Philipp Arras

Context

Mathematical Tools

Basic Object

.

Corresponder

Decompositi

Application

Computation

Observe Fib

Using Algebraic Geometry in F-Theory

Philipp Arras

Institute for Theoretical Physics Heidelberg

October 7, 2014

1 Physical Context

Mathematical Tools

- Basic Objects
- Ideal-Variety Correspondence
- Decomposition
- Dimension

3 Application

- Computations
- Observe Fibre Enhancement

Philipp Arras

Physical Context

Mathematica Tools

Rasic Objects

Ideal-Variety

Corresponder

Decompositi

Application

Computation

Observe Fibre

Section 1

Physical Context

Physical Context

Using Algebraic Geometry in F-Theory

Philipp Arra

Physical Context

Tools

Basic Object

Ideal-Varie

Corresponde

Dimension

Application

Computation Observe Fibre

- Type-IIB String Theory: all classical Lie Groups
- F-Theory: all classical Lie Groups as Gauge Group possible

F-Theory

Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematica Tools

Tools

Corresponde

Dimension

Computations
Observe Fibre

Internal space: 6-real-dimensional (B_6) + elliptic fibre (2-real-dimensional)

| Locus | Physical content | $\dim_{\mathbb{C}}$ |
|--|--------------------|---------------------|
| Torus degenerates into one \mathbb{P}^1 | 7-branes | 2 |
| 7-branes intersect $/$ degeneration into two \mathbb{P}^1 s | matter curves | 1 |
| matter curves intersect $/$ degeneration into three \mathbb{P}^1 s | intersection locus | |

F-Theory

Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematical Tools

Basic Objects

Corresponde Decompositi Dimension

Computations

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F-Theory

Using Algebraic Geometry in F-Theory

Philipp Arra

Physical Context

Mathematical Tools

Basic Objects Ideal-Variety

Corresponder Decomposition Dimension

Application
Computations
Observe Fibre
Enhancement

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Given Equations

Using Algebraic Geometry in F-Theory

Philipp Arra

Physical Context

Mathematical

Tools

Ideal-Variety

Decomposit

Dimension

Application

Observe Fibre Enhancement

$$\begin{bmatrix} 0 &=& d_0c_2^2 + b_0^2c_1 - b_0b_1c_2 \\ 0 &=& d_1b_0c_2 - b_0^2b_2 - c_2^2d_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 &=& d_0b_2c_1 - b_0b_2^2 - c_1^2d_2 \\ 0 &=& d_1c_1^2 - b_1b_2c_1 + b_2^2c_2 \end{bmatrix}$$

$$\begin{bmatrix} 0 &=& d_0c_1^3c_2^2 + b_0^2c_1^4 - b_0b_1c_1^3c_2 + c_2^3(b_1b_2c_1 - b_2^2c_2 - c_1^2d_1) \\ 0 &=& d_2c_1^4c_2^2 + (b_0c_1^2 + c_2(-b_1c_1 + b_2c_2)) \times \\ && \times (b_0b_2c_1^2 + c_2(-b_1b_2c_1 + b_2^2c_2 + c_1^2d_1)) \end{bmatrix}$$

→ Algebraic Geometry

Given Equations

Using Algebraic Geometry in F-Theory

Philipp Arra

Physical Context

Mathematical

Tools

Ideal-Variety Corresponde

Decomposit Dimension

Application

Computations
Observe Fibre
Enhancement

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 \rightarrow Algebraic Geometry

Mathematical Tools

Section 2

Mathematical Tools

Ideals

Using Algebraic Geometry in F-Theory

Philipp Arras

Physica Context

Mathema Tools

Basic Objects

Corresponden

Decomposition

Application

Computation

Definition

- **1** $0 \in I$.
- $2 f, g \in I \Rightarrow f + g \in I.$
- $f \in I \text{ and } h \in k[x_1, \dots, x_n] \Rightarrow hf \in I.$
 - Example:
 - $I := \langle x, y + 1 \rangle \equiv \{ h_1 \cdot x + h_2 \cdot (y + 1) : h_1, h_2 \in \mathbb{C}[x, y] \}$
- lacksquare Combine ideals: $(f \in I, g \in J)$

Ideals

Using Algebraic Geometry in F-Theory

Philipp Arras

Physica Contex

Tools

Basic Objects

Ideal-Variety

Decomposition Dimension

Applicatio

Computations
Observe Fibre

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- Combine ideals: $(f \in I, g \in J)$
 - $I+J=\{f+g\}$
 - $\blacksquare IJ = \langle f \cdot g \rangle$
 - $I \cap J$

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Radical Ideals

Using Algebraic Geometry in F-Theory

Philipp Arra

Physica Contex

Tools

Basic Objects

Corresponden

e ...

