Privacy and Statistical Discrimination with an Application to Personalized Pricing

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Gerzensee Summer Course

July 28, 2025

Introduction

Overview

- Statistical Discrimination
- Regulating Privacy & Discrimination
  - → Blackwell experiments
  - → Privacy preserving signals
- Regulating Personalize Pricing Discrimination (optimal transport)
  - → Optimal transport
- A self-centered theory of discrimination
  - → Misspecified learning
  - → Semi-definite quadratic programming
- Rational attention & Cost of information
  - Information cost
  - Entropy, LLR cost, Rényi entropy
- Asymptotic Blackwell order / Extreme point methods

Introduction: Signals and the Value of Information

Fix a probability space: $(\Omega \times [0,1], \mathscr{F} \otimes \mathscr{B}, \mathbb{P})$; $(\omega, r) \in \Omega \times [0,1]$.
$(\Omega, \mathscr{F})$: Standard Borel;
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Can treat $\mathbb{P}[\cdot \mid s]$ as a random variable from $\Omega \times [0,1]$ to $\Delta(\Omega)$.

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Definition (Garbling)

A signal s' is a **garbling** of a signal s if there exists a random variable $\tilde{\varepsilon}$, independent of (ω, s), and a measurable function $g : S \times \mathbb{R} \to S'$ such that

$$s' = g(s, \tilde{\varepsilon})$$

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Definition (Convex Order)

Posterior $\mathbb{P}[\cdot \mid s]$ dominates posterior $\mathbb{P}[\cdot \mid s']$ in the **convex order** if

$$\mathbb{E}[C(\mathbb{P}[\cdot \mid s'])] \leq \mathbb{E}[C(\mathbb{P}[\cdot \mid s])]$$

for all convex function $C : \Delta(\Omega) \to \mathbb{R}$. In this case, we also say that the distribution of $\mathbb{P}[\cdot \mid s']$ is a **mean-preserving contraction** of the distribution of $\mathbb{P}[\cdot \mid s']$.

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Definition (Blackwell Order)

A signal s is **Blackwell more informative** than s', denoted by $s' \preceq s$, if for any decision problem (u, A),

$$V_{(u,A)}(s) \geq V_{(u,A)}(s').$$

Two signals s and s' are **(Blackwell) equivalent**, denoted by $s \sim s'$, if $s \preceq s'$ and $s' \preceq s$.

Blackwell's Theorem

Theorem (Blackwell, 1953; Strassen, 1965; Green and Stokey, 2022)

Given any pair of signals s, s', the following are equivalent

(i) *s is Blackwell more informative than s'.*

(ii) $\mathbb{P}[\cdot \mid s]$ *dominates* $\mathbb{P}[\cdot \mid s']$ *in the convex order.*

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That is, t' is a garbling of $s \Rightarrow s$ is Blackwell more informative than s'.

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However, s, s' can be linked through the garbling characterization (iii), which is also known as "couplings" or "embeddings" (c.f., Strassen, 1965; Green and Stokey, 2022).

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What information can be given to Ann and Bob separately without them knowing each other's information?

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for $\theta \in \{B, R\}$.

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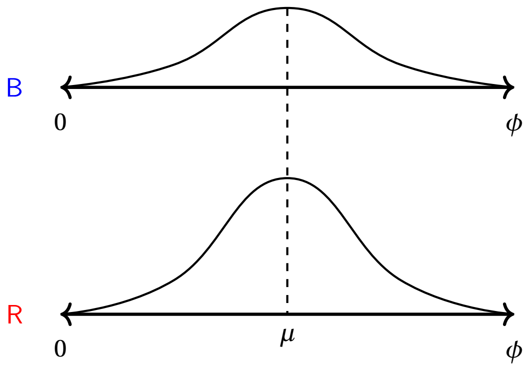

$\phi := \mathbb{E}[\gamma \mid x, \theta] = \frac{\sigma^{-2}\mu + \sigma_\theta^{-2} x}{\sigma^{-2} + \sigma_\theta^{-2}}$ and thus

$$\mathrm{var}(\phi) = \left(\frac{\sigma_\theta^{-2}}{\sigma^{-2} + \sigma_\theta^{-2}} \right)^2 (\sigma^2 + \sigma_\theta^2) = \frac{\sigma^2}{\sigma_\theta^2 / \sigma^2 + 1}$$

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Statistical discrimination prevails if there exists $W : \mathbb{R} \to \mathbb{R}$ such that

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Two groups with the same productivity distribution,
but different signal precisions
end up having different expected wages,
even if wages are only functions of expected productivity.

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- **Renting Out an Apartment**
  - Here the land-lord is concerned with downward risk κ is small.
  - Group member's about which the land-lord has more precise information are accepted less often

Equilibrium Statistical Discrimination

Two groups $\theta \in \{B, R\}$, worker types $\gamma \in \{0,1\}$. Firms observe signal $s \in \{H, L\}$:

$$\Pr(s = H \mid \gamma = 1) = p_H, \quad \Pr(s = H \mid \gamma = 0) = p_L, \; p_H > p_L.$$

Posterior hiring rule:

$$\Pr(\gamma = 1 \mid s = H, \theta) = \frac{p_H \pi_\theta}{p_H \pi_\theta + p_L(1 - \pi_\theta)} \geq \kappa$$

where $\pi_\theta = \Pr(\gamma = 1 \mid \theta)$.

Worker gets high payoff only when hired.

Self-fulfilling: If $\theta = B$ expects lower hiring, fewer invest $\Rightarrow$ lower π_B, validating the firm's belief.

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- Note that in all the above people with the same realization of x (same CV, credit score, etc) are treated differently.

Statistical Discrimination More Generally

- In general distributions of underlying "productivity" will not be the same accross groups.
- This will amplify differences in treatment.
- Statistical discrimination can be a pure equilibrium phenomenon:
  1. People of different groups are treated differently.
  2. Leads to different actions across groups (education choice etc..).
  3. This creates group differences which justify different treatment.
- (aside: statistical discrimination largely mispredicts beliefs, probably biases play a large role in discrimination).
- Note that in all the above people with the same realization of x (same CV, credit score, etc) are treated differently.
- Such differences in treatment of viewed as undesirable and in many contexts illegal.

Contexts Where Differential Treatment is Illegal

- **Employment:** Under laws such as Title VII of the U.S. Civil Rights Act and EU Equal Treatment Directives, discrimination based on race, gender, or ethnicity in hiring, pay, or promotion is prohibited.
- **Housing:** Fair housing laws (e.g., U.S. Fair Housing Act, EU anti-discrimination rules) make it illegal to refuse rental or sale based on race, gender, or other protected characteristics.
- **Education:** Schools receiving public funding are prohibited from discriminating based on gender, race, or ethnicity under laws like Title IX in the U.S. or EU equality regulations.
- **Access to Services:** Public accommodations (restaurants, hotels, transportation) are often barred from treating customers differently based on protected traits.
- **Credit and Lending:** Equal Credit Opportunity laws prohibit banks and lenders from using gender or race in deciding creditworthiness.

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Are there better ways?

(Selected) Related Literature

Privacy: Dowark et al. (2006); Ichihashi (2020); Eliat, Eliaz and Mu (2021); Echenique and He (2021); Hidir and Vellodi (2021); Bergemann and Bonatti (2023); Galperti and Perego (2023); Vaidya (2023); Rhodes and Zhou (2024); Schmutte and Yoder (2024).

Statistical discrimination: Coute and Loury (1993); Chan and Eyster (2003); Hardt et al, (2016); Borhen, Imas and Rosenberg (2019); Chambers and Echenique (2021); Escudé, et al. (2022); Martin and Marx (2022); Onuchic and Ray (2023); Onuchic (2024); Bardhi, Guo and Strulovici (2024); Bohren, Hull and Imas (2024); Zhu (2024); Liang, et al. (2024); Doval and Smolin (2024); Bharadwaj, Deb and Renou (2024); Che, Kim and Zhong (2024);

Non-discriminatory personalized pricing: Cowen (2016); Kallus and Zhou (2021); Cohen, Elmachtoub and Lei (2022); Farboodi, Haghpanah and Shourideh (2025).

This lecture: Strack and Yang (2024; 2025); He, Sandomirskiy and Tamuz (2024).

Privacy-Preserving Signals (Strack and Yang, 2024)

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Questions:

- Which signals are privacy-preserving?
- What are the "most informative" privacy-preserving signals?

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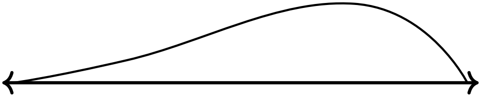

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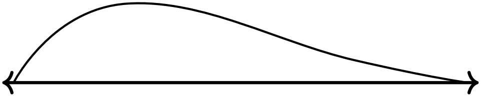

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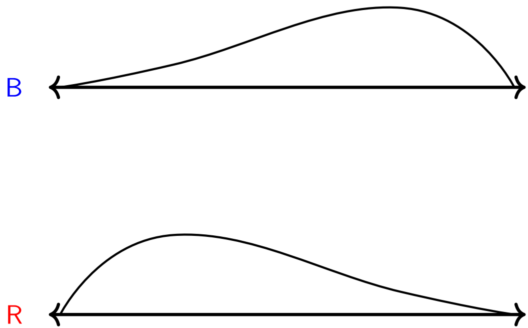

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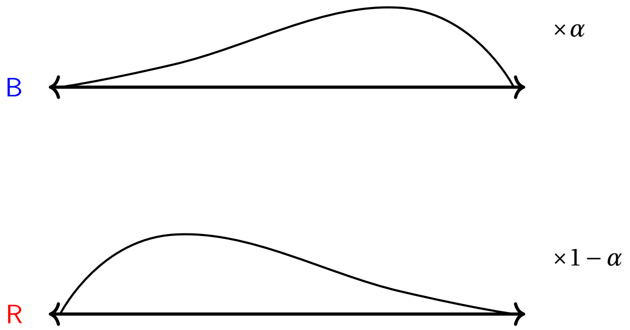

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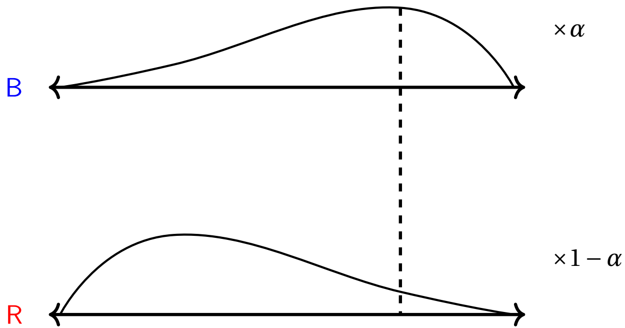

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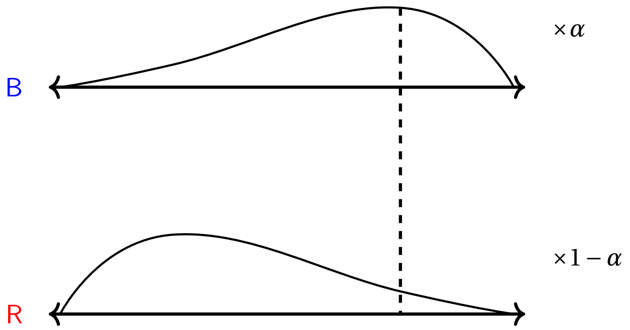

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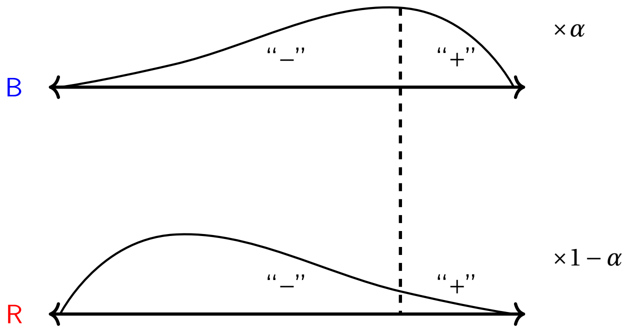

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Not privacy-preserving

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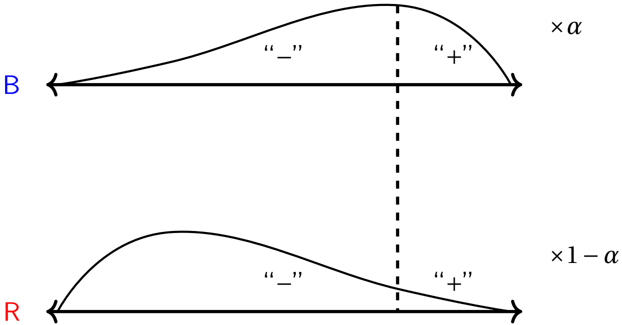

