Using Algebraic Geometry in F-Theory

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Section 1

Physical Context

Physical Context

- Type-IIB String Theory: ~~all classical Lie Groups~~
- F-Theory: all classical Lie Groups as Gauge Group possible

F-Theory

Internal space: 6-real-dimensional (B_6) + elliptic fibre (2-real-dimensional)

Locus	Physical content	$\dim_{\mathbb{C}}$
Torus degenerates into one $\mathbb{P}^1$	7-branes	2
7-branes intersect / degeneration into two $\mathbb{P}^1$s	matter curves	1
matter curves intersect / degeneration into three $\mathbb{P}^1$s	intersection locus	0

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Given Equations

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$\rightarrow$ Algebraic Geometry

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$\to$ Algebraic Geometry

Section 2

Mathematical Tools

Ideals

Definition

A subset $I \subset k[x_1, \dots, x_n]$ *is an ideal if:*

1. $0 \in I$.
2. $f, g \in I \;\Rightarrow\; f + g \in I$.
3. $f \in I$ *and* $h \in k[x_1, \dots, x_n] \;\Rightarrow\; hf \in I$.

- Example:

$$I := \langle x, y+1 \rangle \equiv \{h_1 \cdot x + h_2 \cdot (y+1) \;:\; h_1, h_2 \in \mathbb{C}[x, y]\}$$

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- Definition: $(f^m \in I \Rightarrow f \in I)$
- Counterexample: $J = \langle x^2 \rangle \subset \mathbb{C}[x]$
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Varieties

- Zero-set of polynomials
- $V := \mathbf{V}(I) \equiv \mathbf{V}(x, y+1) := \{p \in \mathbb{R}^2 \ : \ f(p) = 0 \, \forall \, p \in I\}$

- $W := \mathbf{V}(J) \equiv \mathbf{V}(x^2)$

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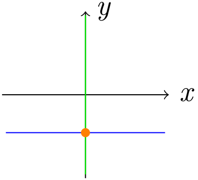

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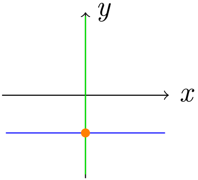

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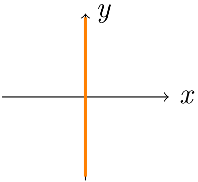

Combining Varieties

$$V \cap W = \mathbf{V}(f_1, \ldots, f_s, g_1, \ldots, g_t),$$
$$V \cup W = \mathbf{V}(f_i g_j \;:\; 1 \leq i \leq s, 1 \leq j \leq t).$$

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Towards Ideal-Variety Correspondence

Have:

$$\mathbf{V} : \{\text{Ideals}\} \to \{\text{Varieties}\}$$

Need:

$$\{\text{Varieties}\} \to \{\text{Ideals}\}$$

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The map $\mathbf{I}$

Definition

Let $V \subset k^n$ be a variety. Then:

$$\mathbf{I}(V) := \{f \in k[x_1, \ldots, x_n] \; : \; f(p) = 0 \, \forall \, p \in V\}.$$

$\mathbf{I}(V)$ *is the ideal of* V.

Then:

$$\mathbf{I} : \{\text{Varieties}\} \to \{\text{Ideals}\}$$

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Are $\mathbf{I}$ and $\mathbf{V}$ inverse maps?

- $\mathbf{V}^{-1} = \mathbf{I}$?
- $\mathbf{I}^{-1} = \mathbf{V}$?

No.

In $\mathbb{C}^2$:

$$\mathbf{I}(\mathbf{V}(\langle x^2 \rangle)) = \mathbf{I}(\{(0, y)\}) = \langle x \rangle$$

But: This is the worst that can happen.

