Numerical Solution of Pion Bethe-Salpeter-Equation

Project in: Computational Methods of Particle Physics

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Contents

1 Introduction and Project Outline

Using methods of Quantum Chromodynamics (QCD) the goal of this project is to calculate the mass of the pion, the lightest hadron, which is a quark-anti-quark bound state. To do so, we will first formulate the quark propagator Dyson-Schwinger equation (DSE) which we need afterwards to solve the bound state equation, specifically the Bethe-Salpeter equation (BSE). Due to the complexity of the problem a number of approximations is needed to solve the BSE numerically. However, we will see that even with crude approximations the experimental value of the pion mass can be reproduced within a few MeV.

Conventions

Throughout this project we will use the Euclidean metric

$$g_{\mu\nu} = \delta_{\mu\nu} = \mathrm{diag}(1,1,1,1)$$

and consequently also the Euclidean gamma matrices

$$\gamma_i = \begin{pmatrix} 0 & i\sigma^i \\ -i\sigma^i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$$

where σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will use Greek letters from the middle of the alphabet $(\mu, \nu, ...)$ for Lorentz indices, from the beginning of the alphabet $(\alpha, \beta, ...)$ for Dirac indices, Latin letters from the middle of the alphabet $(i, j, ...)$ for fundamental $SU(3)_c$ indices and from the beginning of the alphabet $(a, b, ...)$ for adjoint color indices. Identical indices appearing twice in an expression are implicitly summed over. At some point, we will also use $i, j, ...$ as summation indices of the discretized problem together with an explicit $\sum_i$ sign but there, it should be unambiguous.

1.1 Quantum Chromodynamics

Quantum Chromodynamics, the strongly interacting theory, underlies the interaction between quarks which is mediated by gluons. It is a non-Abelian gauge theory based on the group $SU(3)_c$ and the newly introduced degrees of freedom are called *color* which is the analogue of electric charge in quantum electrodynamics. The QCD Lagrangian consists of a part describing the dynamics of quarks and the gauge field (whose elementary excitations are called *gluons*) and a part describing their interaction through the covariant derivative

$$\mathcal{L}_{\mathrm{QCD}} = \frac{1}{2}\mathrm{tr}_c\ F_{\mu\nu}F_{\mu\nu} + \sum_{\text{flavors}} \bar{q}^{(f)} \left(-i\not{D} + m^{(f)}\right) q^{(f)}.$$

The quantities shown are defined by

$$\not{D} = \gamma_\mu D_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\left[A_\mu, A_\nu\right]$$
$$D_\mu = \partial_\mu + igA_\mu \qquad A_\mu = A_\mu^a t^a.$$

A_μ^a are the eight gluon fields and t^a are the generators of the $SU(3)_c$ group with structure constants f^{abc} satisfying

$$\left[t^a, t^b\right] = if^{abc}t^c.$$

Here, they are used in their defining representation, i.e. t^a are the Gell-Mann matrices. Using ϕ as a short-hand notation for quark and gluon fields, we can define the generating functional of vertex functions

$$\Gamma[\Phi(x)] = -W[J(x)] + \int d^4x\ \Phi(x)J(x)$$

where $\Phi(x) = \langle\phi\rangle_J = \delta W/\delta J(x)$ and $W[J]$ is the generating functional of connected correlation functions depending on some externel source J. It is in turn defined via

$$Z[J] = e^{W[J]} = \int \mathcal{D}\phi\ e^{-S[\phi]+\int d^4x J(x)\phi(x)}$$

where S is the Euclidean QCD action $\int d^4x\ \mathcal{L}_{\text{QCD}}(x)$. The gluon propagator, quark propagator and quark-gluon vertex can then be obtained by performing functional derivatives on Γ:

$$D_{\mu\nu}^{ab}(x,y) = \left(\frac{\delta^2\Gamma}{\delta A_\mu^a(x)\delta A_\nu^b(y)}\right)^{-1}\Bigg|_{\phi=0}$$
$$S^{ij}(x,y) = \left(\frac{\delta^2\Gamma}{\delta\bar{q}^i(x)\delta q^j(y)}\right)^{-1}\Bigg|_{\phi=0}$$
$$\Gamma_\mu^{a,ij}(x,y,z) = -\left.\frac{\delta^3\Gamma}{\delta A_\mu^a(x)\delta\bar{q}^i(y)\delta q^j(z)}\right|_{\phi=0}$$

In momentum space and on tree-level, i.e. without taking quantum corrections into account, those three function correspond to the simple Feynman rules (in Landau gauge) of Fig. 1.

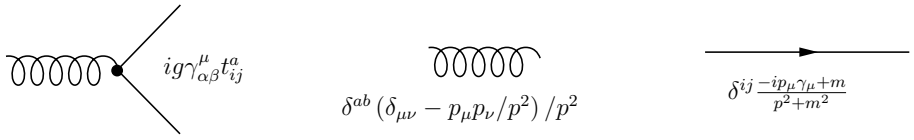

Figure 1: Tree-level (bare) Feynman rules

However, we are interested in the quantum corrected *dressed* quantities. It turns out that the gluon and quark propagator can be dressed by simply introducing dressing

functions $Z(p^2)$, $A(p^2)$ and $B(p^2)$. For the quark-gluon vertex on the other hand, this is not possible and one has to make a general ansatz:

$$D^{ab}_{\mu\nu}(p) = \delta^{ab}\left[\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{Z(p^2)}{p^2}\right] \tag{1}$$

$$S^{ij}(p) = \delta^{ij}\frac{-iA(p^2)\not{p} + B(p^2)}{p^2A^2(p^2) + B^2(p^2)} \tag{2}$$

$$\Gamma^{a\ ij}_{\mu}(k;p,q) = igt^a_{ij}\sum_{i=1}^{12}\tau^i_\mu(k;p,q)h^i(k;p,q). \tag{3}$$

These are the three building blocks which encode all the information about the behaviour of quarks and gluons yet they tell us nothing because we don't know what the dressing functions are. Later on, we will perform some approximations and solve for them numerically.

Remark: Quark Flavors

There are 6 known quark flavors (u, d, s, c, t, b) but for our purposes (calculating the pion mass) only the two lightest quarks (u, d) are important. Even though stating a rest mass for objects that aren't observable in isolation is somewhat delicate, our best guess for their masses are [1] $m_u = 2.16^{+0.49}_{-0.26}$ MeV and $m_d = 4.67^{+0.48}_{-0.17}$ MeV. Seen from bound state energy scales of $m_\pi \sim \mathcal{O}(100\text{ MeV})$ those masses are roughly identical and we will treat them as if they were, which is called the *isospin limit* because once we set their masses to be equal our Lagrangian exhibits an $SU(2)$ flavor symmetry and the corresponding quantum number is called *isospin*.

1.2 Dyson-Schwinger-Equations (DSE)

The behaviour of quarks and gluons is encoded in their correlation functions listed above, namely the quark-gluon vertex and the quark and gluon propagator. Their dynamics are described by the Dyson-Schwinger equations which are a set of infinitely many equations and are often called equations of motion of correlation functions in analogy to classical mechanics. All DSEs for connected correlation functions can be obtained from the master equation

$$-\left.\frac{\delta S}{\delta\Phi(x)}\right|_{\Phi(x')=\frac{\partial W[J]}{\partial J(x')}+\frac{\partial}{\partial J(x')}} + J(x) = 0$$

by differentiation with respect to the sources J. There also exists a version for vertex functions which is of greater value for us:

$$-\left.\frac{\delta S}{\delta\phi(x)}\right|_{\phi(x')=\Phi(x')+\int dz D(x',z)^J\delta/\delta\Phi(z)} + \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = 0$$

As it is not necessary for the present project to write out all the details as we will only need the explicit form of the quark propagator DSE we refer the interested reader to a more detailed discussion of DSEs, e.g. in [2].

1.3 Bound-State-Equations

The only thing that is accessible in a quantum field theory are its correlation functions. Therefore, to learn about a two-quark bound state we have to investigate four-point-functions describing the propagation of two quarks. Usually, one thinks about four-point-functions as being some sort of $2 \to 2$ scattering process. However, a bound state is merely two particles coming together, interacting with each other for some time and (depending on whether the resulting particle is stable or not) eventually decaying into two particles again. Just like the propagator carries information about elementary particles in its pole structure, the poles of this four-point-function tells us the mass of the bound state[1].This should make the usage of a four-point-function somewhat intuitive. The generic structure of such a function is given by

$$G^{(4)} = G_0^{(4)} + G_0^{(4)} T^{(4)} G_0^{(4)} \tag{4}$$

where $G_0^{(4)}$ is the product of two quark-propagators

$$G_0^{(4)}(p_1, p_2, p_3, p_4) = S(p_1)\delta(p_1 - p_4)S(p_2)\delta(p_2 - p_3)$$

and $T^{(4)}$ contains all $2 \to 2$ scattering process. Therefore, the first part of Eq. (4) describes side-by-side propagation of two quarks but the (interesting) interactions between the two are completely carried by the second part. The scattering matrix can be related to the four-quark scattering kernel via

$$\begin{aligned} T^{(4)} &= K^{(4)} + K^{(4)} G_0^{(4)} K^{(4)} + K^{(4)} G_0^{(4)} K^{(4)} G_0^{(4)} K^{(4)} + ... \\ &= K^{(4)}(1 + G_0^{(4)} T^{(4)}) \end{aligned} \tag{5}$$

which is called the *inhomogeneous Bethe-Salpeter equation* (BSE) and we will later investigate it to calculate the pion mass.

