Particle Creation

Joachim Pomper, Philipp Schreiner, Tristan Feyer

February 22, 2022

Contents

1. Introduction

In our project we will be considering real scalar fields and how they behave in a background metric which varies with time. Because we're not dealing with a time-independent system, energy will not be conserved. In particular, we will see that this leads to particle creation.

1.1. Scalar field in gravitational field

Consider a free real scalar field ϕ in usual Minkowski space with Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{1.1}$$

where $\eta^{\mu\nu}$ is the metric tensor of Minkowski space[1]. The equation of motion for ϕ is then obtained by minimizing the action

$$S = \int \mathrm{d}^4x \left(\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{1.2}$$

To couple ϕ *minimally* to a gravitational background field we generalize this result by introducing an arbitrary metric $g_{\mu\nu}$ instead of $\eta_{\mu\nu}$ and to make the integration measure covariant again[2] we further add a factor $\sqrt{-g}$ where g is the determinant of $g_{\mu\nu}$[3]. Our action then becomes

$$S_{\text{min}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right]. \tag{1.3}$$

Another way to couple ϕ is *non-minimally* via an additional term $-\frac{1}{12}R\phi^2$ in the action, where R is the Ricci scalar:[4]

$$S_{\text{non-min}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left(m^2 + \frac{R}{6}\right)\phi^2\right] \tag{1.4}$$

While $S_{\text{non-min}}$ looks a bit more complicated (remember that R also depends on the metric), it has the nice property of being *conformally invariant* for $m = 0$, i.e. invariant under transformations

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu}.$$

Therefore, this type of coupling is also referred to as *conformal coupling.* Even though we're not going to set $m = 0$, this form of the action will nevertheless have a pleasant effect.

[1] We use the particle physics convention $\eta = \mathrm{diag}(+1, -1, -1, -1)$

[2] d^4x would not be in a space with arbitrary metric $g_{\mu\nu}$

[3] The minus sign under the square root is necessary to make g positive. g is always negative if we use the metric grading $(+, -, -, -)$.

[4] Notice that that violates the the strong equivalence principle [1].

Notice that both (1.3) and (1.4) reduce to (1.2) for $g_{\mu\nu} = \eta_{\mu\nu}$. Minimizing (1.3) and (1.4) yields the equation of motion [2]

$$\Box\phi + (m^2 + \xi R)\phi = 0 \tag{1.5}$$

where ξ is either 0 or $\frac{1}{6}$, depending on whether one uses minimal or non-minimal coupling, respectively, and $\Box$ is the covariant generalization of the d'Alembertian, explicitly[5]:

$$\Box\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu \left(g^{\mu\nu}\sqrt{-g}\partial_\nu\phi\right) \tag{1.6}$$

For the rest of the project we will employ the assumption of a homogeneous and isotropic universe for which Einstein's field equations can be solved exactly. The solution thereof is the so-called *Friedmann–Lemaître–Robertson–Walker-Metric* (FLRW). This solution admits both curved and flat metrics but we will only be interested in the flat ($k = 0$) solution with infinitesimal volume

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{1.7}$$

Notice that the scaling factor $a(t)$ only depends on time but not on the spatial part of x. After introducing so-called *conformal time*

$$\eta(t) \equiv \int^t \frac{\mathrm{d}t}{a(t)} \tag{1.8}$$

we see, that (1.7) is conformally equivalent to the Minkowski metric $\eta_{\mu\nu}$

$$\mathrm{d}s^2 = a^2(\eta)\left[\mathrm{d}\eta^2 - \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right] = a^2(\eta)\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \equiv g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{1.9}$$

which will greatly simplify our calculations[6].

We can now write (1.5) for metric (1.9) for which we calculate (recall that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$ and R can be calculated by performing a coordinate transformation on the well known result for the Ricci scalar curvature of the FLRW-metric in FLRW coordinates).

$$\begin{aligned} g^{\mu\nu} &= a^{-2}\eta^{\mu\nu} \\ \sqrt{-g} &= a^4 \\ R &= 6\frac{a''}{a^3} \end{aligned}$$

[5] Notice again, that for $g_{\mu\nu} = \eta_{\mu\nu}$ this reduces to the usual d'Alembertian and (1.5) reduces to the Klein-Gordon-Equation.

[6] Remark: For a massless field this means that we could perform a conformal transformation in (1.4), i.e. effectively using the Minkowski metric $\eta_{\mu\nu}$, while leaving the action invariant. But $R = 0$ for $\eta_{\mu\nu}$, hence, (1.4) reduces to (1.2) with $m = 0$ which means that the dynamics of a massless scalar field do not change if it's conformally coupled to gravity. (This can also be seen mathematically in (6.13) in [2].)

where the double prime denotes the second derivative with respect to η.Plugging this into (1.5) yields

$$\frac{1}{a^4}\partial_\mu\left(a^2\partial^\mu\phi\right)+\left(m^2+6\xi\frac{a''}{a^3}\right)\phi=0$$

which, after defining

$$\chi(x)\equiv a(\eta)\phi(x) \tag{1.10}$$

and exploiting that $\partial_\mu\partial^\mu a=a''$ (as a only depends on η), takes the simple form

$$0=\left[\partial_\mu\partial^\mu+m_{\text{eff}}^2(\eta)\right]\chi(x) \tag{1.11}$$

$$m_{\text{eff}}^2(\eta)\equiv m^2a^2-\frac{a''}{a}+6\xi\frac{a''}{a}. \tag{1.12}$$

Decomposing $\chi(x)=\chi(\eta,\boldsymbol{x})$ into Fourier modes[7]

$$\chi(\eta,\boldsymbol{x})=\int\frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\chi_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}$$

results in a differential equation for the modes:

$$0=\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}+\omega_k^2(\eta)\right]\chi_{\mathbf{k}}(\eta) \tag{1.13}$$

$$\omega_k^2(\eta)\equiv k^2+m_{\text{eff}}^2(\eta) \tag{1.14}$$

where $k^2\equiv|\boldsymbol{k}|^2$. Upon specifying ξ we obtain explicit expressions for the effective mass for the two possibilities to couple the scalar field to gravity[8]:

$$\begin{aligned}\text{minimally coupled }(\xi=0):&\quad m_{\text{eff}}^2(\eta)=m^2a^2-\frac{a''}{a}\\ \text{conformally coupled }(\xi=1/6):&\quad m_{\text{eff}}^2(\eta)=m^2a^2\end{aligned}$$

Notice that for massless fields χ we find that the conformally coupled field behaves just like a free field while the minimally coupled one feels gravity's effect even in the massless case.

From a mathematical point of view, the actual form of $m_{\text{eff}}^2(\eta)$ is irrelevant: Our task is to solve the wave equation (1.13) with a time-dependent mass term. Nevertheless, we will see that the conformally coupled one allows for an easier interpretation in terms of what happens physically: In the actual calculation we will specify $m_{\text{eff}}^2(\eta)$ by some function

$$m_{\text{eff}}^2(\eta)=m^2F(\eta) \tag{1.15}$$

[7] We denote 3-vectors by boldface symbols.

[8] The first one agrees with (6.8) in [1], where the authors discuss the situation of a minimally coupled field. The second one is what the authors of [3] are working with (c.f.(9,10)).

where $F(\eta)$ is such that it approaches constants $c_\pm^2$ for $\eta \to \pm\infty$, i.e. the effective mass (and therefore gravitational effects) is only relevant in some time interval and infinitely far in the past/future our field behaves like a free field of mass $mc_\pm$.
Consequently, we can interpret $F(\eta)$ as the scaling function $a^2(\eta)$ **only** if we couple scalar field and gravity *conformally*! If we coupled them *minimally*, on the other hand, we'd have to solve the differential equation

$$F(\eta) = a^2(\eta) - \frac{a''(\eta)}{m^2 a(\eta)}$$

to find the scaling function $a^2(\eta)$ and to understand how the universe expands/contracts. But since this differential equation contains m explicitly, we expect $a^2(\eta)$ to also depend on m which means that the scaling of the universe depends on the mass of the scalar field it is coupled to. In our project, though, we want the gravitational field to be a background field and in particular, we do not want the scalar field to have any influence on the background.
Obviously, one could just go the other way around: Specify $a^2(\eta)$, calculate $m^2_{\text{eff}}(\eta)$ for the minimally and conformally coupled field and solve (1.13) for that effective mass. Notice, that with a scaling function having the same asymptotic behaviour as discussed above, both fields would behave as if they were free fields of mass $mc_\pm$ in the distant future and past: The second derivative $a''(\eta)$ which appears in the effective mass of the minimally coupled field vanishes in those limits and for the conformally coupled one it is trivial. The initial and final conditions thus seem to be the same in both cases. Though the crucial thing is that the *way to go from past infinity to future infinity* is different. And the actual evolution of the universe's size has a huge influence on which and how many particles are produced during the expansion (as we shall see later). In general, the effective mass expressions for the minimally coupled field in those regions would be a lot more complicated and we didn't manage to find an appropriate $a^2(\eta)$ such that it takes a simple form. Therefore, we will not follow this approach in our project but rather choose the function $F(\eta)$ (because this makes calculations easier) and consequently understand the scalar field to be conformally coupled to gravity (because then we can understand $F(\eta)$ to be the scaling function of the universe and treat gravity as being just a classical background).

1.2. Particle creation in expanding universe

1.2.1. Vacua

Consider a Lagrange density of a scalar field in curved spacetime, which can be quantized according to cite equation of motion. For a spacelike hypersurface Σ one can define an inner product on the solutions of this equation that is independent of the choice of Σ.
In flat spacetime, one then defines a complete orthonormal set of positive and negative frequency modes which form a basis for the solutions, such that one can expand the field operator in terms of these solutions and the creation and annihilation operators. In flat Minkowski space one finds a timelike Killing vector and can this define a positive

frequency mode with respect to it. Hence, one can define a vacuum state and by that an entire Fock space. Characteristically, the Poincare group of symmetries is privileges and therefore, any inertial time coordinate is related by a Lorentz transformation. This does also count for the number operator.
In general spacetime, however, there does not have to be any timelike Killing vector and therefore, one will generally not be able to find mode solutions that separate into time-dependent and space-dependent factors. Consequently, one cannot find sets of basis modes, since there are no distinguished ones. This shows that the notation of the vacuum as well as the number operator depends on the chosen set. Hence, different observers will observe a different number of particles.

