Functional renormalization group approach for interacting Dirac fermions

Wetterich equation applied to the Gross-Neveu-Model

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1. Fermionic-Pathintegral formalism

For the derivation of the Wetterich equation, we will be interested in the generating functional of the n-point correlation functions. These can be expressed in terms of fermionic path integrals. To understand these integrals, we define them by a grid regularization, which introduces an ultraviolet cutoff Λ. For each point x_i of the d dimensional lattice, we define $4d_\gamma$ Grassmann variables, $\phi_\alpha(x_i)$, $\bar{\phi}_\alpha(x_i)$, $\eta_\alpha(x_i)$, $\bar{\eta}_\alpha(x_i)$, acting as independent generators of a Grassmann algebra $\mathcal{A}$, where d_γ is the dimensionality of the representation used to describe the dirac fermions. We need these to model the fermionic behavior of the fields. In the continuum limit they become 4 independent Grassmann fields

$$\begin{aligned}
\phi : \{0, 1, \ldots, d_\gamma\} \times \mathbb{R}^d &\to \mathcal{A} & \bar{\phi} : \{0, 1, \ldots, d_\gamma\} \times \mathbb{R}^d &\to \mathcal{A} \\
(\alpha, x) &\mapsto \phi_\alpha(x) & (\alpha, x) &\mapsto \bar{\phi}_\alpha(x) \\
\eta : \{0, 1, \ldots, d_\gamma\} \times \mathbb{R}^d &\to \mathcal{A} & \bar{\eta} : \{0, 1, \ldots, d_\gamma\} \times \mathbb{R}^d &\to \mathcal{A} \\
(\alpha, x) &\mapsto \eta_\alpha(x) & (\alpha, x) &\mapsto \bar{\eta}_\alpha(x)
\end{aligned} \tag{1.1}$$

spanning an unaccountably infinite Grassmann algebra $\mathcal{A}$. The integral operator is also defined as the limit of such a continuum process (if the considered volume is infinite we the thermodynamic limit is performed after the continuum limit).

$$\int \mathcal{D}\bar{\phi}\mathcal{D}\phi \ := \lim_{N\to\infty} \int \prod_{i=1}^{N} \prod_{\alpha=0}^{d_\gamma} \mathrm{d}\bar{\phi}_\alpha(x_i)\, \mathrm{d}\phi_\alpha(x_i) \tag{1.2}$$

The generating functional Z is then given by the expression

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \ \mathrm{e}^{-S[\phi,\bar{\phi}] + \int \bar{\eta}\phi - \int \bar{\phi}\eta} \tag{1.3}$$

where

$$\begin{aligned}
\int \bar{\eta}\phi &:= \int \mathrm{d}^d x \sum_{\alpha=0}^{d_\gamma} \bar{\eta}_\alpha(x)\phi_\alpha(x) &&= \int \mathrm{d}^d x\, \bar{\eta}_\alpha(x)\phi_\alpha(x) \\
\int \bar{\phi}\eta &:= \int \mathrm{d}^d x \sum_{\alpha=0}^{d_\gamma} \bar{\phi}_\alpha(x)\eta_\alpha(x) &&= \int \mathrm{d}^d x\, \bar{\phi}_\alpha(x)\eta_\alpha(x)
\end{aligned} \tag{1.4}$$

and for the last equality sign, we used a summation convention (See Appendix). The Quantity $S[\phi, \bar{\phi}]$ is the action of our theory, depending on the independent Dirac spinor-fields ϕ, $\bar{\phi}$. The source terms are modeled by η, $\bar{\eta}$.
For the calculation with Grassmann valued expressions it is often handy to be aware that their algebra can be written as the direct sum of two subspaces $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$, where $\mathcal{A}^+$ is the maximal commutative sub-algebra of $\mathcal{A}$. Elements of $\mathcal{A}^+$ commute with all elements of $\mathcal{A}$ and we refer to them as c-numbers. Elements of $\mathcal{A}^-$ anti-commute with all other elements in $\mathcal{A}^-$and we sometimes refer to the them as a-numbers.

We would like to mention some noteworthy points about the generating functional $Z[\eta, \bar{\eta}]$ with respect to Grassmann calculus:

i The order of grid points and spinor indices is irrelevant for the integration, as long as the terms $\mathrm{d}\bar{\phi}_\alpha(x_i)\,\mathrm{d}\phi_\alpha(x_i)$ stay paired. Also, because it is an even dimensional grid, it holds that $\int \mathcal{D}\bar{\phi}\mathcal{D}\phi \;=\int \mathcal{D}\phi\mathcal{D}\bar{\phi}$.

ii Differentiation with respect to Grassmann variables commutes with path integration. This is due to the fact that for each finite grid, there is an even number of integration operations. We therefore assume that this also holds true for the infinite dimensional case.

iii If $S[\phi,\bar{\phi}]$ contains only field-monomials of even degree then $Z[\eta,\bar{\eta}]\in\mathcal{A}^+$ is a c-number. For a finite dimensional grid this follows from the fact that on $\mathrm{e}^{-S[\phi,\bar{\phi}]+\int\bar{\eta}\phi-\int\bar{\phi}\eta}\in\mathcal{A}^+$ an even number of integration operations is performed. We assume that this also holds true for the infinite dimensional case.

iv $Z[\eta,\bar{\eta}]$ is assumed to posses an inverse element $Z^{-1}[\eta,\bar{\eta}]\in\mathcal{A}^+$. This can be made plausible from the finite dimensional case.

v If $Z[\eta,\bar{\eta}]\in\mathcal{A}^+$ holds, single functional derivatives of $Z[\eta,\bar{\eta}]$ with respect to a single Grassmann variable live in $\mathcal{A}^-$.

After having discussed all this, quantities such as the connected generating functional

$$W[\eta,\bar{\eta}] := \ln\left(Z[\eta,\bar{\eta}]\right) \;, \tag{1.5}$$

their functional derivatives (with respect to single Grassmann variables) and the corresponding n-point functions are well defined. Furthermore, we can introduce an effective action

$$\Gamma[\psi,\bar{\psi}] := \int\bar{\eta}\psi - \int\bar{\psi}\eta \;-W[\eta,\eta] \tag{1.6}$$

with

$$\psi_\alpha(x) := \frac{\partial W[\eta,\bar{\eta}]}{\partial\bar{\eta}_\alpha(x)} \qquad\qquad \bar{\psi}_\alpha(x) := \frac{\partial W[\eta,\bar{\eta}]}{\partial\eta_\alpha(x)} \tag{1.7}$$

Due to the definition of $W[\eta,\bar{\eta}]$ one concludes that $\psi,\bar{\psi}\in\mathcal{A}^-$, which qualifies them as generators of the Algebra (this is only necessary, not sufficient). We want to assume that there is a bijective correspondence between the fields $\psi,\bar{\psi}$ and $\eta,\bar{\eta}$. Therefore the fields $\psi,\bar{\psi}$ are families of independent (alternative) generators of the sub-algebra generated by the fields $\eta,\bar{\eta}$. From this it follows that

$$\frac{\partial\psi_\alpha(x)}{\partial\psi_\beta(y)} = \frac{\partial\bar{\psi}_\alpha(x)}{\partial\bar{\psi}_\beta(y)} = \delta_{\alpha\beta}\,\delta(x-y) \tag{1.8}$$

$$\frac{\partial\bar{\psi}_\alpha(x)}{\partial\psi_\beta(y)} = \frac{\partial\psi_\alpha(x)}{\partial\bar{\psi}_\beta(y)} = 0 \tag{1.9}$$

which expresses the independence of the fields and in general

$$\frac{\partial\eta_\alpha(x)}{\partial\psi_\beta(y)} \neq 0 \qquad \frac{\partial\eta_\alpha(x)}{\partial\bar{\psi}_\beta(y)} \neq 0$$

$$\frac{\partial\bar{\eta}_\alpha(x)}{\partial\psi_\beta(y)} \neq 0 \qquad \frac{\partial\bar{\eta}_\alpha(x)}{\partial\bar{\psi}_\beta(y)} \neq 0$$

2. Derivation of the Wetterich equation for fermionic systems

2.1. RegularizedGeneratingFunctionals

To see the connection between two-point functions and the connected generating functional, an explicit derivation is given below. In order to evaluate the functional derivatives appearing, we recall definition (1.4) and obtain

$$\begin{aligned}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\int\bar{\eta}\phi &= \int \mathrm{d}^d x' \frac{\delta}{\delta\bar{\eta}_\alpha(x)}\sum_{\alpha'}\bar{\eta}_{\alpha'}(x)\phi_{\alpha'}(x') \\ &= \int \mathrm{d}^d x' \sum_{\alpha'}\delta_{\alpha\alpha'}\delta(x-x')\phi_{\alpha'}(x') = \phi_\alpha(x) \end{aligned} \tag{2.1}$$

for the derivative. Similarly, for the second derivative we have

$$\begin{aligned}\frac{\delta}{\delta\eta_\alpha(x)}\int\bar{\phi}\eta &= \int \mathrm{d}^d x' \frac{\delta}{\delta\eta_\alpha(x)}\sum_{\alpha'}\bar{\phi}_{\alpha'}(x)\eta_{\alpha'}(x') \\ &= -\int \mathrm{d}^d x' \sum_{\alpha'}\delta_{\alpha\alpha'}\delta(x-x')\bar{\phi}_{\alpha'}(x') = -\bar{\phi}_\alpha(x) \end{aligned} \tag{2.2}$$

The minus sign appears due to the changed order of Grassmann fields. Using these results we obtain for our connected generating functional.

$$\begin{aligned}\frac{\delta^2 W_k[\eta,\bar{\eta}]}{\delta\eta_\beta(y)\bar{\eta}_\alpha(x)} &= \frac{\delta}{\delta\eta_\beta(y)}\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\ln\left(Z_k[\eta,\bar{\eta}]\right) \\ &= \frac{\delta}{\delta\eta_\beta(y)}Z_k[\eta,\bar{\eta}]^{-1}\int\mathcal{D}\bar{\phi}\mathcal{D}\phi\;\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\mathrm{e}^{-S[\phi,\bar{\phi}]-\Delta S_k[\psi,\bar{\psi}]+\int\bar{\eta}\phi-\int\bar{\phi}\eta} \\ &= \frac{\delta}{\delta\eta_\beta(y)}Z_k[\eta,\bar{\eta}]^{-1}\int\mathcal{D}\bar{\phi}\mathcal{D}\phi\;\left(\frac{\delta}{\delta\bar{\eta}_\alpha(x)}\int\bar{\eta}_\alpha\phi\right)\mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\ &= \left(\frac{\delta}{\delta\eta_\beta(y)}Z_k[\eta,\bar{\eta}]^{-1}\right)\int\mathcal{D}\bar{\phi}\mathcal{D}\phi\;\phi_\alpha(x)\mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\ &\quad + Z_k[\eta,\bar{\eta}]^{-1}\frac{\delta}{\delta\eta_\beta(y)}\int\mathcal{D}\bar{\phi}\mathcal{D}\phi\;\phi_\alpha(x)\mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \end{aligned} \tag{2.3}$$

Where from the second to the third line we introduced the abbreviation

$$\tilde{S}[\phi,\bar{\phi},\eta,\bar{\eta},k] := -S[\phi,\bar{\phi}] - \Delta S_k[\psi,\bar{\psi}] + \textstyle\int\bar{\eta}\phi - \int\bar{\phi}\eta \tag{2.4}$$

The first term in brackets can be calculated via product rule and reads:

$$\frac{\delta}{\delta\eta_\beta(y)}Z_k[\eta,\bar{\eta}]^{-1} = -Z_k[\eta,\bar{\eta}]^{-2}\frac{\delta}{\delta\eta_\beta(y)}Z_k[\eta,\bar{\eta}] \tag{2.5}$$

The reversed order is due to Grassmann rules for chain differentiation, but since $Z_k[\eta,\bar{\eta}]^{-1} \in \mathcal{A}^+$ it commutes with the second term and no sign caution is necessary here.

Continuing our original derivation we obtain:

$$
\begin{aligned}
\frac{\delta^2 W_k[\eta,\bar\eta]}{\delta\eta_\beta(y)\bar\eta_\alpha(x)} &= -Z_k[\eta,\bar\eta]^{-2}\left(\int \mathcal{D}\bar\phi\mathcal{D}\phi\ \bar\phi_\beta(y)\mathrm{e}^{\tilde S[\psi,\bar\psi,\eta,\bar\eta,k]}\right)\left(\int \mathcal{D}\bar\phi\mathcal{D}\phi\ \phi_\beta(x)\mathrm{e}^{\tilde S[\psi,\bar\psi,\eta,\bar\eta,k]}\right)\\
&\quad + Z_k[\eta,\bar\eta]^{-1}\int \mathcal{D}\bar\phi\mathcal{D}\phi\ (-\phi_\alpha(x))\frac{\delta}{\delta\bar\eta_\alpha(y)}\mathrm{e}^{\tilde S[\psi,\bar\psi,\eta,\bar\eta,k]}\\
&= -Z_k[\eta,\bar\eta]^{-1}\int \mathcal{D}\bar\phi\mathcal{D}\phi\ \bar\phi_\beta(y)\mathrm{e}^{\tilde S[\psi,\bar\psi,\eta,\bar\eta,k]}\\
&\quad \times Z_k[\eta,\bar\eta]^{-1}\int \mathcal{D}\bar\phi\mathcal{D}\phi\ \phi_\alpha(x)\mathrm{e}^{\tilde S[\psi,\bar\psi,\eta,\bar\eta,k]}\\
&\quad - Z_k[\eta,\bar\eta]^{-1}\int \mathcal{D}\bar\phi\mathcal{D}\phi\ \phi_\alpha(x)\bar\phi_\beta(y)\mathrm{e}^{\tilde S[\psi,\bar\psi,\eta,\bar\eta,k]}
\end{aligned}
\tag{2.6}
$$

The resulting three terms consist of the product of one-point functions and the two-point function.

$$
\begin{aligned}
\frac{\delta^2 W_k[\eta,\bar\eta]}{\delta\eta_\beta(y)\bar\eta_\alpha(x)} &= -\left\langle\bar\phi_\beta(y)\right\rangle_k\langle\phi_\alpha(x)\rangle_k - \left\langle\phi_\beta(x)\bar\phi_\alpha(y)\right\rangle_k\\
&= \left\langle\bar\phi_\beta(y)\phi_\alpha(x)\right\rangle_k - \left\langle\bar\phi_\beta(y)\right\rangle_k\langle\phi_\alpha(x)\rangle_k
\end{aligned}
\tag{2.7}
$$

Similarly, one can derive all other combinations of derivatives with respect to regular and adjoint fields.

