Optimal Stopping & Cumulative Prospect Theory

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Question:

optimal stopping when preferences are not expected utility?

Static Preferences

- $\Delta(\mathbb{R})$ the set of (regular) CDF's on $\mathbb{R}$
- Preference is a function $C : \Delta(\mathbb{R}) \to \mathbb{R}$
- Agent prefers $F \in \Delta(\mathbb{R})$ over $G \in \Delta(\mathbb{R})$ iff

$$C(F) > C(G)$$

Examples

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- Prospect Theory:
$$C(F) = \int_{\mathbb{R}_+} w^{+}(1 - F(x))u'(x)\mathrm{d}x - \int_{\mathbb{R}_-} w^{-}(F(x))u'(x)\mathrm{d}x$$

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- Mean Variance:

$$C(F) = \int x dF(x) - \alpha \mathsf{Var}(F)$$

From Preferences to Behavior

There are three ways to derive behavior from preferences

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3. **no commitment & sophistication**
   - agent continues whenever it is better than stopping given the behavior of future selves
   - behavior of future selves is correctly predicted

Environment

- "Nice" diffusion $X = (X_t)_t$

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$
$$X_0 = x\,.$$

- Agent chooses stopping rule τ
- Preference only over distribution of final values $X_\tau \sim F_{\tau,x}$
- Skorokhod embedding gives us the set of distributions $\mathcal{P}_x$ which can be embedded by some stopping time

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   - assume <u>no randomization</u>
   - Markovian strategies $s : \mathbb{R} \to \{0,1\}$
   - $\tau_s = \inf\{t\colon s(X_t) = 1\}$
   - Equilibrium Conditions:
     - $S = \{x \in \mathbb{R}\colon C(F_{\tau_s,x}) \leq C(\delta_x)\}$
     - $C = \{x \in \mathbb{R}\colon C(F_{\tau_s,x}) \geq C(\delta_x)\}$
     - $s(x) = 1 \Leftrightarrow x \in S$ and $s(x) = 0 \Leftrightarrow x \in C$

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- today's self wants future selves to act differently than they want
  1. commitment allows today's self to bind future selves
  2. naïvié assumes that today's self acts as if future selves where committed
  3. sophistication imposes correct expectation about future behavior

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  - I am seeking commitment, which is a sign of sophistication
- A lot of empirical evidence for time-inconsistent behavior

Prospect Theory

Decision Making Under Risk

Two theories:

- EUT (expected utility theory): *normative, rational* theory
- CPT (cumulative prospect theory): *descriptive, behavioral* theory
- prospect theory papers: Kahnemann and Tversky (1979, E), Tversky and Kahnemann (1992, JRU)
- Nobel Prize 2002:

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Applications to:

- static decision problems
- dynamic decision problems

Four Elements of CPT

CPT functional:

$$CPT(X) = \int_{\mathbb{R}_+} \mathbf{w}^+(\mathbb{P}(\mathbf{u}(X-\mathbf{r}) > y))\mathrm{d}y - \int_{\mathbb{R}_-} \mathbf{w}^-(\mathbb{P}(\mathbf{u}(X-\mathbf{r}) < y))\mathrm{d}y$$

1. Reference Point: $\mathbf{r}$
2. Value function $\mathbf{u} : \mathbb{R} \to \mathbb{R}$ distorts outcomes: $\mathbf{u}(x-r)$
3. Weighting function distorts (objective) probabilities: $\mathbf{w}^+(p)$, $\mathbf{w}^-(p)$
4. Outcome loss aversion: $\lambda = \frac{\partial_- \mathbf{u}(0)}{\partial_+ \mathbf{u}(0)}$