Discussion of relevance of plane waves

1.2.2. Bogolubov transformations and coefficients

Let $\phi(x)$ be a general quantized scalar field that can be decomposed in a complete orthonormal set of mode solutions of field equation cite field equation which reads

$$\phi(x) = \sum_i [a_i u_i(x) + a_i^\top u_i^*(x)], \tag{1.16}$$

where the vacuum state $|0>$ is defined by

$$a_i|0>= 0 \forall i \tag{1.17}$$

and a Fock space.
As shown in Birrell, one next considers a second complete orthonormal set of modes $\bar{u}_j(x)$ and expands the aforementioned scalar field $\phi(x)$ in terms of the new set as

$$\phi(x) = \sum_j [\bar{a}_j \bar{u}_j(x) + \bar{a}_j^\top \bar{u}_j^*(x)]. \tag{1.18}$$

For this orthonormal set of modes the vacuum state $|\bar{0}>$ is given by

$$\bar{a}_j|\bar{0}>= 0 \forall j. \tag{1.19}$$

Thus, one works in a new Fock space.
Since both of the sets are complete it is valid to expand the new modes $\bar{u}_j$ in terms of the old ones u_i:

$$\bar{u}_j = \sum_i [\alpha_{ji} u_i + \beta_{ji} u_i^*]. \tag{1.20}$$

Respectively,

$$u_i = \sum_j [\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*]. \tag{1.21}$$

1.20 and 1.21 are called Bogolubov transformations. These isomorphisms are linear transformations of the creation and annihilation operators and have the speciality to preserve the algebraic relations among those. The coefficients α_{ji} and β_{ji} are known as

Bogolubov coefficients. some more on them with respect to the Sauter pulse
α_{ji} and β_{ji} connect the creation and annihilation operators by

$$\begin{aligned} a_i &= \sum_j [\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}_j^\top], \\ \bar{a}_j &= \sum_i [\alpha^*_{ji} a_i - \beta^*_{ji} a_i^\top]. \end{aligned} \tag{1.22}$$

By this, one finds that the two Fock spaces generally differ from each other as long as $\beta_{ji} \neq 0$. It is then straight forward to show that $a_i|0>$ is not a vacuum:

$$\begin{aligned} a_i|0> &= \sum_j [\alpha_{ji}\bar{a}_j + \beta^*_{ji}\bar{a}^*_j]|\bar{0}> \\ &= \sum_j \beta^*_{ji}|\bar{1}_j> \\ &\neq 0. \end{aligned} \tag{1.23}$$

Consequently, the expectation value of the operator for the number of u_i modes, $N_i = a_i^\top a_i$, gives a non-zero value in $|\bar{0}>$:

$$<\bar{0}|N_i|\bar{0}>= \sum_j |\beta_{ji}|^2. \tag{1.24}$$

From this, one reads off that the vacuum of the $\bar{u}_j$ modes contains $\sum_j |\beta_{ji}|^2$ particles in the u_i mode. This is what one refers to as particle creation[9].

1.3. The physical vacuum

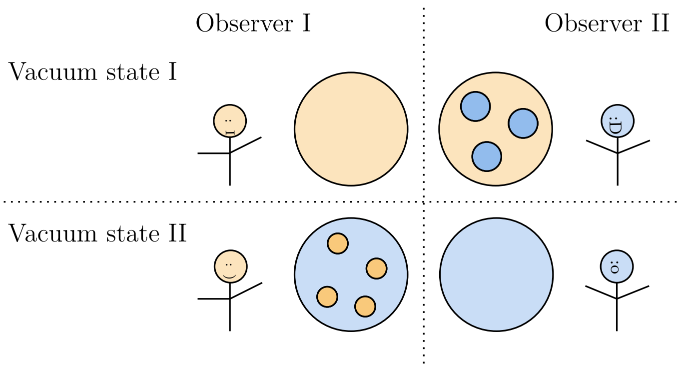


Figure 1: Observer I defines a vacuum state which is the lowest energy state to him and also contains no particles. When observer II measures the *same* state (as indicated by the color) they find what to their perception is referred to as particles. Likewise, Observer II defines their vacuum state as having no particles from their point of view. In that exact state, though, Observer I measures particles.

discussion pending ...

[9]An illustrative calculation is done in appendix A.2.

2. Particle creation due to expansion

2.1. The model

Remember that in section 1.1 we restricted ourselves to a scalar field which is coupled to gravity conformally. The effective mass of the field in that case was

$$m_{\text{eff}}^2(\eta) = m^2 F(\eta)$$

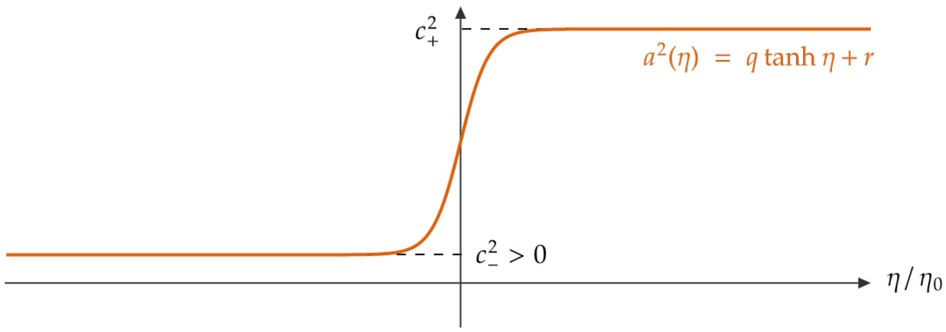
where $F(\eta)$ is also the scaling function of the universe. To specify a model, our only task left is to specify the function $F(\eta)$: In order to have a well-defined notion of particles and vacua before and after the expansion, we want the model to exhibit a static universe behaviour in the remote past and future (at least asymptotically). This was previously alluded to by requiring that $F(\eta)$ asymptotically approaching constants $c_\pm^2$ for $\eta \to \pm\infty$. Such a behaviour can for example be modelled by[10]

$$F(\eta) = a^2(\eta) = q \tanh \frac{\eta}{\eta_0} + r \tag{2.1}$$

$$q = \frac{c_+^2 - c_-^2}{2} \tag{2.2}$$

$$r = \frac{c_+^2 + c_-^2}{2} \tag{2.3}$$

where η_0 sets a scale on which the expansion happens, smaller η_0 meaning more rapid expansion.
Furthermore, we want that the universe starts out at finite size, i.e. we set $c_-^2 > 0$ and we want it to expand, i.e. $c_+^2 > c_-^2$. Such a function looks like the following:



Disclaimer: This model is not meant to be highly physical or something that could actually exist without contradicting General Relativity. This choice is merely instructive as it allows for an analytic solution.
Our goal is thus to solve (1.13), i.e.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + \omega_k^2(\eta)\right] \chi_{\boldsymbol{k}}(\eta) = 0 \quad \text{where} \quad \omega_k^2(\eta) = k^2 + m_{\text{eff}}^2(\eta) \tag{2.4}$$

[10]see appendix A.1 for a version also including a $\tanh^2$ term

with an effective mass given by

$$m_{\text{eff}}^2(\eta) = m^2 \left(q \tanh \frac{\eta}{\eta_0} + r \right). \tag{2.5}$$

We can already see that for asymptotic times $\eta \to \mp\infty$, ω_k approaches a constant value

$$\omega_{\text{in}} \equiv \omega_k(\eta \to -\infty) = \sqrt{k^2 + (r - q)m^2} = \sqrt{k^2 + c_-^2 m^2} \tag{2.6}$$

$$\omega_{\text{out}} \equiv \omega_k(\eta \to +\infty) = \sqrt{k^2 + (r + q)m^2} = \sqrt{k^2 + c_+^2 m^2} \tag{2.7}$$

and therefore, in the distant past and future, the solutions χ will behave like plane waves of those frequencies as (2.4) asymptotically reduces to a simple wave equation. Notice that this is exactly what we need to define the notion of a vaccum.
The functions $\chi_{\boldsymbol{k}}(\eta)$ are the modes of the auxiliary field $\chi(\eta, \boldsymbol{x})$ which relates to the initial real scalar field $\phi(x)$ as (c.f. (1.10))

$$\chi(x) \equiv a(\eta)\phi(x)$$

with $a^2(\eta)$ being the conformal factor. Since $\omega_{\mathbf{k}}^2(\eta) = \omega_k^2(\eta)$ is direction-independent, one can find solutions $v_{\mathbf{k}}^*$ and $v_{\mathbf{k}}$ to the ODE[11] which also depend on $k = |\mathbf{k}|$ only. Because χ is a real field, the complex conjugate components are related by $\chi_{\mathbf{k}}^* = \chi_{-\mathbf{k}}$. The general solution can hence be written as[12]

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}}[a_{\mathbf{k}}^- v_{\mathbf{k}}^*(\eta) + a_{-\mathbf{k}}^+ v_{\mathbf{k}}(\eta)] \tag{2.8}$$

where $a_{\mathbf{k}}^-$ and $a_{-\mathbf{k}}^+$ are integration constants and are related to each other by

$$a_{\mathbf{k}}^+ = (a_{\mathbf{k}}^-)^*. \tag{2.9}$$

For the linearly independent functions $v_{\mathbf{k}}^*$ and $v_{\mathbf{k}}$ (which satisfy the ODE) one can define a scalar product using the Wronski-determinant

$$W[v_{\mathbf{k}}, v_{\mathbf{k}}^*] \equiv v_{\mathbf{k}}' v_{\mathbf{k}}^* - v_{\mathbf{k}}(v_{\mathbf{k}}^*)' = 2i\,\text{Im}(v_{\mathbf{k}}' v_{\mathbf{k}}^{*2}) \tag{2.10}$$

which can be shown to be η-independent (a prime denotes the derivative with respect to η). One can use this to *normalize* the component functions:

$$\begin{aligned} W[v_{\mathbf{k}}, v_{\mathbf{k}}^*] &\overset{!}{=} 2i \\ \Leftrightarrow \text{Im}(v_{\mathbf{k}}' v_{\mathbf{k}}^*) &= 1 \end{aligned} \tag{2.11}$$

which then holds for all time. Once this holds[13], one calls $v_{\mathbf{k}}$ and $v_{\mathbf{k}}^*$ mode functions. To give a short summary: Our final task is to find mode functions satisfying

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + \omega_k^2(\eta) \right] v_{\mathbf{k}}(\eta) = 0 \qquad \text{where} \qquad \text{Im}(v_{\mathbf{k}}' v_{\mathbf{k}}^*) = 1. \tag{2.12}$$

[11]For a complex scalar field one would have to find two independent mode functions $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$ but since we shall continue treating real fields only, the two mode functions are connected by complex conjugation.

[12]see [1] for details

[13]which is apparently linked to current conservation of the Klein-Gordon field [3] (can we please discuss this?)