2 Quark Propagator

First things first: To start talking about mesons, i.e. bound states of quarks, we first have to get quarks under control. Graphically, the DSE for the quark propagator is depicted in Fig. 2. After applying the appropriate Feynman Rules it reads

$$(S_{ij})^{-1}(p) = (S_{ij}^{(0)})^{-1} - \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Gamma_\mu^{(0)a\ ki}(k; -q, p) S^{kl}(q) \Gamma_\nu^{b\ jl}(-k; -p, q) D_{\mu\nu}^{ab}(k) \tag{6}$$

where S, D and Γ are the fully dressed quark-propagator, gluon-propagator and quark-gluon-vertex (1-3) whereas $S^{(0)}$ and $\Gamma^{(0)}$ are the respective bare quantities.

[1]Since the four-point-function is related to the $2 \to 2$ scattering cross section one can understand this rather nicely: By tuning the external energy one will see a sharp increase once the energy

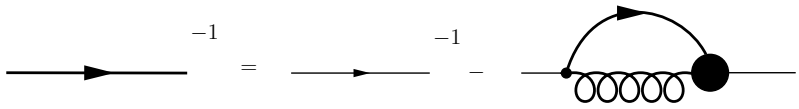

Figure 2: Pictorial representation of Eq. (6). Note that the bold face lines correspond to the fully dressed propagators and a large circle on a vertex also indicates that it's the corresponding dressed version.

2.1 Equations for Quark Propagator Dressing Functions

As shown in the introduction, the bare vertex is simply given by $\Gamma_\mu^{(0)a\ ki} = ig\gamma_\mu t^a_{ki}$. The dressed one, however, is much more complicated, see Eq. (3). To keep the computations rather simple we therefore employ the approximation

$$\Gamma_\mu^{a\ ki} = ig\gamma_\mu t^a_{ki}\Gamma(k;p,q)$$

where Γ is now sort of an effective dressing function of the quark-gluon-vertex (do *not* confuse it with the actual vertex). After inserting all quantities and the mentioned approximation into (6) we can extract two (coupled) integral equations for the dressing functions $A(p^2)$ and $B(p^2)$

$$A(p^2) = 1 + \frac{4g^2}{3p^2}\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{A(p^2)}{q^2A^2(p^2)+B^2(p^2)}\left(p\cdot q + 2\frac{(k\cdot p)^2}{k^2}\right)\frac{Z(k^2)}{k^2}\Gamma(-k;-p,q)$$
$$B(p^2) = m + 4g^2\int \frac{\mathrm{d}^4q}{(2\pi)^4}\frac{B(p^2)}{q^2A^2(p^2)+B^2(p^2)}\frac{Z(k^2)}{k^2}\Gamma(-k;-p,q).$$

As an additional approximation we collect the gluon dressing function $Z(k^2)$ and the effective quark-gluon dressing function Γ into an effective interaction parameterized by a strength D and a scale ω

$$g^2\frac{Z(k^2)}{k^2}\Gamma(-k;-p,q) \to 4\pi^2 D\frac{k^2}{\omega^2}e^{-k^2/\omega^2}. \tag{7}$$

After introducing hyperspherical coordinates and dimensionless units the two equations finally turn into

$$A(x) = 1 + D\omega^2\int_0^\infty \frac{\mathrm{d}y\ yA(y)}{yA^2(y)+B^2(y)/\omega^2}\times$$
$$\times\frac{2}{\pi}\int_{-1}^1 \mathrm{d}z\sqrt{1-z^2}\left[-\frac{2y}{3}+\left(1+\frac{y}{x}\right)\sqrt{xy}z-\frac{4yz^2}{3}\right]e^{-x-y+2\sqrt{xy}z} \tag{8}$$
$$B(x) = m + D\omega^2\int_0^\infty \frac{\mathrm{d}y\ yB(y)}{yA^2(y)+B^2(y)/\omega^2}\times$$
$$\times\frac{2}{\pi}\int_{-1}^1 \mathrm{d}z\sqrt{1-z^2}\,(x+y-2\sqrt{xy}z)\,e^{-x-y+2\sqrt{xy}z} \tag{9}$$

to form the bound state is met. Therefore, the poles which show up in theoretical considerations correspond to actual physical conditions.

with $x \equiv p^2/\omega^2$, $y \equiv q^2/\omega^2$ and $z \equiv p \cdot q/\sqrt{p^2 q^2}$. The next step is to solve these equations which then ultimately yield us the fully dressed quark propagator (at least within the employed approximations).

2.2 Solving for the Quark Propagator Dressing Functions

This chapter presents our analytical and numerical results for exercise 3 in the lecture notes. The tasks were to analytically solve the radial parts of (8) and (9), which will turn out to be a sum of modified Bessel functions, and to ultimately determine the quark propagator dressing functions $A(x)$ and $B(x)$ by numerically approximating the solution to the coupled integral equations for A and B. Initially, those solutions are calculated on the positive real half-axis which is sufficient to analytically extend it to the region bounded by the parabola in the complex plane, as discussed in [3]. This continuation will be discussed in Sec. 3.4 and covers the content of exercise 4 in the lecture notes.

Analytical Considerations

The quark propagator functions $A(x)$ and $B(x)$, which were derived in the lecture are analytically solved here by integrating the angular z component. First we must argue why it is legitimate to perform an analytical continuation of the external variable x: When specifically looking at the exponential functions and Taylor expanding therms, one realizes that they all have an exponent of $\sqrt{x}$ in them and are therefore odd, leading to them disappearing when performing the angular integration. We start from the integral equations (8) and (9) for the dressing functions $A(x)$ and $B(x)$. The steps of the analytical solution will be portrayed in more detail for the dressing function $A(x)$ and can then be used, with only slight modifications to do the equivalent derivation for $B(x)$. First of all, consider the following term of the integrals which we label as C for convenience

$$C = \int_{-1}^{1} \sqrt{1-z^2} e^{-(x+y-2\sqrt{xy}z)}\, dz.$$

After performing partial integration we are left with:

$$C = \frac{1}{2\sqrt{xy}} e^{x+y} \int_{-1}^{1} e^{2\sqrt{xy}z} \frac{z}{\sqrt{1-z^2}}\, dz$$

where we can identify

$$I_1(s) = \frac{1}{\pi} \int_{-1}^{1} e^{sx} \frac{x}{\sqrt{1-x^2}}\, dx$$

as the first modified Bessel function which allows us to rewrite C as

$$C = \frac{\pi}{2\sqrt{xy}} e^{x+y} I_1(2\sqrt{xy}).$$

Next, we look at the part of the original integral of A containing C multiplied with the first term in the square brackets, $-2y/3$:

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-z^2}e^{-(x+y-2\sqrt{xy}z)}\left(-\frac{2}{3}y\right)\,dz\;=-\frac{2y}{3\sqrt{xy}}e^{-x-y}I_1(2\sqrt{xy}).$$

The same is done for the term second term in the square brackets containing a multiple of z:

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-z^2}e^{-(x+y-2\sqrt{xy}z)}\left(1+\frac{y}{x}\right)\sqrt{xy}z\,dz\;=e^{-x-y}\left(1+\frac{y}{x}\right)\frac{1}{\pi}\int_{-1}^{1}e^{2\sqrt{xy}z}\frac{2z^2-1}{\sqrt{1-z^2}}\,dz.$$

Here, the second modified Bessel function can be identified as the following:

$$I_2(s)=\frac{1}{\pi}\int_{-1}^{1}e^{sx}\frac{2x^2-1}{\sqrt{1-x^2}}\,dx.$$

We can therefore continue writing the integral using the modified Bessel function:

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-z^2}e^{-(x+y-2\sqrt{xy}z)}\left(1+\frac{y}{x}\right)\sqrt{xy}z\,dz\;=e^{-x-y}\left(1+\frac{y}{x}\right)I_2(2\sqrt{xy})$$

Finally, the last term in the square brackets, $-4yz^2/3$, will be evaluated with the integral C:

$$\frac{2}{\pi}\int_{-1}^{1}\sqrt{1-z^2}e^{-(x+y-2\sqrt{xy}z)}\frac{-4}{3}yz^2\,dz\;=-\frac{1}{3}e^{(-x-y)}\frac{1}{\sqrt{xy}}(I_1(2\sqrt{xy})+3I_3(2\sqrt{xy}))$$

Here, the third modified Bessel function was used, which is defined by the integral

$$I_3(s)=\frac{1}{\pi}\int_{-1}^{1}e^{sx}\frac{4x^3-1}{\sqrt{1-x^2}}\,dx.$$

Putting all three parts of the integral together we obtain the following result for the integral equation

$$\begin{aligned}A(x)=1+D\omega^2\int_0^\infty\frac{dy\;yA(y)}{yA(x)^2+\frac{1}{\omega^2}B(y)^2}e^{-x-y}\frac{1}{\sqrt{xy}}\times\\ \times\left[-\frac{2}{3}I_1(2\sqrt{xy})+\left(1+\frac{x}{y}\right)\sqrt{xy}I_2(2\sqrt{xy})-I_3(2\sqrt{xy})-\frac{1}{3}I_1(2\sqrt{xy})\right].\end{aligned}$$

