

## First Order versus Second Order Risk Aversion\*

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This paper defines a new concept of attitude towards risk. For an actuarially fair random variable  $\tilde{\epsilon}$ ,  $\pi(t)$  is the risk premium the decisionmaker is willing to pay to avoid  $t\tilde{\epsilon}$ . In expected utility, and as it turns out, in the case of smooth Fréchet differentiability of the representation functional,  $\pi'(0)=0$ . There are models (e.g., rank dependent probabilities) in which  $\partial\pi/\partial t|_{t=0^+} \neq 0$ . We call the latter attitude as being of order 1, and we call the first one attitude of order 2. These concepts are then applied to analyze the problem of full insurance. *Journal of Economic Literature* Classification Number: 026. © 1990 Academic Press, Inc.

### 1. INTRODUCTION

Since Pratt [8] it has been known that in expected utility theory the risk premium that a risk averse decisionmaker with a differentiable utility function is willing to pay to avoid an actuarially fair random variable  $t\tilde{\epsilon}$  is proportional, for small  $t$ , to  $t^2$  and to the variance of  $\tilde{\epsilon}$ . Formally, let  $E[\tilde{\epsilon}]=0$ . The risk premium  $\pi$  is that amount of money that makes the decisionmaker indifferent between paying  $\pi$  and the lottery  $x + t\tilde{\epsilon}$ . For a sufficiently small  $t$ ,  $\pi \approx -(t^2/2) \sigma_{\tilde{\epsilon}}^2 u''(x)/u'(x)$ . The risk premium  $\pi$  is proportional to  $t^2$ , rather than  $t$ , and therefore approaches zero faster than  $t$ . For small risks, the decisionmaker is thus almost risk neutral, and his major concern will be the expected value of the risk. This is the case because for small risks a differentiable utility function is almost linear. This

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property of expected utility theory implies Samuelson's [13] revision of Ulam's definition of a coward—a person who will not make a sufficiently small bet when you offer him two-to-one odds and let him choose his side. For a formal exposition of this discussion see Proposition 1 below.

The supposition that people are not cowards (a la Ulam–Samuelson's definition) has some clear implications to financial markets. It implies that a risk averse decisionmaker will not buy full insurance, unless there is no marginal loading. If all but the last unit are already insured, the decisionmaker will consider this last unit almost only by its expected value and will insure it only if the insurance premium at the margin does not exceed its expected loss. This prediction is considered by Borch [3] as being "against all observation".

In Proposition 1 we show that these properties of expected utility theory hold whenever the risk premium  $\pi$  of a small actuarially fair random variable  $t\tilde{\varepsilon}$  is proportional to  $t^2$ . On the other hand, we show that if  $\pi$  is proportional to  $t$ , these results are no longer valid and people may buy full insurance even though they have to pay some marginal loading. The existence of such functionals is proved in Section 4. We call such an attitude risk aversion of order 1, while the expected utility attitude is naturally regarded as being of order 2. We show in Section 4 that under several smoothness assumptions, Machina's [6] Fréchet differentiable functionals also exhibit attitudes of order 2. We thus attach economic meaning to the technical assumption of differentiable utility functions.<sup>1</sup> Section 5 concludes the paper with an analysis of attitude towards risk when the decisionmaker receives a transformation of the random variable, rather than the random variable itself.

## 2. ORDERS OF RISK AVERSION

Let  $M$  be a bounded interval in  $\mathbb{R}$  and let  $D$  be the set of random variables (or lotteries) with outcomes in  $M$ . For a lottery  $\tilde{x} \in D$ , let  $F_{\tilde{x}}(x) = \Pr(\tilde{x} \leq x)$  be the cumulative distribution function of  $\tilde{x}$ . Lotteries with a finite number of outcomes are sometimes written as vectors of the form  $(x_1, p_1; \dots; x_n, p_n)$ , where  $\sum p_i = 1$  and  $p_1, \dots, p_n \geq 0$ . Such a lottery yields  $x_i$  dollars with probability  $p_i$ ,  $i = 1, \dots, n$ .  $\delta_x$  stands for the degenerate lottery  $(x, 1)$ .

On  $D$  there exists a complete and transitive preference relation  $\succsim$ . We assume that  $\succsim$  is continuous with respect to the topology of weak convergence, and monotonic with respect to first order stochastic dominance (i.e.,  $\forall x F_{\tilde{x}}(x) \leq F_{\tilde{y}}(x) \Rightarrow \tilde{x} \succsim \tilde{y}$ ).  $V: D \rightarrow \mathbb{R}$  represents the relation  $\succsim$  if

<sup>1</sup> Or of the local utility functions in the case of Fréchet differentiability.

$V(\tilde{x}) \geq V(\tilde{y}) \Leftrightarrow \tilde{x} \succsim \tilde{y}$ . The certainty equivalent of  $\tilde{x}$ ,  $CE(\tilde{x})$ , is defined implicitly by  $\delta_{CE(\tilde{x})} \sim \tilde{x}$ . Its existence is guaranteed by the continuity and monotonicity assumptions, and it can be used as a representation of  $\succsim$ . Special attention is given below to the difference between the certainty equivalent of a lottery and its expected value.

DEFINITION 1. The risk premium of a lottery  $\tilde{x}$  is given by  $\pi(\tilde{x}) = E[\tilde{x}] - CE(\tilde{x})$ .

The risk premium  $\pi$  may be positive, which is the case when the decisionmaker is risk averse, or negative, the case of risk loving. In both cases,  $\pi(\delta_x) = 0$ . If  $E[\tilde{x}] = 0$ , then the risk premium is that amount of money the decisionmaker is willing to pay (or to receive, if  $\pi < 0$ ) to avoid participating in the lottery  $\tilde{x}$ .