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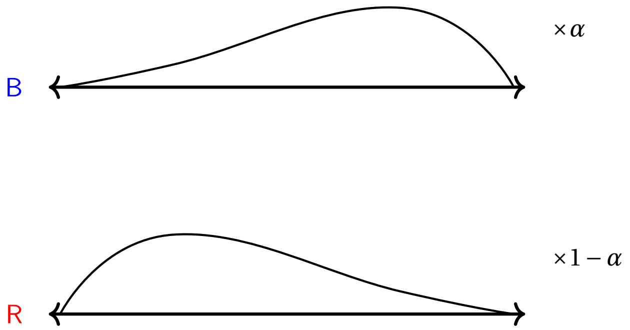

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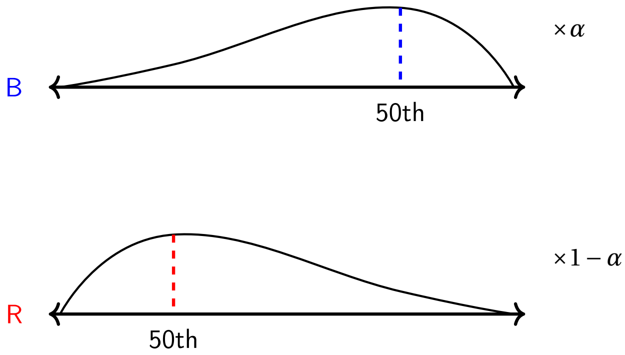

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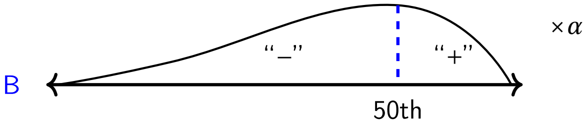

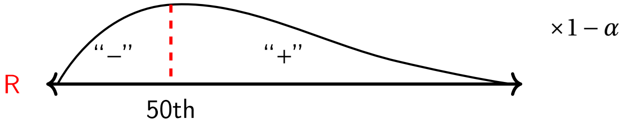

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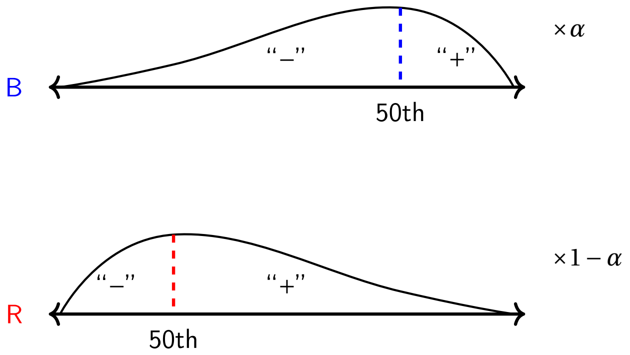

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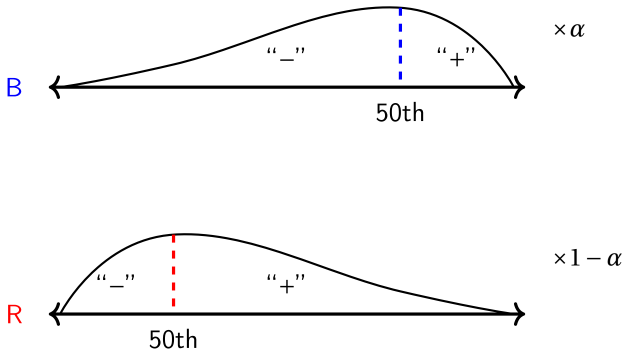

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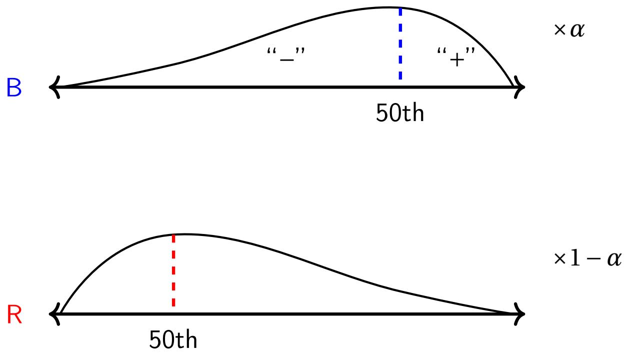

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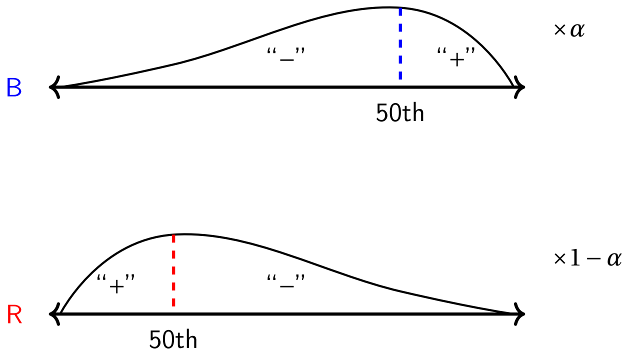

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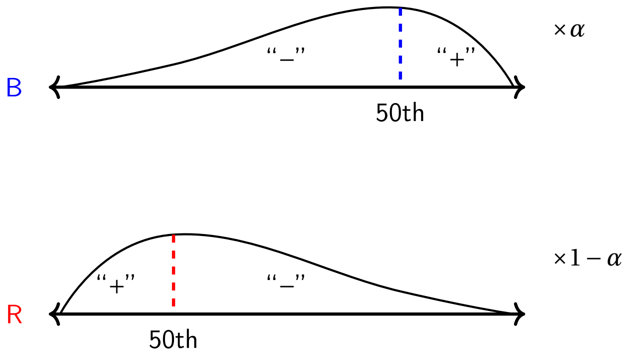

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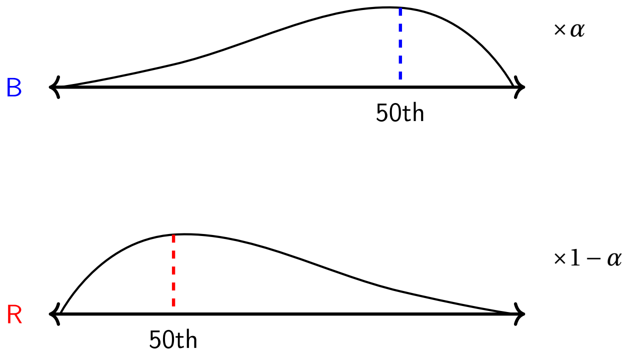

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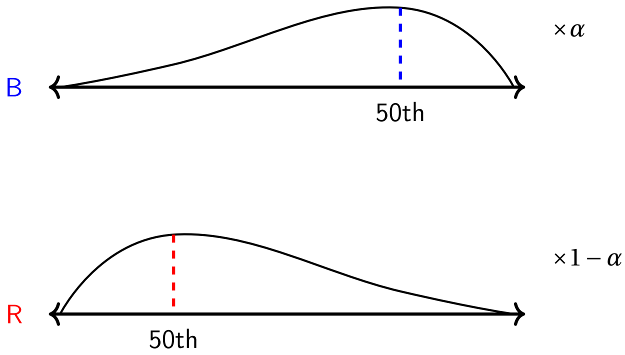

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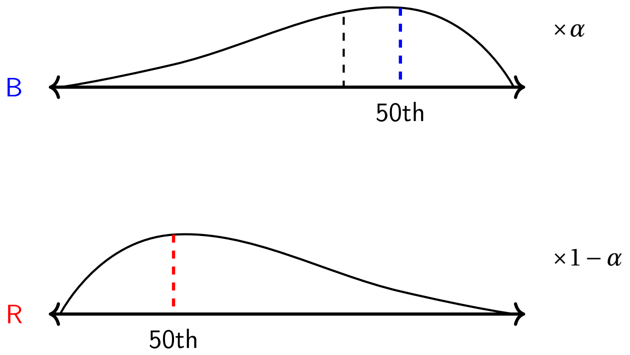

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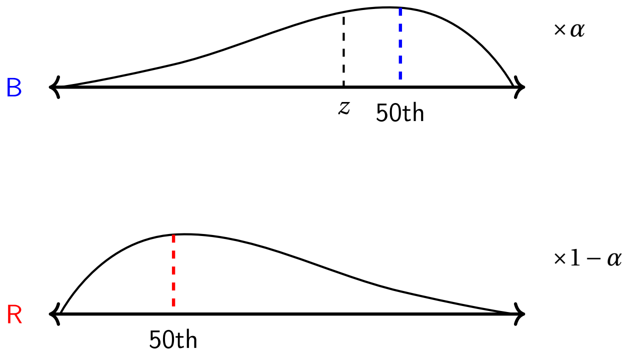

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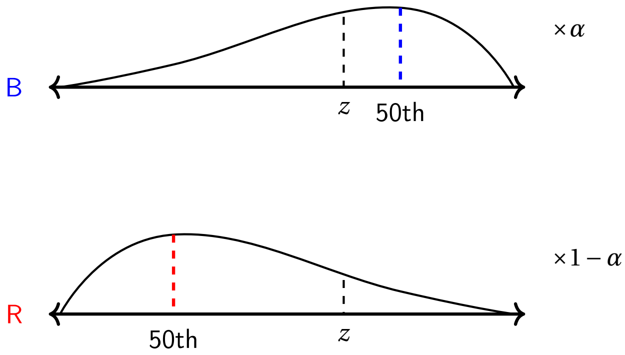

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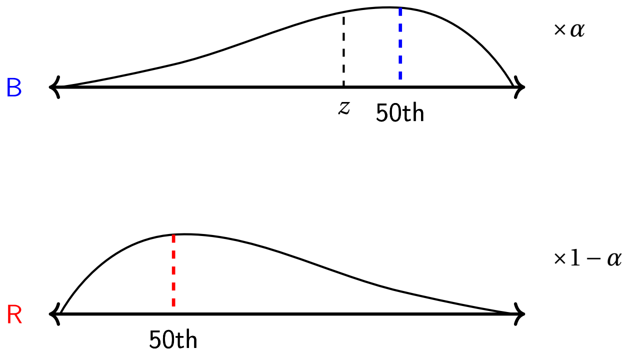

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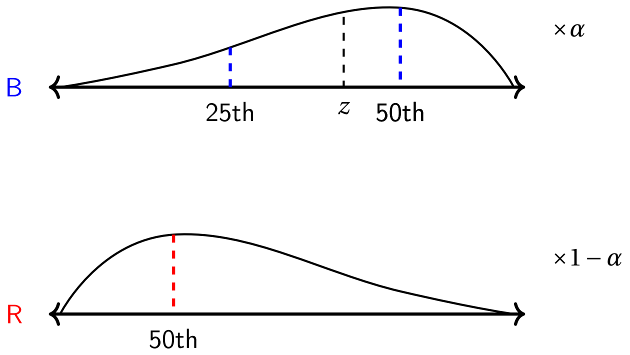

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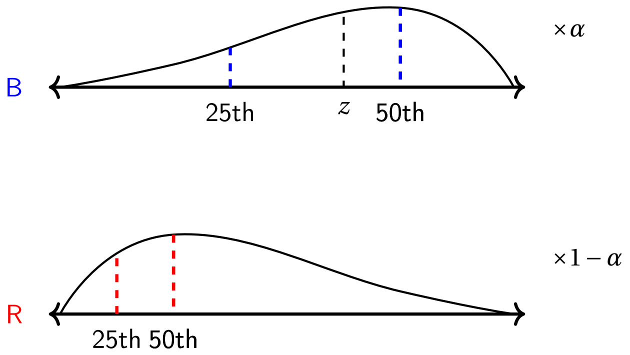

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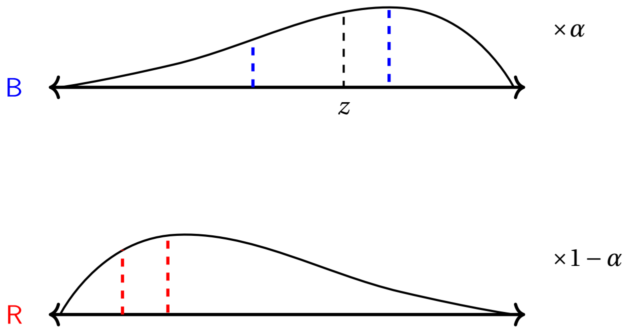

