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Observe Fibr

# Using Algebraic Geometry in F-Theory

Philipp Arras

Institute for Theoretical Physics Heidelberg

October 10, 2014

### 1 Physical Context

### Mathematical Tools

- Basic Objects
- Ideal-Variety Correspondence
- Decomposition
- Dimension

### 3 Application

- Computations
- Observe Fibre Enhancement

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# Physical Context

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- Type-IIB String Theory: all classical Lie Groups
- F-Theory: all classical Lie Groups as Gauge Group possible

# F-Theory

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Internal space: 6-real-dimensional  $(B_6)$  + elliptic fibre (2-real-dimensional)

Locus	Physical content	$\dim_{\mathbb{C}}$
Torus degenerates into one $\mathbb{P}^1$	7-branes	2
7-branes intersect $/$ degeneration into two $\mathbb{P}^1$ s	matter curves	1
matter curves intersect $/$ degeneration into three $\mathbb{P}^1$ s	intersection locus	

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# Given Equations

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$$\begin{bmatrix} 0 &=& d_0c_2^2 + b_0^2c_1 - b_0b_1c_2 \\ 0 &=& d_1b_0c_2 - b_0^2b_2 - c_2^2d_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 &=& d_0b_2c_1 - b_0b_2^2 - c_1^2d_2 \\ 0 &=& d_1c_1^2 - b_1b_2c_1 + b_2^2c_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 &=& d_0c_1^3c_2^2 + b_0^2c_1^4 - b_0b_1c_1^3c_2 + c_2^3(b_1b_2c_1 - b_2^2c_2 - c_1^2d_1) \\ 0 &=& d_2c_1^4c_2^2 + (b_0c_1^2 + c_2(-b_1c_1 + b_2c_2)) \times \\ && \times (b_0b_2c_1^2 + c_2(-b_1b_2c_1 + b_2^2c_2 + c_1^2d_1)) \end{bmatrix}$$

→ Algebraic Geometry

# Given Equations

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 $\rightarrow$  Algebraic Geometry

#### Mathematical Tools

# Section 2

# Mathematical Tools

## Ideals

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### Definition

- **1**  $0 \in I$ .
- $2 f, g \in I \Rightarrow f + g \in I.$
- $f \in I \text{ and } h \in k[x_1, \dots, x_n] \Rightarrow hf \in I.$ 
  - Example:
    - $I := \langle x, y + 1 \rangle \equiv \{ h_1 \cdot x + h_2 \cdot (y + 1) : h_1, h_2 \in \mathbb{C}[x, y] \}$
- lacksquare Combine ideals:  $(f \in I, g \in J)$

### Ideals

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- Combine ideals:  $(f \in I, g \in J)$ 
  - $I+J=\{f+g\}$
  - $\blacksquare IJ = \langle f \cdot g \rangle$
  - $I \cap J$

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### Radical Ideals

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■ Definition:  $(f^m \in I \Rightarrow f \in I)$ 

■ Counterexample:  $J = \langle x^2 \rangle \subset \mathbb{C}[x]$ 

■ Produce radical:  $\sqrt{\langle (x-2)^2 \cdot y^3 \rangle} := \langle (x-2)y \rangle$ 

### Radical Ideals

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## **Varieties**

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### Zero-set of polynomials

$$V := \mathbf{V}(I) \equiv \mathbf{V}(x, y+1) := \{ p \in \mathbb{R}^2 \, : \, f(p) = 0 \, \forall \, p \in I \}$$



$$W := V(J) \equiv V(x^2)$$



## **Varieties**

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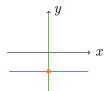
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### Zero-set of polynomials



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## **Varieties**

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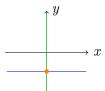
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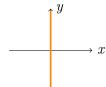
#### Application

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Zero-set of polynomials



$$\mathbf{W} := \mathbf{V}(J) \equiv \mathbf{V}(x^2)$$



# **Combining Varieties**

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$$V \cap W = \mathbf{V}(f_1, \dots, f_s, g_1, \dots, g_t),$$
  
 $V \cup W = \mathbb{V}(f_i g_j : 1 \le i \le s, 1 \le j \le t).$ 

# **Combining Varieties**

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# Towards Ideal-Variety Correspondence

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Have:

 $\mathbf{V}: \{\mathsf{Ideals}\} \to \{\mathsf{Varieties}\}$ 

Veed:

 $\{\mathsf{Varieties}\} o \{\mathsf{Ideals}\}$ 

# Towards Ideal-Variety Correspondence

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Have:

 $V : {\mathsf{Ideals}} \to {\mathsf{Varieties}}$ 

Need:

 ${Varieties} \rightarrow {Ideals}$ 

# The map ${f I}$

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### Definition

Let  $V \subset k^n$  be a variety. Then:

$$\mathbf{I}(V) := \{ f \in k[x_1, \dots, x_n] : f(p) = 0 \,\forall \, p \in V \}.$$

 $\mathbf{I}(V)$  is the ideal of V.

Then

$$\mathbf{I}: \{\mathsf{Varieties}\} o \{\mathsf{Ideals}\}$$

# The map ${f I}$

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### Are I and V inverse maps?

• 
$$V^{-1} = I$$
?

$$I^{-1} = V?$$

No.

In  $\mathbb{C}^2$ 

$$\mathbf{I}(\mathbf{V}(\langle x^2 \rangle)) = \mathbf{I}(\{(0, y)\}) = \langle x \rangle$$

But: This is the worst that can happen

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# Strong Nullstellensatz

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### **Theorem**

k algebraically closed. Then:

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

# Algebra-Geometry Dictionary

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### for k algebraically closed

ALGEBRA		GEOMETRY
radical ideals $I \ I(V)$	$\overset{\rightarrow}{\leftarrow}$	$\begin{array}{c} {\rm varieties} \\ V(I) \\ V \end{array}$
addition of ideals $I+J$	$\rightarrow$	intersection of varieties $V(I) \cap V(J)$
product / inter- section of ideals		union of varieties
$IJ$ or $I \cap J$	$\rightarrow$	$V(I) \cup V(J)$

# Algebra-Geometry Dictionary

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for k algebraically closed

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	→ ← →

# Prime Decomposition

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### Definition

A variety  $V \subset k[x_1,\ldots,x_n]$  is irreducible if whenever V is written in the form  $V=V_1\cup V_2$  ( $V_1,V_2$  varieties), then either  $V_1=V$  or  $V_2=V$ .

- lacksquare minimal decomposition:  $V_i \not\subset V_j$  for  $i \neq j$
- A variety has a unique minimal decomposition.

### Example:

$$\begin{array}{cccc} \mathbf{V}(x(y-1)) & = & \mathbf{V}(x) \cup \mathbf{V}(y-1) \\ & & & \downarrow y \\ & & & \downarrow y \\ & & & \downarrow x \end{array}$$

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# Prime Decomposition: Example

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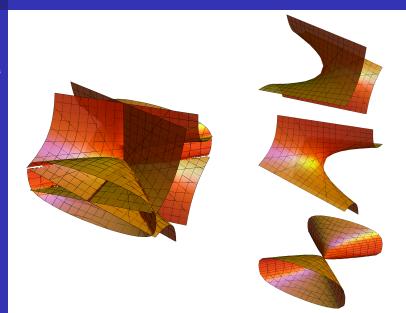
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ALGEBRA	GEOMETRY		
prime ideal	$\leftrightarrow$	irreducible variety	

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Dimension

## Global concepts (same for irreducible varieties)

$$T_p V = \ker \left( \frac{\partial F_i}{\partial x_j} \Big|_p \right)_i$$

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

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Dimension

## Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded) irreducible varieties)
- Krull dimension (longest chain of prime ideals in the ring

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Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded irreducible varieties)
- Krull dimension (longest chain of prime ideals in the ring  $\mathcal{O}(V) = {}^{k[x_1,...,x_n]}/\mathbf{I}(V)$ )

Local concepts

Tangent space:

$$T_p V = \ker \left( \frac{\partial F_i}{\partial x_j} \Big|_p \right)_{ij}$$

lacksquare Zariski tangent space  $(\mathfrak{m}_p:=\{f\in\mathcal{O}(V)\,:\,f(p)=0\})$ 

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

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**Z**ariski tangent space  $(\mathfrak{m}_p := \{ f \in \mathcal{O}(V) : f(p) = 0 \})$ :

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

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Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded irreducible varieties)
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### Local concepts

Tangent space:

$$T_p V = \ker \left( \frac{\partial F_i}{\partial x_j} \Big|_p \right)_{ij}$$

■ Zariski tangent space  $(\mathfrak{m}_p := \{ f \in \mathcal{O}(V) : f(p) = 0 \})$ :

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

# Singular Locus

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## Definition (for irreducible varieties)

If local dimension in p= global dimension, then p is regular. If local dimension in  $p\neq$  global dimension, then p is singular .