$$
\frac{\delta^2 W_k[\eta,\bar\eta]}{\delta\bar\eta_\beta(y)\eta_\alpha(x)} = \left\langle\phi_\beta(y)\bar\phi_\alpha(x)\right\rangle_k - \left\langle\phi_\beta(y)\right\rangle_k\langle\bar\phi_\alpha(x)\rangle_k =: G^{k,\phi\bar\phi}_{\alpha\beta}(x,y) \tag{2.8}
$$

$$
\frac{\delta^2 W_k[\eta,\bar\eta]}{\delta\eta_\beta(y)\eta_\alpha(x)} = \left\langle\bar\phi_\beta(y)\bar\phi_\alpha(x)\right\rangle_k - \left\langle\bar\phi_\beta(y)\right\rangle_k\langle\bar\phi_\alpha(x)\rangle_k =: G^{k,\bar\phi\bar\phi}_{\alpha\beta}(x,y) \tag{2.9}
$$

$$
\frac{\delta^2 W_k[\eta,\bar\eta]}{\delta\bar\eta_\beta(y)\bar\eta_\alpha(x)} = \left\langle\phi_\beta(y)\phi_\alpha(x)\right\rangle_k - \left\langle\phi_\beta(y)\right\rangle_k\langle\phi_\alpha(x)\rangle_k =: G^{k,\phi\phi}_{\alpha\beta}(x,y) \tag{2.10}
$$

Here we also defined the connected two-point correlation functions $G^{k,\phi\bar\phi}_{\alpha\beta}(x,y)$, $G^{k,\phi\phi}_{\alpha\beta}(x,y)$ and $G^{k,\bar\phi\bar\phi}_{\alpha\beta}(x,y)$. One can then observe:

$$
\frac{\delta^2 W_k[\eta,\bar\eta]}{\delta\eta_\beta(y)\bar\eta_\alpha(x)} \equiv -G^{k,\phi\bar\phi}_{\beta\alpha}(y,x) \tag{2.11}
$$

This can be summarized in a matrix notation

$$
\boldsymbol{W}^{(2)}_k[\eta,\bar\eta] = \begin{pmatrix} \dfrac{\delta^2 W_k[\eta,\bar\eta]}{\delta\bar\eta_\beta(y)\eta_\alpha(x)} & \dfrac{\delta^2 W_k[\eta,\bar\eta]}{\delta\eta_\beta(y)\bar\eta_\alpha(x)} \\ \dfrac{\delta^2 W_k[\eta,\bar\eta]}{\delta\bar\eta_\beta(y)\bar\eta_\alpha(x)} & \dfrac{\delta^2 W_k[\eta,\bar\eta]}{\delta\eta_\beta(y)\eta_\alpha(x)} \end{pmatrix} = \begin{pmatrix} G^{k,\phi\phi}_{\alpha\beta}(x,y) & -G^{k,\phi\bar\phi}_{\beta\alpha}(y,x) \\ G^{k,\phi\bar\phi}_{\alpha\beta}(x,y) & G^{k,\bar\phi\bar\phi}_{\alpha\beta}(x,y) \end{pmatrix} =: \boldsymbol{G}_k(x,y) \tag{2.12}
$$

defining the Hessian matrix of $W_k[\eta,\bar\eta]$ and the connected two-point correlator matrix $\boldsymbol{G}_k$, which is written as an 8×8 block matrix.

2.2. Wetterich-Equation

We define the generating functional $Z_k[\eta,\bar{\eta}]$ with an IR regulator term depending on the momentum shell parameter k

$$Z_k[\eta,\bar{\eta}] := \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \; \mathrm{e}^{-S[\phi,\bar{\phi}]-\Delta S_k[\psi,\bar{\psi}]+\int\bar{\eta}\phi-\int\bar{\phi}\eta} \tag{2.13}$$

where the IR regulator term is given by

$$\Delta S_k[\psi,\bar{\psi}] := \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \sum_{\alpha\beta} \bar{\phi}_\alpha(-p) R^k_{\alpha\beta}(p)\phi_\beta(p) \tag{2.14}$$

$R^k_{\alpha\beta}$ fulfills (13)-(15) in [1] (will later be investigated further). Secondly, we define the regulated effective action

$$\Gamma_k[\psi,\bar{\psi}] := \textstyle\int \bar{\eta}\psi - \int \bar{\psi}\eta - W_k[\eta,\bar{\eta}] - \Delta S_k[\psi,\bar{\psi}] \tag{2.15}$$

where

$$\psi_\alpha := \frac{\delta W_k[\eta,\bar{\eta}]}{\delta\bar{\eta}_\alpha} = \langle\phi_\alpha\rangle_k \qquad \bar{\psi}_\alpha := \frac{\delta W_k[\eta,\bar{\eta}]}{\delta\eta_\alpha} = \langle\bar{\phi}_\alpha\rangle_k \tag{2.16}$$

In the above, $\langle\cdot\rangle_k$ has to be understood as the expectation value with sources and regulator term unequal zero, hence the additional terms in the functional integral
$\exp\left(-S[\phi,\bar{\phi}] - \Delta S_k[\psi,\bar{\psi}] + \int\bar{\eta}\phi - \int\bar{\phi}\eta\right)$.

In analogy to [1] we define a logarithmic scale

$$t := \ln\left(\frac{k}{\Lambda}\right) \qquad \partial_t := k\frac{\mathrm{d}}{\mathrm{d}k} \tag{2.17}$$

Using this notation we perform the differentiation of the regulated connected generating functional:

$$\begin{aligned}
\partial_t W_k[\eta,\bar{\eta}] &= \partial_t \ln Z_k[\eta,\bar{\eta}] \\
&= Z_k[\eta,\bar{\eta}]^{-1}\partial_t Z_k[\eta,\bar{\eta}] \\
&= Z_k[\eta,\bar{\eta}]^{-1}\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; \partial_t \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\
&= Z_k[\eta,\bar{\eta}]^{-1}\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]}\, \partial_t(-\underbrace{S[\phi,\bar{\phi}]}_{\text{(i)}} - \Delta S_k[\psi,\bar{\psi}] + \underbrace{\textstyle\int\bar{\eta}\phi - \int\bar{\phi}\eta}_{\text{(ii)}}) \\
&= \left.\quad Z_k[\eta,\bar{\eta}]^{-1}\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; (-\partial_t \Delta S_k[\psi,\bar{\psi}])\, \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \quad\right\}_{\text{(I)}} \\
&\quad \left. + Z_k[\eta,\bar{\eta}]^{-1}\int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; \partial_t(\textstyle\int\bar{\eta}\phi - \int\bar{\phi}\eta)\, \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \quad\right\}_{\text{(II)}}
\end{aligned} \tag{2.18}$$

The term (i) does not depend on t (nor k respectively) but we permit a k-dependence of (ii). Note that the exponential term is a c-number, therefore we can move it to the right without worrying about additional minus signs. We Fourier transform (2.14)

$$\begin{aligned}
\Delta S_k &= \int \frac{d^4p}{(2\pi)^4} \sum_{\alpha\beta} \bar{\phi}_\alpha(-p) R^k_{\alpha\beta}(p) \phi_\beta(p) \\
&= \int \frac{\mathrm{d}^4p}{(2\pi)^4} \sum_{\alpha\beta} \left(\int \mathrm{d}^4x\, \bar{\phi}_\alpha(x)\, \mathrm{e}^{ipx} \right) R^k_{\alpha\beta}(p) \left(\int \mathrm{d}^4y\, \phi_\beta(y)\, \mathrm{e}^{-ipy} \right) \\
&= \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \bar{\phi}_\alpha(x) \left(\int \frac{d^4p}{(2\pi)^4} R^k_{\alpha\beta}(p)\, \mathrm{e}^{ip(x-y)} \right) \phi_\beta(y) \\
&= \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \bar{\phi}_\alpha(x)\, R^k_{\alpha\beta}(x-y)\, \phi_\beta(y)
\end{aligned} \tag{2.19}$$

and evaluate the two terms in (2.18) separately:

$$\begin{aligned}
(\mathrm{I}) &= -Z_k[\eta,\bar{\eta}]^{-1} \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \bar{\phi}_\alpha(x) \partial_t R^k_{\alpha\beta}(x-y)\, \phi_\beta(y)\, \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\
&= -\int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y)\, Z_k[\eta,\bar{\eta}]^{-1} \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; \bar{\phi}_\alpha(x)\, \phi_\beta(y)\, \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\
&= \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y) \left\langle \phi_\beta(y) \bar{\phi}_\alpha(x) \right\rangle_k \\
&= \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y) \left(G^{k,\phi\bar{\phi}}_{\alpha\beta}(x,y) + \left\langle \phi_\beta(y) \right\rangle_k \langle \bar{\phi}_\alpha(x) \rangle_k \right) \\
&= \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y)\, G^{k,\phi\bar{\phi}}_{\alpha\beta}(y,x) - \partial_t \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \langle \bar{\phi}_\alpha(x) \rangle_k\; R^k_{\alpha\beta}(x-y) \left\langle \phi_\beta(y) \right\rangle_k \\
&\overset{(2.14)}{\underset{(2.16)}{=}} \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y)\, G^{k,\phi\bar{\phi}}_{\alpha\beta}(x,y) - \partial_t \Delta S_k[\psi,\bar{\psi}] \\
(\mathrm{II}) &= Z_k[\eta,\bar{\eta}]^{-1} \int \mathcal{D}\bar{\phi}\mathcal{D}\phi \left(\textstyle\int \bar{\eta}\phi - \int \bar{\phi}\eta \right) \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\
&= \int \partial_t\bar{\eta}\, Z_k[\eta,\bar{\eta}]^{-1} \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; \phi\; \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} + \int \partial_t\eta\, Z_k[\eta,\bar{\eta}]^{-1} \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\; \bar{\phi}\; \mathrm{e}^{\tilde{S}[\psi,\bar{\psi},\eta,\bar{\eta},k]} \\
&\overset{(2.16)}{=} \textstyle\int (\partial_t\bar{\eta})\, \psi - \int \bar{\psi}\, (\partial_t\eta)
\end{aligned}$$

Putting everything back into equation (2.18) we obtain

$$\partial_t W_k[\eta,\bar{\eta}] = \int \mathrm{d}^dx\, \mathrm{d}^dy \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y)\, G^{k,\phi\bar{\phi}}_{\alpha\beta}(x,y) - \partial_t \Delta S_k[\psi,\bar{\psi}] + \textstyle\int (\partial_t\bar{\eta})\, \psi - \int \bar{\psi}\, (\partial_t\eta) \tag{2.20}$$

which we can finally use to derive the flow equation for the effective action:

$$
\begin{aligned}
\partial_t \Gamma_k[\psi,\bar\psi] &\overset{(2.15)}{=} \partial_t \left\{\textstyle\int \bar\eta\psi - \int \bar\psi\eta - W_k[\eta,\bar\eta] - \Delta S_k[\psi,\bar\psi]\right\} \\
&= \textstyle\int (\partial_t\bar\eta)\,\psi - \int \bar\psi\,(\partial_t\eta) - \partial_t W_k[\eta,\bar\eta] - \partial_t \Delta S_k[\psi,\bar\psi] \\
&\overset{(2.20)}{=} -\int \mathrm{d}^d x\, \mathrm{d}^d y \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x-y)\, G^{k,\phi\bar\phi}_{\alpha\beta}(x,y)
\end{aligned}
$$

$$
\partial_t \Gamma_k[\psi,\bar\psi] = -\int \mathrm{d}^d x\, \mathrm{d}^d y \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x,y)\, G^{k,\phi\bar\phi}_{\alpha\beta}(y,x) \tag{2.21}
$$

Next, we want to establish a connection between the connected 2-point function $G^{\alpha\beta}_{k,\phi\bar\phi}(y,x)$ and the effective action. To do so, we first investigate the first derivatives

$$
\frac{\delta\Gamma_k[\psi,\bar\psi]}{\delta\psi_\alpha(x)} \qquad \frac{\delta\Gamma_k[\psi,\bar\psi]}{\delta\bar\psi_\alpha(x)}
$$

of the effective action. To do so we consider consecutive derivatives

$$
\frac{\delta\Gamma_k[\psi,\bar\psi]}{\delta\psi_\alpha(x)} = \frac{\delta}{\delta\psi_\alpha(x)}\int \bar\eta\psi - \frac{\delta}{\delta\psi_\alpha(x)}\int \bar\psi\eta - \frac{\delta W_k[\eta,\bar\eta]}{\delta\psi_\alpha(x)} - \frac{\delta\Delta S_k[\psi,\bar\psi]}{\delta\psi_\alpha(x)}
$$

of the effective action separately.

$$
\begin{aligned}
\frac{\delta}{\delta\psi_\alpha(x)}\int \bar\eta\psi &= \int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \frac{\delta}{\delta\psi_\alpha(x)} \bar\eta_{\alpha'}(x')\psi_{\alpha'}(x') \\
&= \int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \left(\frac{\delta\bar\eta_{\alpha'}(x')}{\delta\psi_\alpha(x)}\psi_{\alpha'}(x') - \bar\eta_{\alpha'}(x')\delta_{\alpha\alpha'}\delta(x-x') \right) \\
&= -\bar\eta_\alpha(x) + \int \frac{\delta\bar\eta}{\delta\psi_\alpha(x)}\psi
\end{aligned} \tag{2.22}
$$

For the next term we use that ψ and $\bar\psi$ are independent generators.

$$
\begin{aligned}
\frac{\delta}{\delta\psi_\alpha(x)}\int \bar\psi\eta &= \int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \frac{\delta}{\delta\psi_\alpha(x)} \bar\psi_{\alpha'}(x')\eta_{\alpha'}(x') \\
&= -\int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \bar\psi_{\alpha'}(x') \frac{\delta\eta_{\alpha'}(x')}{\delta\psi_\alpha(x)} \\
&= -\int \bar\psi \frac{\delta\eta}{\delta\psi_\alpha(x)}
\end{aligned} \tag{2.23}
$$

Using the chain rule for (Grassmann valued) functionals, we rewrite the derivative of the connected generating functional.

$$\frac{\delta}{\delta\psi_\alpha(x)}W_k[\eta,\bar\eta] = \int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \frac{\delta\eta'_\alpha(x')}{\delta\psi_\alpha(x)}\frac{\delta W_k[\eta,\bar\eta]}{\delta\eta_\alpha(x')} + \int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \frac{\delta\bar\eta'_\alpha(x')}{\delta\psi_\alpha(x)}\frac{\delta W_k[\eta,\bar\eta]}{\delta\bar\eta_\alpha(x')}$$
$$\overset{(2.16)}{=}\int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \frac{\delta\eta'_\alpha(x')}{\delta\psi_\alpha(x)}\bar\psi_{\alpha'}(x') + \int \mathrm{d}^d x' \sum_{\alpha'=0}^{3} \frac{\delta\bar\eta'_\alpha(x')}{\delta\psi_\alpha(x)}\psi_{\alpha'}(x')$$