2.2. Hypergeometric differential equation

Equation (2.12) can be solved by substituting

$$\xi(\eta) \equiv \frac{1}{2}\left(1+\tanh\frac{\eta}{\eta_0}\right), \tag{2.13}$$

leading to[14]

$$0 = \left[\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}+\omega_k^2\right]v_{\mathbf{k}} = \left[\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{\partial\xi}{\partial\eta}\frac{\partial}{\partial\xi}+\omega_k^2\right]v_{\mathbf{k}} = \left[\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{\partial\xi}{\partial\eta}\right)\frac{\partial}{\partial\xi}+\left(\frac{\partial\xi}{\partial\eta}\frac{\partial\xi}{\partial\eta}\right)\frac{\partial^2}{\partial\xi^2}+\omega_k^2\right]v_{\mathbf{k}}.$$

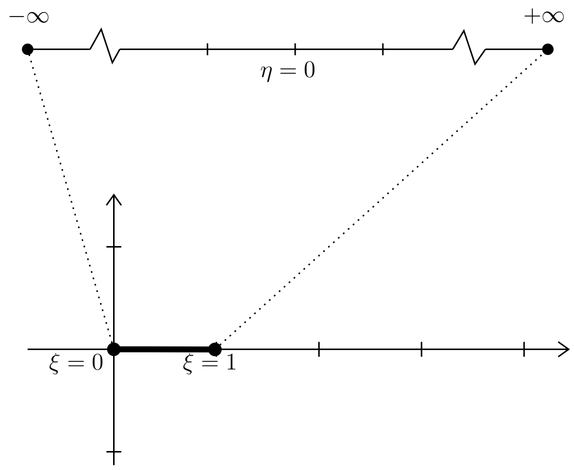


Figure 2: Substitution (2.13) maps the real axis to the unit interval, i.e. formally, the points $\eta = \pm\infty$ are mapped to $\xi = 0, 1$ and the whole real axis fits in between. The differential equation can then be analytically continued to the whole complex plane (actually, this is what we will exploint) but keep in mind that in the end we will just be interested in the solution for $\xi \in [0, 1]$ i.e. $\eta \in (-\infty, \infty)$.

Now using that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\eta}\frac{\partial\xi}{\partial\eta} &= \frac{\tanh\frac{\eta}{\eta_0}}{\eta_0^2}\left(1-\tanh^2\frac{\eta}{\eta_0}\right) = \frac{4}{\eta_0^2}(2\xi-1)(\xi^2-\xi) \quad \text{and}\\ \frac{\partial\xi}{\partial\eta}\frac{\partial\xi}{\partial\eta} &= \left[\frac{1}{2\eta_0}\left(1-\tanh^2\frac{\eta}{\eta_0}\right)\right]^2 = \left(\frac{2}{\eta_0}(\xi^2-\xi)\right)^2\end{aligned}$$

one obtains

$$\begin{aligned}\left[\frac{4}{\eta_0^2}\left(\xi(\xi-1)\right)^2\frac{\partial^2}{\partial\xi^2}+\frac{4}{\eta_0^2}\xi(\xi-1)(2\xi-1)\frac{\partial}{\partial\xi}+\omega_k^2\right]v_{\mathbf{k}} &= 0\\ \left[\xi^2(\xi-1)^2\frac{\partial^2}{\partial\xi^2}+\xi(\xi-1)(2\xi-1)\frac{\partial}{\partial\xi}+\frac{\eta_0^2}{4}\omega_k^2\right]v_{\mathbf{k}} &= 0.\end{aligned}$$

[14] With an abuse of notation, i.e. keeping the same symbol for $v_{\mathbf{k}}(\xi(\eta))$ and $v_{\mathbf{k}}(\eta)$. Furthermore, we omit the arguments of $\omega_k^2(\xi(\eta))$ and $v_{\mathbf{k}}(\xi(\eta))$ to enhance readability.

We now need an explicit expression for ω_k^2:

$$\begin{aligned}
\omega_k^2 &= k^2 + m^2 \left[q \tanh \frac{\eta}{\eta_0} + r \right] \\
&= k^2 + m^2 \left(r - q(1-2\xi) \right) \\
&= \omega_{\text{in}}^2 + 2qm^2\xi \\
&\equiv \omega_{\text{in}}^2 + \tilde{c}\xi
\end{aligned}$$

where we see (using (2.6) and (2.7)) that

$$\omega_{\text{in}}^2 + \tilde{c} = \omega_{\text{out}}^2.$$

If we further define

$$c \equiv \frac{\eta_0^2}{4}\tilde{c} \tag{2.14}$$

$$\Omega_{\text{in}}^2 \equiv \frac{\eta_0^2}{4}\omega_{\text{in}}^2 \tag{2.15}$$

$$\Omega_{\text{out}}^2 \equiv \frac{\eta_0^2}{4}\omega_{\text{out}}^2 = \Omega_{\text{in}}^2 + c \tag{2.16}$$

$$\Omega_\pm \equiv \Omega_{\text{out}} \pm \Omega_{\text{in}} \tag{2.17}$$

the differential equation becomes

$$0 = \left[\xi^2(1-\xi)^2 \frac{\partial^2}{\partial \xi^2} + \xi(1-\xi)(1-2\xi)\frac{\partial}{\partial \xi} + \Omega_{\text{in}}^2 + c\xi \right] v_{\mathbf{k}} \tag{2.18}$$

Since we want our solutions to behave like plane waves for $\eta \to \pm\infty$ (i.e. $\xi \to 0^+, 1^-$) we make the ansatz[15]

$$v_\epsilon(\xi) = N\xi^{-\epsilon i \Omega_{\text{in}}}(1-\xi)^{\epsilon i \Omega_{\text{out}}} h_\epsilon(\xi) \tag{2.19}$$

with $\epsilon = \pm 1$ just a sign to compensate the fact that $\sqrt{\Omega^2} = \pm\Omega = \epsilon\Omega$ and N a normalization constant to be determined by (2.11). The demonstration that this ansatz indeed asymptotically behaves like plane waves is postponed to the end of section 2.3. Substituting the ansatz into (2.18) gives[16]

$$\left[\xi(1-\xi)\frac{\partial^2}{\partial \xi^2} + (1 - 2i\epsilon\Omega_{\text{in}} - 2(1+i\epsilon\Omega_-)\xi)\frac{\partial}{\partial \xi} - (i\epsilon\Omega_- - \Omega_-^2) \right] h_\epsilon(\xi) = 0$$

which has the form of the hypergeometric differential equation

$$\left[x(1-x)\frac{\partial^2}{\partial x^2} + (\gamma - (\alpha+\beta+1)x)\frac{\partial}{\partial x} - \alpha\beta \right] h(x) = 0 \tag{2.20}$$

[15] We will omit the $\boldsymbol{k}$-dependence for now to enhance readability.

[16] don't try this by hand

if we set

$$\gamma = 1 - 2i\epsilon\Omega_{\text{in}}$$
$$\alpha + \beta + 1 = 2(1 + i\epsilon\Omega_-)$$
$$\alpha\beta = (i\epsilon\Omega_- - \Omega_-^2).$$

Explicit expressions can be obtained by solving this system of equations:

$$\alpha = i\epsilon\Omega_- \tag{2.21}$$
$$\beta = 1 + i\epsilon\Omega_- \tag{2.22}$$
$$\gamma = 1 - 2i\epsilon\Omega_{\text{in}} \tag{2.23}$$

for which the following very convenient relations hold

$$1 - \alpha = \beta^* \tag{2.24}$$
$$1 - \beta = \alpha^* \tag{2.25}$$
$$2 - \gamma = \gamma^* \tag{2.26}$$
$$1 - \gamma = 2i\epsilon\Omega_{\text{in}} \tag{2.27}$$
$$\gamma - \alpha - \beta = -2i\epsilon\Omega_{\text{out}}. \tag{2.28}$$

Now that we know the coefficients of the hypergeometric differential equations the problem is basically solved.

2.3. Solutions to the Hypergeometric differential equation

We can now easily find the solutions of the hypergeometric differential equation which are tabulated in [4]. In particular, we are interested in solutions $h_\epsilon(\eta)$ which are asymptotically constant for $\eta \to \pm\infty$, i.e. in the regular singular points $\xi = 0, 1$ of the differential equation (because then our mode functions v's behaviour is determined by the plane wave part of the ansatz (2.19)). For each of those points there exist two linearly independent solutions defined within an open disc of radius 1 in the complex plane, centered around that points (see fig. 3). If neither γ nor $\gamma - \alpha - \beta$ are integers (which is the case here) the solution reads[17]

$$\begin{aligned}
h^{(0)}_{\epsilon,1}(\xi) &= F(\alpha, \beta, \gamma; \xi) \\
h^{(0)}_{\epsilon,2}(\xi) &= \xi^{1-\gamma}(1-\xi)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma; \xi) \\
h^{(1)}_{\epsilon,1}(\xi) &= F(\alpha, \beta, \alpha+\beta+1-\gamma; 1-\xi) \\
h^{(1)}_{\epsilon,2}(\xi) &= \xi^{1-\gamma}(1-\xi)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, \gamma-\alpha-\beta+1; 1-\xi)
\end{aligned} \tag{2.29}$$

which, after using relations (2.24)-(2.28) can be rewritten as

[17]c.f. chapter (15.10(i)) of [5] together with their equation (15.8.1). Notice that the differential equation also has a regular singular point at $\xi = \infty$ but we are only interested in $\xi = 0, 1$, as discussed earlier. This is actually a great relieve because for the solution to have the nice plane wave behaviour at $\xi = \infty$, we'd also need $\alpha - \beta$ to not be an integer. A requirement that would fail here.

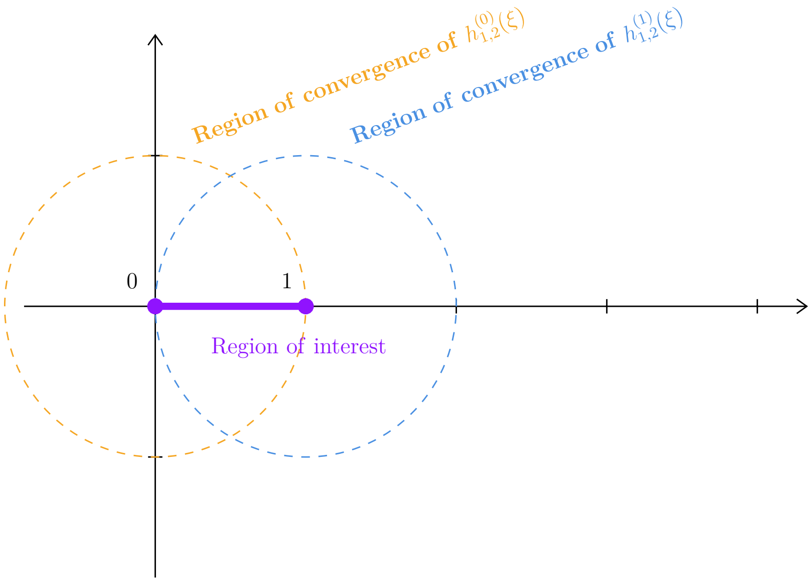


Figure 3: The hypergeometric functions can be expressed as a series with convergence radius 1. The special solutions hold in a neighborhood of 0 and 1, which means (by analycity) that they can be unambigously continued to the whole region of convergence (i.e. in particular to the whole interval $[0,1]$ which is our region of interest). One may as well discuss all the other analytic properties of the solutions outside this interval but we're not interested in that as discussed before. This means, that for our purposes the solutions given in (2.29) may be treated as analytic functions defined everywhere in $[0,1]$. In particular, this also means, that we are allowed to compare the solutions $h^{(0)}$ to $h^{(1)}$.

