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The manifestly gauge-invariant spectrum of the Minimal Supersymmetric Standard Model

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AFFIDAVIT

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ABSTRACT

Despite numerous efforts, no particles predicted by (broken) supersymmetric theories have been observed in experiments. One possible explanation is that the particle spectrum predicted by perturbation theory is not the physical spectrum predicted by the full, non-perturbative theory. In fact, because it is gauge-variant, it *cannot* be from a field theoretical standpoint.

A framework for analyzing the gauge-invariant, non-perturbative spectrum and efficiently contrasting it with perturbation theory results is provided by the Fröhlich-Morchio-Strocchi mechanism. For the standard model, this description shows remarkable agreement between the two spectra, supporting the results of perturbation theory. However, for many grand-unified-theory-like scenarios, qualitative differences between the two emerge. These have been thoroughly investigated and are confirmed by lattice findings. The absence of experimental evidence may be explained by the fact that one would *not* expect the perturbative spectrum to be observable in such circumstances.

For the first time, the investigations are extended to supersymmetric theories in this thesis. It constructs the minimal supersymmetric standard model's inherently gauge-invariant spectrum and shows that no qualitative differences occur. This finding supports the results of the standard perturbative treatment, which means that the actually physical spectrum would indeed include the undiscovered particles.

KURZZUSAMMENFASSUNG

Trotz zahlreicher Versuche ist es bisher nicht gelungen, die von (gebrochen-) supersymmetrischen Theorien vorhergesagten Teilchen im Experiment nachzuweisen. Eine Erklärung dafür könnte sein, dass das von Störungstheorie vorhergesagte Spektrum womöglich nicht dem physikalischen Spektrum der vollen Theorie entspricht. Aus feldtheoretischer Sicht ist dies sogar ausgeschlossen, da es eichabhängig ist.

Zur Untersuchung des nicht-störungstheoretischen, eichunabhängigen Spektrums eignet sich der Fröhlich-Morchio-Strocchi Mechanismus, welcher darüber hinaus eine direkte Verbindung mit dem bekannten, störungstheoretischen Spektrum herstellt. Im Falle des Standardmodells kann damit gezeigt werden, dass die beiden Spektren hervorragend übereinstimmen und der Formalismus folglich die Ergebnisse der Störungstheorie stützt. Allerdings kann es auch zu qualitativen Unterschieden kommen, was anhand von zahlreichen Szenarien im Bereich von großen vereinheitlichten Theorien demonstriert und in Gittersimulationen bestätigt wurde. In solchen Fällen würde man *nicht* erwarten, dass das störungstheoretische Spektrum physikalischen Gehalt hat, die entsprechenden Teilchen also auch im Experiment nicht beobachtbar sind.

In der vorliegenden Arbeit wird diesbezüglich erstmals eine supersymmetrische Theorie untersucht. Es wird gezeigt, wie das eichunabhängige Spektrum des minimal-supersymmetrischen Standardmodells konstruiert wird, und dass es qualitativ mit den Vorhersagen der Störungstheorie übereinstimmt. Somit werden abermals die störungstheoretischen Ergebnisse untermauert und bestätigt, dass das physikalische Spektrum in der Tat die unentdeckten Teilchen beinhaltet.

Contents

1 Introduction

Our current understanding of fundamental particles and interactions is very well captured by the standard model of particle physics [1–4]. It accurately describes weak and strong interactions as well as quantum electrodynamic processes up to energies of a few TeV. The Brout-Englert-Higgs effect [5, 6], which was proposed to explain the mass origin of the weak gauge bosons and fermions, is an essential component of the standard model, and the 2012 discovery of the corresponding Higgs boson [7, 8] made a huge impact in our understanding of the inner workings of the Universe.

Nevertheless, there are some things that cannot be explained solely by the standard model. Gravity and dark matter are the most obvious missing pieces. Furthermore, physicists work hard to explain seemingly strange parameters in the standard model, such as the small Higgs mass (fine-tuning problem) or the θ-parameter (strong CP problem). More often than not, hypothetical particles and symmetries are proposed to 'elegantly' address these supposedly strange circumstances. Peccei and Quinn [9], for example, suggested a new $U(1)$ symmetry and the axion as a solution to the strong CP problem.

The idea of proposing more hypothetical particles than experimentally verified ones, in addition to a symmetry that alters our perception of spacetime, is admittedly pretty radical. However, these supersymmetric theories [10, 11] are extremely popular scenarios of beyond the standard model physics. And for good reason: Already the most straight-forward supersymmetric extension of the standard model, the minimal supersymmetric standard model (MSSM), contains a 'light' Higgs boson [12] in addition to offering better gauge-coupling unification [13]. Furthermore, it suggests that electroweak symmetry breaking is ultimately connected to the breaking of supersymmetry at low energies [11], and it yields a dark matter candidate without much effort: the lightest supersymmetric particle (LSP) [14]. Finally, promoting supersymmetry to a local symmetry results in supergravity models which could be the long sought-after theory of quantum gravity [15].

Despite the arguably beautiful maths and implications of (broken) supersymmetry, we cannot deny the fact that none of its many predicted particles have yet been discovered in experiments [16, 17]. Indeed, many constraint versions of the MSSM are highly restricted by now [18]. Similarly, there is growing evidence that the MSSM's LSP is not a suitable dark matter candidate, as direct detection experiment results [19] do not well agree with astrophysical data on the relic density [20]. The full MSSM, however, contains plenty of parameters that, if left unconstrained, can be used to fit it to observational data. Instead of extending the parameter space for decades to come, one might ask a provoking question: Are we even looking for the right particles? What if the particle spectrum we *calculated* from the MSSM was incorrect, rather than the MSSM itself? Is the spectrum obtained by standard perturbation theory the actual *physical* spectrum?

The MSSM, like the standard model and many of its proposed extensions, is formulated as a gauge theory. At the same time, only truly gauge-invariant fields can have physical meaning from a conceptual standpoint. Manifest gauge-invariance in Abelian gauge theories such as quantum electrodynamics (QED) can be easily achieved by dressing fields with a Dirac phase ('photon cloud') [21]. This, for example, converts the gauge-dependent elementary 'electron' field into a truly physical electron. It shows that the physical spectrum is in one-to-one correspondence with the elementary spectrum, even though the latter is, strictly speaking, nonphysical. Non-Abelian theories, on the other hand, require more involved constructions: In quantum chromodynamics (QCD), quarks and gluons cannot be the physical degrees of freedom because they are gauge-dependent. Unlike before, adding a dressing phase is insufficient, and bound states such as mesons and baryons are required to obtain gauge-invariant objects [2]. As a result, the physical spectrum includes these bound states and deviates significantly from the elementary spectrum of quarks and gluons.

Interestingly, nothing along these lines is found in the usual treatment [3] of the standard model weak-Higgs sector even though it is a non-Abelian theory as well and the same reasoning applies [22–24]. In contrast to QCD, the small weak coupling constant seems to motivate the use of perturbation theory (PT) which cannot treat bound states in the first place. As a result, the possibility of a physical spectrum that differs qualitatively from the elementary one is dismissed. The Faddeev-Popov gauge-fixing procedure and the BRST construction [2] are widely believed to adequately account for gauge-invariance in this case, even though they are only known to work reliably within PT. The Gribov-Singer ambiguity [25, 26] impedes their non-perturbative construction. Moreover, they only guarantee *gauge-parameter*-independence within this specific framework but not gauge invariance of the full, non-perturbative spectrum. If these 'subtleties' are ignored, this procedure yields a Higgs field and weak gauge bosons which carry an open gauge index. As a result, despite being BRST singlets, they cannot be truly gauge-invariant quantities of the full theory. Thus, it is strictly speaking incorrect to attribute any sort of physical reality to them. On the other hand, it is undeniable that the usual description 'works', in the sense that it produces accurate predictions [2–4]. Therefore, it should not come as a surprise that it appears in many textbooks, despite the fact that, given the discussion above, this description seems fundamentally flawed.

As it turns out, its success is not entirely coincidental. The FMS mechanism, developed by Fröhlich, Morchio and Strocchi [23, 24], establishes a connection between the perturbative spectrum and the theory's inherently gauge-invariant, non-perturbative spectrum. The latter is indeed made up of 'bound state' objects, but unlike in QCD, they are in one-to-one correspondence with the elementary fields and their masses and decay widths are identical [27]. The FMS mechanism hence shows, that *in the standard model weak-Higgs sector* no qualitative differences between the two spectra arise, and therefore explains perturbation theory's apparent success. For a review, see [28]. Thus, demanding full gauge-invariance appears to be an esoteric issue. As indicated by the emphasize, however, this result is special for the standard model weak-Higgs sector and should not be taken for granted: For many grand-unified-theory-like (GUT) models, the FMS mechanism actually indicates a mismatch between the non-perturbative and the perturbative spectrum [29, 30], which is also found in lattice calculations [31]. It is possible that some particles present in the elementary spectrum are completely *absent* from the physical spectrum. Searching for evidence of the former can thus be highly misleading because it might contain particles which are not expected to be observable in the first place. In such cases, the true power of the FMS mechanism shows: Whereas it merely *supported* the use of perturbation theory in the standard model, it is now *critical* to investigate potential discrepancies and arrive at truly physical statements. Furthermore, even when the spectra agree, the FMS mechanism predicts sub-leading corrections to off-shell properties, form-factors, cross-sections, etc., that are in principle detectable under the right kinematical conditions [32–34].

Such considerations are especially appealing for the MSSM, which predicts a plethora of new particles. Its Higgs sector is essentially a two-Higgs-doublet-model (2HDM), and previous research found that the FMS mechanism predicts no qualitative spectrum changes there [35]. The more important and as-yet unanswered question is whether the superpartners, particularly the LSP (the dark matter candidate), are part of the physical spectrum and thus, at least in principle, observable.

This thesis starts out with a brief introduction to supersymmetry in Chapter 2: The Weyl spinor formalism which is very practical for its description will be presented first. Following that, we will look at how a symmetry between bosons and fermions can be established in the first place. This leads to the Super-Poincaré-Algebra and its irreducible representations, the supermultiplets, which are eventually used to formulate supersymmetric Lagrangians. We will see that supersymmetry imposes severe constraints on the form of the Lagrangian terms, and we will conclude the chapter by stating the most general supersymmetric Lagrangian as a starting point for later model building.

Chapter 3 introduces the FMS mechanism which is eventually required to relate the physical spectrum of the MSSM to the already known elementary spectrum. We will also review the standard perturbative approach to non-Abelian gauge theories, which includes the Faddeev-Popov and BRST construction. Beyond PT, we will see that there is no reason to expect these concepts to be well-defined. Motivated by manifest gauge-invariance, we will then consider composite bound state operators as a replacement for elementary fields, as well as how PT can be used in conjunction with the FMS mechanism to make statements about their relationship to the perturbative spectrum. This will ultimately lead to the concept of augmented perturbation theory (APT).

Chapter 4 illustrates how APT works in practice. We go over the most important steps in the usual treatment of the standard model weak-Higgs sector and point out common misconceptions and subtleties. We show how switching to a bound state operator language results in a conceptually much more satisfying description that is automatically in agreement with the standard treatment using the FMS mechanism and APT.

The minimal supersymmetric standard model is introduced in Chapter 5 as a special case of the general theory presented in Chapter 2, and the importance of its custodial symmetry for the FMS mechanism is highlighted. Both the elementary and the physical spectra of the MSSM must then be constructed to eventually determine if they agree (as in the standard model) or if qualitative discrepancies emerge (as in many GUT-like theories). First, the perturbative spectrum of the weak-Higgs(ino) sector and one (s)lepton generation is calculated. Keeping custodial symmetry intact makes the calculations slightly unconventional, but extremely insightful. The physical spectrum is then constructed using gauge-invariant operators, and their relationship to the elementary fields is established again using the FMS mechanism and APT. Finally, the results are generalized to the case of broken custodial symmetry and multiple lepton generations, and the addition of quarks and hypercharge completes the MSSM's inherently gauge-invariant description.

Remark on terminology: Throughout this thesis, the terms 'elementary spectrum', 'perturbative spectrum' and 'gauge-variant spectrum' are used interchangeably. Likewise, 'gauge-invariant spectrum', 'non-perturbative spectrum' and 'physical spectrum' denote the same concept. The elements of the latter are called 'bound states operators', 'inherently gauge-invariant operators' or 'composite operators'. We freely choose from those options to make the text sound less robotic.

2 Supersymmetric Theories

Supersymmetry is a popular scenario for beyond the standard model theories and its most prominent feature is that it predicts a symmetry between bosons and fermions. Even though it still lacks experimental evidence, it is worthwhile to explore because to the best of our knowledge, supersymmetry would be the largest possible symmetry of spacetime (see Sec. 2.2). Therefore, finding evidence would be exciting, yet not very surprising. On the other hand, it would be equally interesting to learn why Nature has *not* chosen to realize it.

If found true, supersymmetric theories could address numerous issues of the standard model: An often brought up motivation [10, 11] is the *fine-tuning problem*. In short, it revolves around the unnaturally low mass of the standard model (SM) Higgs boson. It can be immediately remedied by partnering every SM fermion with a boson of equal mass and vice versa. Furthermore, they are way better at unifying the gauge couplings than the SM [13] and adding more particles results in a richer phenomenology, which usually comes with dark matter candidates, too. Finally, promoting supersymmetry to a local symmetry naturally leads to a theory of quantum gravity [15].

In this chapter, we will review the basics of supersymmetry and closely follow [10, 11, 36], to which the reader is referred for more detailed derivations and explanations. In Sec. 2.2 we will see how the Poincaré algebra is enlarged to allow for a symmetry that connects bosons and fermions. The irreducible representations (irreps) of this enlarged symmetry will be *supermultiplets* which are built from an equal number of bosonic and fermionic degrees of freedom. In a next step, we will see how these irreps are used as building blocks for supersymmetric Lagrangians in Sec. 2.3. So long as supersymmetry is exact, those Lagrangians will turn out to be extremely restrictive with very few independent parameters. On the other hand, we know that it cannot be realized at currently explored energies. Thus, every phenomenologically viable theory of supersymmetry must include a breaking mechanism or at least an effective ('soft') parametrization at low energies. Before we get started, however, we must introduce a bit of notation.

2.1 Weyl Spinor Notation

Describing fermions in a supersymmetric setting is a bit different from what most people are used to. We will not use Dirac spinors to describe them but the more basic left- and right-handed *Weyl spinors*. They are more fundamental in the sense that they are the fundamental representations of $SL(2, \mathbb{C})$ while Dirac spinors are *reducible* representations. Furthermore, the fact that they only entail two degrees of freedom instead of four will be convenient for constructing supersymmetric Lagrangians later. Even though left- and right-handed spinors are distinct objects, there is a natural mapping between the two. It turns out, that only one type of spinor is needed to capture the full information [36]. By convention, they are chosen to be the left-handed spinors ξ. Right-handed spinors are then obtained by 'conjugation', i.e. if χ is left-handed, $\chi^\dagger$ is right-handed[1]. Nevertheless, they are still understood to live in different representation spaces and to not accidentally contract them, we introduce separate indices $a = 1, 2$ and $\dot{a} = \dot{1}, \dot{2}$ as follows:

$$\xi_a \qquad \text{(left-handed Weyl spinors)}$$
$$\chi^{\dagger\dot{a}} \qquad \text{(right-handed Weyl spinors)}$$

[1] See [36] and Appendix B.1 for a careful explanation of that statement.

These indices can be raised and lowered using the metric tensor ϵ, defined by

$$\epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = -\epsilon_{ab} = -\epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2 \tag{2.1}$$

$$\xi^a = \epsilon^{ab}\xi_b \qquad \xi_a = \epsilon_{ab}\xi^b \qquad \chi^{\dagger\dot{a}} = \epsilon^{\dot{a}\dot{b}}\chi^\dagger_{\dot{b}} \qquad \chi^\dagger_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\chi^{\dagger\dot{b}}.$$

The metric tensor can also be used to construct a Lorentz-invariant spinor product

$$\begin{aligned} \xi\psi &\equiv \xi^a\psi_a = \epsilon^{ab}\xi_b\psi_a && \text{(left-handed, convention } {}^a{}_a) \\ \chi^\dagger\zeta^\dagger &\equiv \chi^\dagger_{\dot{a}}\zeta^{\dagger\dot{a}} = \epsilon_{\dot{a}\dot{b}}\chi^{\dagger\dot{b}}\zeta^{\dagger\dot{a}} && \text{(right-handed, convention } {}_{\dot{a}}{}^{\dot{a}}) \end{aligned} \tag{2.2}$$

where the index structure is important and chosen by convention to be descending for left-handed spinors and ascending for right-handed spinors. Notice that due to the anti-commutativity of spinors and the anti-symmetry of ϵ, this product is *symmetric*, i.e. $\xi\psi = \xi^a\psi_a = \psi^a\xi_a = \psi\xi$. It makes working with Weyl spinors very convenient. In fact, we can completely forget about the index structure in most calculations because we only work in, e.g., gauge space which is transparent to the spinor structure.

Sometimes, we will reintroduce Dirac spinors for convenience which can be easily built from a left- and right-handed Weyl spinor as

$$\Psi = \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix} \equiv \begin{pmatrix} \xi \\ \chi^\dagger \end{pmatrix}. \tag{2.3}$$

Again, we will drop the indices but keep in mind, that the index structure is important and as indicated. This ensures that our 'careless' calculations work out in terms of spinor indices, e.g., that we can combine Weyl mass terms into Dirac mass terms as

$$\chi\xi + \xi^\dagger\chi^\dagger = \begin{pmatrix} \chi^a & \xi^\dagger_{\dot{a}} \end{pmatrix} \begin{pmatrix} \xi_a \\ \chi^{\dagger\dot{a}} \end{pmatrix} \equiv \bar{\Psi}\Psi. \tag{2.4}$$

The bar here denotes the usual Dirac adjoint $\bar{\Psi} = \Psi^\dagger\gamma^0$. By setting $\chi = \xi$, this extends to Majorana spinors, too.

If we abandon Dirac spinors, we also need to replace the usual γ-matrices by the σ-matrices

$$\begin{aligned} (\bar{\sigma}^\mu)^{\dot{a}b} &\equiv (\mathbb{1}, -\sigma^1, -\sigma^2 - \sigma^3) \\ (\sigma^\mu)_{a\dot{b}} &\equiv (\mathbb{1}, \sigma^1, \sigma^2, \sigma^3). \end{aligned} \tag{2.5}$$

We can use them to build 4-vectors via $\chi^\dagger\bar{\sigma}^\mu\xi$, just like with γ-matrices. Likewise, we can express the generators of homogeneous Lorentz transformations in the Weyl spinor representation as [36]

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu) \\ \bar{\sigma}^{\mu\nu} &= \frac{i}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu), \end{aligned} \tag{2.6}$$

in complete analogy to $\gamma^{\mu\nu} = -\frac{i}{4}[\gamma^\mu, \gamma^\nu]$ as the generators of rotations for Dirac spinors [1].

Above, we only presented the bare minimum of the Weyl spinor formalism in a form that seems most convenient for this thesis. For more details and many useful relations, see [10, 11, 36, 37]. Be aware, though, that definitions might differ slightly across the literature. For example, our definition (2.3) matches the one of [10, 36] while some authors [11] consider right-handed spinors $\chi^\dagger$ to have a lower index by default. Consequently, a Dirac spinor might be written as

$$\Psi = (\xi, i\sigma^2\chi^\dagger)^T = (\xi, \chi^c)^T \qquad \text{(attention: different notation)}$$

in index-free notation. This is completely equivalent if done consistently as $i\sigma^2$ is just the metric (2.1) that pulls indices accordingly.

2.2 Super-Poincaré-Algebra

Since we want to relate bosons and fermions, i.e. particles of different spin, we have to alter the Poincaré algebra in one way or the other. After all, spin is a spacetime property and any transformation which changes spin will definitely not be completely isolated from or independent of Poincaré transformations. However, one of the most famous no-go-theorems, the *Coleman-Mandula theorem* [38], seems to forbid exactly this: According to that theorem, the full symmetry of the S-matrix of a consistent four-dimensional quantum field theory (satisfying locality, causality, finiteness of particles, etc.) has to be

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, P_\lambda] &= i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu) \\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho} M_{\nu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \\
[B_l, B_m] &= i c^k_{lm} B_k \\
[B_l, P_\mu] &= [B_l, M_{\mu\nu}] = 0.
\end{aligned} \tag{2.7}$$

I.e. it consists of the Poincaré algebra (first three lines) and some internal symmetry algebra (fourth line) with structure constants c^k_{lm}. It is important for the two to be combined *trivially*, i.e. the operators B_l must commute with the Poincare generators (last line), and can therefore never effect the spin of a particle. Luckily for us, *Haag*, *Łopuszański* and *Sohnius* [39] discovered that the maximal symmetry is larger if one not only allows bosonic generators, but also some number $\mathcal{N}$ of *anti-commuting* spinor charges. This upgrades the Poincaré algebra to a *graded algebra*. Moving forward, we will only be considering the case $\mathcal{N} = 1$ as $\mathcal{N} > 1$ are scenarios of extended supersymmetry. The new generator Q satisfies the following (anti-)commutation relations

$$\begin{aligned}
\{Q_a, Q_b\} &= \{Q^\dagger_{\dot{a}}, Q^\dagger_{\dot{b}}\} = 0 \\
\{Q_a, Q^\dagger_{\dot{a}}\} &= 2(\sigma^\mu)_{a\dot{a}} P_\mu \\
[Q_a, P_\mu] &= [Q^\dagger_{\dot{a}}, P_\mu] = 0 \\
[Q_a, M^{\mu\nu}] &= i(\sigma^{\mu\nu})_a{}^b Q_b.
\end{aligned} \tag{2.8}$$

We can see that the spinor components of the left-handed (right-handed) version of Q anti-commute but if we mix them, the result is a spacetime translation, generated by P_μ. Second, while Q commutes with P_μ, the commutator with $M^{\mu\nu}$ explicitly demonstrates that Q transforms like a Weyl spinor under homogeneous Lorentz transformations, c.f. Eq. (2.6). Finally, the charge Q is also assumed to commute with all internal symmetries like gauge symmetries in this thesis[2]. The Poincaré algebra and Eq. (2.8) together are called *Super-Poincaré-Algebra* or *SUSY algebra*.

Just like our usual notion of 'particles' is tied to the irreducible representations (irreps) of the Poincaré algebra [1], we can also construct irreps of the SUSY algebra to arrive at the notion of 'superparticles'. However, SUSY algebra irreps are *reducible* with respect to the Poincaré algebra. Consequently, it is also appropriate to call them 'supermultiplets' instead of 'superparticles'. These supermultiplets then contain regular particles.

Earlier, the squares of the 4-momentum and the Pauli-Lubanski-pseudovector were appropriate Casimir operators to classify irreps of the Poincaré algebra according to their mass and spin. The former is still a Casimir of the SUSY algebra, but the latter has to be modified slightly (see [36] for details). It is therefore clear, that members of a supermultiplet have the same mass, but *not* the same spin. The most important irreps turn out to be the so-called massless[3] *chiral* and *vector/gauge supermultiplet*. They host a fermion and a boson, each, which are called *superpartners*. From the standpoint of SUSY, they are completely equivalent and indistinguishable,

[2] The Haag-Łopuszański-Sohnius theorem actually allows Q to be charged under internal symmetries which are then called *R-symmetries* [10].

[3] Massive versions do exist but are discarded here as we will introduce mass via a Brout-Englert-Higgs mechanism.

Supermultiplet	Superpartners		on-shell dof
Chiral	complex scalar (2 real dof)	Weyl spinor (2 spin states)	$2+2$
Gauge	Weyl spinor (2 spin states)	vector (2 helicity states)	$2+2$

Table 2.1: Particle content (irreps of the Poincaré algebra) of the chiral and gauge supermultiplet (irreps of the SUSY algebra). Evidently, bosonic and fermionic on-shell degrees of freedom (dof) match inside each supermultiplet.

and the charge Q induces transformations which freely rotate them into each other. The contents of the supermultiplets are shown in Tab. 2.1.

Those two supermultiplets will serve as the basic building blocks in constructing supersymmetric theories. Notice that this automatically guarantees that we have as many fermionic as bosonic on-shell[4] degrees of freedom in our theory, hence addressing the fine-tuning-problem automatically. It should now become clear, why the Weyl spinor description is so convenient. We also learn that superpartners are identical in mass as long as SUSY is intact, which is bad from a phenomenological point of view. Hence, SUSY has to be broken at least at low energies. Finally, since gauge transformations and SUSY transformations commute, the fields in a supermultiplet must both carry the same gauge charges and live in the same representation. E.g. the gauge bosons of the SM would get fermionic superpartners which live in the adjoint representation of the gauge group.

Speaking of the SM, this is a good place to introduce a very common naming convention for superpartners. Usually, all the particles contained in the SM are called 'gauge bosons', 'fermions' and 'Higgs', as normal. Their superpartners either get a trailing 'ino' if it is fermionic, or a leading 's' if it is scalar. In this way, gauge bosons are partnered with *gauginos* like the *gluinos* and *winos* and for the Higgs we get the *Higgsino*. The scalar superpartners of the SM fermions are referred to as *sfermions* like *selectron*, *sneutrino* (collectively *sleptons*) and *squarks*. Notation-wise, superpartners receive a tilde, e.g., H and $\widetilde{H}$ denote the Higgs and the Higgsino.

2.3 Supersymmetric Lagrangians

Demanding our Lagrangian to be Lorentz- and gauge-invariant as well as renormalizable already poses strong restrictions on the terms we can possibly write down. Supersymmetry turns out to be yet more restrictive as we will see in a bit. The following is a condensed version of the derivation found in [10].

We start off by considering chiral supermultiplets containing a complex scalar field ϕ_i and their Weyl spinor superpartner $\widetilde{\phi}_i$. The index i indicates that we already consider an arbitrary number of chiral supermultiplets. Summation over repeated indices is implied. The kinetic Lagrangian is then simply given by the free Klein-Gordon and left-handed Weyl Lagrangian

$$\mathcal{L}_{\text{kin}}^{\text{chiral}} = \partial^\mu \phi_i^\dagger \partial_\mu \phi_i + i\widetilde{\phi}_i^\dagger \bar{\sigma}^\mu \partial_\mu \widetilde{\phi}_i.$$

Intuitively, one would think that a plethora of interaction terms between the ϕ_i and $\widetilde{\phi}_j$ is possible. And there are a lot, but they are not independent. In fact, SUSY forces the interaction

[4]In order for the supersymmetry algebra to also close *off-shell* and to match off-shell dof, one has to include *auxiliary fields*. We will not go into this but the interested reader is referred to the literature [10, 11, 36].

Lagrangian to be of the form

$$\mathcal{L}_{\text{int}}^{\text{chiral}} = -\frac{1}{2}\left[\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\widetilde{\phi}_i\widetilde{\phi}_j + h.c.\right] - \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2,$$

where W is the so-called *superpotential*. It is a polynomial

$$W = L_i\phi_i + \frac{1}{2}M_{ij}\phi_i\phi_j + \frac{1}{6}y_{ijk}\phi_i\phi_j\phi_k \tag{2.9}$$

in the scalar fields with coefficients L_i, M_{ij} and y_{ijk}. We can freely choose those coefficients for model building but by doing so, we lock the form of the Lagrangian. No terms may be added, nor omitted. And the factors in front of the interaction terms are not independent. Notice, that W may only contain scalar fields but *no field adjoints*! This will lead to the introduction of a second Higgs boson in Sec. 5.1. The Lagrangian we just built is that of the interacting *Wess-Zumino model* [40].

A gauge supermultiplet includes a gauge boson A_μ and the gaugino $\widetilde{A}$, its fermionic superpartner. Their kinetic Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{kin}}^{\text{gauge}} &= -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + i\widetilde{A}^{\dagger a}\bar{\sigma}^\mu D_\mu \widetilde{A}_a \quad \text{with}\\ F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c.\end{aligned}$$

Here, a are adjoint gauge-indices, f^{abc} are the structure constants of the gauge group and D_μ is the gauge covariant derivative in the adjoint representation which couples $\widetilde{A}$ to the gauge bosons. Notice that it is better to think of the gauginos as being matter fields and not some weird sort of 'fermionic gauge boson': A gauge transformation on $\widetilde{A}$ acts as $\widetilde{A}^a \to \widetilde{A}^a - f^{abc}\theta^b\widetilde{A}^c$, i.e. it does not include the '$g^{-1}\partial_\mu\theta$' part of the gauge boson transformation which is only required for the description of (massless) vector bosons [1]. Furthermore, they also do not show up in the covariant derivative for that very reason.

Coupling the two supermultiplets together is done analogously to Yang-Mills theory: We choose a gauge group representation with generators T^a for the chiral supermultiplet (superpartners must live in the same representation, as discussed above) and replace ∂_μ by D_μ (in the appropriate representation) in its Lagrangian. This automatically introduces interaction terms between A_μ, ϕ and $\widetilde{\phi}$ but one can show that to preserve SUSY, the interaction terms

$$-\sqrt{2}g\left(\phi_i^\dagger T^a\widetilde{\phi}_i\right)\widetilde{A}_a - \sqrt{2}g\widetilde{A}^{\dagger a}\left(\widetilde{\phi}_i^\dagger T^a\phi_i\right) - \frac{g^2}{2}\left(\phi_i^\dagger T^a\phi_i\right)\left(\phi_j^\dagger T^a\phi_j\right)$$

are needed in addition. Notice that the coefficients of these terms are *not* independent but fixed by the gauge coupling g. This also means that we cannot omit them.