Using the fact that derivatives of a-numbers behave like c-numbers, we can interchange the terms in the first integral. Reintroducing the shorthand notation, we obtain.

$$\frac{\delta}{\delta\psi_\alpha(x)}W_k[\eta,\bar\eta] = \int \bar\psi\frac{\delta\eta}{\delta\psi_\alpha(x)} + \int \frac{\delta\bar\eta}{\delta\psi_\alpha(x)}\psi \tag{2.24}$$

Taking on the regulator term yields

$$\begin{aligned}
\frac{\delta}{\delta\psi_\alpha(x)}\Delta S_k[\psi,\bar\psi] &= \int \mathrm{d}^d x'\, \mathrm{d}^d y' \sum_{\alpha'\beta'} \frac{\delta}{\delta\psi_\alpha(x)}\bar\psi_{\alpha'}(x')\, R^k_{\alpha'\beta'}(x',y')\, \psi_{\beta'}(y') \\
&= -\int \mathrm{d}^d x'\, \mathrm{d}^d y' \sum_{\alpha'\beta'} \bar\psi_{\alpha'}(x')\, R^k_{\alpha'\beta'}(x',y')\, \frac{\delta\psi_{\beta'}(y')}{\delta\psi_\alpha(x)} \\
&= -\int \mathrm{d}^d x'\, \mathrm{d}^d y' \sum_{\alpha'\beta'} \bar\psi_{\alpha'}(x')\, R^k_{\alpha'\beta'}(x',y')\, \delta_{\beta'\alpha}\delta(y'-x) \\
&= -\int \mathrm{d}^d x' \sum_{\alpha'} \bar\psi_{\alpha'}(x')\, R^k_{\alpha'\alpha}(x',x)
\end{aligned} \tag{2.25}$$

where again, the independence of the generators $\psi,\bar\psi$ was used. Putting all this together gives the fairly simple result

$$\frac{\delta\Gamma_k[\psi,\bar\psi]}{\delta\psi_\alpha(x)} = -\bar\eta_\alpha(x) + \int \mathrm{d}^d x' \sum_{\alpha'} \bar\psi_{\alpha'}(x')\, R^k_{\alpha'\alpha}(x',x) \tag{2.26}$$

In a similar fashion, the result for the other derivative is obtained:

$$\frac{\delta\Gamma_k[\psi,\bar\psi]}{\delta\bar\psi_\alpha(x)} = -\eta_\alpha(x) - \int \mathrm{d}^d y' \sum_{\beta'} R^k_{\alpha\beta'}(x,y')\, \psi_{\beta'}(y') \tag{2.27}$$

These equations can be interpreted as the quantum equations of motion. With this result, we can now calculate the derivative of the source terms with respect to the fields $\psi,\bar\psi$.

$$\begin{aligned}
\frac{\delta\bar\eta_\alpha(x)}{\delta\bar\psi_\beta(y)} &= -\frac{\delta^2\Gamma_k[\psi,\bar\psi]}{\delta\bar\psi_\beta(y)\delta\psi_\alpha(x)} + \int \mathrm{d}^d x' \sum_{\alpha'} \frac{\delta\bar\psi_{\alpha'}(x')}{\bar\psi_\beta(y)}\, R^k_{\alpha'\alpha}(x',x) \\
&= -\frac{\delta^2\Gamma_k[\psi,\bar\psi]}{\delta\bar\psi_\beta(y)\delta\psi_\alpha(x)} + \int \mathrm{d}^d x' \sum_{\alpha'} \delta_{\alpha'\beta}\, \delta(y-x')\, R^k_{\alpha'\alpha}(x',x) \\
&= -\frac{\delta^2\Gamma_k[\psi,\bar\psi]}{\delta\bar\psi_\beta(y)\delta\psi_\alpha(x)} + R^k_{\beta\alpha}(y,x)
\end{aligned} \tag{2.28}$$

Similar results hold for the other derivatives

$$\frac{\delta\bar{\eta}_\alpha(x)}{\delta\psi_\beta(y)} = -\frac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\psi_\beta(y)\delta\psi_\alpha(x)} \tag{2.29}$$

$$\frac{\delta\eta_\alpha(x)}{\delta\bar{\psi}_\beta(y)} = -\frac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\bar{\psi}_\beta(y)\delta\bar{\psi}_\alpha(x)} \tag{2.30}$$

$$\frac{\delta\eta_\alpha(x)}{\delta\psi_\beta(y)} = -\frac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\psi_\beta(y)\delta\bar{\psi}_\alpha(x)} - R^k_{\alpha\beta}(x,y)\ . \tag{2.31}$$

Note the order of spinor indices and arguments in the regulator terms. Introducing the Hessian (block) matrix of $\Gamma_k[\psi,\bar{\psi}]$

$$\mathbf{\Gamma}^{(2)}_k[\eta,\bar{\eta}](x,y) := \begin{pmatrix} \dfrac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\psi_\beta(y)\delta\psi_\alpha(x)} & \dfrac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\bar{\psi}_\beta(y)\delta\psi_\alpha(x)} \\ \dfrac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\psi_\beta(y)\delta\bar{\psi}_\alpha(x)} & \dfrac{\delta^2\Gamma_k[\psi,\bar{\psi}]}{\delta\bar{\psi}_\beta(y)\delta\bar{\psi}_\alpha(x)} \end{pmatrix} \tag{2.32}$$

and the regulator matrix

$$\boldsymbol{R}_k(x,y) := \begin{pmatrix} 0 & -R^k_{\beta\alpha}(y,x) \\ R^k_{\alpha\beta}(x,y) & 0 \end{pmatrix} \tag{2.33}$$

the equations (2.28) - (2.31) can be written as the 8×8 matrix equation

$$\mathbf{\Gamma}^{(2)}_k[\eta,\bar{\eta}](x,y) + \boldsymbol{R}_k(x,y) = -\begin{pmatrix} \dfrac{\delta\bar{\eta}_\alpha(x)}{\delta\psi_\beta(y)} & \dfrac{\delta\bar{\eta}_\alpha(x)}{\delta\bar{\psi}_\beta(y)} \\ \dfrac{\delta\eta_\alpha(x)}{\delta\psi_\beta(y)} & \dfrac{\delta\eta_\alpha(x)}{\delta\bar{\psi}_\beta(y)} \end{pmatrix} . \tag{2.34}$$

Note that we have three types of indices now: the discrete spinor indices, the continuous position index and the new column and row indices in the Hessian, which we shall call field indices. By the use of (2.16) one can write

$$\boldsymbol{W}^{(2)}_k[\eta,\bar{\eta}] = \begin{pmatrix} \dfrac{\delta\psi_\alpha(x)}{\delta\bar{\eta}_\beta(y)} & \dfrac{\delta\psi_\alpha(x)}{\delta\eta_\beta(y)} \\ \dfrac{\delta\bar{\psi}_\alpha(x)}{\delta\bar{\eta}_\beta(y)} & \dfrac{\delta\bar{\psi}_\alpha(x)}{\delta\eta_\beta(y)} \end{pmatrix} = \begin{pmatrix} G^{k,\phi\phi}_{\alpha\beta}(x,y) & -G^{k,\phi\bar{\phi}}_{\beta\alpha}(y,x) \\ G^{k,\phi\bar{\phi}}_{\alpha\beta}(x,y) & G^{k,\bar{\phi}\bar{\phi}}_{\alpha\beta}(x,y) \end{pmatrix} = \boldsymbol{G}_k(x,y) \tag{2.35}$$

which already hints that $-\boldsymbol{G}_k(x,y)$ might be the inverse of $\mathbf{\Gamma}^{(2)}_k[\eta,\bar{\eta}](x,y) + \boldsymbol{R}_k(x,y)$. That this indeed is the case follows from the relations

$$\begin{aligned}\delta_{\alpha\beta}\,\delta(x-y) &= \frac{\delta\bar{\eta}_\alpha(x)}{\delta\bar{\eta}_\beta(y)} \\ &= \int \mathrm{d}^d z \sum_{\alpha'=0}^{3} \frac{\delta\bar{\eta}_\alpha(x)}{\delta\psi_{\alpha'}(z)}\,\frac{\delta\psi'_\alpha(z)}{\delta\bar{\eta}_\beta(y)} + \sum_{\alpha'=0}^{3} \frac{\delta\bar{\eta}_\alpha(x)}{\delta\bar{\psi}_{\alpha'}(z)}\,\frac{\delta\bar{\psi}'_\alpha(z)}{\delta\bar{\eta}_\beta(y)} \\ &= -\int \mathrm{d}^d z \left\{\left(\boldsymbol{\Gamma}_k^{(2)}[\eta,\bar{\eta}] + \boldsymbol{R}_k\right)(x,z)\,\boldsymbol{G}_k(z,y)\right\}^{11}_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}\delta_{\alpha\beta}\,\delta(x-y) &= \frac{\delta\bar{\eta}_\alpha(x)}{\delta\bar{\eta}_\beta(y)} \\ &= -\int \mathrm{d}^d z \left\{\left(\boldsymbol{\Gamma}_k^{(2)}[\eta,\bar{\eta}] + \boldsymbol{R}_k\right)(x,z)\,\boldsymbol{G}_k(z,y)\right\}^{22}_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}0 &= \frac{\delta\bar{\eta}_\alpha(x)}{\delta\eta_\beta(y)} \\ &= -\int \mathrm{d}^d z \left\{\left(\boldsymbol{\Gamma}_k^{(2)}[\eta,\bar{\eta}] + \boldsymbol{R}_k\right)(x,z)\,\boldsymbol{G}_k(z,y)\right\}^{12}_{\alpha\beta}\end{aligned}$$

$$\begin{aligned}0 &= \frac{\delta\eta_\alpha(x)}{\delta\bar{\eta}_\beta(y)} \\ &= -\int \mathrm{d}^d z \left\{\left(\boldsymbol{\Gamma}_k^{(2)}[\eta,\bar{\eta}] + \boldsymbol{R}_k\right)(x,z)\,\boldsymbol{G}_k(z,y)\right\}^{21}_{\alpha\beta}\end{aligned}$$

which can be summarized into the matrix equation

$$\delta(x-y)\begin{pmatrix} -\delta_{\alpha\beta} & 0 \\ 0 & -\delta_{\alpha\beta}\end{pmatrix} = \int \mathrm{d}^d z\,\left(\boldsymbol{\Gamma}_k^{(2)}[\eta,\bar{\eta}] + \boldsymbol{R}_k\right)(x,z)\,\boldsymbol{G}_k(z,y) \tag{2.36}$$

or the operator equation

$$\boldsymbol{G}_k = -\left(\boldsymbol{\Gamma}_k^{(2)}[\eta,\bar{\eta}] + \boldsymbol{R}_k\right)^{-1} \tag{2.37}$$

If we now exploit

$$\partial_t \boldsymbol{R}_k(x,y)\,\boldsymbol{G}_k(x,y) = \begin{pmatrix} -\sum_{\alpha'} \partial_t R^k_{\alpha'\alpha}(y,x)\,G^{k,\phi\bar{\phi}}_{\alpha'\beta}(x,y) & * \\ * & -\sum_{\alpha'} \partial_t R^k_{\alpha\alpha'}(x,y)\,G^{k,\phi\bar{\phi}}_{\beta\alpha'}(y,x)\end{pmatrix} \tag{2.38}$$

and

$$\begin{aligned}\int \mathrm{d}^d x\,\mathrm{d}^d y\,\mathrm{tr}\left\{\partial_t \boldsymbol{R}_k(x,y)\,\boldsymbol{G}_k(x,y)\right\} &= -\int \mathrm{d}^d x\,\mathrm{d}^d y\,\sum_{\alpha'\alpha} \partial_t R^k_{\alpha'\alpha}(y,x)\,G^{k,\phi\bar{\phi}}_{\alpha'\alpha}(x,y) \\ &\quad -\int \mathrm{d}^d x\,\mathrm{d}^d y\,\sum_{\alpha'\alpha} \partial_t R^k_{\alpha\alpha'}(x,y)\,G^{k,\phi\bar{\phi}}_{\alpha\alpha'}(y,x) \\ &= -2\int \mathrm{d}^d x\,\mathrm{d}^d y \sum_{\alpha'\alpha}\,\partial_t R^k_{\alpha'\alpha}(y,x)\,G^{k,\phi\bar{\phi}}_{\alpha'\alpha}(x,y)\end{aligned} \tag{2.39}$$

then we get from (2.21) the following result

$$\begin{aligned}
\partial_t \Gamma_k[\psi, \bar{\psi}] &= -\int \mathrm{d}^d x\, \mathrm{d}^d y \sum_{\alpha\beta} \partial_t R^k_{\alpha\beta}(x,y)\, G^{k,\phi\bar{\phi}}_{\alpha\beta}(y,x) \\
&= \frac{1}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{tr}\,\{\partial_t \boldsymbol{R}_k(x,y)\, \boldsymbol{G}_k(x,y)\} \\
&= -\frac{1}{2} \int \mathrm{d}^d x\, \mathrm{d}^d y\, \mathrm{tr}\left\{\partial_t \boldsymbol{R}_k(x,y)\, \left(\boldsymbol{\Gamma}^{(2)}_k[\eta, \bar{\eta}] + \boldsymbol{R}_k\right)^{-1}\right\} \\
&= -\frac{1}{2} \mathrm{Str}\left\{\partial_t \boldsymbol{R}_k(x,y)\, \left(\boldsymbol{\Gamma}^{(2)}_k[\eta, \bar{\eta}] + \boldsymbol{R}_k\right)^{-1}\right\}
\end{aligned}$$

which is know as the Wetterich equation.

$$\partial_t \Gamma_k[\psi, \bar{\psi}] = -\frac{1}{2} \mathrm{Str}\left\{\partial_t \boldsymbol{R}_k(x,y)\, \left(\boldsymbol{\Gamma}^{(2)}_k[\eta, \bar{\eta}] + \boldsymbol{R}_k\right)^{-1}\right\} \tag{2.40}$$

The super trace operation $\mathrm{Str}\,\{\cdot\}$ takes on all indices and all degrees of freedom, the continuous position (or momentum) degrees, the field indices and the spinor indices.