Let  $\tilde{\varepsilon}$  be a random variable such that  $E[\tilde{\varepsilon}] = 0$ , and consider the lottery  $x + t\tilde{\varepsilon}$ . Its risk premium  $\pi$  is a function of  $t$ , and it is defined by  $\delta_{x - \pi(t)} \sim x + t\tilde{\varepsilon}$ . Of course,  $\pi(0) = 0$ . We assume throughout this paper that  $\pi$  is continuously twice differentiable with respect to  $t$  around  $t = 0$ , except maybe at  $t = 0$ , where it may happen that only right and left derivatives exist. (In Section 4 we discuss several examples satisfying these properties.) Unless otherwise stated, we assume, in addition, that all these derivatives are continuous in  $x$ .

If the decisionmaker is risk averse, then for every nondegenerate  $\tilde{\varepsilon}$  such that  $E[\tilde{\varepsilon}] = 0$ , and for every  $t \neq 0$ ,  $\delta_x \succ x + t\tilde{\varepsilon}$ . It thus follows that  $\partial\pi/\partial t|_{t=0^+} \geq 0$  and  $\partial\pi/\partial t|_{t=0^-} \leq 0$ . If the decisionmaker is risk loving, these inequalities are reversed. It may of course happen that these two side derivatives have the same strict sign.<sup>2</sup> However, if  $\tilde{\varepsilon}$  is symmetric around zero, then  $\partial\pi/\partial t|_{t=0^-} = -\partial\pi/\partial t|_{t=0^+}$ . It turns out that the signs of these side derivatives, and in particular their being nonzero, have some interesting economic applications. For reasons of clarity, we will concentrate below on right-hand derivatives only.

DEFINITION 2. The decisionmaker's attitude towards risk at  $x$  is of order 1 if for every  $\tilde{\varepsilon} \neq \delta_0$  such that  $E[\tilde{\varepsilon}] = 0$ ,  $\partial\pi/\partial t|_{t=0^+} \neq 0$ . It is of order 2 if for every such  $\tilde{\varepsilon}$ ,  $\partial\pi/\partial t|_{t=0} = 0$ , but  $\partial^2\pi/\partial t^2|_{t=0^+} \neq 0$ .<sup>3</sup> The sign of his attitude is opposite to the sign of the first nonzero derivative at  $0^+$ . (See Fig. 1.)

This definition actually says that the decisionmaker's attitude towards risk is of order one if  $\lim_{t \rightarrow 0^+} \pi(t)/t \neq 0$ , that is, if  $\pi(t)$  is not  $o(t)$ . The

<sup>2</sup> Suppose, for example, that for every  $t > 0$ ,  $(x + 9t, 0.1; x - t, 0.9) \succ (x, 1) \succ (x - 9t, 0.1; x + t, 0.9)$ . These preferences result in buying both insurance and lottery tickets.

<sup>3</sup> As  $E[-\tilde{\varepsilon}] = 0$  it follows that an attitude of order 1 implies  $\partial\pi/\partial t|_{t=0^-} \neq 0$ , and an attitude of order 2 implies  $\partial^2\pi/\partial t^2|_{t=0^-} \neq 0$ .

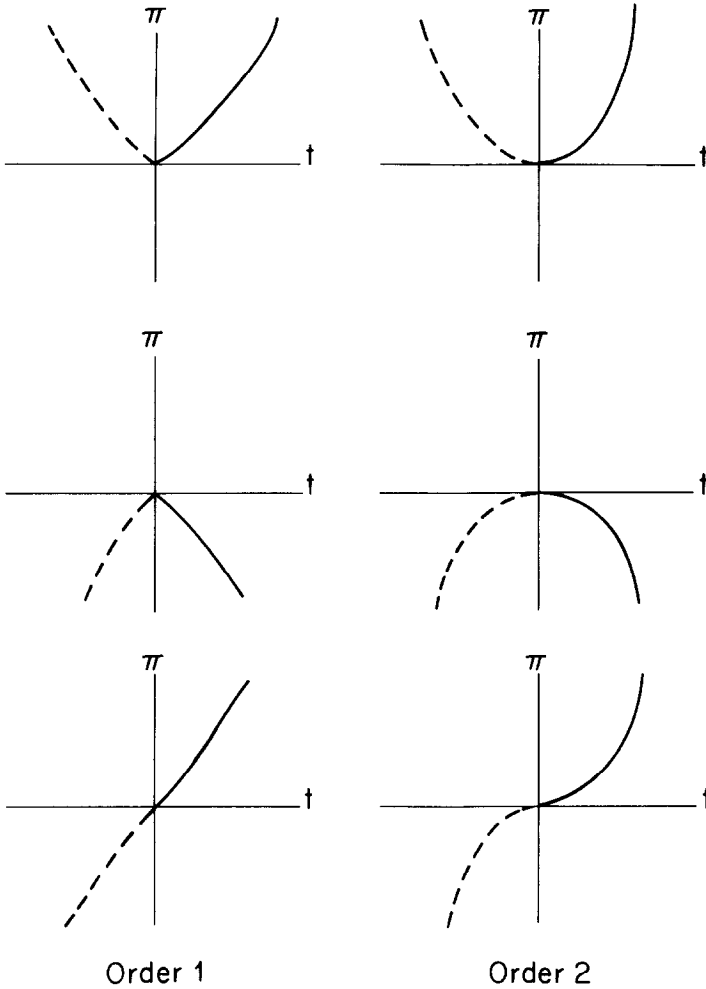


FIGURE 1

attitude is of order 2 if  $\lim_{t \rightarrow 0} \pi(t)/t = 0$ , but  $\lim_{t \rightarrow 0^+} \pi(t)/t^2 \neq 0$ , that is,  $\pi(t) = o(t)$  but is not  $o(t^2)$ .

In the next section we derive some conclusions concerning decision-makers' behavior based on the notions of Definition 2. In Section 4 we calculate the orders of attitude towards risk for several theories of decision-making under uncertainty. It turns out that examples for both orders of attitude can readily be found; therefore, the discussion in the next section is not vacuous.