Collecting everything together, we arrive at the starting point of any supersymmetric gauge theory:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + i\widetilde{A}^{\dagger a}\bar{\sigma}^\mu (D_\mu\widetilde{A})_a && \text{(gauge field kinetic)}\\ &+ (D_\mu\phi)_i^\dagger (D^\mu\phi)_i + i\widetilde{\phi}_i^\dagger\bar{\sigma}^\mu D_\mu\widetilde{\phi}_i && \text{(matter field kinetic)}\\ &- \sum_i\left|\frac{\partial W}{\partial\phi_i}\right|^2 - \frac{g^2}{2}(\phi_i^\dagger T^a\phi_i)(\phi_j^\dagger T_a\phi_j) && \text{(F- and D-terms)}\\ &- \frac{1}{2}\left[\left(\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\right)\widetilde{\phi}_i\widetilde{\phi}_j + \text{h.c.}\right] && \text{(Yukawa-type)}\\ &- \sqrt{2}g\left[(\phi_i^\dagger T^a\widetilde{\phi}_i)\widetilde{A}_a + \widetilde{A}^{\dagger a}(\widetilde{\phi}_i^\dagger T_a\phi_i)\right] && \text{(Yukawa-type)}\\ &+ \mathcal{L}_{\text{soft}}. && \text{(soft breaking terms)}\end{aligned} \tag{2.10}$$

The first two lines consist of the gauge and chiral supermultiplets' kinetic Lagrangians. The two terms in the third line are called F- and D-terms and they constitute the scalar potential. The fourth and fifth line corresponds to Yukawa-type fermion-fermion-scalar interactions. Finally, soft breaking terms may be included to make the model phenomenologically viable. Those terms parametrize the unknown origin of SUSY breaking and vanish at high energies, thus restoring supersymmetry. Possible terms are scalar and Majorana mass terms or trilinear scalar couplings. See Sec. 5.1 for the particular $\mathcal{L}_{\text{soft}}$ that will be used in this work, or e.g. [10, 11] for a detailed discussion. Notice that those terms are not subject to any SUSY constraints and therefore host a huge number of free parameters.

We can clearly see that building a supersymmetric Lagrangian is quite straight-forward: The only things we can choose are

- the gauge groups, possibly more than one which requires additional sums over gauge groups in the Lagrangian above,
- the field content, i.e. the participating supermultiplets together with their gauge group representation and charge assignments,
- the superpotential W, or rather the parameters L_i, M_{ij} and y_{ijk}, and
- the soft SUSY breaking terms.

The exact form of the Lagrangian then follows uniquely. Note that even without the soft breaking terms, Eq. (2.10) is not invariant under SUSY transformations but only the action $S = \int d^4x \mathcal{L}$ is. Explicit expressions for how such transformations act on $\mathcal{L}$ can be found in the literature [10, 11]. They are omitted here because they are quite overwhelming and not relevant for this thesis.

3 Fröhlich-Morchio-Strocchi Mechanism and Augmented Perturbation Theory

Gauge theories like the SM are exceptionally successful in describing fundamental particles and interactions [2, 3]. They have many desirable mathematical properties and serve as a construction principle for modern theories, including theories beyond the SM. Nevertheless, we must not forget that the gauge-principle, *by its very nature*, introduces *redundant* degrees of freedom. Rotating (gauge transforming) in the space of these degrees of freedom has no effect on the physical behavior of a system, *by definition.* After all, changing the gauge amounts to a coordinate transformation and Physics better not depends on our coordinate choice.

Although introducing redundancies as quality of life improvements is perfectly legitimate, the procedure nevertheless calls for a sound prescription on how to extract the physical *gauge-invariant* information of such theories. Within the framework of perturbation theory (PT), the issue is addressed by the so-called *Faddeev-Popov procedure* and the *BRST construction* which we briefly review in Sec. 3.1. As we will discuss, problems and subtleties arise when going beyond PT: Suddenly, it becomes possible that regular PT and non-perturbative methods (lattice field theory in this case) disagree on what the physical degrees of freedom of a theory are. The mismatch is resolved by the so-called *Fröhlich-Morchio-Strocchi mechanism* (FMS) which is introduced in Sec. 3.2. It leads to a slightly modified version of PT which we call *augmented perturbation theory* (APT). This will be the key concept of the remaining thesis.

3.1 Gauge-Fixing, BRST Symmetry and a subtle Issue

Apart from the conceptual idea of getting rid of redundant degrees of freedom, fixing the gauge also has very practical purposes: First, in the usual perturbative approach, a saddle-point approximation is performed, which renders the path integral divergent for gauge theories. The reason are flat directions in field space and gauge-fixing strips off exactly those directions[1]. Secondly, gauge-*variant* correlation functions of elementary fields vanish without fixing the gauge properly [41]. In particular, the Higgs vev $\langle\phi\rangle$ would always be 0, spoiling the usual description of the BEH effect[2] (see Sec. 4.1). The intuitive reason is that all gauge degrees of freedom would average out if no ‘asymmetry’ is introduced during gauge-fixing.

A rigorous and very general derivation of the perturbative gauge-fixing procedure first described by Faddeev and Popov can, e.g., be found in [2]. In practice, however, it can be done in a recipe-like fashion [3]. It amounts to formulating a local gauge-fixing condition

$$C^a(\Psi; x) = 0 \tag{3.1}$$

which depends on any of the fields in the theory (summarized by Ψ) and carries a gauge-index a. Examples for popular choices are[3]

$$\begin{aligned} C^a &= \partial_\mu W^a_\mu && \text{(Landau gauge)} \\ C^a &= \partial_\mu W^a_\mu + ig\xi\phi_i T^a_{ij} v_j && \text{('t Hooft gauge, } R_\xi\text{-gauge).} \end{aligned}$$

[1]In a lattice field theory approach, however, the lattice regularization is sufficient to make the path integral well defined. Nevertheless, the second point persists.

[2]The vev serves as an expansion point for the fluctuation fields. A vanishing vev would not support the argument of ‘small’ fluctuations compared to the vev.

[3]Notice that the ’t Hooft gauge condition comes in different forms.

In a second step, the condition (3.1) is used to build the *gauge-fixing* and *ghost Lagrangian*

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi} C^a(\Psi; x) C^a(\Psi; x)$$

$$\mathcal{L}_{\text{ghost}} = -g \int d^4 y \bar{c}^a(x) M^{ab}(x,y) c^b(y), \quad M^{ab}(x,y) = \left. \frac{\delta C^a \left({}^\theta\Psi; x\right)}{\delta \theta^b(y)} \right|_{\theta=0}$$

which are added to the Lagrangian $\mathcal{L}$ of the theory. Through the Euler-Lagrange equations, the gauge-fixing condition then places restrictions on the equations of motion. In the above expressions, ξ is the gauge-fixing parameter, g is the gauge coupling of the theory, $c, \bar{c}$ are the anti-commuting (anti-)ghost fields and ${}^\theta\Psi$ are the fields transformed by a gauge-transformation parametrized by θ. The effective Lagrangian $\mathcal{L}_{\text{eff}} = \mathcal{L} + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}$ is then used to perform subsequent calculations, derive Feynman rules, etc.

Conceptually, it is irrelevant which particular gauge-condition we choose. Nevertheless, the underlying gauge-invariance of the theory is hidden after gauge-fixing and just by looking at $\mathcal{L}_{\text{eff}}$, one could be lead to believe that inherent gauge-invariance and all of its implications are lost. That this is not the case has been shown by Becchi, Rouet and Stora, and independently by Tyutin. They discovered a symmetry of $\mathcal{L}_{\text{eff}}$ which is essentially the initial gauge symmetry acting on gauge and matter fields, but it is extended also to the (anti-)ghost fields. It is now known as *BRST symmetry* and its key feature is that it allows to define what *physical* states (i.e. states which can be used as initial and final states) are. This is achieved by constructing the conserved, nilpotent BRST charge Q_{BRST}, which can be used to divide the state space into physical and nonphysical subspaces. States with vanishing BRST charge

$$Q_{\text{BRST}} \left|\text{phys}\right\rangle = 0$$

are found to be physical because only for those, matrix elements do not depend on the gauge-fixing procedure. In a more precise language, the physical subspace corresponds to the *cohomology* of Q_{BRST}. Details can be found in [2, 3, 42]. For us, the most important point is that inherently gauge-invariant states are automatically BRST singlets (because BRST acts on gauge and matter fields as a (global) gauge transformation), but the reverse is not necessarily true (because BRST is just a special form of a general gauge transformation).

The subtle problem

Within the framework of PT, gauge-fixing and the BRST construction are well understood and legitimate ways to arrive at physical predictions. If we go beyond PT and employ, e.g., lattice field theory, however, we have to be very carful in carrying over these concepts. It has been shown that in a non-perturbative setting, fixing the gauge *uniquely*, with any of the local gauge-fixing conditions from before, is impossible. The reason for this inconvenience are so-called *Gribov copies* [25, 26]. How gauge-fixing can be approached non-perturbatively is e.g. discussed in [43].

Gribov copies also affect the BRST construction and demand a non-perturbative definition of BRST symmetry [44]. The details are yet to be fully understood but there are hints that such a construction is indeed possible [41, 43]. Nevertheless, this altered BRST symmetry might lead to a different notion of *physical* states. In other words, the physical state space of the theory could be changed considerably when going beyond PT.

Many questions still remain unanswered but the important thing for us is to realize that the straight-forward perturbative description of non-Abelian gauge theories might be severely incomplete. Much work has been done on toy theories beyond the SM which explicitly demonstrate the issue: In [31], the authors studied an $SU(3)$ gauge theory with a fundamental scalar field and found a mismatch between the spectrum predicted by PT and their lattice results[4]. Moreover,

[4] Their lattice calculations found a massive vector singlet for the theory's ground state while PT predicted a massless vector triplet.

large classes of GUT-like scenarios have been studied and showed similar mismatches [29, 30]. One could be skeptical and refer to the apparent success of the SM, which made accurate predictions using PT. However, there are attempts to explain why Gribov copies might only have a tiny impact on the SM and therefore have not been noticed yet [28, 45]. Nevertheless, we conclude that careful model building is required, especially for theories beyond the SM.

Detached from Gribov copies, we want to point out yet another subtle issue which is of some importance to us: Classifying the spectrum into multiplets with respect to some symmetry group. It must be emphasized that only global symmetries have physically observable implications like mass-degeneracies. The global part of a gauge symmetry, in general, does not make physical predictions, because after gauge-fixing, the 'conserved charge' that corresponds to this symmetry is in general gauge-*variant* [46]. The only symmetry that has anything to do with the gauge group and is still physical (within PT at least) is the aforementioned BRST symmetry. Yet all physical states are BRST *singlets*. Nevertheless, in standard treatments, the SM W-triplet[5] appears to be a consequence of a (partially) gauged symmetry. In Sec. 4.2 we will see, how this apparent paradox is resolved.

3.2 The Framework of FMS and APT

An elegant way out of these troublesome topics exists in theories with a Brout-Englert-Higgs (BEH) effect like the SM. The so-called *Fröhlich-Morchio-Strocchi mechanism* (FMS) [23, 24] in combination with *augmented perturbation theory*[6] (APT) [28] allows for an inherently gauge-invariant formulation of such theories. The key idea is to define the physical (i.e. inherently gauge-invariant) quantities first and use a formal expansion to compare it to the results obtained from usual PT afterwards. Since this procedure will be actively performed in the present work, we shall lay out the details in the following.

Given any correlation function of a quantum field theory, e.g. the propagator of the SM Higgs field $\langle \phi(x)\phi^\dagger(y) \rangle$, we shift our view away from the elementary (gauge-variant) objects (ϕ in this case) and adopt a bound state operator language. These bound states must possess two key properties:

- They must be inherently gauge-invariant, i.e. we have to make sure that whatever operator combination we write down is a gauge singlet. This automatically means, that they are physical in the perturbative BRST sense, but also manifestly physical in the full, non-perturbative theory.
- They must carry defined quantum numbers (if applicable) like spin, (charge)parity, etc. Those quantum numbers may *not* be gauge indices but have to correspond to global (and therefore observable) symmetries.

An example for such an operator would be $\mathcal{O}_{0^+}(x) = (\phi^\dagger\phi)(x)$ [23] which has the same quantum numbers as the SM Higgs ($J^P = 0^+$) and can be interpreted as a bound state object of two elementary Higgs fields. The notion of 'bound state' should be taken with a grain of salt, though. After all, the 'constituents' of these 'bound states' (by their nature gauge-variant, nonphysical objects) do not exist in isolation. Secondly, one should not imagine some mechanism that 'binds' the elementary fields together. It is merely a vivid picture of how we build inherently gauge-invariant objects out of these elementary fields.

The aforementioned shift in perspective is now to realize, that experiments probe specific quantum number channels, and that the participants of a scattering process could as well be composite operators encoding these quantum numbers. We can therefore use them to write

[5]In absence of hypercharge, the weak gauge bosons are mass degenerate.

[6]Previously often called *gauge-invariant perturbation theory* but this term was prone to confusion.

down manifestly gauge-invariant correlation functions. The correlator

$$\left\langle \mathcal{O}_{0^+}(x)\mathcal{O}^\dagger_{0^+}(y)\right\rangle = \left\langle (\phi^\dagger\phi)(x)(\phi^\dagger\phi)(y)\right\rangle,$$

for example, describes the propagation of a spin-0, parity even object. Just like the object $\langle\phi(x)\phi^\dagger(y)\rangle$ did. The difference now is that we made sure that we are only working with physical objects to begin with, hence we do not have to worry about how to get rid of any nonphysical gauge-leftovers afterwards. Unfortunately, this correlation function would have to be evaluated using non-perturbative methods as PT alone cannot treat bound states. Remembering the discussion of Sec. 3.1, however, we already know that the situation beyond PT might be substantially different and *has* to be explored!

We now perform a trick: We expand the scalar field as

$$\phi(x) = vn + \varphi(x),$$

where v (chosen to be real) and n are the magnitude and direction of its spacetime independent vacuum expectation value (vev) and φ are the relative fluctuations. Notice that ϕ, vn and φ are *not* invariant under gauge transformations and vn is *only constant* in the (fixed) gauge where this splitting is defined. Choosing this gauge is completely arbitrary and will have no effect on the inherently invariant operators. One could just as well choose a gauge where the vev vanishes. For what we are about to do, however, it is advisable to have a non-vanishing vev, which is e.g. guaranteed in 't Hooft gauge. The expansion of ϕ allows for a formal expansion of the composite object

$$\mathcal{O}_{0^+} = v^2 + 2v\,\mathrm{Re}(n^\dagger\varphi) + \varphi^\dagger\varphi.$$

The first term is just an irrelevant constant. The second term describes the fluctuation along the direction of the vev, which is usually called *the* Higgs field $h \equiv \sqrt{2}\,\mathrm{Re}(n^\dagger\varphi)$. If the gauge is now chosen such that $v \neq 0$ and $||\varphi|| \ll v$, the properties of the gauge-invariant, non-perturbative object $\mathcal{O}_{0^+}$ are well captured by the properties of the gauge-dependent elementary field h. All properties are encoded in the correlation functions and thanks to the linearity of the expectation value, we may perform this split in the correlators as well. E.g. the connected 2-point-function (propagator) of $\mathcal{O}_{0^+}$ is given by

$$\left\langle \mathcal{O}_{0^+}(x)\mathcal{O}^\dagger_{0^+}(y)\right\rangle_c = 2v^2\left\langle h(x)h(y)\right\rangle_c + 2\sqrt{2}v\left\langle h(x)(\varphi^\dagger\varphi)(y)\right\rangle_c + \left\langle (\varphi^\dagger\varphi)(x)(\varphi^\dagger\varphi)(y)\right\rangle_c.$$

Here we used that the propagator only depends on $|x-y|$ [27]. Notice that while the left-hand side is inherently gauge-invariant, the individual terms on the right-hand side are *not*. Only their sum is, by construction. We can see that the dominant contribution is just the propagator of the elementary Higgs field h, i.e. the propagator of $\mathcal{O}_{0^+}$ has the same pole as the elementary Higgs, and both objects therefore share the same mass. That the remaining terms are practically irrelevant has been demonstrated in [27]. The advantage of this approach, called the *FMS mechanism* [23, 24], is that non-perturbative bound state information can be obtained by calculating the elementary n-point functions on the right in an appropriate gauge. They can be expanded and systematically calculated using regular PT. The whole process of formulating composite objects, applying the FMS mechanism and eventually standard PT is called *augmented perturbation theory* (APT). It is an expansion both in v and the gauge coupling. It is important to emphasize again, that the gauge we choose is completely irrelevant for the left-hand side of the equation and the FMS expansion works regardless of the vev splitting that we use. However, in order to establish the connection to the elementary fields, it is crucial to choose a gauge where $||\varphi|| \ll v$ because only then, lower orders in v (or equivalently higher orders in φ) are suppressed [23] and APT is a useful device. Therefore, choosing a gauge where the vev vanishes, although valid, is not very insightful.

In practice, none of this has to be done in such great detail. What we will be doing is write down gauge-invariant bound state operators for the desired channels and perform the FMS expansion by replacing all appearing Higgs fields by their vev and fluctuation field in 't Hooft gauge. We will solely be interested in contributions to leading order in v (i.e. leading order of the FMS expansion) and on tree-level (i.e. leading order of PT). We will further neglect constant terms in the expansions right as they appear without explicitly stating that we consider only connected correlators, etc. In case a bound state object behaves like the tree-level mass eigenstate of regular PT in leading order, we will use the notation

$$\mathcal{O}_{0^+}(x) = v^2 + \sqrt{2}vh(x) + (\varphi^\dagger\varphi)(x) \overset{\text{FMS}}{\sim} vh(x) \tag{3.2}$$

which strips off all further contributions as well as constants other than powers of v.

At this point the reader might think that any arbitrary perturbative mass eigenstate can be *augmented* by such a bound state object, yet this does not have to be the case: In the toy theory mentioned earlier, it turns out that gauge-invariant operators can only be built for *some* of the states predicted by regular PT, i.e. not all of them can be truly physical. By carrying through the formalism of APT, the authors found that those states are indeed removed and the remaining spectrum is finally in agreement with their lattice results [31]. Similar things can be said about a large class of GUT-like scenarios [30].

As already mentioned, actually observing such a mismatch in Nature is very hard. By now it is understood that this is due to the very special structure of the SM, specifically its Higgs sector [27]. In fact, it has been shown that in the SM, one can find a one-to-one mapping between elementary fields and gauge-invariant operators in the spirit of APT. See [28] for a detailed review or Sec. 4 for an overview. Even though it causes no qualitative difference within the SM, the FMS procedure can nevertheless be used to justify the usual approach of calculating the spectrum using gauge-dependent states. Furthermore, the sub-leading corrections neglected above could in principle become relevant under the right kinematical conditions [32–34].

Regardless of what the 'correct' theory beyond the SM might be, we can definitely not expect it to behave as nicely and we should be carefully examining the spectra of all proposed extensions using APT. Putting the Higgs in a different representation as in GUT-like theories or having a different relation between gauge and custodial group as in 2HDMs, e.g., has been shown to have a huge impact on the validity of the 'naive' spectrum [28, 30, 31, 35]. A previously unexplored case are supersymmetric theories like the MSSM, and it will be highly interesting to investigate, whether undiscovered particles like the lightest supersymmetric particle (LSP) are even part of the physical spectrum.

4 Review of the Standard Model in light of APT

In this chapter we apply the Fröhlich-Morchio-Strocchi (FMS) mechanism and augmented perturbation theory (APT) to the standard model (SM). It will help us draw connections later on. A good overview of perturbative Brout-Englert-Higgs (BEH) physics in the SM can, e.g. be found in [47]. Its manifestly gauge-invariant treatment is discussed thoroughly in [28]. The most important results will be presented below. Notice that some of the notation is slightly altered from the initial authors to better fit the notation used in this thesis.

As mentioned before, manifest gauge-invariance is not a problem in the electromagnetic and (pure) strong sector of the SM. Furthermore, the formalism developed in Chapter 3 makes it clear that the interesting subsectors of the SM are those which couple to the Higgs directly. We will therefore only consider the relevant parts of the SM Lagrangian which are given by

$$\mathcal{L}_{\text{SM}} \supset (D^\mu\phi)^\dagger(D_\mu\phi) - V(\phi) + \mathcal{L}_{\text{Yukawa}}, \tag{4.1}$$

i.e. the kinetic term of the Higgs field ϕ, its coupling to the gauge bosons W^a_μ and B_μ via the gauge-covariant derivative

$$D_\mu \equiv \partial_\mu\mathbb{1} - igW^a_\mu\frac{\sigma^a}{2} - ig'B_\mu\frac{\mathbb{1}}{2}, \tag{4.2}$$

the scalar potential

$$V(\phi) \equiv -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{4.3}$$

as well as the Yukawa couplings to the quarks and leptons

$$\mathcal{L}_{\text{Yukawa}} \equiv -\bar{d}\boldsymbol{y}_d\phi^\dagger Q - \bar{u}\boldsymbol{y}_u\widetilde{\phi}^\dagger Q - \bar{e}\boldsymbol{y}_e\phi^\dagger L + h.c. \tag{4.4}$$

In the last line, family indices have been suppressed ($\boldsymbol{y}$ are 3×3 matrices in family space) and $\widetilde{\phi} \equiv i\sigma^2\phi^{\dagger T}$ is the charge conjugated Higgs field (carrying opposite hypercharge to ensure $U(1)_Y$ invariance of the Yukawa terms).

We will build up (4.1) step by step, starting with the pure weak-Higgs sector in Sec. 4.1. We will discuss the usual description of the BEH mechanism and point out subtleties along the way. In Sec. 4.2 we demonstrate how the FMS mechanism and APT can be used to formulate the Higgs particle, weak gauge bosons and left-handed leptons/quarks gauge-invariantly. Finally, we will include Yukawa couplings and hypercharge in Sec. 4.3, where we will also see how ‘electroweak-symmetry-breaking’ should be understood.

4.1 Weak-Higgs Sector

Ignoring the Yukawa terms in Eq. (4.1) and setting $g' = 0$ for now, the theory reduces to

$$\mathcal{L} = -\frac{1}{4}W^{\mu\nu}_a W^a_{\mu\nu} + (D^\mu\phi)^\dagger(D_\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{4.5}$$

where the covariant derivative simplifies to $D_\mu = \partial_\mu\mathbb{1} - igW^a_\mu\frac{\sigma^a}{2}$ as well. This Lagrangian is inherently invariant under $SU(2)_L$ gauge transformations but it also exhibits a slightly less obvious *global* $SU(2)$ symmetry which connects the Higgs to its charge conjugate (see Appendix B.2

for details). Both symmetries can be made explicit by introducing the *bidoublet*

$$\Phi \equiv (i\sigma^2\phi^{\dagger T}, \phi) = \begin{pmatrix} \phi_2^\dagger & \phi_1 \\ -\phi_1^\dagger & \phi_2 \end{pmatrix} \tag{4.6}$$

and rewriting Eq. (4.5) in the perfectly equivalent form

$$\mathcal{L} = -\frac{1}{4}W_a^{\mu\nu}W_{\mu\nu}^a + \frac{1}{2}\mathrm{tr}\left[(D^\mu\Phi)^\dagger(D_\mu\Phi)\right] + \frac{\mu^2}{2}\mathrm{tr}\left[\Phi^\dagger\Phi\right] - \frac{\lambda}{4}\mathrm{tr}\left[\Phi^\dagger\Phi\right]^2. \tag{4.7}$$

This version is superior because it is now obvious (due to the cyclic property of the trace) that the Lagrangian is invariant under $SU(2)_L \times SU(2)_\mathcal{C}$ transformations of the form

$$\Phi \to \Phi' = L(x)\Phi R^\dagger, \quad L(x) \in SU(2)_L,\ R \in SU(2)_\mathcal{C}.$$

Following [28], we shall call $SU(2)_\mathcal{C}$ the *custodial symmetry* group but it should be mentioned that the term is used slightly differently by other authors (see later). $SU(2)_\mathcal{C}$ will play a central role in the FMS construction and it also has important phenomenological implications. We postpone this discussion to Sec. 5.2.

The description of the BEH mechanism then goes as follows [47]: Given that $-\mu^2 < 0$ and $\lambda > 0$, we realize that the potential (4.3) has a non-trivial (classical) minimum when the field 'length' is $||\phi||^2 = \phi^\dagger\phi = \mu^2/(2\lambda) \equiv v^2/2$ or $\mathrm{tr}\left[\Phi^\dagger\Phi\right] = \mu^2/\lambda$, respectively. Since $\mathrm{tr}\left[\Phi^\dagger\Phi\right]$ is an $SU(2)_L \times SU(2)_\mathcal{C}$ invariant, this is a gauge and custodial invariant statement. Even though all ϕ satisfying this condition minimize the potential, we have to (arbitrarily) pick one particular field configuration on the 3-sphere of constant $\phi^\dagger\phi$ to do perturbative phenomenology. A common choice is

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \eta_1 + i\eta_2 \\ v + h + i\eta_3 \end{pmatrix}$$

with the real scalar fields h and η_i which are the fluctuation fields around the classical minimum and they are supposed to be 'small' compared to v in the spirit of perturbation theory. Eventually, h becomes the massive Higgs mode and the remaining three constitute the (would-be) Goldstone modes. For the bidoublet description, this translates to

$$\Phi = V + \eta \tag{4.8}$$

where $V \equiv v\mathbb{1}/\sqrt{2}$ is the vev and η is a complex 2×2 fluctuation matrix of the form (4.6). It is important to mention that both η *and* V transform under gauge and custodial transformations, just like Φ. Notice that even though the theory and minimum are perfectly $SU(2)_L \times SU(2)_\mathcal{C}$ symmetric, our *choice* of a *specific* vacuum (which we need to do calculations) 'breaks' the symmetry down to their common diagonal subgroup[1]

$$SU(2)_L \times SU(2)_\mathcal{C} \to SU(2)_\mathrm{diag}.$$

In other words, the little group of our chosen minimum V is built from all $SU(2)_L \times SU(2)_\mathcal{C}$ rotations for which $L(x) = L = R$ because then

$$V \to V' = LVR^\dagger = RVR^\dagger = VRR^\dagger = V.$$

If we considered $SU(2)_L \times SU(2)_\mathcal{C}$ to be the global symmetry of a classical system this choice would correspond to us spontaneously breaking the symmetry. Nevertheless, it has to be emphasized that this is *not* 'spontaneous gauge symmetry breaking' as this view only works classically.

[1]It is this diagonal subgroup that many authors refer to as 'custodial symmetry'.

Quantum theoretically, this process has to be understood from a gauge-fixing point of view and we will shortly see what happens to the symmetry in this case. We perform the usual gauge-fixing procedure by Faddeev and Popov introduced in Sec. 3.1. Following [28], we employ the 't Hooft gauge condition

$$C^a = \partial^\mu W_\mu^a + g\xi \operatorname{Im} \operatorname{tr}\left[V^\dagger \sigma^a \Phi\right] \tag{4.9}$$

with the *constant* (i.e. it does not transform under symmetry operations) matrix V setting the direction of the vev[2]. Again, both the direction and the particular gauge-fixing condition are a *choice*. And it is now this choice which breaks the $SU(2)_L \times SU(2)_\mathcal{C}$ symmetry on a quantum level: In order to leave $\mathcal{L}_{\text{gf}}$ invariant, we have to compensate every custodial transformation with an equal (global) gauge-transformation, showing that the full symmetry is 'broken' down to the diagonal subgroup. One could call this 'spontaneous gauge symmetry breaking' but the term is a bit misleading as a particular choice of gauge obviously 'breaks' the symmetry – *explicitly*. Additionally, true spontaneous breaking of a local symmetry is forbidden by *Elitzur's Theorem* [48]. Notice that the outcome seems to be identical to the classical case but the reasoning is different!

If one collects all the Lagrangian terms together and plugs in the splitting (4.8) for Φ we recover the well-known result that the gauge bosons receive mass terms

$$\mathcal{L} \supset \frac{1}{2} m_W^2 W_a^\mu W_\mu^a, \quad m_W \equiv \frac{gv}{2}.$$

Their mass degeneracy is protected by the remaining diagonal $SU(2)$ subgroup. We also find that the field h acquires the mass $m_h = \sqrt{2\lambda v^2}$ and that the masses of the (would-be) Goldstone modes depend on the gauge-fixing parameter ξ. This is a clear signal that those modes are nonphysical as no physical mass should depend on our gauge choice. They are often said to be 'eaten by the gauge bosons' as a pictorial explanation of why they become massive but this picture should be treated with utmost care.

4.2 Problems of and Resolution to the usual Treatment

After examining the spectrum in view of the BRST construction we conclude that the Higgs field h as well as the massive Ws are physical fields [3]. Even though this is fine in a PT setting, we nevertheless remark the following:

- As discussed in Sec. 3, the BRST construction might break down beyond perturbation theory, i.e. the mass eigenstates that we just found might not be the *physical* fields in a full treatment of the theory.
- We used $SU(2)_{\text{diag}}$ as a reason for why the W bosons should be mass degenerate in absence of hypercharge. In a second step, we argued that those fields are physical, hence the mass degeneracy should also be physical. Looking at it closer, we realize that $SU(2)_{\text{diag}}$ contains both parts of the custodial as well as the gauge symmetry. And gauge symmetries are (by their very nature) *not observable*. The argument seems to fall apart and asks for a gauge-invariant reason for the degeneracy.
- Similar things pop up once we introduce left-handed leptons and quarks. They are charged under $SU(2)_L$ and are often denoted as $L = (\nu, e)^T$ and $Q = (u, d)^T$ which implies a flavor symmetry between left-handed neutrinos and electrons (up-type and down-type quarks)[3]. Unfortunately, this is highly misleading: The components of these $SU(2)_L$ doublets can be

[2] It only sets *the direction* of the vev but should not be associated with the vev itself, which is a dynamical quantity of the theory. In a more pedantic language it would make sense to introduce a different symbol (like the author of [28]) but we do not do that here as the V here and before are identical in their entries.