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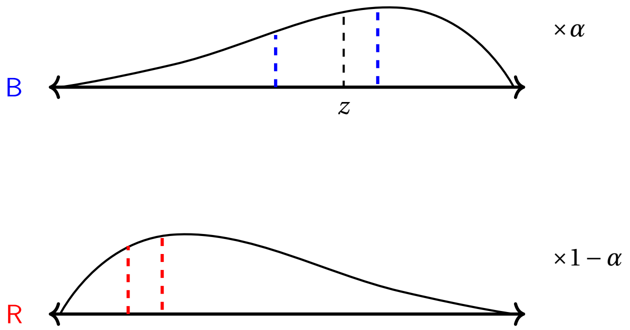

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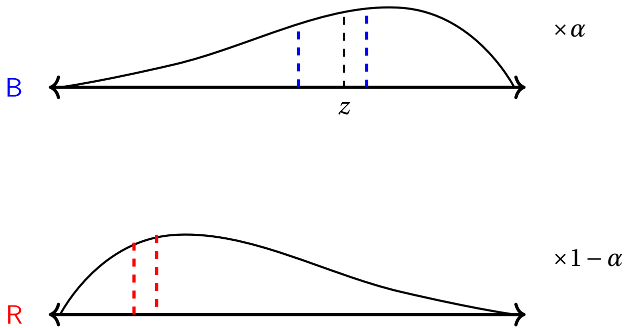

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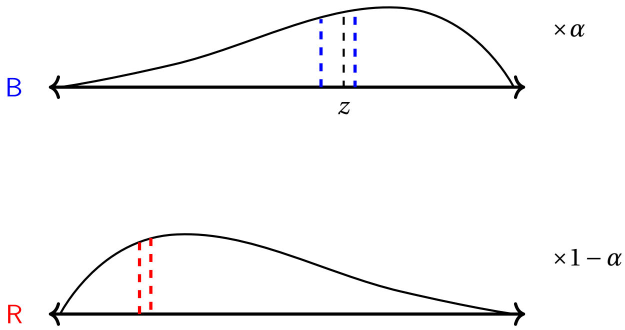

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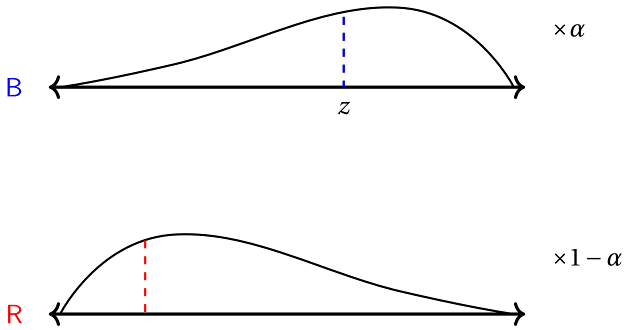

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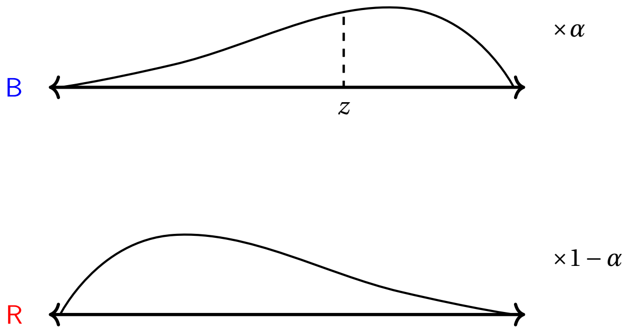

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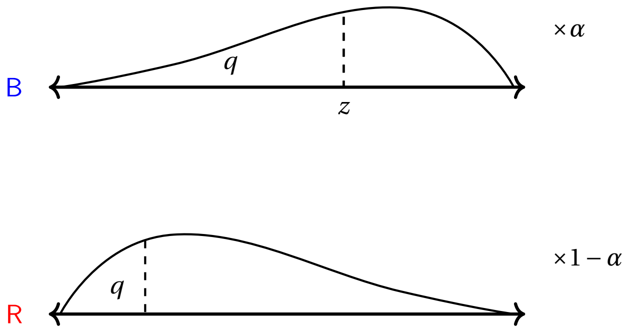

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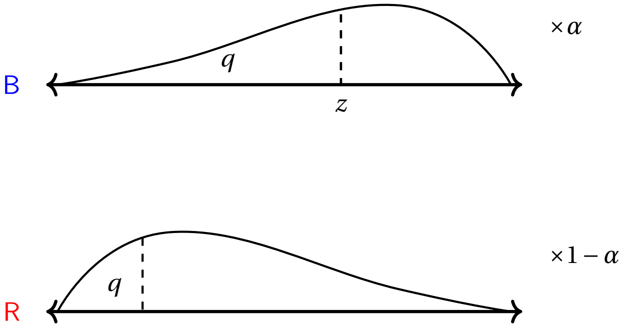

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Quantile signal (conditionally revealing)

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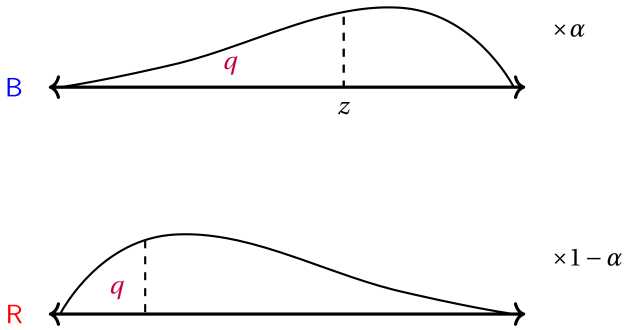

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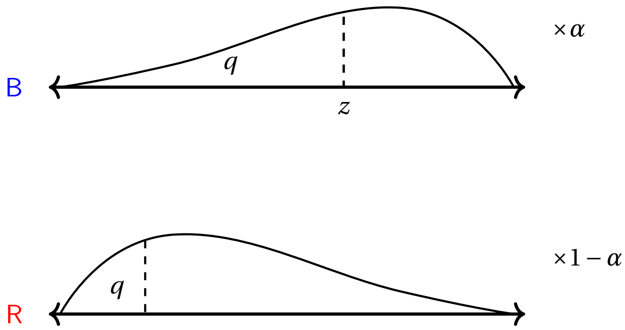

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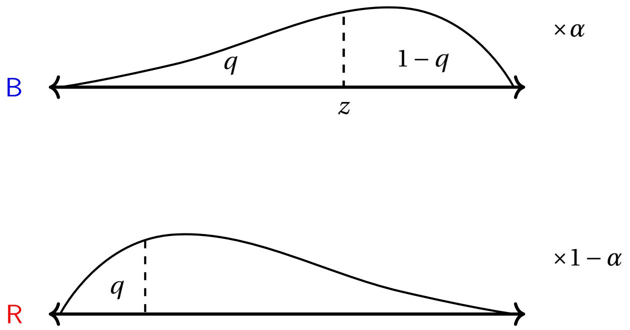

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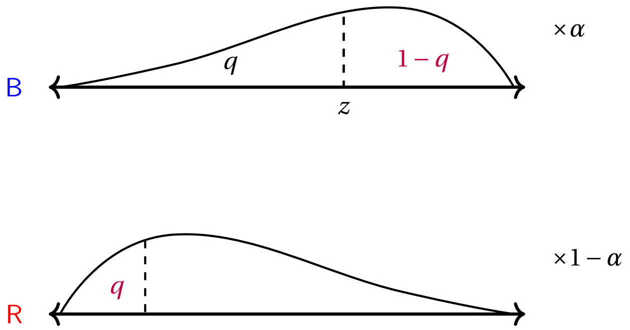

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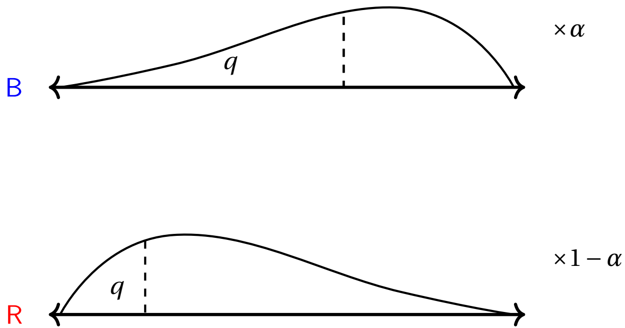

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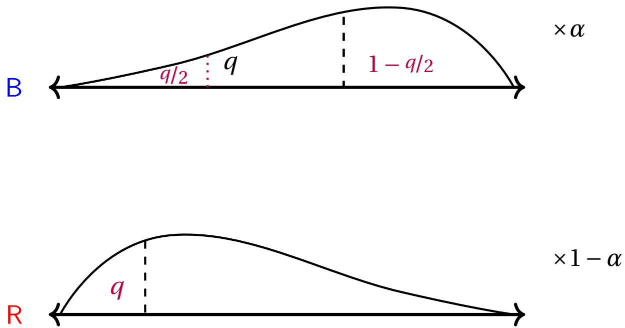

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Reordered Quantile Signals (also conditionally revealing)

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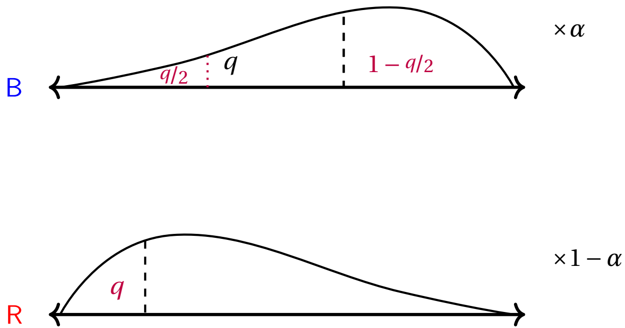

Characterization of Privacy-Preserving Signals

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Fix $\theta : \Omega \to \Theta$ from now on and use that to represent $\mathscr{P}$.

Conditionally Revealing Signals

A signal s is **conditionally revealing** if (s, θ) reveals ω.
(i.e., for all $A \in \mathscr{F}$, $\mathbb{P}[\omega \in A \mid s, \theta] \in \{0, 1\}$ a.s.).

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A signal is privacy-preserving if and only if it is Blackwell dominated by some conditionally revealing privacy-preserving signal.

Proof

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- Proof also works for Θ countable, but breaks down for the uncountable case.

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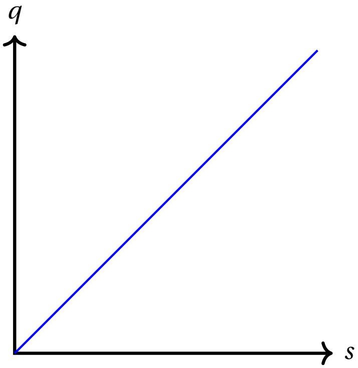

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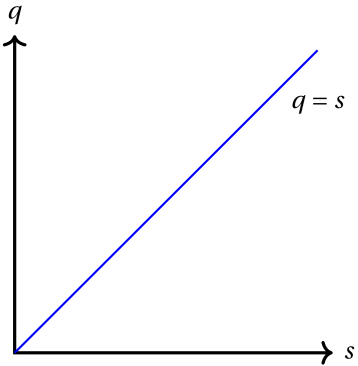

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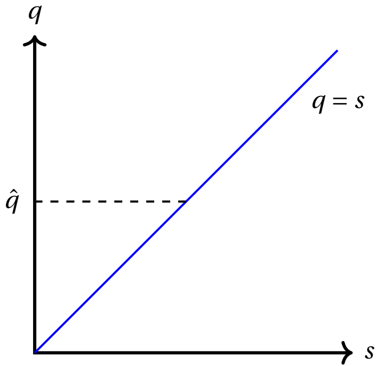

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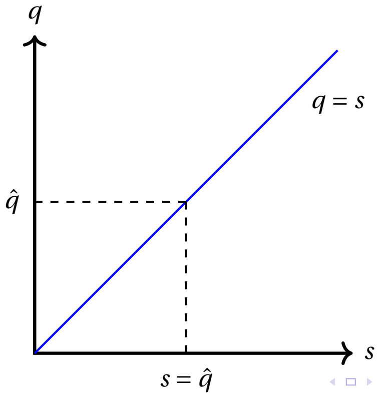

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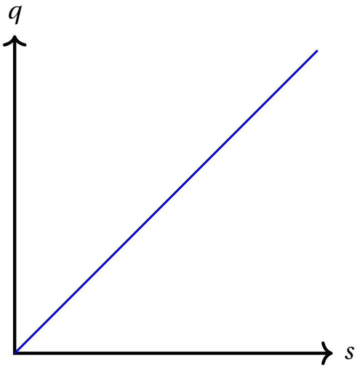