## 3. OPTIMAL INSURANCE

Since Mossin [7], it has been known that an expected utility consumer will prefer to buy less than full insurance if his cost of insurance include positive marginal loading, i.e., if purchasing one more unit of insurance increases his premium by more than the fair actuarial cost of this insurance. As claimed by Borch [3, pp. 27–28], this result is at variance with common observation. In this section we show that this result holds whenever the decisionmaker's attitude towards risk is of order 2, but not when it is of order 1.<sup>4</sup>

We start with a discussion of the lottery  $x + t\tilde{e}$  when  $E[\tilde{e}] > 0$ .

**PROPOSITION 1.** *Let  $E[\tilde{e}] > 0$ . If the decisionmaker's attitude towards risk is of order 2, then for a sufficiently small  $t > 0$ ,  $x + t\tilde{e} \succ \delta_x$ . If his attitude towards risk is of order 1 and is negative (i.e.,  $\partial\pi/\partial t|_{t=0^+} > 0$ ), and if  $E[\tilde{e}]$  is small enough, then for a sufficiently small  $t > 0$ ,  $\delta_x \succ x + t\tilde{e}$ .*

As mentioned in the Introduction, Samuelson claimed that a decisionmaker is to be considered irrational if he rejects very small gambles that are in his favor. According to Proposition 1, this suggests considering attitude towards risk of order 1 irrational. In the next section we show that this implies that expected utility maximizers with nondifferentiable utility functions are to be considered irrational. (The importance of the differentiability was recognized by Samuelson [13].) In this section we present another argument in favor of not abolishing attitudes of order 1.

Consider a consumer having an initial wealth  $W$  and facing a risk of losing  $X$  units with a probability  $p$ . The insurance premium in the market is  $c$  for each insured unit,<sup>5</sup> and the decisionmaker chooses the optimal number of units he wants to insure. It follows immediately from Proposition 1 that if the decisionmaker's attitude towards risk is of order 2, he will not buy full insurance (i.e., he will insure less than  $X$  units), unless  $c \leq p$ . Suppose that he already insured all but  $t$  units of the possible loss. If he insures these last  $t$  units, his certain wealth level will be  $W - cX$ . If he does not insure these last units, his wealth will be  $W - c(X - t)$ , and he will face the lottery  $(-t, p, 0, 1 - p)$ . Let  $\tilde{e} = (c - 1, p; c, 1 - p)$ . Not insuring the last  $t$  units can be written as  $W - cX + t\tilde{e}$ .  $E[\tilde{e}] = c - p$ ; hence, by Proposition 1, if the decisionmaker's attitude towards risk is of order 2, and if  $c > p$ , then he will leave some  $t$  units uninsured. On the other hand, if his attitude towards risk is of order 1 and negative, and  $c$ , although

<sup>4</sup> This analysis follows Arrow [1] and Raviv [10] in assuming away moral hazards, so the probabilities and the outcome values of the risk are independent of the consumers' actions.

<sup>5</sup> We assume that  $c$  does not depend on the number of units already insured.

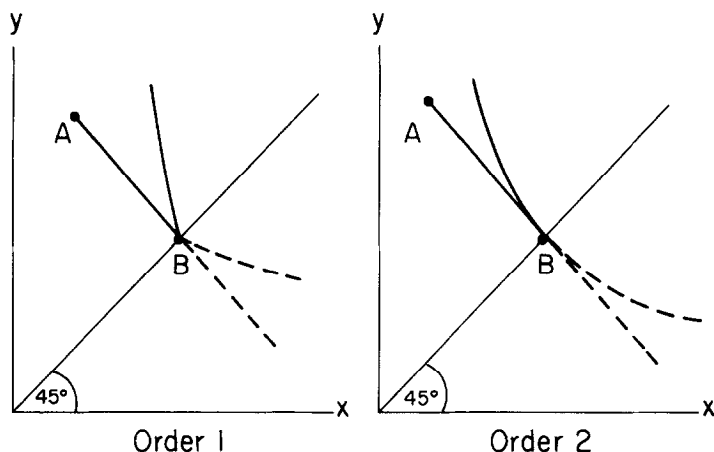


FIGURE 2

greater than  $p$ , is not too large, then the decisionmaker will buy full insurance.

This discussion is illustrated in the two panels of Fig. 2. In this Yaari [17]-type diagrams, a point  $(x, y)$  represents outcome  $x$  if the damage occurs, and  $y$  if not. That is, it stands for the lottery  $(x, p; y, 1 - p)$ . No insurance is represented by  $A = (W - X, W)$ , while full insurance is represented by  $B = (W - cX, W - cX)$ . The chord  $AB$  is the decisionmaker's budget line, and its slope is  $-c/(1 - c)$ .

We now derive the slope of the indifference curve at  $B$ . Let  $\tilde{\varepsilon} = (-1, p; p/(1 - p), 1 - p)$  (note that  $E[\tilde{\varepsilon}] = 0$ ). The degenerate lottery  $\delta_{x - \pi(t)}$  is represented by the point  $(x - \pi(t), x - \pi(t))$ , and the lottery  $x + t\tilde{\varepsilon}$  is represented by the point  $(x - t, x + tp/(1 - p))$ . The slope from the left of the indifference curve at  $(x, x)$  is given by

$$\lim_{t \rightarrow 0^+} -\frac{tp/(1 - p) + \pi(t)}{t - \pi(t)} = -\frac{p/(1 - p) + (\partial\pi/\partial t)|_{t=0^+}}{1 - (\partial\pi/\partial t)|_{t=0^+}}.$$

If the decisionmaker's attitude towards risk is of order 2, then this expression equals  $-p/(1 - p)$ . If his attitude is of order 1 and negative, i.e.,  $\partial\pi/\partial t|_{t=0^+} > 0$ , then the slope from the left of the indifference curve at  $(x, x)$  is steeper than  $-p/(1 - p)$ . Therefore, if  $c > p$  and the attitude is of order 2, the optimal point is strictly between  $A$  and  $B$ , but if the attitude is of order 1 and negative,  $B$  may be the optimal point. In other words, there exists a range of values of  $c$  for which the marginal loading is positive, but the optimal insurance is complete and there is no deductible.