Relation to Expected Utility

Consider a lottery with bounded outcomes $\underline{x} \leq X \leq \overline{x}$ and an absolutely continuous distribution F:

$$
\begin{aligned}
\mathbb{E}(u(X)) &= \int_{\underline{x}}^{\overline{x}} u(z)\mathrm{d}F(z) = [u(z)F(z)]_{z=\underline{x}}^{z=\overline{x}} - \int_{\underline{x}}^{\overline{x}} u'(z)F(z)\mathrm{d}z \\
&= u(\underline{x}) + \int_{\underline{x}}^{\overline{x}} (1-F(z))u'(z)\mathrm{d}z \\
&= u(\underline{x}) + \int_{u(\underline{x})}^{u(\overline{x})} (1-F(u^{-1}(y)))\mathrm{d}y \\
&= u(\underline{x}) + \int_{u(\underline{x})}^{u(\overline{x})} \mathbb{P}(u(X)>y)\mathrm{d}y \\
&= u(\underline{x}) + \int_{0}^{u(\overline{x})} \mathbb{P}(u(X)>y)\mathrm{d}y + \int_{u(\underline{x})}^{0} 1-\mathbb{P}(u(X)<y)\mathrm{d}y \\
&= u(0) + \int_{\mathbb{R}_+} \mathbb{P}(u(X)>y)\mathrm{d}y - \int_{\mathbb{R}_-} \mathbb{P}(u(X)<y)\mathrm{d}y
\end{aligned}
$$

CPT Value of a Binary Lottery

Binary lottery with outcoms $a < b$ and $p \in (0,1)$

$$X = \begin{cases} b & \text{with probability } p \\ a & \text{with probability } 1-p \end{cases} .$$

The value of a CPT agent equals:

$$CPT(L) = \begin{cases} w^+(p)u(b-r) + (1-w^+(p))u(a-r), & \text{if } 0 \leq a-r \\ w^+(p)u(b-r) + w^-(1-p)u(a-r), & \text{if } a-r < 0 \leq b-r \\ (1-w^-(1-p))u(b-r) + w^-(1-p)u(a-r) & \text{if } b-r < 0. \end{cases}$$

Tversky and Kahnemann (1992) Parametrization

$$w^{+}(p) = \frac{p^{\delta}}{\left(p^{\delta} + (1-p)^{\delta}\right)^{\frac{1}{\delta}}}$$

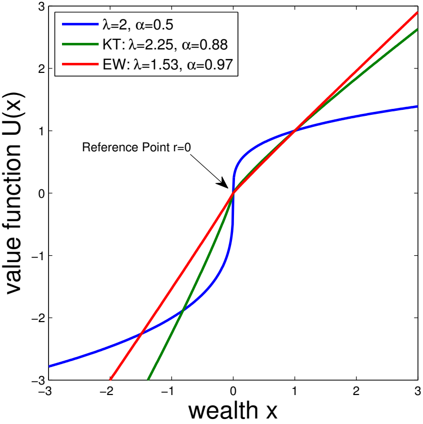
$$w^{-}(p) = \frac{p^{\gamma}}{\left(p^{\gamma} + (1-p)^{\gamma}\right)^{\frac{1}{\gamma}}}$$

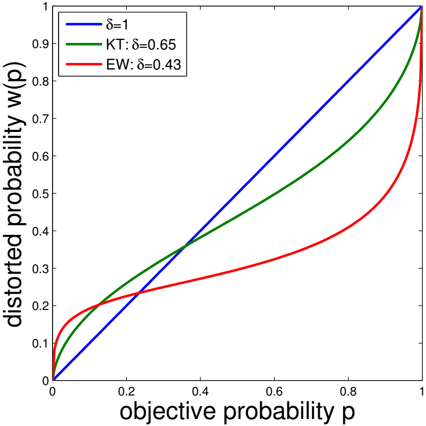
$$u(x) = \begin{cases} x^{\alpha}, & \text{if } x \geq 0 \\ -\lambda\,(-x)^{\beta}\,, & \text{if } x < 0. \end{cases}$$

Often assumed: $\beta = \alpha$ and $\gamma = \delta$, thus parameter triple

$$(\alpha, \lambda, \delta)$$

Tversky and Kahnemann (1992) Parametrization





Our Definition of CPT

1. Value function $u : \mathbb{R} \to \mathbb{R}$
   (a) absolutely continuous, strictly increasing
   (b) $\lambda = \sup_{x \in \mathbb{R}} \frac{\partial_- u(x)}{\partial_+ u(x)} < \infty$
2. Weighting functions $w^+, w^- : [0,1] \to [0,1]$, strictly increasing with $w^+(0) = w^-(0) = 0$ and $w^+(1) = w^-(1) = 1$.
   (a)
   $$\frac{w^+(p)}{p} \to \infty \text{ for } p \to 0 \text{ or } {w^+}'(0) > \lambda\,.$$
   (b) similar condition for w^-

Note:

- All commonly used parametrizations (Kahnemann and Tversky (1979), Prelec (1998), Goldstein and Einhorn (1987), neo-additive function Wakker) suffice our assumptions.