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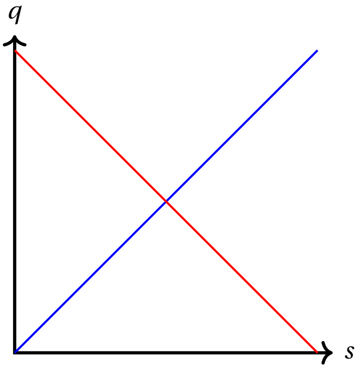

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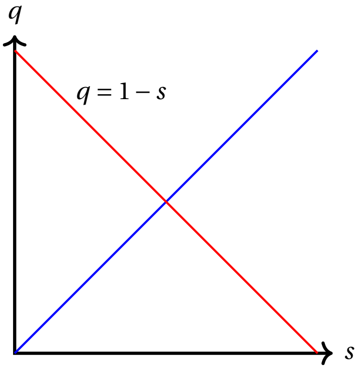

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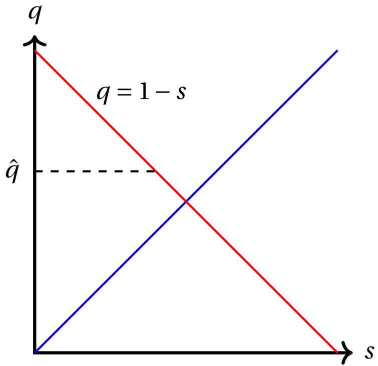

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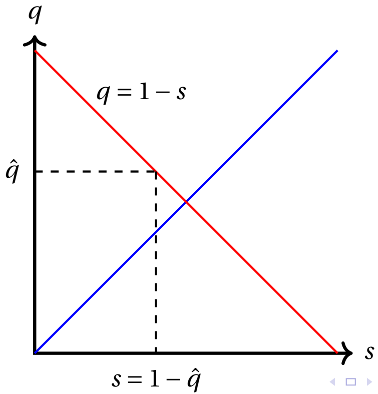

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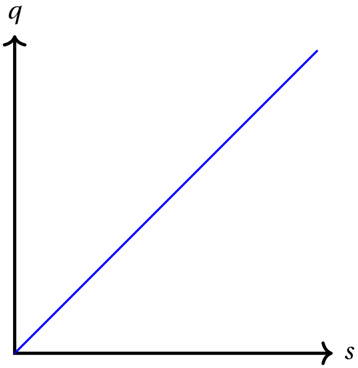

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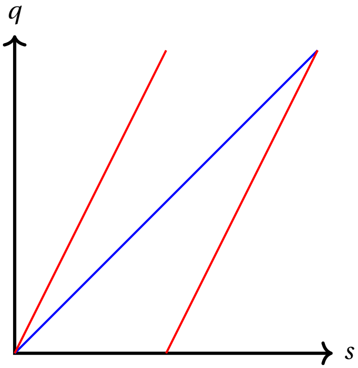

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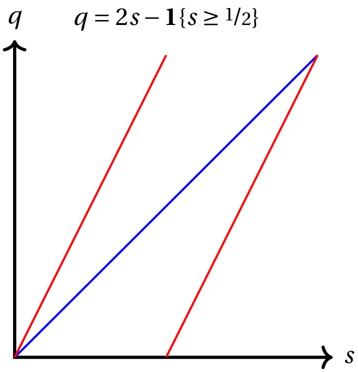

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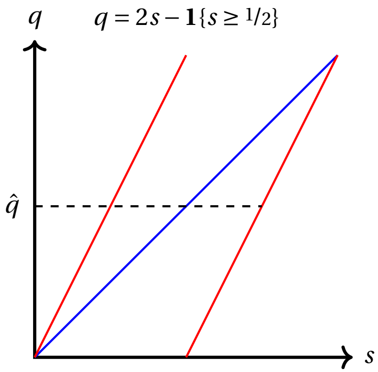

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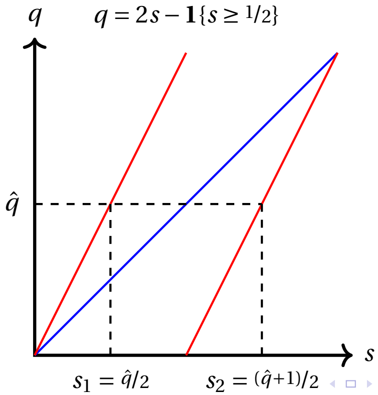

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Theorem (Characterization of Privacy-Preserving Signals)

Fix any conditionally revealing quantile signal $q^\star$.

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Examples

Example #1: $\Omega = X \times \Theta$, $X \subseteq \mathbb{R}$.
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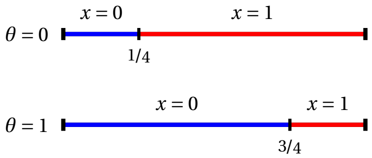

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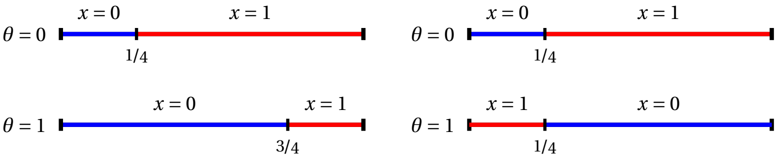

$M_0(s) = M_1(s) = s$ $\qquad$ $M_0(s) = s; M_1(s) = 1 - s$

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Existing knowledge in the mean measurable setting can be readily extended to accommodate privacy constraints.

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Proof

$\overline{F}_\phi(z) := \inf\{y \in [0,1] : \mathbb{E}[F_\phi^{-1}(y \mid \theta)] \geq z\}$.

($\Leftarrow$): Suffice to show: $\mathbb{E}[\phi \mid s] \leq_{\text{cv}} \mathbb{E}[\phi \mid q_\phi]$ for any reordering s of q_ϕ.

Take any $(M_{\hat{\theta}})_{\hat{\theta}\in\Theta}$-reordering s of q_ϕ.

Let G be the CDF of $\mathbb{E}[\phi \mid s] = \mathbb{E}[F_\phi^{-1}(M_\theta(s) \mid \theta)]$.

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Can replicate the analysis pointwise for each realization of y.

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But how? This is equivalent to an optimal transport problem.

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Let $\mathscr{R}$ be the set of joint distributions $\rho \in \Delta(\Omega^J)$ with marginals being $\mathbb{P}[\cdot \mid \theta = j]$.

Let $V : \Omega^J \to \mathbb{R}$ be

$$V(\omega_1, \ldots, \omega_J) := \max_{a \in A} \left[\sum_{j=1}^{J} u(\omega_j, a) \mathbb{P}[\theta = j] \right]$$

Optimal Privacy-Preserving Signals

Proposition

The decision-maker's optimal value among all privacy-preserving signals is given by

$$\sup_{s:\, s \perp \theta} \mathbb{E}\left[\sup_{a \in A} \mathbb{E}[u(\omega, a) \mid s]\right] = \sup_{\rho \in \mathcal{R}} \int_{\Omega^J} V(\omega_1, \ldots, \omega_J)\, \mathrm{d}\rho\,, \tag{T}$$

Moreover, fix any conditionally revealing quantile signal $q^\star$, there exists an optimal privacy-preserving signal that is Blackwell-equivalent to a reordering s of $q^\star$ such that the distribution of $(\eta_j(s))_{j=1}^J$ is a solution of (T).

Remark: Privacy vs Discrimination

Optimal privacy-preserving signal $\iff$ optimal transport.

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Corollary

$$\sup_{a: a \perp \theta} \mathbb{E}[u(\omega, a)] = \sup_{s:\, s \perp \theta} \mathbb{E}\left[\sup_{a \in A} \mathbb{E}[u(\omega, a) \mid s]\right] = \sup_{\rho \in \mathscr{R}} \int_{\Omega^J} V(\omega_1, \ldots, \omega_J)\, \mathrm{d}\rho\,.$$

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Proof:

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Privacy-preserving signal $\iff$ Non-discriminatory decisions.

Supermodular Payoff

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Proposition

Suppose that A is a totally ordered set and that there exists a random variable $\phi : \Omega \to \mathbb{R}$ such that

$$u(\omega, a) = h(\phi(\omega), \theta(\omega), a),$$

for some $h : \mathbb{R} \times \Theta \to \mathbb{R}$ that is supermodular in (ϕ, a). Then the ϕ-quantile signal q_ϕ is optimal.

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e.g., $u(\omega, a) = -|\phi(\omega) - a|^p, \; p > 1.$

Binary Action

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Proposition

Suppose that $A = \{0, 1\}$. *Let* $\phi(\omega) := u(\omega, 1) - u(\omega, 0)$. *Then the* ϕ*-quantile signal* q_ϕ *is optimal.*

Separable Problems and Binary States

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Proposition

Suppose that there exists $\phi : \Omega \to \mathbb{R}$ and functions $f : \mathbb{R} \times A \to \mathbb{R}$, $g : A \to \mathbb{R}$, $h : A \times \Theta \to \mathbb{R}$ such that

- $u(\omega, a) = \phi(\omega) g(a) + h(a, \theta(\omega))$, *or*
- $u(\omega, a) = f(\phi(\omega), a)$ *and* $\phi(\omega) \in \{0, 1\}$.

Then the ϕ-quantile signal q_ϕ is optimal.

Information Design

State space: Ω, compact;

Action space: A, compact;

Sender's payoff $u_S : \Omega \times A \to \mathbb{R}$, upper-semicontinuous;

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Question: What is the sender's optimal signal?

Information Design

Step 1: Find the optimal garbling of each reordered quantile signal (concavafication).

Step 2: Solve for the optimal reordered quantile signal (optimal transport).

Information Design: Mean-Measurable Case

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Suppose that the sender's indirect utility U_S is measurable with respect to posterior means. The sender's value is given by

$$\sup_{G \preceq_{\mathrm{MPS}} \overline{F}_\phi} \int_{\mathbb{R}} U_S(x)\, \mathrm{d}G,$$

Regulating Statistical Discrimination

Phelps (1972) Revisited

Recall:

- $\gamma \sim N(\mu, \sigma^2)$: productivity; $\theta \in \{B, R\}$: color.
- $x = \gamma + \varepsilon_\theta$, $\varepsilon_\theta \sim N(0, \sigma_\theta^2)$; $\sigma_B < \sigma_R$.
- $\phi := \mathbb{E}[\gamma \mid x, \theta]$.

ϕ is correlated with θ in general, and hence $\exists W$ such that

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The "orthogonalization procedure".

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Informational Environment

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Informational Environment

- Underlying outcome: γ (default probability)
- Materially relevant covariates: y (credit score)
- Protected characteristics: θ (race)
- Other observable covariates: z (zipcode)

A statistical model takes inputs (y, θ, z) and generates outputs $s \in S$ for outcome γ.

A decision maker observes the output s and chooses an action $a \in A$ to maximize payoff $\mathbb{E}[\hat{u}(\gamma, a) \mid s]$.

Statistical Discrimination: θ correlated with (γ, y, z), might be used in producing score s.

Non-discriminatory actions: The action taken has to be independent of θ conditional on y.

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Literature: $A = \{0, 1\}$, $\omega \in [0, 1]$, and (mostly) $\hat{u}(\gamma, a) = a(1 - \gamma - c)$.

Statistical Discrimination and Algorithmic Fairness

Let $\Phi := \Delta(\Gamma)$, where ϕ denotes the distribution of γ conditional on (y, θ, z). Let $\Omega := \Phi \times \Theta$.

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If $\gamma \in \{0, 1\}$ and $\hat{u}$ does not depend of θ:
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4. Decision-makers take an action based on the quantiles.

Procedure is detail-free, applies to all decision problems.

Separation among legislation (step 1), estimation (step 2), regulation (step 3), and decision (step 4)

Privacy-Preserving Disclosure in Auctions

Optimal Privacy-Preserving Disclosure in Auctions

A publisher runs a second-price auction to sell ad audience.

N Advertisers have (unknown) IPV $\{v_i\}_{i=1}^N \sim F$.