It is worthwhile to note that first order risk aversion implies a kink in

the indifference curve along the certainty line. To see that, note first that the right-hand side slope of the indifference curve is

$$-\frac{p/(1-p) - (\partial\pi/\partial t)|_{t=0}}{1 + (\partial\pi/\partial t)|_{t=0}}.$$

This expression is smaller in its absolute value than the absolute value of the left-hand side slope. The set of possible supporting slopes at the certainty line includes the slope  $-p/(1-p)$ . This is always true by risk aversion, as moving from point  $B$  on a line with a slope of  $-p/(1-p)$  is a mean preserving spread, and hence is worse than  $B$ . It follows that if the indifference curve is differentiable at  $B$ , the only supporting slope is  $-p/(1-p)$ . In other words, first order attitude towards risk is equivalent to nondifferentiability of the indifference curve on the  $45^\circ$  line.

One of Arrow's [1] famous results is that when a risk averse decision-maker can invest in a riskless, profitless asset (say money) and in a risky asset, then he will invest some positive amount in the latter one if and only if it has a positive mean. Our analysis shows that this is true whenever the decisionmaker's attitude towards risk is of order 2, but fails when it is of order 1 (and negative). In that case we get the much more plausible result that if the expected yield is positive but sufficiently small, a risk averse decisionmaker will not invest in the risky asset. The proof of this claim is essentially the same as that of the case of partial insurance.

#### 4. SOME FUNCTIONAL FORMS

The discussion so far analyzed properties of preference relations that exhibit different orders of attitudes towards risk. The discussion is vacuous if such preference relations do not exist. It is our aim in this section to compute the order of attitude of several well-known representation functionals, all of which fall into one of the two categories of Definition 2.

We start the discussion with the case of expected utility, i.e.,  $V(\tilde{x}) = \int u(x) dF_{\tilde{x}}$ . Pratt [8] proved that in that case, if  $\delta_{x-\pi} \sim x + t\tilde{\varepsilon}$ , where  $E[\tilde{\varepsilon}] = 0$ , then

$$\pi(t) \approx -\frac{t^2}{2} \sigma_{\tilde{\varepsilon}}^2 \frac{u''(x)}{u'(x)}.$$

Hence  $\partial\pi/\partial t|_{t=0} = 0$ , and  $\partial^2\pi/\partial t^2 = -\sigma_{\tilde{\varepsilon}}^2 u''(x)/u'(x) \neq 0$  (unless, of course,  $u''(x) = 0$ ). As emphasized by Samuelson [13], this analysis strongly depends on  $u$  being (twice) differentiable. Suppose that  $u$  is not differen-

tiable at  $x$ , but has right and left (different) derivatives at that point. The risk premium  $\pi$  is defined by

$$\begin{aligned} u(x - \pi(t)) &= \int u(x + t\varepsilon) dF_{\tilde{\varepsilon}}(\varepsilon) \\ &= \int_{\varepsilon \leq 0} u(x + t\varepsilon) dF_{\tilde{\varepsilon}}(\varepsilon) + \int_{\varepsilon > 0} u(x + t\varepsilon) dF_{\tilde{\varepsilon}}(\varepsilon). \end{aligned}$$

Differentiate both sides with respect to  $t$  and obtain that at  $t = 0^+$ ,

$$-u'_-(x) \frac{\partial \pi}{\partial t} \Big|_{t=0^+} = u'_-(x) \int_{\varepsilon \leq 0} \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon) + u'_+(x) \int_{\varepsilon > 0} \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon);$$

$E[\tilde{\varepsilon}] = 0$ , hence

$$\frac{\partial \pi}{\partial t} \Big|_{t=0^+} = \left[ 1 - \frac{u'_+(x)}{u'_-(x)} \right] \int_{\varepsilon > 0} \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon)$$

and the decisionmaker's attitude towards risk is of order 1. If the decisionmaker is risk averse then  $u'_+(x) < u'_-(x)$  and his attitude is negative, and if he is risk loving, his attitude is positive.

If  $u$  is differentiable, the risk premium  $\pi$  is proportional to the variance of  $\tilde{\varepsilon}$ ,  $\sigma_{\tilde{\varepsilon}}^2$ . It would be wrong to conclude that when  $u$  is nondifferentiable,  $\pi$  is proportional to the standard deviation  $\sigma_{\tilde{\varepsilon}}$ . Consider the random variables  $\tilde{\varepsilon}_x = (-\alpha/(\alpha-1), (\alpha-1)/\alpha; \alpha, 1/\alpha)$ ,  $\alpha > 1$ . For every  $\alpha$ ,  $\int_{\varepsilon > 0} \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon) = 1$ , but  $\sigma_{\tilde{\varepsilon}_x} = \alpha/\sqrt{\alpha-1}$ , which changes with  $\alpha$ .

The above discussion is summarized by the following Proposition.