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Strong Nullstellensatz

Theorem

k algebraically closed. Then:

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Algebra-Geometry Dictionary

for k algebraically closed

ALGEBRA		GEOMETRY
radical ideals I $I(V)$	 $\rightarrow$ $\leftarrow$	varieties $V(I)$ V
addition of ideals $I + J$	 $\rightarrow$	intersection of varieties $V(I) \cap V(J)$
product / intersection of ideals IJ or $I \cap J$	 $\rightarrow$	union of varieties $V(I) \cup V(J)$

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Prime Decomposition

Definition

A variety $V \subset k[x_1, \dots, x_n]$ is irreducible if whenever V is written in the form $V = V_1 \cup V_2$ (V_1, V_2 varieties), then either $V_1 = V$ or $V_2 = V$.

- minimal decomposition: $V_i \not\subset V_j$ for $i \neq j$
- A variety has a unique minimal decomposition.

Example:

$$\mathbf{V}(x(y-1)) \quad = \quad \mathbf{V}(x) \cup \mathbf{V}(y-1)$$

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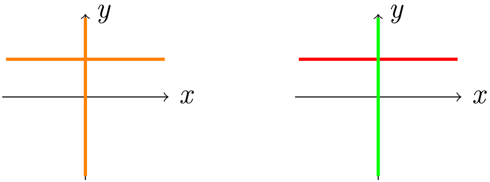

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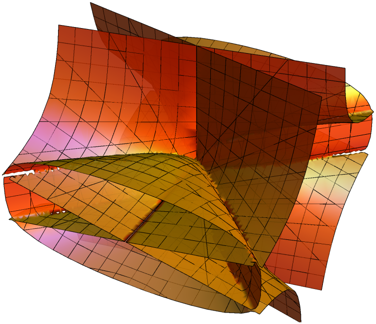

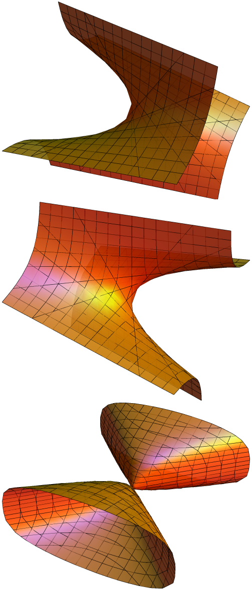

Prime Decomposition

ALGEBRA		GEOMETRY
prime ideal	$\leftrightarrow$	irreducible variety

Notions of Dimension

Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded irreducible varieties)
- Krull dimension (longest chain of prime ideals in the ring $\mathcal{O}(V) = k[x_1,\dots,x_n]/\mathbf{I}(V)$)

Local concepts

- Tangent space:

$$T_p V = \ker \left(\left. \frac{\partial F_i}{\partial x_j} \right|_p \right)_{ij}$$

- Zariski tangent space ($\mathfrak{m}_p := \{ f \in \mathcal{O}(V) \ : \ f(p) = 0 \}$):

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Singular Locus

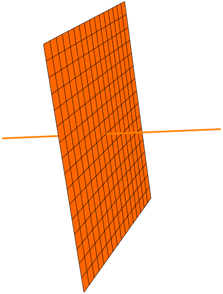
Definition (for irreducible varieties)

If local dimension in $p =$ global dimension, then p is regular.
If local dimension in $p \neq$ global dimension, then p is singular .

Singular Locus: Example

- Defining equations:
  $\mathbf{V}(x) \cup \mathbf{V}(y, z) = \mathbf{V}(xy, xz)$
- Jacobian matrix

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$



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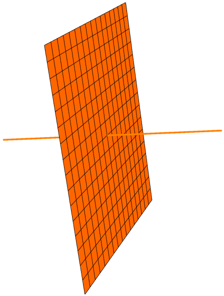

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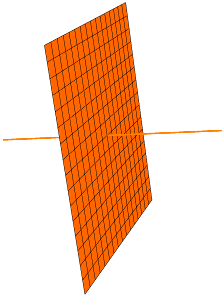

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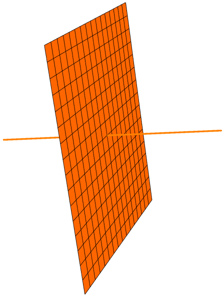