$$
\begin{aligned}
h^{(0)}_{\epsilon,1}(\xi) &= F(\alpha,\beta,\gamma;\xi)\\
h^{(0)}_{\epsilon,2}(\xi) &= \xi^{2i\epsilon\Omega_{\text{in}}}(1-\xi)^{-2i\epsilon\Omega_{\text{out}}}F(\beta^*,\alpha^*,\gamma^*;\xi)\\
h^{(1)}_{\epsilon,1}(\xi) &= F(\alpha,\beta,\alpha+\beta+1-\gamma;1-\xi)\\
h^{(1)}_{\epsilon,2}(\xi) &= \xi^{2i\epsilon\Omega_{\text{in}}}(1-\xi)^{-2i\epsilon\Omega_{\text{out}}}F(\beta^*,\alpha^*,\alpha^*+\beta^*+1-\gamma^*;1-\xi).
\end{aligned}
\tag{2.30}
$$

Inserting this into ansatz (2.19) we obtain[18]

$$
\begin{aligned}
v^{(0)}_{\epsilon,1}(\xi) &= N^{(0)}\xi^{-\epsilon i\Omega_{\text{in}}}(1-\xi)^{\epsilon i\Omega_{\text{out}}}F(\alpha,\beta,\gamma;\xi)\\
v^{(0)}_{\epsilon,2}(\xi) &= N^{(0)}\xi^{\epsilon i\Omega_{\text{in}}}(1-\xi)^{-\epsilon i\Omega_{\text{out}}}F(\alpha^*,\beta^*,\gamma^*;\xi)\\
v^{(1)}_{\epsilon,1}(\xi) &= N^{(1)}\xi^{-\epsilon i\Omega_{\text{in}}}(1-\xi)^{\epsilon i\Omega_{\text{out}}}F(\alpha,\beta,\alpha+\beta+1-\gamma;1-\xi)\\
v^{(1)}_{\epsilon,2}(\xi) &= N^{(1)}\xi^{\epsilon i\Omega_{\text{in}}}(1-\xi)^{-\epsilon i\Omega_{\text{out}}}F(\alpha^*,\beta^*,\alpha^*+\beta^*+1-\gamma^*;1-\xi)
\end{aligned}
\tag{2.31}
$$

with unspecified normalization constants (which w.l.o.g. were assumed to be real). Notice that those are just two functions and their complex conjugates (since one can easily see from (2.20) that the hypergeometric function with complex conjugated arguments

[18]we also exploit that F is completely symmetric in α and β (can be seen from differential equation)

is just the complex conjugated function, i.e. $F(\alpha^*, \beta^*, \gamma^*, x) = F^*(\alpha, \beta, \gamma, x)$). Therefore, we only have to calculate two normalization constants $N^{(0,1)}$. Those can be found by condition (2.11). But since it turns out that this is rather tedious and we do not explicitly need them here, we apologize to the completionist reader and leave them undetermined for now.

We can now freely choose $\epsilon = +1$ and call the first and third equation *positive energy solutions* v^+. Thus one obtains the following expressions for the mode functions ($v_{\text{in/out}}$ behaving like a plane wave at $\mp\infty$):

$$\begin{aligned}
v_{\text{in}}^+ &= N^{(0)} \xi^{-i\Omega_{\text{in}}} (1-\xi)^{i\Omega_{\text{out}}} F(\alpha, \beta, \gamma; \xi) \\
v_{\text{in}}^- &= N^{(0)} \xi^{i\Omega_{\text{in}}} (1-\xi)^{-i\Omega_{\text{out}}} F(\alpha^*, \beta^*, \gamma^*; \xi) \\
v_{\text{out}}^+ &= N^{(1)} \xi^{-i\Omega_{\text{in}}} (1-\xi)^{i\Omega_{\text{out}}} F(\alpha, \beta, \alpha+\beta+1-\gamma; 1-\xi) \\
v_{\text{out}}^- &= N^{(1)} \xi^{i\Omega_{\text{in}}} (1-\xi)^{-i\Omega_{\text{out}}} F(\alpha^*, \beta^*, \alpha^*+\beta^*+1-\gamma^*; 1-\xi).
\end{aligned} \tag{2.32}$$

from which it is apparent that we have pairs of complex conjugate functions (as we expected the mode functions to be as our initial field was real). The asymptotic behavior of a positive or negative frequency solution, indicated by the superscripts + and −, can be read off by the asymptotic behavior for the in- and outgoing solutions, respectively. First, consider the ingoing mode functions: They are supposed to behave like plane waves for $\eta \to -\infty$ or, equivalently, $\xi \to 0^-$. In order to see that this is the case let's remind ourselves of the definition of ξ in (2.13). Then we have

$$\xi = \frac{1}{2}\left(1 + \tanh\frac{\eta}{\eta_0}\right) \tag{2.33}$$

$$1 - \xi = \frac{1}{2}\left(1 - \tanh\frac{\eta}{\eta_0}\right). \tag{2.34}$$

This can be rewritten in terms of exponential functions:

$$1 \pm \tanh\frac{\eta}{\eta_0} = \frac{2\,\mathrm{e}^{\pm\frac{\eta}{\eta_0}}}{\mathrm{e}^{+\frac{\eta}{\eta_0}} + \mathrm{e}^{-\frac{\eta}{\eta_0}}} = \frac{2}{1 + \mathrm{e}^{\mp\frac{\eta}{\eta_0}}}$$

Hence we obtain

$$\begin{aligned}
\xi^{-i\Omega_{in}} (1-\xi)^{i\Omega_{out}} \quad &= \left(\frac{1-\xi}{\xi}\right)^{i\Omega_{in}} (1-\xi)^{i\Omega_-} \\
&= \left(\mathrm{e}^{-2\frac{\eta}{\eta_0}}\right)^{i\Omega_{in}} \left(\frac{1}{1+\mathrm{e}^{\frac{\eta}{\eta_0}}}\right)^{i\Omega_{out}} \\
&\underset{\eta\to-\infty}{\sim} \mathrm{e}^{-i\frac{2\Omega_{in}}{\eta_0}\eta} = \mathrm{e}^{-i\omega_{\text{in}}\eta}.
\end{aligned} \tag{2.35}$$

By complex conjugation it follows

$$\xi^{i\Omega_{in}} (1-\xi)^{-i\Omega_{out}} \underset{\eta\to-\infty}{\sim} \mathrm{e}^{i\frac{2\Omega_{in}}{\eta_0}\eta} = \mathrm{e}^{i\omega_{\text{in}}\eta}. \tag{2.36}$$

Taking into account that the ingoing solutions v_{in}^+ and v_{in}^- in (2.32) are constructed such that $F(\alpha,\beta,\gamma;\xi) \to 1$ for $\xi \to 0^-$, one can now easily read off that v_{in}^+ asymptotically behaves like a positive frequency plane wave at $\eta \to -\infty$ and v_{in}^- like a negative frequency one. A similar calculation for $\eta \to +\infty$ yields that

$$\xi^{-i\Omega_{in}}(1-\xi)^{i\Omega_{out}} \underset{\eta\to\infty}{\sim} \mathrm{e}^{-i\frac{2\Omega_{out}}{\eta_0}\eta} = \mathrm{e}^{-i\omega_{\mathrm{out}}\eta} \tag{2.37}$$

$$\xi^{i\Omega_{in}}(1-\xi)^{-i\Omega_{out}} \underset{\eta\to\infty}{\sim} \mathrm{e}^{i\frac{2\Omega_{out}}{\eta_0}\eta} = \mathrm{e}^{i\omega_{\mathrm{out}}\eta} \tag{2.38}$$

and thus again an easy identification of positive and negative frequency solution for the outgoing modes.
We now have two equivalent pairs of solutions of the initial differential equation. Those can be related by a Bogolyubov transformation as explained in section 1.2 and the Bogolyubov coefficients can be used to calculate the density of particles created during the expansion.

2.4. Particle creation (Bogolyubov transformation)

Writing out the dependencies again, the Bogolyubov transformation reads as

$$\begin{aligned} v_{\mathrm{in}}^+(k,\eta) &= a_k v_{\mathrm{out}}^+(k,\eta) + b_k v_{\mathrm{out}}^-(k,\eta) \\ v_{\mathrm{in}}^-(k,\eta) &= b_k^* v_{\mathrm{out}}^+(k,\eta) + a_k^* v_{\mathrm{out}}^-(k,\eta) \\ |a_k|^2 - |b_k|^2 &= 1. \end{aligned}$$

In our case, the two equations are the same (because we treat a real field) so we only have to find $a_k,\ b_k$ such that v_{in}^+ can be expressed via $v_{\mathrm{out}}^\pm$. For that we only need to consider the following three equations of (2.32) (which we write in an earlier form using again relations (2.24)-(2.28)):

$$\begin{aligned} v_{\mathrm{in}}^+ &= N^{(0)}\xi^{-i\Omega_{\mathrm{in}}}(1-\xi)^{i\Omega_{\mathrm{out}}}F(\alpha,\beta,\gamma;\xi) \\ v_{\mathrm{out}}^+ &= N^{(1)}\xi^{-i\Omega_{\mathrm{in}}}(1-\xi)^{i\Omega_{\mathrm{out}}}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-\xi) \\ v_{\mathrm{out}}^- &= N^{(1)}\xi^{i\Omega_{\mathrm{in}}}(1-\xi)^{-i\Omega_{\mathrm{out}}}F(1-\alpha,1-\beta,\gamma-\alpha-\beta+1;1-\xi). \end{aligned} \tag{2.39}$$

Now using the transformation formula[19]

$$\begin{aligned} F(\alpha,\beta,\gamma;\xi) = \kappa_1 F(\alpha,\beta,\alpha+\beta-\gamma+1;1-\xi) \\ + \kappa_2 \xi^{1-\gamma}(1-\xi)^{\gamma-\alpha-\beta}F(1-\alpha,1-\beta,\gamma-\alpha-\beta+1;1-\xi) \end{aligned} \tag{2.40}$$

with

$$\begin{aligned} \kappa_1 &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \\ \kappa_2 &= \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \end{aligned} \tag{2.41}$$

[19] as arises from (9.131 1/2) in [4]. Notice that we also exploited $F(a,b,c;x) = F(b,a,c;x)$ again.