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Proposition (Bergemann et al., 2022)

Any optimal symmetric signal is equivalent to a partitional signal s

$$s(v_i, r_i) = \begin{cases} v_i, & \text{if } F(v_i) < q_N^\star \\ \mathbb{E}[v_i \mid F(v_i) \geq q_N^\star], & \text{if } F(v_i) \geq q_N^\star \end{cases}.$$

in terms of the advertisers' posterior expected values $\mathbb{E}[v_i \mid s]$*, where* $q_N^\star$ *is the unique root in* $(0,1)$ *of a* N*-degree polynomial.*

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Since any feasible distribution of posterior means is a MPS of $\overline{F}$, garbling each q_i in the same way as Bergemann et al. (2022) is then optimal

Optimal Privacy-Preserving Disclosure in Auction

Proposition

Any optimal symmetric privacy-preserving signal is equivalent to a partitional signal s

$$s(v_i,\theta,r_i) = \begin{cases} \mathbb{E}[v_i \mid F(v_i \mid \theta)], & \text{if } F(v_i \mid \theta) < q_N^\star \\ \mathbb{E}[v_i \mid F(v_i \mid \theta) \geq q_N^\star], & \text{if } F(v_i \mid \theta) \geq q_N^\star \end{cases}.$$

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Remark: Privacy-Preserving vs Differential Privacy

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For any $\varepsilon \geq 0$, a signal s satisfies ε-**differential privacy** if for all $\hat{\omega}, \hat{\omega}' \in \Omega$ such that $\hat{\omega} \neq \hat{\omega}'$ and $\hat{\omega}_{-i} = \hat{\omega}'_{-i}$ for some i,

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A signal privacy-preserving signal s satisfies ε-differential privacy if and only if $\varepsilon = 0$ and $\mathscr{P} = \mathscr{F}$.

The notions of privacy-preserving and differential privacy capture different aspects of privacy.

- Privacy-preserving does not allow updates on **any** privacy events (privacy for characteristics).
- Differential privacy allows for updates, as long as they are small enough for "similar" states (privacy for identities).

Non-Discriminatory Personalized Pricing (Strack and Yang, 2025)

Motivation

Price discrimination:
Charge different prices to different segments of consumers.

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How should a seller set prices given that some consumer characteristics cannot be discriminated?

What are the welfare implications?

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"*Big data naturally raises concerns among groups that have historically been victims of discrimination. Given hundreds of variables to choose from, it is easy to imagine that statistical models could be used to hide more explicit forms of discrimination by generating customer segments that are* ***closely correlated with race, gender, ethnicity, or religion*** *[...], even if the profit motive is different from [...] the sort of prejudice that our antidiscrimination laws seek to prohibit.*"

— The White House, 2015, Big Data and Differential Pricing.

Motivation

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FINANCE

Regulators caught Wells Fargo, other banks in probe over mortgage pricing discrimination

PUBLISHED MON, DEC 11 2023·10:00 AM EST | UPDATED MON, DEC 11 2023·1:20 PM EST

Hugh Son
@HUGH_SON

KEY POINTS

- Wells Fargo received an official notice from the Consumer Financial Protection Bureau on problems with its use of mortgage rate discounts, sources said.

Motivation

Chicago Area Disparities in Car Insurance Premiums

By Al Shaw, Jeff Larson and Julia Angwin, ProPublica, April 5, 2017

Some car insurers charge higher premiums in Chicago's minority neighborhoods than in predominantly white neighborhoods with similar insurance losses. The areas outlined in black are more than 50 percent minority. Many insurers charge the same premiums throughout Chicago, but quote higher prices than in suburbs with similar risk. | Read the methodology » | Related story: Minority Neighborhoods Pay Higher Car Insurance Premiums Than White Areas With the Same Risk »

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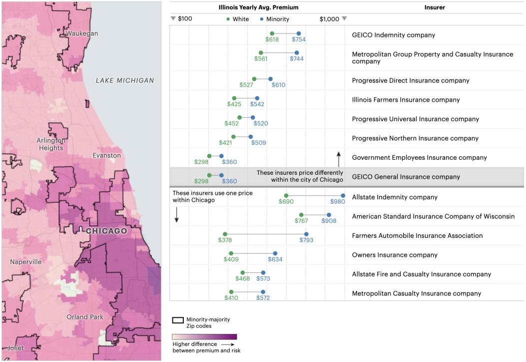

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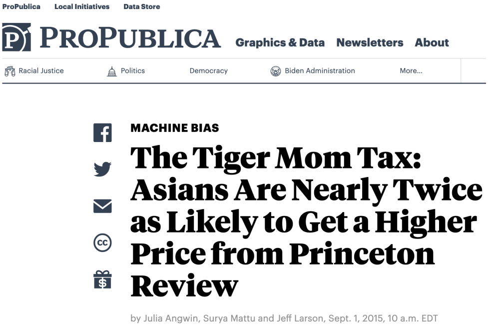
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The Tiger Mom Tax: Asians Are Nearly Twice as Likely to Get a Higher Price from Princeton Review

by Julia Angwin, Surya Mattu and Jeff Larson, Sept. 1, 2015, 10 a.m. EDT

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Which **non-discriminatory pricing rule** maximizes the seller's revenue?

What are the welfare implications of anti-discrimination regulation?

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Let $F_{c,\theta}(x) := \mathbb{P}[v \leq x \mid c, \theta]$; $\alpha_c := \mathbb{P}[\theta = h \mid c]$; and $G(x) := \mathbb{P}[c \leq x]$.

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- $F_{c,h} \geq F_{c,l}$ in FOSD.
- Demand of $\theta = l$ consumers more elastic than $\theta = h$.

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Seller's profit given pricing rule p is

$$\Pi(p) := \mathbb{E}\left[(p - c) \cdot \mathbf{1}\{v \geq p\}\right].$$

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A pricing rule p is **non-discriminatory** if

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Goal: Find a non-discriminatory pricing rule that maximizes the seller's profit.

Optimal Non-Discriminatory Pricing Rule

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Focus on undominated optimal pricing rules: $\nexists$ other pricing rules that induce higher profit, higher average surplus among consumers $\theta = l$ and $\theta = h$.

Optimal Pricing as Optimal Transport

$$\pi_c(v_l, v_h) := \max\{\min\{v_l, v_h\} - c, \alpha_c(v_h - c)^+, (1 - \alpha_c)(v_l - c)^+\}.$$

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Proposition (Optimal Transport Representation)

For each $c \in C$, let $\pi^\star$ be the value of the optimal transport problem:

$$\pi^\star := \int_C \left(\max_{\rho_c \in \mathscr{R}_c} \int_{V^2} \pi_c(v_l, v_h)\, \mathrm{d}\rho_c(v_l, v_h) \right) \mathrm{d}G(c). \tag{T}$$

Then $\pi^\star = \Pi^\star$. Moreover, any solution of (P) induces a solution of (T), and any solution of (T) induces a solution of (P).

Optimal Non-Discriminatory Pricing

Let $\|\cdot\|$ denote the total variation norm.

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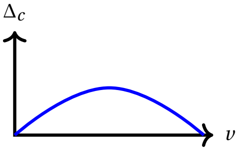

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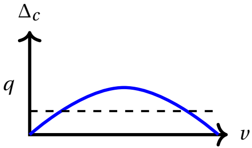

Optimal Non-Discriminatory Pricing

Let $\|\cdot\|$ denote the total variation norm.

Assumption

$\mathbb{P}[v \leq c \mid c, \theta = l] < \|F_{c,l} - F_{c,h}\|$ for almost all c.

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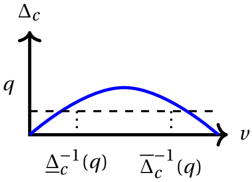

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Lemma

For any c, there exists a unique increasing vector $\kappa_c \in \mathbb{R}^5_+$

$$F_{c,l}(\kappa_c^2) = \Delta_c(\kappa_c^3) + F_{c,h}(\kappa_c^1) = \Delta_c(\kappa_c^4) = \Delta_c(\kappa_c^5)$$
$$\kappa_c^1 - c = (1-\alpha_c)\cdot(\kappa_c^3 - c) = \alpha_c \cdot (\kappa_c^5 - \kappa_c^4).$$

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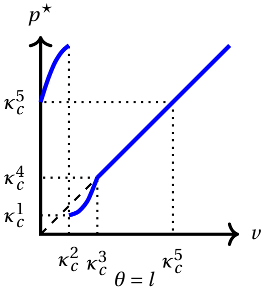

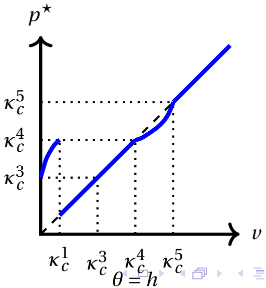

Optimal Pricing Rule

Theorem (Optimal Pricing)

$p^\star$ is a profit-maximizing non-discriminatory pricing rule. Moreover, every undominated profit-maximizing non-discriminatory pricing rule p induces the same average surplus $\mathbb{E}[(v-p)^+ \mid \theta]$ for consumer of each protected characteristic θ.

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Fix a c, $p^\star$ corresponds to the following matching scheme $\rho_c^\star$:

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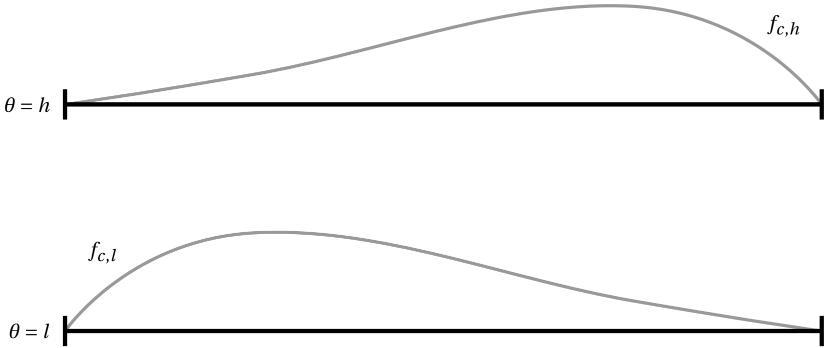

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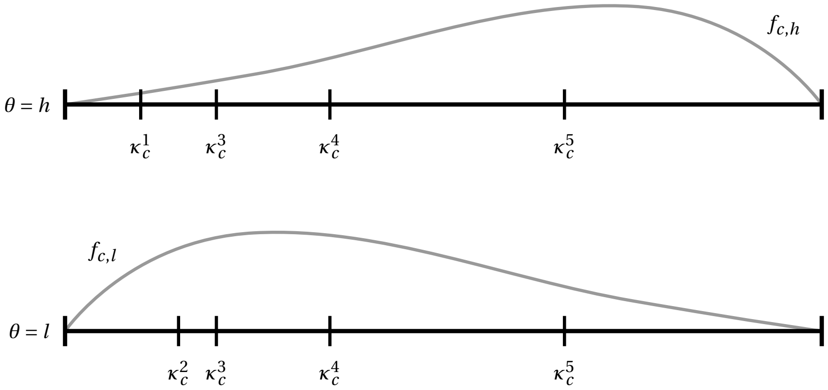

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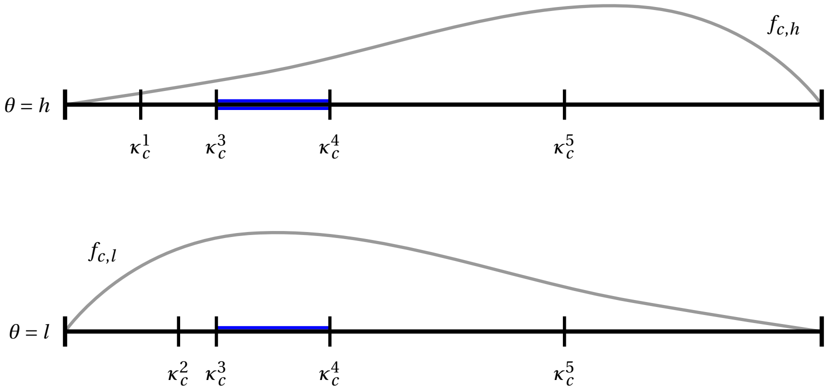

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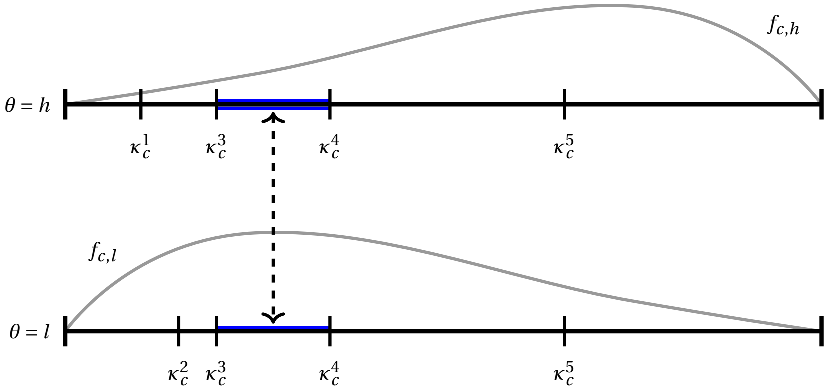