[3] Remember that this is still in absence of hypercharge and Yukawa couplings.

freely rotated into each other by a weak gauge transformation, i.e. the symbols ν, e, u, d do *not* correspond to physical fields as they still carry a gauge index.

Luckily, we have already built up the formalism to work around those intricacies. Instead of jumping right into the usual treatment, we take a step back and construct manifestly gauge-invariant objects first [23, 24]. Those objects are by construction gauge singlets but may be classified into multiplets according to their spin, parity and the global custodial symmetry, $J^P_{\mathcal{C}}$. The bidoublet form is of great help in that respect as $SU(2)_{\mathcal{C}}$ acts linearly upon Φ which makes assigning custodial charges easy. A scalar singlet can easily be built, e.g. as $\mathcal{O}_{0^+_{\mathbf{1}}} = \operatorname{tr}\left[\Phi^\dagger\Phi\right]$. We already met this operator in its non-bidoublet form in Sec. 3.2. To leading order in the vev, it augments the elementary Higgs field

$$\mathcal{O}_{0^+_{\mathbf{1}}} \overset{\text{FMS}}{\sim} vh.$$

A vector triplet is given by $\mathcal{O}^A_{1^-_{\mathbf{3}},\mu} = \operatorname{tr}\left[\Phi^\dagger D_\mu \Phi \sigma^A\right]$. Notice that the uppercase A is a reminder that those are custodial indices, not gauge indices. It FMS-expands as

$$\mathcal{O}^A_{1^-_{\mathbf{3}},\mu} \overset{\text{FMS}}{\sim} v^2 \delta^{Aa} W^a_\mu,$$

i.e. $SU(2)_{\text{diag}}$ indices a are traded for truly physical $SU(2)_{\mathcal{C}}$ indices A, and the elementary vector degrees of freedom are mapped to the inherently gauge-invariant vector triplet operator. As long as custodial symmetry is exact, this operator predicts a mass degenerate vector triplet, and due to the FMS mechanism this is finally a *physical* argument for the mass degeneracy of the elementary W^a_μ states. Notice that this is only possible because the little group and the custodial group are both $SU(2)$ in the SM!

Gauge-invariant operators with appropriate quantum numbers can also be built for the left-handed fermions. For example, the operators $\phi^\dagger L$ and $\det \phi L$ correspond to the physical electron and neutrino, respectively [23]. They can further be combined into one operator using the bidoublet Φ. The bound state

$$\Psi_L \equiv \Phi^\dagger L \overset{\text{FMS}}{\sim} v \begin{pmatrix} \nu \\ e \end{pmatrix} = vL$$

is a gauge singlet, but a (physical) doublet with respect to custodial transformations. By means of the FMS mechanism, it reduces to the elementary 'doublet' L and gauge indices are again traded for custodial indices. We conclude that as long as $SU(2)_{\mathcal{C}}$ is intact, the left-handed neutrino and electron indeed form a physical doublet when dressed with a Higgs. L is still not a physical doublet, but the FMS mechanism yields an explanation to why it is a good approximation still. Note that the right-handed electron is an $SU(2)_L$ singlet and hence does not need to be dressed with a Higgs.

Evidently, the same can be applied to quarks as well as the remaining two fermion generations by constructing analogous bound state operators $\Phi^\dagger Q^{1,2,3}$, $\Phi^\dagger L^{1,2,3}$. Those can eventually also be combined into mesons and hadrons in a fully gauge-invariant way [32]. Within our bound state language, the extension to the second and third generation could possibly also be established by regarding them as excited states of the first generation [32]. Even though it is highly speculative at the moment and the only hint in this direction are exploratory calculations [49], it is nevertheless interesting to think about operators like $\Phi^\dagger\Phi\Phi^\dagger L$, i.e. just the leptonic bound state $\Phi^\dagger L$ with internal Higgs excitations. Those internal excitations could be speculated to provide the mass discrepancy between fermion generations.

4.3 The Effect of Hypercharge and Yukawa terms

In the usual treatment [3], B and $g' \neq 0$ are included from the very beginning. The BEH effect induces electroweak symmetry breaking and it is presented as a 'gauge symmetry breaking' $SU(2)_L \times U(1)_Y \to U(1)_{\text{EM}}$. It results in the formation of three massive gauge bosons, $W^\pm$ and Z, as well as the massless photon. Eventually, this also leads to the definition of the electric charge and coupling constant

$$Q = T^3 + Y, \qquad\qquad e \equiv \frac{gg'}{\sqrt{g^2 + g'^2}}. \tag{4.10}$$

We will now show how the same phenomenology can be described gauge-invariantly by exploiting the intricate structure of $SU(2)_\mathcal{C}$.

Custodial symmetry is not exact in the full SM which becomes apparent after reintroducing B and setting $g' \neq 0$. Writing the gauge-kinetic part of Eq. (4.1) in the bidoublet form

$$\mathcal{L} \supset \frac{1}{2}\text{tr}\left[\left(\partial^\mu\Phi - igW_a^\mu\frac{\sigma^a}{2}\Phi + 2ig'B^\mu\Phi\frac{\sigma^3}{2}\right)^\dagger\left(\partial_\mu\Phi - igW_\mu^a\frac{\sigma^a}{2}\Phi + 2ig'B_\mu\Phi\frac{\sigma^3}{2}\right)\right]$$

makes it clear that for $g' \neq 0$, $SU(2)_\mathcal{C}$ is partially broken. What survives is its Abelian subgroup and since σ^3 acts on Φ from the right (just like $SU(2)_\mathcal{C}$), we see that this subgroup is actually the hypercharge group $U(1)_Y$. It acts on the doublets ϕ as $\phi \to \exp(i\alpha(x)/2)\phi$ which translates to the bidoublets as

$$\Phi \xrightarrow{U(1)_Y} \Phi' = \Phi\ \exp\left[-i\alpha(x)\frac{\sigma^3}{2}\right]. \tag{4.11}$$

In other words, introducing hypercharge amounts to gauging the $U(1)$ subgroup of $SU(2)_\mathcal{C}$ and since $SU(2)/U(1)$ is merely a coset, this is all the symmetry that remains. A second source of custodial symmetry violation are the Yukawa terms as can be clearly seen from (4.4). However, the $U(1)$ subgroup of $SU(2)_\mathcal{C}$ does *not* further break by introducing Yukawa terms as long as we simultaneously perform a $U(1)$ transformation of the fermions according to their hypercharge assignments. This combined (local) $U(1)$ transformation persists in the full SM and naturally results in the $U(1)_{\text{EM}}$ gauge theory of quantum electrodynamic. Completely without using the wrong [48] notion of 'spontaneous gauge-symmetry-breaking'.

Furthermore, in our picture it is very obvious how the weak bosons obtain their electric charge, despite them having zero hypercharge assigned: Consider again the vector triplet bound state operator $\mathcal{O}^A_{1^-_\mathbf{3},\mu} = \text{tr}\left[\Phi^\dagger D_\mu\Phi\sigma^A\right]$. Trivial linear combinations can be used to construct $\mathcal{O}_{W^{\pm,0}}$ and performing the $U(1)_{\text{EM}}$ (former custodial) transformation (4.11) reveals that they have the same electric charge as their elementary counterparts

$$\begin{pmatrix}\mathcal{O}_{W^+}\\ \mathcal{O}_{W^-}\\ \mathcal{O}_{W^0}\end{pmatrix} = \begin{pmatrix}\text{tr}\left[\Phi^\dagger D_\mu\Phi(\sigma^2 + i\sigma^1)\right]\\ \text{tr}\left[\Phi^\dagger D_\mu\Phi(\sigma^2 - i\sigma^1)\right]\\ \text{tr}\left[\Phi^\dagger D_\mu\Phi\sigma^3\right]\end{pmatrix} \xrightarrow{U(1)_{\text{EM}}} \begin{pmatrix}\mathcal{O}'_{W^+}\\ \mathcal{O}'_{W^-}\\ \mathcal{O}'_{W^0}\end{pmatrix} = \begin{pmatrix}e^{i\alpha} & & \\ & e^{-i\alpha} & \\ & & 1\end{pmatrix}\begin{pmatrix}\mathcal{O}_{W^+}\\ \mathcal{O}_{W^-}\\ \mathcal{O}_{W^0}\end{pmatrix}.$$

Since $SU(2)_\mathcal{C}$ is broken, there is nothing protecting their mass degeneracy anymore, though, and the neutral component $\mathcal{O}_{W^0}$ is allowed to mix with the neutral $\mathcal{O}_B$ operator to create operators $\mathcal{O}_Z$ and $\mathcal{O}_\Gamma$ for the physical Z and photon in an inherently gauge-invariant way. See [28] for details. Likewise, lepton operators receive the correct electric charges through

$$\Phi^\dagger L \xrightarrow{U(1)_{\text{EM}}} \begin{pmatrix}1 & \\ & e^{-i\alpha}\end{pmatrix}\Phi^\dagger L \overset{\text{FMS}}{\sim} v\begin{pmatrix}\nu\\ e^{-i\alpha}e\end{pmatrix},$$

and the Higgs singlet tr $\left[\Phi^\dagger\Phi\right]$ is neutral, as required. We see that electric charge directly follows from combining the custodial $U(1)$ subgroup acting on Φ and $U(1)_Y$ acting on the fermions, and in some sense it replaces (4.10).

We will discuss this in more detail for the case of the MSSM in Sec. 5.4.2.

5 Manifestly gauge-invariant Spectrum of the MSSM

The minimal supersymmetric standard model (MSSM) has been extensively studied over the past couple of decades. See e.g. [10, 11] for a review. Additionally, huge effort went into attempts to detect the particles it predicted, without questioning the validity of the (perturbatively) calculated spectrum. As mentioned multiple times by now, one should be extremely careful in assigning any physical reality to these calculations, as numerous examples beyond the SM have illustrated, c.f. Chapter 3. Only the inherently gauge-invariant spectrum of a theory can be used to make reliable *physical* statements. The FMS mechanism has proved to be an incredibly powerful and convenient tool to investigate the physical spectrum of the SM, GUT-like theories and the 2HDM. For the first time, these investigations will be extended to a supersymmetric theory, the MSSM, in this chapter. We will answer the question, whether the truly physical spectrum of the MSSM agrees with the one calculated from perturbation theory or not. A particularly interesting point will be whether the lightest supersymmetric particle (LSP) is expected to be observable.

This Chapter is structured as follows: First, the MSSM will be introduced in Sec. 5.1. Just like in the SM, we will again look at the weak-Higgs sector in detail in Sec. 5.3.1. Once more, custodial symmetry plays a major role which is why we spend some time discussing how it is realized in the MSSM in Sec. 5.2. Afterwards, we include one lepton family in Sec. 5.3.2, where we will also lay out the details on how to obtain particular supersymmetric Lagrangians from the general form (2.10) for once. To make things easier and more transparent, we will make some simplifications along the way. In Sec. 5.4 those restrictions are lifted and the calculations are generalized to the full MSSM. Finally, Sec. 5.5 summarizes its physical spectrum.

5.1 The Minimal Supersymmetric Standard Model

We have established that every supersymmetric theory is subject to the constraints set by SUSY itself which is why we have to start from the general Lagrangian (2.10) in any case. Additionally, we may choose the particle content, gauge groups and charges, the superpotential and soft breaking terms as explained in Sec. 2.3.

Particle content

The MSSM is *minimal* in the sense that it reproduces the SM interactions while introducing as few additional particles as possible to make it supersymmetric. Hence, the SM particles and all its gauge groups and charges are kept the same but they receive a superpartner: Leptons and quarks are placed into chiral supermultiplets and are accompanied by spin-0 *sleptons* and *squarks* (short for *scalar* leptons/quarks). The Higgs shares a chiral supermultiplet with the spin-1/2 *Higgsino*. Finally, the gauge bosons go into gauge supermultiplets with *gauginos* (also spin-1/2) as their superpartners. A detailed discussion can be found, e.g. in [10, 11]. The particle content of the MSSM is summarized in Tab. 5.1. There, you will find the (possibly) surprising fact that we also have to include a second Higgs field in addition to the SM Higgs. It has opposite hypercharge and is needed to properly cancel the anomalies introduced by just a single Higgsino [10][1]. The

[1] There is yet another reason for why we need two Higgs which we will discuss later.

Names	Boson	Fermion	$[SU(3)_c, SU(2)_L, U(1)_Y]$
l.h. (s)quarks	$\widetilde{Q} = (\widetilde{u}, \widetilde{d})$	$Q = (u, d)$	$[\mathbf{3}, \mathbf{2}, \ \frac{1}{3}]$
r.h. up (s)quark	$\widetilde{\bar{u}}$	$\bar{u}$	$[\bar{\mathbf{3}}, \mathbf{1}, \text{-}\frac{4}{3}]$
r.h. down (s)quark	$\widetilde{\bar{d}}$	$\bar{d}$	$[\bar{\mathbf{3}}, \mathbf{1}, \ \frac{2}{3}]$
l.h. (s)leptons	$\widetilde{L} = (\widetilde{\nu}, \widetilde{e})$	$L = (\nu, e)$	$[\mathbf{1}, \mathbf{2}, \text{-}1]$
r.h. (s)electron	$\widetilde{\bar{e}}$	$\bar{e}$	$[\mathbf{1}, \mathbf{1}, \ 2]$
Higgs(inos)	$H_u = (H_u^{(1)}, H_u^{(2)})$	$\widetilde{H}_u = (\widetilde{H}_u^{(1)}, \widetilde{H}_u^{(2)})$	$[\mathbf{1}, \mathbf{2}, \ 1]$
	$H_d = (H_d^{(1)}, H_d^{(2)})$	$\widetilde{H}_d = (\widetilde{H}_d^{(1)}, \widetilde{H}_d^{(2)})$	$[\mathbf{1}, \mathbf{2}, \text{-}1]$
gluons, gluinos	g	$\widetilde{g}$	$[\mathbf{8}, \mathbf{1}, \ 0]$
W bosons, winos	$W^{\pm}, W^0$	$\widetilde{W}^{\pm}, \widetilde{W}^0$	$[\mathbf{1}, \mathbf{3}, \ 0]$
B boson, bino	B	$\widetilde{B}$	$[\mathbf{1}, \mathbf{1}, \ 0]$

Table 5.1: Field content of the MSSM. The convention is as follows: The fields are denoted by their SM counterpart and the superpartners receive a tilde above their name. Notice that only the first family of quarks and leptons is listed explicitly. The charge assignments follow the ones used in [11].

Higgs sector of the MSSM is therefore essentially a two-Higgs-doublet-model (2HDM) which has been extensively studied in the literature [50]. We further expect its gauge-invariant description and FMS behavior to be similar to the one of the 2HDM [35], yet it is totally unclear what happens to their superpartners.

In Sec. 2.1 we established a formalism for describing theories whose fermionic fields are *left-handed*. We therefore stress that $\bar{u}$, $\bar{d}$ and $\bar{e}$ in Tab. 5.1 are left-handed fields (the bar signals that it is the anti-particle field)! In the SM they are treated as right-handed fields and one could also do so in the MSSM but it would easily lead to confusion and we already established that supersymmetric theories are formulated with only one type of handedness. The reason for why one may use left-handed fields instead is that one can consider them to be the charge-conjugate fields of their right-handed counterparts. E.g. a left-handed positron $\bar{e}$ would be used to describe a right-handed electron $e^\dagger$. Be aware that the term *charge-conjugation* might be a bit misleading here as it is often used to just flip conserved charges, i.e. turn a particle into its anti-particle. However, we define the charge-conjugation operation to also act on spinor space, i.e. we have the correspondence

$$\bar{e}_a \longleftrightarrow e^{\dagger\dot{a}}$$

with different index structure (for details see Appendix B.1). In other words: The dagger changes all the conserved charges as required for charge-conjugation and the different index position ensures the correct spinor structure as can be seen from Eq. (2.3). As mentioned in Sec. 2.1, this identification might be slightly different across the literature. For example the relation $\bar{e} = e^c = i\sigma^2 e^\dagger$ is also sometimes found, e.g. in [11].

Superpotential

A crucial part of the SM Lagrangian are the Yukawa couplings which eventually generate all the fermion masses once the Higgs acquires its vev. Due to the restrictive nature of the SUSY

Lagrangian, however, we may not just add them freely, but have to introduce them through the superpotential. The MSSM superpotential is chosen as [11]

$$W_{\mathrm{MSSM}} = \widetilde{\bar{u}} \boldsymbol{y}_u \widetilde{Q} \cdot H_u - \widetilde{\bar{d}} \boldsymbol{y}_d \widetilde{Q} \cdot H_d - \widetilde{\bar{e}} \boldsymbol{y}_e \widetilde{L} \cdot H_d + \mu H_u \cdot H_d \tag{5.1}$$

where $\boldsymbol{y}$ are again 3×3 matrices in family space and they are equal to the Yukawa matrices in Eq. (4.4). The dot product is an $SU(2)$ invariant, anti-symmetric product defined by

$$X \cdot Y \equiv X^T (i\sigma^2) Y \quad \xrightarrow[Y \to UY]{X \to UX} \quad X \cdot Y, \quad U \in SU(2). \tag{5.2}$$

From W_{MSSM} we can see the second reason for why we need an additional copy of the Higgs in the MSSM: In the SM, the down-type quarks couple to the Higgs field itself, while up-type quarks couple to its charge conjugate. As mentioned in Sec. 2.3, however, the superpotential must not include conjugated fields! To construct a hypercharge invariant superpotential, we therefore need two Higgs fields of opposite hypercharge.

One can then check [10] that this choice of superpotential recovers all the Yukawa terms as well as the Higgs potential of the SM (and also introduces many new terms). At this point, the SM has been fully supersymmetrized and μ is the only new parameter.

Soft breaking terms

If we have a closer look at Eq. (5.1) and (2.10), we realize that all of the scalar Lagrangian terms which result from W_{MSSM} are non-negative. In particular, this means that the Higgs potential is of the form $A(\phi^\dagger \phi) + B(\phi^\dagger \phi)^2$ with $A, B \geq 0$ or in other words, it has no non-trivial minimum. This means that electroweak symmetry breaking (EWSB) is off the table. Additionally, supersymmetry is not realized in nature (or otherwise we would have long found the superpartners which would be mass degenerate with the SM particles). Clearly, we have to address this fact. In the MSSM, broken supersymmetry at low energies is parametrized using soft breaking terms. They have coupling constants of positive mass dimension which means they are suppressed at high energies and SUSY gets restored. Therefore, they can at most be cubic in fields. A detailed discussion can, e.g. be found in [10, 11] and the soft breaking Lagrangian of the MSSM turns out to be

$$\begin{aligned}
\mathcal{L}_{\mathrm{MSSM}}^{\mathrm{soft}} = &-\frac{1}{2}\left[M_3 \widetilde{g}\widetilde{g} + M_2 \widetilde{W}\widetilde{W} + M_1 \widetilde{B}\widetilde{B} + h.c.\right] && \text{(gaugino masses)} \\
&- \left[\widetilde{\bar{u}} \boldsymbol{a}_u \widetilde{Q} \cdot H_u - \widetilde{\bar{d}} \boldsymbol{a}_d \widetilde{Q} \cdot H_d - \widetilde{\bar{e}} \boldsymbol{a}_e \widetilde{L} \cdot H_d + h.c.\right] && \text{(triple scalar couplings)} \\
&- \widetilde{Q}^\dagger \boldsymbol{m}_Q^2 \widetilde{Q} - \widetilde{L}^\dagger \boldsymbol{m}_L^2 \widetilde{L} - \widetilde{\bar{u}} \boldsymbol{m}_{\bar{u}}^2 \widetilde{\bar{u}}^\dagger - \widetilde{\bar{d}} \boldsymbol{m}_{\bar{d}}^2 \widetilde{\bar{d}}^\dagger - \widetilde{\bar{e}} \boldsymbol{m}_{\bar{e}}^2 \widetilde{\bar{e}}^\dagger && \text{(squark/slepton masses)} \\
&- m_u^2 H_u^\dagger H_u - m_d^2 H_d^\dagger H_d - \left[m_{ud}^2 H_u \cdot H_d + h.c.\right] && \text{(Higgs masses)}
\end{aligned} \tag{5.3}$$

Notice that the gauginos are not chiral, which means that we can write down Majorana mass terms for them. Scalar mass terms can always be written down gauge-invariantly and for the triple couplings we again used the dot product to build appropriate scalars. The most important mass terms are the ones for the Higgs doublets as those will eventually allow us to get non-zero vevs. In principle, there could be more soft breaking terms but (5.3) is the most general breaking Lagrangian that respects all of the gauge symmetries and conserves R-parity (see below). Nevertheless, we have just introduced a huge number of new parameters to make the model phenomenologically viable.

Eventually, a theory of spontaneous SUSY breaking could explain how these parameters arise. It is by now understood, that such a breaking cannot be described from within the MSSM and additional sectors are required, which mediate SUSY breaking to the MSSM. The most popular options are *gravity-mediated* and *gauge-mediated* spontaneous breaking mechanisms.

For a review, see [10]. The number of free parameters gets heavily reduced after proposing such extensions. For example minimal supergravity (mSUGRA) predicts the unification of all scalar mass terms, gaugino mass terms and trilinear couplings at some unification scale M_X [51]

$$\begin{aligned} \boldsymbol{m}_Q^2 = \boldsymbol{m}_L^2 = \boldsymbol{m}_{\bar{u}}^2 = \boldsymbol{m}_{\bar{d}}^2 = \boldsymbol{m}_{\bar{e}}^2 = m_u^2 = m_d^2 \equiv m_0^2 \\ M_1 = M_2 = M_3 \equiv m_{1/2} \\ \boldsymbol{a}_u = A_0 \boldsymbol{y}_u,\ \boldsymbol{a}_d = A_0 \boldsymbol{y}_d,\ \boldsymbol{a}_e = A_0 \boldsymbol{y}_e. \end{aligned} \tag{5.4}$$

The soft breaking terms in (5.3) would then be obtained by running them down to low energies. In particular, the runnings of the Higgs mass parameters favor EWSB which also suggests that SUSY breaking and EWSB could be linked [11].

R-parity

In principle, the superpotential (5.1) could include additional terms and still not be in conflict with SUSY, gauge-invariance or renormalizability. However, we wish not to include them as they would not only introduce a huge amount of new coupling constants but more importantly, they lead to lepton and baryon number violations which are experimentally highly constrained, e.g. by the non-observation of proton decay.

Simply leaving such terms out of the superpotential feels ad hoc, which is why the MSSM is assumed to possess a $\mathbb{Z}_2$ symmetry which forbids them: We assign to each particle an *R-parity* given by

$$P_R \equiv (-1)^{3(B-L)+2s}, \tag{5.5}$$

where B/L are its baryon/lepton number and s its spin. This definition conveniently assigns $P_R = +1$ to all SM particles and the Higgs bosons, and $P_R = -1$ to all their superpartners. Any interaction must conserve R-parity which immediately guarantees that there is no mixing between sparticles and the SM particles, sparticles can only be produced in even numbers at colliders, the lightest $P_R = -1$ particle, called the *lightest supersymmetric particle (LSP)*, is absolutely stable and all the other sparticles must eventually decay into an odd number of LSPs [10].

5.2 Custodial Symmetry

A subtle but crucial feature of the SM weak-Higgs sector is its custodial symmetry which causes the parameter

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} \tag{5.6}$$

to be exactly 1 on tree-level. This quantity, expressing a relation between the W-mass, Z-mass and the Weinberg angle θ_W, is tightly constrained by electroweak precision measurements and appears to deviate from 1 only slightly. Therefore, models which feature a custodial symmetry are phenomenologically favorable. A review of custodial symmetry and its effects on the experimentally accessible S, T, U parameters[2] can e.g. be found in [52]. For our purposes, the custodial symmetry plays yet another important role: It is a global symmetry which we can use to classify gauge-invariant bound state operators. Furthermore, as discussed at the end of Sec. 3.2, the form of the custodial symmetry is very important for the FMS mechanism and APT.

It is therefore important to investigate how it arises in the MSSM. To see it more clearly, we revisit the situation in the SM: The pure weak-Higgs sector of the SM is both symmetric under local $SU(2)_L$ and global $SU(2)_\mathcal{C}$ custodial transformations of the Higgs field ϕ. $SU(2)_\mathcal{C}$ relates ϕ to its charge conjugate field $i\sigma^2\phi^{\dagger T}$. Hypercharge transformations are a subgroup of $SU(2)_\mathcal{C}$

[2]Eventually, ρ can be calculated from T and the fine-structure-constant α.

which is why $SU(2)_\mathcal{C}$ is merely approximate in the full SM. Additionally, the Yukawa couplings explicitly distinguish between Higgs and charge conjugated Higgs which couple to down- and up-type quarks, respectively. Hence $SU(2)_\mathcal{C}$ is further broken once (non-degenerate) Yukawa couplings are switched on. We remind ourselves that the Lagrangian of the SM weak-Higgs sector could be brought into manifestly $SU(2)_L \times SU(2)_\mathcal{C}$ invariant form by placing ϕ and its charge conjugate into a bidoublet $\Phi = (i\sigma^2\phi^{\dagger T}, \phi)$.

The situation in the MSSM is more involved because we have two Higgs doublets. Consequently, one would naturally expect a larger symmetry to be present: The straight-forward generalization of the SM case would be to build two bidoublets for both H_u and H_d and the theory would be expected to exhibit an $SU(2)_L \times SU(2)_{\mathcal{C}_1} \times SU(2)_{\mathcal{C}_2}$ symmetry. As it turns out, the special structure of the MSSM potential (which is *enforced* by SUSY) breaks most of this enlarged symmetry, the troublesome term being $(H_d^\dagger H_u)(H_u^\dagger H_d)$. We will later see that this term shows up in the MSSM Higgs potential. Since our motivation for such a symmetry is driven by SM phenomenology, we do not need an enlarged symmetry, though. Just one $SU(2)_\mathcal{C}$ is totally enough to protect the ρ-parameter and to classify doublet and triplet states gauge-invariantly. Luckily, we are able to find such an $SU(2)_\mathcal{C}$ and in Appendix B.3 we show that it is in fact the only additional global symmetry of the MSSM Higgs potential (The argumentation follows the work of [53] on the 2HDM). We find that $SU(2)_\mathcal{C}$ is a Higgs flavor symmetry connecting $H_u \leftrightarrow -H_d$ and that we can again establish a manifestly symmetric form using the bidoublets[3]

$$H \equiv (H_u, -H_d) = \begin{pmatrix} H_u^{(1)} & -H_d^{(1)} \\ H_u^{(2)} & -H_d^{(2)} \end{pmatrix} \quad \text{and} \quad \widetilde{H} \equiv (\widetilde{H}_u, -\widetilde{H}_d) = \begin{pmatrix} \widetilde{H}_u^{(1)} & -\widetilde{H}_d^{(1)} \\ \widetilde{H}_u^{(2)} & -\widetilde{H}_d^{(2)} \end{pmatrix} \tag{5.7}$$

to describe the degrees of freedom of the Higgs(inos). The action of gauge and custodial transformations is then linear

$$H \to L(x) H R^\dagger, \qquad \widetilde{H} \to L(x) \widetilde{H} R^\dagger,$$

with $L(x) \in SU(2)_L$ and $R \in SU(2)_\mathcal{C}$. The fact that a Higgs flavor symmetry plays the role of the custodial symmetry comes up very naturally: In the SM, $SU(2)_\mathcal{C}$ connected ϕ with its charge conjugate. Due to SUSY, conjugate fields are not allowed in the superpotential, yet we explicitly introduced a second Higgs field to act as the charge conjugated SM Higgs in the Yukawa sector. Therefore, it is no surprise that the MSSM $SU(2)_\mathcal{C}$ rotates between the two Higgs fields now. It is noteworthy, that in the MSSM the rotation also includes a sign flip: This is absolutely necessary and can be seen from the Higgs potential (which we will calculate in Sec. 5.3.1) as well as the superpotential (5.1).

The expression for the superpotential makes it clear that $SU(2)_\mathcal{C}$ is only approximate due to the Yukawa couplings (just like in the SM). Furthermore, hypercharge transformations

$$H_u \to H_u' = e^{i\alpha/2} H_u \qquad H_d \to H_d' = e^{-i\alpha/2} H_d \tag{5.8}$$

translate to the bidoublet as

$$H \to H' = H \exp\left(i\alpha \frac{\sigma^3}{2}\right), \tag{5.9}$$

just like in the SM. $U(1)_Y$ is therefore again a subgroup of $SU(2)_\mathcal{C}$ and the global symmetry breaks upon gauging hypercharge. From all that we may already suspect that the gauge-invariant description of the MSSM will follow similar lines as in the SM.

Finally, there are two more sources of custodial symmetry violation in the MSSM: For the MSSM weak-Higgs sector to be symmetric under $SU(2)_\mathcal{C}$

[3] See Appendix B.2 to learn how this bidoublet relates to the SM bidoublet.