Using the recurrence relation for Bessel functions, this can be rewritten in terms of I_1 and I_2 only. The final equation for $A(x)$ is then given by

$$A(x)=1+D\omega^2\int_0^\infty\frac{dy\;yA(y)}{yA(x)^2+\frac{1}{\omega^2}B(y)^2}e^{-x-y}\left(-2\frac{\sqrt{x}}{\sqrt{y}}I_1(2\sqrt{xy})+\frac{2+x+y}{x}I_2(2\sqrt{xy})\right).$$

Similar steps can be performed to rewrite the second equation for the dressing function $B(x)$. Again, separately the terms in the square brackets are evaluated and sorted by order of z that is contained in them. This yields

$$B(x) = m + D\omega^2 \int_0^\infty \frac{dy\ yB(y)}{yA(x)^2 + \frac{1}{\omega^2}B(y)^2} e^{-x-y} \left(\frac{x+y}{\sqrt{xy}} I_1(2\sqrt{xy}) - 2I_2(2\sqrt{xy}) \right).$$

Since the integral runs over all of $\mathbb{R}^+$ we realize that we have to evaluate the modified Bessel functions for very large arguments. Numerically, this is very prone to large errors. However, we can rewrite the integral equations as

$$\begin{aligned} A(x) = 1 + D\omega^2 \int_0^\infty & \frac{dy\ yA(y)}{yA(x)^2 + \frac{1}{\omega^2}B(y)^2} \times \\ & \times e^{-x-y+2\sqrt{xy}} \left(-2\frac{\sqrt{x}}{\sqrt{y}} \tilde{I}_1(2\sqrt{xy}) + \frac{2+x+y}{x} \tilde{I}_2(2\sqrt{xy}) \right) \end{aligned} \tag{10}$$

$$\begin{aligned} B(x) = m + D\omega^2 \int_0^\infty & \frac{dy\ yB(y)}{yA(x)^2 + \frac{1}{\omega^2}B(y)^2} \times \\ & \times e^{-x-y+2\sqrt{xy}} \left(\frac{x+y}{\sqrt{xy}} \tilde{I}_1(2\sqrt{xy}) - 2\tilde{I}_2(2\sqrt{xy}) \right) \end{aligned} \tag{11}$$

with the exponentially modified Bessel functions

$$\tilde{I}_j(x) \equiv e^{-x} I_j(x)$$

which can be well approximated by their power series for small x and by their asymptotic behaviour for large x.

Numerical Results

The above integral equations (10,11) for $A(x)$ and $B(x)$ have to be solved numerically (iteratively). The problem we now face is two-fold: On the one hand, we have to discretize the functions $A(x)$ and $B(x)$ to treat them on a computer. On the other hand, we need to know those functions on $[0, \infty]$ in order to perform the integration in each step. The key observation is that we can choose the discretization nodes to be the same on both sides of the equations, i.e. $x_i = y_i$. Additionally, we may choose those to be the roots of some special polynomial, e.g. a Laguerre, Legendre or Chebyshev polynomial. Since the numerical integration (as explained in Appendix 5.1) only needs the function values at these points and in each step we again calculate a new approximation for the function *at these points* we never have to interpolate the function or anything. The downside to this is, that we may not arbitrarily choose the nodes as those are prescribed by the roots of said polynomials. The difference between some methods is discussed in Appendix 5.2 and it turns out that for the present task Gauss-Chebyshev integration after performing the variable transformation[2] $y = (1+u)/(1-u)$ yields the

[2] To have the right integration domain

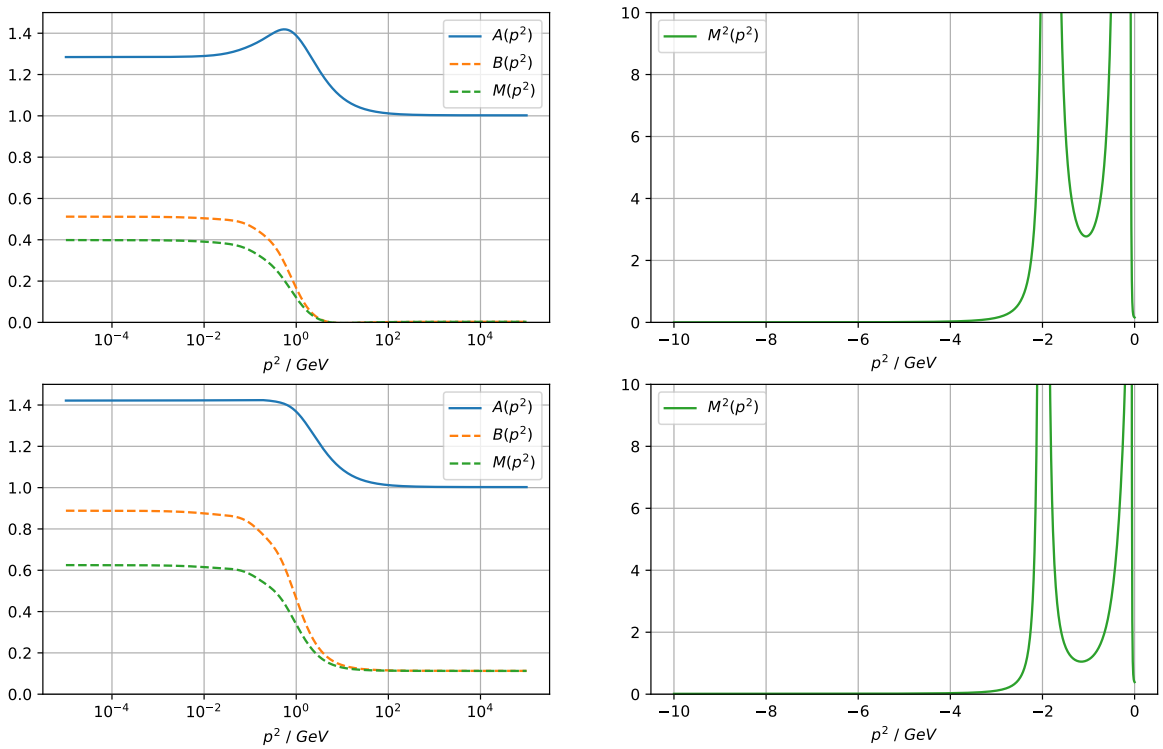

Figure 3: *Left*: Plot of the space-like quark propagator dressing functions A, B and the mass function $M = B/A$ as a function of the momentum squared for $\omega = 0.5$ GeV, $D = 16.0$ GeV^{-2} and $m = 0$ (top) or $m = 0.115$ GeV (bottom), respectively. *Right*: Time-like mass-squared function for identical parameters. Note that this behaviour undergoes huge changes when we alter the interpolation procedure.

best results. Using Chebyshev polynomials also permits an easy extrapolation afterwards (explained in Appendix 5.3). Finally, we arrive at our first intermediate result (Fig. 3) which is in agreement with [3] for spacelike momenta p^2. The behaviour for timelike momenta matches their results qualitatively in the sense that there is definitely non-trivial behaviour going on to the left of the origin, however this behaviour strongly depends on the degree of the polynomials used for extrapolation in our case. As we will later argue, this has no effect on the end result as we will make sure that we only pick values from regions in the complex plane which are robust to changes in the interpolation procedure. The physical meaning of the mass function $M(p^2) = B(p^2)/A(p^2)$ is discussed in [3] and we will not go into it here. Nevertheless, we want to point out that our calculations for $M(p^2 = 0)$ (which is relevant for dynamical chiral symmetry breaking) yield results which are in very good agreement with theirs, see Tab. 1.

With the knowledge of $A(p^2)$ and $B(p^2)$ we can reconstruct the quark propagator which we will need in the meson bound state equation in the following section.

ω / GeV	D / GeV^{-2}	m / GeV	$M(p^2=0)$	$M(p^2=0)$ in [3]
0.40	45	0	0.524	0.520
		5×10^{-3}	0.535	0.531
		0.12	0.714	0.709
0.45	25	0	0.452	0.450
		5×10^{-3}	0.465	0.462
		0.12	0.661	0.657
0.45	16	0	0.399	0.397
		5×10^{-3}	0.414	0.413
		0.12	0.632	0.622

Table 1: Benchmark results for $M(p^2=0)$ for various values of ω, D and m.

3 Bound State Equation

By applying a few approximations we managed to calculate the momentum dependent quark propagator which is an important first step to describe bound states. As we will see in the following, though, it needs quite some additional approximations to make the task of solving the bound state equations feasible.