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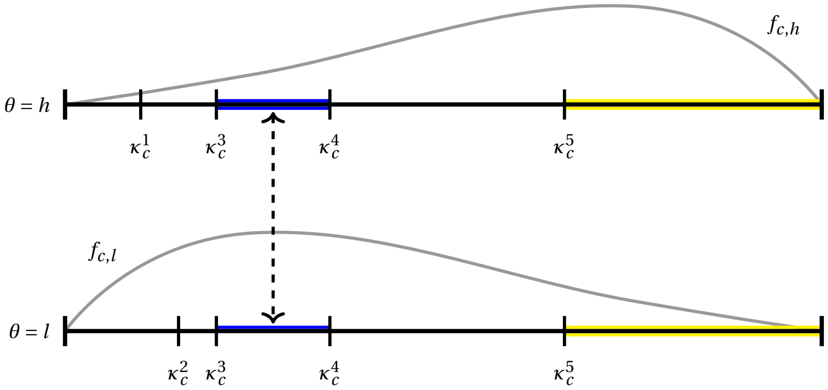

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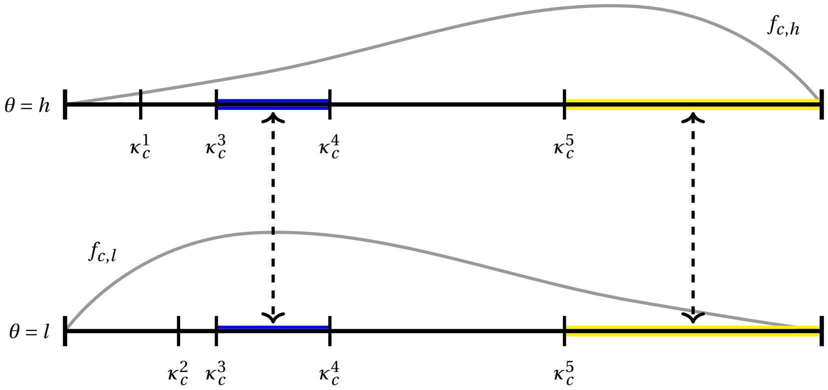

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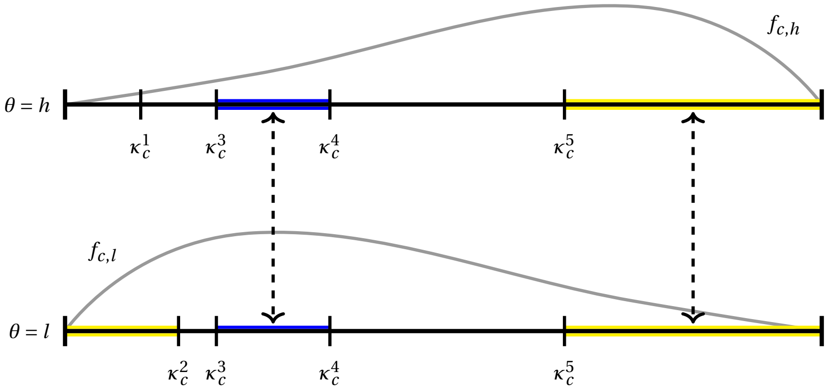

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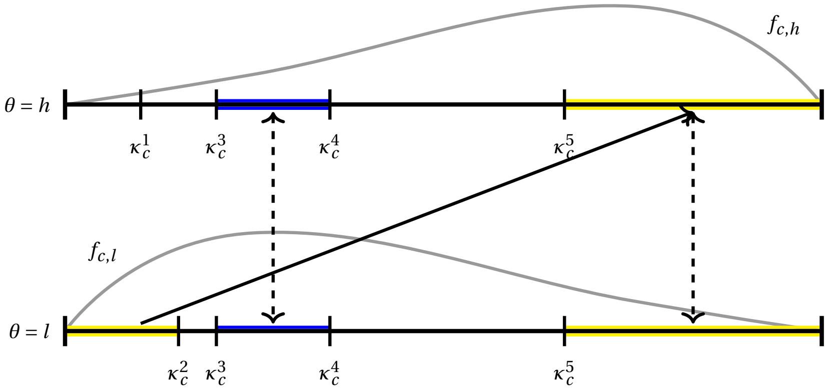

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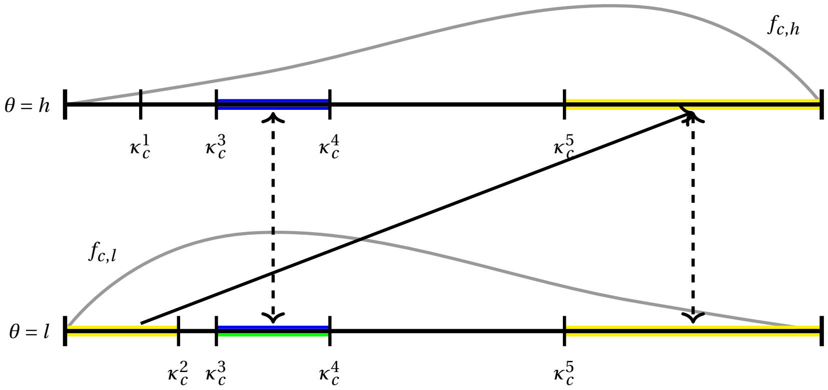

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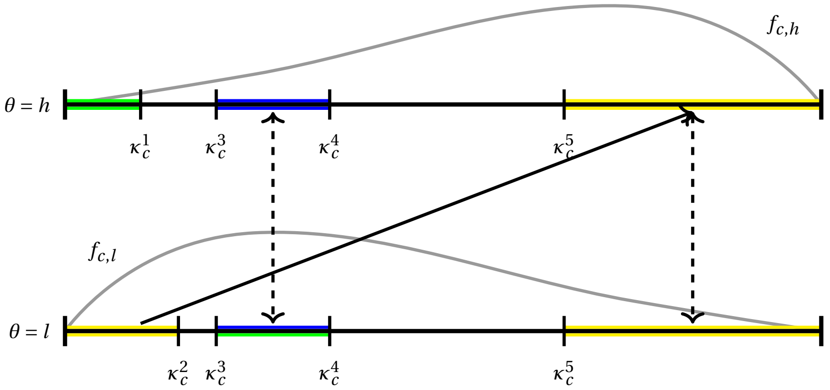

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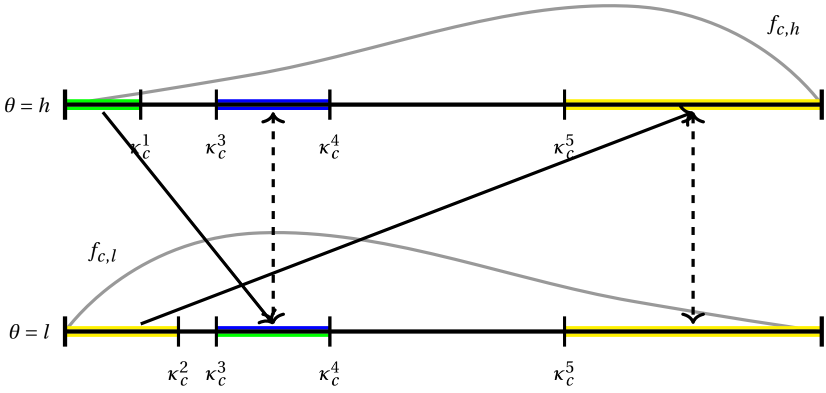

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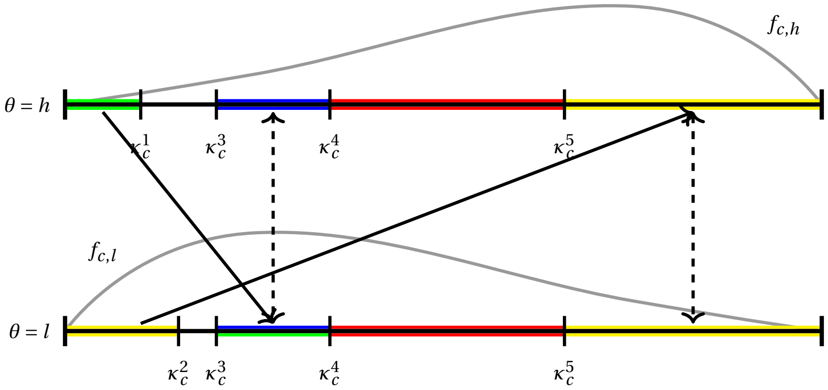

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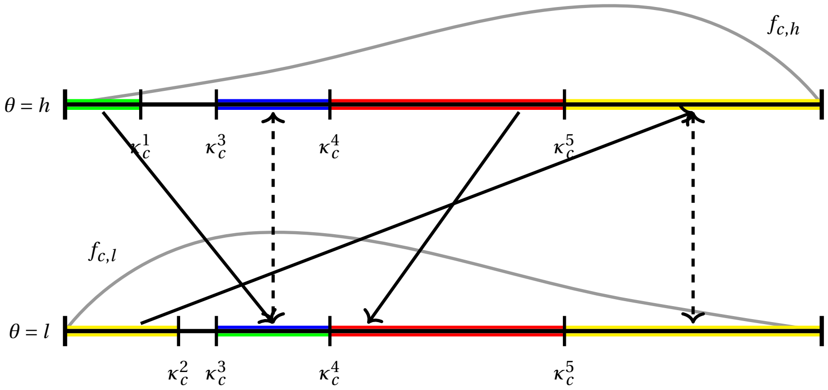

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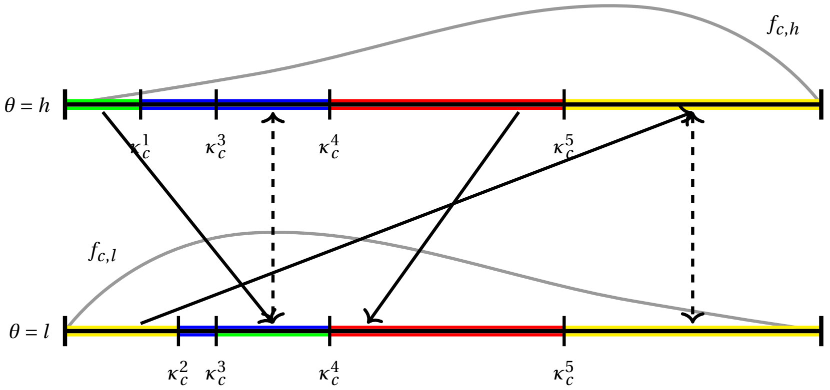

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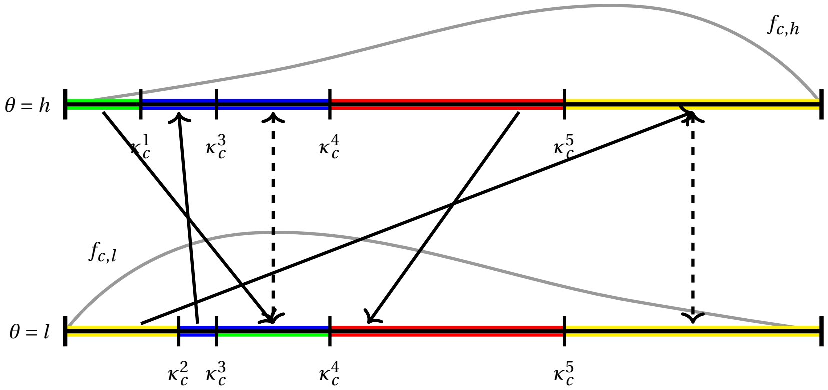

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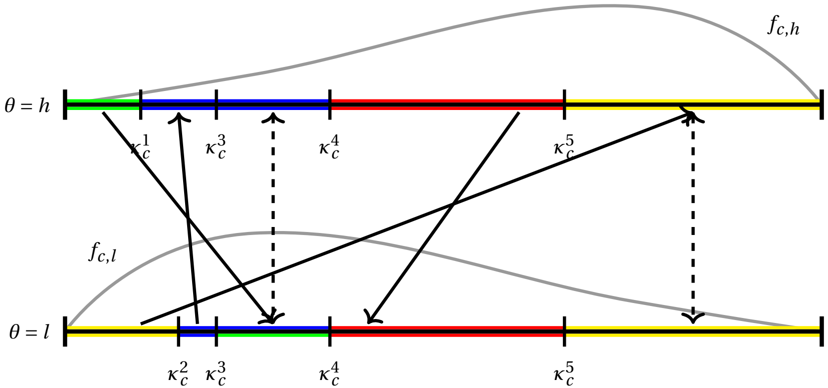

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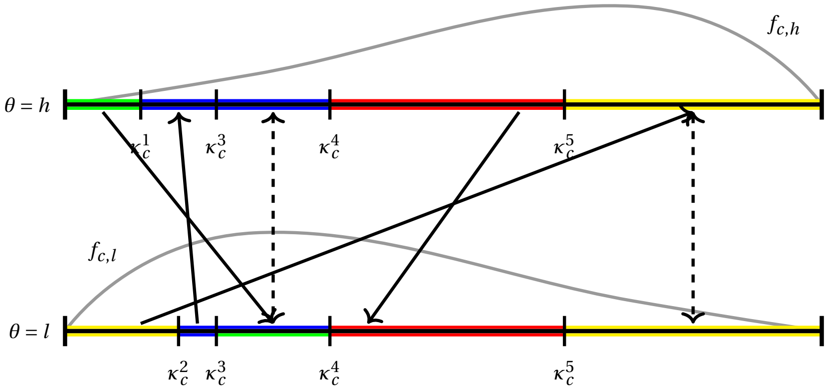

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Insurance market:

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Assumption holds iff

$$1 - \exp\left(\frac{-1}{\lambda_l}\right) < \left(\frac{\lambda_h}{\lambda_l}\right)^{\frac{\lambda_l}{\lambda_h - \lambda_l}} - \left(\frac{\lambda_h}{\lambda_l}\right)^{\frac{\lambda_h}{\lambda_h - \lambda_l}}.$$

Proof

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$\mathscr{R}_c$: set of joint distributions $\rho_c \in \Delta(V^2)$ with marginals $F_{c,l}$ and $F_{c,h}$.