we can pretty much read off the Bogolyubov coefficients: Inserting our relations for $\alpha,\ \beta,\ \gamma$ and multiplying by $N^{(0)}N^{(1)}\xi^{-i\Omega_{\text{in}}}(1-\xi)^{i\Omega_{\text{out}}}$ gives

$$\begin{aligned}
&N^{(0)}N^{(1)}\xi^{-i\Omega_{\text{in}}}(1-\xi)^{i\Omega_{\text{out}}}F(\alpha,\beta,\gamma;\xi)\\
&\quad=\kappa_1 N^{(0)}N^{(1)}\xi^{-i\Omega_{\text{in}}}(1-\xi)^{i\Omega_{\text{out}}}F(\alpha,\beta,\alpha+\beta-\gamma+1;1-\xi)\\
&\qquad+\kappa_2 N^{(0)}N^{(1)}\xi^{i\Omega_{\text{in}}}(1-\xi)^{-i\Omega_{\text{out}}}F(1-\alpha,1-\beta,\gamma-\alpha-\beta+1;1-\xi)
\end{aligned}$$

or, looking at (2.39),

$$N^{(1)}v^+_{\text{in}}=\kappa_1 N^{(0)}v^+_{\text{out}}+\kappa_2 N^{(0)}v^-_{\text{out}} \tag{2.42}$$

which means that our Bogolyubov coefficients are just given by $\kappa_{1,2}$ and the normalization constants. Our final result is

$$a_k=\frac{N^{(0)}}{N^{(1)}}\frac{\Gamma(1-2i\Omega_{\text{in}})\Gamma(-2i\Omega_{\text{out}})}{\Gamma\left(1-i\Omega_+\right)\Gamma\left(-i\Omega_+\right)} \tag{2.43}$$

$$b_k=\frac{N^{(0)}}{N^{(1)}}\frac{\Gamma(1-2i\Omega_{\text{in}})\Gamma(2i\Omega_{\text{out}})}{\Gamma\left(i\Omega_-\right)\Gamma\left(1+i\Omega_-\right)}. \tag{2.44}$$

Now is a good point to talk about the normalization constants: As you can see, we only need the ratio $N^{(0)}/N^{(1)}$ and we know that

$$\begin{aligned}
&|a_k|^2-|b_k|^2=1\\
\Leftrightarrow\ &|\kappa_1|^2-|\kappa_2|^2=\left|\frac{N^{(1)}}{N^{(0)}}\right|^2
\end{aligned} \tag{2.45}$$

must hold. Therefore, it's totally sufficient for our purposes to calculate only this quantity: Using some relations for the Γ-function[20] we obtain

$$\begin{aligned}
|\kappa_2|^2&=\frac{|-2i\Omega_{\text{in}}|^2\,|\Gamma(-2i\Omega_{\text{in}})|^2\,|\Gamma(2i\Omega_{\text{out}})|^2}{|i\Omega_-|^2\,|\Gamma\left(i\Omega_-\right)|^2\,|\Gamma\left(i\Omega_-\right)|^2}\\
&=\frac{4\Omega_{\text{in}}^2}{\Omega_-^2}\frac{\Omega_-^2\left(\sinh\pi\Omega_-\right)^2}{4\Omega_{\text{in}}\Omega_{\text{out}}\sinh(2\pi\Omega_{\text{in}})\sinh(2\pi\Omega_{\text{out}})}\\
&=\frac{\Omega_{\text{in}}}{\Omega_{\text{out}}}\frac{\left(\sinh\pi\Omega_-\right)^2}{\sinh(2\pi\Omega_{\text{in}})\sinh(2\pi\Omega_{\text{out}})}
\end{aligned} \tag{2.46}$$

$$|\kappa_1|^2=\frac{\Omega_{\text{in}}}{\Omega_{\text{out}}}\frac{\left(\sinh\pi\Omega_+\right)^2}{\sinh(2\pi\Omega_{\text{in}})\sinh(2\pi\Omega_{\text{out}})} \tag{2.47}$$

and after applying some basic relations for the hyperbolic functions the final result is

$$|\kappa_2|^2=\frac{\Omega_{\text{in}}}{\Omega_{\text{out}}}\frac{\cosh(2\pi\Omega_-)-1}{\cosh(2\pi\Omega_+)-\cosh(2\pi\Omega_-)} \tag{2.48}$$

$$|\kappa_1|^2=\frac{\Omega_{\text{in}}}{\Omega_{\text{out}}}\frac{\cosh(2\pi\Omega_+)-1}{\cosh(2\pi\Omega_+)-\cosh(2\pi\Omega_-)}. \tag{2.49}$$

[20] $\Gamma(x+1)=x\Gamma(x)$, $|\Gamma(iy)|^2=\pi/(y\sinh\pi y)$, for $y\in\mathbb{R}$ (see [4] sec. 8.33)

Then (2.45) reveals

$$\left|\frac{N^{(1)}}{N^{(0)}}\right|^2 = \frac{\Omega_{\text{in}}}{\Omega_{\text{out}}}. \tag{2.50}$$

For finding the density of particles created during the expansion we are interested in $|b_k|^2$:

$$n(k) = |b_k|^2 = \left|\frac{N^{(0)}}{N^{(1)}}\right|^2 |\kappa_2|^2 = \frac{\cosh(2\pi\Omega_-) - 1}{\cosh(2\pi\Omega_+) - \cosh(2\pi\Omega_-)} \tag{2.51}$$

where we remind the reader of some definitions from earlier:

$$\begin{aligned}
\Omega_{\text{in}} &= \frac{\eta_0}{2}\sqrt{k^2 + (r-q)m^2} = \frac{\eta_0}{2}\sqrt{k^2 + c_-^2 m^2} \\
\Omega_{\text{out}} &= \frac{\eta_0}{2}\sqrt{k^2 + (r+q)m^2} = \frac{\eta_0}{2}\sqrt{k^2 + c_+^2 m^2} \\
\Omega_{\pm} &= \Omega_{\text{out}} \pm \Omega_{\text{in}}
\end{aligned}$$

This result is in agreement with [3][21].
Notice that for $m = 0$ we have that $\Omega_- = 0$ and therefore $n(k, m = 0) = 0$ as well, i.e. no *massless* particles are produced during the expansion.
Finally, one can find the total number of produced particles for different values of η_0 and m by integrating the density:

$$N(\eta_0, m) = \int_{\mathbb{R}} \mathrm{d}^3\boldsymbol{k}\ n(k; \eta_0, m) = 4\pi \int_0^\infty \mathrm{d}k\ k^2\ n(k; \eta_0, m) \tag{2.52}$$

2.5. Discussion

Since we're dealing with three parameters m, k and η_0 which still have dimensions our first goal is to define dimensionless parameters. This can be achieved by defining

$$\tilde{m} \equiv c_+\eta_0 m \tag{2.53}$$
$$\tilde{k} \equiv \eta_0 k. \tag{2.54}$$

It can be understood as using k and m in units of η_0^{-1} and we further scaled m such that it relates to the mass *after* the expansion of the universe (remember that $m_{\text{eff}}^2 = a^2 m^2$, i.e. the notion of mass *scales* together with the universe). Because in order to be able to later relate results for different expansion scenarios we need to normalize $\tilde{m}$ such that it relates to the respective mass scale *after* the universe stopped expanding[22]. With these

[21] Set $a = r$, $b = q$, $c = 0$, $\lambda = \eta_0^{-1}$

[22] Measuring a particle's mass to be $\tilde{m} = 1$ then amounts to reading the value $1/\eta_0$ off of a scale in the remote future, long after the expansion of the universe ended. Normalizing it like this makes it comparable to different expansion scenarios: After all, this is the value you measure and say *the expansion created that many particles of that specific mass.*

definitions, (2.51) becomes

$$
\begin{aligned}
n(\tilde{k}) &= \frac{\cosh \pi \left(\sqrt{k^2\eta_0^2 + c_+^2\eta_0^2 m^2} - \sqrt{k^2\eta_0^2 + c_-^2\eta_0^2 m^2} \right) - 1}{\cosh \pi \left(\sqrt{k^2\eta_0^2 + c_+^2\eta_0^2 m^2} + \sqrt{k^2\eta_0^2 + c_-^2\eta_0^2 m^2} \right) - \cosh \pi \left(\sqrt{k^2\eta_0^2 + c_+^2\eta_0^2 m^2} - \sqrt{k^2\eta_0^2 + c_-^2\eta_0^2 m^2} \right)} \\
&= \frac{\cosh \pi \left(\tilde{\omega}_{\text{out}} - \tilde{\omega}_{\text{in}} \right) - 1}{\cosh \pi \left(\tilde{\omega}_{\text{out}} + \tilde{\omega}_{\text{in}} \right) - \cosh \pi \left(\tilde{\omega}_{\text{out}} - \tilde{\omega}_{\text{in}} \right)}
\end{aligned} \tag{2.55}
$$

with[23]

$$
\begin{aligned}
\tilde{\omega}_{\text{out}} &\equiv \sqrt{\tilde{k}^2 + \tilde{m}^2} \\
\tilde{\omega}_{\text{in}} &\equiv \sqrt{\tilde{k}^2 + \frac{c_-^2}{c_+^2}\tilde{m}^2}
\end{aligned}
$$

which do *not* depend on η_0 anymore and is therefore independent of the timescale of the expansion. It also doesn't depend on the absolute values of c_+ and c_- but just on their ratio (which is always positive and smaller than 1). We can also infer that the closer c_-/c_+ is to one, the smaller $n(\tilde{k})$ is, i.e. the bigger the universe ends up compared to its size in the remote past, the more particles will be created.
In addition to the particle density, we find for the total number of particles (2.52)

$$
N(\eta_0, \tilde{m}) = \frac{4\pi}{\eta_0^3} \int_0^\infty \mathrm{d}\tilde{k}\ \tilde{k}^2\ n(\tilde{k}; \tilde{m}) \tag{2.56}
$$

where η_0 only enters via the prefactor η_0^{-3}, meaning that a relative increase in how rapid the expansion happens will have a cubic effect on the number of produced particles! We

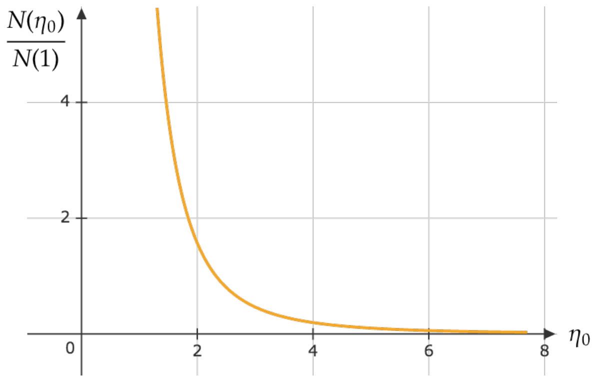


Figure 4: Dependence of the total number of produced particles on the characteristic time scale of the expansion of the universe: For infinitely fast expansions, infinitely many particles are produced

[23]Notice that $\tilde{\omega}_{\text{out}}$ and $\tilde{\omega}_{\text{in}}$ are the frequencies (now in terms of the dimensionless quantities $\tilde{k}$ and $\tilde{m}$) of the plane waves which we use to define vacua in the remote future and past, respectively. It is here where we can clearly see, that $\tilde{m}$ corresponds to the rest mass of a particle as seen from an observer in the remote future.