3.1 Bethe-Salpeter equation for four-point-function

We start from the generic BSE for a four-point-function (5). Remember that we are merely interested in the poles of $G^{(4)}$: Since $G_0^{(4)}$ only has trivial poles at the quark masses, the poles relevant for bound states hide in $T^{(4)}$. We now make the assumption that $T^{(4)}$ has a simple pole at the external momentum (squared sum of initial/final state energy) $P^2 = -m_\mathrm{B}^2$, i.e. at the bound state mass (which we eventually want to calculate), and is analytic anywhere else. This allows us to expand $T^{(4)}$ into a Laurent series around that pole

$$T^{(4)} = \frac{c_{-1}}{P^2 + m_\mathrm{B}^2} + c_0 + c_1(P^2 + m_\mathrm{B}^2) + ...$$

The coefficient c_{-1} of course still carries the index structure of $T^{(4)}$ and it is furthermore still momentum dependent. However, this dependence can be factorized (see chapter 6.1.1 in [2]) as

$$c_{-1}(p_1, p_2, p_3, p_4) \propto \Gamma(P; p_1 - p_2)\bar{\Gamma}(P; p_3 - p_4)$$

which intuitively describes some object Γ entering the interaction vertex and $\bar{\Gamma}$ leaving it. Overall, they are characterized by the external momentum $P^2 = (p_1 + p_2)^2$ as well as the relative momenta of the initial, respectively final quarks $(p_1 - p_2, p_3 - p_4)$. Consequently, close to the pole, the behaviour of the scattering matrix is

$$T^{(4)} \xrightarrow{P^2 \to -m_\mathrm{B}^2} \mathcal{N} \frac{\Gamma\bar{\Gamma}}{P^2 + m_\mathrm{B}^2}$$

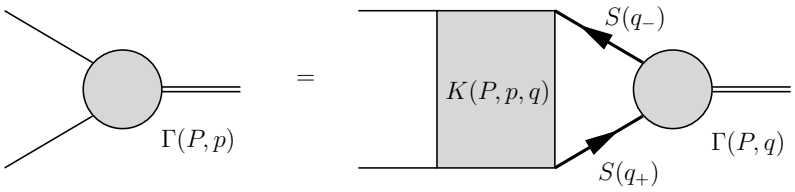

Figure 4: Pictorial representation of Eq. (13)

and as we approach that pole Eq. (5) becomes

$$\Gamma = K^{(4)} G_0^{(4)} \Gamma. \tag{12}$$

This might sound dubious on first sight but remember that the only thing we're interested in is that very pole as it will tell us about the bound state energy. Knowing the function in a neighborhood of that pole is therefore completely sufficient. We're now at a point where we cannot simply suppress all indices and momentum dependencies anymore because we ought to be careful: All quantities in Eq. (12) depend on the external momenta $p_{1,2,3,4}$. But not independently: While $K^{(4)}$ depends on the total momentum $P^2 = (p_1 + p_2)^2 = (p_3 + p_4)^2$ as well as the relative momenta $p_1 - p_2$ and $p_3 - p_4$, Γ and $G_0^{(4)}$ only depend on the total momentum and *one* relative momentum. Now, P^2 is the only quantity fixed by the experiment (so it definitely has to be the same on both sides of the equation), however, the relative momenta should remain unspecified and integrated out. Therefore, the equation reads with explicit momentum dependence

$$\Gamma(P;p) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} K^{(4)}(P;p,q) G_0^{(4)}(P;q) \Gamma(P;q) \tag{13}$$

where p and q are relative momenta. The structure of this equation becomes more apparent in Fig. 4. We stress again that this equation is *only* valid when P is the same on both sides of the equation. One can show (see chapter 6.1.1 in [2]) that

$$G_0^{(4)}(P;q) = S(q_+) \otimes S(q_-)$$

with $q_\pm = q \pm P/2$ which can be immediately inserted such that (13) turns into

$$\Gamma_{\alpha\beta}(P;p) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} K^{(4)}_{\alpha\beta,\rho\sigma}(P;p,q) \left[S(q_+) \Gamma(P;q) S(q_-) \right]_{\rho\sigma} \tag{14}$$

where we explicitly reintroduced Dirac indices. S is an object that we already have control over so we are done with those parts for now and focus on Γ next: Since we are only interested in pions which are pseudo-scalar objects we will now restrict the possible operators for Γ to a subspace which contains only pseudo-scalar isospin triplets but no other channels. This is of course possible because the other channels should not influence the poles which is the only thing we're interested in. In other words: Just because there are baryonic channels as well doesn't change the meson masses. A full

basis of pseudo-scalar operators[3] is given by

$$\begin{aligned}\tau_1(P,p) &= \gamma_5 \\ \tau_2(P,p) &= \gamma_5 \hat{\not{P}} \\ \tau_3(P,p) &= \gamma_5 \hat{\not{p}}_t \\ \tau_4(P,p) &= \gamma_5 \frac{i}{2}\left[\hat{\not{P}}, \hat{\not{p}}_t\right]\end{aligned}$$

where $\hat{P}_\mu = P_\mu/\sqrt{2} = P_\mu/im_{\mathrm{B}}$, $\hat{p}_\mu = p_\mu/\sqrt{2}$ and $p_{t,\mu}$ is the transverse momentum

$$p_{t,\mu} = \left(\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right) p_\nu \equiv P_{\mu\nu} p_\nu. \tag{15}$$

We can then use this basis to make a generic ansatz for Γ

$$\Gamma_{\alpha\beta}(P;p) = \sum_{i=1}^{4} f_i(p^2, P^2, \cos\theta)\, [\tau_i(p,P)]_{\alpha\beta} \tag{16}$$

with Lorentz- and parity-invariant coefficient functions f_i with depend on the external momentum P^2, the relative quark momentum p^2 as well as the cosine of the relative angle $\cos\theta$ between the two. The angle dependence can be expanded in terms of Chebyshev polynomials of second kind U_i which will have a technical advantage later on

$$f_i(p^2, P^2, \cos\theta) = \sum_{j=0}^{\infty} d_{i,j}(p^2, P^2) U_j(\cos\theta). \tag{17}$$

This guarantees that Γ transforms like a pion under Lorentz transformations. We also have to model its isospin behaviour, though. Notice, however, that we are working in the isospin limit where the up and down quark are mass degenerate. We can then express the pions as[4] $\pi^a = \bar{q}\sigma^a q/\sqrt{2}$ with a an adjoint $SU(2)_I$ index and σ^a the usual Pauli matrices. As long as we're in this limit, the pion triplet members are indistinguishable and therefore it is sufficient to restrict our analysis to just one of them. We choose the physical positive pion $\pi^+ = (\pi^1 + i\pi^2)/\sqrt{2}$ for which we should in principle include the defining operator

$$\frac{1}{2}\left(\sigma^1 + i\sigma^2\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

in our general ansatz (16) but it's not hard to see that it's actually redundant because it merely projects out the product $S(q_+)S(q_-)$ in (14) and removes all isospin indices from the DSE. Hence, it is safe to just neglect the isospin part. We are therefore a step closer to figuring out all the parts of the DSE. The only thing in (14) that we have no idea about yet is $K^{(4)}$. In the next section we will find an approximate form for it and turn the whole equation into something that can be solved numerically.

[3] Which is additionally also orthonormal with respect to the scalar product $\langle \tau_i, \tau_j \rangle \equiv \frac{1}{4} \operatorname{tr} \tau_i^\dagger \tau_j$

[4] There are various ways to normalize them but we chose them here to be orthonormal with respect to the scalar product $\langle x, y \rangle \equiv \operatorname{tr} x^\dagger y$

3.2 Approximations

The kernel K describes all possible interactions between two quarks via gluons (within pure QCD the only interaction vertex is the quark-gluon-vertex) yet their number could be arbitrary and all of those interactions interfere. A popular simplification to add one gluon at a time as long as complexity allows it. This is the so-called *Rainbow-Ladder-Approximation* and we shall only be considering the easiest version where the kernel is approximated by a single (dressed) quark exchange, c.f. Fig. 5, yet we include (just like when we treated the quark propagator) an effective dressing function Γ for the bare quark-gluon vertex which must not be confused with the BS amplitude and. One can

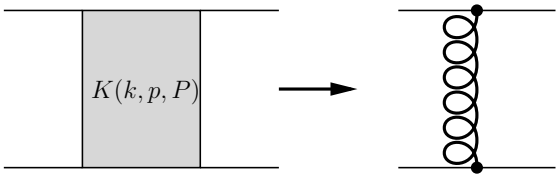

Figure 5: Rainbow Ladder approximation of the kernel. This models the kernel as an effective one-gluon exchange, i.e. for K we insert two bare quark-gluon-vertices and a dressed quark propagator.

then simply use the Feynman rules of Fig. 6 to get to the explicit form

$$K^{(4)\ ij,kl}_{\alpha\beta,\gamma\delta}(P;p,q) \to -g^2 \frac{Z(k^2)}{k^2} k_{\mu\nu}(k) \Gamma(-k;-p,q) \gamma_\mu^{\alpha\gamma} \gamma_\nu^{\delta\beta} t^a_{ik} t^b_{lj} \delta^{ab}$$

and insert again the effective model interaction (7) as well as the explicit form of the transverse momentum projector (15) to arrive at

$$K^{(4)\ ij,kl}_{\alpha\beta,\gamma\delta}(P;p,q) = -4\pi^2 D \frac{k^2}{\omega^2} e^{-k^2/\omega^2} \left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \gamma_\mu^{\alpha\gamma} \gamma_\nu^{\delta\beta} t^a_{ik} t^b_{lj} \delta^{ab}.$$

The second approximation is to not use all Lorentz operators in our ansatz for Γ, but start with just one to illustrate the procedure, i.e. we only include $\tau_1 = \gamma_5 \neq \tau_1(P,p)$ for now. Additionally, only the zero order Chebyshev polynomial is included to describe the radial part of Γ, i.e. overall we make the simplification

$$\Gamma_{\alpha\beta}(P;p) = d_{1,0}(p^2,P^2) \gamma_5^{\alpha\beta}.$$

Inserting all that into (14) we obtain

$$d_{1,0}(p^2,P^2)\gamma_5^{\alpha\beta} = -4\pi^2 \frac{D}{\omega^2} C_F \int \frac{\mathrm{d}^4 q}{(2\pi)^4} k^2 e^{-k^2/\omega^2} d_{1,0}(q^2,P^2) k_{\mu\nu} \left[\gamma_\mu S(q_+) \gamma_5 S(q_-) \gamma_\nu \right]_{\alpha\beta}$$