Dimensi

Application

Observe Fibr

■ Definition: $(f^m \in I \Rightarrow f \in I)$

■ Counterexample: $J = \langle x^2 \rangle \subset \mathbb{C}[x]$

■ Produce radical: $\sqrt{\langle (x-2)^2 \cdot y^3 \rangle} := \langle (x-2)y \rangle$

Radical Ideals

Using Algebraic Geometry in F-Theory

Philipp Arra

Physica Context

Tools

Basic Objects

Corresponden

Decompositio

Application

Computations
Observe Fibre

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Radical Ideals

Using Algebraic Geometry in F-Theory

Basic Objects

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Varieties

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Philipp Arras

Physical Context

Mathematica

Tools

Basic Objects

Corresponder

Corresponder

Decomposition

Applicatio

Computat

Observe Fibr

Zero-set of polynomials

$$V := \mathbf{V}(I) \equiv \mathbf{V}(x, y+1) := \{ p \in \mathbb{R}^2 \, : \, f(p) = 0 \, \forall \, p \in I \}$$



$$W := V(J) \equiv V(x^2)$$



Varieties

Using Algebraic Geometry in F-Theory

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Physical Context

Mathematica

Basic Objects

Corresponden

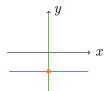
Decomposition

Application

Computation

Observe Fibr

Zero-set of polynomials



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Varieties

Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematica Tools

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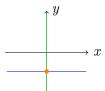
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Decomposition

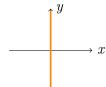
Application

Computations
Observe Fibre

Zero-set of polynomials



$$\mathbf{W} := \mathbf{V}(J) \equiv \mathbf{V}(x^2)$$



Combining Varieties

Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematic

Basic Objects

Dusic Objec

Corresponden

Decomposition

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Computation

Observe Fibr

$$V \cap W = \mathbf{V}(f_1, \dots, f_s, g_1, \dots, g_t),$$

 $V \cup W = \mathbb{V}(f_i g_j : 1 \le i \le s, 1 \le j \le t).$

Combining Varieties

Using Algebraic Geometry in F-Theory

Philipp Arras

Context

Mathematica

Basic Objects

Corresponden

Decomposition

A 11 .1

Observe Fibre

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Towards Ideal-Variety Correspondence

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Philipp Arras

Physica Context

Mathematica Tools

Basic Objects

Ideal-Variety

Correspondence

Decomposition

Application

Computation
Observe Fibro

Have:

 $\mathbf{V}: \{\mathsf{Ideals}\} \to \{\mathsf{Varieties}\}$

Veed:

 $\{\mathsf{Varieties}\} o \{\mathsf{Ideals}\}$

Towards Ideal-Variety Correspondence

Using Algebraic Geometry in F-Theory

Ideal-Variety

Correspondence

Have:

 $V : {\mathsf{Ideals}} \to {\mathsf{Varieties}}$

Need:

 ${Varieties} \rightarrow {Ideals}$

The map ${f I}$

Using Algebraic Geometry in F-Theory

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Physica Context

Tools

Basic Objects

Ideal-Variety Correspondence

Decomposit

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Applicatio

Observe Fibr

Definition

Let $V \subset k^n$ be a variety. Then:

$$\mathbf{I}(V) := \{ f \in k[x_1, \dots, x_n] : f(p) = 0 \,\forall \, p \in V \}.$$

 $\mathbf{I}(V)$ is the ideal of V.

Then

$$\mathbf{I}: \{\mathsf{Varieties}\} o \{\mathsf{Ideals}\}$$

The map ${f I}$

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Philipp Arra

Physica Context

Tools

Basic Objects

Ideal-Variety Correspondence

Decompositi

Application

Observe Fibre

Definition

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Then:

$$\mathbf{I}: \{\mathsf{Varieties}\} \to \{\mathsf{Ideals}\}$$

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Physica Contex

Mathema Tools

Basic Objec

Ideal-Variety Correspondence

Corresponder

Dimension

Application

Observe Fibre

Are I and V inverse maps?

•
$$V^{-1} = I$$
?

$$I^{-1} = V?$$

No.