3. Gross-Neveu-Model

For our studies, we consider the so called Gross-Neveu-Model, a self-interaction model consisting of N different flavours of massless fermions. This model is characterized by the bare action

$$\Gamma = \int \mathrm{d}^d x \left\{ \bar{\psi} i \mathrm{D} \psi + \frac{1}{2} \lambda \left(\bar{\psi} \psi \right)^2 \right\} \tag{3.1}$$

with the expressions

$$\psi = \left(\psi_1^{(1)}, \ldots, \psi_{d_\gamma}^{(1)}, \psi_1^{(2)}, \ldots, \psi_{d_\gamma}^{(N_f)} \right)^T = \oplus_{i=1}^{N_f} \psi^i \tag{3.2}$$

$$\mathrm{D} = \begin{pmatrix} \partial\!\!\!/ & & \\ & \ddots & \\ & & \partial\!\!\!/ \end{pmatrix} \tag{3.3}$$

$$\bar{\psi}\psi = \sum_{k=1}^{N_f} \bar{\psi}^{(k)} \psi^{(k)} = \sum_{k=1}^{N_f} \sum_{\alpha=1}^{d_\gamma} \bar{\psi}_\alpha^{(k)} \psi_\alpha^{(k)} \tag{3.4}$$

including all summations over spinor, flavor and Lorentz indices. Conventions for the different degrees of freedom are chosen as follows:

1. Spinor indices: $\alpha, \beta, \gamma \in \{1, ..., d_\gamma\}$, Summation over lower-lower pairs.
2. Flavor indices: $a, b, c \in \{1, ..., N_f\}$, Summation over upper-upper pairs.
3. Lorentz indices: $\mu, \nu, \eta \in \{1, ..., d\}$, Summation over upper-lower pairs .

Using Einstein convention, writing out the first term in the action reads explicitly:

$$\bar{\psi} i \mathrm{D} \psi = i \bar{\psi}_\alpha^a \gamma_{\alpha\beta}^\mu \partial_\mu \psi_\beta^a = i \bar{\psi}_\alpha^a \partial\!\!\!/_{\alpha\beta} \psi_\beta^a = i \bar{\psi}^a \partial\!\!\!/ \psi^a \tag{3.5}$$

Before transitioning to a scale-dependent effective action for our Wetterich equation, a few points about relevant phase transformations of the action are due, as they heavily impact approximations in the effective action. Firstly, the action of the Gross-Neveu model is in invariant under global $U(N)$ transformations and individual phase transformations $U(1)^{\bigotimes N}$,

$$\psi^{(k)} \mapsto e^{i\alpha_k} \psi^{(k)}, \; \bar{\psi}^{(k)} \mapsto \bar{\psi}^{(k)} e^{-i\alpha_k}, \; k = 1, ..., N_f \tag{3.6}$$

conserving charge in each individual flavor-sector separately. Furthermore, there exists a discrete invariance under synchronous $\mathbb{Z}_2^5 = \{\mathbb{1}, \gamma_5\}$ transformations.

$$\psi \mapsto \gamma_5 \psi, \; \bar{\psi} \mapsto -\bar{\psi} \gamma_5 \tag{3.7}$$

We now want to change to an effective action, introducing a scale dependence on the system. The following considerations are motivated by the desire to obtain a Wetterich equation that can be solved analytically and expanding on the bare action. This motivation is also reflected in the specific choice of the regulator later on.

Starting off with the potential term, we assume that it can be formulated as a Taylor expansion:

$$V_k[\bar{\psi}, \psi] = \sum_n \frac{V_k^{(n)}}{n!}(\bar{\psi}\psi - \bar{\psi}_0\psi_0)^n \tag{3.8}$$

The terms $V_k^{(n)}$ are proportional to the couplings, $\bar{\psi}_0\psi_0$ represents the scale-independent point of expansion we immediately set to zero for convenience purposes. We will only use terms up to the second order, providing us with a mechanism of chiral symmetry breaking. With $V_k^{(2)} = \lambda_k$ we have

$$V_k(x) = \frac{1}{2}\lambda_k(\bar{\psi}\psi)^2 + O\left((\bar{\psi}\psi)^4\right) \tag{3.9}$$

with our scale-dependent four-fermion coupling λ_k entering into the flow equation later. As far as the kinetic term is concerned, it is associated with a first order derivative expansion in the so called anomalous fermion field dimension $\eta_k = -\partial_t lnZ_k$, justified for small values. This implies that higher order derivative terms $\propto (\bar{\psi}iD\psi)^n$ for $n \geq 2$ are neglected. Within our consideration of point-like interactions, the wave-function renormalization Z_k turns out to be constant (see [2]) and we can later set $Z_k = 1$.

One may investigate if additional four-fermion couplings arising from e.g. vector interaction $(\bar{\psi}\gamma_\nu\psi)$ are compatible with our assumed symmetries. Care is necessary in this study, as Fierz transformations allow transformation of combinations of interaction channels into other ones. The idea is that the Clifford algebra can be decomposed into a basis such that corresponding bilinear forms in the fields transform in specific ways under Lorentz transformations. A certain ambiguity can arise, where e.g the combination of scalar and pseudoscalar channels can be transformed into vector and pseudovector channels in the NJL model for $d = 4$. This is a purely algebraic problem, heavily dependent on the dimension of our effective action, and its study and implications are omitted due to time constraints and the practicality of a scalar interaction in analytical calculations.

To sum up, all considerations so far lead to the following effective action compatible with phase and discrete invariances $U(N)$ and $\mathbb{Z}_2^5$, Lorentz invariance and minimal channels in lowest order potential and derivative expansion:

$$\Gamma_k[\bar{\psi}, \psi] = \int \mathrm{d}^d x \left\{ Z_k\bar{\psi}i\mathrm{D}\psi + \frac{1}{2}\lambda_k\left(\bar{\psi}\psi\right)^2 \right\} \tag{3.10}$$

4. Derivation of the flow equation

4.1. Second functional derivative of effective action

We want to calculate the momentum representation of our Hessian of Γ_k. In doing so, we explicitly split Γ_k into its kinetic term $\Gamma_{k,kin}$ and potential term $\Gamma_{k,pot}$. For arbitrary spacetime-dimension d, the kinetic part is given by

$$\Gamma_{k,kin}[\bar{\psi},\psi] = \int \mathrm{d}^d x\, \bar{\psi}(x)\, \mathrm{iD}\, \psi(x) \tag{4.1}$$

$$= \int \mathrm{d}^d x\, \bar{\psi}^a_\alpha(x)\, \mathrm{i}\partial\!\!\!/_{\alpha\alpha'}\, \psi^a_{\alpha'}(x) \tag{4.2}$$

$$= \int \mathrm{d}^d x \int \frac{\mathrm{d}^d q}{(2\pi)^d} \int \frac{\mathrm{d}^d q'}{(2\pi)^d}\, \mathrm{i}\, \bar{\psi}^a_\alpha(q) \mathrm{e}^{iqx}\, \mathrm{i}\partial\!\!\!/_{\alpha\alpha'}\, \psi^a_{\alpha'}(q') \mathrm{e}^{iq'x} \tag{4.3}$$

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d} \int \frac{\mathrm{d}^d q'}{(2\pi)^d}\, \bar{\psi}^a_\alpha(q)\, (\mathrm{i})^2 q\!\!\!/'_{\alpha\alpha'}\, \psi^a_{\alpha'}(q') \int \mathrm{d}^d x\, \mathrm{e}^{i(q+q')x} \tag{4.4}$$

$$= -\int \frac{\mathrm{d}^d q}{(2\pi)^d} \int \frac{\mathrm{d}^d q'}{(2\pi)^d}\, \bar{\psi}^a_\alpha(q)\, q\!\!\!/'_{\alpha\alpha'}\, \psi^a_{\alpha'}(q')\, (2\pi)^d \delta(q+q') \tag{4.5}$$

$$= \int \frac{\mathrm{d}^d q}{(2\pi)^d}\, \bar{\psi}^a_\alpha(q)\, q\!\!\!/_{\alpha,\alpha'}\, \psi^a_{\alpha'}(-q) \tag{4.6}$$

This term can now be differentiated twice easily. Note that the diagonal terms of $\Gamma^{(2)}_{k,kin}$ necessarily vanish. We are left with

$$\frac{\delta \Gamma_{k,kin}[\bar{\psi},\psi]}{\delta \psi^c_\gamma(p)} = -\int \frac{\mathrm{d}^d q}{(2\pi)^d}\, \bar{\psi}^a_\alpha(q)\, q\!\!\!/_{\alpha\alpha'}\, \frac{\delta \psi^a_{\alpha'}(-q)}{\delta \psi^c_\gamma(p)} \tag{4.7}$$

$$= -\int \frac{\mathrm{d}^d q}{(2\pi)^d}\, \bar{\psi}^a_\alpha(q)\, q\!\!\!/_{\alpha\alpha'}\, \delta^{ac} \delta_{\gamma\alpha'}\, (2\pi)^d \delta(p+q) \tag{4.8}$$

$$= \bar{\psi}^c_\alpha(-p)\, p\!\!\!/_{\alpha\gamma} \tag{4.9}$$

$$\frac{\delta^2 \Gamma_{k,kin}[\bar{\psi},\psi]}{\delta \bar{\psi}^{c'}_{\gamma'}(p') \delta \psi^c_\gamma(p)} = \frac{\delta \bar{\psi}^c_\alpha(-p)}{\delta \bar{\psi}^{c'}_{\gamma'}(p')}\, p\!\!\!/_{\alpha\gamma} \tag{4.10}$$

$$= (2\pi)^d \delta(p+p')\, \delta^{cc'} \delta_{\alpha\gamma'}\, p\!\!\!/_{\alpha\gamma} \tag{4.11}$$

$$= \delta^{cc'}\, p\!\!\!/_{\gamma'\gamma}\, (2\pi)^d \delta(p+p') \tag{4.12}$$

Note that we have used the convention

$$\frac{\delta \bar{\psi}^c_\gamma(p)}{\delta \bar{\psi}^{c'}_{\gamma'}(p')} = \delta^{cc'}\, \delta_{\gamma\gamma'}\, (2\pi)^d \delta(p-p') \tag{4.13}$$

where the factor $(2\pi)^d$ arises from our conventional definition of Fourier transformations of functionals, see Appendix[].

To obtain the missing derivative we use the anti-commutation relation for Graßmann derivatives.

$$\frac{\delta^2\Gamma_{k,kin}[\bar{\psi},\psi]}{\delta\psi^{c'}_{\gamma'}(p')\delta\bar{\psi}^{c}_{\gamma}(p)} = -\frac{\delta^2\Gamma_{k,kin}[\bar{\psi},\psi]}{\delta\psi^{c}_{\gamma}(p)\delta\bar{\psi}^{c'}_{\gamma'}(p')} \tag{4.14}$$

$$= -\,\delta^{c'c}\ \not{p}'_{\gamma\gamma'}\ (2\pi)^d\delta(p'+p) \tag{4.15}$$

$$= \delta^{c'c}\ \not{p}_{\gamma\gamma'}\ (2\pi)^d\delta(p+p') \tag{4.16}$$

This can be written in matrix form as

$$\boldsymbol{\Gamma}^{(2)}_{k,kin}[\bar{\psi},\psi] = \begin{pmatrix} \dfrac{\delta^2\Gamma_{k,kin}[\bar{\psi},\psi]}{\delta\psi^{c'}_{\gamma'}(p')\delta\psi^{c}_{\gamma}(p)} & \dfrac{\delta^2\Gamma_{k,kin}[\bar{\psi},\psi]}{\delta\bar{\psi}^{c'}_{\gamma'}(p')\delta\psi^{c}_{\gamma}(p)} \\ \dfrac{\delta^2\Gamma_{k,kin}[\bar{\psi},\psi]}{\delta\psi^{c'}_{\gamma'}(p')\delta\bar{\psi}^{c}_{\gamma}(p)} & \dfrac{\delta^2\Gamma_{k,kin}[\bar{\psi},\psi]}{\delta\bar{\psi}^{c'}_{\gamma'}(p')\delta\bar{\psi}^{c}_{\gamma}(p)} \end{pmatrix} \tag{4.17}$$

$$= \begin{pmatrix} 0 & \delta^{cc'}\ \not{p}_{\gamma'\gamma} \\ \delta^{c'c}\ \not{p}_{\gamma\gamma'} & 0 \end{pmatrix} (2\pi)^d\delta(p+p') \tag{4.18}$$

$$\tag{4.19}$$

We can reformulate this using the unity matrix in the flavor-subspace and by introducing the matrix operator acting on the spinor-flavor-product-space

$$P = \not{p} \otimes \mathbb{1}_{N_f} \tag{4.20}$$

by the usage of the Kronecker product for matrices. With this we can write

$$\Gamma^{(2)}_{k,kin}[\bar{\psi},\psi] = \begin{pmatrix} 0 & P^T \\ P & 0 \end{pmatrix} (2\pi)^d\delta(p+p') \tag{4.21}$$

In this expression one can clearly see the structure in terms of the sub-spaces corresponding to the different degrees of freedoms. The operator P factors in terms of the sub-spaces for flavor and spinor degrees of freedoms. Its dimensionality depends on the dimension of the representation of the Dirac algebra and the number of flavors. Since the spinor and the adjoint spinor are independent objects, the degrees of freedoms are doubled. We see that we cannot factor the operator P out of the operator acting on the spinor-flavor-Dirac-subspace due to the transpositon. Nevertheless, we can factor out the infinite-dimensional contribution from momentum space. This expression is also sort of diagonal in momentum space (it becomes diagonal if we index the adjoint components with $-P'$ instead, which is also meaningful from a physical point of view and might be used later).