Name	Boson	Fermion	$[SU(2)_L, U(1)_Y]$
Higgs(inos)	$H_u = (H_u^{(1)}, H_u^{(2)})$	$\widetilde{H}_u = (\widetilde{H}_u^{(1)}, \widetilde{H}_u^{(2)})$	$[\mathbf{2},\ 1]$
	$H_d = (H_d^{(1)}, H_d^{(2)})$	$\widetilde{H}_d = (\widetilde{H}_d^{(1)}, \widetilde{H}_d^{(2)})$	$[\mathbf{2},\ \text{-}1]$
W(ino)	$W_\mu = W_\mu^a T^a$	$\widetilde{W} = \widetilde{W}^a T^a$	$[\mathbf{3},\ 0]$

Table 5.2: Field content of the weak-Higgs(ino) sector of the MSSM.

- the soft breaking parameters cannot be arbitrary: We must require $m_u^2 = m_d^2 \equiv m^2$. This can actually be motivated by the discussion prior to Eq. (5.4), i.e. $SU(2)_\mathcal{C}$ could be thought of as being realized only above some unification scale.
- both Higgs must acquire the same vev. However, $\tan\beta \equiv v_u/v_d$ most likely is larger than 1 [11]. Luckily, this has little consequences for the ρ-parameter as even the extreme cases of $\tan\beta \to 0, \infty$ barely affect its value [54].

5.3 Gauge-invariant Spectrum of a simplified Model

Now that we know how custodial symmetry works in the MSSM we can start working on a bound state operator description just like for the SM in Chapter 4. Unlike in the SM, where the custodial symmetry was exact as long as hypercharge was not gauged or Yukawa couplings introduced, the situation in the MSSM is slightly different. As argued in Sec. 5.2, custodial symmetry is already broken in the weak-Higgs sector of the MSSM by the soft breaking masses. Subsequently, the (phenomenologically viable) situation where we have two different vevs for the two Higgs fields also explicitly violates this symmetry. For simplicity, we will make sure to keep the symmetry intact for now. The case of broken symmetry is addressed in Sec. 5.4.1.

Like in the case of the SM, we start out with the description of just the weak-Higgs(ino) sector in Sec. 5.3.1. Our main interest will be to investigate the special role of the second Higgs field in the MSSM. Afterwards we will include a single generation of leptons in Sec. 5.3.2.

5.3.1 Weak-Higgs(ino) Sector

First of all, we have to find the Lagrangian which describes our reduced model. The relevant parts of the MSSM are the $SU(2)_L$ gauge bosons W_μ^a, their superpartners, the winos $\widetilde{W}^a$, as well as two chiral supermultiplets containing the Higgs fields $H_{u,d}$ and their respective Higgsinos $\widetilde{H}_{u,d}$. For the $SU(2)_L$ generators, the convention $T^a = \frac{\sigma^a}{2}$ is used. The field content is summarized in Tab. 5.2. For the superpotential we choose the currently relevant part of the MSSM superpotential (5.1), i.e.

$$W = \mu H_u \cdot H_d. \tag{5.10}$$

The relevant soft breaking terms contained in (5.3) are

$$\mathcal{L}_{\text{soft}} = -m_u^2 H_u^\dagger H_u - m_d^2 H_d^\dagger H_d - \left[m_{ud}^2 H_u \cdot H_d + h.c.\right] - \frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right].$$

As discussed in Sec. 5.2, we restrict the diagonal mass parameters to $m_u^2 = m_d^2 \equiv m^2$ for the weak-Higgs sector to become custodial symmetric. The parameter m_{ud}^2 can always be chosen to be real by redefining either of the Higgs fields to absorb its phase [55]. Usually, it is also assumed that $\mu, M_2 \in \mathbb{R}$ to avoid additional CP violations [10, 11]. We will follow this reasoning here. Finally, one can easily check that for the resulting scalar potential to have a non-zero stationary point, m_{ud}^2 cannot be arbitrary but has to equal $|\mu|^2 + m^2$ in order to avoid contradictions[4]. The soft breaking Lagrangian we are going to work with hence simplifies to

$$\mathcal{L}_{\text{soft}} = -m^2(H_d^\dagger H_d + H_u^\dagger H_u) - (\mu^2 + m^2)\left[H_u \cdot H_d + h.c.\right] - \frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right]. \tag{5.11}$$

Using (5.10) and (5.11) in the general SUSY Lagrangian (2.10) results in ($i = u, d$)

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}W_{\mu\nu}^a W_a^{\mu\nu} + i\widetilde{W}^{\dagger a}\bar{\sigma}^\mu(D_\mu\widetilde{W})_a + (D_\mu H_i)^\dagger(D^\mu H_i) + i\widetilde{H}_i^\dagger\bar{\sigma}^\mu D_\mu\widetilde{H}_i\\ &-\frac{g}{\sqrt{2}}\left[(H_i^\dagger\sigma^a\widetilde{H}_i)\widetilde{W}_a + \widetilde{W}^{\dagger a}(\widetilde{H}_i^\dagger\sigma^a H_i)\right] - \mu\left[\widetilde{H}_u\cdot\widetilde{H}_d + \text{h.c.}\right]\\ &-\frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right] - V(H_d, H_u)\end{aligned} \tag{5.12}$$

with the scalar potential

$$\begin{aligned}V(H_d, H_u) = &(|\mu|^2 + m^2)\left[H_d^\dagger H_d + H_u^\dagger H_u + (H_u\cdot H_d + h.c.)\right]\\ &+\frac{g^2}{8}\left[(H_d^\dagger H_d)^2 + (H_u^\dagger H_u)^2\right] - \frac{g^2}{4}(H_d^\dagger H_d)(H_u^\dagger H_u) + \frac{g^2}{2}(H_d^\dagger H_u)(H_u^\dagger H_d).\end{aligned} \tag{5.13}$$

This MSSM subsector with the appropriately chosen soft breaking parameters has an $SU(2)_L \times SU(2)_{\mathcal{C}} \times \mathbb{Z}_{2,R}$ symmetry and it is made manifest by introducing the Higgs(ino) bidoublets from Eq. (5.7). This allows us to rewrite the model Lagrangian as

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}W_{\mu\nu}^a W_a^{\mu\nu} + i\widetilde{W}^{\dagger a}\bar{\sigma}^\mu(D_\mu\widetilde{W})_a + \text{tr}\left[(D_\mu H)^\dagger(D^\mu H)\right] + \text{tr}\left[i\widetilde{H}^\dagger\bar{\sigma}^\mu D_\mu\widetilde{H}\right]\\ &-\frac{g}{\sqrt{2}}\left[\text{tr}\left[H^\dagger\sigma^a\widetilde{H}\right]\widetilde{W}_a + \widetilde{W}^{\dagger a}\text{tr}\left[\widetilde{H}^\dagger\sigma^a H\right]\right] + \mu\left[\det\widetilde{H} + \text{h.c.}\right]\\ &-\frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right] - V(H)\end{aligned} \tag{5.14}$$

$$V(H) = (|\mu|^2 + m^2)\left[\text{tr}\left[H^\dagger H\right] - 2\ \text{Re}\det H^\dagger\right] + \frac{g^2}{8}\text{tr}\left[H^\dagger H\right]^2 - \frac{g^2}{2}\det H^\dagger H. \tag{5.15}$$

It is clearly symmetric upon local $SU(2)_L$, global $SU(2)_{\mathcal{C}}$ as well as R-parity transformations of the form

$$\begin{aligned}H &\xrightarrow{SU(2)_{L,\mathcal{C}}} L(x)HR^\dagger & W_\mu^a &\xrightarrow{P_R} W_\mu^a\\ \widetilde{H} &\xrightarrow{SU(2)_{L,\mathcal{C}}} L(x)\widetilde{H}R^\dagger & H &\xrightarrow{P_R} H\\ W_\mu &\xrightarrow{SU(2)_L} L(x)W_\mu L^\dagger(x) - ig^{-1}(\partial_\mu L(x))L^\dagger(x) & \widetilde{W}^a &\xrightarrow{P_R} -\widetilde{W}^a\\ \widetilde{W} &\xrightarrow{SU(2)_L} L(x)\widetilde{W}L^\dagger(x) & \widetilde{H} &\xrightarrow{P_R} -\widetilde{H}.\end{aligned}$$

Just like in the SM, our next step is to calculate what mass eigenstates are predicted by conventional perturbation theory.

Tree-level spectrum (bosonic masses)

After making sure that $v_u = v_d \equiv v \neq 0$, we let the Higgs doublets acquire a vev in the following way[5]

$$\begin{aligned}H_u &\to \left(\begin{smallmatrix}0\\v\end{smallmatrix}\right) + \eta_u\\ H_d &\to \left(\begin{smallmatrix}v\\0\end{smallmatrix}\right) + \eta_d\end{aligned}$$

[4] Looking for stationary points of the potential where m_{ud}^2 is an independent parameter either leads to $v_d = v_u = 0$ (which makes a perturbative treatment of the Higgs effect impossible) or $v_d \neq v_u$. The latter option, however, violates $SU(2)_{\mathcal{C}}$ which we want to avoid for now.

[5] Choosing the vev with a factor of $\frac{1}{\sqrt{2}}$ is also widely spread in the literature but we refrain to do so here to avoid clutter.

with η_i the 8 real degrees of freedom of the fluctuation fields. Notice that the vevs are in the first and second component, respectively, just like in the full MSSM, yet they have the same magnitude. This choice translates to the bidoublet as

$$H \to \begin{pmatrix} 0 & -v \\ v & 0 \end{pmatrix} + \begin{pmatrix} \eta_u^{(1)} & -\eta_d^{(1)} \\ \eta_u^{(2)} & -\eta_d^{(2)} \end{pmatrix} \equiv V + \eta, \qquad V = -v(i\sigma^2). \tag{5.16}$$

Plugging this into the potential (5.15), one immediately finds that it points along a flat direction, i.e. the value of v cannot be expressed in terms of the other tree-level parameters of the theory. One therefore has to treat it as a free parameter for now.

It is clear that V is again *not* invariant under the full $SU(2)_L \times SU(2)_\mathcal{C}$ group but for every custodial transformation R we can find a corresponding gauge transformation L_R such that the vev respects the combined symmetry

$$V \to L_\mathrm{R} V R^\dagger = V.$$

The specific form of L_R can easily be worked out and gives

$$L_\mathrm{R} = V R V^{-1} = (-i\sigma^2) R (i\sigma^2) = R^*, \tag{5.17}$$

since R is an $SU(2)$ matrix. After the Higgs fields acquire this vev, all fields will fall into multiplets of this remaining symmetry which we shall call $SU(2)_m$. To make this process as transparent as possible it is useful to introduce the basis

$$b_i = \left\{ i\sigma^2, -\sigma^3, i\mathbb{1}, \sigma^1 \right\} = \sigma^i (i\sigma^2) \qquad (i = 0, 1, 2, 3) \tag{5.18}$$

which is orthonormal with respect to the scalar product

$$\langle x, y \rangle \equiv \frac{1}{2} \mathrm{tr}\left[x^\dagger y \right]. \tag{5.19}$$

Any bidoublet Y can then be expressed in terms of this basis and the field bilinears y^i via

$$Y = \begin{pmatrix} Y_2^{(1)} & -Y_1^{(1)} \\ Y_2^{(2)} & -Y_1^{(2)} \end{pmatrix} = y^i b_i \quad \text{with} \quad y = -\frac{1}{2} \begin{pmatrix} Y_1^{(1)} + Y_2^{(2)} \\ Y_1^{(2)} + Y_2^{(1)} \\ i\left(-Y_1^{(2)} + Y_2^{(1)}\right) \\ Y_1^{(1)} - Y_2^{(2)} \end{pmatrix}. \tag{5.20}$$

The transformation behavior for the components of y under $SU(2)_m$ transformations can be easily inferred using (5.17) and (5.18):

$$\begin{aligned} y_i' = \left\langle b_i, L_\mathrm{R} Y R^\dagger \right\rangle &= \frac{1}{2} \mathrm{tr}\left[b_i^\dagger L_\mathrm{R} Y R^\dagger \right] \\ &= \frac{1}{2} \mathrm{tr}\left[b_i^\dagger L_\mathrm{R} b_j (i\sigma^2) L_\mathrm{R}^\dagger (-i\sigma^2) \right] y^j = \frac{1}{2} \mathrm{tr}\left[\sigma_i L_\mathrm{R} \sigma_j L_\mathrm{R}^\dagger \right] y_j \equiv \tilde{T}(L_\mathrm{R})_{ij} y^j. \end{aligned}$$

The matrix $\tilde{T}(L_\mathrm{R})$ decomposes as

$$\tilde{T}(L_\mathrm{R}) = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & T(L_\mathrm{R}) \end{pmatrix}$$

with $T(L_\mathrm{R})$ the adjoint $SU(2)_L$ rotation matrix induced by L_R. The adjoint transformations are now of course restricted to $L_\mathrm{R} = R^*$ but the important thing to realize is that y^0 is a singlet under said transformation and the y^k $(k = 1, 2, 3)$ form a triplet, *and* this transformation is *identical* to how the gauge fields and gauginos transform under the remaining symmetry group. Therefore, we will also use the same indices from now on. If we make use of this decomposition for the Higgs fluctuations $\eta = h^i b_i$ and Higgsinos $\widetilde{H} = \widetilde{h}^i b_i$, we can summarize the transformation behavior of all fields as follows:

$$\begin{array}{ccc} h^0 & \xrightarrow{SU(2)_m} & h^0 \\ \widetilde{h}^0 & & \widetilde{h}^0 \end{array} \qquad \begin{array}{ccc} W^a_\mu & & T(L_\mathrm{R})^{ab} W^b_\mu \\ \widetilde{W}^a & \xrightarrow{SU(2)_m} & T(L_\mathrm{R})^{ab} \widetilde{W}^b \\ h^a & & T(L_\mathrm{R})^{ab} h^b \\ \widetilde{h}^a & & T(L_\mathrm{R})^{ab} \widetilde{h}^b. \end{array}$$

We now insert the expansion (5.16) into the Lagrangian and fix to 't Hooft gauge

$$C^a = \partial_\mu W^\mu_a + g\xi \,\mathrm{Im}\,\mathrm{tr}\left[V^\dagger \sigma^a H\right].$$

Mass terms for the gauge bosons emerge from the kinetic Higgs Lagrangian

$$\mathrm{tr}\left[(D_\mu H)^\dagger (D^\mu H)\right] \supset \mathrm{tr}\left[\partial_\mu \eta^\dagger \partial^\mu \eta\right] + g(\partial_\mu W^\mu_a)\,\mathrm{Im}\,\mathrm{tr}\left[V^\dagger \sigma^a \eta\right] + \frac{g^2}{4}\mathrm{tr}\left[V^\dagger \sigma^a \sigma^b V\right] W^a_\mu W^\mu_b.$$

The second term is canceled by the mixing terms of the gauge-fixing Lagrangian

$$\mathcal{L}_\mathrm{gf} = -\frac{1}{2\xi} C^a C_a,$$

and the third term[6] yields the gauge boson masses $m_W^2 \equiv g^2 v^2$:

$$\begin{aligned} \frac{g^2}{4}\mathrm{tr}\left[V^\dagger \sigma^a \sigma^b V\right] W^a_\mu W^\mu_b &= \frac{1}{2}\left\{\frac{g^2}{4}\mathrm{tr}\left[V^\dagger \sigma^a \sigma^b V\right] W^a_\mu W^\mu_b + \frac{g^2}{4}\mathrm{tr}\left[V^\dagger \sigma^b \sigma^a V\right] W^a_\mu W^\mu_b\right\} \\ &= \frac{1}{2}\frac{g^2}{4}\mathrm{tr}\left[V^\dagger(\sigma^a\sigma^b + \sigma^b\sigma^a)V\right] W^a_\mu W^\mu_b = \frac{1}{2} m_W^2 W^a_\mu W^\mu_a \end{aligned}$$

Do not get confused when comparing this to the SM W-mass: Since we have two vevs now and refused to carry along the factor of $\sqrt{2}$, a rescaling would be needed to match the previous result. Mass terms for the scalar fields are contained in the gauge-fixing Lagrangian

$$\mathcal{L}_\mathrm{gf} \supset -\frac{g^2\xi}{2}\left(\mathrm{Im}\,\mathrm{tr}\left[V^\dagger \sigma^a H\right]\right)^2 = -2\xi g^2 v^2 (\mathrm{Im}\,h^a)^2,$$

and the scalar potential after the vev expansion. Using

$$\begin{aligned} \mathrm{tr}\left[H^\dagger H\right] &\to 2\left[v^2 + h^0 h^{0\dagger} + h^a h^{a\dagger} - 2v\,\mathrm{Re}\,h^0\right] \\ \det H^\dagger H &\to (v^2 - 2vh^0 + h^0h^0 - h^ah^a)(v^2 - 2vh^{0\dagger} + h^{0\dagger}h^{0\dagger} - h^{a\dagger}h^{a\dagger}) \\ \mathrm{Re}\det H^\dagger &\to v^2 - 2v\,\mathrm{Re}\,h^0 + (\mathrm{Re}\,h^0)^2 - (\mathrm{Im}\,h^0)^2 - (\mathrm{Re}\,h^a)^2 + (\mathrm{Im}\,h^a)^2 \end{aligned}$$

we write the relevant parts of the potential as

$$\begin{aligned} V(H) \supset\ & 4(|\mu|^2 + m^2)\left[(\mathrm{Im}\,h^0)^2 + (\mathrm{Re}\,h^a)^2\right] \\ & + g^2v^2\left[3(\mathrm{Re}\,h^0)^2 + (\mathrm{Im}\,h^0)^2 + (\mathrm{Re}\,h^a)^2 + (\mathrm{Im}\,h^a)^2\right] \\ & - g^2v^2\left[3(\mathrm{Re}\,h^0)^2 + (\mathrm{Im}\,h^0)^2 - (\mathrm{Re}\,h^a)^2 + (\mathrm{Im}\,h^a)^2\right] \\ =\ & 4(|\mu|^2 + m^2)\left[(\mathrm{Im}\,h^0)^2 + (\mathrm{Re}\,h^a)^2\right] + 2g^2v^2(\mathrm{Re}\,h^a)^2. \end{aligned}$$

[6]Note that since $W^a_\mu W^\mu_b$ is symmetric in a and b we remove the anti-symmetric parts of the matrix by symmetrization.

Field	Decomposition	$SU(2)_m$	T^3	squared mass
W^0	W^3	**3**	0	m_W^2
W^+	$\frac{1}{\sqrt{2}}\left(W^2+iW^1\right)$	**3**	+1	m_W^2
W^-	$\frac{1}{\sqrt{2}}\left(W^2-iW^1\right)$	**3**	−1	m_W^2
h	$-\operatorname{Re} H_d^{(1)} - \operatorname{Re} H_u^{(2)}$	**1**	0	0
A	$-\operatorname{Im} H_d^{(1)} - \operatorname{Im} H_u^{(2)}$	**1**	0	$2(\lvert\mu\rvert^2+m^2)$
H^0	$-\operatorname{Re} H_d^{(1)} + \operatorname{Re} H_u^{(2)}$	**3**	0	$m_A^2+m_W^2$
H^-	$\frac{i}{\sqrt{2}}\left(H_d^{(2)}+H_u^{(1)\dagger}\right)$	**3**	−1	$m_A^2+m_W^2$
H^+	$-\frac{i}{\sqrt{2}}\left(H_u^{(1)}+H_d^{(2)\dagger}\right)$	**3**	+1	$m_A^2+m_W^2$
G^0	$-\operatorname{Im} H_d^{(1)} + \operatorname{Im} H_u^{(2)}$	**3**	0	ξm_W^2
G^-	$\frac{1}{\sqrt{2}}\left(H_d^{(2)}-H_u^{(1)\dagger}\right)$	**3**	−1	ξm_W^2
G^+	$-\frac{1}{\sqrt{2}}\left(H_u^{(1)}+H_d^{(2)\dagger}\right)$	**3**	+1	ξm_W^2

Table 5.3: Bosonic mass spectrum: Literature names for the mass eigenstates as well as their decomposition into initial degrees of freedom. Notice that $m_{H^\pm}^2 = m_A^2 + m_W^2$ and $m_h^2 + m_{H^0}^2 = m_A^2 + m_Z^2$ which are MSSM tree-level relations enforced by SUSY [56].

Expressing the kinetic Lagrangian in terms of these new fields is also easy because of the orthonormality of the basis:

$$\operatorname{tr}\left[\partial^\mu\eta^\dagger\partial_\mu\eta\right] = 2\ \partial^\mu h^{i\dagger}\partial_\mu h^i = \frac{1}{2}\partial^\mu(2\operatorname{Re}h^i)\partial_\mu(2\operatorname{Re}h^i) + \frac{1}{2}\partial^\mu(2\operatorname{Im}h^i)\partial_\mu(2\operatorname{Im}h^i)$$

Here we see the appropriate normalization of the (now real) scalar fields $2\operatorname{Re}h^i$ and $2\operatorname{Im}h^i$. In total the relevant parts are therefore

$$\begin{aligned}\mathcal{L} \supset\ & \frac{1}{2}\partial^\mu(2\operatorname{Re}h^0)\partial_\mu(2\operatorname{Re}h^0)\\ &+\frac{1}{2}\partial^\mu(2\operatorname{Im}h^0)\partial_\mu(2\operatorname{Im}h^0) - \frac{2(|\mu|^2+m^2)}{2}(2\operatorname{Im}h^0)^2\\ &+\frac{1}{2}\partial^\mu(2\operatorname{Re}h^a)\partial_\mu(2\operatorname{Re}h^a) - \frac{1}{2}\left[2(|\mu|^2+m^2)+m_W^2\right](2\operatorname{Re}h^a)^2\\ &+\frac{1}{2}\partial^\mu(2\operatorname{Im}h^a)\partial_\mu(2\operatorname{Im}h^a) - \xi\frac{m_W^2}{2}(2\operatorname{Im}h^a)^2,\end{aligned}$$

from which we can immediately read off that our scalar spectrum contains a massless scalar field $h \equiv 2\operatorname{Re}h^0$, a pseudoscalar $A \equiv 2\operatorname{Im}h^0$ of mass $m_A^2 \equiv 2(|\mu|^2+m^2)$ and a mass-degenerate scalar triplet $H^a \equiv 2\operatorname{Re}h^a$ of mass $m_H^2 = m_A^2+m_W^2$. The remaining fields are the would-be Goldstone bosons $G^a \equiv 2\operatorname{Im}h^a$ which are also an $SU(2)_m$ triplet and have the gauge-parameter dependent mass $m_G^2 = \xi m_W^2$. The fact that their mass is gauge-dependent again reveals that they are nonphysical degrees of freedom. We can further build linear combinations of the members of each triplet to get eigenstates of the $(T^3)^{ab} = i\epsilon^{3ab}$ operator. We find that $H^\pm \equiv (H^2 \pm iH^1)/\sqrt{2}$ and $H^0 \equiv H^3$ are eigenstates of definite $T^3 = \pm1, 0$ and analogously for G^a. Finally, the bosonic spectrum is summarized in Tab. 5.3.

Tree-level spectrum (fermionic masses)

Next, we calculate the fermionic mass eigenstates. Again, we look only at the relevant parts which are in this case the explicit Higgsino and wino mass terms and the Higgs-Higgsino-wino vertices of (5.14)

$$\mathcal{L} \supset -\frac{g}{\sqrt{2}}\left[\text{tr}\left[H^\dagger\sigma^a\widetilde{H}\right]\widetilde{W}_a + \widetilde{W}^{\dagger a}\text{tr}\left[\widetilde{H}^\dagger\sigma^a H\right]\right] + \mu\left[\det\left(\widetilde{H}\right) + \text{h.c.}\right] - \frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right].$$

After expanding $\widetilde{H} = \widetilde{h}^i b_i$ and $H = V + h^i b_i$, using $\det\widetilde{H} = \widetilde{h}^0\widetilde{h}^0 - \widetilde{h}^a\widetilde{h}^a$ and remembering that $b_j \neq b_j^\dagger$, this turns into

$$\begin{aligned}\mathcal{L} &\supset -\frac{gv}{\sqrt{2}}\left[\text{tr}\left[(i\sigma^2)\sigma^a b_j\right]\widetilde{h}^j\widetilde{W}_a + h.c.\right] + \mu\left[\widetilde{h}^0\widetilde{h}^0 - \widetilde{h}^a\widetilde{h}^a + h.c.\right] - \frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right] \\ &= \sqrt{2}gv\widetilde{h}^a\widetilde{W}_a + \mu\left[\widetilde{h}^0\widetilde{h}^0 - \widetilde{h}^a\widetilde{h}^a\right] - \frac{M_2}{2}\widetilde{W}^a\widetilde{W}^a + h.c.\end{aligned}$$

if we again only keep terms quadratic in the fields. Since the mass terms already look suspicious (c.f. Eq. (2.4)), we proceed by introducing the Majorana spinors

$$\Psi_{\widetilde{\chi}_3^0} \equiv \sqrt{2}\begin{pmatrix} i\widetilde{h}^0 \\ -i\widetilde{h}^{0\dagger}\end{pmatrix} \qquad \Psi_{\widetilde{h}^a} \equiv \sqrt{2}\begin{pmatrix}\widetilde{h}^a \\ \widetilde{h}^{a\dagger}\end{pmatrix} \qquad \Psi_{\widetilde{W}^a} \equiv \sqrt{2}\begin{pmatrix}\widetilde{W}^a \\ \widetilde{W}^{a\dagger}\end{pmatrix}.$$

The relevant Higgsino-wino Lagrangian can then be expressed as

$$\begin{aligned}\mathcal{L} \supset\ & \frac{i}{2}\bar{\Psi}_{\widetilde{\chi}_3^0}\gamma^\mu\partial_\mu\Psi_{\widetilde{\chi}_3^0} + \frac{i}{2}\bar{\Psi}_{\widetilde{h}^a}\gamma^\mu\partial_\mu\Psi_{\widetilde{h}^a} + \frac{i}{2}\bar{\Psi}_{\widetilde{W}^a}\gamma^\mu\partial_\mu\Psi_{\widetilde{W}^a} \\ & - \frac{\mu}{2}\bar{\Psi}_{\widetilde{\chi}_3^0}\Psi_{\widetilde{\chi}_3^0} - \frac{\mu}{2}\bar{\Psi}_{\widetilde{h}^a}\Psi_{\widetilde{h}^a} - \frac{M_2}{4}\bar{\Psi}_{\widetilde{W}^a}\Psi_{\widetilde{W}^a} + \frac{gv}{2\sqrt{2}}\left(\bar{\Psi}_{\widetilde{h}^a}\Psi_{\widetilde{W}^a} + \bar{\Psi}_{\widetilde{W}^a}\Psi_{\widetilde{h}^a}\right).\end{aligned}$$

We can already read off that $\Psi_{\widetilde{\chi}_3^0}$ is a singlet Majorana mass eigenstate[7] with $m_{\widetilde{\chi}_3^0} = \mu$ but the triplet states still mix. Luckily, this mixing is transparent to $SU(2)_m$, i.e. independent of a, and we can write it in the $(\Psi_{\widetilde{h}^a}, \Psi_{\widetilde{W}^a})$ basis as

$$-\frac{1}{2}\begin{pmatrix}\bar{\Psi}_{\widetilde{h}^a} & \bar{\Psi}_{\widetilde{W}^a}\end{pmatrix}\begin{pmatrix}\mu & -\frac{gv}{\sqrt{2}} \\ -\frac{gv}{\sqrt{2}} & \frac{M_2}{2}\end{pmatrix}\begin{pmatrix}\Psi_{\widetilde{h}^a} \\ \Psi_{\widetilde{W}^a}\end{pmatrix}. \tag{5.21}$$

The diagonalization is straight-forward[8]

$$\begin{aligned}(5.21) &= -\frac{1}{2}\begin{pmatrix}\bar{\Psi}_{\widetilde{h}^a} & \bar{\Psi}_{\widetilde{W}^a}\end{pmatrix} SS^T\begin{pmatrix}\mu & -\frac{gv}{\sqrt{2}} \\ -\frac{gv}{\sqrt{2}} & \frac{M_2}{2}\end{pmatrix} SS^T\begin{pmatrix}\Psi_{\widetilde{h}^a} \\ \Psi_{\widetilde{W}^a}\end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix}\bar{X}_1^a & \bar{X}_2^a\end{pmatrix}\begin{pmatrix}m_1 & 0 \\ 0 & m_2\end{pmatrix}\begin{pmatrix}X_1^a \\ X_2^a\end{pmatrix}\end{aligned}$$

and we obtain the mass eigenvalues (remember $m_W^2 = g^2v^2$)

$$m_{1,2} = \frac{1}{4}\left(M_2 + 2\mu \pm \sqrt{(M_2 - 2\mu)^2 + 8m_W^2}\right). \tag{5.22}$$

[7] This will turn out to be the LSP.

[8] Notice that S is not a similarity transformation as Majorana mass matrices cannot be unitarily diagonalized in general [57].