will now choose two specific scaling factors and try to verify our findings:

$$F(\eta) = a^2(\eta) = q \tanh \frac{\eta}{\eta_0} + r$$

with

$$q \in \{1, 4\} \qquad c_-^2 = 1$$
$$r \in \{2, 5\} \qquad c_+^2 \in \{3, 9\}.$$

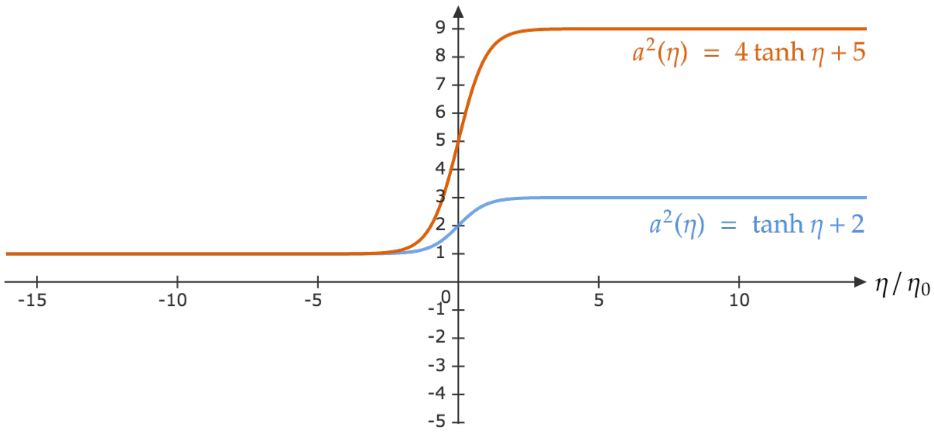


Figure 5: Two universes starting out at the same size but expanding differently. The orange one ends up larger than the blue one. The time scale on which the expansion happens is identical in both cases.

We can use (2.56) to calculate the number of produced particles as a function of their mass $\tilde{m}$ (which is the mass *after* the expansion) and plot $N(\tilde{m})$ for both cases. This is shown in figure 6. We observe that for some particular mass, which lies at $\sim (\eta_0)^{-1}$, the number of produced particles is maximal. This means that the timescale of the expansion sets the mass scale of the preferably produced particles! As expected, we also see that for smaller c_-/c_+, i.e. further away from 1, more particles are produced[24]. Indeed, one also finds, that for identical ratios c_-/c_+, the number of produced particles is identical, as expected and seen in figure 7. If we keep the expansion function the same but just shift it up, i.e. make the constant term larger, we effectively bring c_-/c_+ closer to 1 and following the intuition established earlier we'd expect less particles to be produced. That this is indeed the case can be seen in figure 8[25].

So far we only used a constant η_0 for all our scenarios. What happens if we change it, though? The result is shown in figure 9: As expected, the number of produced particles

[24] They're 1/3 for the blue curve and 1/9 for the orange one in this example

[25] This has a rather nice consequence when thinking about the early universe: If it starts out very small, a tiny expansion (in absolute terms) already creates a lot of particles, whereas the same (absolute) expansion happening today, for example, would be negligible in terms of particle production.

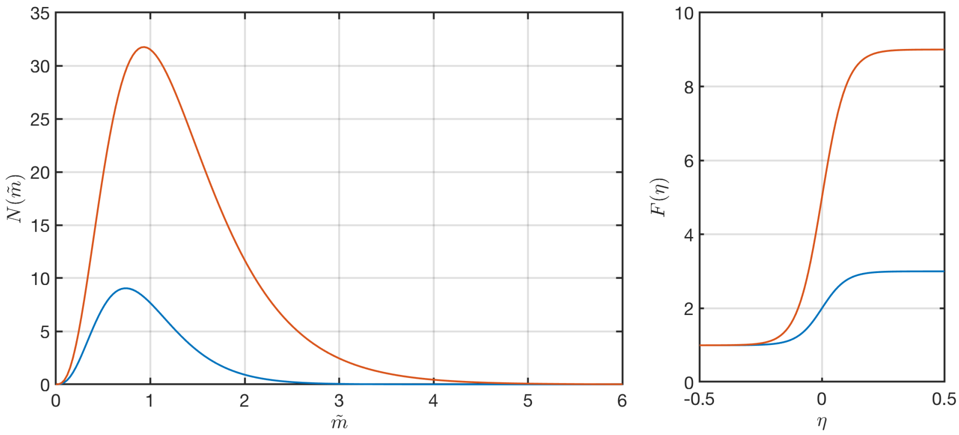


Figure 6: Total number of produced particles for two different ratios c_-/c_+, starting out at the same universe *size*. The right plot shows the two pulses that the left graphic corresponds to (in this case those are the functions shown at the beginning of this chapter).

increases. In fact, it increases just like we predicted it: For a ratio of 0.8 between the two η_0s, i.e. making the expansion 20% more rapid, we expect a relative increase of $(0.8)^{-3} \approx 2$.

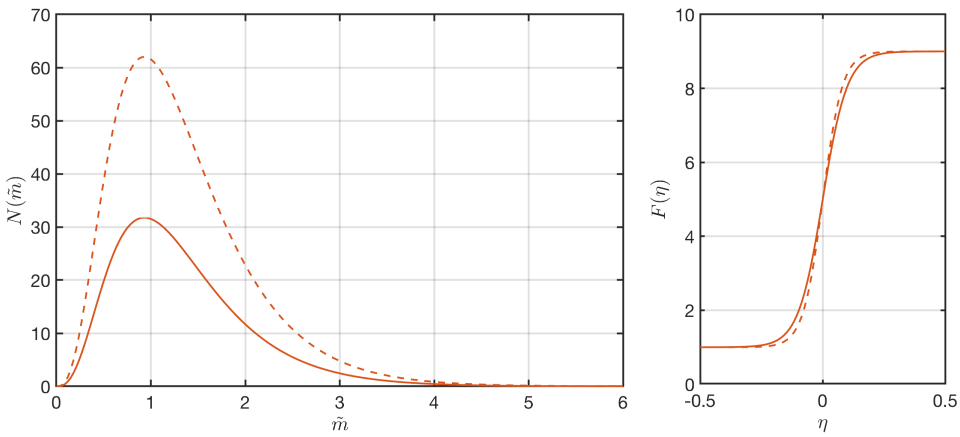


Figure 9: Total number of produced particles for two different expansion rates. The dashed curve represents the case where $\eta_0^{\text{dashed}} = 0.8\eta_0^{\text{solid}}$, i.e. expansion happens 20% more rapidly. This leads to a curve about twice as high as before. The expansion functions are again shown on the right.

Let's now look at the spectra themselves, i.e. $n(\tilde{k})$. For that we keep the same expansion functions from above. Figure 10 shows the difference for the two expansion scenarios: For each $\tilde{k}$, more particles are produced for the *stronger* expansion. For large $\tilde{k}$ the particle density falls off exponentially and we can see that more massive modes *survive longer*[26].

[26] how does this make sense?

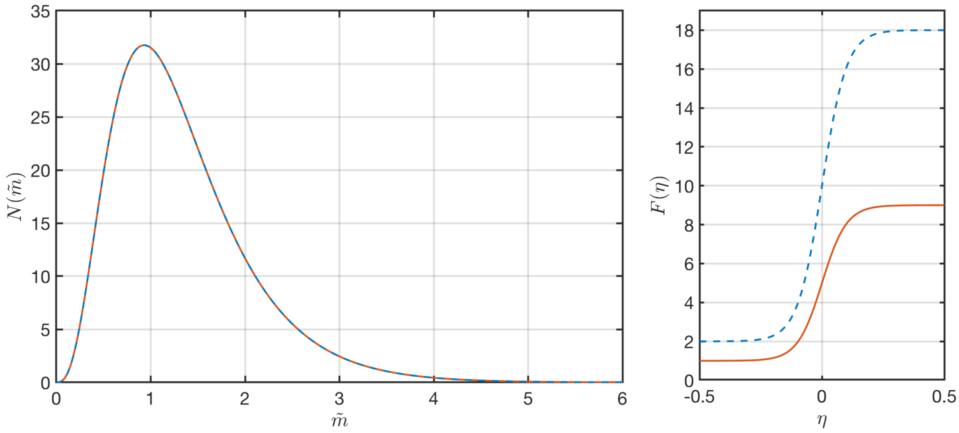


Figure 7: Total number of produced particles for two expansions with *identical* ratio c_-/c_+. The right plot shows the two pulses that the left graphic corresponds to.

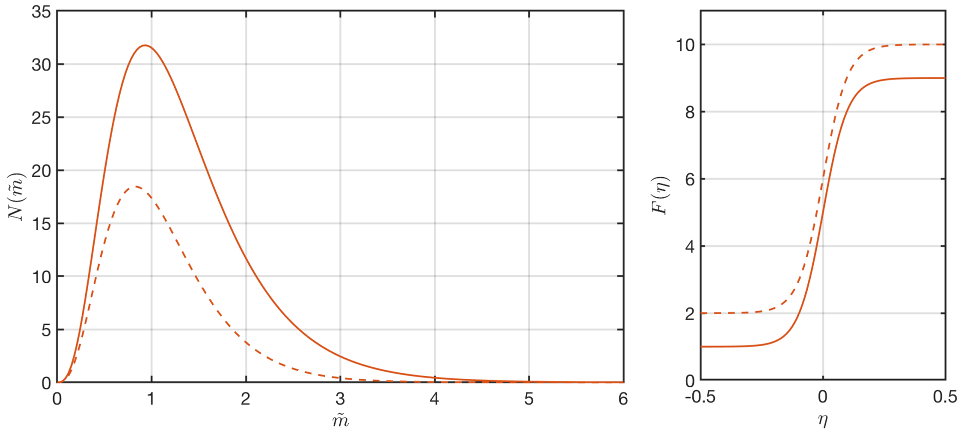


Figure 8: Total number of produced particles for two identical expansions with *different* ratios c_-/c_+. The right plot shows the two pulses that the left graphic corresponds to.

As explained above, the particle density does not depend on η_0 and therefore it makes no sense to examine the difference for different η_0.
pending: physical aspects of those results.