Next we investigate the potential term $\Gamma_{k,pot}$ of the effective action:

$$\begin{aligned} \frac{\delta\Gamma_{k,pot}[\bar{\psi},\psi]}{\delta\psi^{c}_{\gamma}(y)} &= \int \mathrm{d}^d x\ \frac{\delta}{\delta\psi^{c}_{\gamma}(y)}\frac{1}{2}(\bar{\psi}^a_\alpha\psi^a_\alpha)^2(x) \\ &= -\int \mathrm{d}^d x\ \bar{\psi}^c_\gamma(x)\delta(x-y)\ (\bar{\psi}^a_\alpha\psi^a_\alpha)(x) \\ &= -\,\bar{\psi}^c_\gamma(y)\ (\bar{\psi}^a_\alpha\psi^a_\alpha)(y) \end{aligned} \tag{4.22}$$

For the adjoint fields we proceed similarly, with a sign change due to interchange of Grassmann fields:

$$\begin{aligned}\frac{\delta\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\bar\psi^c_\gamma(y)} &= \int \mathrm{d}^d x\ \frac{\delta}{\delta\bar\psi^c_\gamma(y)}\frac{1}{2}(\bar\psi^a_\alpha\psi^a_\alpha)^2(x)\\ &= \int \mathrm{d}^d x\ \psi^c_\gamma(x)\delta(x-y)\ (\bar\psi^a_\alpha\psi^a_\alpha)(x)\\ &= \psi^c_\gamma(y)\ (\bar\psi^a_\alpha\psi^a_\alpha)(y)\end{aligned} \tag{4.23}$$

For second derivatives we obtain:

$$\begin{aligned}\frac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\psi^{c'}_{\gamma'}(y')\delta\psi^c_\gamma(y)} &= -\ \frac{\delta}{\delta\psi^{c'}_{\gamma'}(y')}\ \bar\psi^c_\gamma(y)\ (\bar\psi^a_\alpha\psi^a_\alpha)(y)\\ &= -\ \bar\psi^c_\gamma\bar\psi^a_\alpha(y)\ \delta^{ac'}\delta_{\alpha\gamma'}\ \delta(y-y')\\ &= -\ \bar\psi^c_\gamma\bar\psi^{c'}_{\gamma'}(y)\ \delta(y-y')\end{aligned} \tag{4.24}$$

$$\begin{aligned}\frac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\bar\psi^{c'}_{\gamma'}(y')\delta\bar\psi^c_\gamma(y)} &= \frac{\delta}{\delta\bar\psi^{c'}_{\gamma'}(y')}\ \psi^c_\gamma(y)\ (\bar\psi^a_\alpha\psi^a_\alpha)(y)\\ &= -\ \psi^c_\gamma(y)\ \delta^{ac'}\delta_{\alpha\gamma'}\ \delta(y-y')\ \psi^a_\alpha(y)\\ &= -\ \psi^c_\gamma\psi^{c'}_{\gamma'}(y)\ \delta(y-y')\end{aligned} \tag{4.25}$$

$$\begin{aligned}\frac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\bar\psi^{c'}_{\gamma'}(y')\delta\psi^c_\gamma(y)} &= -\ \frac{\delta}{\delta\bar\psi^{c'}_{\gamma'}(y')}\ \bar\psi^c_\gamma(y)\ (\bar\psi^a_\alpha\psi^a_\alpha)(y)\\ &= -\ \delta^{cc'}\delta_{\gamma\gamma'}\ \delta(y-y')\ (\bar\psi^a_\alpha\psi^a_\alpha)(y) + \bar\psi^c_\gamma(y)\ \delta^{ac'}\delta_{\alpha\gamma'}\ \delta(y-y')\ \psi^a_\alpha(y)\\ &= -\ \delta^{cc'}\delta_{\gamma\gamma'}\ (\bar\psi^a_\alpha\psi^a_\alpha)(y)\ \delta(y-y') + \bar\psi^c_\gamma\psi^{c'}_{\gamma'}(y)\ \delta(y-y')\end{aligned} \tag{4.26}$$

The missing term we again obtain by using the anti-commuting property of the derivatives

$$\begin{aligned}\frac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\psi^{c'}_{\gamma'}(y')\delta\bar\psi^c_\gamma(y)} &= -\frac{\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\bar\psi^c_\gamma(y)\delta\psi^{c'}_{\gamma'}(y')}\\ &= \delta^{c'c}\delta_{\gamma'\gamma}\ (\bar\psi^a_\alpha\psi^a_\alpha)(y')\ \delta(y'-y) - \bar\psi^{c'}_{\gamma'}\psi^c_\gamma(y')\ \delta(y'-y)\\ &= \delta^{cc'}\delta_{\gamma\gamma'}\ (\bar\psi^a_\alpha\psi^a_\alpha)(y)\ \delta(y-y') - \bar\psi^{c'}_{\gamma'}\psi^c_\gamma(y)\ \delta(y-y')\end{aligned} \tag{4.27}$$

This can again be formulated in a matrix equation

$$\mathbf{\Gamma}^{(2)}_{k,pot}[\bar\psi,\psi] = \begin{pmatrix} \dfrac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\psi^{c'}_{\gamma'}(y')\delta\psi^c_\gamma(y)} & \dfrac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\bar\psi^{c'}_{\gamma'}(y')\delta\psi^c_\gamma(y)} \\ \dfrac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\psi^{c'}_{\gamma'}(y')\delta\bar\psi^c_\gamma(y)} & \dfrac{\delta^2\Gamma_{k,pot}[\bar\psi,\psi]}{\delta\bar\psi^{c'}_{\gamma'}(y')\delta\bar\psi^c_\gamma(y)} \end{pmatrix}$$

$$= \begin{pmatrix} -\bar{\psi}^c_\gamma \bar{\psi}^{c'}_{\gamma'}(y) & -\delta^{cc'}\delta_{\gamma\gamma'}\,(\bar{\psi}^a_\alpha \psi^a_\alpha)(y) + \bar{\psi}^c_\gamma \psi^{c'}_{\gamma'}(y) \\ \delta^{cc'}\delta_{\gamma\gamma'}\,(\bar{\psi}^a_\alpha \psi^a_\alpha)(y) - \bar{\psi}^{c'}_{\gamma'}\psi^c_\gamma(y) & -\psi^c_\gamma \psi^{c'}_{\gamma'}(y) \end{pmatrix} \delta(y-y') \tag{4.28}$$

We would like to have a closer look on the structure of the matrix. Introducing the abbreviations ${}^-\Phi^\circ$, ${}^-\Phi^\circ$, ${}^-\Phi^\circ$, ${}^-\Phi^\circ$ for the $(d_\gamma N_f)\times(d_\gamma N_f)$ matrices, defined by

$$\left\{{}^\circ\Phi^-\right\}^{c'c}_{\gamma'\gamma} := \psi^{c'}_{\gamma'}\bar{\psi}^c_\gamma \tag{4.29}$$

gives the possibility to write (4.27) as the following matrix

$$\boldsymbol{\Gamma}^{(2)}_{k,pot}[\bar{\psi},\psi] = \bar{\psi}\psi \begin{pmatrix} 0 & -\mathbb{1}_{d_\gamma}\otimes\mathbb{1}_{N_f} \\ \mathbb{1}_{d_\gamma}\otimes\mathbb{1}_{N_f} & 0 \end{pmatrix} \delta(y-y') + \begin{pmatrix} {}^-\Phi^- & -{}^\circ\Phi^- \\ -{}^-\Phi^\circ & {}^\circ\Phi^\circ \end{pmatrix} \delta(y-y') \tag{4.30}$$

This structural analysis will help us to invert the matrix eventually. We also recognize that if we assume constant fields ψ, $\bar{\psi}$, that our matrices are translation invariant and therefore trivial to transform to momentum space. Since arbitrary but constant fields are sufficient to project the flow equation onto the local potential, we would like to restrict ourselves to this case from now on. Then we get for the momentum representation

$$\begin{aligned} \boldsymbol{\Gamma}^{(2)}_{k,pot}[\bar{\psi},\psi] &= \bar{\psi}\psi \begin{pmatrix} 0 & -\mathbb{1}_{d_\gamma}\otimes\mathbb{1}_{N_f} \\ \mathbb{1}_{d_\gamma}\otimes\mathbb{1}_{N_f} & 0 \end{pmatrix} (2\pi)^d\delta(p+p') + \begin{pmatrix} {}^-\Phi^- & -{}^\circ\Phi^- \\ -{}^-\Phi^\circ & {}^\circ\Phi^\circ \end{pmatrix} (2\pi)^d\delta(p+p') \\ &= \begin{pmatrix} -\bar{\psi}^c_\gamma \bar{\psi}^{c'}_{\gamma'} & -\delta^{cc'}\delta_{\gamma\gamma'}\,(\bar{\psi}^a_\alpha \psi^a_\alpha) + \bar{\psi}^c_\gamma \psi^{c'}_{\gamma'} \\ \delta^{cc'}\delta_{\gamma\gamma'}\,(\bar{\psi}^a_\alpha \psi^a_\alpha) - \bar{\psi}^{c'}_{\gamma'}\psi^c_\gamma & -\psi^c_\gamma \psi^{c'}_{\gamma'} \end{pmatrix} (2\pi)^d\delta(p+p') \end{aligned} \tag{4.31}$$

At last, we have to inspect the regulator term which is given in position space by

$$\boldsymbol{R}_k(x,y) = \begin{pmatrix} 0 & -\delta^{cc'}R^k_{\gamma'\gamma}(y-x) \\ \delta^{c'c}R^k_{\gamma\gamma'}(x-y) & 0 \end{pmatrix} \tag{4.32}$$

$$\tag{4.33}$$

With the Fourier transformations

$$\begin{aligned} \int\int \mathrm{d}^d x\, \mathrm{d}^d y\, R^k_{\gamma'\gamma}(y-x)\, \mathrm{e}^{\mathrm{i}px}\mathrm{e}^{\mathrm{i}qy} &= \int\int \mathrm{d}^d x\, \mathrm{d}^d z\, R^k_{\gamma'\gamma}(z)\, \mathrm{e}^{\mathrm{i}px}\mathrm{e}^{\mathrm{i}q(z+x)} \\ &= R^k_{\gamma'\gamma}(q) \int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}(p+q)x} \\ &= R^k_{\gamma'\gamma}(q)\, (2\pi)^d\delta(p+q) \\ &= R^k_{\gamma'\gamma}(-p)\, (2\pi)^d\delta(p+q) \end{aligned} \tag{4.34}$$

$$\begin{aligned} \int\int \mathrm{d}^d x\, \mathrm{d}^d y\, R^k_{\gamma\gamma'}(x-y)\, \mathrm{e}^{\mathrm{i}px}\mathrm{e}^{\mathrm{i}qy} &= \int\int \mathrm{d}^d z\, \mathrm{d}^d y\, R^k_{\gamma\gamma'}(z)\, \mathrm{e}^{\mathrm{i}p(z+y)}\mathrm{e}^{\mathrm{i}qy} \\ &= R^k_{\gamma\gamma'}(p) \int \mathrm{d}^d x\, \mathrm{e}^{\mathrm{i}(p+q)y} \\ &= R^k_{\gamma\gamma'}(p)\, (2\pi)^d\delta(p+q) \end{aligned} \tag{4.35}$$

one obtains the momentum representation of the regulator

$$\boldsymbol{R}_k(p,p') = \begin{pmatrix} 0 & -\delta^{cc'} R^k_{\gamma'\gamma}(-p) \\ \delta^{c'c} R^k_{\gamma\gamma'}(p) & 0 \end{pmatrix} (2\pi)^d \delta(p+p'). \tag{4.36}$$

We choose the regulator function in momentum space to be of the form

$$R^k_{\gamma\gamma'}(p) = \not{p}_{\gamma\gamma'}\ r_k \tag{4.37}$$

where $r_k := r\left(\frac{p^2}{k^2}\right)$ is some real-valued function such that all the properties of a regulator are satisfied. Then the regulator term takes on a similar form as the matrix in eq. (**??**).

$$\boldsymbol{R}_k(p,p') = \begin{pmatrix} 0 & \delta^{cc'} \not{p}_{\gamma'\gamma}\ r_k \\ \delta^{cc'} \not{p}_{\gamma\gamma'}\ r_k & 0 \end{pmatrix} (2\pi)^d \delta(p+p') \tag{4.38}$$

$$= \begin{pmatrix} 0 & P^T \\ P & 0 \end{pmatrix} r_k\ (2\pi)^d \delta(p+p') \tag{4.39}$$

To obtain an expression for $\boldsymbol{\Gamma}^{(2)}_k + \boldsymbol{R}_k$, we now can, under the assumption of field-independent flow parameter Z_ψ, λ_k, put all this together.

$$\begin{aligned} \left(\boldsymbol{\Gamma}^{(2)}_k + Z_\psi \boldsymbol{R}_k\right)(p,p') &= Z_\psi \boldsymbol{\Gamma}^{(2)}_{k,kin}[\bar{\psi},\psi](p,p') + \lambda_k \boldsymbol{\Gamma}^{(2)}_{k,pot}[\bar{\psi},\psi](p,p') + Z_\psi \boldsymbol{R}_k(p,p') \\ &= Z_\psi \begin{pmatrix} 0 & P^T \\ P & 0 \end{pmatrix} (1+r_k)\ (2\pi)^d \delta(p+p') \\ &+ \lambda_k\ \bar{\psi}\psi \begin{pmatrix} 0 & -\mathbb{1}_{d_\gamma N_f} \\ \mathbb{1}_{d_\gamma N_f} & 0 \end{pmatrix} (2\pi)^d \delta(p+p') \\ &+ \lambda_k \begin{pmatrix} {}^-\Phi^- & -{}^\circ\Phi^- \\ -{}^-\Phi^\circ & {}^\circ\Phi^\circ \end{pmatrix} (2\pi)^d \delta(p+p') \end{aligned} \tag{4.40}$$

4.2. General considerations concerning the inversion of the matrix

Inverting the matrix yields a couple of technical difficulties, the biggest being the fact the matrix is Grassmann valued. Since we restricted to constant fields we reduced the problem down to a algebra generated by $2\,N_f\,d_\gamma$ Grassman variables, nevertheless the inversion of such matrices is a bit tricky since we work, from a multiplicative point of view, over a non zero devisor free ring. In general a Grassmann-valued matrix

$$M = M_0 + M_\psi, \tag{4.41}$$

where M_0 denotes the part containing the zeroth component of all matrix entries, is invertible if and only if M_0 is invertible. The inverse is given by

$$M^{-1} = \left(\mathbb{1} + M_0^{-1}\,M_\psi\right)^{-1}\ M_0^{-1} = M_0^{-1}\ \left(\mathbb{1} +\ M_\psi M_0^{-1}\right)^{-1} \tag{4.42}$$

where we use the Neumann-series to define the inverse of the Grassmann-valued matrix

$$\left(\mathbb{1} + \tilde{M}_\psi\right)^{-1} = \sum_{k=0}(-1)^k \left(\tilde{M}_\psi\right)^k \tag{4.43}$$

In the case of a finite flavor number N_f, this series terminates after $k > N_f\, d_\gamma$ terms. To make the inversion of (4.40) easier we next consider first the inversion of the part that corresponds to M_0. Note that, due to the delta function in the momentum depence, we can simply restrict to matrix inversion to calculate the inverse of this expression. The matrix

$$M_0 = Z_\psi \begin{pmatrix} 0 & P^T \\ P & 0 \end{pmatrix} (1 + r_k) \tag{4.44}$$

can be inverted, by using

$$P^{-1} = \left(\not{p} \otimes \mathbb{1}_{N_f}\right)^{-1} = \not{p}^{-1} \otimes \mathbb{1}_{N_f} = \frac{\not{p}}{p^2} \otimes \mathbb{1}_{N_f} = \frac{1}{p^2} P\,. \tag{4.45}$$

