phi_1(\tau,u,v)(x)$ are satisfied by Equations (C.7)–(C.10):

$$\int_0^\infty \int_0^t |\partial_s f_s^x - x f_s^x|(1 \wedge x^{-\frac{1}{2}}) ds\mu(dx)$$
$$= \int_0^\infty \int_0^t x e^{-\tau x}\big(1 + \tau p(x)\big)(1 \wedge x^{-\frac{1}{2}}) ds\mu(dx) < \infty,$$
$$\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds}\mu(dx) \le \int_0^\infty \sqrt{\int_0^t 2e^{-2\tau x} ds}\mu(dx)$$
$$+ \int_0^\infty \sqrt{\int_0^t 2\tau^2 e^{-2\tau x} ds}\nu(dx) < \infty.$$

Conditions (2.8) and (2.9) for $g_t^x = \partial_\tau \Phi_2(\tau,u,v)(x)$ are satisfied by Equation (C.10):

$$\int_0^\infty \int_0^t |\partial_s g_s^x - x g_s^x|(1 \wedge x^{-\frac{3}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t x e^{-\tau x}(1 \wedge x^{-\frac{3}{2}}) ds\nu(dx) < \infty,$$
$$\int_0^\infty \int_0^t |g_s^x|(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) = \int_0^\infty \int_0^t x e^{-\tau x}(1 \wedge x^{-\frac{1}{2}}) ds\nu(dx) < \infty.$$

Thus, we have verified the conditions of Theorem 2.10 and the statement of the lemma follows. □

Appendix F. Auxiliary results for Section 4.3

Lemma F.1. *Under Assumption 2.3, the expressions* $\langle Y_t, \partial_\tau \phi_1(\tau, -u, -v)\rangle_\mu$ *and* $\langle Z_t, \partial_\tau \phi_2(\tau, -u, -v)\rangle_\nu$ *are continuous semimartingales in* $t \in [0, \infty)$ *for each fixed* $\tau > 0$ *and* $(u, v) \in L^\infty(\mu) \times L^\infty(\nu)$*.*

Proof. We verify the conditions of Theorem 2.10. As u and v are bounded we may assume without loss of generality in the following estimates that $u = v = 1$. Conditions (2.6)–(2.9) for $f_t^x = \partial_\tau \phi_1(\tau, -u, -v)(x)$ and $g_t^x = \partial_\tau \phi_2(\tau, -u, -v)(x)$ are satisfied by Equations (C.7)–(C.10):

$$\begin{aligned}
&\int_0^\infty \int_0^t |\partial_s f_s^x - x f_s^x|(1 \wedge x^{-\frac{1}{2}}) ds \mu(dx) = t \int_0^\infty |x \partial_\tau \phi_1(\tau, -u, -v)| (1 \wedge x^{-\frac{1}{2}}) \mu(dx) \\
&\qquad \le t \int_0^\infty x^2 e^{-x\tau} \mu(dx) + t \int_0^\infty x e^{-x\tau} \nu(dx) + t\tau \int_0^\infty x^2 e^{-x\tau} \nu(dx) < \infty, \\
&\int_0^\infty \sqrt{\int_0^t (f_s^x)^2 ds} \mu(dx) = \sqrt{t} \int_0^\infty |\partial_\tau \phi_1(\tau, -u, -v)| \mu(dx) \\
&\qquad \le \sqrt{t} \int_0^\infty x e^{-x\tau} \mu(dx) + \sqrt{t} \int_0^t e^{-x\tau} \nu(dx) + \sqrt{t}\tau \int_0^t x e^{-x\tau} \nu(dx) < \infty, \\
&\int_0^\infty \int_0^t |\partial_s g_s^x - x g_s^x|(1 \wedge x^{-\frac{3}{2}}) ds \nu(dx) = t \int_0^\infty |x \partial_\tau \phi_2(\tau, -u, -v)| (1 \wedge x^{-\frac{3}{2}}) \nu(dx) \\
&\qquad = t \int_0^\infty x^2 e^{-x\tau} \nu(dx) < \infty, \\
&\int_0^\infty \int_0^t |g_s^x|(1 \wedge x^{-\frac{1}{2}}) ds \nu(dx) = t \int_0^\infty |\partial_\tau \phi_2(\tau, -u, -v)| \nu(dx) \\
&\qquad \le t \int_0^\infty e^{-x\tau} \nu(dx) < \infty.
\end{aligned}$$

□

Appendix G. Auxiliary results for Section 5

Lemma G.1. *For each* $\tau \ge 0$ *any symmetric two-tensor* $w \in L^\infty(\mu; \mathbb{C}) \otimes L^\infty(\mu; \mathbb{C})$ *has a representation as a sum of squares*

$$w = \sum_{k=1}^n \vartheta_k u_k \otimes u_k, \quad \text{with } \vartheta_k \in \mathbb{C} \text{ and } u_k \in L^\infty(\mu; \mathbb{C}),$$

such that

$$\langle P_\tau u_k, u_l \rangle_\mu = \delta_{kl},$$

where P_τ *is the covariance operator defined in Lemma D.4.*

Proof. Let $\mathsf{H}_\tau \subseteq L^1(\mu)$ be the reproducing kernel Hilbert space associated to the covariance operator $P_\tau \colon L^\infty(\mu; \mathbb{C}) \to L^1(\mu; \mathbb{C})$ (see Appendix B). The symmetric tensor $(P_\tau \otimes P_\tau)w \in \mathsf{H}_\tau \otimes \mathsf{H}_\tau$ can be diagonalized with respect to the scalar product of H_τ, giving the representation

$$(P_\tau \otimes P_\tau)w = \sum_{k=1}^{n} \vartheta_k P_\tau u_k \otimes P_\tau u_k = (P_\tau \otimes P_\tau) \sum_{k=1}^{n} \vartheta_k u_k \otimes u_k.$$

The mapping $P_\tau \otimes P_\tau$ is injective because P_τ is injective by Lemma G.2, and the desired representation of w follows. $\square$

Lemma G.2. *For any $\tau > 0$, the mapping $P_\tau \colon L^\infty(\mu; \mathbb{C}) \to \mathsf{H}_\tau$ is injective.*

Proof. If $P_\tau u = 0$ for some $u \in L^\infty(\mu; \mathbb{C})$, then

$$0 = \langle P_\tau u, P_\tau u \rangle_{\mathsf{H}_\tau} = \langle P_\tau u, u \rangle_\mu = \int_0^\tau \left(\int_0^\infty u(x) e^{-sx} \mu(dx) \right) ds.$$

Therefore, the Laplace transform $\mathcal{L}(u\mu)(s)$ of the signed measure $u\mu$ vanishes at almost all $s \in [0, \tau]$. As $\mathcal{L}(u\mu)(s)$ is analytic in s, it vanishes identically. By the injectivity of the Laplace transform [9, Section 3.8], the signed measure $u\mu$ vanishes, which is equivalent to $u = 0$ in $L^\infty(\mu; \mathbb{C})$. $\square$

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