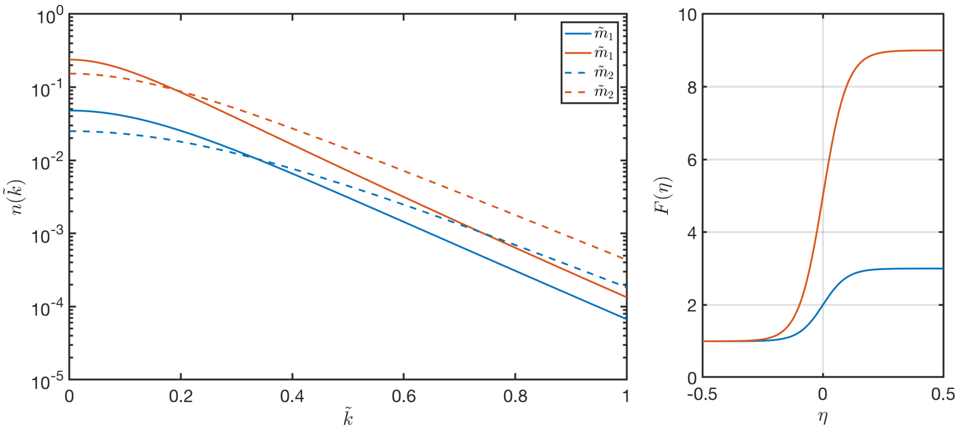


Figure 10: Density of created particles as a function of $\tilde{k}$ for two different masses ($0.5 = \tilde{m}_1 < \tilde{m}_2 = 0.8$) and two different expansion scenarios (which are again shown on the right and are the same as in fig. 6)

A. Additional calculations

A.1. More general form of scaling function and Sauter pulse

We treated scaling functions

$$a^2(\eta) = q \tanh \frac{\eta}{\eta_0} + r$$

before but it is actually little work to generalize everything to the form form

$$a^2(\eta) = p \tanh^2 \frac{\eta}{\eta_0} + q \tanh \frac{\eta}{\eta_0} + r.$$

In that case, in- and outgoing plane waves have frequencies

$$\omega_{\text{in}} \equiv \omega_k(\eta \to -\infty) = \sqrt{k^2 + (p + r - q)m^2}$$
$$\omega_{\text{out}} \equiv \omega_k(\eta \to +\infty) = \sqrt{k^2 + (p + r + q)m^2}.$$

The same substitution for ξ still works, but we have to generalize the expression for ω_k^2 as follows:

$$\begin{aligned}
\omega_k^2 &= k^2 + m^2 \left[p \tanh^2 \frac{\eta}{\eta_0} + q \tanh \frac{\eta}{\eta_0} + r \right] \\
&= k^2 + m^2 \left(p + r - q(1 - 2\xi) - 4p\xi(1 - \xi) \right) \\
&= \omega_{\text{in}}^2 + 2(q - 2p)m^2\xi + 4pm^2\xi^2 \\
&\equiv \omega_{\text{in}}^2 + \tilde{c}_1\xi + \tilde{c}_2\xi^2
\end{aligned}$$

where we see

$$\begin{aligned}
\tilde{c}_1 + \tilde{c}_2 &= 2qm^2 \\
\omega_{\text{in}}^2 + \tilde{c}_1 + \tilde{c}_2 &= \omega_{\text{out}}^2.
\end{aligned}$$

Again, we define

$$
\begin{aligned}
c_1 &\equiv \eta_0^2 \tilde{c}_1/4 & \Omega_{\text{out}}^2 &\equiv \frac{\eta_0^2}{4}\omega_{\text{out}}^2 = \Omega_{\text{in}}^2 + c_1 + c_2 \\
c_2 &\equiv \eta_0^2 \tilde{c}_2/4 & \Omega_\pm &\equiv \Omega_{\text{out}} \pm \Omega_{\text{in}} \\
\Omega_{\text{in}}^2 &\equiv \eta_0^2 \omega_{\text{in}}^2/4
\end{aligned}
$$

and the differential equation becomes

$$
0 = \left[\xi^2(1-\xi)^2\frac{\partial^2}{\partial\xi^2} + \xi(1-\xi)(1-2\xi)\frac{\partial}{\partial\xi} + \Omega_{\text{in}}^2 + c_1\xi + c_2\xi^2\right] v_{\mathbf{k}}
$$

(the term $\sim \xi^2$ is new). We make the same ansatz once again:

$$
v_\epsilon(\xi) = N\xi^{-\epsilon i\Omega_{\text{in}}}(1-\xi)^{\epsilon i\Omega_{\text{out}}} h_\epsilon(\xi)
$$

which leads to

$$
\left[\xi(1-\xi)\frac{\partial^2}{\partial\xi^2} + (1 - 2i\epsilon\Omega_{\text{in}} - 2(1+i\epsilon\Omega_-)\xi)\frac{\partial}{\partial\xi} - (c_2 + i\epsilon\Omega_- - \Omega_-^2)\right] h_\epsilon(\xi) = 0
$$

which has the form of the hypergeometric differential equation for

$$
\begin{aligned}
\gamma &= 1 - 2i\epsilon\Omega_{\text{in}} \\
\alpha + \beta + 1 &= 2(1 + i\epsilon\Omega_-) \\
\alpha\beta &= (c_2 + i\epsilon\Omega_- - \Omega_-^2).
\end{aligned}
$$

Explicit expressions can be obtained by solving this system of equations:

$$
\begin{aligned}
\alpha &= \frac{1}{2} + i\epsilon\Omega_- + \frac{i}{2}\sqrt{4c_2 - 1} \\
\beta &= \frac{1}{2} + i\epsilon\Omega_- - \frac{i}{2}\sqrt{4c_2 - 1} \\
\gamma &= 1 - 2i\epsilon\Omega_{\text{in}}
\end{aligned}
$$

for which we find the relations

$$
\begin{aligned}
1 - \alpha &= \alpha^* & 1 - \gamma &= 2i\epsilon\Omega_{\text{in}} \\
1 - \beta &= \beta^* & \gamma - \alpha - \beta &= -2i\epsilon\Omega_{\text{out}}. \\
2 - \gamma &= \gamma^*
\end{aligned}
$$

Now the Bogolyubov transformation done earlier is still valid, we only need to update the explicit expressions for $\alpha,\ \beta,\ \gamma$ and get

$$
a_k = \frac{N^{(0)}}{N^{(1)}} \frac{\Gamma(1 - 2i\Omega_{\text{in}})\Gamma(-2i\Omega_{\text{out}})}{\Gamma\left(\frac{1}{2} - i\Omega_+ - \frac{i}{2}\sqrt{4c_2 - 1}\right)\Gamma\left(\frac{1}{2} - i\Omega_+ + \frac{i}{2}\sqrt{4c_2 - 1}\right)}
$$

$$b_k = \frac{N^{(0)}}{N^{(1)}} \frac{\Gamma(1-2i\Omega_{\text{in}})\Gamma(2i\Omega_{\text{out}})}{\Gamma\left(\frac{1}{2}+i\Omega_- + \frac{i}{2}\sqrt{4c_2-1}\right)\Gamma\left(\frac{1}{2}+i\Omega_- - \frac{i}{2}\sqrt{4c_2-1}\right)}.$$

To find the normalization constants we calculate again

$$\begin{aligned}
|\kappa_2|^2 &= \frac{|-2i\Omega_{\text{in}}|^2\,|\Gamma(-2i\Omega_{\text{in}})|^2\,|\Gamma(2i\Omega_{\text{out}})|^2}{\left|\Gamma\left(\frac{1}{2}+i\left(\Omega_- + \frac{1}{2}\sqrt{4c_2-1}\right)\right)\right|^2\left|\Gamma\left(\frac{1}{2}+i\left(\Omega_- - \frac{1}{2}\sqrt{4c_2-1}\right)\right)\right|^2} \\
&= 4\Omega_{\text{in}}^2 \frac{\cosh\left(\pi\Omega_- + \frac{\pi}{2}\sqrt{4c_2-1}\right)\cosh\left(\pi\Omega_- - \frac{\pi}{2}\sqrt{4c_2-1}\right)}{4\Omega_{\text{in}}\Omega_{\text{out}}\sinh(2\pi\Omega_{\text{in}})\sinh(2\pi\Omega_{\text{out}})} \\
&= \frac{\Omega_{\text{in}}}{\Omega_{\text{out}}} \frac{\cosh\left(\pi\Omega_- + \frac{\pi}{2}\sqrt{4c_2-1}\right)\cosh\left(\pi\Omega_- - \frac{\pi}{2}\sqrt{4c_2-1}\right)}{\sinh(2\pi\Omega_{\text{in}})\sinh(2\pi\Omega_{\text{out}})} \\
&= \frac{\Omega_{\text{in}}}{\Omega_{\text{out}}} \frac{\cosh(2\pi\Omega_-) + \cosh\left(\pi\sqrt{4m^2\eta_0^2 p - 1}\right)}{\cosh(2\pi\Omega_+) - \cosh(2\pi\Omega_-)}
\end{aligned}$$

$$\begin{aligned}
|\kappa_1|^2 &= \frac{\Omega_{\text{in}}}{\Omega_{\text{out}}} \frac{\cosh\left(\pi\Omega_+ + \frac{\pi}{2}\sqrt{4c_2-1}\right)\cosh\left(\pi\Omega_+ - \frac{\pi}{2}\sqrt{4c_2-1}\right)}{\sinh(2\pi\Omega_{\text{in}})\sinh(2\pi\Omega_{\text{out}})} \\
&= \frac{\Omega_{\text{in}}}{\Omega_{\text{out}}} \frac{\cosh(2\pi\Omega_+) + \cosh\left(\pi\sqrt{4m^2\eta_0^2 p - 1}\right)}{\cosh(2\pi\Omega_+) - \cosh(2\pi\Omega_-)}
\end{aligned}$$

and find (as previously)

$$\left|\frac{N^{(1)}}{N^{(0)}}\right|^2 = \frac{\Omega_{\text{in}}}{\Omega_{\text{out}}}.$$

The particle density $|b_k|^2$ is thus modified for the more general case and reads:

$$n(k) = |b_k|^2 = \left|\frac{N^{(0)}}{N^{(1)}}\right|^2 |\kappa_2|^2 = \frac{\cosh(2\pi\Omega_-) + \cosh\left(\pi\sqrt{4m^2\eta_0^2 p - 1}\right)}{\cosh(2\pi\Omega_+) - \cosh(2\pi\Omega_-)}$$

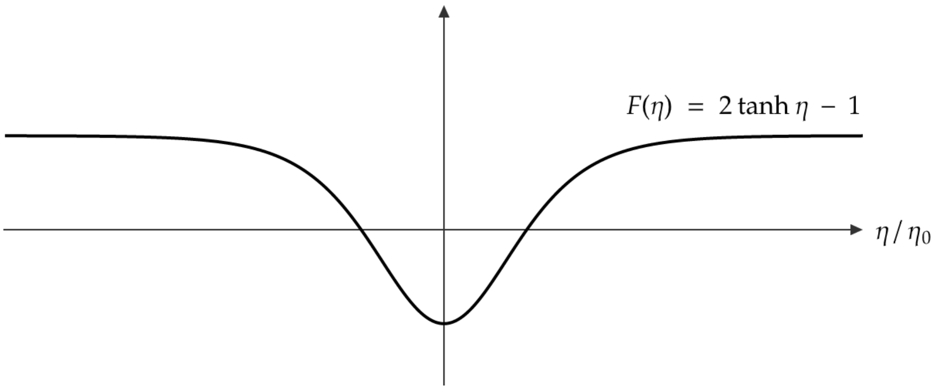
where

$$\begin{aligned}
\Omega_{\text{in}} &= \frac{\eta_0}{2}\sqrt{k^2 + (p+r-q)m^2} \\
\Omega_{\text{out}} &= \frac{\eta_0}{2}\sqrt{k^2 + (p+r+q)m^2} \\
\Omega_\pm &= \Omega_{\text{out}} \pm \Omega_{\text{in}} \\
c_2 &= m^2\eta_0^2 p
\end{aligned}$$

This case now includes the initial idea of using the Sauter pulse

$$m_{\text{eff}}^2 \equiv m^2\left[2\tanh\frac{\eta}{\eta_0} - 1\right]$$

as an effective action term. Notice, however, that this can only be interpreted as the contraction/expansion of the universe for *conformally* coupled fields. In that case, though, the given function wouldn't make any physical sense, as it switches signs. Therefore (if we use the Sauter pulse to describe the time-dependent effective action), we should be considering minimally coupled fields only. In that case, we may of course have negative effective masses. But then the bigger annoyance is that we do not know how the universe expands/contracts anymore, which is why we didn't choose the Sauter pulse in our model earlier.