**PROPOSITION 2.** *Let the decisionmaker be an expected utility maximizer. At the points where his utility function is differentiable and  $u'' \neq 0$  his attitude towards risk is of order 2, and at the points where the utility function is not differentiable but has (different) side derivatives, his attitude is of order 1.*

We next discuss some nonexpected utility theories. Machina [6] showed that if  $V: D \rightarrow \mathbb{R}$  is Fréchet differentiable, then for every distribution function  $F$  there is a local utility function  $U(\cdot; F)$  such that

$$V(G) - V(F) = \int U(x; F) dG - \int U(x; F) dF + o\|G - F\|. \quad (1)$$

If the decisionmaker is risk averse, then  $U(\cdot; F)$  is concave (Machina, Theorem 2), hence  $U_{11}(\cdot; F)$  exists almost everywhere. The next proposition restricts attention to Fréchet differentiable functionals with smooth local utility functions. We assume that  $U_{11}(\cdot; F)$  exists everywhere



and, moreover,  $U_1(\cdot; F)$  and  $U_{11}(\cdot; F)$  are continuous in  $F$ . That is,  $\|F_n - F\| \rightarrow 0 \Rightarrow U_1(\cdot; F_n) \rightarrow U_1(\cdot; F)$ , and  $U_{11}(\cdot; F_n) \rightarrow U_{11}(\cdot; F)$  pointwise.<sup>6</sup> Assume further that for every  $x$ ,  $U_{11}(x; \delta_x) \neq 0$ .

**PROPOSITION 3.** *If the decisionmaker's preference relation can be represented by a Fréchet Differentiable functional, with the above-mentioned smoothness assumptions, then at every  $x$  his attitude towards risk is of order 2, and  $\partial^2 \pi / \partial t^2|_{t=0} = -U_{11}(x; \delta_x) \sigma_{\delta}^2 / U_1(x; \delta_x)$ .<sup>7</sup>*

A weaker version of this proposition, with respect to smooth preferences over finite lotteries was given by Samuelson [12] (see also Machina [6], footnote 44). The above proposition gives a behavioral interpretation to the smoothness of the (local) utility functions. It follows from Proposition 1 that for sufficiently small risk, decisionmakers with smooth local utilities behave as though they are risk neutral. The discussion of full insurance with marginal loading in the previous section indicates that this approach implies some doubtful result.

First order attitude towards risk is obtained with expected utility at points of nondifferentiability, but the set of these points may be empty. It turns out, however, that there is an alternative to expected utility, called anticipated utility (or expected utility with rank dependent probabilities), such that the decisionmaker's attitude towards risk is always of order 1. According to this theory, first presented by Quiggin [9], there is a strictly increasing, onto, continuous function  $f: [0, 1] \rightarrow [0, 1]$ , such that

$$V(\tilde{x}) = \int u(x) df(F_{\tilde{x}}(x)).$$

Note that when  $f$  is the identity function,  $V$  reduces to the expected utility representation. Otherwise, this functional is not Fréchet differentiable (see Chew, Karni, and Safra [5]). We will assume that  $u$  and  $f$  are differentiable, and moreover, that their derivatives are always strictly positive.

Due to Chew, Karni, and Safra [5], Yaari [18], Røell [11], and Segal [14] we now know that in this model risk aversion (in the sense of aversion to a mean preserving increase in risk) implies that  $u$  and  $f$  are both concave. Risk loving implies that they are both convex.

<sup>6</sup> Let  $\succsim$  be represented by Machina's [6] quadratic form  $V(F) = \int R dF + \frac{1}{2} [\int S dF]^2$ . The local utility  $U(x; F) = R(x) + S(x) \int S dF$  satisfies these assumptions for twice differentiable  $R$  and  $S$ . If  $\succsim$  can be represented by Chew's [4] weighted utility functional  $V(F) = \int u dF / \int w dF$ , then the local utility  $U(x; F) = (u(x) \int w dF - w(x) \int u dF) / [\int w dF]^2$  satisfies our assumptions for twice differentiable  $u$  and  $w$ .

<sup>7</sup>  $-U_{11}(x; F) / U_1(x; F)$  is Machina's [6] "Arrow-Pratt" term. See the proof of this proposition for a comment on the necessary assumptions for the analysis of this expression.

**PROPOSITION 4.** *If the decisionmaker's preference relation can be represented by the anticipated utility functional with either strictly concave or strictly convex  $f$ , then his attitude towards risk is of order 1. If  $f$  is concave his attitude is negative ( $\partial\pi/\partial t|_{t=0^+} > 0$ ), and if  $f$  is convex his attitude is positive ( $\partial\pi/\partial t|_{t=0^+} < 0$ ). Furthermore,  $\partial\pi/\partial t|_{t=0^+} = -\int \varepsilon df(F_\varepsilon(\varepsilon))$ .*

This proposition asserts that if the risk is sufficiently small, then within the anticipated utility model the attitude towards risk (order and direction) is determined by the distribution transformation function  $f$ . Of course, if  $f$  is neither concave nor convex, it may happen that  $\int \varepsilon df(F_\varepsilon(\varepsilon)) = 0$  even when  $f$  is not linear, and Proposition 4 will not hold.

It follows immediately from Definition 2 that if the decisionmaker is risk averse and his attitude towards risk is of order 1, then  $\pi$  as a function of  $t$  is nondifferentiable at  $t=0$  (see Fig. 1). Similarly, his indifference curves are nondifferentiable around the certainty line (Fig. 2). To obtain such nondifferentiability, Bewley [2] removed the completeness assumption. Proposition 4 shows that this nondifferentiability can be obtained in a model where the preference relation is complete, transitive, and continuous.