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Section 3

Application

Given Equations

$$
\begin{aligned}
0 &= d_0\, c_2^2 + b_0^2\, c_1 - b_0\, b_1\, c_2 \\
0 &= d_1\, b_0\, c_2 - b_0^2\, b_2 - c_2^2\, d_2
\end{aligned}
$$

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\begin{aligned}
0 &= d_0\, b_2\, c_1 - b_0\, b_2^2 - c_1^2\, d_2 \\
0 &= d_1\, c_1^2 - b_1\, b_2\, c_1 + b_2^2\, c_2
\end{aligned}
$$

$$
\begin{aligned}
0 &= d_0\, c_1^3\, c_2^2 + b_0^2\, c_1^4 - b_0\, b_1\, c_1^3\, c_2 + c_2^3 (b_1\, b_2\, c_1 - b_2^2\, c_2 - c_1^2\, d_1) \\
0 &= d_2\, c_1^4\, c_2^2 + (b_0\, c_1^2 + c_2(-b_1\, c_1 + b_2\, c_2)) \times \\
&\quad \times (b_0\, b_2\, c_1^2 + c_2(-b_1\, b_2\, c_1 + b_2^2\, c_2 + c_1^2\, d_1))
\end{aligned}
$$

Extract Matter Curves


```
singular.lib('primdec.lib');
singular.lib('sing.lib');
R = singular.ring(0, '(b0,b1,b2,c1,c2,d0,d1,d2)', '
    dp');
I1 = singular.ideal('d0*c2^2+b0^2*c1-b0*b1*c2','d1*
    b0*c2-b0^2*b2-c2^2*d2').std();
I2 = singular.ideal('d0*b2*c1-b0*b2^2-c1^2*d2','d1*
    c1^2-b1*b2*c1+b2^2*c2').std();
I3 = singular.ideal('d0*c1^3*c2^2-(-b0^2*c1^4+b0*b1*
    c1^3*c2+c2^3*(-b1*b2*c1+b2^2*c2+c1^2*d1))','d2*
    c1^4*c2^2+(b0*c1^2+c2*(-b1*c1+b2*c2))*(b0*b2*c1
    ^2+c2*(-b1*b2*c1+b2^2*c2+c1^2*d1))').std();

# Extract the matter curves
I1pr = I1.minAssGTZ();
I2pr = I2.minAssGTZ();
I3pr = I3.minAssGTZ();
```

Matter Curves

Every pair of the above equations defines two matter curves.

```
# I1
CI1 = I1pr[2].std(); # c2 = 0 = b0
CI2 = I1pr[1].std(); # complicated

# I2
CI3 = I2pr[2].std(); # b2 = 0 = c1
CI4 = I2pr[1].std(); # complicated

# I3
CI5 = I3pr[2].std(); # c2 = 0 = c1
CI6 = I3pr[1].std(); # complicated
```

Allowed Yukawa Interactions

- $\mathbf{1}^{(1)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(5)}$
- $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(5)}$
- $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(6)}$
- $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$
- $\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}$
- $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$.

```
inters1 = (CI1+CI4+CI5).std();
inters2 = (CI2+CI3+CI5).std();
inters3 = (CI2+CI4+CI6).std();
inters4 = (CI1+CI2+CI6).std();
inters5 = (CI3+CI4+CI6).std();
singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

Allowed Yukawa Interactions

- $\mathbf{1}^{(1)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(5)}$
- $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(5)}$
- $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(6)}$
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```
inters1 = (CI1+CI4+CI5).std();
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inters4 = (CI1+CI2+CI6).std();
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singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

Compute Codimension