Field	Decomposition	$SU(2)_m$	T^3	Mass
$\widetilde{\chi}^0_3$	$\frac{-i}{\sqrt{2}}\left(\widetilde{H}^{(1)}_d + \widetilde{H}^{(2)}_u\right)$	$\mathbf{1}$	0	μ
$\widetilde{\chi}^0_1$	$\frac{1}{\mathcal{N}_1}\left[\kappa_1\left(-\widetilde{H}^{(1)}_d + \widetilde{H}^{(2)}_u\right) + \sqrt{2}\,\widetilde{W}^0\right]$	$\mathbf{3}$	0	m_1
$\widetilde{\chi}^-_1$	$\frac{\sqrt{2}i}{\mathcal{N}_1}\left[\kappa_1\widetilde{H}^{(2)}_d - i\widetilde{W}^-\right]$	$\mathbf{3}$	-1	m_1
$\widetilde{\chi}^+_1$	$\frac{-\sqrt{2}i}{\mathcal{N}_1}\left[\kappa_1\widetilde{H}^{(1)}_u + i\widetilde{W}^+\right]$	$\mathbf{3}$	+1	m_1
$\widetilde{\chi}^0_2$	$\frac{1}{\mathcal{N}_2}\left[\kappa_2\left(-\widetilde{H}^{(1)}_d + \widetilde{H}^{(2)}_u\right) + \sqrt{2}\,\widetilde{W}^0\right]$	$\mathbf{3}$	0	m_2
$\widetilde{\chi}^-_2$	$\frac{\sqrt{2}i}{\mathcal{N}_2}\left[\kappa_2\widetilde{H}^{(2)}_d - i\widetilde{W}^-\right]$	$\mathbf{3}$	-1	m_2
$\widetilde{\chi}^+_2$	$\frac{-\sqrt{2}i}{\mathcal{N}_2}\left[\kappa_2\widetilde{H}^{(1)}_u + i\widetilde{W}^+\right]$	$\mathbf{3}$	+1	m_2

Table 5.4: Fermionic mass spectrum: Literature names as well as the decomposition into initial Higgsino and wino degrees of freedom. The Higgsino and wino components mix to form three *neutralinos* and two *charginos*. The masses $m_{1,2}$ are given in (5.22) and we have introduced the common notations $\widetilde{W}^\pm \equiv (\widetilde{W}^2 \pm i\widetilde{W}^1)/\sqrt{2}$ and $\widetilde{W}^0 \equiv \widetilde{W}^3$.

By defining

$$\kappa_{1,2} = \frac{1}{2}\frac{M_2}{m_W} - \frac{m_{1,2}}{m_W} \qquad \mathcal{N}_{1,2} = \sqrt{1 + 2\kappa^2_{1,2}} \tag{5.23}$$

we can then also express the mass eigenstates

$$S = \begin{pmatrix} \sqrt{2}\frac{\kappa_1}{\mathcal{N}_1} & \sqrt{2}\frac{\kappa_2}{\mathcal{N}_2} \\ \frac{1}{\mathcal{N}_1} & \frac{1}{\mathcal{N}_2} \end{pmatrix}, \qquad \begin{pmatrix} X^a_1 \\ X^a_2 \end{pmatrix} = S^T \begin{pmatrix} \Psi_{\widetilde{h}^a} \\ \Psi_{\widetilde{W}^a} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathcal{N}_1}(\sqrt{2}\kappa_1\Psi_{\widetilde{h}^a} + \Psi_{\widetilde{W}^a}) \\ \frac{1}{\mathcal{N}_2}(\sqrt{2}\kappa_2\Psi_{\widetilde{h}^a} + \Psi_{\widetilde{W}^a}) \end{pmatrix}$$

which are still $SU(2)_m$ triplets. Just like for the scalar fields, we build linear combinations to also get T^3 eigenstates:

$$\Psi_{\widetilde{\chi}^\pm_i} \equiv \frac{1}{\sqrt{2}}(X^2_i \pm iX^1_i), \quad \Psi_{\widetilde{\chi}^0_i} \equiv X^3_i \qquad (i = 1, 2) \tag{5.24}$$

Those mass eigenstates have famous names in the literature: $\Psi_{\widetilde{\chi}^\pm_{1,2}}$ are the *charginos* (Dirac fermions) and $\Psi_{\widetilde{\chi}^0_{1,2,3}}$ are called *neutralinos* (Majorana fermions). In Weyl components they can be written as

$$\Psi_{\widetilde{\chi}^\pm_{1,2}} = \begin{pmatrix} \widetilde{\chi}^\pm_{1,2} \\ \widetilde{\chi}^{\mp\dagger}_{1,2} \end{pmatrix} \qquad\qquad \Psi_{\widetilde{\chi}^0_{1,2,3}} = \begin{pmatrix} \widetilde{\chi}^0_{1,2,3} \\ \widetilde{\chi}^{0\dagger}_{1,2,3} \end{pmatrix}$$

and we will for simplicity denote them by their upper Weyl component only. It is nevertheless good to remember their 4-spinor form, though, because this way it is clear that $\widetilde{\chi}^+_i$ and $\widetilde{\chi}^-_i$ are anti-particles, for example.

We conclude that the mass spectrum contains three Majorana fermions. One of them ($\widetilde{\chi}^0_3$) has mass μ and is the lightest of the undiscovered particles (LSP). The other two are part of the $SU(2)_m$ triplets of mass $m_{1,2}$ which also host one Dirac fermion $\widetilde{\chi}^\pm_{1,2}$ each. The fermionic spectrum is summarized in Tab. 5.4.

Operator	Spin	$SU(2)_\mathcal{C}$	P_R	$\overset{\text{FMS}}{\sim}$
$\operatorname{tr}\left[H^\dagger H\right]$	0	$\mathbf{1}$	+1	vh
$\operatorname{Im}\det H$	0	$\mathbf{1}$	+1	vA
$\operatorname{tr}\left[H^\dagger H\sigma^A\right]$	0	$\mathbf{3}$	+1	$vc^{Aa}H^a$
$\operatorname{tr}\left[H^\dagger \widetilde{H}\right]$	$\frac{1}{2}$	$\mathbf{1}$	-1	$v\widetilde{\chi}_3^0$
$\operatorname{tr}\left[H^\dagger \widetilde{H}\sigma^A\right]$	$\frac{1}{2}$	$\mathbf{3}$	-1	$vc^{Aa}\widetilde{h}^a$
$\operatorname{tr}\left[H^\dagger\sigma^a H\sigma^A\right]\widetilde{W}_a$	$\frac{1}{2}$	$\mathbf{3}$	-1	$v^2c^{Aa}\widetilde{W}^a$
$\operatorname{tr}\left[H^\dagger D_\mu H\sigma^A\right]$	1	$\mathbf{3}$	+1	$v^2c^{Aa}W_\mu^a$

Table 5.5: Gauge-invariant bound state operators with minimal field content up to spin 1 for the custodial symmetric special case of the MSSM weak-Higgs(ino) sector. The last column contains the corresponding leading order FMS contributions.

Gauge-invariant operators

Now that we know what the mass eigenstates predicted by PT look like, we can start to investigate the non-perturbative, gauge-invariant spectrum. At first, we have to construct gauge-invariant operators which is quite easily done by using the bidoublet formulation. Obviously, all those operators will be gauge singlets by construction but we also have two *global* quantum numbers that we can assign: The custodial quantum number and R-parity. In Tab. 5.5 those operators are listed. Notice that we explicitly distinguish between gauge indices (lower case) and custodial indices (upper case) just like before. We find both scalar and fermionic singlets and triplets as in the tree-level spectrum and the vector triplet is present as well. In particular, we find an operator which has the quantum numbers of the LSP. The remaining question is whether or not those physical (custodial) triplets map to the triplets found in standard perturbation theory in the spirit of the FMS mechanism. The last column in Tab. 5.5 states the leading order FMS expansion of the operators, i.e. the fields with highest powers of v after performing the split $H \to -v(i\sigma^2) + \eta$. For example the first operator decomposes as

$$\operatorname{tr}\left[H^\dagger H\right] = \operatorname{tr}\left[(v(i\sigma^2)+\eta^\dagger)(-v(i\sigma^2)+\eta)\right] \supset -2v\ \operatorname{tr}\left[\sigma^i\right]\operatorname{Re}h^i = -4v\operatorname{Re}h^0 \sim vh,$$

i.e. it reduces to the elementary scalar singlet. Likewise, the fermionic singlet operator reduces to $\widetilde{\chi}_3^0$, i.e. the LSP as described by PT is *indeed* part of the physical spectrum. The matrix c^{Aa} maps gauge indices a to custodial (physical) indices A. Because of its special form

$$c^{Aa} = \operatorname{diag}(1,-1,1) \tag{5.25}$$

this mapping is one-to-one but not trivial like in the SM. This shows that the nonphysical $SU(2)_m$ triplets can be augmented by $SU(2)_\mathcal{C}$ triplets and their mass degeneracy is hence physical. The (pseudo)scalar singlet and fermionic singlet operators already map to perturbative mass and T^3 eigenstates so no more work has to be done here. For the rest, we can easily construct linear combinations to augment all other eigenstates. E.g. the charginos can be constructed as

$$\sqrt{2}\kappa_i v\operatorname{tr}\left[H^\dagger\widetilde{H}(\sigma^2 - i\sigma^1)\right] + \operatorname{tr}\left[H^\dagger\sigma^a H(\sigma^2 - i\sigma^1)\right]\widetilde{W}_a \overset{\text{FMS}}{\sim} \widetilde{\chi}_i^+ .$$

Writing the bound state operators for $\widetilde{\chi}_i^+$ and $\widetilde{\chi}_i^{-\dagger}$ into a 4-spinor is also straight-forward in case one prefers a Dirac description. Notice that the + sign between the Pauli matrices is due to the special structure of (5.25) and *opposite* to the convention in Eq. (5.24) for example.

Name	Boson	Fermion	$[SU(2)_L, U(1)_Y]$
l.h. (s)leptons	$\widetilde{L} = (\widetilde{\nu}, \widetilde{e})$	$L = (\nu, e)$	[**2**, -1]
r.h. (s)electron	$\widetilde{\bar{e}}$	$\bar{e}$	[**1**, 2]
r.h. (s)neutrino	$\widetilde{\bar{\nu}}$	$\bar{\nu}$	[**1**, 0]
Higgs(inos)	$H_u = (H_u^{(1)}, H_u^{(2)})$	$\widetilde{H}_u = (\widetilde{H}_u^{(1)}, \widetilde{H}_u^{(2)})$	[**2**, 1]
	$H_d = (H_d^{(1)}, H_d^{(2)})$	$\widetilde{H}_d = (\widetilde{H}_d^{(1)}, \widetilde{H}_d^{(2)})$	[**2**, -1]
W(ino)	W^a	$\widetilde{W}^a$	[**3**, 0]

Table 5.6: Field content of the toy model including one lepton generation and (for simplicity) right-handed neutrinos. In terms of charge assignment we follow [11] and assign zero hypercharge to the additional $\bar{\nu}$.

It is important to realize that there is *no* gauge-invariant operator describing the (nonphysical) would-be Goldstone bosons $G^{0,\pm}$! They are 'projected' out of the physical spectrum automatically and we do not have to use their gauge-parameter dependent mass to argue their nonphysical nature.

Those operators can now be used to build correlation functions. In particular, we can investigate the propagators and find that to leading order in v, the propagators of the (physical) bound state objects reduce to the (gauge-variant) elementary fields, e.g.

$$\left\langle \mathcal{O}_\mu^A(x) \mathcal{O}_\nu^B(y) \right\rangle \overset{\text{FMS}}{\sim} v^4 c^{Aa} c^{Bb} \left\langle W_\mu^a(x) W_\nu^b(y) \right\rangle ,$$

for the vector triplet operators $\mathcal{O}_\mu^A = \mathrm{tr}\left[H^\dagger D_\mu H \sigma^A\right]$. Therefore, to leading order, the propagation of the gauge-invariant bound state object $\mathcal{O}_\mu^A$ is well described by the propagation of the elementary W_μ^a. In particular, both sides of the equation must have the same pole structure and therefore an identical masses.

We have herewith shown that the weak-Higgs sector of the MSSM can just as well be described by gauge-invariant objects and the differences to the usual perturbative treatment are merely quantitative.

5.3.2 Leptons

Now that we have the weak-Higgs(ino) sector under control we include our first lepton generation in an MSSM-like toy model described in the following. Practically, this means that we consider two additional chiral supermultiplets in our Lagrangian: One hosting the left-handed part of a lepton which is charged under $SU(2)_L$ and the other contributing the respective right-handed degrees of freedom. For simplicity, we assume neutrinos to be Dirac fermions and hence also the existence of right-handed neutrinos here. This is different from the pure MSSM case and it means that we have to include a third new chiral supermultiplet. We emphasize that this has a completely exploratory reason motivated by the work done in [58] and the FMS description would work just fine without them. The field content of the theory we are going to discuss in the following is summarized in Tab. 5.6. The hypercharge assignment of $\bar{\nu}$ is not part of the MSSM and we assign it a value of zero. We stress that $\bar{e}$ and $\bar{\nu}$ in Tab. 5.6 are left-handed fields, as explained in Sec. 5.1.

As a first step, we derive the Lagrangian for this theory and formulate it in a custodial symmetric way for we want to eventually be able to follow the construction done in [58] for the

SM. Writing down the kinetic parts is straight-forward

$$\begin{aligned}\mathcal{L} \supset & -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu}_a + i\widetilde{W}^{\dagger a} \bar{\sigma}^\mu (D_\mu \widetilde{W})_a \\ & + (D_\mu H_i)^\dagger (D^\mu H_i) + i\widetilde{H}_i^\dagger \bar{\sigma}^\mu D_\mu \widetilde{H}_i \\ & + (D_\mu \widetilde{L})^\dagger (D^\mu \widetilde{L}) + iL^\dagger \bar{\sigma}^\mu D_\mu L \\ & + (\partial_\mu \widetilde{\bar{e}})^\dagger (\partial^\mu \widetilde{\bar{e}}) + i\bar{e}^\dagger \bar{\sigma}^\mu \partial_\mu \bar{e} + (\partial_\mu \widetilde{\bar{\nu}})^\dagger (\partial^\mu \widetilde{\bar{\nu}}) + i\bar{\nu}^\dagger \bar{\sigma}^\mu \partial_\mu \bar{\nu}.\end{aligned} \tag{5.26}$$

We only gauge the left-handed lepton doublet under $SU(2)_L$ which is why their right-handed counterparts get a normal derivative. The interaction part is dictated by the superpotential

$$W = \mu H_u \cdot H_d - y_e \widetilde{\bar{e}} \widetilde{L} \cdot H_d + y_\nu \widetilde{\bar{\nu}} \widetilde{L} \cdot H_u. \tag{5.27}$$

Again, note that the right-handed neutrino is usually not part of the MSSM superpotential (5.1). We include it analogously to the MSSM up- and down-type quarks, i.e. with opposite sign[9]. For the soft breaking terms, we adopt all the soft breaking terms from before in Eq. (5.11)

$$\mathcal{L}_{\text{soft}} \supset -m^2 (H_d^\dagger H_d + H_u^\dagger H_u) - (|\mu|^2 + m^2) \left[H_u \cdot H_d + h.c. \right] - \frac{M_2}{2} \left[\widetilde{W}^a \widetilde{W}^a + h.c. \right], \tag{5.28}$$

and additionally add (one generation of) the trilinear and mass terms of Eq. (5.3) which are relevant for the current particle content. Once again, we extend the description to right-handed neutrinos and arrive at

$$\mathcal{L}_{\text{soft}} \supset a_e \widetilde{\bar{e}} \widetilde{L} \cdot H_d - a_\nu \widetilde{\bar{\nu}} \widetilde{L} \cdot H_u - m_L^2 \widetilde{L}^\dagger \widetilde{L} - m_{\bar{e}}^2 \widetilde{\bar{e}}^\dagger \widetilde{\bar{e}} - m_{\bar{\nu}}^2 \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{\nu}}. \tag{5.29}$$

We can now start from the general SUSY Lagrangian (2.10) and work our way through some calculations to get the remaining mass and interaction terms from the superpotential. Since it is helpful to see the calculations done at least once, we go over them in great detail:

1. We calculate the F-term contribution $|\partial W / \partial \phi_i|^2$ for $\phi_i \in \left\{ H_d, H_u, \widetilde{L}, \widetilde{\bar{e}}, \widetilde{\bar{\nu}} \right\}$:

$$\begin{aligned}\left| \frac{\partial W}{\partial H_d} \right|^2 &= |\mu|^2 H_u^\dagger H_u + |y_e|^2 \widetilde{\bar{e}}^\dagger \widetilde{\bar{e}} \widetilde{L}^\dagger \widetilde{L} - \left[\mu y_e^* \widetilde{\bar{e}}^\dagger \widetilde{L}^\dagger H_u + h.c. \right] \\ \left| \frac{\partial W}{\partial H_u} \right|^2 &= |\mu|^2 H_d^\dagger H_d + |y_\nu|^2 \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{\nu}} \widetilde{L}^\dagger \widetilde{L} - \left[\mu y_\nu^* \widetilde{\bar{\nu}}^\dagger \widetilde{L}^\dagger H_d + h.c. \right] \\ \left| \frac{\partial W}{\partial \widetilde{L}} \right|^2 &= |y_e|^2 \widetilde{\bar{e}}^\dagger \widetilde{\bar{e}} H_d^\dagger H_d + |y_\nu|^2 \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{\nu}} H_u^\dagger H_u - \left[y_e y_\nu^* \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{e}} H_u^\dagger H_d + h.c. \right] \\ \left| \frac{\partial W}{\partial \widetilde{\bar{e}}} \right|^2 &= |y_e|^2 \left| \widetilde{L} \cdot H_d \right|^2 \\ \left| \frac{\partial W}{\partial \widetilde{\bar{\nu}}} \right|^2 &= |y_\nu|^2 \left| \widetilde{L} \cdot H_u \right|^2\end{aligned} \tag{5.30}$$

The minus sign in the second line is not a mistake and actually very important, though it is easy to overlook when calculating the derivative.

2. The D-term contributions are fairly simple because we only have to sandwich the $SU(2)_L$ generators $T^a = \sigma^a/2$ with all the left-handed fields and multiply them. There are no D-terms for right-handed fields as they carry no gauge-charge [10].

$$\begin{aligned}-\frac{g^2}{8} \Big[& (H_d^\dagger \sigma^a H_d)^2 + (H_u^\dagger \sigma^a H_u)^2 + (H_d^\dagger \sigma^a H_d)(H_u^\dagger \sigma^a H_u) \\ & + (\widetilde{L}^\dagger \sigma^a \widetilde{L})^2 + (\widetilde{L}^\dagger \sigma^a \widetilde{L})(H_d^\dagger \sigma^a H_d) + (\widetilde{L}^\dagger \sigma^a \widetilde{L})(H_u^\dagger \sigma^a H_u) \Big]\end{aligned} \tag{5.31}$$

[9]This is also well motivated by our attempt to keep custodial symmetry manifest (remember that it includes a sign flip).

3. Similarly, the gaugino Yukawa-like interaction terms only emerge from the winos and the left-handed fields:

$$-\frac{g}{\sqrt{2}}\left[(H_d^\dagger\sigma^a\widetilde{H}_d)\widetilde{W}_a+(H_u^\dagger\sigma^a\widetilde{H}_u)\widetilde{W}_a+(\widetilde{L}^\dagger\sigma^a L)\widetilde{W}_a+h.c.\right] \tag{5.32}$$

4. Finally, the Yukawa contributions from $\partial^2 W/(\partial\phi_i\partial\phi_j)$ are more involved:

Luckily, not all 25 combinations of (i,j) have to be calculated because, first of all, the derivative is symmetric. Secondly, we find that many combinations vanish just because there are no respective couplings in the superpotential (5.27). For example H_d does not couple to $\widetilde{\bar{\nu}}$ and $\partial^2 W/(\partial H_d\partial\widetilde{\bar{\nu}})$ thus vanishes. The table to the right visualizes the non-zero combinations. Those contributions are calculated below.

	H_d	H_u	$\widetilde{L}$	$\widetilde{\bar{e}}$	$\widetilde{\bar{\nu}}$
H_d	0	x	x	x	0
H_u		0	x	0	x
$\widetilde{L}$			0	x	x
$\widetilde{\bar{e}}$				0	0
$\widetilde{\bar{\nu}}$					0

$$\begin{aligned}
&\left(\frac{\partial^2 W}{\partial H_d\partial H_u}\right)\widetilde{H}_d\widetilde{H}_u=\mu\widetilde{H}_u\cdot\widetilde{H}_d && \left(\frac{\partial^2 W}{\partial H_u\partial\widetilde{\bar{\nu}}}\right)\widetilde{H}_u\bar{\nu}=y_\nu\bar{\nu}\widetilde{L}\cdot\widetilde{H}_u\\
&\left(\frac{\partial^2 W}{\partial H_d\partial\widetilde{L}}\right)\widetilde{H}_d L=-y_e\widetilde{\bar{e}}L\cdot\widetilde{H}_d && \left(\frac{\partial^2 W}{\partial\widetilde{L}\partial\widetilde{\bar{e}}}\right)L\bar{e}=-y_e\bar{e}L\cdot H_d\\
&\left(\frac{\partial^2 W}{\partial H_d\partial\widetilde{\bar{e}}}\right)\widetilde{H}_d\bar{e}=-y_e\bar{e}\widetilde{L}\cdot\widetilde{H}_d && \left(\frac{\partial^2 W}{\partial\widetilde{L}\partial\widetilde{\bar{\nu}}}\right)L\bar{\nu}=y_\nu\bar{\nu}L\cdot H_u\\
&\left(\frac{\partial^2 W}{\partial H_u\partial\widetilde{L}}\right)\widetilde{H}_u L=y_\nu\widetilde{\bar{\nu}}L\cdot\widetilde{H}_u
\end{aligned} \tag{5.33}$$

Combining Eq. (5.26) and (5.28) to (5.33) finally gives the Lagrangian of our toy model. Putting all of this together as is seems tedious and little insightful. However, remember that the pure weak-Higgs sector (with our assumptions on the soft breaking parameters) is symmetric under $SU(2)_\mathcal{C}$ transformations connecting H_u and $-H_d$. Our current model is obviously still symmetric under this rotation when $y_e=y_\nu=a_e=a_\nu=0$. But not only that: From the terms derived above we get a hint that they could be transformed into each other as well. Even for non-zero parameters.

Lagrangian in manifestly symmetric form

We now want to establish the supersymmetric analogy to the theory discussed in [58]. There, the authors considered one lepton generation, assumed degenerate Yukawa couplings and the existence of right-handed neutrinos. They show that the perturbative electron-neutrino-doublet can be described via a suitable bound state operator which is a physical doublet under a custodial-flavor subgroup.

Since the MSSM has more parameters (due to the soft breaking trilinear and mass terms), our situation is slightly different. In analogy, we assume

$$\begin{aligned}
y_e=y_\nu&\equiv y &&\text{(degenerate Yukawa couplings)}\\
a_e=a_\nu&\equiv a &&\text{(degenerate trilinear parameters)}\\
m_{\bar{e}}=m_{\bar{\nu}}&\equiv m_{\bar{\lambda}} &&\text{(degenerate soft breaking masses).}
\end{aligned} \tag{5.34}$$

Next, we place $\bar{e}$ and $\bar{\nu}$ (as well as their superpartners) into $SU(2)_F$ flavor doublets

$$\bar{\lambda} \equiv \begin{pmatrix} \bar{\nu} \\ \bar{e} \end{pmatrix} \qquad\qquad \widetilde{\bar{\lambda}} \equiv \begin{pmatrix} \widetilde{\bar{\nu}} \\ \widetilde{\bar{e}} \end{pmatrix} \tag{5.35}$$

and rewrite the terms (5.26) and (5.28) to (5.33) in terms of these doublets. In the pure weak-Higgs(ino) sector this obviously makes no difference which is why we only write down the new terms in the following expressions. The Lagrangian of the weak-Higgs(ino) sector $\mathcal{L}_{\text{WH}}$ is given by Eq. (5.14).

First, the new kinetic terms (5.26) can be expressed as

$$\mathcal{L} \supset (D_\mu \widetilde{L})^\dagger (D^\mu \widetilde{L}) + i L^\dagger \bar{\sigma}^\mu D_\mu L + (\partial_\mu \widetilde{\bar{\lambda}})^\dagger (\partial^\mu \widetilde{\bar{\lambda}}) + i \bar{\lambda}^\dagger \bar{\sigma}^\mu \partial_\mu \bar{\lambda}.$$

The contributions to the scalar potential F-terms (5.30) which are not yet contained in $\mathcal{L}_{\text{WH}}$ are

$$\begin{aligned}
\mathcal{L} \supset & -|y|^2 \left[\widetilde{\bar{e}}^\dagger \widetilde{\bar{e}} + \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{\nu}}\right] \widetilde{L}^\dagger \widetilde{L} + \left[\mu y^* \left(\widetilde{\bar{e}}^\dagger \widetilde{L}^\dagger H_u + \widetilde{\bar{\nu}}^\dagger \widetilde{L}^\dagger H_d\right) + h.c.\right] \\
& - |y|^2 \left[\widetilde{\bar{e}}^\dagger \widetilde{\bar{e}} H_d^\dagger H_d + \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{\nu}} H_u^\dagger H_u - \left(\widetilde{\bar{\nu}}^\dagger \widetilde{\bar{e}} H_u^\dagger H_d + h.c.\right)\right] - |y|^2 \left[\left|\widetilde{L} \cdot H_d\right|^2 + \left|\widetilde{L} \cdot H_u\right|^2\right] \\
= & -|y|^2 (\widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}})(\widetilde{L}^\dagger \widetilde{L}) + \left[\mu y^* \left(\widetilde{\bar{e}}^\dagger \widetilde{L}^\dagger H_u + \widetilde{\bar{\nu}}^\dagger \widetilde{L}^\dagger H_d\right) + h.c.\right] \\
& - |y|^2 \left(\widetilde{\bar{e}}^\dagger H_d^\dagger - \widetilde{\bar{\nu}}^\dagger H_u^\dagger\right) \left(\widetilde{\bar{e}} H_d - \widetilde{\bar{\nu}} H_u\right) - |y|^2 \widetilde{L}^T (i\sigma^2) \left[H_d H_d^\dagger + H_u H_u^\dagger\right] (-i\sigma^2) \widetilde{L}^* \\
= & -|y|^2 (\widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}})(\widetilde{L}^\dagger \widetilde{L}) + \left[\mu^* y \widetilde{\bar{\lambda}}^T (-i\sigma^2) H^\dagger \widetilde{L} + h.c.\right] - |y|^2 \left(H \widetilde{\bar{\lambda}}\right)^\dagger \left(H \widetilde{\bar{\lambda}}\right) - |y|^2 \widetilde{L}^T (i\sigma^2) H H^\dagger (-i\sigma^2) \widetilde{L}^* \\
= & -|y|^2 (\widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}})(\widetilde{L}^\dagger \widetilde{L}) - \left[\mu^* y \widetilde{\bar{\lambda}} \cdot H^\dagger \widetilde{L} + h.c.\right] - |y|^2 \left(H \widetilde{\bar{\lambda}}\right)^\dagger \left(H \widetilde{\bar{\lambda}}\right) - |y|^2 (\widetilde{L} \cdot H)(\widetilde{L} \cdot H)^\dagger.
\end{aligned}$$

In the second to last step we reintroduced the Higgs(ino) bidoublets from Eq. (5.7). Next, we look at the additional D-terms (5.31) and gaugino interactions (5.32):

$$\begin{aligned}
\mathcal{L} \supset & -\frac{g^2}{8} \left[(\widetilde{L}^\dagger \sigma^a \widetilde{L})^2 + (\widetilde{L}^\dagger \sigma^a \widetilde{L})(H_d^\dagger \sigma^a H_d + H_u^\dagger \sigma^a H_u)\right] - \frac{g}{\sqrt{2}} \left[(\widetilde{L}^\dagger \sigma^a L) \widetilde{W}_a + h.c.\right] \\
= & -\frac{g^2}{8} \left[(\widetilde{L}^\dagger \sigma^a \widetilde{L})^2 + (\widetilde{L}^\dagger \sigma^a \widetilde{L}) \operatorname{tr}\left[H^\dagger \sigma^a H\right]\right] - \frac{g}{\sqrt{2}} \left[(\widetilde{L}^\dagger \sigma^a L) \widetilde{W}_a + h.c.\right]
\end{aligned} \tag{5.36}$$

The Yukawa terms (5.33) can be rewritten as

$$\begin{aligned}
\mathcal{L} \supset & \; y \left[\widetilde{\bar{e}} L \cdot \widetilde{H}_d - \widetilde{\bar{\nu}} L \cdot \widetilde{H}_u + \bar{e} \widetilde{L} \cdot \widetilde{H}_d - \bar{\nu} \widetilde{L} \cdot \widetilde{H}_u + \bar{e} L \cdot H_d - \bar{\nu} L \cdot H_u\right] + h.c. \\
= & -y \left[(L \cdot \widetilde{H}) \widetilde{\bar{\lambda}} + (\widetilde{L} \cdot \widetilde{H}) \bar{\lambda} + (L \cdot H) \bar{\lambda}\right] + h.c.
\end{aligned}$$

Finally, the soft breaking terms (5.29) contribute

$$\begin{aligned}
\mathcal{L} \supset & -a \left(\widetilde{\bar{\nu}} \widetilde{L} \cdot H_u - \widetilde{\bar{e}} \widetilde{L} \cdot H_d + h.c.\right) - m_L^2 \widetilde{L}^\dagger \widetilde{L} - m_{\bar{\lambda}}^2 \left(\widetilde{\bar{e}}^\dagger \widetilde{\bar{e}} + \widetilde{\bar{\nu}}^\dagger \widetilde{\bar{\nu}}\right) \\
= & -\left[a (\widetilde{L} \cdot H) \widetilde{\bar{\lambda}} + h.c.\right] - m_L^2 \widetilde{L}^\dagger \widetilde{L} - m_{\bar{\lambda}}^2 \widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}}.
\end{aligned}$$

Altogether, we find that the Lagrangian of our toy theory is

$$\begin{aligned}
\mathcal{L} = & \; \mathcal{L}_{\text{WH}} + (D_\mu \widetilde{L})^\dagger (D^\mu \widetilde{L}) + i L^\dagger \bar{\sigma}^\mu D_\mu L + (\partial_\mu \widetilde{\bar{\lambda}})^\dagger (\partial^\mu \widetilde{\bar{\lambda}}) + i \bar{\lambda}^\dagger \bar{\sigma}^\mu \partial_\mu \bar{\lambda} \\
& - \frac{g^2}{8} \left[(\widetilde{L}^\dagger \sigma^a \widetilde{L})^2 + (\widetilde{L}^\dagger \sigma^a \widetilde{L}) \operatorname{tr}\left[H^\dagger \sigma^a H\right]\right] - \frac{g}{\sqrt{2}} \left[(\widetilde{L}^\dagger \sigma^a L) \widetilde{W}_a + h.c.\right] \\
& - |y|^2 (\widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}})(\widetilde{L}^\dagger \widetilde{L}) - \left[\mu^* y \widetilde{\bar{\lambda}} \cdot H^\dagger \widetilde{L} + h.c.\right] - |y|^2 \left(H \widetilde{\bar{\lambda}}\right)^\dagger \left(H \widetilde{\bar{\lambda}}\right) - |y|^2 (\widetilde{L} \cdot H)(\widetilde{L} \cdot H)^\dagger \\
& - \left[y (L \cdot \widetilde{H}) \widetilde{\bar{\lambda}} + y (\widetilde{L} \cdot \widetilde{H}) \bar{\lambda} + y (L \cdot H) \bar{\lambda} + h.c.\right] - \left[a (\widetilde{L} \cdot H) \widetilde{\bar{\lambda}} + h.c.\right] - m_L^2 \widetilde{L}^\dagger \widetilde{L} - m_{\bar{\lambda}}^2 \widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}}.
\end{aligned} \tag{5.37}$$

In this form, it is easy to see that Eq. (5.37) exhibits a number of symmetries for $y = a = 0$: First of all, it is invariant under $SU(2)_L$ gauge-transformations. Second, the custodial symmetry $SU(2)_\mathcal{C}$ acting on the Higgs(ino) bidoublets is still present and finally, there is an $SU(2)_F$ flavor symmetry of the right-handed (s)leptons

$$\begin{aligned} &H \to U_L(x) H U_\mathcal{C}^\dagger & &L \to U_L(x) L & &\bar{\lambda} \to U_F \bar{\lambda} \\ &\widetilde{H} \to U_L(x) \widetilde{H} U_\mathcal{C}^\dagger & &\widetilde{L} \to U_L(x) \widetilde{L} & &\widetilde{\bar{\lambda}} \to U_F \widetilde{\bar{\lambda}}. \end{aligned}$$

When the (degenerate) Yukawa coupling y and/or the soft breaking parameter a is non-zero, the custodial and flavor symmetry break down to their diagonal subgroup

$$SU(2)_\mathcal{C} \times SU(2)_F \to SU(2)_f.$$

We can easily convince ourselves that this is indeed a symmetry using the special form of the Lagrangian (5.37), e.g.

$$\widetilde{\bar{\lambda}} \cdot H^\dagger \widetilde{L} \to (U_f \widetilde{\bar{\lambda}}) \cdot (H U_f^\dagger)^\dagger \widetilde{L} = (U_f \widetilde{\bar{\lambda}}) \cdot (U_f H^\dagger) \widetilde{L} = \widetilde{\bar{\lambda}} \cdot H^\dagger \widetilde{L}$$

by the invariance property of the $\cdot$ product or

$$H \widetilde{\bar{\lambda}} \to H U_f^\dagger U_f \widetilde{\bar{\lambda}} = H \widetilde{\bar{\lambda}}$$

because U_f is unitary. To see that $(\widetilde{L} \cdot H)(\widetilde{L} \cdot H)^\dagger$ is invariant it is actually easier to go back a few steps in the derivation where we find the product

$$H H^\dagger \to H U_f^\dagger (H U_f^\dagger)^\dagger = H H^\dagger.$$

Because all terms contain an even number of SUSY particles it is also easy to see that R-parity is still a symmetry, i.e. the Lagrangian is invariant if we flip the sign of all SUSY particles but not of the SM particles.