In the case of anticipated utility  $\pi$  is not differentiable at  $t=0$ , and moreover, it is not necessarily symmetric. For example, let  $u(x) = x^{1/2}$ ,  $f(p) = p^3$ ,  $\bar{\varepsilon} = (-1, \frac{2}{3}; 2, \frac{1}{3})$ , and  $x = 9$ . For  $t > 0$  we obtain

$$\begin{aligned}\delta_{9-\pi(t)} &\sim (9-t, \tfrac{2}{3}; 9+2t, \tfrac{1}{3}) \Rightarrow \\ (9-\pi(t))^{1/2} &= \frac{8(9-t)^{1/2}}{27} + \frac{19}{27}(9+2t)^{1/2} \Rightarrow \\ \frac{\partial\pi}{\partial t} \Big|_{t=0^+} &= -\frac{10}{9}.\end{aligned}$$

However, when  $t < 0$  it follows that

$$\begin{aligned}(9-\pi(t))^{1/2} &= \frac{1}{27}(9+2t)^{1/2} + \frac{26}{27}(9-t)^{1/2} \Rightarrow \\ \frac{\partial\pi}{\partial t} \Big|_{t=0^-} &= \frac{8}{9}.\end{aligned}$$

Proposition 1 implies that if the decisionmaker is a risk averse anticipated utility maximizer, then for small positive marginal loading he will still prefer to buy a full insurance. This remains true even if his (differentiable) utility function  $u$  is convex. Of course, the marginal loading he is willing to tolerate depends on  $u$  (and  $f$ ), but his attitude towards risk depends only on  $f$ .

## 5. TRANSFORMATIONS OF RANDOM VARIABLES

There are situations in which the decisionmaker does not receive the random variable itself, but a transformation  $\phi$  of it. For example, the random variable may be his income  $I$  while  $\phi(I)$  is his net income. The random variable may be the price  $p$  of a certain good, and  $\phi(p)$  is the quantity demanded of that good. We will assume that  $\phi$  is twice differentiable with  $\phi' \neq 0$ . The risk premium  $\pi_\phi$  is paid out of the random variable itself, and the basic relationship between  $\pi_\phi$  and  $t$  is given by  $\delta_{\phi(x - \pi_\phi(t))} \sim \phi(x + t\tilde{\varepsilon})$ . As before,  $\pi_\phi(0) = 0$ .

Assume that for  $t > 0$ ,  $\pi_\phi$  is differentiable, and obtain that  $\phi(x - \pi_\phi(t)) = \phi(x) - t\phi'(x) \cdot \partial\pi_\phi/\partial t|_{t=0^+} + o(t)$ , while  $\phi(x + t\varepsilon) = \phi(x) + t\phi'(x)\varepsilon + o(t)$ . It thus follows that

$$\delta_{\phi(x) - t\phi'(x) \cdot \partial\pi_\phi/\partial t|_{t=0^+} + o(t)} \sim \phi(x) + o(t) + t\phi'(x)\tilde{\varepsilon}.$$

In other words, both the random variable  $\tilde{\varepsilon}$  and the risk premium  $\pi$  are multiplied by  $\phi'(x)$ . Therefore, if the decisionmaker's attitude towards risk at  $\phi(x)$  is of order 1, then  $\partial\pi_\phi/\partial t|_{t=0^+} \neq 0$ . Moreover, its sign is opposite to that of his attitude at  $\phi(x)$ . In other words, the sign of the derivative with respect to  $t$  of the risk premium  $\pi_\phi$  out of  $x$  is determined by the decisionmaker's attitude at  $\phi(x)$ . In particular, if the first order attitude does not depend on  $x$  (e.g., expected utility with differentiable  $u$ , anticipated utility with concave or convex  $f$ , etc.), then this attitude will not change when the decisionmaker receives  $\phi(x)$  rather than  $x$ . This is true regardless of the shape of  $\phi$ , provided it is differentiable. If  $\phi$  is not differentiable his attitude may change. For example, when the decisionmaker is an expected utility maximizer with a differentiable utility function, and there is an  $x$  for which  $\phi'_+(x) \neq \phi'_-(x)$ .

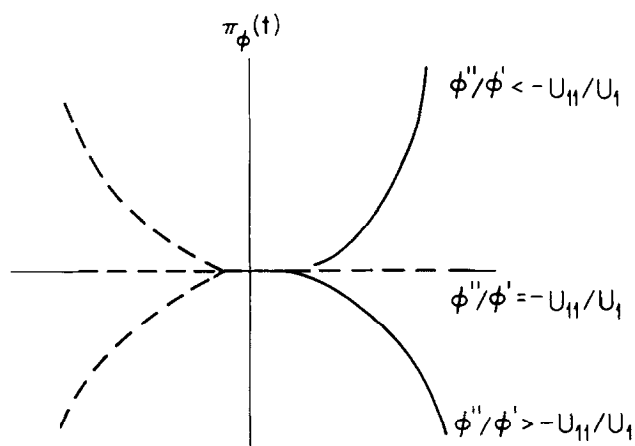
Consider now the case of Fréchet differentiability with smooth local utility functions. Following the same technique used in the proof of Proposition 3, and with the same smoothness assumptions, it follows that  $\partial\pi_\phi/\partial t|_{t=0} = 0$  and

$$\begin{aligned} \frac{\partial^2 \pi_\phi}{\partial t^2} \Big|_{t=0} &= -\sigma_\varepsilon^2 \left[ \frac{U_{11}(\phi(x); \delta_{\phi(x)}) \phi'(x)}{U_1(\phi(x); \delta_{\phi(x)})} + \frac{\phi''(x)}{\phi'(x)} \right] \cong 0 \Leftrightarrow \\ \frac{\phi''(x)}{[\phi'(x)]^2} &\cong - \frac{U_{11}(\phi(x); \delta_{\phi(x)})}{U_1(\phi(x), \delta_{\phi(x)})}. \end{aligned}$$