```
inters1pr = inters1.minAssGTZ();
inters2pr = inters2.minAssGTZ();
inters3pr = inters3.minAssGTZ();
inters4pr = inters4.minAssGTZ();
inters5pr = inters5.minAssGTZ();
inters6pr = inters6.minAssGTZ();
inters61pr = inters61.minAssGTZ();

inters1pr[1].std().dim()  # -> codim 3
inters2pr[1].std().dim()  # -> codim 3
inters3pr[1].std().dim()  # -> codim 3
inters3pr[2].std().dim()  # -> codim 4 # wrong codim
inters4pr[1].std().dim()  # -> codim 3
inters5pr[1].std().dim()  # -> codim 3
inters6pr[1].std().dim()  # -> codim 3
inters61pr[1].std().dim() # -> codim 3
inters61pr[2].std().dim() # -> codim 4 # wrong codim
```

Enhancement over Matter Curves

- Hypersurface equation:

$$\begin{aligned} P_{T^2} = vw(c_1 w s_1 + c_2 v s_0) + u(b_0 v_2 s_0^2 + b_1 v w s_0 s_1 + b_2 w_2 s_1^2) \\ + u_2(d_0 v s_0^2 s_1 + d_1 w s_0 s_1^2 + d_2 u s_0^2 s_1^2) = 0. \end{aligned}$$

- Consider e.g. $\mathbf{1}^{(1)}$: $c_2 = 0 = b_0$.
- Then: 2-fold factorization

$$\begin{aligned} P_{T^2}\big|_{\mathbf{1}^{(1)}} &\equiv P_{T^2}\big|_{b_0=0, c_2=0} \\ &= s_1(d_2 s_0^2 s_1 u^3 + d_0 s_0^2 u^2 v + \ldots). \end{aligned}$$

- Meaning: Torus $\rightarrow$ two $\mathbb{P}^1$

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$$P_{T^2} = vw(c_1 ws_1 + c_2 vs_0) + u(b_0 v_2 s_0^2 + b_1 vws_0 s_1 + b_2 w_2 s_1^2) + u_2(d_0 vs_0^2 s_1 + d_1 ws_0 s_1^2 + d_2 us_0^2 s_1^2) = 0.$$

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Enhancement over Intersection Loci

- Consider e.g. $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$: $c_2 = 0 = b_0$ and $d_1 = \frac{b_1\, b_2\, d_0 + b_1\, c_1\, d_2 \pm \sqrt{b_1^2 - 4 c_1\, d_0}\,(b_2\, d_0 - c_1\, d_2)}{2\, c_1\, d_0}$.
- Then: 3-fold factorization

$$P_{T^2}\big|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4 c_1\, d_0} s_1 \left[\left(b_1 \pm \sqrt{b_1^2 - 4 c_1\, d_0}\right) s_0 u + 2 c_1 w\right] \times \left[\left(b_1 \mp \sqrt{b_1^2 - 4 c_1\, d_0}\right) d_2 s_0 s_1 u^2 + \dots\right]$$

- Meaning: Torus $\rightarrow$ three $\mathbb{P}^1$

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- Then: 3-fold factorization

$$P_{T^2}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4 c_1 d_0} s_1 \left[\left(b_1 \pm \sqrt{b_1^2 - 4 c_1 d_0}\right) s_0 u + 2 c_1 w\right] \times \left[\left(b_1 \mp \sqrt{b_1^2 - 4 c_1 d_0}\right) d_2 s_0 s_1 u^2 + \dots\right]$$

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- Then: 3-fold factorization

$$
\begin{aligned}
P_{T^2}\big|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = & \frac{1}{4 c_1 d_0} s_1 \left[\left(b_1 \pm \sqrt{b_1^2 - 4 c_1 d_0}\right) s_0 u + 2 c_1 w\right] \\
& \times \left[\left(b_1 \mp \sqrt{b_1^2 - 4 c_1 d_0}\right) d_2 s_0 s_1 u^2 + \ldots\right]
\end{aligned}
$$

- Meaning: Torus $\to$ three $\mathbb{P}^1$

Enhancement over Intersection Loci

Locus	Physical content	$\dim_{\mathbb{C}}$
Torus degenerates into one $\mathbb{P}^1$	7-branes	2
7-branes intersect / degeneration into two $\mathbb{P}^1$s	matter curves	1
matter curves intersect / degeneration into three $\mathbb{P}^1$s	intersection locus	0

✓

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✓