Tree-level masses

Just like before, we assume $\mu, v \in \mathbb{R}$. Furthermore, we assume y and a to be real. After the Higgs acquires its vev $H = v(-i\sigma^2) + \eta$ the relevant parts of the Lagrangian are[10]

$$\begin{aligned} \mathcal{L} \supset& -v(a - y\mu)\left(\widetilde{\bar{\lambda}}^T \widetilde{L} + h.c.\right) - (v^2y^2 + m_{\bar{\lambda}}^2)\widetilde{\bar{\lambda}}^\dagger \widetilde{\bar{\lambda}} - (v^2y^2 + m_L^2)\widetilde{L}^\dagger \widetilde{L} - vy\left(L^T \bar{\lambda} + h.c.\right) \\ =& -\begin{pmatrix} \widetilde{L}^T & \widetilde{\bar{\lambda}}^\dagger \end{pmatrix} \begin{pmatrix} v^2y^2 + m_L^2 & v(a - y\mu) \\ v(a - y\mu) & v^2y^2 + m_{\bar{\lambda}}^2 \end{pmatrix} \begin{pmatrix} \widetilde{L}^{\dagger T} \\ \widetilde{\bar{\lambda}} \end{pmatrix} - vy\left[(\nu\bar{\nu} + e\bar{e}) + h.c.\right]. \end{aligned} \tag{5.38}$$

The last bracket contains Dirac mass terms of the electron and neutrino (c.f. Eq. (2.4)). We therefore combine their left- and right-handed components into Dirac spinors ψ^e and ψ^ν, and since they have the same mass vy, we collect them into a doublet ψ

$$\psi \equiv \begin{pmatrix} \psi^\nu \\ \psi^e \end{pmatrix}, \quad \psi^e \equiv \begin{pmatrix} e \\ \bar{e}^\dagger \end{pmatrix}, \quad \psi^\nu \equiv \begin{pmatrix} \nu \\ \bar{\nu}^\dagger \end{pmatrix}. \tag{5.39}$$

[10] The vev cancels the $\cdot$ product. Notice further that the term proportional to tr $\left[H^\dagger \sigma^a H\right]$ in (5.36), which would in principle contribute a mass term, vanishes for equal vevs.

Operator	Spin	$SU(2)_\mathcal{C}$	$SU(2)_F$	$SU(2)_f$	P_R
$H^\dagger L$	$\frac{1}{2}$	**2**	**1**	**2**	$+1$
$H^\dagger \widetilde{L}$	0	**2**	**1**	**2**	-1
$\bar{\lambda}$	$\frac{1}{2}$	**1**	**2**	**2**	$+1$
$\widetilde{\bar{\lambda}}$	0	**1**	**2**	**2**	-1

Table 5.7: Gauge-invariant operators in the lepton toy model and their quantum numbers.

Do not get confused by the fact that both look like doublets: ψ is a doublet hosting two Dirac spinors ψ^e and ψ^ν which in turn contain two Weyl spinors each (the upper component is left-handed, the lower component right-handed and the spinor index structure is as indicated in Eq. (2.3)).

To determine the scalar masses, we have to diagonalize the mass matrix in (5.38). A straightforward calculation yields the mass Lagrangian

$$\begin{aligned}
\mathcal{L} &\supset -m_{\phi_1}^2 \phi_1^\dagger \phi_1 - m_{\phi_2}^2 \phi_2^\dagger \phi_2 - m_\psi \bar{\psi}\psi \\
m_{\phi_{1,2}}^2 &= \frac{1}{2}\left(m_L^2 + m_{\bar{\lambda}}^2 + 2v^2y^2 \pm \sqrt{\left(m_L^2 - m_{\bar{\lambda}}^2\right)^2 + 4v^2(a - y\mu)^2}\right) \\
m_\psi &= vy.
\end{aligned}$$

The fields $\phi_{1,2}$ are linear combinations of $\widetilde{L}^{\dagger T}$ and $\widetilde{\bar{\lambda}}$, i.e. they still contain selectrons and sneutrinos which form mass-degenerate doublets. From that we can read off that our mass spectrum consists of two scalar doublets of mass $m_{\phi_{1,2}}^2$ and a doublet of Dirac fermions of mass $m_\psi = vy$. However, the 'doublets' are not really doublets under a single symmetry group but they actually mix $SU(2)_L$ and $SU(2)_F$ rotations, e.g. $\widetilde{L}$ transforms under $SU(2)_L$ while $\widetilde{\bar{\lambda}}$ transforms under $SU(2)_F$ and $\phi_{1,2}$ transform under neither of them. Even if we mixed the two transformations, the result would contain parts of the gauge group and hence be nonphysical. This is a strong hint that what we just found following the usual perturbative procedure cannot be physical doublet states.

Gauge-invariant operators

The right-handed leptons are already gauge-invariant (physical) objects because they form multiplets of $SU(2)_F$ or $SU(2)_f$, respectively. The left-handed lepton doublet on the other hand has an open gauge-index and is therefore nonphysical. However, we can just write down a gauge-invariant bound state operator of a left-handed lepton and the Higgs bidoublet. This is very similar to what we did earlier in the pure weak-Higgs sector and the only new thing is that we can now build custodial doublets, too, which was previously impossible because we were missing $SU(2)_L$ doublets which are also $SU(2)_\mathcal{C}$ singlets (a role now played by L). In Tab. 5.7 we state the possible operators (with minimal field content) and both the $SU(2)_\mathcal{C}$ and $SU(2)_F$ multiplet structure to drive home the point that both are important and their interplay is crucial. Nonetheless, the only remaining (global) symmetry is $SU(2)_f$ and all states listed above are doublets with respect to that group. Both $H^\dagger L$ and $\bar{\lambda}$ are Weyl spinors and can readily be combined into Dirac spinors

$$\Psi^e \equiv \begin{pmatrix} (H^\dagger L)_1 \\ v(\bar{\lambda}^c)_1 \end{pmatrix}, \quad \Psi^\nu \equiv \begin{pmatrix} (H^\dagger L)_2 \\ v(\bar{\lambda}^c)_2 \end{pmatrix}. \tag{5.40}$$

Notice that the charge conjugation now acts on the doublet of Weyl spinors as

$$\bar{\lambda}^c = i\sigma^2 \begin{pmatrix} \bar{\nu}^\dagger \\ \bar{e}^\dagger \end{pmatrix} = \begin{pmatrix} \bar{e}^\dagger \\ -\bar{\nu}^\dagger \end{pmatrix}$$

and that $\bar{\lambda}^c$ transforms identical to $\bar{\lambda}$ under $SU(2)_F$, resp. $SU(2)_f$. Furthermore, $\bar{\lambda}^c$ carries upper dotted spinor indices. We can now use those elements to build a gauge-invariant lepton doublet

$$\Psi = \begin{pmatrix} \Psi^\nu \\ \Psi^e \end{pmatrix} = \begin{pmatrix} [(H^\dagger L)_2, -v\bar{\nu}^\dagger]^T \\ [(H^\dagger L)_1,\ v\bar{e}^\dagger]^T \end{pmatrix} \overset{\text{FMS}}{\sim} v \begin{pmatrix} -[\nu,\ \bar{\nu}^\dagger]^T \\ [e,\ \bar{e}^\dagger]^T \end{pmatrix} = v \begin{pmatrix} -\psi^\nu \\ \psi^e \end{pmatrix}$$

which transforms like a proper doublet under $SU(2)_f$ and *almost* reduces to the elementary lepton doublet in leading order of the FMS expansion. The difference, however, is that the global (physical) $SU(2)$ transformation mixes the elementary electron and neutrino with different signs (which does not change arguments about mass degeneracy and the like).

Having a supersymmetric theory at hand, we already know that the scalar partners of the leptons will also form a doublet. They are readily constructed from the operators in the table as

$$\Phi_{1,2} \equiv \alpha_{1,2} i\sigma^2 (H^\dagger \widetilde{L})^* + \beta_{1,2} v\widetilde{\bar{\lambda}} \overset{\text{FMS}}{\sim} v\left(\alpha_{1,2}\widetilde{L}^{\dagger T} + \beta_{1,2}\widetilde{\bar{\lambda}}\right) = v\phi_{1,2} \tag{5.41}$$

where α and β encode the relative phases obtained by diagonalizing the mass matrix of (5.38). $\Phi_{1,2}$ now truly transform as

$$\Phi_{1,2} \xrightarrow{U\in SU(2)_f} \Phi'_{1,2} = U\Phi_{1,2}$$

and we once again observe that the perturbative 'doublet' can be mapped to a proper doublet using non-perturbative bound state operators and APT. In total, we have shown that the physical spectrum indeed contains a doublet of Dirac fermions Ψ as well as two scalar doublets $\Phi_{1,2}$, the description (and in particular masses) of which reduces to the perturbative results in leading order of the FMS expansion. In the totally symmetric case the members of the doublets are mass degenerate, which is now a physical statement.

Next, we will see how those degeneracies are lifted once we turn to the more realistic case of $v_d \neq v_u$, $y_e \neq y_\nu$, $a_e \neq a_\nu$ and $m_{\bar{e}} \neq m_{\bar{\nu}}$. All those changes break both the custodial as well as the flavor symmetry from before. Consequently, $SU(2)_f$ will not be an exact symmetry of the theory anymore and the degeneracies of the bound state objects will be lifted. Via the FMS mechanism, this in turn explains how the elementary 'doublets' obtain different masses in a completely gauge-invariant fashion.

Important note: If we did not include right-handed neutrinos, we could have still formulated the gauge-invariant operators and applied the FMS mechanism. However, we could not have established the additional flavor symmetry and consequently, custodial symmetry would have been the only global symmetry. It would be broken even for degenerate couplings, yet for zero couplings at least, the left-handed leptons would constitute a proper $SU(2)_C$ doublet.

5.4 Extension to the full MSSM

Up until now, we intentionally kept custodial symmetry intact and many parameters degenerate in order to make the calculations of the tree-level spectra easier and to see the structure behind how the different multiplets are mapped to gauge-invariant operator multiplets. Lifting these restrictions is straight-forward in the sense that we can easily generalize the calculations done so far. In Sec. 5.4.1 we will see, how explicitly breaking custodial symmetry and having non-degenerate Yukawa couplings and soft breaking terms (which is the case in the full MSSM) merely mixes both the elementary spectra as well as the corresponding bound state operators. Including hypercharge into the description is analogous to the SM (c.f. Sec. 4.3) and discussed in Sec. 5.4.2. Finally, Sec. 5.4.3 discusses the extension to quarks/mesons as well as multiple fermion generations.

5.4.1 Broken Custodial Symmetry

So far, we had the two Higgs fields acquire the same vacuum expectation value which is nice to work with and it actually gave us a lot of structural insight. However, it is phenomenologically not viable. We should therefore finally discuss the theory presented above for non-degenerate Higgs vevs, masses and soft breaking parameters.

Consider first the pure weak-Higgs(ino) sector: Introducing separate soft breaking masses for the two Higgs, making their off-diagonal mass an independent parameter and letting them acquire different vevs, explicitly breaks the custodial symmetry. We therefore expect the previously found mass eigenstates to mix with each other. Unfortunately, this also means that the bidoublet formalism does not quite work anymore as mass terms like $\mathrm{tr}\left[H^\dagger H\right]$ would always enforce a symmetry between H_u and H_d. By introducing $M^2 \equiv \mathrm{diag}(m_u^2 + |\mu|^2, m_d^2 + |\mu|^2)$ as an auxiliary mass matrix, however, we can still write down the Lagrangian in matrix form. The broken custodial symmetry becomes manifest. In the weak-Higgs(ino) sector only the scalar potential has to be modified, which now reads

$$V(H) = \mathrm{tr}\left[H^\dagger H M\right] - 2m_{ud}^2 \ \mathrm{Re}\det H^\dagger + \frac{g^2}{8}\mathrm{tr}\left[H^\dagger H\right]^2 - \frac{g^2}{2}\det H^\dagger H.$$

A detailed discussion of this potential including the calculation of the scalar masses can, e.g., be found in [11]. However, we want to take a slightly different route here as we want to see the connection to the calculation we did previously.

Minimizing the potential leads to two independent vevs, v_u and v_d, and we align their directions in the bidoublet language as follows

$$H = \begin{pmatrix} 0 & -v_d \\ v_u & 0 \end{pmatrix} + \eta = \frac{v_u + v_d}{2}(-i\sigma^2) + \frac{v_u - v_d}{2}\sigma^1 + \eta. \tag{5.42}$$

Without loss of generality, we choose $v_u > v_d$ and we define the famous parameter

$$\tan\beta \equiv \frac{v_u}{v_d}. \tag{5.43}$$

After inserting the split (5.42) into the Lagrangian we find the gauge-boson mass

$$m_W^2 = \frac{g^2}{2}(v_u^2 + v_d^2).$$

After expressing the fluctuation fields again via the bilinears defined in Eq. (5.20), i.e. $\eta = h^i b_i$, we can write the scalar mass matrix in the $(\mathrm{Re}\,h^i, \mathrm{Im}\,h^i)$ basis in block-diagonal form

$$\begin{aligned}
\mathcal{L} \supset & \begin{pmatrix} \mathrm{Re}\,h^0 & \mathrm{Re}\,h^3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,h^0 \\ \mathrm{Re}\,h^3 \end{pmatrix} + \begin{pmatrix} \mathrm{Im}\,h^0 & \mathrm{Im}\,h^3 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ b_2 & c_2 \end{pmatrix} \begin{pmatrix} \mathrm{Im}\,h^0 \\ \mathrm{Im}\,h^3 \end{pmatrix} \\
& + \begin{pmatrix} \mathrm{Re}\,h^1 & \mathrm{Im}\,h^2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ b_3 & c_3 \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,h^1 \\ \mathrm{Im}\,h^2 \end{pmatrix} + \begin{pmatrix} \mathrm{Re}\,h^2 & \mathrm{Im}\,h^1 \end{pmatrix} \begin{pmatrix} a_3 & -b_3 \\ -b_3 & c_3 \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,h^2 \\ \mathrm{Im}\,h^1 \end{pmatrix} \\
= & \frac{1}{2} \begin{pmatrix} h & H^0 \end{pmatrix} \begin{pmatrix} a_1/2 & b_1/2 \\ b_1/2 & c_1/2 \end{pmatrix} \begin{pmatrix} h \\ H^0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} A & G^0 \end{pmatrix} \begin{pmatrix} a_2/2 & b_2/2 \\ b_2/2 & c_2/2 \end{pmatrix} \begin{pmatrix} A \\ G^0 \end{pmatrix} \\
& + \begin{pmatrix} H^+ & G^+ \end{pmatrix} \begin{pmatrix} a_3/2 & -ib_3/2 \\ ib_3/2 & c_3/2 \end{pmatrix} \begin{pmatrix} H^- \\ G^- \end{pmatrix}, \qquad (5.44)
\end{aligned}$$

where the matrix entries a_i, b_i and c_i are overwhelming constants. In the second step we already renamed the fields (with the proper normalization) to match the mass eigenstates from Sec. 5.3.1. We can clearly see that the fields only mix pair-wise as compared to the previous (symmetric) case. Solving the eigenvalue problem yields the explicit mixing for the pseudoscalar and charged scalars

$$\begin{aligned}
A' &= \frac{(v_u + v_d)A + (v_u - v_d)G^0}{\sqrt{2}\sqrt{v_u^2 + v_d^2}} = \frac{1}{\sqrt{2}} \left[(\cos\beta + \sin\beta)A + (\sin\beta - \cos\beta)G^0 \right] \\
H^{-\prime} &= \frac{i(v_u + v_d)H^- + (v_u - v_d)G^-}{\sqrt{2}\sqrt{v_u^2 + v_d^2}} = \frac{1}{\sqrt{2}} \left[i(\cos\beta + \sin\beta)H^- + (\sin\beta - \cos\beta)G^- \right].
\end{aligned}$$

Primed fields correspond to mass eigenstates of the non-custodial-symmetric case. We see that for $v_d = v_u$ the relations reduce to the previous pseudoscalar A and charged scalars H, even though the entire model does *not* approach the fully symmetric case in that limit. This becomes apparent when looking at the neutral Higgs scalars h' and $H^{0'}$ where the mixing is not just because v_u and v_d are different but also due to the newly introduced parameters (m_u^2, m_d^2, m_{ud}^2 all independent). They do not reduce nicely to the previous states which is not surprising at all. Nevertheless, we know from (5.44), that they will be linear combinations of the previously found fields h and H^0. Furthermore, we can still state the corresponding mass eigenvalues

$$\begin{aligned}
m_{A'}^2 &= m_{ud}^2 \frac{v_u^2 + v_d^2}{v_u v_d} \\
m_{h',H^{0'}}^2 &= \frac{1}{2} \left(m_{A'}^2 + m_W^2 \mp \sqrt{(m_{A'}^2 + m_W^2)^2 - 4m_A^2 m_W^2 \cos^2 2\beta} \right) \\
m_{H^{\pm\prime}}^2 &= m_{A'}^2 + m_W^2 \\
m_{G^{0,\pm\prime}}^2 &= \xi m_W^2.
\end{aligned}$$

As for the gauge-invariant operators we find that the ones corresponding to A and $H^\pm$ from before (c.f. Tab. 5.5) now automatically reduce to the new mass eigenstates, i.e.

$$\begin{aligned}
\mathrm{Im}\det H &\overset{\text{FMS}}{\sim} (v_u + v_d)A + (v_u - v_d)G^0 \sim A' \\
\mathrm{tr}\left[H^\dagger H(\sigma^2 \mp i\sigma^1) \right] &\overset{\text{FMS}}{\sim} (v_u + v_d)H^\pm \pm i(v_u - v_d)G^\pm \sim H^{\pm\prime}.
\end{aligned}$$

As already mentioned, this is not the case for h and H^0. However, we can express h and H^0 via the operators

$$
\begin{aligned}
(v_u + v_d)\mathrm{tr}\left[H^\dagger H\right] - (v_u - v_d)\mathrm{tr}\left[H^\dagger H\sigma^3\right] &\overset{\text{FMS}}{\sim} v_d v_u \,\mathrm{Re}\, h^0 \sim h \\
(v_u - v_d)\mathrm{tr}\left[H^\dagger H\right] - (v_u + v_d)\mathrm{tr}\left[H^\dagger H\sigma^3\right] &\overset{\text{FMS}}{\sim} v_d v_u \,\mathrm{Re}\, h^3 \sim H^0
\end{aligned}
$$

which in leading order correspond to the elementary fields of the fully symmetric case. Appropriate linear combinations then reduce to the corresponding primed fields.

Next, we turn to the higgsinos and winos: The relevant parts of the Lagrangian are unchanged

$$
\mathcal{L} \supset -\frac{g}{\sqrt{2}}\left[\mathrm{tr}\left[H^\dagger\sigma^a\widetilde{H}\right]\widetilde{W}_a + \widetilde{W}^{\dagger a}\mathrm{tr}\left[\widetilde{H}^\dagger\sigma^a H\right]\right] + \mu\left[\det\left(\widetilde{H}\right) + \text{h.c.}\right] - \frac{M_2}{2}\left[\widetilde{W}^a\widetilde{W}^a + h.c.\right],
$$

however, the new vev $V = \left(\begin{smallmatrix} 0 & -v_d \\ v_u & 0\end{smallmatrix}\right)$ introduces additional mixing terms

$$
\begin{aligned}
\mathcal{L} \supset& -\frac{g}{\sqrt{2}}\left[\mathrm{tr}\left[V^\dagger\sigma^a b_j\right]\widetilde{h}^j\widetilde{W}_a + \mathrm{tr}\left[b_j^\dagger\sigma^a V\right]\widetilde{W}^{a\dagger}\widetilde{h}^{j\dagger}\right] + \mu\left[\widetilde{h}^0\widetilde{h}^0 + \widetilde{h}^{0\dagger}\widetilde{h}^{0\dagger} - \widetilde{h}^a\widetilde{h}^a - \widetilde{h}^{a\dagger}\widetilde{h}^{a\dagger}\right] \\
=& \frac{g}{\sqrt{2}}(v_u + v_d)\left[\widetilde{h}^a\widetilde{W}_a + \widetilde{W}^{a\dagger}\widetilde{h}^{a\dagger}\right] - \frac{g}{\sqrt{2}}(v_u - v_d)\left[\widetilde{h}^0\widetilde{W}^3 + i\widetilde{h}^2\widetilde{W}^1 - i\widetilde{h}^1\widetilde{W}^2 + h.c.\right] \\
&+ \mu\left[\widetilde{h}^0\widetilde{h}^0 + \widetilde{h}^{0\dagger}\widetilde{h}^{0\dagger} - \widetilde{h}^a\widetilde{h}^a - \widetilde{h}^{a\dagger}\widetilde{h}^{a\dagger}\right].
\end{aligned}
$$

Notice that we can read off that for $v_d = v_u$, the Higgsino-wino mass Lagrangian reduces to the fully symmetric case. We introduce the fields $\widetilde{h}^\pm = (\widetilde{h}^2 \pm i\widetilde{h}^1)/\sqrt{2}$ and $\widetilde{W}^\pm = (\widetilde{W}^2 \pm i\widetilde{W}^1)/\sqrt{2}$ and rewrite the Lagrangian as

$$
\begin{aligned}
\mathcal{L} \supset& \frac{g}{\sqrt{2}}(v_u + v_d)\left[\widetilde{W}^3\widetilde{h}^3\right] - \frac{g}{\sqrt{2}}(v_u - v_d)\left[\widetilde{W}^3\widetilde{h}^0\right] + \sqrt{2}v_d g\left[\widetilde{W}^+\widetilde{h}^-\right] + \sqrt{2}v_u g\left[\widetilde{W}^-\widetilde{h}^+\right] \\
&+ \mu\left[\widetilde{h}^0\widetilde{h}^0 - \widetilde{h}^3\widetilde{h}^3 - 2\widetilde{h}^+\widetilde{h}^-\right] - \frac{M_2}{2}\left[\widetilde{W}^3\widetilde{W}^3 + 2\widetilde{W}^+\widetilde{W}^-\right] + h.c.
\end{aligned}
$$

To make contact with our previous calculations, we express the fields by the previous mass eigenstates of Tab. 5.4

$$
\widetilde{h}^0 = -\frac{i}{\sqrt{2}}\widetilde{\chi}^0_3, \qquad \begin{pmatrix}\widetilde{h}^{3,\pm} \\ \widetilde{W}^{3,\pm}\end{pmatrix} = B^{-1}\begin{pmatrix}\widetilde{\chi}^{0,\pm}_1 \\ \widetilde{\chi}^{0,\pm}_2\end{pmatrix}, \qquad B = \begin{pmatrix} 2\frac{\kappa_1}{\mathcal{N}_1} & \frac{\sqrt{2}}{\mathcal{N}_1} \\ 2\frac{\kappa_2}{\mathcal{N}_2} & \frac{\sqrt{2}}{\mathcal{N}_2}\end{pmatrix},
$$

with $\kappa_i, \mathcal{N}_i$ as defined in Eq. (5.23). This yields the mass terms

$$
\begin{pmatrix}\widetilde{\chi}^0_1 & \widetilde{\chi}^0_2 & \widetilde{\chi}^0_3\end{pmatrix}\begin{pmatrix}\cdot & * & * \\ * & \cdot & * \\ * & * & \cdot\end{pmatrix}\begin{pmatrix}\widetilde{\chi}^0_1 \\ \widetilde{\chi}^0_2 \\ \widetilde{\chi}^0_3\end{pmatrix} + \begin{pmatrix}\widetilde{\chi}^-_1 & \widetilde{\chi}^-_2\end{pmatrix}\begin{pmatrix}\cdot & * \\ * & \cdot\end{pmatrix}\begin{pmatrix}\widetilde{\chi}^+_1 \\ \widetilde{\chi}^+_2\end{pmatrix} + h.c.
$$

where the off-diagonal elements $*$ are terms of the form $(v_u^2+v_d^2)^{1/2} - (v_u+v_d)/\sqrt{2}$ or $(v_d - v_u)$ and therefore vanish when $v_u = v_d$. We have thus demonstrated that the neutral (charged) fermions mix once custodial symmetry is violated. As expected and explained above, we furthermore see, that the new mass eigenstates approach the previous result when the symmetry is restored. Once again, appropriate linear combinations of the operators found in Sec. 5.3.1 can be used to build gauge-invariant bound states. Those in turn reduce to the elementary mass eigenstates in leading order of the FMS mechanism. In particular, an operator which augments the lightest of the uncharged fermions can be constructed, i.e. the LSP is part of the physical spectrum.