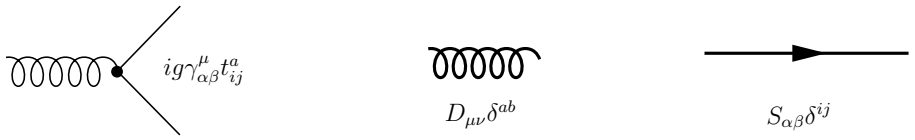

Figure 6: Feynman rules for bare quark-gluon vertex and fully dressed gluon and quark propagator. Remember that $D_{\mu\nu} = (\delta_{\mu\nu} - k_\mu k_\nu/k^2) Z(k^2)/k^2$.

with the gluon momentum $k \equiv q - p$. Notice that we omitted color indices and immediately replaced them by a color factor C_F which can be derived as follows: The quark-gluon vertices contribute color matrices $t^a_{ik} t^b_{lj} \delta^{ab}$ and the quark propagators a trivial color dependency δ^{ij} each:

$$C_F = t^a_{ik} t^b_{lj} \delta^{ab} \delta^{ij} \delta^{kl} = \operatorname{tr} t^a t^a = \frac{4}{3}.$$

Γ has no color structure (pions are color singlets) so it is transparent to those considerations. To get rid of the gamma matrix on the left side, we take the scalar product with γ_5 on both sides, i.e. multiply by γ_5 and trace over Dirac indices:

$$d_{1,0}(p^2, P^2) = -\pi^2 \frac{D}{\omega^2} C_F \int \frac{\mathrm{d}^4 q}{(2\pi)^4} k^2 e^{-k^2/\omega^2} k_{\mu\nu} \operatorname{tr}_D \left[\gamma_5 \gamma_\mu S(q_+) \gamma_5 S(q_-) \gamma_\nu\right] d_{1,0}(q^2, P^2). \tag{18}$$

Notice that we exploited that γ_5 is hermitian. As a next step we will introduce hyperspherical coordinates and bring this integral equation into a form that can be treated numerically.

3.3 Reformulation as eigenvalue problem

To effectively reduce the dimension of the integration (18) we introduce hyperspherical coordinates

$$q = |q| \begin{pmatrix} \sin\theta_2 \sin\theta_1 \cos\phi \\ \sin\theta_2 \sin\theta_1 \sin\phi \\ \sin\theta_2 \cos\theta_1 \\ \cos\theta_2 \end{pmatrix}, \qquad d^4 q = \frac{d(q^2)}{2} d(\cos\theta_1) d(\cos\theta_2) d\phi \; q^2 \sin\theta_2$$

and choose the external momenta to be

$$P = |P| \; (0,0,0,1)^T, \qquad p = |p| \; (0,0,\cos\theta, \sin\theta)^T.$$

To make things a bit easier to read we first define

$$\tilde{K}(P^2, q^2, p^2, c_1, c_2) \equiv -\frac{\pi^2}{(2\pi)^4} \frac{D}{\omega^2} C_F k^2 e^{-k^2/\omega^2} k_{\mu\nu} \operatorname{tr}_D \left[\gamma_5 \gamma_\mu S(q_+) \gamma_5 S(q_-) \gamma_\nu\right]$$

where the dependence on $c_i \equiv \cos\theta_i$ hides in $q^2_\pm$ and k^2. Explicitly, we have

$$k^2(p^2, q^2, c_1, c_2) = (q-p)^2 = q^2 + p^2 - 2\sqrt{q^2 p^2} \left(\sqrt{1 - c_2^2} \; c_1 \sin\theta + c_2 \cos\theta \right)$$

$$q^2_\pm(P^2, q^2, c_2) = (q \pm P/2)^2 = q^2 - \frac{m_\mathrm{B}^2}{4} \pm i m_\mathrm{B} \sqrt{q^2} c_2$$

where we already inserted $P^2 = -m_\mathrm{B}^2$. Now we observe, that the left hand side of (18) does not depend on θ due to our approximation of only including the 0-th Chebyshev

polynomial. Therefore, also the right hand side has to be θ-independent which means that we may choose a θ which simplifies our calculations. For that we take $\theta = 0$ which simplifies the relation for k^2 to

$$k^2(p^2, q^2, c_2) = (q-p)^2 = q^2 + p^2 - 2\sqrt{q^2 p^2}\ c_2$$

This of course also means, that the θ_1 integration can be performed trivially (yielding 2) and, additionally, also the ϕ-integration (2π) and can write the rest of (18) as

$$d_{1,0}(p^2, P^2) = 2\pi \int_0^\infty d(q^2) \int_{-1}^{1} dc_2\ q^2 \sqrt{1-c_2^2}\ \tilde{K}(P^2, q^2, p^2, c_2) d_{1,0}(q^2, P^2).$$

The c_2-integration already has a domain suitable for Gauss-Chebyshev integration, however the one over $[0, \infty)$ has to be transformed as we did for the quark propagator. We substitute $q^2 = (1+u)/(1-u)$ and multiply with unity to get the weight functions right

$$\begin{aligned} d_{1,0}(p^2, P^2) &= 4\pi \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} \int_{-1}^{1} \frac{dc_2}{\sqrt{1-c_2^2}} \times \\ &\quad \times \frac{(1+u)^{3/2}}{(1-u)^{5/2}} (1-c_2^2)\ \tilde{K}(P^2, q^2(u), p^2, c_1, c_2)\ d_{1,0}(q^2(u), P^2) \\ &\approx 4\pi \sum_j \sum_k w_j w_k \times \\ &\quad \times \frac{(1+u_j)^{3/2}}{(1-u_j)^{5/2}} (1-c_{2,k}^2)\ \tilde{K}(P^2, q^2(u_j), p^2, c_{2,k}) d_{1,0}(q^2(u_j), P^2). \end{aligned}$$

Here, u_j, c_k and w_j, w_k are roots and the corresponding integration weights of the chosen Chebyshev polynomials of the first kind. If we also discretize p^2 on the left (with the same nodes as q^2 on the right) and employ the short-hand notation $d^{(i)} = d(p_i^2)$ we can rewrite this as a matrix equation

$$d_{1,0}^{(i)}(P^2) = \sum_j \mathcal{K}_{ij}(P^2) d_{1,0}^{(j)}(P^2) \tag{19}$$

with the matrix

$$\mathcal{K}_{ij}(P^2) \equiv 4\pi w_j \frac{(1+u_j)^{3/2}}{(1-u_j)^{5/2}} \sum_k w_k (1-c_{2,k}^2)\ \tilde{K}(P^2, q^2(u_j), p_i^2, c_{2,k}).$$

We remind the reader, that the initial premise when deriving this equation was that it is only valid for $P^2 = -m_{\mathrm{B}}^2$. In principle, however, we could set an arbitrary value of P^2 in our calculations. The only problem with this is that the solution which we might find by doing so (if we find any) is not a solution of our initial problem. To work around this issue we perform a mathematical trick: We can also understand (19) as an eigenvalue equation

$$\lambda(P^2) d_{1,0}^{(i)}(P^2) = \sum_j \mathcal{K}_{ij}(P^2) d_{1,0}^{(j)}(P^2).$$

Now, instead of randomly trying out values for P^2 in the initial equation we can calculate $\mathcal{K}$ for different values of P^2 and see if it possesses an eigenvalue $\lambda(P^2) = 1$. If it does, we know that the eigenvalue equation is equivalent to our initial equation *and* the value of P^2 for which it happens is the bound state energy $P^2 = -m_{\mathrm{B}}^2$.

3.4 Calculating the kernel

The only thing left to do before we can solve the eigenvalue equation on a PC is to calculate the explicit form of $\tilde{K}(P^2, q^2, p^2, c_2)$ which pretty much boils down to calculating $k_{\mu\nu}\,\mathrm{tr}_D\left[\gamma_5\gamma_\mu S(q_+)\gamma_5 S(q_-)\gamma_\nu\right]$. First, we have to insert the explicit definition (2) for the quark propagators[5]

$$\frac{k_{\mu\nu}\,\mathrm{tr}_D\left[\gamma_5\gamma_\mu(-iA(q_+^2)\not{q}_+ + B(q_+^2))\gamma_5(-iA(q_-^2)\not{q}_- + B(q_-^2))\gamma_\nu\right]}{(q_+^2A^2(q_+^2) + B^2(q_+^2))(q_-^2A^2(q_-^2) + B^2(q_-^2))}.$$

The cross-terms contain an odd number of gamma matrices, hence their trace vanishes and we are left with

$$\frac{-k_{\mu\nu}q_+^\rho q_-^\sigma A(q_+^2)A(q_-^2)\,\mathrm{tr}_D\left[\gamma_5\gamma_\mu\gamma_\rho\gamma_5\gamma_\sigma\gamma_\nu\right] + k_{\mu\nu}B(q_+^2)B(q_-^2)\,\mathrm{tr}_D\left[\gamma_5\gamma_\mu\gamma_5\gamma_\nu\right]}{\left[q_+^2A^2(q_+^2) + B^2(q_+^2)\right]\left[q_-^2A^2(q_-^2) + B^2(q_-^2)\right]}.$$