Then we have

$$M_0^{-1} = \frac{1}{q} \begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix} \tag{4.46}$$

where

$$q := p^2\ (1 + r_k)\, Z_\psi \tag{4.47}$$

Hence the inverse takes the form

$$\left(\boldsymbol{\Gamma}_k^{(2)} + Z_\psi \boldsymbol{R}_k\right)^{-1} = \left(\mathbb{1} + \tilde{M}_\psi\right)^{-1} M_0^{-1}\, (2\pi)^d\, \delta(p + p') \tag{4.48}$$

with

$$\tilde{M}_\psi := \frac{\lambda_k}{q}\, \bar{\psi}\psi \begin{pmatrix} P & 0 \\ 0 & -P^T \end{pmatrix} + \frac{\lambda_k}{q} \begin{pmatrix} -P\, {}^-\Phi^\circ & P\, {}^\circ\Phi^\circ \\ P^T\, {}^-\Phi^- & -P^T\, {}^\circ\Phi^- \end{pmatrix} =: \tilde{M}_1 + \tilde{M}_2 \tag{4.49}$$

This form will proof to be very useful when calculating the super trace. Also note that this matrix is proportional to $\frac{\lambda_k}{q}$

4.3. Calculation of supertrace

We are now interested in calculating the expression

$$
\begin{aligned}
&\mathrm{STr}\left\{\left(\Gamma_k^{(2)}[\psi,\bar{\psi}] + Z_\psi R_k\right)^{-1} \circ \partial_t \left(Z_\psi R_k\right)\right\} \\
&= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \left\{\left(\boldsymbol{\Gamma}_k^{(2)}[\psi,\bar{\psi}] + Z_\psi \boldsymbol{R}_k\right)^{-1}\right\}_{ij} (p,p') \left\{\partial_t \left(Z_\psi \boldsymbol{R}_k\right)\right\}_{ji} (p',p) \qquad (4.50)
\end{aligned}
$$

where the integration over p' and the inner contraction over j stems from the composition $\left(\Gamma_k^{(2)}[\psi,\bar{\psi}] + R\right)^{-1} \circ \partial_t R_k$ and the integration over p and the outer contraction of i originates from the supertrace. By using the expression eq. (4.48) and the fact that

$$
\begin{aligned}
M_0^{-1}\, \partial_t \left(Z_\psi \boldsymbol{R}_k\right) &= \frac{1}{q} \begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} 0 & P^T \\ P & 0 \end{pmatrix} \partial_t \left(Z_\phi\, r_k\right)\, (2\pi)^d \delta(p+p') \\
&= \begin{pmatrix} \mathbb{1}_{N_f d_\gamma} & 0 \\ 0 & \mathbb{1}_{N_f d_\gamma} \end{pmatrix} \frac{p^2 \partial_t \left(Z_\phi\, r_k\right)}{q}\, (2\pi)^d \delta(p+p') \qquad (4.51)
\end{aligned}
$$

we obtain for the supertrace

$$
\begin{aligned}
&\mathrm{STr}\left\{\left(\Gamma_k^{(2)}[\psi,\bar{\psi}] + Z_\psi R_k\right)^{-1} \circ \partial_t \left(Z_\psi R_k\right)\right\} \\
&= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \mathrm{Tr}\left\{\left(1 + \tilde{M}_\psi\right)^{-1} M_0^{-1}\, \partial_t \left(Z_\psi \boldsymbol{R}_k\right)\right\}\, (2\pi)^d\, \delta(p+p') \\
&= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \mathrm{Tr}\left\{\left(1 + \tilde{M}_\psi\right)^{-1}\right\} \frac{p^2 \partial_t \left(Z_\psi\, r_k\right)}{q}\, (2\pi)^{2d}\, \delta(p+p')\, \delta(p+p') \qquad (4.52)
\end{aligned}
$$

where we wrote the matrix-trace instead of the index contraction. As no explicit dependence on p' is given, one can perform the transformation $p' \to -p'$ preserving integration except for a sign change in the delta distribution, emphasizing the diagonal structure in momentum representation.

$$
= \int \frac{\mathrm{d}^d p}{(2\pi)^d} \int \frac{\mathrm{d}^d p'}{(2\pi)^d} \mathrm{Tr}\left\{\left(1 + \tilde{M}_\psi\right)^{-1}\right\} \frac{p^2 \partial_t \left(Z_\psi\, r_k\right)}{q}\, (2\pi)^{2d}\, \delta(p-p')\, \delta(p-p') \qquad (4.53)
$$

We therefore see that it is only necessary to calculate the trace of the inverse of $1 + \tilde{M}_\psi$. This can be done easily for second order of the von-Neumann series, which, as it turns out, are the only term's we are interested in to obtain the flow equation for the running couplings. This can be seen in the following way: $\tilde{M}_\psi$ is a Graßmann valued matrix, where every entry is a sum of degree 2 monomials. Therefore, $\tilde{M}_\psi^n$ is a graßmann matrix, where every entry is a sum of degree $2\,n$ monomials. Hence the matrix is $N_f d\gamma$ nilpotent and only the second order term can contain terms of the form $(\bar{\psi}\psi)^2$ which are needed to project out the flow equation for the coupling $\tilde{\lambda}_k$. In order to calculate the inverse of the Grassmann-valued matrix we mobilize the von-Neumann-series

$$\begin{aligned}\mathrm{Tr}\left\{\left(1+\tilde{M}_\psi(p)\right)^{-1}\right\} =& \mathrm{Tr}\left\{\sum_{k=0}^{\infty}(-1)^k(\tilde{M}_\psi)^k\right\}\\ & 2N_f d_\gamma - \mathrm{Tr}\left\{\tilde{M}_\psi(p)\right\} + \mathrm{Tr}\left\{(\tilde{M}_\psi(p))^2\right\} - ...\end{aligned} \tag{4.54}$$

and investigate the quadratic term. The zero order term represents exactly the degrees of freedom. Noticed that in the aforementioned notation

$$\begin{aligned}\{{}^\circ\Phi^\circ\}^{c'c}_{\gamma'\gamma} &= \Psi^{c'}_{\gamma'}\Psi^{c}_{\gamma}\\ \{{}^\circ\Phi^-\}^{c'c}_{\gamma'\gamma} &= \Psi^{c'}_{\gamma'}\bar{\Psi}^{c}_{\gamma}\\ \{{}^-\Phi^\circ\}^{c'c}_{\gamma'\gamma} &= \bar{\Psi}^{c'}_{\gamma'}\Psi^{c}_{\gamma}\\ \{{}^-\Phi^-\}^{c'c}_{\gamma'\gamma} &= \bar{\Psi}^{c'}_{\gamma'}\bar{\Psi}^{c}_{\gamma}\end{aligned}$$

the following relation holds

$$\left({}^-\Phi^\circ\right)^T = -{}^\circ\Phi^- \tag{4.55}$$

Which in particular means $\left({}^-\Phi^-\right)^T = -{}^-\Phi^-$ etc. and can be easily checked as follows:

$$\left\{{}^-\Phi^{\circ T}\right\}^{c'c}_{\gamma'\gamma} = \{{}^-\Phi^\circ\}^{cc'}_{\gamma\gamma'} = \bar{\Psi}^{c}_{\gamma}\Psi^{c'}_{\gamma'} = -\Psi^{c'}_{\gamma'}\bar{\Psi}^{c}_{\gamma} = \{-{}^\circ\Phi^-\}^{c'c}_{\gamma'\gamma}$$

We will use this relation heavily in the following: In shorthand notation, the second order term using (4.49) reads

$$\tilde{M}^2_\psi = \tilde{M}^2_1 + \tilde{M}_1\tilde{M}_2 + \tilde{M}_2\tilde{M}_1 + \tilde{M}^2_2 \tag{4.56}$$

with

$$\tilde{M}^2_1 = \frac{\tilde{\lambda}^2_k}{q^2}(\bar{\psi}\psi)^2\begin{pmatrix} PP & 0\\ 0 & -P^TP^T\end{pmatrix} \tag{4.57}$$

$$\tilde{M}_1\tilde{M}_2 = \frac{\tilde{\lambda}^2_k}{q^2}\bar{\psi}\psi\begin{pmatrix} -PP{}^-\Phi^\circ & PP^0\Phi^0\\ -P^TP^T{}^-\Phi^- & P^TP^T{}^\circ\Phi^-\end{pmatrix} \tag{4.58}$$

$$\tilde{M}_2\tilde{M}_1 = \frac{\tilde{\lambda}^2_k}{q^2}\bar{\psi}\psi\begin{pmatrix} -P{}^-\Phi^\circ P & -P{}^\circ\Phi^\circ\\ P^T{}^-\Phi^-P & P^T{}^\circ\Phi^-P^T\end{pmatrix} \tag{4.59}$$

$$\tilde{M}^2_2 = \frac{\tilde{\lambda}^2_k}{q^2}\begin{pmatrix} (P{}^-\Phi^\circ)^2 + P{}^\circ\Phi^\circ P^T{}^-\Phi^- & -P{}^-\Phi^\circ P{}^\circ\Phi^\circ - P{}^\circ\Phi^\circ P^T{}^\circ\Phi^-\\ -P^T{}^-\Phi^-P{}^-\Phi^\circ - P^T{}^\circ\Phi^-P^T{}^-\Phi^- & P^T{}^-\Phi^-P{}^\circ\Phi^\circ + (P^T{}^\circ\Phi^-)^2\end{pmatrix} \tag{4.60}$$

We calculate the traces of all terms separately:

$$\begin{aligned}\frac{q^2}{\tilde{\lambda}^2_k}\mathrm{Tr}\left\{\tilde{M}^2_1\right\} &= (\bar{\psi}\psi)^2\,\mathrm{Tr}\left\{p^2\begin{pmatrix}\mathbb{1}_{N_f d_\gamma} & 0\\ 0 & \mathbb{1}_{N_f d_\gamma}\end{pmatrix}\right\}\\ &= 2N_f d_\gamma\,(\bar{\psi}\psi)^2\,p^2\end{aligned}$$

$$
\begin{aligned}
\frac{q^2}{\tilde{\lambda}_k^2}\mathrm{Tr}\left\{\tilde{M}_1\tilde{M}_2\right\} = \frac{q^2}{\tilde{\lambda}_k^2}\mathrm{Tr}\left\{\tilde{M}_2\tilde{M}_1\right\} &= (\bar{\psi}\psi)\left(\mathrm{Tr}\left\{P^T P^{T\,\circ}\Phi^-\right\} + \mathrm{Tr}\left\{PP^-\Phi^\circ\right\}\right)\\
&= (\bar{\psi}\psi)\left(\mathrm{Tr}\left\{{}^\circ\Phi^{-T}PP\right\} + \mathrm{Tr}\left\{PP^-\Phi^\circ\right\}\right)\\
&= (\bar{\psi}\psi)\left(\mathrm{Tr}\left\{-{}^-\Phi^\circ p^2 \mathbb{1}_{N_f d_\gamma}\right\} - \mathrm{Tr}\left\{p^2\mathbb{1}_{N_f d_\gamma}{}^-\Phi^\circ\right\}\right)\\
&= -2\,(\bar{\psi}\psi)\,\mathrm{Tr}\left\{{}^-\Phi^\circ\right\}p^2\\
&= -2\,(\bar{\psi}\psi)^2\,p^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{q^2}{\tilde{\lambda}_k^2}\mathrm{Tr}\left\{\tilde{M}_2{}^2\right\} &= \mathrm{Tr}\left\{P^-\Phi^\circ P^-\Phi^\circ\right\} + \mathrm{Tr}\left\{P^\circ\Phi^\circ P^{T\,-}\Phi^-\right\} + \mathrm{Tr}\left\{P^{T\,-}\Phi^- P^\circ\Phi^\circ\right\} + \mathrm{Tr}\left\{P^{T\,\circ}\Phi^- P^{T\,\circ}\Phi^-\right\}\\
&= 2\mathrm{Tr}\left\{P^-\Phi^\circ P^-\Phi^\circ + P^\circ\Phi^\circ P^{T\,-}\Phi^-\right\}\\
&= 2\mathrm{Tr}\left\{P^\circ\Phi^\circ P^{T\,-}\Phi^- - P^{T\,\circ}\Phi^\circ P^-\Phi^-\right\}\\
&= 2\mathrm{Tr}\left\{P^\circ\Phi^\circ P^{T\,-}\Phi^- - P^{T\,\circ}\Phi^\circ P^-\Phi^- - P^\circ\Phi^\circ P^-\Phi^- + P^\circ\Phi^\circ P^-\Phi^-\right\}\\
&= 2\mathrm{Tr}\left\{P^\circ\Phi^\circ P^{T\,-}\Phi^- - P^{T\,\circ}\Phi^\circ P^-\Phi^- - P^\circ\Phi^\circ P^-\Phi^- + {}^-\Phi^- P^\circ\Phi^\circ P\right\}\\
&= 2\mathrm{Tr}\left\{P^\circ\Phi^\circ P^{T\,-}\Phi^- - P^{T\,\circ}\Phi^\circ P^-\Phi^- - P^\circ\Phi^\circ P^-\Phi^- + {}^-\Phi^{-T} P^\circ\Phi^{\circ T} P\right\}\\
&= 2\mathrm{Tr}\left\{P^\circ\Phi^\circ P^{T\,-}\Phi^- - P^{T\,\circ}\Phi^\circ P^-\Phi^- - P^\circ\Phi^\circ P^-\Phi^- + P^{T\,\circ}\Phi^\circ P^{T\,-}\Phi^-\right\}\\
&= -2\mathrm{Tr}\left\{\left(P + P^T\right){}^\circ\Phi^\circ\left(P - P^T\right){}^-\Phi^-\right\}
\end{aligned}
$$

Where from the second to third line we used

$$
\begin{aligned}
\mathrm{Tr}\left\{P^-\Phi^\circ P^-\Phi^\circ\right\} &= P^{ab}_{\alpha\beta}\,\bar{\psi}^b_\beta\,\psi^c_\gamma\,P^{cd}_{\gamma\delta}\,\bar{\psi}^d_\delta\,\psi^a_\alpha\\
&= -P^{ab}_{\alpha\beta}\,\psi^c_\gamma\,\bar{\psi}^b_\beta\,P^{cd}_{\gamma\delta}\,\bar{\psi}^d_\delta\,\psi^a_\alpha\\
&= -P^{ab}_{\alpha\beta}\,\psi^c_\gamma\,\psi^a_\alpha\,P^{cd}_{\gamma\delta}\,\bar{\psi}^b_\beta\,\bar{\psi}^d_\delta\\
&= -P^{ab}_{\alpha\beta}\,\psi^a_\alpha\,\psi^c_\gamma\,P^{cd}_{\gamma\delta}\,\bar{\psi}^d_\delta\,\bar{\psi}^b_\beta\\
&= -\mathrm{Tr}\left\{P^{T\,\circ}\Phi^\circ P^-\Phi^-\right\}
\end{aligned}
$$