In \mathbb{C}^2

$$\mathbf{I}(\mathbf{V}(\langle x^2 \rangle)) = \mathbf{I}(\{(0, y)\}) = \langle x \rangle$$

But: This is the worst that can happen

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Tools

Basic Object

Ideal-Variety

Correspondence

Decomposition Dimension

Application

Observe Fibre

Are ${f I}$ and ${f V}$ inverse maps?

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Physica Contex

Tools

Basic Objects

Ideal-Variety

Correspondence

A!!!!

Application

Observe Fibre Enhancement Are I and V inverse maps?

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In \mathbb{C}^2 :

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But: This is the worst that can happen.

Strong Nullstellensatz

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Physica Context

Mathematica Tools

Rasic Objects

Ideal-Variety Correspondence

Decompositi

Application

Computation

Theorem

k algebraically closed. Then:

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Algebra-Geometry Dictionary

Using Algebraic Geometry in F-Theory

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Physica Context

Tools

Basic Objec

Ideal-Variety Correspondence

Decompositi

Applicatio

Computation Observe Fibre

for k algebraically closed

| ALGEBRA | | GEOMETRY |
|---------------------------------------|-------------------------------------|---|
| radical ideals $I \ I(V)$ | $\overset{\rightarrow}{\leftarrow}$ | $\begin{array}{c} {\rm varieties} \\ V(I) \\ V \end{array}$ |
| addition of ideals $I+J$ | \rightarrow | intersection of varieties $V(I) \cap V(J)$ |
| product / inter- section of ideals | | union of varieties |
| IJ or $I \cap J$ | \rightarrow | $V(I) \cup V(J)$ |

Algebra-Geometry Dictionary

Using Algebraic Geometry in F-Theory

Ideal-Variety

Correspondence

for k algebraically closed

| ALGEBRA | | GEOMETRY |
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Algebra-Geometry Dictionary

Using Algebraic Geometry in F-Theory

Philipp Arra

Physica Context

Tools

Basic Objects

Ideal-Variety

Correspondence

Dimension

Application Computations

for k algebraically closed

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| | → ← → |

Prime Decomposition

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Physical Context

Mathematica Tools

Tools

Corresponde

Decomposition

Applicatio

Observe Fibre

Definition

A variety $V \subset k[x_1,\ldots,x_n]$ is irreducible if whenever V is written in the form $V=V_1\cup V_2$ (V_1,V_2 varieties), then either $V_1=V$ or $V_2=V$.

- lacksquare minimal decomposition: $V_i \not\subset V_j$ for $i \neq j$
- A variety has a unique minimal decomposition.

Example:

$$\begin{array}{cccc} \mathbf{V}(x(y-1)) & = & \mathbf{V}(x) \cup \mathbf{V}(y-1) \\ & & & \downarrow y \\ & & & \downarrow y \\ & & & \downarrow x \end{array}$$

Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematica Tools

Tools

Ideal-Variety

Decomposition

Decompositi Dimension

Application

Computation
Observe Fibre

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Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematica Tools

Tools

Ideal-Variety

Decomposition

Decompositi Dimension

Application

Computation
Observe Fibre

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Using Algebraic Geometry in F-Theory

Philipp Arras

Physical Context

Mathematica Tools

l ools

Basic Objects

Corresponde

Decomposition Dimension

Application

Computation
Observe Fibre

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Prime Decomposition: Example

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Physica Context

Mathematica Tools

Basic Object

Ideal-Varie

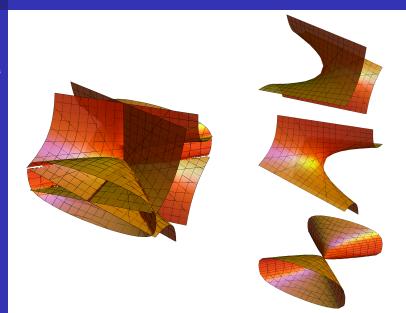
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Applicatio

Computation

Observe Fibr Enhancement



Using Algebraic Geometry in F-Theory

Philipp Arra

Physica Context

Tools

Racio Obioete

Basic Objects

- Correspond

Dimonsion

Applicatio

Observe Fib

Observe Fibre

| ALGEBRA | GEOMETRY | | |
|-------------|-------------------|---------------------|--|
| prime ideal | \leftrightarrow | irreducible variety | |