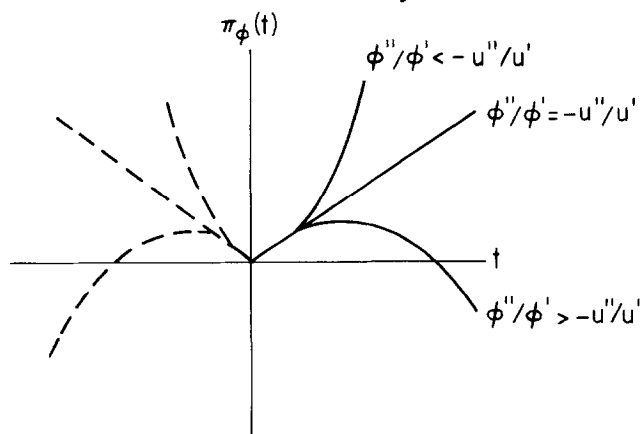
This last inequality implies that although a transformation  $\phi$  cannot change the decisionmaker's order of attitude towards risk (provided  $V$  and  $\phi$  are smooth), it can nevertheless change the direction of his attitude towards risk, and transform him from risk averse to risk loving and vice

versa. For this,  $\phi''/\phi'^2$  has to be sufficiently large. Similar results hold for expected utility. It is well known that a risk averse decision-maker may act as though he loves risk in prices, provided he is not too risk averse (see Turnovsky, Shalit, and Schmitz [16] and Segal and Spivak [15]). This result is described in the upper panel of Fig. 3 (we assume that the underlying preference relation exhibits risk aversion).

If the decisionmaker's attitude towards risk is always of order 1, then a differentiable transformation  $\phi$  will not change the sign of his first order attitude. This implies that at least for sufficiently small  $t$  (which may



### Fréchet Differentiability



### Anticipated Utility

FIGURE 3

depend on  $\phi$  and  $\tilde{\varepsilon}$ ,  $\delta_{\phi(x)} > \phi(x + t\tilde{\varepsilon})$ . In the case of anticipated utility we obtain, using the proof of Proposition 4, that

$$\begin{aligned}\frac{\partial \pi_\phi}{\partial t} \Big|_{t=0^+} &= - \int \varepsilon df(F_{\tilde{\varepsilon}}(\varepsilon)) \\ \frac{\partial^2 \pi_\phi}{\partial t^2} \Big|_{t=0^+} &= - \left[ \frac{u''(\phi(x)) \phi'(x)}{u'(\phi(x))} + \frac{\phi''(x)}{\phi'(x)} \right] \int (\varepsilon - \varepsilon^2) df(F_{\tilde{\varepsilon}}(\varepsilon)).\end{aligned}$$

If the decisionmaker is risk averse,  $f$  and  $u$  are concave. In that case  $\int (\varepsilon - \varepsilon^2) df(F_{\tilde{\varepsilon}}(\varepsilon)) < 0$ , and

$$\frac{\partial^2 \pi_\phi}{\partial t^2} \Big|_{t=0^+} \cong 0 \Leftrightarrow \frac{\phi''(x)}{[\phi'(x)]^2} \cong - \frac{u''(\phi(x))}{u'(\phi(x))}.$$

Note the similarity between these conditions and the conditions for the case of Fréchet differentiability. These results are depicted in the lower panel of Figure 3.

#### APPENDIX

*Proof of Proposition 1.* Let  $\tilde{\varepsilon}' = \tilde{\varepsilon} - E[\tilde{\varepsilon}]$ . Define  $\pi_t(s)$  implicitly by  $\delta_{x+tE[\tilde{\varepsilon}] - \pi_t(s)} \sim x + tE[\tilde{\varepsilon}] + s\tilde{\varepsilon}'$ , and define  $\pi^*(t)$  by  $\delta_{x+\pi^*(t)} \sim x + t\tilde{\varepsilon}$ . Obviously,  $\partial \pi^*/\partial t = E[\tilde{\varepsilon}] - \partial \pi_t/\partial s|_{s=t}$ . By the continuity of  $\partial \pi_t/\partial s$  in  $s$  and  $t$  it follows that  $\lim_{t \rightarrow 0^+} \partial \pi_t/\partial s|_{s=t} = \partial \pi_0/\partial t|_{t=0^+}$ ; hence  $\partial \pi^*/\partial t|_{t=0^+} = E[\tilde{\varepsilon}] - \partial \pi_0/\partial t|_{t=0^+}$ . If the decisionmaker's attitude towards risk is of order 2, then  $\partial \pi_0/\partial t|_{t=0^+} = 0$ , and  $\partial \pi^*/\partial t|_{t=0^+} = E[\tilde{\varepsilon}] > 0$ . If his attitude is of order 1 and negative, then  $\partial \pi_0/\partial t|_{t=0^+} > 0$ . For  $E[\tilde{\varepsilon}] < \partial \pi_0/\partial t|_{t=0^+}$ ,  $\partial \pi^*/\partial t|_{t=0^+} < 0$ . Q.E.D.

*Proof of Proposition 3.* We first find the first order derivative of  $V(x + t\tilde{\varepsilon})$  with respect to  $t$ :

$$\begin{aligned}\frac{\partial V}{\partial t}(x + t\tilde{\varepsilon}) &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [V(x + (t + \alpha)\tilde{\varepsilon}) - V(x + t\tilde{\varepsilon})] \\ &= \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} \left[ \int_{\varepsilon} U(x + (t + \alpha)\varepsilon; F_{s+t\tilde{\varepsilon}}) dF_{\tilde{\varepsilon}}(\varepsilon) \right. \\ &\quad \left. - \int_{\varepsilon} U(x + t\varepsilon; F_{x+t\tilde{\varepsilon}}) dF_{\tilde{\varepsilon}}(\varepsilon) + \|F_{\tilde{\varepsilon}} - \delta_0\| \cdot o(\alpha) \right] \\ &= \int_{\varepsilon} U_1(x + t\varepsilon; F_{x+t\tilde{\varepsilon}}) \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon).\end{aligned}$$

For  $t=0$  we obtain

$$\left. \frac{\partial V}{\partial t}(x + t\tilde{\varepsilon}) \right|_{t=0} = U_1(x; \delta_x) \int_{\varepsilon} \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon) = 0.$$

We now turn to the second order derivative at  $t=0$ :

$$\left. \frac{\partial^2 V}{\partial t^2}(x + t\tilde{\varepsilon}) \right|_{t=0} = \lim_{t \rightarrow 0} \frac{1}{t} \int_{\varepsilon} U_1(x + t\varepsilon; F_{x+t\tilde{\varepsilon}}) \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon).$$