The traces yield

$$\begin{aligned}\mathrm{tr}_D\left[\gamma_5\gamma_\mu\gamma_5\gamma_\nu\right] &= -4\delta_{\mu\nu}\\ \mathrm{tr}_D\left[\gamma_5\gamma_\mu\gamma_\rho\gamma_5\gamma_\sigma\gamma_\nu\right] &= 4\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right),\end{aligned}$$

which immediately simplifies things a lot: The contraction with the gluon propagator just gives

$$-4\delta_{\mu\nu}k_{\mu\nu} = -4\delta_{\mu\nu}(\delta_{\mu\nu} - k_\mu k_\nu/k^2) = -12$$

and for the contraction with $q_+^\rho q_-^\sigma$ we can exploit the symmetry $k_{\mu\nu} = k_{\nu\mu}$ to arrive at

$$-4k_{\mu\nu}q_+^\rho q_-^\sigma\left(\delta_{\mu\nu}\delta_{\rho\sigma} + \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}\right) = -4k_{\mu\nu}q_+^\rho q_-^\sigma\delta_{\mu\nu}\delta_{\rho\sigma} = -12(q^2 - P^2/4).$$

This considerably simplifies our expression to

$$\tilde{K}(P^2, q^2, p^2, c_2) = \frac{D}{\pi^2}\frac{k^2}{\omega^2}e^{-k^2/\omega^2}\frac{(q^2 - P^2/4)A(q_+^2)A(q_-^2) + B(q_+^2)B(q_-^2)}{\left[q_+^2A^2(q_+^2) + B^2(q_+^2)\right]\left[q_-^2A^2(q_-^2) + B^2(q_-^2)\right]}$$

where we refrained from inserting

$$\begin{aligned}k^2(p^2, q^2, c_2) &= (q - p)^2 = q^2 + p^2 - 2\sqrt{q^2p^2}c_2\\ q_\pm^2(P^2, q^2, c_2) &= (q \pm P/2)^2 = q^2 - \frac{m_{\mathrm{B}}^2}{4} \pm im_{\mathrm{B}}\sqrt{q^2}c_2\end{aligned}$$

[5]Notice that the color part was already taken care of earlier.

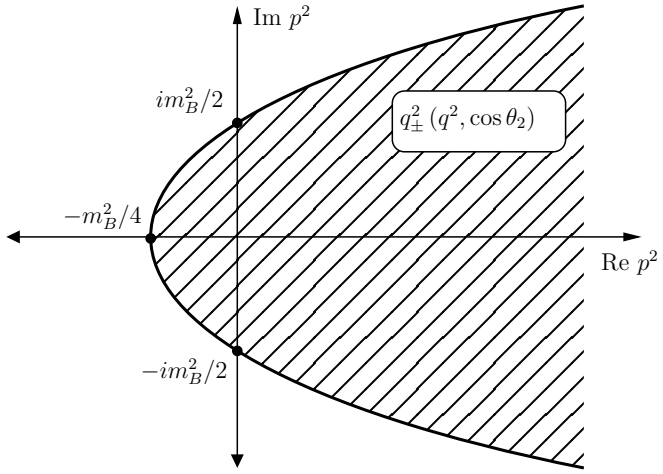

Figure 7: Domain of $q_\pm^2$ needed to calculate the kernel.

explicitly to avoid clutter. We can see that to calculate the final kernel we have to evaluate the quark propagator dressing functions for complex momenta $q_\pm$. Initially, we calculated A and B only on the positive real axis and argued that we can analytically continue the solution into the whole complex plane. This works to some extend but it is questionable whether the extrapolated values are usable for two reasons: First, extrapolated solutions usually have poles away from the real axis and second, we already mentioned that the solution away from the real axis heavily depends on what degree of Chebyshev polynomials we use to extrapolate. To settle the discomfort, we make the following observation: The required values of $q_\pm$ to calculate the kernel lie within a parabola region in the complex q^2 plane. We write

$$x \equiv \mathrm{Re}\, q_\pm^2 = q^2 - \frac{m_\mathrm{B}^2}{4} \qquad\qquad y \equiv \mathrm{Im}\, q_\pm^2 = \pm m_\mathrm{B}\sqrt{q^2}c_2,$$

and conclude that

$$|y| = m_\mathrm{B}\sqrt{q^2}|c_2| \leq m_\mathrm{B}\sqrt{q^2} = m_\mathrm{B}\sqrt{x + \frac{m_\mathrm{B}^2}{4}},$$

which is an inequality describing the shaded area in Fig. 7. Within this region, there are no poles and also the solution is stable with respect to changes in the polynomial degree. A typical value for the bound state mass is $m_\mathrm{B} \approx B(0)/10$. In Fig. 8 we show the functions A and B (actually, only their absolute values) in the complex plane. Additionally, we added the parabola region described above. As we can see, most of the interesting behaviour, e.g. the appearance of poles, happens *outside* of that region. In particular, by changing the degree of the interpolation the parts inside the parabola do not change (unless we go to unreasonably high/low polynomial degrees). The plot might look overwhelming at first but together with Fig. 3 it can easily be understood: For $m = 0$ GeV the function A is has a bump whereas it only has a plateau for $m = 0.115$ GeV. The function B moves between two plateaus. This can exactly be seen from the contour plot as well where those plateaus correspond to different solid

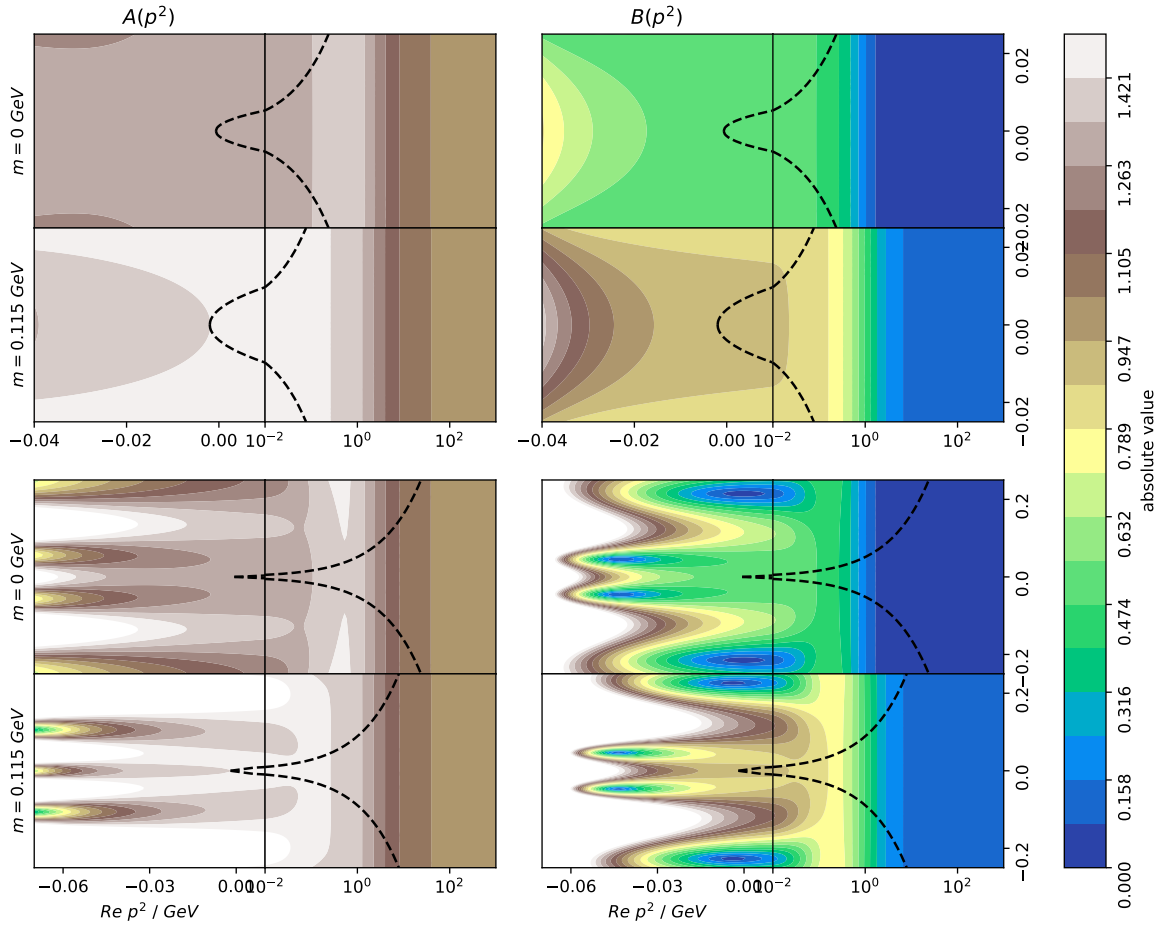

Figure 8: *Columns*: Abs. val. of dressing functions $A(p^2)$ and $B(p^2)$. *Rows*: Different values of bare quark mass m. Top and bottom row are differently framed versions of the same plot. The dashed curve is the boundary of the parabola region Fig. 7. **Important**: Notice that the real axis is divided into a linear and logarithmic part! White areas have absolute values greater than 1.5

colors.
We conclude that it is save to just use our previously calculated quark propagator dressing functions as long as we only probe values for m_{B}, respectively P^2, which do not push the parabola inside troublesome regimes. E.g. taking values

$$P^2 \in \mathcal{I} = [-B(0)/10, 0]\,.$$

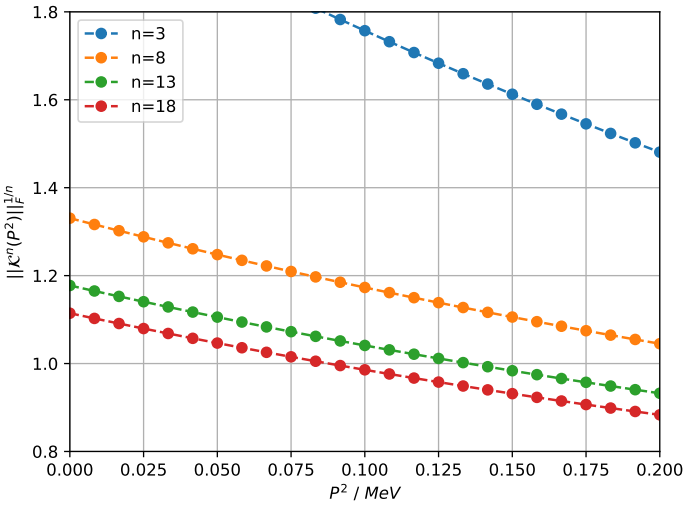

Figure 9: Upper bounds for the spectral radius of $\mathcal{K}$ given by its matrix norm. The P^2 dependent eigenvalues have to lie below the lowest curve.