I The Commitment Case

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- Consider $\mu \equiv 0$ otherwise rescale $\hat{X}_t = S(X_t), \hat{U}(x) = (U \circ S)^{-1}(x)$

$$S(x) = \int_{\hat{x}}^{x} \exp\left(- \int_{\hat{x}}^{y} \frac{2\mu(z)}{\sigma^2(z)} \mathrm{d}z \right) \mathrm{d}y \,.$$

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$$\max_F \; CPT(F) \text{ subject to } \; x = \int z dF(z) \tag{1}$$

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- Given the optimal distribution $F^\star$ solve Skorokhod's embedding problem.

Features of the Solution

- Optimal solution will (often) involve randomization (Wakker, 2010; Henderson, Hobson, and Tse, 2016)
- Strategy is not unique
- Strategy is sometimes not first leaving time of a set
- If the support contains at least three points it will be
  - either randomized
  - or non-Markov
- Contrasts with EU where Markov and deterministic are wlog.

II The Naive Case

Skewness Preference in the Small

Proposition: For every $x \in \mathbb{R}$ and every $\epsilon > 0$ there exists a binary lottery

$$L = (1-p)\,\delta_{x-a} + p\,\delta_{x+b}$$

with a, $b \in (-\epsilon, +\epsilon)$ and $pb - (1-p)a = 0$ that a CPT agent is strictly willing to take

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Proof:

1. three cases: $x > r$, $x = r$, $x < r$
2. construction: $a_n = -\frac{p}{n}$ and $b_n = \frac{1-p}{n}$
3. pick sequence p_n such that $p_n \to 0$ as $n \to \infty$

Corollary I: Unfair Gambles

For every wealth level $x \in \mathbb{R}$ and every $\epsilon > 0$ there exist two arbitrarily small binary lotteries with negative and positive mean, respectively, that a CPT agent wants to take.

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Proof:

- follows from continuity of CPT functional

Corollary II: Risk Aversion in CPT

A CPT agent is risk-averse and risk-seeking nowhere.

- forget about "risk aversion over gains and risk-seeking over losses"
- http://prospect-theory.behaviouralfinance.net: *"The value function is defined on deviations from a reference point and is normally concave for gains (implying risk aversion), commonly convex for losses (risk seeking)..."*
- cf. Schmidt and Zank (2008, MS), Chateauneuf and Cohen (1994, JRU); curvature arguments

Stopping Behavior of the Naive

- The naive stops at x with positive probability only if there exists

$$\tau^{\star} \in \arg\max_{\tau} CPT(F_{\tau,x})$$

with $\mathbb{P}[\tau^{\star} = 0] > 0$ or $\frac{d}{dt}\mathbb{P}[\tau^{\star} \leq t]|_{t=0} > 0$

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for $\tau_{[a,b]} = \inf\{t \colon X_t \notin [a,b]\}$
- Thus the naive agent never stops!

Theorem (Ebert & Strack, 2015)

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  - Arguably too extreme
  - Easy to reject in experiments/data
  - Useful to understand limitations of the theory

What if the agent can randomize?

Two recent papers:

- He, Hu, Obloj, and Zhou (2017): pre-commitment, discrete case (like in Barberis 2012)
- Henderson, Hobson, Tse (2017): naïvité, continuous time

Three questions:

1. Can humans randomize? (philosophical or biological question)
2. Is the ability to randomize a good behavioral assumption? (economic/ psychological question)
3. Is the ability to randomize changing the predictions of the model? (mathematical question)

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  - when non Markov strategies are allowed...