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Dimension

Defining equations:  $\mathbf{V}(x) \cup \mathbf{V}(y, z) = \mathbf{V}(xy, xz)$ 

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$



Locus	Rank	Dimension
x = y = z = 0	0	3
$x = 0$ and $(y \neq 0 \text{ or } z \neq 0)$	1	2
$x \neq 0, y = z = 0$	2	1

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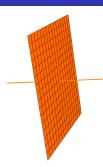
Corresponden

Dimension

### Applicatio

Computation Observe Fibr ■ Defining equations:  $\mathbf{V}(x) \cup \mathbf{V}(y, z) = \mathbf{V}(xy, xz)$ 

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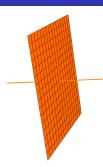
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Dimension

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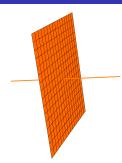
Dimension

Applications

Defining equations: V(x) + V(x, x) = V(x, x)

$$\mathbf{V}(x) \cup \mathbf{V}(y,z) = \mathbf{V}(xy,xz)$$

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$



Locus	Rank	Dimension
x = y = z = 0	0	3
$x=0$ and $(y \neq 0 \text{ or } z \neq 0)$	1	2
$x \neq 0, y = z = 0$	2	1

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# Section 3

# Application

# Given Equations

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$$0 = d_0 c_2^2 + b_0^2 c_1 - b_0 b_1 c_2$$
  
$$0 = d_1 b_0 c_2 - b_0^2 b_2 - c_2^2 d_2$$

$$0 = d_0 b_2 c_1 - b_0 b_2^2 - c_1^2 d_2$$
  
$$0 = d_1 c_1^2 - b_1 b_2 c_1 + b_2^2 c_2$$

$$0 = d_0 c_1^3 c_2^2 + b_0^2 c_1^4 - b_0 b_1 c_1^3 c_2 + c_2^3 (b_1 b_2 c_1 - b_2^2 c_2 - c_1^2 d_1)$$
  

$$0 = d_2 c_1^4 c_2^2 + (b_0 c_1^2 + c_2 (-b_1 c_1 + b_2 c_2)) \times$$
  

$$\times (b_0 b_2 c_1^2 + c_2 (-b_1 b_2 c_1 + b_2^2 c_2 + c_1^2 d_1))$$

### Extract Matter Curves

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```
singular.lib('primdec.lib');
singular.lib('sing.lib');
R = singular.ring(0, '(b0, b1, b2, c1, c2, d0, d1, d2)', '
   dp');
I1 = singular.ideal('d0*c2^2+b0^2*c1-b0*b1*c2','d1*
   b0*c2-b0^2*b2-c2^2*d2').std();
12 = singular.ideal('d0*b2*c1-b0*b2^2-c1^2*d2','d1*
   c1^2-b1*b2*c1+b2^2*c2').std();
13 = singular.ideal('d0*c1^3*c2^2-(-b0^2*c1^4+b0*b1*)
   c1^3*c2+c2^3*(-b1*b2*c1+b2^2*c2+c1^2*d1))','d2*
   c1^4*c2^2+(b0*c1^2+c2*(-b1*c1+b2*c2))*(b0*b2*c1
   ^2+c2*(-b1*b2*c1+b2^2*c2+c1^2*d1))').std();
# Extract the matter curves
I1pr = I1.minAssGTZ();
12pr = 12.minAssGTZ();
13pr = 13.minAssGTZ();
```

# Matter Curves

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Every pair of the above equations defines two matter curves.