3.5 Numerical solution

Previously, we discussed the strategy to solve the problem and we already discretized it. We found that our final task is to calculate the matrix $\mathcal{K}$

$$\mathcal{K}_{ij}(P^2) \equiv 4\pi w_j \frac{(1+u_j)^{3/2}}{(1-u_j)^{5/2}} \sum_k w_k (1-c_{2,k}^2)\ \tilde{K}(P^2, q^2(u_j), p_i^2, c_{2,k}) \quad \text{with}$$

$$\tilde{K}(P^2, q^2, p^2, c_2) = \frac{D}{\pi^2}\frac{k^2}{\omega^2} e^{-k^2/\omega^2} \frac{(q^2 - P^2/4)A(q_+^2)A(q_-^2) + B(q_+^2)B(q_-^2)}{[q_+^2 A^2(q_+^2) + B^2(q_+^2)]\,[q_-^2 A^2(q_-^2) + B^2(q_-^2)]}$$

$$k^2(p^2, q^2, c_2) = (q-p)^2 = q^2 + p^2 - 2\sqrt{q^2 p^2}\ c_2$$

$$q_\pm^2(P^2, q^2, c_2) = (q \pm P/2)^2 = q^2 + \frac{P^2}{4} \pm i\sqrt{-q^2 P^2}\ c_2$$

for a given value $P^2 \in \mathcal{I}$ and calculate its eigenvalues and eigenvectors. The spectral radius of $\mathcal{K}_{ij}$ can be bounded by any submultiplicative matrix norm, e.g. the Frobenius norm

$$|\lambda(P^2)| \leq ||\mathcal{K}^n(P^2)||_F^{1/n}$$

and the inequality holds for any integer n. Fig. 9 shows the upper bounds for the spectral radius for some values of n as a function of P^2. The largest modulus of the eigenvalues of $\mathcal{K}$ are certainly below the lowest bound. From the figure we can conclude that there are definitely values for P^2 for which the eigenvalues are all smaller than one and for all P^2 in the interval $\mathcal{I}$ the eigenvalues at least lie in a neighborhood of 1. This is very comforting as we are precisely looking for $\lambda(P^2) = 1$. This justifies to use the von-Mises iteration method (power method) which by design find the largest eigenvalue (modulus) because we are not interested in any other. The method is briefly explained in Appendix 5.4. The bound state mass is found once we have a $\lambda(P^2)$ equal to one.

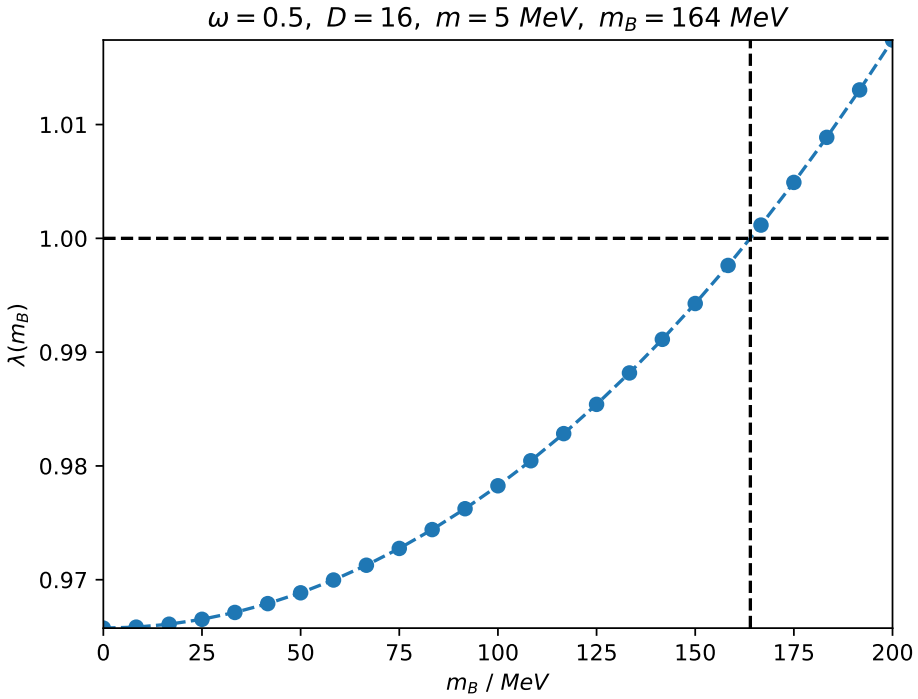

Figure 10: Eigenvalue of $\mathcal{K}$ as a function of $m_\mathrm{B} = \sqrt{-P^2}$. The point where it crosses $\lambda = 1$ determines the bound state mass.

To narrow this value down we search for solutions to the equation $\lambda(P^2) - 1 = 0$ using a bisection scheme in the interval $\mathcal{I}$. This is justified as well because the eigenvalues depend continuously on the matrix entries[6], hence bisection can be performed safely. The results for the momentum dependent eigenvalues are visualized in Fig. 10 which also shows the pion mass of $m_\pi = 164$ MeV. The calculation can be performed for different values of the energy scale ω and the effective interaction strength D and one obtains very similar results which are collected in Tab. 2.

ω / GeV	D / GeV^{-2}	m / MeV	m_π / MeV	m_π / MeV of [3]
0.40	45	5	153	135
0.45	25	5	168	135
0.50	16	5	164	137

Table 2: Calculated pion masses m_π for given quark masses m, energy scales ω and interaction strengths D with reference values from [3]

[6]Polynomial roots are continuous in the polynomial coefficients.

4 Discussion and Conclusion

The aim of this project within the scope of the course *Computational Methods in Particle Physics* was to find an estimate for the mass of the pion. As described in this report in consecutive order, we started by solving the quark propagator dressing functions analytically for the angular integral (demonstrated in Section 2.2). These integral equations were then solved numerically, using Gauss-Chebyshev integration, since this method showed the best results (see Appendix 5.2). The integral equations for the dressing functions A and B were solved on the real positive half-axis and analytically extended to the parabola-bounded region in the complex plane (see Fig. 7). We compared our results for the dressing functions to those of [3] and found excellent agreement (see Tab. 1).
Afterwards, the Bethe-Salpeter equation for the 4-point correlation function, resp. the pion bound state amplitude was constructed (see Section 3). We introduced the ladder approximation and only considered the simplest version in which the kernel is approximated by single quark exchange. Furthermore, we approximated the Lorentz structure of the bound state amplitude Γ by a single (momentum-independent) operator as well as neglected all of its angle-dependencies in our ansatz ansatz (previously described as only considering the zeroth order in Chebyshev polynomials). After various algebraic reformulations of the BSE we arrived at an eigenvalue problem which is described in Section 3.3 and solved using the von-Mises iteration method (power method). Using this particular method is justified by Fig. 9 and the fact that we are precisely looking for an eigenvalue of 1. Fig. 10 finally shows the bound state mass of the pion which, according to our construction, happens for an eigenvalue of 1. Bound state masses for various values of the input parameters (quark mass, energy scale and interaction strength) are collected in Tab. 2. These particular values correspond to those used by the authors of [3] which allows for an easy comparison. We see that they are in good agreement. However, our results are much further from the experimentally acquired value [1] $m_{\pi^0} = 134.9768 \pm 0.0005$ MeV than the ones in the cited article. This is not surprising at all as the authors of [3] not only included the entire possible Lorentz structure of Γ but also took angle-dependencies up to the fourth Chebyshev polynomial into account.
Including more operators is just a matter of clever data rearrangements and a steep increase in computational effort but conceptually not too difficult. Including more Chebyshev polynomials on the other hand seems more tedious because this means that the trick of Sec. 3.3 (where we explicitly exploited the angle independence) doesn't work anymore. We would therefore have to reintroduce a third numerical integration and another grid with discrete θ_1 points which is definitely beyond the present work.