A.2. Unruh effect

Let us now have a closer look at the Bogolubov coefficients between the expansion of the massive scalar fields in Minkowski spacetime and by that derive the Unruh effect, i.e. the circumstance that the Minkowski vacuum state is a thermal state with temperature $T = \frac{a}{2\pi}$ on the left and right Rindler wedge.
Let $\mathbf{k}_\perp \equiv (k_x, k_y)$. We know that the Bogolubov coefficients between modes with different $\mathbf{k}_\perp$ are 0 and can thus write the general expressions for the $v^R_{\omega k_\perp}$ and $v^L_{\omega_{k_\perp}}$ modes, which denote the right and left Rindler wedge:

$$v^R_{\omega k_\perp} = \int_{-\infty}^{\infty} \frac{dk_z}{\sqrt{4\pi k_0}} \left[\alpha^R_{\omega k_z \mathbf{k}_\perp} \mathrm{e}^{-ik_0 t + ik_z z} + \beta^R_{\omega k_z \mathbf{k}_\perp} \mathrm{ik_0 t - ik_z z}\right] \frac{\mathbf{ik}_\perp \cdot \mathbf{x}_\perp}{2\pi}, \tag{A.1}$$

$$v^L_{\omega k_\perp} = \int_{-\infty}^{\infty} \frac{dk_z}{\sqrt{4\pi k_0}} \left[\alpha^L_{\omega k_z \mathbf{k}_\perp} \mathrm{e}^{-ik_0 t + ik_z z} + \beta^L_{\omega k_z \mathbf{k}_\perp} \mathrm{ik_0 t - ik_z z}\right] \frac{\mathbf{ik}_\perp \cdot \mathbf{x}_\perp}{2\pi}. \tag{A.2}$$

Due to the transformation rules implied by the Rindler wedge we find that we can obtain $v^L_{\omega k_\perp}$ from $v^R_{\omega k_\perp}$ by $z \to -z$ and therefore,

$$\alpha^L_{\omega k_z \mathbf{k}_\perp} = \alpha^R_{\omega - k_z \mathbf{k}_\perp}, \tag{A.3}$$

$$\beta^L_{\omega k_z \mathbf{k}_\perp} = \beta^R_{\omega - k_z \mathbf{k}_\perp}. \tag{A.4}$$

For a massless scalar field the Unruh effect will thus follow if

$$(\alpha^R_{\omega \mathbf{k}_\perp} - \mathrm{e}^{-\frac{\pi\omega}{a}} \alpha^L_{\omega - \mathbf{k}_\perp}) \left|0_M\right\rangle = 0, \tag{A.5}$$

$$(\alpha^L_{\omega\mathbf{k}_\perp} - \mathrm{e}^{-\frac{\pi\omega}{a}}\alpha^R_{\omega-\mathbf{k}_\perp})\,|0_M\rangle = 0. \tag{A.6}$$

We find these relations fulfilled if the modes

$$w_{-\omega\mathbf{k}_\perp} = \frac{v^R_{\omega\mathbf{k}_\perp} + \mathrm{e}^{-\frac{\pi\omega}{a}}v^{L*}_{\omega-\mathbf{k}_\perp}}{\sqrt{1-\mathrm{e}^{-\frac{2\pi\omega}{a}}}}, \tag{A.7}$$

$$w_{+\omega\mathbf{k}_\perp} = \frac{v^L_{\omega\mathbf{k}_\perp} + \mathrm{e}^{-\frac{\pi\omega}{a}}v^{R*}_{\omega-\mathbf{k}_\perp}}{\sqrt{1-\mathrm{e}^{-\frac{2\pi\omega}{a}}}} \tag{A.8}$$

are purely positive frequency modes in Minkowski spacetime. This results if

$$\beta^R_{\omega k_z\mathbf{k}_\perp} = -\mathrm{e}^{-\frac{\pi\omega}{a}}\alpha^{L*}_{\omega k_z\mathbf{k}_\perp}, \tag{A.9}$$

$$\beta^L_{\omega k_z\mathbf{k}_\perp} = -\mathrm{e}^{-\frac{\pi\omega}{a}}\alpha^{R*}_{\omega k_z\mathbf{k}_\perp}. \tag{A.10}$$

These relations can be shown by explicitly computing the Bogolubov coefficients. Consider the solutions on the future Killing horizon, $t = z$, $t > 0$, where we have

$$v^R_{\omega\mathbf{k}_\perp} \to \int_{-\infty}^{\infty}\frac{dk_z}{\sqrt{4\pi k_0}}\left[\alpha^R_{\omega k_z\mathbf{k}_\perp}\mathrm{e}^{-\frac{i(k_0-k_z)V}{2}} + \beta^R_{\omega k_z\mathbf{k}_\perp}\mathrm{e}^{-\frac{i(k_0-k_z)V}{2}}\right]\frac{\mathrm{e}^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}}{2\pi}. \tag{A.11}$$

The small-argument approximation for the modified Bessel function for $\zeta \to \infty$, however, yields

$$v^R_{\omega\mathbf{k}_\perp} \to \frac{i}{4\pi}\left[a\sinh(\frac{\pi\omega}{a})\right]^{-\frac{1}{2}}\mathrm{e}^{i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\left(\frac{\frac{\kappa}{2a}^{\frac{i\omega}{a}}\mathrm{e}^{-i\omega u}}{\Gamma(1+\frac{i\omega}{a})} - \frac{-\frac{\kappa}{2a}^{\frac{i\omega}{a}}\mathrm{e}^{-i\omega v}}{\Gamma(1-\frac{i\omega}{a})}\right), \tag{A.12}$$

with $\kappa = (\mathbf{k}_\perp^2 + m^2)^{\frac{1}{2}}$. We find, that the first term inside the parentheses here oscillates infinitely many times as $u \to \infty$ where the future Killing horizon is bounded. We can consider this term to be zero. Consequently, by multiplying this equation by $\mathrm{e}^{\frac{i(k_0-k_z)V}{2}}$ and integrating over V we find

$$\begin{aligned}\alpha^R_{\omega k_z\mathbf{k}_\perp} &= -\frac{i\frac{\kappa}{2a})^{-\frac{i\omega}{a}}(k_0-k_z)}{4\sqrt{\pi a k_0\sinh(\frac{\pi\omega}{a})}\Gamma(1-\frac{i\omega}{a})}\int_0^\infty dV(aV)^{-\frac{i\omega}{a}}\mathrm{e}^{\frac{i(k_0-k_z)V}{2}}\\ &= \frac{\mathrm{e}^{\frac{\pi\omega}{2a}}}{\sqrt{4\pi k_0 a\sinh(\frac{\pi\omega}{a})}}\left(\frac{k_0+k_z}{k_0-k_z}\right)^{-\frac{i\omega}{2a}},\end{aligned} \tag{A.13}$$

with $\kappa = \sqrt{(k_0-k_z)(k_0+k_z)}$. Keep in mind that we have hereby implicitly selected a particular extension of the modes $v^R_{\omega\mathbf{k}_\perp}$ to the entire Minkowski spacetime. Otherwise, it was not possible to find the Bogolubov coefficients. We have particularly excluded any δ-function contribution at $V = 0$.

Multiplying A.12 by $\mathrm{e}^{-\frac{i(k_0-k_z)V}{2}}$ instead and integrating over V yields

$$\beta^R_{\omega k_z\mathbf{k}_\perp} = -\frac{\mathrm{e}^{-\frac{\pi\omega}{2a}}}{\sqrt{4\pi k_0 a\sinh(\frac{\pi\omega}{a})}}\left(\frac{k_0+k_z}{k_0-k_z}\right)^{-\frac{i\omega}{2a}}. \tag{A.14}$$

By introducing the rapidity $\theta(k_z)$ which is by definition given as

$$\theta(k_z) = \frac{1}{2}\log(\frac{k_0 - k_z}{k_0 - k_z}) \tag{A.15}$$

we find the results to be

$$\begin{aligned} \alpha^R_{\omega k_z \mathbf{k}_\perp} &= \alpha^L_{\omega - k_z \mathbf{k}_\perp} \\ &= \frac{\mathrm{e}^{i\frac{\theta(k_z)}{a}\omega}}{\sqrt{2\pi k_0 a(1 - \mathrm{e}^{-\frac{2\pi\omega}{a}})}}, \end{aligned} \tag{A.16}$$

$$\begin{aligned} \beta^R_{\omega k_z \mathbf{k}_\perp} &= \beta^L_{\omega - k_z \mathbf{k}_\perp} \\ &= -\frac{e^{-\frac{\pi\omega}{a}}\mathrm{e}^{i\frac{\theta(k_z)}{a}\omega}}{\sqrt{2\pi k_0 a(1 - \mathrm{e}^{-\frac{2\pi\omega}{a}})}} \end{aligned} \tag{A.17}$$

Thus the Unruh effect is established.

References

[1] V. Mukhanov, S. Winitzki. *Introduction to Quantum Effects in Gravity.* Cambridge University Press, 2007.
doi:10.1017/CBO9780511809149

[2] S. A. Fulling. *Quantum field theory in general space-times*, S. 116–133. London Mathematical Society Student Texts. Cambridge University Press, 1989.
doi:10.1017/CBO9781139172073.007

[3] S. HAOUAT, R. CHEKIREB. *On the creation of scalar particles in a flat Robertson-Walker Spacetime.* Modern Physics Letters A **26** (2011) 2639–2651.
doi:10.1142/s0217732311037017

[4] I. S. Gradshteyn, I. M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh Auflage, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[5] F. W. J. Olver, N. I. of Standards and Technology (U.S.) (Herausgeber). *NIST handbook of mathematical functions.* Cambridge University Press : NIST, Cambridge ; New York, 2010. OCLC: ocn502037224.