At last, we investigate the case of $v_d \neq v_u$ in the (s)lepton sector. Additionally, we set $y_e \neq y_\nu$, $a_e \neq a_\nu$ and $m^2_{\tilde{\bar{e}}} \neq m^2_{\tilde{\bar{\nu}}}$ but still assume that they are real[11]. Inserting the new split (5.42) into the lepton Lagrangian yields

$$\begin{aligned}
\mathcal{L} \supset\ & \frac{g^2}{8}(v_u^2 - v_d^2)\left[\widetilde{\nu}^\dagger\widetilde{\nu} - \widetilde{e}^\dagger\widetilde{e}\right] - \left[(v_u a_\nu - \mu v_d y_\nu)\widetilde{\nu}\widetilde{\bar{\nu}} + (v_d a_e - \mu v_u y_e)\widetilde{e}\widetilde{\bar{e}} + h.c.\right] \\
& - \left[(v_d^2 y_e^2 + m_{\tilde{\bar{e}}}^2)\widetilde{\bar{e}}^\dagger\widetilde{\bar{e}} + (v_d^2 y_e^2 + m_L^2)\widetilde{e}^\dagger\widetilde{e} + (v_u^2 y_\nu^2 + m_{\tilde{\bar{\nu}}}^2)\widetilde{\bar{\nu}}^\dagger\widetilde{\bar{\nu}} + (v_u^2 y_\nu^2 + m_L^2)\widetilde{\nu}^\dagger\widetilde{\nu}\right] \\
& - \left[v_d y_e e\bar{e} + v_u y_\nu \nu\bar{\nu} + h.c.\right] \\
=\ & -\xi_1^\dagger M_1 \xi_1 - \xi_2^\dagger M_2 \xi_2 - v_d y_e \bar{\psi}^e\psi^e - v_u y_\nu \bar{\psi}^\nu\psi^\nu .
\end{aligned}$$

Notice that now a term proportional to the gauge coupling appears because the term $\mathrm{tr}\left[H^\dagger\sigma^a H\right]$ in Eq. (5.36) does not disappear anymore. We immediately see that the lepton doublet splits, with masses proportional to the different vevs. The slepton masses are currently written in the basis

$$\xi_1 = \begin{pmatrix} \widetilde{\nu} \\ \widetilde{\bar{\nu}}^\dagger \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \widetilde{e} \\ \widetilde{\bar{e}}^\dagger \end{pmatrix}$$

with the matrices

$$\begin{aligned}
M_1 &= \begin{pmatrix} \left(y_\nu^2 - \frac{g^2}{8}\right)v_u^2 + \frac{g^2}{8}v_d^2 + m_L^2 & v_u a_\nu - \mu v_d y_\nu \\ v_u a_\nu - \mu v_d y_\nu & v_u^2 y_\nu^2 + m_{\tilde{\bar{\nu}}}^2 \end{pmatrix} \\
M_2 &= \begin{pmatrix} \left(y_e^2 - \frac{g^2}{8}\right)v_d^2 + \frac{g^2}{8}v_u^2 + m_L^2 & v_d a_e - \mu v_u y_e \\ v_d a_e - \mu v_u y_e & v_d^2 y_e^2 + m_{\tilde{\bar{e}}}^2 \end{pmatrix}
\end{aligned}$$

which are yet to be diagonalized. It is straight-forward to do so but adds nothing new apart from four different slepton mass eigenstates. We notice, however, that for the case of equal vevs and degenerate $y, a, m_{\tilde{\bar{\lambda}}}$, both mass matrices reduce to the one found in the fully symmetric case, Eq. (5.38), which restores the mass-degenerate doublets.

For the leptons, we can immediately write down gauge-invariant bound state operators

$$\begin{aligned}
\Psi^e &= \begin{pmatrix} (H^\dagger L)_1 \\ v_u(\bar{\lambda}^c)_1 \end{pmatrix} \overset{\text{FMS}}{\sim} v_u\psi^e \\
\Psi^\nu &= \begin{pmatrix} (H^\dagger L)_2 \\ v_d(\bar{\lambda}^c)_2 \end{pmatrix} \overset{\text{FMS}}{\sim} v_d\psi^\nu
\end{aligned} \tag{5.45}$$

which are essentially the lepton operators found in the fully symmetric case, Eq. (5.40). Now, they merely expand with the different vevs. The slepton mass eigenstates will be linear combinations of $\widetilde{\nu}$ and $\widetilde{\bar{\nu}}^\dagger$ ($\widetilde{e}$ and $\widetilde{\bar{e}}^\dagger$, respectively) which is why it is sufficient to know that

$$\begin{aligned}
(H^\dagger\widetilde{L})_1 &\overset{\text{FMS}}{\sim} v_u\widetilde{e} & \qquad (H^\dagger\widetilde{L})_2 &\overset{\text{FMS}}{\sim} v_d\widetilde{\nu} \\
v_u(\widetilde{\bar{\lambda}}^\dagger)_2 &\overset{\text{FMS}}{\sim} v_u\widetilde{\bar{e}}^\dagger & \qquad v_d(\widetilde{\bar{\lambda}}^\dagger)_1 &\overset{\text{FMS}}{\sim} v_d\widetilde{\bar{\nu}}^\dagger .
\end{aligned}$$

Those operators are all gauge-invariant and can simply be combined such that they match whatever form the explicit mass eigenstates have. Notice that the linear combinations are formed

[11] Just like in the weak-Higgs sector, we can implement the non-degeneracy within the bidoublet formalism via the left-/right-multiplication of H with a matrix $\mathrm{diag}(y_e, y_\nu)$.

between $(H^\dagger\widetilde{L})_1$ and $(\widetilde{\widetilde{\lambda}}^\dagger)_2$, i.e. with the components reverse. This is not a mistake and also present in the fully symmetric case, Eq. (5.41), where this 'mixing' is slightly hidden by $i\sigma^2$. Furthermore, do not get confused by the fact that the sneutrino FMS-expands with v_d whereas the selectron expands with v_u. This is indeed opposite to the masses, which are proportional to v_u and v_d, respectively.

We conclude, that a mapping between perturbative mass eigenstates and non-perturbative bound states is possible, even for different Higgs vevs and non-degenerate couplings. The mixing of the perturbative mass eigenstates is completely dual to the mixing of the gauge-invariant bound state operators. Therefore, we can understand the mixing of elementary fields completely gauge-invariantly by means of the mixing of composite operators once they are not protected by a global symmetry anymore.

5.4.2 Electric Charge and QED

In Sec. 5.2 we learnt, that $U(1)_Y$ is a subgroup of $SU(2)_\mathcal{C}$. As a result, fields that have no explicit hypercharge assignment in the elementary field description (e.g. W_μ^a) nevertheless acquire a non-zero electric charge in the composite operator language. This is in contrast to the usual definition of electric charge (4.10). We should therefore check whether the operators we constructed carry the same electric charge as their elementary counterparts. For that it is sufficient to investigate what effect the (global) hypercharge transformations[12]

$$H \to H' = H\ \exp\left(i\alpha\frac{\sigma^3}{2}\right), \quad L \to L' = e^{-i\alpha/2}L, \quad \bar{e} \to \bar{e}' = e^{2\alpha/2}\bar{e}$$

of the elementary fields have on the bound state operators. The neutrino has 0 hypercharge and therefore does not transform. The scalar and pseudo-scalar singlet operators $\mathrm{tr}\left[H^\dagger H\right]$ and $\mathrm{Im}\det H$ are invariant under such transformation because of the properties of trace and determinant. Hence, they are charge neutral just like their elementary counterparts h and A. Likewise, the LSP operator $\mathrm{tr}\left[H^\dagger\widetilde{H}\right]$ is charge neutral. The Higgs triplet is investigated in more detail:

$$\mathrm{tr}\left[H^\dagger H\sigma^A\right] \to \mathrm{tr}\left[H^\dagger H e^{i\alpha\sigma^3/2}\sigma^A e^{-i\alpha\sigma^3/2}\right] = R^{AB}\ \mathrm{tr}\left[H^\dagger H\sigma^B\right]$$

with the $SO(3)$ matrix

$$R = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Diagonalization reveals the charge eigenstates

$$\begin{pmatrix} \mathcal{O}_{H^+} \\ \mathcal{O}_{H^-} \\ \mathcal{O}_{H^0} \end{pmatrix} = \begin{pmatrix} \mathrm{tr}\left[H^\dagger H(\sigma^2 - i\sigma^1)\right] \\ \mathrm{tr}\left[H^\dagger H(\sigma^2 + i\sigma^1)\right] \\ \mathrm{tr}\left[H^\dagger H\sigma^3\right] \end{pmatrix} \to \begin{pmatrix} \mathcal{O}'_{H^+} \\ \mathcal{O}'_{H^-} \\ \mathcal{O}'_{H^0} \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & & \\ & e^{-i\alpha} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}_{H^+} \\ \mathcal{O}_{H^-} \\ \mathcal{O}_{H^0} \end{pmatrix},$$

which are identical to the T^3 eigenstates found earlier. This confirms that they indeed carry electric charges of $0, \pm 1$, just like the corresponding $H^{0,\pm}$. The same can be done for the remaining triplet operators of the pure weak-Higgs sector as all of them boil down to a rotation with R. We showed earlier, that the perturbative mass eigenstates are linear combinations of these operators and that they combine in a way which is transparent to this rotation[13]. Therefore, we

[12]See Tab. 5.1 for hypercharge assignments

[13]I.e. only states with the same T^3 quantum number mix.

can conclude that the bound state object corresponding to the physical W^+, e.g., has indeed the same charge as the elementary object.

The left-handed (s)leptonic operators[14] transform as

$$\begin{pmatrix} \mathcal{O}_e \\ \mathcal{O}_\nu \end{pmatrix} = H^\dagger L \to \begin{pmatrix} \mathcal{O}'_e \\ \mathcal{O}'_\nu \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} & \\ & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}_e \\ \mathcal{O}_\nu \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{O}_{\widetilde{e}} \\ \mathcal{O}_{\widetilde{\nu}} \end{pmatrix} = H^\dagger \widetilde{L} \to \begin{pmatrix} \mathcal{O}'_{\widetilde{e}} \\ \mathcal{O}'_{\widetilde{\nu}} \end{pmatrix} = \begin{pmatrix} e^{-i\alpha} & \\ & 1 \end{pmatrix} \begin{pmatrix} \mathcal{O}_{\widetilde{e}} \\ \mathcal{O}_{\widetilde{\nu}} \end{pmatrix}$$

and we have therefore shown that their charge assignment is correct as well. Notice that right-handed fields automatically adopt their hypercharge as electric charge.

If we now gauge hypercharge, i.e. include B_μ in the covariant derivative, $SU(2)_\mathcal{C}$ breaks down to its $U(1)$ subgroup and becomes local (see also Sec. 4.3). This subgroup is *not* further broken by any other custodial symmetry offending term. If we now simultaneously perform the $U(1) \subset SU(2)_\mathcal{C}$ transformation on the Higgs and a $U(1)_Y$ transformation on the (s)fermions according to their hypercharge assignments of Tab. 5.1, we naturally obtain a theory which is locally $U(1)_\mathrm{EM}$ symmetric. In other words, the gauge theory of (supersymmetric) quantum electrodynamics arises, just like in the SM in Sec. 4.3.

Since $U(1)_\mathrm{EM}$ is a local symmetry, we again have to discuss gauge-invariance. Luckily, the situation for an Abelian gauge-group is straight-forward: A *Dirac phase factor* can be used to dress fields with open $U(1)$ gauge index to make them inherently gauge-invariant [21]. This can be thought of as charged (physical) particles to only exist as surrounded by a *photon cloud*, constituting their Coulomb field. Likewise, the gauge field can be made inherently gauge-invariant by projecting out the physical (transversal) part. The $U(1)$ gaugino neither carries charge nor is it a 4-vector, hence, no dressing or projecting is required to make it gauge-invariant.

Due to the introduction of B_μ and the breaking of the W^a_μ triplet we now have two fields in the neutral vector singlet channel which mix to create the Z boson and the photon. This holds both in a perturbative sense as well as in the operator language, where the breaking of the triplet happens naturally upon breaking $SU(2)_\mathcal{C}$. For the superpartners, the situation is similar: Introducing the bino $\widetilde{B}$ (which is neutral) does not affect the charginos and merely mixes with the neutralinos. Non-perturbatively, this mixing would manifest itself as the existence of an additional mass pole in the respective channels. Neither (the transversal part of) B_μ nor $\widetilde{B}$ have to be combined with the Higgs to create invariant bound states as they are not charged under $SU(2)_L$. They can therefore readily be used as physical fields. Matching the new mass eigenstates with inherently gauge-invariant bound state operators again reduces to a task of linear combination.

5.4.3 Multiple Generations and Quarks

Including all three lepton generations substantially increases the complexity but changes nothing about our construction. Inter-generation mixing is completely transparent to our composite operator construction as one could just introduce operators for each generation (c.f. Tab. 5.7)

$$\begin{aligned} &H^\dagger L_e,\ H^\dagger L_\mu,\ H^\dagger L_\tau && \bar{\lambda}_e,\ \bar{\lambda}_\mu,\ \bar{\lambda}_\tau \\ &H^\dagger \widetilde{L}_e,\ H^\dagger \widetilde{L}_\mu,\ H^\dagger \widetilde{L}_\tau && \widetilde{\bar{\lambda}}_e,\ \widetilde{\bar{\lambda}}_\mu,\ \widetilde{\bar{\lambda}}_\tau. \end{aligned}$$

Both components of these operators are inherently gauge-invariant and can be linearly combined and rotated in generation space to augment all resulting mass eigenstates. Note however, that we again assumed the existence of right-handed neutrinos for all generations.

[14] With $\bar{\lambda}$ and L as defined in Eq. (5.35) and Tab. 5.1.

In the context of (S)QCD, speaking about bound states is much more generally established: The low energy description is all about objects which are built from elementary quarks and gluons to form color neutral (i.e. gauge-invariant with respect to $SU(3)_c$) bound states, e.g. Mesons. Nevertheless, fields like the pion still carry $SU(2)_L$ charge which has to be taken care of. Luckily, in terms of electroweak and Higgs physics, the description of quarks is completely analogous to leptons[15], i.e. we can readily write down[16]

$$\Psi^d = \begin{pmatrix} (H^\dagger Q)_1 \\ v_u \bar{d}^\dagger \end{pmatrix} \overset{\text{FMS}}{\sim} v_u \begin{pmatrix} d \\ \bar{d}^\dagger \end{pmatrix} = v_u \psi^d$$

$$\Psi^u = \begin{pmatrix} (H^\dagger Q)_2 \\ v_d \bar{u}^\dagger \end{pmatrix} \overset{\text{FMS}}{\sim} v_d \psi^u,$$

in analogy to Eq. (5.45) for leptons. The only difference is, that $\Psi^{u,d}$ are *not yet* physical as they still carry color charge. Even though these 'quark-Higgs bound states' cannot exist in isolation, such contractions are still important when building the usual color singlets. An inherently gauge-invariant operator for π^+ would e.g. be[17]

$$\Pi^+ \equiv \bar{\Psi}^d \Psi^u = \begin{pmatrix} v_u \bar{d} & (H^\dagger Q)_1^\dagger \end{pmatrix} \begin{pmatrix} (H^\dagger Q)_2 \\ v_d \bar{u}^\dagger \end{pmatrix} \overset{\text{FMS}}{\sim} v_u v_d \left(\bar{d} u + d^\dagger \bar{u}^\dagger \right) \sim v_u v_d \pi^+. \tag{5.46}$$

Notice that the expression on the right hand side is a color singlet but *not* an $SU(2)_L$ singlet which makes it apparent that the FMS mechanism is also important in (S)QCD. Finally, just like before, we can use the $U(1)$ subgroup of $SU(2)_\mathcal{C}$ as well as the hypercharge assignments of the quarks to find that $\Psi^{d,u}$ carry electric charges $-1/3$ and $2/3$, respectively. Consequently, Π^+ carries the correct electric charge of $+1$.

Squarks and gluinos are no gauge-invariant fields either and have to be treated in a bound state language with respect to $SU(3)_c$, too. Nevertheless, this has no effect on our construction. Gluinos carry no weak charge and are therefore trivial in terms of the FMS mechanism. 'Left-handed' squarks get an appropriate Higgs dressing, just like in the leptonic case.

This concludes the generalization of the FMS mechanism and augmented perturbation theory to the full MSSM.

5.5 Summary of the Physical Spectrum

Finally, we collect all the results obtained in this thesis in Tab. 5.8. It shows the inherently gauge-invariant spectrum of the MSSM which looks qualitatively identical to the spectrum obtained by performing standard perturbation theory. The FMS mechanism establishes a one-to-one mapping (c.f. last column), i.e. just like in the standard model, all mass eigenstates can be augmented by appropriate composite operators. We only list one generation of (s)leptons, but as argued in Sec. 5.4.3, the generalization is trivial. Furthermore, we excluded the right-handed (s)neutrinos in the table which also makes the flavor symmetry constructed in Sec. 5.3.2 obsolete. Therefore, the only remaining global quantum number (apart from spin, electric charge and (charge)parity) is given by the custodial symmetry. As discussed before, this symmetry is only approximate in the full MSSM which is why identical quantum number channels mix, as can be also seen in the table.

[15] In a sense it is even better as we have right-handed up- and down- type quarks, and we therefore do not have to assume the existence of additional particles like we did for the right-handed neutrinos.

[16] Remember that both d and $\bar{d}$ are taken to be left-handed Weyl spinors in the MSSM. See discussion in Sec. 5.1. In SM notation, $(d, \bar{d}^\dagger)^T$ would read $(d_L, d_R)^T$.

[17] Again, in the usual SM notation the expansion would read $\bar{d}_R u_L + \bar{d}_L u_R \sim \pi^+$.

Name	Operator(s)	$SU(2)_{\mathcal{C}}$	$\overset{\text{FMS}}{\sim}$
light Higgs	$\operatorname{tr}\left[H^\dagger H\right]$ $\operatorname{tr}\left[H^\dagger H\sigma^3\right]$	$\mathbf{1}$	h
pseudoscalar Higgs	$\operatorname{Im}\det H$	$\mathbf{1}$	A
heavy Higgs	$\operatorname{tr}\left[H^\dagger H\sigma^3\right]$ $\operatorname{tr}\left[H^\dagger H\right]$	$\mathbf{3}$, 0	H^0
charged Higgs	$\varphi\ \operatorname{tr}\left[H^\dagger H\sigma^\mp\right]$	$\mathbf{3}$, $\pm$	$H^\pm$
Z boson	$\operatorname{tr}\left[H^\dagger D_\mu H\sigma^3\right]$ DB_μ	$\mathbf{3}$, 0	Z_μ
charged W	$\varphi\ \operatorname{tr}\left[H^\dagger D_\mu H\sigma^\pm\right]$	$\mathbf{3}$, $\pm$	$W^\pm_\mu$
photon	DB_μ $\operatorname{tr}\left[H^\dagger D_\mu H\sigma^3\right]$	$\mathbf{1}$	Γ_μ
neutralino (LSP)	$\operatorname{tr}\left[H^\dagger\widetilde{H}\right]$ $\operatorname{tr}\left[H^\dagger\widetilde{H}\sigma^3\right]$, $\operatorname{tr}\left[H^\dagger\sigma^a H\sigma^3\right]\widetilde{W}_a$ $\widetilde{B}$	$\mathbf{1}$	$\widetilde{\chi}^0_3$
other neutralinos	$\operatorname{tr}\left[H^\dagger\widetilde{H}\sigma^3\right]$, $\operatorname{tr}\left[H^\dagger\sigma^a H\sigma^3\right]\widetilde{W}_a$ $\operatorname{tr}\left[H^\dagger\widetilde{H}\right]$ $\widetilde{B}$	$\mathbf{3}$, 0	$\widetilde{\chi}^0_{1,2}$
charginos	$\varphi\ \operatorname{tr}\left[H^\dagger\widetilde{H}\sigma^\mp\right]$, $\varphi\ \operatorname{tr}\left[H^\dagger\sigma^a H\sigma^\mp\right]\widetilde{W}_a$	$\mathbf{3}$, $\pm$	$\widetilde{\chi}^\pm_{1,2}$
left-handed leptons	$\varphi(H^\dagger L)_1$, $(H^\dagger L)_2$	$\mathbf{2}$	e, ν
'left-handed' sleptons	$\varphi(H^\dagger\widetilde{L})_1$, $(H^\dagger\widetilde{L})_2$	$\mathbf{2}$	$\widetilde{e}$, $\widetilde{\nu}$
right-handed electron	$\varphi\bar{e}$	$\mathbf{1}$	$\bar{e}$
'right-handed' selectron	$\varphi\widetilde{\bar{e}}$	$\mathbf{1}$	$\widetilde{\bar{e}}$
SQCD bound states	example: $\varphi\Pi^+$ (Eq. (5.46))	-	example: π^+

Table 5.8: Summary of the physical, gauge-invariant spectrum of the MSSM. The second column shows possible operators describing the respective quantum number channels. If multiple are listed vertically, the top one corresponds to the custodial symmetric case. If $SU(2)_{\mathcal{C}}$ is broken, mixing of the operators occurs. If multiple are listed horizontally, they already mix in the $SU(2)_{\mathcal{C}}$ symmetric case. Invariance under $U(1)_{\text{EM}}$ transformations is ensured by the (not necessarily universal) Dirac phase φ. Likewise, D is used to project the transversal part out of vector fields for the same reason. We introduced $\sigma^\pm \equiv \sigma^2 \pm i\sigma^1$ for brevity. The third column denotes the multiplicity of the respective operators while $SU(2)_{\mathcal{C}}$ is intact. The last column shows the corresponding fields of the perturbative, elementary spectrum. Analogous to the SM, the FMS mechanism establishes the duality between the two spectra.

6 Summary and Conclusion

The goal of this thesis was to both motivate and construct the manifestly gauge-invariant spectrum of the minimal supersymmetric standard model (MSSM) and compare it to the known perturbative spectrum using the Fröhlich-Morchio-Strocchi (FMS) mechanism. For reasons discussed in Chapter 3, such a description is not only desirable theoretically, but also crucial in answering the non-trivial question of whether the particle spectrum predicted by perturbation theory is the same as the *physical* spectrum predicted by the full theory. In our case this could be rephrased as 'whether particles predicted by the standard MSSM treatment, like the lightest supersymmetric particle (LSP), are even observable *in principle*'.

Previous research on the standard model, GUT-like theories, and the 2-Higgs-doublet-model has revealed that this is indeed a delicate issue: While the FMS mechanism in the standard model justifies the use of perturbation theory by establishing a one-to-one correspondence between the elementary and physical spectrum (c.f. Chapter 4), many GUT-like models display a qualitative mismatch. It is therefore crucial to test the perturbative MSSM spectrum as well and see if there are any discrepancies.

To answer this question, the FMS framework was applied to the MSSM in this thesis. It is the first time that such investigations have been extended to the realm of supersymmetric theories. It was discovered that, as in the standard model, there is a duality between the elementary and the inherently gauge-invariant spectrum. The findings are summarized in Table 5.8. We have gained a thorough understanding of the structure of custodial symmetry in the MSSM: It can be used to gauge-invariantly explain mass degeneracies of weak bosons and left-handed (s)leptons while intact (Sec. 5.3.1 and 5.3.2). When it breaks, it reveals how the degeneracies of the elementary states are lifted by mixing the corresponding bound state operators (Sec. 5.4). Furthermore, it appears that SUSY and the FMS construction do not interfere. This is an important result because it suggests that supersymmetric GUTs behave like regular GUTs in terms of the FMS mechanism.

Finally, this thesis does *not* impose any new constraints or limits on previously known results for the MSSM. It rather places its usual description on theoretically sound ground, confirming the 'naive' results obtained decades ago. It tells us, in particular, that experiments are searching the right quantum number channels, and particles like the LSP are indeed expected to be part of the MSSM's physical spectrum. Furthermore, as an immediate result of the FMS mechanism, the masses of the physical spectrum would remain unchanged in comparison to the masses of the elementary spectrum. Nonetheless, sub-leading FMS contributions to, e.g., cross-sections are expected, which may become relevant if any of the new particles are discovered.

Appendix A

Conventions

Pauli Matrices

$$\sigma^0 = \mathbb{1} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

Yang-Mills Theory

Below, T^a, t^a and $\mathcal{T}^a$ are generators of the fundamental, adjoint and unspecified representation, respectively. f^{abc} are the Lie algebra's structure constant (in the totally anti-symmetric basis), ϕ and Φ are fields in the fundamental and adjoint representation, g is the gauge coupling and A^a_μ and $F^a_{\mu\nu}$ are the components of the gauge field and field strength tensor. Notice that in the derivation, one has some freedom in absorbing constants and choosing signs which leads to slightly different conventions. The following is what is used in this thesis.

$$\begin{aligned}
&[\mathcal{T}^a, \mathcal{T}^b] = i f^{abc} \mathcal{T}^c && \text{(Lie algebra)} \\
&(t^a)^{bc} = -i f^{abc} && \text{(adjoint representation)} \\
&D^{ab}_\mu = \delta^{ab}\partial_\mu - igA^c_\mu(\mathcal{T}^c)^{ab} && \text{(covariant derivative)} \\
&D^{ab}_\mu \phi_b = \partial_\mu \phi_a - igA^c_\mu (T^c)^{ab}\phi_b && \text{(acting on field in fund. rep.)} \\
&D^{ab}_\mu \Phi_b = \partial_\mu \Phi_a + g f^{abc} A^b_\mu \Phi_c && \text{(acting on field in adj. rep.)} \\
&A_\mu = A^a_\mu T^a && \text{(gauge field)} \\
&F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu && \\
&F_{\mu\nu} = F^a_{\mu\nu} T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] && \text{(field strength)}
\end{aligned}$$

The matter fields, gauge fields and field strength transform under gauge-transformations as follows:

$$\begin{aligned}
&\phi \to V\phi = \exp(i\theta^a \mathcal{T}^a)\phi && \text{(matter fields)} \\
&\phi_a \to \phi_a + i\theta^b (T^b)^{ac}\phi_c && \text{(matter fields in fund. rep., infinitesimal)} \\
&\Phi_a \to \Phi_a - f^{abc}\theta^b \Phi_c && \text{(matter fields in adj. rep., infinitesimal)} \\
&A_\mu \to VA_\mu V^\dagger - \frac{i}{g}(\partial_\mu V)V^\dagger && \text{(gauge fields)} \\
&A^a_\mu \to A^a_\mu + \frac{1}{g}\partial_\mu \theta^a - f^{abc}\theta^b A^c_\mu && \text{(gauge fields, components, infinitesimal)} \\
&F_{\mu\nu} \to V(x) F_{\mu\nu} V^\dagger(x) && \text{(field strength)} \\
&F^a_{\mu\nu} \to F^a_{\mu\nu} - f^{abc}\theta^b F^c_{\mu\nu} && \text{(field strength, components, infinitesimal)}
\end{aligned}$$

Appendix B

Additional Information and Calculations

B.1 Notes on Weyl Spinors

Even though we are not going to be as pedantic in the main work, I want to point out a possible source of confusion which is the ‘sloppy’ language of using a dagger to denote right-handed Weyl spinors. It is ‘sloppy’ because left- and right-handed spinors are elements of two completely unrelated representation spaces of $SL(2,\mathbb{C})$. At least at first. We will see how we can still use this broadly used notation while keeping the exact meaning in the back of our minds.

For this, we shall adopt the precise formalism of [36], i.e. we distinguish between the representations of $SL(2,\mathbb{C})$ as follows. For all $M \in SL(2,\mathbb{C})$ we define two representations

$$\begin{aligned} D(M) &\equiv M && \text{(fundamental representation)} \\ D(M) &\equiv M^* && \text{(conjugate representation)} \end{aligned}$$

which act on representation spaces F and $\dot{F}$, respectively. Elements of these spaces are called

$$\begin{aligned} \psi_a &\in F && \text{(left-handed Weyl spinors)} \\ \bar{\psi}_{\dot{a}} &\in \dot{F} && \text{(right-handed Weyl spinors)} \end{aligned}$$

with $a, b = 1, 2$ and $\dot{a}, \dot{b} = \dot{1}, \dot{2}$. Notice that (unlike in the main text), right-handed spinors have a lower index by default in this Appendix. Furthermore, the bar should not be understood as any kind of operator: It merely serves to distinguish between left- and right-handed spinors[1]. Within their respective representation spaces they transform according to

$$\begin{aligned} \psi'_a &= M_a{}^b \psi_b \\ \bar{\psi}'_{\dot{a}} &= (M^*)_{\dot{a}}{}^{\dot{b}} \bar{\psi}_{\dot{b}} \end{aligned}$$

where summation over repeated indices of different height is performed. Dotted indices may not be contracted with undotted ones and vice versa. This is the main reason for introducing them. One can construct two more two-dimensional representations

$$\begin{aligned} D(M) &\equiv M^{-1T} && \text{(contravariant (dual) representation)} \\ D(M) &\equiv M^{*-1T} && \text{(contra-conjugate representation)} \end{aligned}$$

but it turns out that they are equivalent to the ones introduced above in the sense that they can be related by a similarity transformation

$$\begin{aligned} (M^{-1T})^a{}_d &= \epsilon^{ab} M_b{}^c \epsilon_{cd} \\ (M^{*-1T})^{\dot{a}}{}_{\dot{d}} &= \epsilon^{\dot{a}\dot{b}} (M^*)_{\dot{b}}{}^{\dot{c}} \epsilon_{\dot{c}\dot{d}}, \end{aligned}$$

[1] Later we will see that the components are connected but we will have to pile up some more formalities before we can introduce that relationship without getting caught in confusion and contradictions.