5 Appendix

5.1 Numerical Integration

In this work, numerical integration is performed via a so-called *Gauss Quadrature* which approximates the integral of some function $f(x)$ over a domain $[a, b]$ by a weighted sum over discrete function values $f(x_i)$. In addition to $f(x)$ we can also have some weight function $g(x)$ which is specific to a given type of Gauss Quadrature and doesn't show up in the sum at all. The general formula looks like this

$$\int_a^b dx\ g(x)f(x) \approx \sum_{i=1}^{n} \omega_i f(x_i)$$

with the integration *nodes* x_i and *weights* ω_i. Those aren't chosen arbitrarily but are actually connected to sets of polynomials which are orthogonal with respect to the scalar product

$$\int_a^b dx\ g(x)P_k(x)P_l(x).$$

The set of polynomials gives the integration method its name, e.g. *Gauss-Legendre Integration*. Some examples are

$g(x)$	Polynomials	Domain
1	Legendre	$[-1, 1]$
$1/\sqrt{1-x^2}$	Chebyshev 1^{st} kind	$[-1, 1]$
$\sqrt{1-x^2}$	Chebyshev 2^{nd} kind	$[-1, 1]$
$\exp(-x)$	Laguerre	$[0, \infty]$
$\exp(-x^2)$	Hermite	$[-\infty, \infty]$

The important thing to realize is that one can specifically exploit those weight functions, e.g. for an integral $\int \exp(-x)f(x)$ it is probably best to use Gauss-Laguerre, etc. Once a set of polynomials is chosen, one may only choose n, i.e. the degree of the quadrature, because the nodes and weights are prescribed by the polynomials themselves: The nodes are the roots of the n-th degree polynomial and the weights can be calculated from the polynomials, too, e.g. for Gauss-Legendre

$$\omega_i = \frac{2(1-x_i^2)}{(n+1)^2 P_{n+1}^2(x_i)}.$$

We note, however, that routines for the most common polynomials exist in modern packages like `scipy`. They can be called for a given n and return the nodes and weights.

5.2 Most appropriate integration method for dressing functions

In the following we discuss a few different approaches to the integral equation in Sec. 2.2: The most obvious one would be to use Gauss-Laguerre integration as our integration domain is $[0, \infty]$. Even though this results in a very dense node distribution close to the region of interest it unfortunately has none beyond this region and therefore the method fails to capture the right function behaviour for regions $\gtrsim 10^2$ GeV. The second option would be Gauss-Chebyshev integration. To make use of this method, we first have to transform the integration domain to $[-1, 1]$. This can be achieved by mappings like

$$y = \frac{1+u}{1-u} \qquad \text{or} \qquad y = -\ln \frac{1-u}{2} \qquad u \in [-1, 1].$$

The logarithmic substitution again leads to a similar problem as for the Gauss-Laguerre integration whereas the node distribution for $y = (1+u)/(1-u)$ is satisfying. This discussion is most easily understood by looking at the integration nodes themselves in Fig. 11. At this point, we only know the functions at the integration nodes. To extend

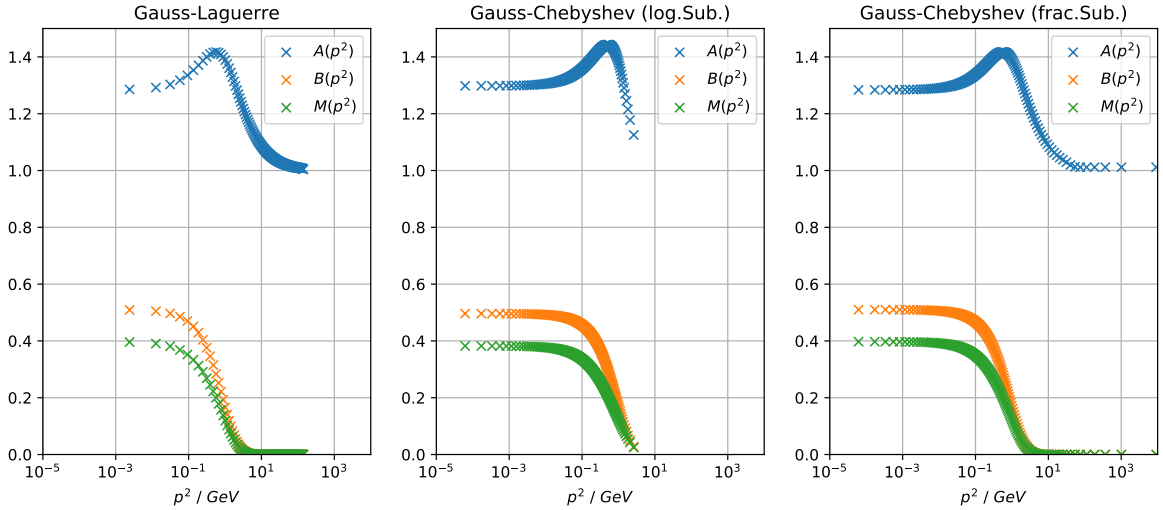

Figure 11: Comparison of different integration methods. The integration nodes are fixed by the choice of orthogonal polynomials and can at most be stretched by a variable transformation. The most appealing option seems to be the right-most one, i.e. Gauss-Chebyshev integration and a variable transformation given by $y = (1+u)/(1-u)$.

them to arbitrary points in the complex plane (the possibility of which is ensured by the discussion in section 2.2) we again have two options: We could just use the converged (pointwise) functions A and B and plug them back into the integral equation with the only difference that we may now choose *any* value of x (earlier, the x_i had to be identical to the integration nodes). This works in principle but unfortunately leads to numerical issues and oscillatory behaviour. Alternatively, we can also exploit that we know the functions *at the roots of a Chebyshev polynomial*! This lets us calculate the

expansion coefficients for Chebyshev interpolation (c.f. Appendix 5.3) easily and by using polynomials just up to degree 7 we manage to get a very smooth function which we can use for extrapolation. The different behaviour is shown in Fig. 12.

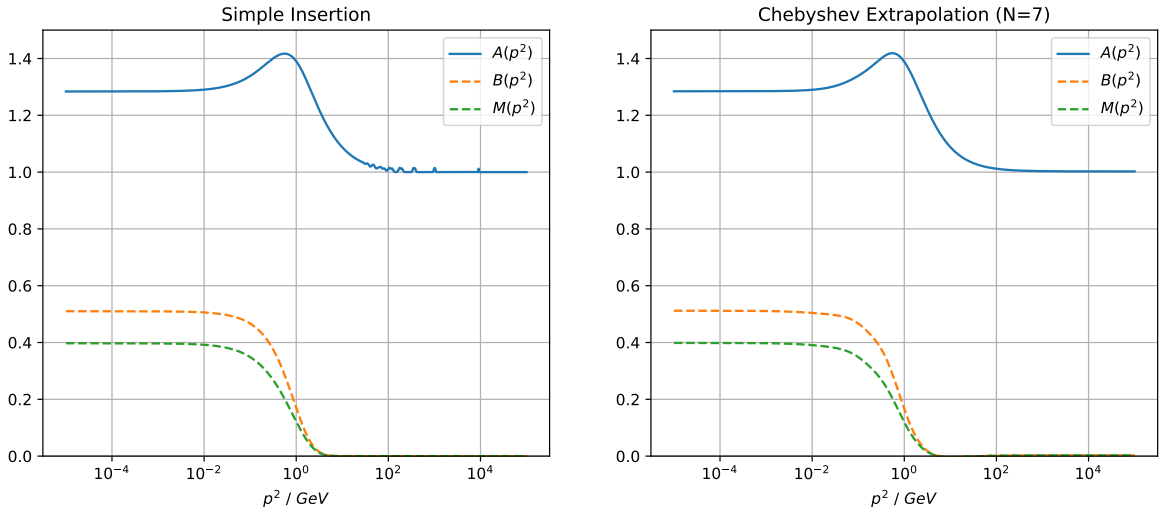

Figure 12: Comparison of pointwise extrapolation by inserting x values back into the integral equation and extrapolation by first interpolating with Chebyshev polynomials up to degree 7

5.3 Chebyshev Interpolation

Given the values of a continuous function at some discrete values, we can always use a polynomial of high enough degree to interpolate it. However, this often results in not-so-smooth interpolating functions. Once again, the use of orthogonal polynomials can be key to a better result. Here, we consider Chebyshev polynomials of the 1^{st} kind $T_k(x)$. Let x_k be the roots of T_n. Then we can approximate any function as

$$f(x) \approx \frac{d_0}{2} + \sum_{i=1}^{m-1} d_i T_i(x)$$

$$d_i = \frac{2}{n} \sum_{k=0}^{n-1} f(x_k) T_i(x_k)$$

with some cutoff m. In our specific case we will use Gauss Chebyshev integration up to some order n which then automatically yields a solution on the roots of the n-th Chebyshev polynomial. Therefore, calculating the weights d_i for Chebyshev interpolation is very easy as the function values are already known.

5.4 Power Method for approximating Eigenvalues

The power method, which is briefly introduced here, is a method to approximate the dominant eigenvalue of a matrix A, meaning the eigenvalue which is largest in absolute

value. It is an iterative method in which some initial vector v_0 is guessed to be a non-zero eigenvector and yields then a first guess for the eigenvalue λ_0

$$\lambda_0 = \frac{b_0^T A b_0}{b_0^T b_0}.$$

An iterative step corresponds to the following update of the eigenvector v_k:

$$v_{n+1} = \frac{Ab_n}{||Ab_n||}$$

In every iteration step the eigenvector is then recalculated and when convergence is reached it yields the dominant eigenvector. When the eigenvalue v and eigenvector λ of the matrix are found the following quotient gives the Rayleigh quotient:

$$\lambda = \frac{b^T A b}{b^T b}$$

For further insight confront the book *Numerical recipes : the art of scientific computing*, [4].

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