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- a strategy that stops with some probability immediately *is equally good* as a strategy that stops with some probability later
- recall that "later" does not matter under naïvité
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- Yes.
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Two recent papers:

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Literature

Optimal Stopping with Prospect Theory
Henderson (2012), Barberris (2012), Xu, and Zhou (2013), Ebert, Strack (2015), Henderson, Hobson, and Tse (2016), He, Hu, Obloj, Zhou, (2017), Henderson, Hobson, and Tse (2017), Ebert, Strack (2017)

Optimal Stopping with Non-EU preferences
Riedel (2009), Pedersen, Peskir (2013, 2016), Pedersen, Peskir (2017), Duraj (2017), Belomestny, .. many others

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Thank You!

III The Sophisticated Case

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  - $\mathcal{C}_s = \{x \in \mathbb{R} \colon C(F_{\tau_s, x}) \geq C(\delta_x)\}$
- Definition: A strategy s is an equilibrium iff

$$s(x) = 1 \Leftrightarrow x \in \mathcal{S}_s \text{ and } s(x) = 0 \Leftrightarrow x \in \mathcal{C}_s$$

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- Definition: An equilibrium s is Pareto-preferred if there does not exists an equilibrium s' which gives weakly higher value to every self and a strictly higher value to at least one self.

$$C(F_{\tau_s,x}) \leq C(F_{\tau_{s'},x}) \text{ for all } x$$
$$C(F_{\tau_s,\hat{x}}) < C(F_{\tau_{s'},\hat{x}}) \text{ for some } \hat{x}$$

Proposition

Let C be an expected utility preference. Suppose an optimal strategy exists. A strategy is optimal if and only if it is a preferred equilibrium.

Proof:

- let $s^\star$ be an optimal strategy
- Optimal $\Rightarrow$ Preferred Eq.
  - $V(x) = C(F_{s^\star,x})$ is the value function
  - thus $C(F_{s^\star,x}) \geq C(F_{s',x})$
- Optimal $\Leftarrow$ Preferred Eq.
  - if s is not optimal
  - $C(F_{s^\star,x}) \geq C(F_{s,x})$ and $C(F_{s^\star,x'}) > C(F_{s,x'})$ for some x'
  - As $s^\star$ is an equilibrium s is not a preferred equilibrium.

Sophisticated Behavior

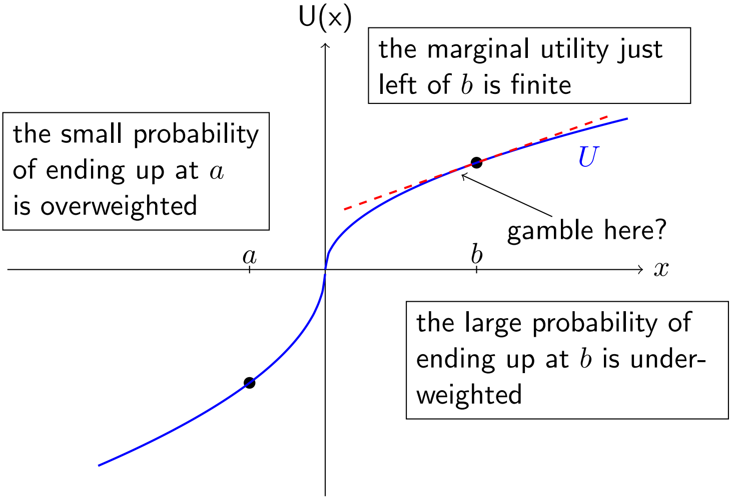
Theorem 2: If ${w^+}'(1) = {w^-}'(0) = \infty$, the sophisticated agent stops any process immediately.

Proof idea:

- let $a = \sup\{z \leq x \colon s(x) = 1\}$ and $b = \inf\{z \leq x \colon s(x) = 1\}$
- for a continuation to be optimal it has to be so at every wealth level and hence for the agent without probability distortion. For $\tau_{[a,b]}$ we need to have that for all $x \in [a, b]$

  $$CPT(F_{\tau_{[a,b]},x}) = w^+(p)U(b-r) + (1-w^+(p))U(a-r) \geq U(x)$$

- if the derivative of the weighting function is infinite than the agent always prefers stopping close to the boundary, i.e. the above equation is not satisfied for $x \to b$.



A sufficient condition for never gambling is:

$$\partial_- U(b) < \frac{U(b)}{b-a} w^{+\prime}(1) - \frac{U(a)}{b-a} w^{-\prime}(0)$$