Representation	Elements	Transformation
Fundamental	$\psi_a \in F$	$\psi'_a = M_a{}^b \psi_b$
Conjugate	$\bar{\psi}_{\dot{a}} \in \dot{F}$	$\bar{\psi}'_{\dot{a}} = (M^*)_{\dot{a}}{}^{\dot{b}} \bar{\psi}_{\dot{b}}$
Contravariant	$\psi^a \in F^*$	$\psi^{a'} = \psi^b (M^{-1})_b{}^a$
Contra-conjugate	$\bar{\psi}^{\dot{a}} \in \dot{F}^*$	$\bar{\psi}^{\dot{a}'} = \bar{\psi}^{\dot{b}} (M^{*-1})_{\dot{b}}{}^{\dot{a}}$

Table B.1: Two-dimensional representations of $SL(2, \mathbb{C})$, their elements and transformations.

where we have introduced the totally anti-symmetric quantity

$$(\epsilon^{ab}) = (\epsilon^{\dot{a}\dot{b}}) = (-\epsilon_{ab}) = (-\epsilon_{\dot{a}\dot{b}}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma^2. \tag{B.1}$$

These representations (despite being equivalent to the initial ones) inhabit yet more states. We denote them and their representation spaces as follows:

$$\psi^a \equiv \epsilon^{ab} \psi_b \in F^*$$
$$\bar{\psi}^{\dot{a}} \equiv \epsilon^{\dot{a}\dot{b}} \bar{\psi}_{\dot{b}} \in \dot{F}^*$$

The stars here do *not* denote complex conjugation but F^* is the dual space of F, etc. The ϵ-symbol can be understood as a map from the space into its dual or (more conveniently) as a tool to raise or lower indices[2]. Finally, the transformation behavior is

$$\psi^{a'} = (M^{-1T})^a{}_b \psi^b = \psi^b (M^{-1})_b{}^a$$
$$\bar{\psi}^{\dot{a}'} = (M^{*-1T})^{\dot{a}}{}_{\dot{b}} \bar{\psi}^{\dot{b}} = \bar{\psi}^{\dot{b}} (M^{*-1})_{\dot{b}}{}^{\dot{a}}.$$

We thus have four different representations and representation spaces, collected in Tab. B.1, but only two are distinct, i.e. the representations on F and F^* are equivalent, and so are those on $\dot{F}$ and $\dot{F}^*$. This is still not the whole story, though. Because also elements of F and $\dot{F}^*$ are related (and $\dot{F}$ with F^*). This can be seen as follows: Define two new matrices

$$(\bar{\sigma}^\mu)^{\dot{a}b} = (\mathbb{1}, -\boldsymbol{\sigma})$$
$$(\sigma^\mu)_{a\dot{b}} = (\mathbb{1}, \boldsymbol{\sigma})$$

i.e. 4-vectors of the identity and the Pauli matrices. The crucial thing here is the index structure. As is, they can be interpreted as maps

$$(\bar{\sigma}^\mu)^{\dot{a}b} : \ F \to \dot{F}^*$$
$$(\sigma^\mu)_{a\dot{b}} : \dot{F}^* \to F.$$

In particular, we may use σ^0 (which is merely the identity) to do the following:

$$\begin{aligned}
\bar{\psi}^{\dot{a}} &= (\bar{\sigma}^0)^{\dot{a}a} (\psi_a)^* & (\psi_a)^* &= (\sigma^0)_{a\dot{a}} \bar{\psi}^{\dot{a}} \\
\psi^a &= (\bar{\psi}_{\dot{a}})^* (\bar{\sigma}^0)^{\dot{a}a} & (\bar{\psi}_{\dot{a}})^* &= \psi^a (\sigma^0)_{a\dot{a}} \\
\bar{\psi}_{\dot{a}} &= (\psi^a)^* (\sigma^0)_{a\dot{a}} & (\psi^a)^* &= \bar{\psi}_{\dot{a}} (\bar{\sigma}^0)^{\dot{a}a} \\
\psi_a &= (\sigma^0)_{a\dot{a}} (\bar{\psi}^{\dot{a}})^* & (\bar{\psi}^{\dot{a}})^* &= (\bar{\sigma}^0)^{\dot{a}a} \psi_a
\end{aligned}$$

[2]Notice that the order is important: By convention indices are raised/lowered with an ϵ from the left. Though one can also apply it from the right. The only thing to keep in mind is that this changes the sign as the ϵ-symbol is anti-symmetric.

This tells us that the complex conjugated elements of $\psi_a \in F$ transform like $\bar{\psi}^{\dot{a}}$, i.e. like an element of $\dot{F}^*$, *if* we apply the additional mapping induced by σ^0. This now finally is the reason why in much of the literature people write things like

$$(\psi_a^*) \equiv \psi^{\dagger\dot{a}}$$

and call $\psi^\dagger$ the right-handed Weyl spinors, never introducing the bar in the first place. While technically correct (in the sense that both objects have the same components) this is rather sloppy and leads to confusion if one does not exactly know what they are doing. First, the complex conjugation is to be replaced by a dagger in QFT (which only acts on the creation-/annihilation-operators in the field but performs a regular complex conjugation on everything else). Therefore, we would have the same symbol for two different things (which, on top, are *almost* identical). Another reason is that the index structure is different on both sides of the equal sign and can easily lead to confusions in calculations without σ^0. What we can do, though (and this is mostly to stay consistent with all the other literature), is the following: For the transition from classical fields to quantum fields, we replace

$$.^* \to .^\dagger$$

where the dagger is to be understood as acting on the creation-/annihilation-operators of the fields while it still acts as complex conjugation on the mode functions for example. Now you can see where it can get messy: The relations above become

$$\begin{aligned}
(\psi_a)^\dagger &= (\sigma^0)_{a\dot{a}}\bar{\psi}^{\dot{a}} && \equiv \bar{\psi}_a \\
(\bar{\psi}_{\dot{a}})^\dagger &= \psi^a(\sigma^0)_{a\dot{a}} && \equiv \psi_{\dot{a}} \\
(\psi^a)^\dagger &= \bar{\psi}_{\dot{a}}(\bar{\sigma}^0)^{\dot{a}a} && \equiv \bar{\psi}^a \\
(\bar{\psi}^{\dot{a}})^\dagger &= (\bar{\sigma}^0)^{\dot{a}a}\psi_a && \equiv \psi^{\dot{a}}
\end{aligned}$$

where we *defined* bar-ed spinors with undotted indices (and vice versa). Surely, we can always define things to be the way we would like them to be, but here it is somewhat intuitive because σ^0 is merely the identity which means that e.g. $(\psi_a)^\dagger$ has *the same entries* as $\bar{\psi}^{\dot{a}}$, just a *different index structure*! Similarly we have

$$\begin{aligned}
\bar{\psi}^{\dot{a}} &= (\bar{\sigma}^0)^{\dot{a}a}(\psi_a)^\dagger && \equiv \psi^{\dagger\dot{a}} \\
\psi^a &= (\bar{\psi}_{\dot{a}})^\dagger(\bar{\sigma}^0)^{\dot{a}a} && \equiv \bar{\psi}^{\dagger a} \\
\bar{\psi}_{\dot{a}} &= (\psi^a)^\dagger(\sigma^0)_{a\dot{a}} && \equiv \psi^\dagger_{\dot{a}} \\
\psi_a &= (\sigma^0)_{a\dot{a}}(\bar{\psi}^{\dot{a}})^\dagger && \equiv \bar{\psi}^\dagger_a
\end{aligned}$$

which is perfectly consistent with the relations above. Here we can clearly see that the information stored in a right-handed spinor $\bar{\psi}^{\dot{a}}$ can equally well be described by the conjugate of a left handed spinor $(\psi_a)^\dagger$. This finally establishes the duality

$$\begin{aligned}
(\psi_a)^\dagger &\longleftrightarrow \psi^{\dagger\dot{a}} \\
(\psi^a)^\dagger &\longleftrightarrow \psi^\dagger_{\dot{a}}
\end{aligned}$$

which is exploited throughout the literature. One might now be lead to *think* 'Why do we even care about the bar if the dagger describers the exact same thing?'. This is probably what most authors think which is why the bar notation is so rarely employed. Nevertheless, it caused me a lot of confusion which is why I wanted to write it down. On the other hand, strictly using the bar notation might also cause problems as the bar is used differently in the standard notation of the MSSM, see Sec. 5.1.

B.2 The Higgs Bidoublet as an SO(4) Representation

Both the SM and MSSM Higgs potential have an $SO(4) \cong SU(2) \times SU(2)$ symmetry[3]. One of those $SU(2)$ symmetries is gauged and becomes $SU(2)_L$, the second one remains a global symmetry called $SU(2)_{\mathcal{C}}$. Both transformations can be considered separately (and acting linearly on the Higgs) once we rewrite the theory in terms of the bidoublet (4.6) or (5.7), respectively. In the following, we will see what representation those bidoublets actually live in.

Consider an arbitrary element $g \in SO(4)$. Every such g can be decomposed as [59]

$$\begin{aligned} g &= \begin{pmatrix} \alpha_1 & -\alpha_2 & \beta_1 & -\beta_2 \\ \alpha_2 & \alpha_1 & \beta_2 & \beta_1 \\ -\beta_1 & -\beta_2 & \alpha_1 & \alpha_2 \\ \beta_2 & -\beta_1 & -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\gamma_2 & -\delta_1 & -\delta_2 \\ \gamma_2 & \gamma_1 & -\delta_2 & \delta_1 \\ \delta_1 & \delta_2 & \gamma_1 & -\gamma_2 \\ \delta_2 & -\delta_1 & \gamma_2 & \gamma_1 \end{pmatrix} \\ &= \left[(\alpha_1 \mathbb{1} + i\beta_1\sigma^2) \otimes \mathbb{1} - i(\alpha_2\sigma^3 + \beta_2\sigma^1) \otimes \sigma^2\right] \\ &\quad \left[\mathbb{1} \otimes (\gamma_1 \mathbb{1} - i\gamma_2\sigma^2) - i\sigma_2 \otimes (\delta_1\sigma^3 + \delta_2\sigma^1)\right] \\ &\equiv g_1 g_2 \end{aligned} \tag{B.2}$$

where σ^i are the Pauli matrices and $\alpha_k,\ \beta_k,\ \gamma_k,\ \delta_k$ are real coefficients satisfying

$$\begin{aligned} \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 &= 1 \\ \gamma_1^2 + \gamma_2^2 + \delta_1^2 + \delta_2^2 &= 1. \end{aligned} \tag{B.3}$$

One can easily check that g_1 and g_2 commute by exploiting the (anti-)commuting property of the Pauli matrices and the special tensor structure. This also means that

$$gg' = (g_1 g_2)(g_1' g_2') = (g_1 g_1')(g_2 g_2') = (gg')_1 (gg')_2,$$

i.e. the structure $g = g_1 g_2$ is preserved within $SO(4)$. We can now define two maps ($i = 1, 2$)

$$\begin{aligned} \lambda_i : SO(4) &\to G_i \subset SO(4) \\ g &\mapsto g_i. \end{aligned}$$

From the explicit form (B.2), one easily sees that g_i both are still orthogonal matrices, which means that both λ_i map to some subset of $SO(4)$. And because of

$$\lambda_i(gg') = (gg')_i = g_i g_i' = \lambda_i(g)\lambda_i(g'),$$

we find that the maps are homomorphisms and, consequently, that G_i are subgroups of $SO(4)$. That those are isomorphic to the two copies of $SU(2)$ that we are after, can be seen as follows: There exist similarity transformations[4], such that

$$\begin{aligned} (S_1 \circ \lambda_1)(g) = s_1 \lambda_1(g) s_1^{-1} &= \begin{pmatrix} \alpha_1{+}i\alpha_2 & \beta_1{+}i\beta_2 & 0 & 0 \\ -\beta_1{+}i\beta_2 & \alpha_1{-}i\alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_1{+}i\alpha_2 & \beta_1{+}i\beta_2 \\ 0 & 0 & -\beta_1{+}i\beta_2 & \alpha_1{-}i\alpha_2 \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta & 0 & 0 \\ -\beta^* & \alpha^* & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta^* & \alpha^* \end{pmatrix} \\ (S_2 \circ \lambda_2)(g) = s_2 \lambda_1(g) s_2^{-1} &= \begin{pmatrix} \gamma_1{+}i\gamma_2 & 0 & \delta_1{+}i\delta_2 & 0 \\ 0 & \gamma_1{+}i\gamma_2 & 0 & \delta_1{+}i\delta_2 \\ -\delta_1{+}i\delta_2 & 0 & \gamma_1{-}i\gamma_2 & 0 \\ 0 & -\delta_1{+}i\delta_2 & 0 & \gamma_1{-}i\gamma_2 \end{pmatrix} \equiv \begin{pmatrix} \gamma & 0 & \delta & 0 \\ 0 & \gamma & 0 & \delta \\ -\delta^* & 0 & \gamma^* & 0 \\ 0 & -\delta^* & 0 & \gamma^* \end{pmatrix} \end{aligned}$$

via two isomorphisms S_1 and S_2. We can now clearly see that two identical $SU(2)$ matrices each are hidden inside g_1 and g_2[5]. Therefore, $G_i \cong SU(2)$ and defining the trivial isomorphisms ($i = 1, 2$)

$$\begin{aligned} I_i : G_i &\to SU(2) \\ (S_i \circ \lambda_i)(g) &\mapsto (I_i \circ S_i \circ \lambda_i)(g) \end{aligned}$$

[3] Strictly speaking, it's even $O(4)$ but we do not worry about that in the present work.

[4] The similarity of two matrices can be checked by comparing their Jordan decomposition

[5] Because $\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$, a straight-forward calculation shows that $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$ is indeed unitary. Likewise for $\begin{pmatrix} \gamma & \delta \\ -\delta^* & \gamma^* \end{pmatrix}$.

allows us to write down two distinct complex representations[6] of $SO(4)$:

$$\begin{aligned}\Lambda_1 : SO(4) &\to SU(2)\\ g &\mapsto (I_1 \circ S_1 \circ \lambda_1)(g) = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}\\ \Lambda_2 : SO(4) &\to SU(2)\\ g &\mapsto (I_2 \circ S_2 \circ \lambda_2)(g) = \begin{pmatrix} \gamma & \delta \\ -\delta^* & \gamma^* \end{pmatrix}\end{aligned}$$

These two representations act on $V_1, V_2 = \mathbb{C}^2$ and we can use them to naturally construct a representation $\rho(g)$ on $\mathrm{Hom}(V_2, V_1) \cong \mathrm{Mat}(2 \times 2, \mathbb{C})$ via [60]

$$\rho(g)A \equiv \Lambda_1(g)\Theta\Lambda_2(g)^{-1}, \quad \Theta \in \mathrm{Mat}(2 \times 2, \mathbb{C}).$$

The bidoublet (5.7) is just a complex (2×2)-matrix like Θ. Hence, the representation space of the MSSM Higgs (which has 8 real degrees of freedom) is precisely $\mathrm{Mat}(2 \times 2, \mathbb{C})$. The $SU(2)$ acting from the left correspond to weak gauge transformations and custodial $SU(2)$ transformations act from the right. Since $\Lambda_i(g)$ are $SU(2)$ matrices, we immediately see that the matrices

$$\mathcal{V} \equiv \left\{ \begin{pmatrix} y & z \\ -z^* & y^* \end{pmatrix} \middle|\, y, z \in \mathbb{C} \right\} = \{\kappa U \mid U \in SU(2),\ \kappa \in \mathbb{R}\}$$

furnish an invariant subspace[7] of $\mathrm{Mat}(2 \times 2, \mathbb{C})$. This is the representation space of the SM Higgs (4.6), which only has 4 real degrees of freedom. Every element in $v \in \mathcal{V}$ can be written as κU with an 'amplitude' κ and a 'phase' (unimodular matrix) U.

B.3 The MSSM Higgs Sector as a Special Case of the 2HDM

Two-Higgs-doublet-models (2HDMs) are usually formulated with both Higgs fields carrying the same hypercharge $Y = +1$ [53, 61]. Therefore, we have to perform a charge conjugation on H_d in order to use results from 2HDMs as explained e.g. in [55]

$$\begin{aligned}\phi_1 &= i\sigma^2 H_d^{\dagger T}\\ \phi_2 &= H_u.\end{aligned} \tag{B.4}$$

We now consider the scalar sector of the MSSM Lagrangian as introduced in Sec. 5.3.1 as well as the parameter restriction $m_u = m_d \equiv m$. Using the relations $\phi_1^\dagger\phi_1 = H_d^\dagger H_d$, $\phi_2^\dagger\phi_2 = H_u^\dagger H_u$ and $\phi_1^\dagger\phi_2 = H_2^T(i\sigma^2)H_d$, we can rephrase it as a valid 2HDM:

$$\begin{aligned}\mathcal{L}' &\supset (D_\mu\phi_i)^\dagger(D^\mu\phi_i) - V(\phi_1, \phi_2)\\ V(\phi_1, \phi_2) &= (|\mu|^2 + m^2)(\phi_1^\dagger\phi_1 + \phi_2^\dagger\phi_2) + m_{ud}^2\left[\phi_1^\dagger\phi_2 + \phi_2^\dagger\phi_1\right]\\ &\quad + \frac{g^2}{8}\left[(\phi_1^\dagger\phi_1)^2 + (\phi_2^\dagger\phi_2)^2\right] + \frac{g^2}{4}(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) - \frac{g^2}{2}(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1).\end{aligned} \tag{B.5}$$

Notice that the last two signs in the potential are opposite to the potential in terms of the fields $H_{u,d}$ in Eq. (5.13). From V we can read off

$$\begin{aligned}&\mu_1^2 = \mu_2^2 = -(|\mu|^2 + m^2) && m_{12}^2 = -m_{ud}^2 \in \mathbb{R} && \lambda_1 = \lambda_2 = \frac{g^2}{8}\\ &\lambda_3 = \frac{g^2}{4} && \lambda_4 = -\frac{g^2}{2} && \lambda_5 = \lambda_6 = \lambda_7 = 0\end{aligned}$$

[6] They are representations because we constructed them from successively applied homo- and isomorphisms.

[7] Because $SU(2)$ is closed under matrix multiplication and the constant κ (which is nothing but the determinant) does not change that behavior. In fact, $\mathcal{V}$ is still a group if one excludes $\kappa = 0$. Notice, that $\kappa \in \mathbb{R}$, since the determinant of such matrices is always real.

as the 2HDM parameters corresponding to the general form in [53][8]. Following this work, we collect both fields into a Majorana type object

$$\Phi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \\ i\sigma^2 \phi_1^{\dagger T} \\ i\sigma^2 \phi_2^{\dagger T} \end{pmatrix}$$

which allows us to rewrite (B.5) as

$$\begin{aligned} \mathcal{L}' \supset \frac{1}{2}(D_\mu \Phi)^\dagger (D^\mu \Phi) + \frac{1}{2} M_A \Phi^\dagger \Sigma^A \Phi - \frac{1}{4} L_{AB} \Phi^\dagger \Sigma^A \Phi \Phi^\dagger \Sigma^B \Phi \quad \text{with} \\ M_A = (-2(|\mu|^2 + m^2), -2m_{ud}^2, 0, 0, 0, 0) \\ L_{AB} = \operatorname{diag}\left(\frac{g^2}{2}, -\frac{g^2}{2}, -\frac{g^2}{2}, 0, 0, 0\right). \end{aligned}$$

Inserting the specific forms of M_A and L_{AB} turns the Lagrangian into

$$\begin{aligned} \mathcal{L}' \supset \frac{1}{2}(D_\mu \Phi)^\dagger (D^\mu \Phi) - (|\mu|^2 + m^2)\Phi^\dagger \Sigma^0 \Phi - m_{12}^2 \Phi^\dagger \Sigma^1 \Phi \\ - \frac{g^2}{8}(\Phi^\dagger \Sigma^0 \Phi)^2 + \frac{g^2}{8}\left[(\Phi^\dagger \Sigma^1 \Phi)^2 + (\Phi^\dagger \Sigma^2 \Phi)^2\right] \end{aligned} \tag{B.6}$$

where

$$\Sigma^0 = \frac{1}{2}\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \qquad \Sigma^1 = \frac{1}{2}\mathbb{1} \otimes \sigma^1 \otimes \mathbb{1} \qquad \Sigma^2 = \frac{1}{2}\sigma^3 \otimes \sigma^2 \otimes \mathbb{1}.$$

Notice that none of the Σ-matrices acts in gauge-space. For completeness, the covariant derivative here is

$$D_\mu \Phi = \left[\mathbb{1} \otimes \mathbb{1} \otimes \left(\mathbb{1}\partial_\mu - igW_\mu^a \frac{\sigma^a}{2}\right)\right]\Phi.$$

The kinetic part $(D_\mu \Phi)^\dagger (D^\mu \Phi)$ is $SU_M(4)$ symmetric [53], i.e. it has a restricted $SU(4)$ symmetry which turns out to be 10-dimensional and generated by

$$\begin{aligned} K^0 &= \frac{1}{2}\sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} & K^5 &= \frac{1}{2}\sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \\ K^1 &= \frac{1}{2}\sigma^3 \otimes \sigma^1 \otimes \mathbb{1} & K^6 &= \frac{1}{2}\sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \\ K^2 &= \frac{1}{2}\mathbb{1} \otimes \sigma^2 \otimes \mathbb{1} & K^7 &= \frac{1}{2}\sigma^2 \otimes \sigma^3 \otimes \mathbb{1} \\ K^3 &= \frac{1}{2}\sigma^3 \otimes \sigma^3 \otimes \mathbb{1} & K^8 &= \frac{1}{2}\sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \\ K^4 &= \frac{1}{2}\sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} & K^9 &= \frac{1}{2}\sigma^2 \otimes \sigma^1 \otimes \mathbb{1} \end{aligned} \tag{B.7}$$

This $SU_M(4)$ symmetry is broken by the scalar potential – but not completely: Our specific set of parameters[9] (almost) falls into category 10 in Table 1 of [53], which is shown in Tab. B.2. Now actually, $\mu_{12}^2 \neq 0$ as long as $m_{ud}^2 \neq 0$. This is not a problem, though, because we only break the $O(2)$ by introducing $m_{ud}^2 \neq 0$. But the theory stays $O(3)$ symmetric even after including a non-vanishing m_{ud}^2. This in turn means, that we may enhance the accidental global symmetry by requiring the absence of the soft SUSY breaking m_{ud}^2-terms.

[8]Notice that this potential is defined *without* factors $\frac{1}{2}$ in front of $\lambda_{1,2}$ but this differs across the literature.

[9]Attention: this classification applies to a (ϕ_1, ϕ_2) basis in which L_{AB} is diagonal *only*! Luckily, this is already the case here.

No.	Symmetry	μ_1^2	μ_2^2	μ_{12}^2	λ_1	λ_2	λ_3	λ_4	$\mathrm{Re}\,\lambda_5$	$\lambda_6 = \lambda_7$
10	$O(2) \times O(3)$	-	μ_1^2	0	-	λ_1	$2\lambda_1$	-	0	0

Table B.2: Excerpt of Table 1 in [53], showing the remaining symmetry for a 2HDM with the given set of parameters.

The generators K^0, K^8, K^9 in (B.7) generate $O(3)$ and K^3 generates $O(2)$

$$\Phi \xrightarrow{O(3)} \Phi' = U\Phi = \exp\bigl(i\theta^A K^A\bigr)\Phi \quad (A = 0, 8, 9)$$
$$\Phi \xrightarrow{O(2)} \Phi' = U_3\Phi = \exp\bigl(i\bar{\theta} K^3\bigr)\Phi,$$

which can be read off from Table 2 in [53]. We can use those to explicitly check, that (B.6) is invariant under the first of these transformations but only for $m_{ud}^2 = 0$ also under the second one. For convenience we define

$$\kappa^1 = -2K^8 \qquad \kappa^2 = 2K^9 \qquad \kappa^3 = -2K^0$$

which can be checked to satisfy the $SU(2)$ algebra just like the Pauli matrices. The structure of U is then as follows:

$$U = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ \gamma & 0 & 0 & \delta \end{pmatrix} \otimes \mathbb{1} = \exp\left(i\eta^C \frac{\kappa^C}{2}\right) \quad \text{where}$$
$$\tilde{U}^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \exp\left(i\eta^C \frac{\sigma^C}{2}\right) \quad (C = 1, 2, 3),$$

i.e. the rescaling had the nice effect that the center of U is already $\tilde{U}$, generated by the Pauli matrices and *the same* η^i! This is nothing more than a nice presentation and has no real practical benefit.

Now, even though the construction was such that (B.6) is invariant under U, we can do a short sanity check here: One can easily calculate that

$$[\Sigma^{0,1,2}, U] = 0,$$

i.e. the potential part of $\mathcal{L}'$ is definitely invariant. On the other hand, U is just $\mathbb{1}$ in gauge space, the only space in which D_μ acts. Consequently, we can pull it through the covariant derivative which also makes the invariance of the kinetic term trivial.

Back to the MSSM

At this point, we are done with the analysis of accidental global symmetries and we can translate the whole theory back from the 2HDM to the MSSM (i.e. have scalar fields of opposite hypercharge). Using (B.4), we can translate Φ as

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ i\sigma^2\phi_1^{\dagger T} \\ i\sigma^2\phi_2^{\dagger T} \end{pmatrix} \rightarrow H \equiv \begin{pmatrix} i\sigma^2 H_d^{\dagger T} \\ H_u \\ -H_d \\ i\sigma^2 H_u^{\dagger T} \end{pmatrix},$$

which we can use to express the Higgs part of the Lagrangian as

$$\mathcal{L} \supset \frac{1}{2}(D_\mu H)^\dagger(D^\mu H) - \frac{1}{2}(|\mu|^2 + m^2)H^\dagger H - m_{ud}^2(H^\dagger\Sigma^1 H)$$
$$- \frac{g^2}{32}(H^\dagger H)^2 + \frac{g^2}{8}[(H^\dagger\Sigma^1 H)^2 + (H^\dagger\Sigma^2 H)^2].$$

This form is concise and manifestly invariant under

$$\begin{aligned} H &\to UH \quad \text{(always)} \\ H &\to U_3 H \quad (\text{if } m_{ud}^2 = 0) \end{aligned}$$

but for what we will eventually do, a different form is more practical. Following [61], we introduce

$$H_{ud} \equiv (\phi_2, i\sigma^2\phi_1^{\dagger T}) = (H_u, i\sigma^2(i\sigma^2 H_d)) = (H_u, -H_d) = \begin{pmatrix} H_u^{(1)} & -H_d^{(1)} \\ H_u^{(2)} & -H_d^{(2)} \end{pmatrix}.$$

Then

$$\begin{aligned} \mathcal{L} &\supset \mathrm{tr}\left[(D_\mu H_{ud})^\dagger (D^\mu H_{ud})\right] - V(H_{ud}) \\ V(H_{ud}) &= (|\mu|^2 + m^2)\mathrm{tr}\left[H_{ud}^\dagger H_{ud}\right] - 2m_{ud}^2(\mathrm{Re}\det H_{ud}^\dagger) \\ &\quad - \frac{g^2}{2}\det H_{ud}^\dagger H_{ud} + \frac{g^2}{8}\mathrm{tr}\left[H_{ud}^\dagger H_{ud}\right]^2 \end{aligned}$$

can be checked to be an equivalent form of the Lagrangian. The nice thing about this representation is that now everything is expressed in a non-tensor-product-fashion and the transformations are just

$$H_{ud} \to L(x) H_{ud} R^\dagger \quad \text{with} \quad L(x) \in SU(2)_L,\ R \in SU(2)_{\mathcal{C}},$$

and the translation to the previous representation is conveniently given (thanks to the careful definition of κ^a) by

$$UH = \exp\left(i\eta^c \frac{\kappa^c}{2}\right) \leftrightarrow H_{ud} R^\dagger = H_{ud}\exp\left(-i\eta^c\frac{\sigma^c}{2}\right) = H_{ud}\tilde{U}^T.$$

And since H_{ud} just includes the initial doublets as columns, the action of $SU(2)_L$ from the left is obviously also equivalent.

Extension to the Higgsinos

Since $\widetilde{H}_i$ are the superpartners of the H_i we expect them to have the same global symmetries. That this is indeed the case can be seen by putting them into a bidoublet $\widetilde{H}_{ud} \equiv (\widetilde{H}_u, -\widetilde{H}_d)$, upon which transformations again act as

$$\widetilde{H}_{ud} \to L(x)\widetilde{H}_{ud} R^\dagger \quad \text{with} \quad L(x) \in SU(2)_L,\ R \in SU(2)_{\mathcal{C}}.$$

With $\widetilde{H}_{ud}$ and H_{ud} we can not only write the two sectors separately, but also their interaction terms in a manifestly $SU(2)_L \times SU(2)_{\mathcal{C}}$ invariant way:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}W_{\mu\nu}^a W_a^{\mu\nu} + i\lambda^{\dagger a}\bar{\sigma}^\mu (D_\mu\lambda)_a + \mathrm{tr}\left[(D_\mu H_{ud})^\dagger (D^\mu H_{ud})\right] + \mathrm{tr}\left[i\widetilde{H}_{ud}^\dagger \bar{\sigma}^\mu D_\mu \widetilde{H}_{ud}\right] \\ &\quad - \frac{g}{\sqrt{2}}\left[\mathrm{tr}\left[H_{ud}^\dagger \sigma^a \widetilde{H}_{ud}\right]\lambda_a + \lambda^{\dagger a}\mathrm{tr}\left[\widetilde{H}_{ud}^\dagger \sigma^a H_{ud}\right]\right] + \mu\left[\det\left(\widetilde{H}_{ud}\right) + \text{h.c.}\right] - V(H_{ud}) \\ V(H_{ud}) &= (|\mu|^2 + m^2)\mathrm{tr}\left[H_{ud}^\dagger H_{ud}\right] - 2m_{ud}^2(\mathrm{Re}\det H_{ud}^\dagger) + \frac{g^2}{8}\mathrm{tr}\left[H_{ud}^\dagger H_{ud}\right]^2 - \frac{g^2}{2}\det H_{ud}^\dagger H_{ud} \end{aligned}$$

Notice that we call the bidoublets H and $\widetilde{H}$ instead of H_{ud} and $\widetilde{H}_{ud}$ in the main work.

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