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$$\max_{\rho_c \in \mathscr{R}_c} \int_{V^2} \pi_c(v_l, v_h)\, \mathrm{d}\rho_c\,.$$

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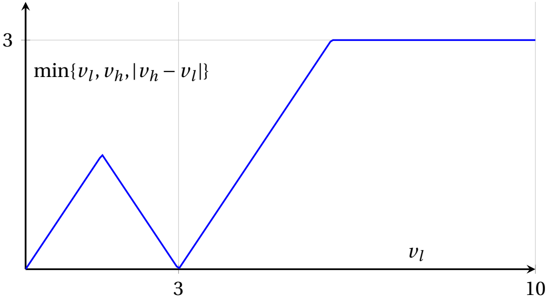

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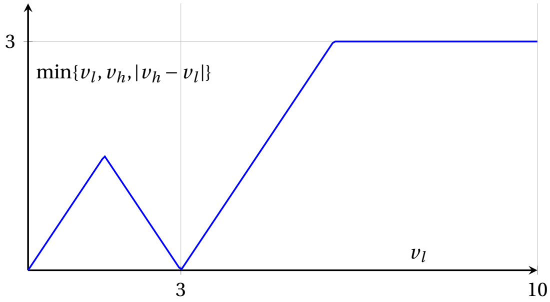

Boerma, Tsyvinski, Wang, and Zhang (2024): $|v_l - v_h|^\beta$ for $\beta \in (0,1)$.

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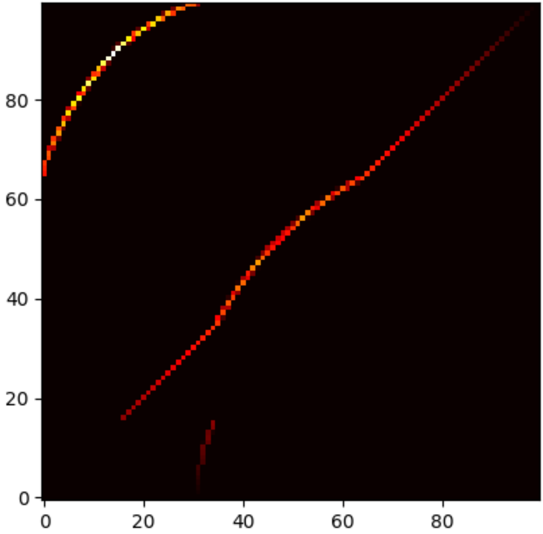

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Dual problem:

$$\pi_\star(c) = \inf_{\phi,\psi} \left[\int_0^\infty \phi(v_l) F_{c,l}(\mathrm{d}v_l) + \int_0^\infty \psi(v_h) F_{c,h}(\mathrm{d}v_h) \right] \tag{D}$$

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Lemma (Kantorovich Duality)

$\pi_\star(c) = \pi^\star(c)$ for all c. Moreover, for any feasible ρ in (P) and for any feasible ϕ_c, ψ_c in (D), ρ_c is a solution of (P) and (ϕ_c, ψ_c) is a solution of (D) if and only if $\phi_c(v_l) + \psi_c(v_h) = \pi_c(v_l, v_h)$ for all $(v_l, v_h) \in \mathrm{supp}(\rho_c)$.

Solution to the Dual Problem

$$\phi_c^\star(v_l) := \begin{cases} \kappa_c^1 - c, & v_l \le \kappa_c^3 \\ (1-\alpha_c)(v_l - c), & v_l \in \left(\kappa_c^3, \kappa_c^4\right] \\ v_l - c - \alpha_c(\kappa_c^4 - c), & v_l \in (\kappa_c^4, \kappa_c^5] \\ (1-\alpha_c)(v_l - c) + \kappa_c^1 - c, & v_l > \kappa_c^5 \end{cases};$$

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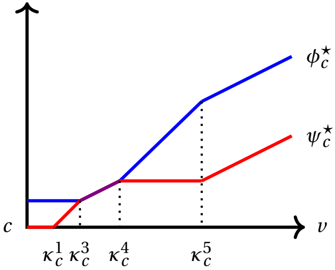

Solution to the Dual Problem

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Lemma

For all c and for all (v_l, v_h),

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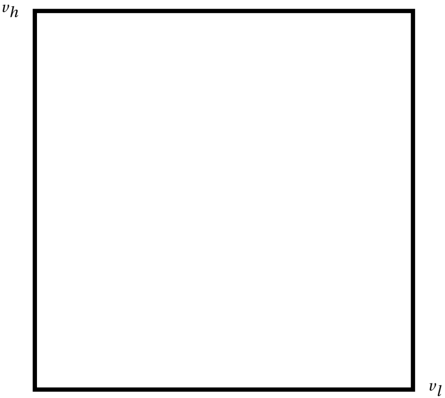

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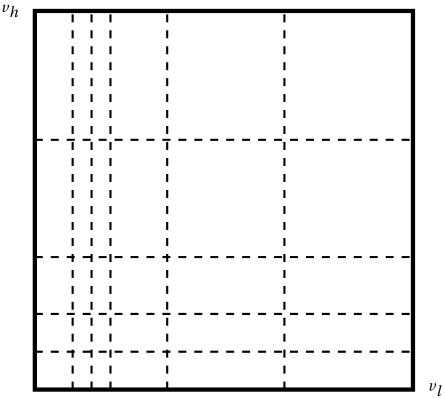

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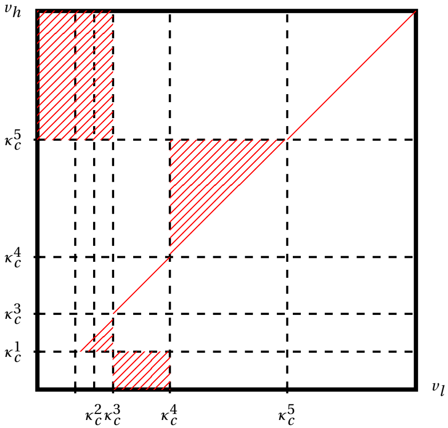

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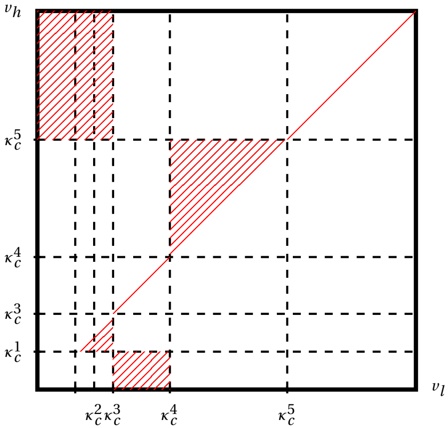

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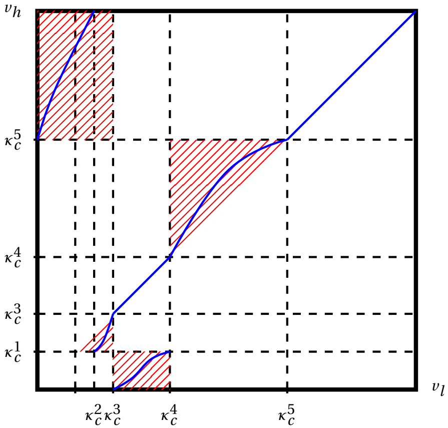

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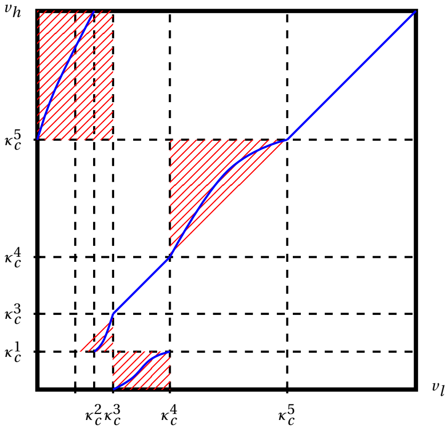

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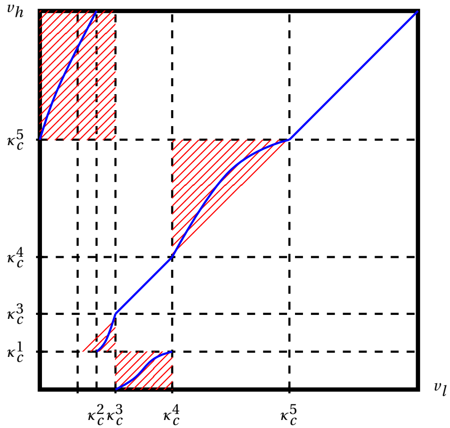

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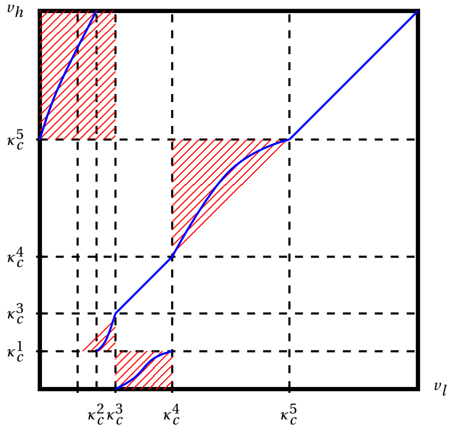

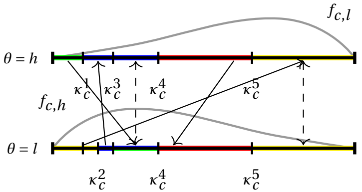

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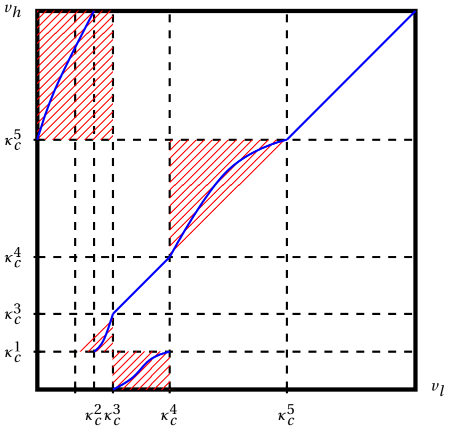

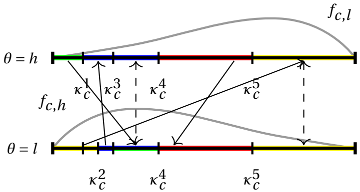

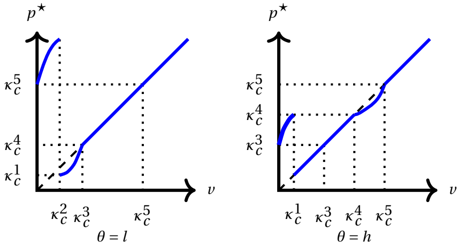

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Proposition

Under the optimal pricing rule $p^\star$,

1. *Consumer surplus for both protected characteristics $\theta \in \{l, h\}$ is positive:*
$$\mathbb{E}[(v - p^\star)^+ \mid \theta = t] > 0, \forall t \in \{l, h\}.$$
2. *Deadweight losses from both protected characteristics are positive:*
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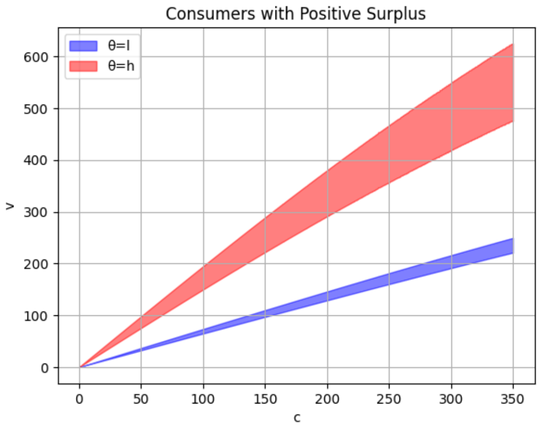
Benefits are differentiated:

- Low-value consumers are priced out of the market.
- High-value consumer surplus extracted.
- Intermediate value consumers enjoy surplus.

