# L-PICOLA Cosmology Equations

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Here we go over the derivation of the cosmology factors within Version 1.1 of L-PICOLA. The terms we need are

$$a^3E(a)\frac{dD_1}{da},\ a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_1}{da}\right],\ a^3E(a)\frac{dD_2}{da},\text{ and } a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_2}{da}\right] \tag{1}$$

In the notation of Tassev et al. (2013) these are $T[D_1]$, $T[D_2]$, $T^2[D_1]$, $T^2[D_2]$ (the notation of Howlett et. al. (in prep), the L-PICOLA paper, is different, sorry ☹). We'll start by going over the form for a flat $\Lambda$CDM cosmology in section 1, before moving on to the non-flat version currently implemented in L-PICOLA in section 2.

# 1 Flat ΛCDM cosmology

Let's start with a definition of the linear growth factor

$$D_1 = D_{1,0}E(a)\int_0^a \frac{da'}{a'^3E^3(a')}. \tag{2}$$

For a flat cosmology $E(a)$ is given by

$$H(a) = H_0E(a) = H_0\sqrt{\Omega_{m,0}a^{-3}+\Omega_{\Lambda,0}}, \tag{3}$$

where $H(a)$ is the Hubble parameter, with present value $H_0$ (at $z=0$ and $a=1$). $a=1/(1+z)$ is the scale factor. The cosmological model is given by $\Omega_{m,0}$, the present day matter density, and $\Omega_{\Lambda,0} = 1-\Omega_{m,0}$ is the present day dark energy density. The linear growth factor is normalised to 1 at z=0, i.e. the normalisation factor $D_{1,0}$ is given by

$$D_{1,0} = \left[\int_0^1 \frac{da'}{a'^3E^3(a')}\right]^{-1}. \tag{4}$$

Strictly speaking there should be an additional factor of $5\Omega_{m,0}/2$ infront of the expression for the linear growth factor, however as we will always be dealing with the normalised version, we will forget about this.

## 1.1 First expression

Taking the derivative of $D_1$ this with respect to $a$

$$\frac{dD_1}{da} = D_{1,0}\frac{dE}{da}\int_0^a \frac{da'}{a'^3E^3(a')} + \frac{D_{1,0}E(a)}{a^3E^3(a)} \tag{5}$$

$$= -\frac{D_{1,0}}{2}\frac{1}{E(a)}\frac{3\Omega_{m,0}}{a^4}\int_0^a \frac{da'}{a'^3E^3(a')} + \frac{D_{1,0}}{a^3E^2(a)} \tag{6}$$

hence

$$a^3E(a)\frac{dD_1}{da} = \frac{1}{E(a)}\left(D_{1,0} - \frac{3\Omega_{m,0}}{2a}D_{1,0}E(a)\int_0^a \frac{da'}{a'^3E^3(a')}\right) \tag{7}$$

and substituting the linear growth factor in again

$$\boxed{a^3E(a)\frac{dD_1}{da} = \frac{1}{E(a)}\left(D_{1,0} - \frac{3\Omega_{m,0}}{2a}D_1(a)\right)} \tag{8}$$

## 1.2 Second expression

To solve the next derivative of this expression, let's take a step back and look at the original differential equation from which the linear growth factor is actually derived. This is easier than actually differentiating the above expression.

$$\frac{d^2D_1}{dt^2} + 2H(t)\frac{dD_1}{dt} = \frac{3}{2}H^2(t)\Omega_m(t)D_1(t) \tag{9}$$

if we then look at changing variables from 't' to 'a'

$$\frac{d}{dt} = \frac{da}{dt}\frac{d}{da} = aH(a)\frac{d}{da} \tag{10}$$

therefore

$$\frac{dD_1}{dt} = aH(a)\frac{dD_1}{da}\,\frac{d^2D_1}{dt^2} = aH(a)\frac{d}{da}\left(aH(a)\frac{dD_1}{da}\right) = aH(a)\left(H(a)\frac{dD_1}{da} + a\frac{dH}{da}\frac{dD_1}{da} + aH(a)\frac{d^2D_1}{da^2}\right) \tag{11}$$

Substituting this into Eq. 9 and rearranging, we find

$$a^2H^2(a)\frac{d^2D_1}{da^2} = \frac{3}{2}H^2(a)\Omega_m aD_1(a) - 3aH^2(a)\frac{dD_1}{da} - a^2H(a)\frac{dH}{da}\frac{dD_1}{da} \tag{12}$$