Finally, we obtain

$$
\frac{q^2}{\tilde{\lambda}_k^2}\mathrm{Tr}\left\{\tilde{M}_\psi^2\right\} = 2\left(N_f d_\gamma - 2\right)p^2\,(\bar{\psi}\psi)^2 - 2\mathrm{Tr}\left\{\left(P + P^T\right){}^\circ\Phi^\circ\left(P - P^T\right){}^-\Phi^-\right\} \tag{4.61}
$$

We inspect the last term further. For that we expand our short-hand-notation:

$$
\begin{aligned}
\left(P \pm P^T\right) &= \left(\not{p}\otimes\mathbb{1}_{N_f} \pm \left(\not{p}\otimes\mathbb{1}_{N_f}\right)^T\right)\\
&= p_\mu\left(\gamma^\mu \pm (\gamma^\mu)^T\right)\otimes\mathbb{1}_{N_f}
\end{aligned} \tag{4.62}
$$

Therefore, if the γ-matrices in a given representation are (anti-)symmetric[1], the last term in eq. (4.61) vanishes, leaving us with the simple expression

$$\mathrm{Tr}\left\{\tilde{M}_{\psi}^{2}\right\} = 2\left(N_f d_\gamma - 2\right)\frac{p^2}{q^2}\tilde{\lambda}_k^2\left(\bar{\psi}\psi\right)^2 \tag{4.63}$$

4.4. Flow equation for the coupling λ_k

To obtain the flow equation, we first calculate the left hand side:

$$\begin{aligned}
\partial_t \Gamma_k[\psi,\bar{\psi}] &= \partial_t \int \mathrm{d}^d x \left(Z_\psi \bar{\psi} i D \psi + \frac{1}{2}\tilde{\lambda}_k \left(\bar{\psi}\psi\right)^2\right) \\
&= \frac{1}{2}\left(\bar{\psi}\psi\right)^2\left(\partial_t \tilde{\lambda}_k\right)\int \mathrm{d}^d x \\
&= \frac{\mathrm{Vol}}{2}\left(\bar{\psi}\psi\right)^2\left(\partial_t \tilde{\lambda}_k\right)
\end{aligned} \tag{4.64}$$

Where we remind that the fields are set to be constant. Notice that by doing so, the term $D\psi$ vanishes and we lose the information about Z_ψ. Since this wave function normalization cannot be treated by our approximations we set it equal to one which removes all of its momentum dependence and corresponds to a point-like limit. We note further, that Vol diverges in the continuum limit and should only be seen as a mathematical tool at this point.
For the right hand side, we just have to collect terms calculated before (remember that we only consider terms quadratic in $(\bar{\psi}\psi)$). Note that there appears a delta function squared wich can be taken care of by

$$(2\pi)^d \delta(p-p')\,\delta(p-p') = (2\pi)^d \delta(p-p') \int \mathrm{d}^d x\, e^{i(p-p')x} = \delta(p-p')\int \mathrm{d}^d x = \delta(p-p')\,\mathrm{Vol}\,, \tag{4.65}$$

which also brings in a factor Vol. Then we can evaluate the right hand side of the flow equation.

$$\begin{aligned}
&-\frac{1}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\int \frac{\mathrm{d}^d p'}{(2\pi)^d}\mathrm{Tr}\left\{\tilde{M}_\psi^2\right\}\frac{p^2(\partial_t r_k)}{q}(2\pi)^{2d}\delta(p-p')\delta(p-p') \\
&= -\frac{\mathrm{Vol}}{2}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\int \frac{\mathrm{d}^d p'}{(2\pi)^d}\mathrm{Tr}\left\{\tilde{M}_\psi^2\right\}\frac{p^2(\partial_t r_k)}{q}(2\pi)^2\delta(p-p') \\
&= -\mathrm{Vol}\left(N_f d_\gamma - 2\right)\left(\bar{\psi}\psi\right)^2\tilde{\lambda}_k^2\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{p^4}{q^3}(\partial_t r_k) \\
&= -\mathrm{Vol}\left(N_f d_\gamma - 2\right)\left(\bar{\psi}\psi\right)^2\tilde{\lambda}_k^2\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^2}\frac{\partial_t r_k}{(1+r_k)^3}
\end{aligned}$$

[1] This is actually not too big of a restriction for $d = 2, 3, 4$ for example because for those it is easy to find a specific choice that satisfies the (anti-)symmetry. From there once can use unitary transformations to show that this then holds, independent of the specific choice. See Appendix B.2

We choose the Litim-regulator

$$r_k = \left(\sqrt{\frac{k^2}{p^2}} - 1\right)\theta(k^2 - p^2) \tag{4.66}$$

for which we find

$$\begin{aligned}
\partial_t\, r_k &= k\, \partial_k r_k \\
&= \sqrt{\frac{k^2}{p^2}}\,\theta(k^2 - p^2) + 2k^2\left(\sqrt{\frac{k^2}{p^2}} - 1\right)\delta(k^2 - p^2) \\
&= \sqrt{\frac{k^2}{p^2}}\,\theta(k^2 - p^2)
\end{aligned}$$

where we used that the δ requires $k = p$ for which the second term vanishes. Further calculating

$$\begin{aligned}
\frac{\partial_t\, r_k}{(1 + r_k)^3} &= \sqrt{\frac{k^2}{p^2}}\frac{\theta(k^2 - p^2)}{(1 + r_k)^3} \\
&= \sqrt{\frac{k^2}{p^2}}\frac{\theta(k^2 - p^2)}{\left(1 + \left(\sqrt{\frac{k^2}{p^2}} - 1\right)\theta(k^2 - p^2)\right)^3} \\
&= \sqrt{\frac{k^2}{p^2}}\left(\sqrt{\frac{p^2}{k^2}}\right)^3\theta(k^2 - p^2) \\
&= \frac{p^2}{k^2}\theta(k^2 - p^2)
\end{aligned}$$

reveals a pleasantly simple expression for the integral above. From the second to the third line we used that the numerator requires $p < k$ while the denominator is always non-zero, hence we can drop the θ in the denominator as it's unity for all cases where the entire expression is non-zero. With this we can finally finish our calculation of the supertrace.

$$\begin{aligned}
&-\frac{1}{2}\mathrm{STr}\left\{\left(\Gamma_k^{(2)} + R\right)^{-1}\partial_t R_k\right\} \\
&= -\mathrm{Vol}\left(N_f d_\gamma - 2\right)(\bar\psi\psi)^2\,\tilde\lambda_k^2\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{p^2}\frac{\partial_t r_k}{(1 + r_k)^3} \\
&= -\mathrm{Vol}\left(N_f d_\gamma - 2\right)(\bar\psi\psi)^2\,\tilde\lambda_k^2\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\frac{1}{k^2}\theta(k^2 - p^2) \\
&= -\mathrm{Vol}\left(N_f d_\gamma - 2\right)(\bar\psi\psi)^2\,\tilde\lambda_k^2\frac{1}{k^2}\frac{1}{(2\pi)^d}\,\mathcal{V}(k, d)
\end{aligned} \tag{4.67}$$

where

$$\mathcal{V}(k, d) = \frac{\pi^{d/2}k^d}{\Gamma(d/2 + 1)} = (2\pi)^d\nu(d)k^d$$

is the volume of the d-ball of radius k, $\nu(d)$ a dimension-dependent constant and the $(2\pi)^d$ just for convenience. Upon comparing eq. (4.64) and (4.67) we arrive at the flow equation

$$\partial_t \tilde{\lambda}_k = -2(N_f d_\gamma - 2)\ \nu(d)\ k^{d-2}\ \tilde{\lambda}_k^2 \tag{4.68}$$

for the coupling $\tilde{\lambda}_k$ of mass dimension $(2-d)$. As a last step, we rewrite eq. (4.68) in terms of the dimensionless coupling $\lambda_k = k^{d-2}\tilde{\lambda}_k$:

$$\begin{aligned}
\partial_t \left(k^{2-d}\lambda_k\right) &= -2(N_f d_\gamma - 2)\ \nu(d)\ k^{d-2}\ \left(k^{2-d}\lambda_k\right)^2 \\
k((2-d)k^{1-d}\lambda_k + k^{2-d}\partial_k\lambda_k) &= -2(N_f d_\gamma - 2)\ \nu(d)\ k^{2-d}\lambda_k^2 \\
(2-d)k^{2-d}\lambda_k + k^{2-d}\partial_t\lambda_k &= -2(N_f d_\gamma - 2)\ \nu(d)\ k^{2-d}\lambda_k^2
\end{aligned}$$

Rearranging the terms of our beta-function, the derivative of λ_k, yields

$$\beta_{\lambda_k} \equiv \partial_t\lambda_k = (d-2)\lambda_k - 2(N_f d_\gamma - 2)\ \nu(d)\ \lambda_k^2 \tag{4.69}$$

At this point, we are interested in the fixed points of our flow equation, found by setting

$$\partial_t\lambda_k = 0 \tag{4.70}$$

This correspond to zeros on a $\partial_t\lambda_k - \lambda_k$ graph. Aside from the trivial Gaussian fixed-point $\lambda_k^{Gauss} = 0$, we obtain a non-trivial solution for $d \neq 2$ and $N_f d_\gamma \neq 2$ of the form:

$$\lambda_k^* = \frac{(d-2)}{2(N_f d_\gamma - 2)\ \nu(d)}, \quad \nu(d) = \frac{1}{(2\pi)^d}\frac{\pi^{d/2}}{\Gamma(d/2+1)} \tag{4.71}$$

The value of the fixed point is dependent on the exact choice of the regulator, which can easily be seen as $\nu(d)$ arose due to the Litim-regulator. The existence, however, is independent of our choice and, as a consequence, universal.
The fixpoints can be visualized (see fig. 1) where arrows indicate the flow towards the infrared. We see that the Gaussian fixpoint is attractive for all initial values smaller than λ^*, whereas the coupling constant diverges for larger initial values.

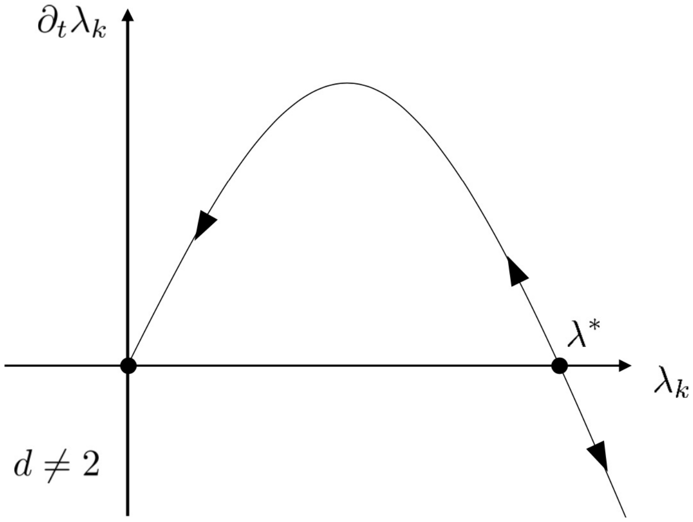

Figure 1: Sketch of the β-function for $d > 2$. Bullets indiacte the Gaussian as well as the nontrivial fixpoint. Arrows indicate the RG flow towards the infrared.

For $d = 2$ the situation is different: We only have the Gaussian fixed point in this case which is not stable either (see fig. 2).

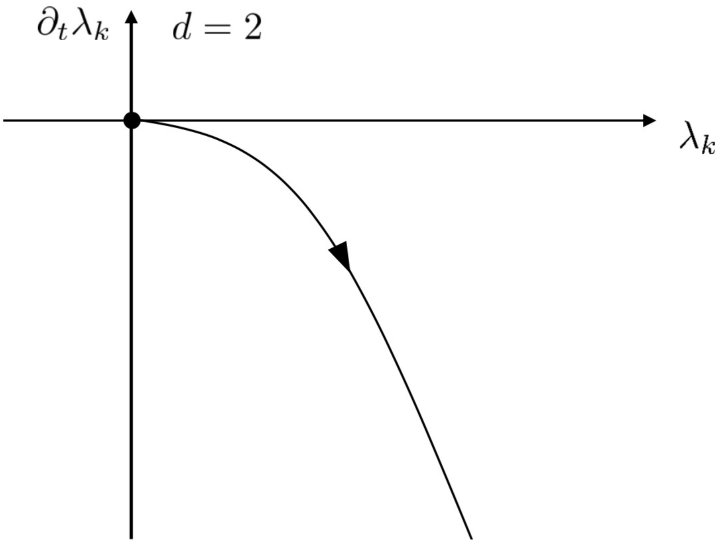

Figure 2: Sketch of the β-function for $d = 2$. The bullet indiactes the Gaussian fixpoint. Arrows indicate the RG flow towards the infrared.

In total we learn (even without actually solving the flow equation) that the $d \neq 2$ case is asymptotically safe with a Gaussian fixed point in the IR while the $d = 2$ case is asymptotically free (similarly to QCD).

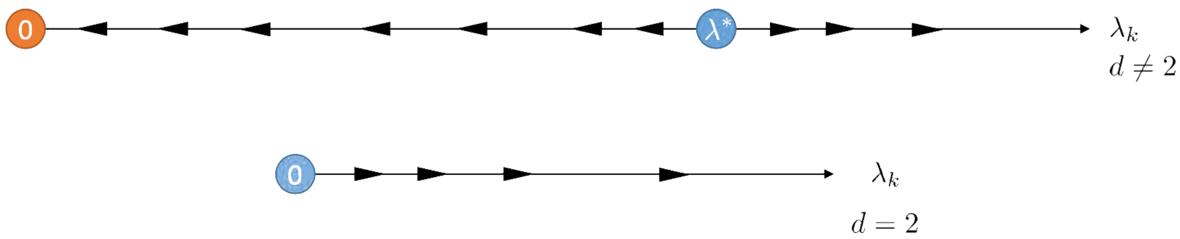

Figure 3: Summary of fixed point structure: For the case $d \neq 2$ we obtain asymptotic safety and an IR Gaussian fixed point. The $d = 2$ case is asymptotically free with no other fixed points. Arrows indicate the RG flow towards the IR.