Let  $G = F_{x+t\tilde{\varepsilon}}$ . Take a Taylor's expansion of  $U_1(x + t\varepsilon; G)$  with respect to  $t$  at  $t=0$  to obtain

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t} \left[ U_1(x; G) \int_{\varepsilon} \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon) + t U_{11}(x; G) \int_{\varepsilon} \varepsilon^2 dF_{\tilde{\varepsilon}}(\varepsilon) + o(t) \right] \\ = U_{11}(x; F_{x+t\tilde{\varepsilon}}) \sigma_{\tilde{\varepsilon}}^2 \xrightarrow{t \rightarrow 0} U_{11}(x; \delta_x) \sigma_{\tilde{\varepsilon}}^2. \end{aligned}$$

This last step required the assumption that  $U_{11}(\cdot; F)$  is continuous in  $F$ . We believe that this is a necessary assumption in Machina's [6] analysis of the expression  $-U_{11}(\cdot; F)/U_1(\cdot; F)$ , although it is not mentioned there.

Consider the equation  $V(\delta_{x-\pi(t)}) = V(x + t\tilde{\varepsilon})$ . Differentiate both sides with respect to  $t$  to obtain

$$\begin{aligned} \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} [U(x - \pi(t + \alpha); \delta_{x-\pi(t)}) - U(x - \pi(t); \delta_{x-\pi(t)}) + o(\alpha)] \\ = \int_{\varepsilon} U_1(x + t\varepsilon; F_{x+t\tilde{\varepsilon}}) \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon) \Rightarrow \\ \frac{\partial \pi(t)}{\partial t} = - \int_{\varepsilon} U_1(x + t\varepsilon; F_{x+t\tilde{\varepsilon}}) \varepsilon dF_{\tilde{\varepsilon}}(\varepsilon) / U_1(x - \pi(t); \delta_{x-\pi(t)}). \end{aligned}$$

Hence  $\partial \pi / \partial t|_{t=0} = 0$ . Differentiate once more to obtain

$$\left. \frac{\partial^2 \pi}{\partial t^2} \right|_{t=0} = - \frac{U_{11}(x; \delta_x) \sigma_{\tilde{\varepsilon}}^2}{U_1(x; \delta_x)}. \quad \text{Q.E.D.}$$

*Proof of Proposition 4.* Let  $t > 0$  and differentiate both sides of

$$u(x - \pi(t)) = \int u(x + t\varepsilon) df(F_{\tilde{\varepsilon}}(\varepsilon))$$

with respect to  $t$  to obtain

$$-u'(x - \pi(t)) \frac{\partial \pi}{\partial t} = \int u'(x + t\varepsilon) \varepsilon df(F_\varepsilon(\varepsilon)) \Rightarrow$$

$$\left. \frac{\partial \pi}{\partial t} \right|_{t=0^+} = - \int \varepsilon df(F_\varepsilon(\varepsilon)).$$

We assumed  $\int \varepsilon dF_\varepsilon(\varepsilon) = 0$ ; hence if  $f$  is concave,  $\partial \pi / \partial t|_{t=0^+} > 0$ , and if  $f$  is convex,  $\partial \pi / \partial t|_{t=0^+} < 0$ . Q.E.D.

### REFERENCES

1. K. J. ARROW, "Essays in the Theory of Risk Bearing," North-Holland, Amsterdam, 1974.
2. T. F. BEWLEY, Knightian decision theory: Part I, Cowles Foundation DP #807, Yale University, 1986.
3. K. BORCH, "The Mathematical Theory of Insurance," Lexington Books, Lexington, MA, 1974.
4. S. H. CHEW, A generalization of the quasilinear mean with application to the measurement of income inequality and decision theory resolving the Allais paradox, *Econometrica* **51** (1983), 1061–1092.
5. S. H. CHEW, E. KARNI, AND Z. SAFRA, Risk aversion in the theory of expected utility with rank dependent probabilities, *J. Econ. Theory* **42** (1987), 370–381.
6. M. J. MACHINA, "Expected utility" analysis without the independence axiom, *Econometrica* **50** (1982), 277–323.
7. J. MOSSIN, Aspects of rational insurance purchasing, *J. Polit. Econ.* **76** (1968), 553–568.
8. J. PRATT, Risk aversion in the small and in the large, *Econometrica* **22** (1964), 122–136.
9. J. C. QUIGGIN, A theory of anticipated utility, *J. Econ. Behav. Organ.* **3** (1982), 323–343.
10. A. RAVIV, The design of an optimal insurance policy, *Amer. Econ. Rev.* **69** (1979), 84–96.
11. A. RÖELL, Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty, *Econ. J.* **97** (1987), Supplement, 149–159.
12. P. A. SAMUELSON, The St. Petersburg paradox as a divergent double limit, *Int. Econ. Rev.* **1** (1961), 31–37.
13. P. A. SAMUELSON, Risk and uncertainty: A fallacy of large numbers, *Scientia*, 6th Series **57** (1963), 1–6.
14. U. SEGAL, Some remarks on Quiggin's anticipated utility, *J. Econ. Behav. Organ.* **8** (1987), 145–154.
15. U. SEGAL AND A. SPIVAK, "Local risk aversion," Working Paper #8806, University of Toronto, 1988.
16. S. J. TURNOVSKY, H. SHALIT, AND A. SCHMITZ, Consumer's surplus, price instability, and consumer welfare, *Econometrica* **48** (1980), 135–152.
17. M. E. YAARI, Some remarks on measures of risk aversion and on their uses, *J. Econ. Theory* **1** (1969), 315–329.
18. M. E. YAARI, The dual theory of choice under risk, *Econometrica* **55** (1987), 95–115.