Want we really want, however, is

$$a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_1}{da}\right] = 3a^5\frac{H^2(a)}{H_0^2}\frac{dD_1}{da} + \frac{a^6H(a)}{H_0^2}\frac{dH}{da}\frac{dD_1}{da} + a^6\frac{H^2(a)}{H_0^2}\frac{d^2D_1}{da^2} \tag{13}$$

Conveniently, we have expressions for some of these terms already. Substituting Eq. (12)

$$\begin{aligned} a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_1}{da}\right] &= 3a^5\frac{H^2(a)}{H_0^2}\frac{dD_1}{da} + \frac{a^6H(a)}{H_0^2}\frac{dH}{da}\frac{dD_1}{da} + \\ &\quad \frac{3}{2}a^4\frac{H^2(a)}{H_0^2}\Omega_m aD_1(a) - 3a^5\frac{H^2}{H_0^2}(a)\frac{dD_1}{da} - a^6\frac{H(a)}{H_0^2}\frac{dH}{da}\frac{dD_1}{da} \end{aligned} \tag{14}$$

and therefore,

$$a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_1}{da}\right] = \frac{3}{2}a^4\frac{H^2(a)}{H_0^2}\Omega_m aD_1(a). \tag{15}$$

Finally, using then the fact

$$\Omega_m(a) = \frac{\Omega_{m,0}}{a^3E^2(a)}, \tag{16}$$

we find

$$\boxed{a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_1}{da}\right] = \frac{3}{2}\Omega_{m,0}aD_1(a)} \tag{17}$$

## 1.3 Third expression

To solve the same expressions for $D_2$ we use the approximation from Bouchet et al. (1995)

$$D_2(a) = -\frac{3}{7}D_1^2(a)\Omega_m^{-1/143}(a) \tag{18}$$

Hence

$$\frac{dD_2}{da} = -\frac{3}{7}\frac{dD_1^2}{da}\Omega_m^{-1/143}(a) + \frac{3}{7}\frac{D_1^2}{143}\frac{\Omega_m^{-1/143}(a)}{\Omega_m(a)}\frac{d\Omega_m}{da} \tag{19}$$

$$= -\frac{6}{7}D_1(a)\frac{dD_1}{da}\Omega_m^{-1/143}(a) - \frac{3}{7}\frac{D_1^2}{143}\frac{\Omega_m^{-1/143}(a)}{\Omega_m(a)}\left(\frac{3\Omega_{m,0}\Omega_{\Lambda,0}}{a^4E^4(a)}\right) \tag{20}$$

therefore, the expression we are looking for is

$$a^3E(a)\frac{dD_2}{da} = \frac{2D_2(a)}{D_1(a)}a^3E(a)\frac{dD_1}{da} + \frac{3}{143}D_2(a)\frac{\Omega_{m,0}\Omega_{\Lambda,0}}{\Omega_m(a)aE^3(a)} \tag{21}$$

Finally, using Eq. (refeq:oma) again

$$\boxed{a^3E(a)\frac{dD_2}{da} = \frac{2D_2(a)}{D_1(a)}a^3E(a)\frac{dD_1}{da} + \frac{3}{143}D_2(a)\frac{\Omega_{\Lambda,0}a^2}{E(a)}} \tag{22}$$

We'll leave it here as we can calculate the required terms from Eqs. (8) and (18).

### 1.4 Fourth expression

We'll derive the fourth and final expression in the same way as the second expression. Instead, however, we use differential equation for the second order growth factor.

$$\frac{d^2D_2}{dt^2} + 2H(t)\frac{dD_2}{dt} = \frac{3}{2}H^2(t)\Omega_m(t)(D_2(t) - D_1^2(t)). \tag{23}$$

Using a change of basis again from 't' to 'a', multiplying out the resultant expression and then substituting this into the expression we actually desire, we arrive at

$$\boxed{a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_2}{da}\right] = \frac{3}{2}\Omega_{m,0}a(D_2(a) - D_1^2(a))} \tag{24}$$

We can simply use Eq. (18) to evaluate this.

## 2 Non-Flat cosmology

The previous expressions only hold under the assumption of a flat cosmology, such that $\Omega_\Lambda = 1 - \Omega_m$. If there is a non-zero value of $\Omega_k$, the curvature density, such that $\Omega_\Lambda = 1 - \Omega_m - \Omega_k$, then we need to reformulate these expression. These are the expressions used in Version 1.1 of L-PICOLA.