4.5. Solution of the flow equation

In order to solve the flow equation in terms of our coupling λ_k we start by denoting our flow equation as

$$k\frac{d\lambda_k}{dk} = a\lambda_k - b\lambda_k^2,\ a = d-2,\ b = 2(N_f d_\gamma - 2)\nu(d)$$

We restrict our analysis to the case $a > 0$ and $b > 0$ which means that we only consider the case of spacetime dimensions greater than 2 with at least 2 fermion flavors. Separation of variables allows an integration over $\mathrm{d}\lambda_k$ and k separately

$$\int \frac{\mathrm{d}\lambda_k}{\lambda_k(a - b\lambda_k)} = \int \frac{\mathrm{d}k}{k}$$

The left side can be decomposed

$$\frac{1}{a}\int \frac{\mathrm{d}\lambda_k}{\lambda_k} - \frac{1}{a}\int \frac{\mathrm{d}\lambda_k}{\lambda_k - \frac{a}{b}} = \ln|k| + \tilde{C}$$

Integration yields

$$\frac{\ln|\lambda_k|}{a} - \frac{\ln|\lambda_k - \frac{a}{b}|}{a} = \ln|k| + \tilde{C}$$

We use that k is always positive and notice that a/b is the fixpoint we found in eq. (4.71). We find

$$\lambda_k = \begin{cases} \frac{\lambda^* C k^a}{C k^a + 1} & 0 < \lambda_k < \lambda^* \\ \\ \frac{\lambda^* C k^a}{C k^a - 1} & \lambda_k > \lambda^* \text{ or } \lambda_k < 0 \end{cases}$$

with $C > 0$. We rewrite this in terms of the UV-cutoff Λ and the initial coupling $\lambda^{\mathrm{UV}} = \lambda_k(\Lambda)$ at that scale respectively:

$$\lambda_k = \frac{\lambda^* C k^a}{C k^a \pm 1} = \frac{\lambda^* C \left(\frac{k}{\Lambda}\right)^a}{C\left(\frac{k}{\Lambda}\right)^a \pm \left(\frac{1}{\Lambda}\right)^a},\quad \lambda^{\mathrm{UV}} = \frac{\lambda^* C}{C \pm \left(\frac{1}{\Lambda}\right)^a}$$

The second equation allows us to express C and insert it into the equation for λ_k. After some algebraic manipulations and inserting the short-hand notation for a we find that the $\pm$ cancels in every expression and we arrive at

$$\lambda_k = \lambda^{\mathrm{UV}}\left[\left(\frac{\Lambda}{k}\right)^{d-2}\left[1 - \frac{\lambda^{\mathrm{UV}}}{\lambda^*}\right] + \frac{\lambda^{\mathrm{UV}}}{\lambda^*}\right]^{-1} \qquad d > 2,\ N_f d_\gamma > 2 \qquad (4.72)$$

We see that for $\lambda^{\text{UV}} = \lambda^*$ the solution stays at λ^* as expected and for all $\lambda^{\text{UV}} < \lambda^*$ it approaches the Gaussian fixpoint for $k \to 0$. For $\lambda^{\text{UV}} > \lambda^*$ on the other hand the term diverges (cf. fig. 1). The solution is plotted for some initial values λ^{UV} in fig. 4.

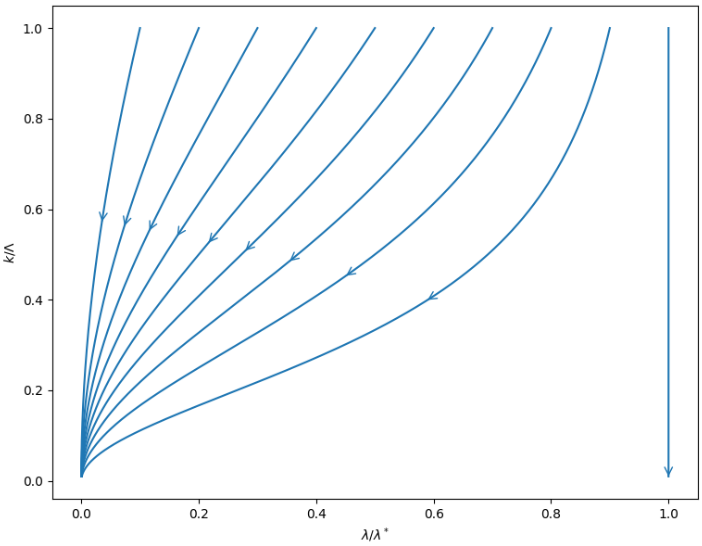

Figure 4: Running coupling for $d = 4, N_f = 6, d_\gamma = 4$. The trajectories start at some λ^{UV} at the cutoff Λ (top) and approach the Gaussian fixpoint towards the infrared ($k \to 0$). Starting at the fixpoint λ^* gives a constant flow. Starting values $\lambda^{\text{UV}} < 0$ also run towards the Gaussian but aren't shown here.

One can now consider all the other cases for a, b but we should only be interested in the additional case $d = 2$ and $N_f d_\gamma > 2$ for which we find after a similar calculation

$$\lambda_k = \lambda^{\text{UV}} \left[2(N_f d_\gamma - 2)\nu(d) \ln\left(\frac{k}{\Lambda}\right) \lambda^{\text{UV}} + 1 \right]^{-1} \qquad d = 2,\ N_f d_\gamma > 2 \tag{4.73}$$

We've see that the only fixpoint in this case is the Gaussian fixpoint (cf. fig. 2). All positive initial values lead to a divergent coupling constant but for negative ones we expect them to approach the Gaussian fixpoint. That this is indeed the case can be seen in fig. 5.

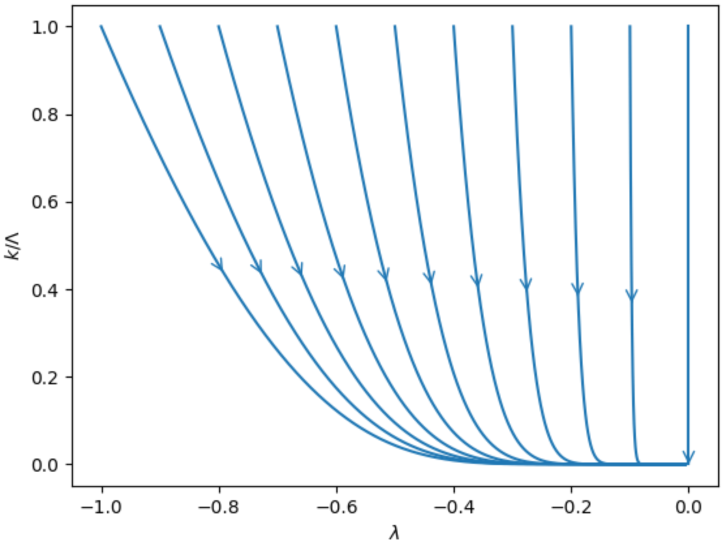

Figure 5: Running coupling for $d = 2, N_f = 2, d_\gamma = 2$. The trajectories start at some λ^{UV} at the cutoff Λ (top) and approach the Gaussian fixpoint towards the infrared ($k \to 0$). Starting at the fixpoint λ^* gives a constant flow.

A. Conventions

A.1. Functionals and Kernels

Functionals are denoted by big letters and square brackets for their arguments. Additional round brackets indicate that this object is in fact the integral kernel of the functional. Arguments x, y, z indicates a coordinate-space representation and p, q, r-indicate momentum space representations. The concrete connection between this quantities is given by

$$\Gamma_k[\psi, \bar{\psi}] = \int\int \mathrm{d}^d x \mathrm{d}^d x' \; \bar{\psi}(x') \, \Gamma_k[\psi, \bar{\psi}](x', x) \, \psi(x) \tag{A.1}$$

$$= \int\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d p'}{(2\pi)^d} \; \bar{\psi}(p') \, \Gamma_k[\psi, \bar{\psi}](p', p) \, \psi(p) \tag{A.2}$$

Note that we here defined a normalization of $(2\pi)^d$ in the intengral measure of the momentum space integral.

A.2. Fourier-transformations

The fourier transformations of fields are definied by

$$\psi(p) := \int \mathrm{d}^d x \; \psi(x) \; \mathrm{e}^{-i\,px} \tag{A.3}$$

$$\psi(x) := \int \frac{\mathrm{d}^d p}{(2\pi)^d} \; \psi(p) \; \mathrm{e}^{i\,px} \tag{A.4}$$

and analogous for the adjoint fields (which are seen as independent fields). Due to this convetion, the kernels of the functionals transform accordingly to

$$\Gamma_k[\psi, \bar{\psi}](p, p') := \int\int \mathrm{d}^d x \; \mathrm{d}^d x' \; \Gamma_k[\psi, \bar{\psi}](x, x') \; \mathrm{e}^{i\,px} \; \mathrm{e}^{i\,p'x'} \tag{A.5}$$

$$\Gamma_k[\psi, \bar{\psi}](x, x') := \int\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{d}^d p'}{(2\pi)^d} \; \Gamma_k[\psi, \bar{\psi}](p, p') \; \mathrm{e}^{-i\,px} \; \mathrm{e}^{-i\,p'x'} \tag{A.6}$$

A.3. Functional and Graßmann derivatives

All functional derivatives are derivatives with respect to graßmann valued fields (representing a continuous indexing of the generators of a non countable graßmann algebra). All derivatives can be calculated taking into account linearity and anticommuting properties (as well as graßmann chain rules etc.) when the relation

$$\frac{\delta\psi_\alpha(x)}{\delta\psi_\beta(y)} = \delta_{\alpha\beta} \; \delta(x - y) \tag{A.7}$$

is fixed. When going to fourier-space one has to add a factor $(2\pi)^d$ due to our convention how fourier transformations and especially the Functionals-Kernel in momentum representaiton are defined.

$$\frac{\delta\psi_\alpha(p)}{\delta\psi_\beta(q)} = (2\pi)^d \delta_{\alpha\beta} \; \delta(p - q) \tag{A.8}$$

A.4. Summation-conventions

We use three different summation conventions for the three different types of indices appearing:

1. Spinor indices: $\alpha, \beta, \gamma \in \{1, ..., d_\gamma\}$, Summation over lower-lower pairs.
2. Flavor indices: $a, b, c \in \{1, ..., N_f\}$, Summation over upper-upper pairs.
3. Lorentz indices: $\mu, \nu, \eta \in \{1, ..., d\}$, Summation over upper-lower pairs .

A.5. Space-Time-convention

We work in a euclidean space time (due to a wick rotation) and therefore

$$g_{\mu,\nu} = \delta_{\mu\nu} \tag{A.9}$$

This ultimatly also influences the clifford algebra and properties of the gamma matrices used. See below.

B. Gamma matrices in arbitrary dimensional euclidean space time

B.1. Explicit choice of gamma matrices

At the end of section 4.3 we argued that we can choose for $d \in \{2, 3, 4\}$ a representation of the gamma matrices such that they are either symmetric or antisymmetric. In Euclidean spacetime, the gamma matrices must satisfy the Clifford algebra defined by

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu} \mathbb{1}_{d_\gamma} \tag{B.1}$$

In $d = 2$ we can choose the matrices

$$\gamma^0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.2}$$

These two matrices are actually both symmetric. For odd dimensions d we can always use a representation of a $d-1$ system and add the chirality operator (which is only defined for even dimensional representations)

$$\gamma^d := \mathrm{i}^{(d-1)/2}\, \gamma^0 \gamma^1 \cdots \gamma^{d-2} \tag{B.3}$$

as additional matrix to complete the representation. Therefore, if we can find a representation where all matrices are either symmetric or antisymmetric for even $d-1$ dimensions, we can also do it in odd d dimensions. This can be seen by the argument

$$\left(\gamma^d\right)^T = \mathrm{i}^{(d-1)/2} \left(\gamma^{d-2}\right)^T \cdots \left(\gamma^1\right)^T \left(\gamma^0\right)^T \tag{B.4}$$

$$= \mathrm{i}^{(d-1)/2} \left(\pm\gamma^{d-2}\right) \cdots \left(\pm\gamma^1\right) \left(\pm\gamma^0\right) \tag{B.5}$$

$$= \mathrm{i}^{(d-1)/2} (\pm) \left(\pm\gamma^0\right) \left(\pm\gamma^1\right) \cdots \left(\pm\gamma^{d-2}\right) \tag{B.6}$$

$$= \pm\gamma^d \tag{B.7}$$

Hence this is also the case for d=3 and the additional matrix is given by

$$\gamma^3 := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{B.8}$$

which is antisymmetric. For $d = 4$ we have the matrices

$$\begin{aligned} \gamma^0 &:= \begin{pmatrix} 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 1 \\ \mathrm{i} & 0 & 0 & 1 \end{pmatrix} & \gamma^1 &:= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \gamma^2 &:= \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} & \gamma^3 &:= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{aligned} \tag{B.9}$$

which also fulfill the property that they are either symmetric or anti-symmetric. Therefore this also holds for $d = 5$ and with certain construction methods it should be possible to construct such representations also for higher dimensions.

B.2. Universality of the argument in section 2.3

Also we argued, at the end of section 4.3, that the mere existence of a representation where the gamma-matrices are either symmetric or anti-symmetric, is sufficient for the vanishing of the trace. This can be seen by simply introducing the following unitary matrix

$$V = U \otimes \mathbb{1}_{N_f} \tag{B.10}$$

where U is a unitary matrix connecting the different representations of the gamma matrices.

$$\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger \tag{B.11}$$

Now, if we have a expression

$$\mathrm{Tr}\left\{\prod_{j=1}^{n} M_i\right\} \tag{B.12}$$

where

$$M_i \in \{P, {}^\circ\Phi^-, {}^\circ\Phi^\circ, {}^-\Phi^-, {}^-\Phi^\circ\} \tag{B.13}$$

we can use the fact the trace is invariant under a unitary transformation

$$\mathrm{Tr}\left\{\prod_{j=1}^{n} M_i\right\} = \mathrm{Tr}\left\{\prod_{j=1}^{n} V M_i V^{\dagger}\right\} \tag{B.14}$$

Now we see

$$VPV\dagger = p_\mu V\left(\gamma^\mu \otimes \mathbb{1}_{N_f}\right)V\dagger = p_\mu\left((U\gamma^\mu U\dagger) \otimes \mathbb{1}_{N_f}\right) \tag{B.15}$$

and therefore that the argument in eq.(4.62) still holds for any representation if the antisymmetry of the matrices $^{\circ}\Phi^{-}$, $^{\circ}\Phi^{\circ}$, $^{-}\Phi^{-}$, $^{-}\Phi^{\circ}$ is preserved under the transformation defined by V. This holds due to the dyadic structure of these matrices. Since V is unitary, and therefore invertible, the vector

$$\begin{pmatrix} \omega_1^1 \\ \vdots \\ \omega_{d_\gamma}^{N_f} \end{pmatrix} := V \begin{pmatrix} \psi_1^1 \\ \vdots \\ \psi_{d_\gamma}^{N_f} \end{pmatrix} \tag{B.16}$$

is also a vector of independent alternative Grassmann generators and has therefor the same algebraic properties. Hence, with

$$^{\circ}\Omega^{-} = V^{\circ}\Phi^{-}V^{\dagger} = V(\psi \oplus \bar{\psi})V^{\dagger} = (V\psi) \oplus (\bar{\psi}V^{\dagger})) \tag{B.17}$$

we realize that the transformed matrix is also a dyadic product of two vectors of independent Grassmann generators and has therefore the same anti-symmetry property.

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