```
# I1
Cl1 = I1pr[2].std(); # c2 = 0 = b0
Cl2 = I1pr[1].std(); # complicated

# I2
Cl3 = I2pr[2].std(); # b2 = 0 = c1
Cl4 = I2pr[1].std(); # complicated

# I3
Cl5 = I3pr[2].std(); # c2 = 0 = c1
Cl6 = I3pr[1].std(); # complicated
```

# Allowed Yukawa Interactions

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$$\begin{array}{c} \mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)} \\ \mathbf{1}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)} \\ \mathbf{1}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}. \end{array}$$

```
inters1 = (CI1+CI4+CI5).std();
inters2 = (CI2+CI3+CI5).std();
inters3 = (CI2+CI4+CI6).std();
inters4 = (CI1+CI2+CI6).std();
inters5 = (CI3+CI4+CI6).std();
singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

# Allowed Yukawa Interactions

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```
inters1 = (CI1+CI4+CI5).std();
inters2 = (CI2+CI3+CI5).std();
inters3 = (CI2+CI4+CI6).std();
inters4 = (CI1+CI2+CI6).std();
inters5 = (CI3+CI4+CI6).std();
singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

# Compute Codimension

```
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```

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```
inters1pr = inters1.minAssGTZ();
inters2pr = inters2.minAssGTZ();
inters3pr = inters3.minAssGTZ();
inters4pr = inters4.minAssGTZ();
inters5pr = inters5.minAssGTZ();
inters6pr = inters6.minAssGTZ();
inters61pr = inters61.minAssGTZ();
inters1pr[1].std().dim() \# -> codim 3
inters2pr[1].std().dim() # -> codim 3
inters3pr[1].std().dim() \# \rightarrow codim 3
inters3pr[2].std().dim() \# -> codim 4 \# wrong codim
inters4pr[1].std().dim() \# \rightarrow codim 3
inters5pr[1].std().dim() \# \rightarrow codim 3
inters6pr[1].std().dim() # -> codim 3
inters61pr[1].std().dim() \# \rightarrow codim 3
inters61pr[2].std().dim() \# -> codim 4 \# wrong codim
```

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### Hypersurface equation:

$$P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0.$$

- Consider e.g.  $\mathbf{1}^{(1)}$ :  $c_2 = 0 = b_0$ .
- Then: 2-fold factorization

$$P_{T^2}|_{\mathbf{1}^{(1)}} \equiv P_{T^2}|_{b_0=0, c_2=0}$$
  
=  $s_1(d_2s_0^2s_1u^3 + d_0s_0^2u^2v + \ldots)$ 

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Hypersurface equation:

$$P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0.$$

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Hypersurface equation:

$$P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0.$$

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Hypersurface equation:

$$P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0.$$

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Consider e.g.  $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$ :  $c_2 = 0 = b_0$  and  $d_1 = \frac{b_1 b_2 d_0 + b_1 c_1 d_2 \pm \sqrt{b_1^2 - 4c_1 d_0}(b_2 d_0 - c_1 d_2)}{2c_1 d_0}$ .

■ Then: 3-fold factorization

$$P_{T^2}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4c_1d_0} s_1 \left[ \left( b_1 \pm \sqrt{b_1^2 - 4c_1d_0} \right) s_0 u + 2c_1 w \right] \times \left[ \left( b_1 \mp \sqrt{b_1^2 - 4c_1d_0} \right) d_2 s_0 s_1 u^2 + \dots \right]$$

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■ Consider e.g.  $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$ :  $c_2 = 0 = b_0$  and  $d_1 = \frac{b_1b_2d_0 + b_1c_1d_2 \pm \sqrt{b_1^2 - 4c_1d_0}(b_2d_0 - c_1d_2)}{2c_1d_0}$ .

■ Then: 3-fold factorization

$$P_{T^{2}}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4c_{1}d_{0}}s_{1}\left[\left(b_{1} \pm \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)s_{0}u + 2c_{1}w\right] \times \left[\left(b_{1} \mp \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)d_{2}s_{0}s_{1}u^{2} + \ldots\right]$$

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$$P_{T^{2}}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4c_{1}d_{0}}s_{1}\left[\left(b_{1} \pm \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)s_{0}u + 2c_{1}w\right] \times \left[\left(b_{1} \mp \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)d_{2}s_{0}s_{1}u^{2} + \ldots\right]$$

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Locus	Physical content	$\dim_{\mathbb{C}}$
Torus degenerates into one $\mathbb{P}^1$	7-branes	2
7-branes intersect $/$ degeneration into two $\mathbb{P}^1$ s	matter curves	1
matter curves intersect $/$ degeneration into three $\mathbb{P}^1$ s	intersection locus	0



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