The expression for the linear growth factor remains the same, except now

$$H(a) = H_0E(a) = H_0\sqrt{\Omega_{m,0}a^{-3} + \Omega_{\Lambda,0} + \Omega_{k,0}a^{-2}}, \tag{25}$$

### 2.1 First expression

The evaluation of the derivative of the linear growth factor can be calculated in a similar way except

$$\frac{dD_1}{da} = -\frac{D_{1,0}}{2}\frac{1}{E(a)}\left(\frac{3\Omega_{m,0}}{a^4} + \frac{2\Omega_{k,0}}{a^3}\right)\int_0^a \frac{da'}{a'^3E^3(a')} + \frac{D_{1,0}}{a^3E^2(a)} \tag{26}$$

therefore

$$\boxed{a^3E(a)\frac{dD_1}{da} = \frac{1}{E(a)}\left(D_{1,0} - \left[\frac{3\Omega_{m,0}}{2a} + \Omega_{k,0}\right]D_1(a)\right)} \tag{27}$$

### 2.2 Second expression

There was actually no cosmology dependence assumed in writing down the differential equation for the linear growth factor and so this expression still holds for a non-flat cosmology. Hence

$$\boxed{a^3E(a)\frac{d}{da}\left[a^3E(a)\frac{dD_1}{da}\right] = \frac{3}{2}\Omega_{m,0}aD_1(a)} \tag{28}$$

### 2.3 Third expression

There is quite a substantial difference for the $a^3E(a)dD_2/da$ expression. We no longer use Eq. (18) as this loses accuracy for non-flat cosmologies. Instead we use the expression from Matsubara (1995)

$$D_2(a) = -D_1^2(a)\left[\frac{\Omega_m}{4} - \frac{\Omega_\Lambda}{2} - \frac{1}{U_{3/2}}\left(1 - \frac{3U_{5/2}}{2U_{3/2}}\right)\right] \tag{29}$$

where

$$U_\alpha(a) = \int_0^1 \frac{da}{(\Omega_m a^{-1} + \Omega_\Lambda a^2 + \Omega_k)^\alpha} = a^{2\alpha-1} E^{2\alpha}(a) \int_0^a \frac{da'}{a'^{2\alpha} E^{2\alpha}(a')} \tag{30}$$

Rather helpfully, Matsubara (1995) provide an expression, $C$, which we can use to ease the calculation of the derivative of the new second order growth factor.

$$C = \frac{dD_2}{dt} \frac{1}{2D_1(t)\frac{dD_1}{dt}} = \frac{1}{8f(a)} \left[ -3\Omega_m(a) + \frac{1}{U_{3/2}^2(a)} \left( 2 + 4U_{3/2}(a) - 3(2 + \Omega_m(a) - 2\Omega_\Lambda(a))U_{5/2}(a) \right) \right] \tag{31}$$

where

$$f = \frac{dD_1}{dt} \frac{1}{H(t)D_1(t)} = \frac{dD_1}{da} a D_1(a) \tag{32}$$

Taking the left hand expression for $C$ and performing a change of variable from 't' to 'a'

$$\frac{dD_2}{da} = 2D_1(a) C \frac{dD_1}{da} \tag{33}$$

Finally substituting $f$ and $C$ into the above expression we find

$$\frac{dD_2}{da} = \frac{D_1^2}{4a} \left[ -3\Omega_m(a) + \frac{1}{U_{3/2}^2(a)} \left( 2 + 4U_{3/2}(a) - 3[2 + \Omega_m(a) - 2\Omega_\Lambda(a)]U_{5/2}(a) \right) \right] \tag{34}$$

such that

$$\boxed{a^3 E(a) \frac{dD_2}{da} = \frac{a^2 E(a) D_1^2(a)}{4a} \left[ -3\Omega_m(a) + \frac{1}{U_{3/2}^2(a)} \left( 2 + 4U_{3/2}(a) - 3[2 + \Omega_m(a) - 2\Omega_\Lambda(a)]U_{5/2}(a) \right) \right]} \tag{35}$$

Although complex looking, we can evaluate $U_{3/2}(a)$ and $U_{5/2}(a)$ using Eq. (30) quite easily and then use the above equation to evaluate this third expression in practice.

## 2.4 Fourth expression

Thankfully, and unlike the previous expression the non-flat version of this term still holds true as the differential equation on which it is based does not depend on the cosmology used. As such we can still use

$$\boxed{a^3 E(a) \frac{d}{da} \left[ a^3 E(a) \frac{dD_2}{da} \right] = \frac{3}{2} \Omega_{m,0} a (D_2(a) - D_1^2(a))} \tag{36}$$

The only difference is that the non-flat versions of $D_1(a)$ and $D_2(a)$ must now be used.

# References


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Tassev, S., Zaldarriaga, M., & Eisenstein, D. J. 2013, J. Cosmo. Astroparticle Phys., 6, 036