Which Values Retain Surplus

$$F_{c,l}(v) = 1 - \exp(-v/\lambda_l c); \; F_{c,h}(v) = 1 - \exp(-v/\lambda_h c).$$

$$\lambda_l = 3, \lambda_h = 15; \; \alpha_c = 1/2.$$



Which Protected Characteristic Benefits More

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Consumer Surplus vs Population Size

Proposition

For all $c \in C$, suppose that $F_{c,l}$ and $F_{c,h}$ are fixed and the share α_c of $\theta = h$ consumers increases. Then, under the optimal pricing rule $p^\star$,

1. *$\theta = h$ consumer surplus $\mathbb{E}[(v - p^\star)^+ \mid c, \theta = h]$ decreases.*
2. *$\theta = l$ consumer surplus $\mathbb{E}[(v - p^\star)^+ \mid c, \theta = l]$ increases.*
3. *$\mathbb{E}[(v - p^\star)^+ \mid c, \theta = h] \to 0$ as $\alpha_c \to 1$.*
4. *$\mathbb{E}[(v - p^\star)^+ \mid c, \theta = l] \to 0$ as $\alpha_c \to 0$.*

Consumer Surplus vs Population Size

Corollary

For any $c \in C$, suppose that $F_{c,l}$ and $F_{c,h}$ are fixed. Then there exists α_c, α'_c such that

$$\mathbb{E}[(v - p^\star)^+ \mid c, \theta = h] < \mathbb{E}[(v - p^\star)^+ \mid c, \theta = l]$$

when the fraction of $\theta = h$ consumers conditional on c equals α_c, and

$$\mathbb{E}[(v - p^\star)^+ \mid c, \theta = h] > \mathbb{E}[(v - p^\star)^+ \mid c, \theta = l]$$

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Proposition

1. *The unique optimal pricing rule with non-discriminatory outcomes corresponds to the assortative matching.*
2. *The profit of the seller is lower under equal outcomes than under non-discriminatory prices.*
3. $\theta = l$ *consumers are worse off under non-discriminatory outcomes than under non-discriminatory prices.*
4. $\theta = h$ *consumers are better off under non-discriminatory outcomes than under non-discriminatory prices.*

Private Private Information (He, Sandomirskiy and Tamuz, 2024)

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Can express private private signals $(s_1, \ldots, s_n)$ by $f : [0,1]^n \to \Delta(\Omega)$, where $f_j(\hat{s}) = \mathbb{P}[\omega = j \mid \hat{s}_1, \ldots, \hat{s}_n]$ for all $j \in \Omega$.

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Agent i's posterior belief given realization $\hat{s}_i$ is

$$\mathbb{P}[\omega = j \mid \hat{s}_i] := q^i_j(\hat{s}_i) = \int_{[0,1]^{n-1}} f_j(\hat{s})\, \mathrm{d}\hat{s}_{-i}\,.$$

Partition of Uniqueness

Private private signals f are jointly fully revealing if $f_j(x) \in \{0,1\}$ for all $x \in [0,1]^n$ and for all $j \in \Omega$.

Lemma

Any undominated private private signals must be jointly fully revealing.

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Lemma

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Any jointly fully revealing f can be identified by a partition $\mathscr{A} = \{A_0, \ldots, A_{m-1}\}$ of $[0,1]^n$:

$$f_j(x) = 1 \iff x \in A_j\,.$$

Partition of Uniqueness

Definition

A jointly fully revealing $f^\star$ is a **partition of uniqueness** if there does not exist another jointly fully revealing $f \neq f^\star$ such that

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In other words, agents' posterior belief distributions uniquely determine the signals.

Theorem (He, Sandomirskiy and Tamuz, 2024)

Private private signals f is undominated if and only if it is a partition of uniqueness.

Binary State Environments

When $\Omega = \{0, 1\}$. A private private signal f is a real-valued function, so that $\mathbb{P}[\omega = 1 \mid \hat{s}] = f(\hat{s}) \in [0, 1]$.

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A set $A \subseteq [0, 1]^n$ is a **set of uniqueness** if there does not exist any $A' \subseteq [0, 1]^n$, $A' \neq A$ such that

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Characterizations of sets of uniqueness with $n \geq 2$ is an open question.

- Sufficient condition: $A = \{x : \sum_{i=1}^{n} \psi_i(x_i) \geq 0\}$.
- Necessary condition: A is upward-closed (i.e., $x \in A$ and $x' \geq x$ implies $x' \in A$).

Binary State and Two Agents

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A set B is a **rearrangement** of an upward-closed set A if for each i, the nondecreasing rearrangement of

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Theorem (Lorenz, 1949)

$A \subseteq [0,1]^2$ is a set of uniqueness if and only if it is a rearrangement of an upward-closed set.

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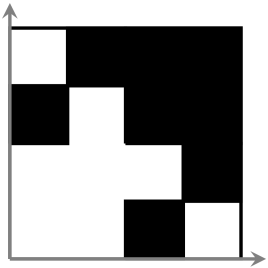
Given a set of uniqueness A, let $q_i(x_i) := \int_0^1 \mathbf{1}_A(x_i, x_{-i})\,\mathrm{d}x_{-i}$. Easy to verify that for all $z \in [0,1]$,

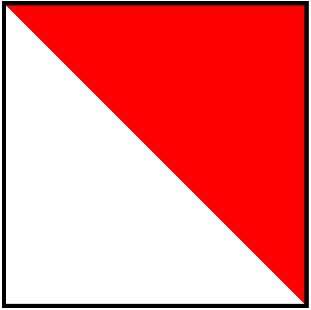
$$q_1(z) = \hat{q}_2(z) = 1 - q_2^{-1}(1-z)\,,$$

where $\hat{q}_2$ is called the **conjugate** of q_2.

Non-Sets of Uniqueness



Set of Uniqueness



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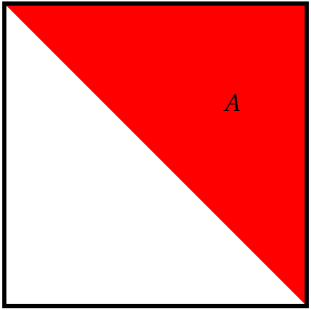

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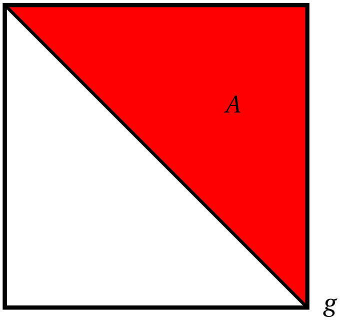

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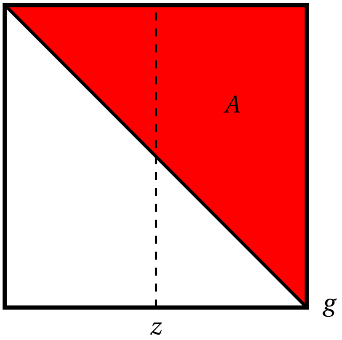

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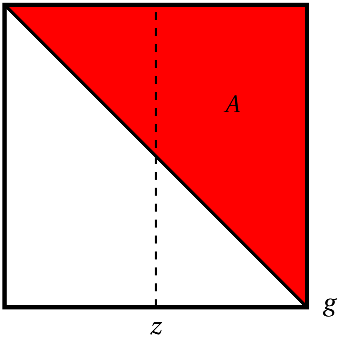

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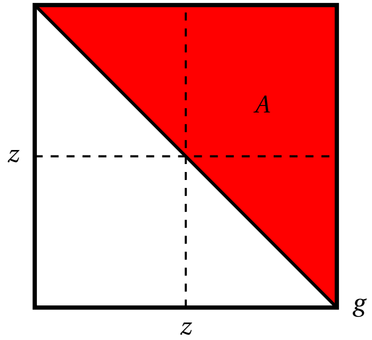

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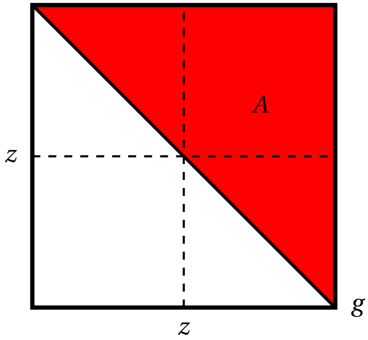

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Binary State and Two Agents

Theorem (He, Sandomirskiy and Tamuz, 2024)

When $n = m = 2$*, private private signals* (s_1, s_2) *are undominated if and only if the distribution of* $\mathbb{P}[\omega \mid s_1]$ *is a mean-preserving contraction of the conjugate of the distribution of* $\mathbb{P}[\omega = 1 \mid s_2]$*.*

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Fix agent 2's signal s_2. Consider agent 1's signal s_1.

Then (s_1, s_2) is private private if and only if s_1 is privacy-preserving (with respect to s_2).

Theorem (He, Sandomirskiy and Tamuz, 2024)

The most informative privacy-preserving signal induces a posterior belief that is the conjugate of the distribution of $\mathbb{P}[\omega = 1 \mid s_2]$.

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Coincides with the characterization for the mean-measurable case.

Extremal Private Private Signals for $\Omega = \{0, 1\}$

Private private signals can be represented by functions $f : [0,1]^n \to [0,1]$.

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Without loss to focus on the set $\mathcal{Q}$ of nondecreasing functions $(q_1, \ldots, q_n)$ such that there exists $f : [0,1]^n \to [0,1]$

$$q_i(x_i) = \int_{[0,1]^{n-1}} f(x)\,\mathrm{d}x_{-i}\,.$$

for all x_i and for all i; and that

$$\int_0^1 q_1(z)\,\mathrm{d}z = \mu := \mathbb{P}[\omega = 1] \in [0,1]\,.$$

Extremal Private Private Signals for $\Omega = \{0,1\}$

Theorem (Yang and Yang, 2025)

Every extreme point of $\overline{\mathrm{Q}}$ is rationalized by $\lambda \mathbf{1}_{A_1} + (1-\lambda)\mathbf{1}_{A_2}$, where $\lambda \in (0,1)$ and $A_1 \subseteq A_2$ are ***nested*** *upward-closed sets.*

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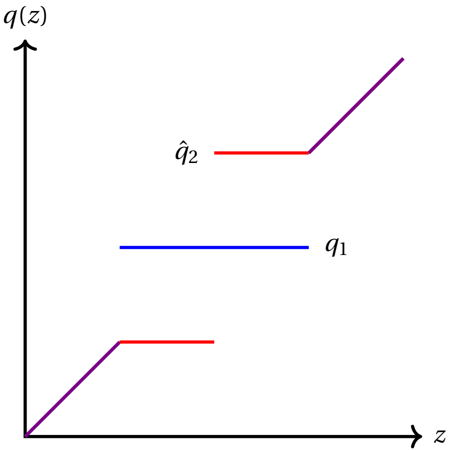
Theorem (Yang and Yang, 2025)

*When $n = 2$, every extreme point of $\overline{Q}$ is uniquely rationalized by $\lambda \mathbf{1}_{A_1} + (1-\lambda)\mathbf{1}_{A_2}$, where $\lambda \in (0,1)$ and $A \subseteq A'$ are nested upward-closed sets with $A_2 \backslash A_1$ being a **rectangle**.*

Extremal Private Private Signals for $n = m = 2$



Discussions

Privacy vs Discrimination.

Non-discriminatory pricing as optimal transport.

Privacy preserving signal s could depend on θ.

What are the signals that are both privacy-preserving and do not depend on θ?

Privacy-Efficiency trade-off?

Extremal privacy-preserving signals? Hard!

Thank You!

Questions & Comments Welcomed:

philipp.strack@gmail.com