Using Algebraic Geometry in F-Theory

Dimension

Global concepts (same for irreducible varieties)

$$T_p V = \ker \left(\frac{\partial F_i}{\partial x_j} \Big|_p \right)_i$$

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

Using Algebraic Geometry in F-Theory

Dimension

Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded) irreducible varieties)
- Krull dimension (longest chain of prime ideals in the ring

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Using Algebraic Geometry in F-Theory

Philipp Arras

Context

Mathematical Tools

Basic Objects

Corresponder

Decomposition Dimension

Computations
Observe Fibre

Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded irreducible varieties)
- Krull dimension (longest chain of prime ideals in the ring $\mathcal{O}(V) = {}^{k[x_1,...,x_n]}/\mathbf{I}(V)$)

Local concepts

Tangent space:

$$T_p V = \ker \left(\frac{\partial F_i}{\partial x_j} \Big|_p \right)_{ij}$$

lacksquare Zariski tangent space $(\mathfrak{m}_p:=\{f\in\mathcal{O}(V)\,:\,f(p)=0\})$

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

Using Algebraic Geometry in F-Theory

Philipp Arras

Context

Mathematical Tools

Basic Objects

Corresponder

Decomposition Dimension

Computations
Observe Fibre

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Using Algebraic Geometry in F-Theory

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- Topological dimension (longest chain of embedded irreducible varieties)
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Using Algebraic Geometry in F-Theory

Philipp Arra

Context

Mathematical Tools

Ideal-Variety
Correspondence
Decomposition
Dimension

Application

Computations

Observe Fibre

Global concepts (same for irreducible varieties)

- Topological dimension (longest chain of embedded irreducible varieties)
- Krull dimension (longest chain of prime ideals in the ring $\mathcal{O}(V) = {}^{k[x_1,...,x_n]}/\mathbf{I}(V)$)

Local concepts

Tangent space:

$$T_p V = \ker \left(\frac{\partial F_i}{\partial x_j} \Big|_p \right)_{ij}$$

■ Zariski tangent space $(\mathfrak{m}_p := \{ f \in \mathcal{O}(V) : f(p) = 0 \})$:

$$(T_p V)^* \cong \mathfrak{m}_p/\mathfrak{m}_p^2$$

Singular Locus

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Context

Mathematical

Tools

Correspond

Decomposi Dimension

Computations
Observe Fibre

Definition (for irreducible varieties)

If local dimension in p= global dimension, then p is regular. If local dimension in $p\neq$ global dimension, then p is singular .

Using Algebraic Geometry in F-Theory

Dimension

Defining equations: $\mathbf{V}(x) \cup \mathbf{V}(y, z) = \mathbf{V}(xy, xz)$

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$



| Locus | Rank | Dimension |
|---|------|-----------|
| x = y = z = 0 | 0 | 3 |
| $x = 0$ and $(y \neq 0 \text{ or } z \neq 0)$ | 1 | 2 |
| $x \neq 0, y = z = 0$ | 2 | 1 |

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Physica Context

Mathematic

Basic Object

Ideal-Variety

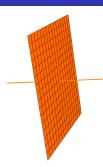
Corresponden

Dimension

Applicatio

Computation Observe Fibr ■ Defining equations: $\mathbf{V}(x) \cup \mathbf{V}(y, z) = \mathbf{V}(xy, xz)$

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$



| Locus | Rank | Dimension |
|---|------|-----------|
| x = y = z = 0 | 0 | 3 |
| $x = 0$ and $(y \neq 0 \text{ or } z \neq 0)$ | 1 | 2 |
| $x \neq 0, y = z = 0$ | 2 | 1 |

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Physica Context

Mathematic

Basic Object

Ideal-Variety

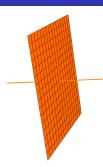
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Physica Context

Mathematic Tools

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Ideal-Variety

Corresponder

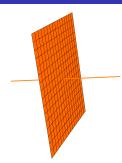
Dimension

Applications

Defining equations: V(x) + V(x, x) = V(x, x)

$$\mathbf{V}(x) \cup \mathbf{V}(y,z) = \mathbf{V}(xy,xz)$$

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} y & x & 0 \\ z & 0 & x \end{pmatrix}$$



| Locus | Rank | Dimension |
|---|------|-----------|
| x = y = z = 0 | 0 | 3 |
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Using Algebraic Geometry in F-Theory

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Physical Context

Mathematica Tools

Basic Objects

Ideal Variet

Corresponde

Decomposit

Application

Computations

Observe Fibre

Section 3

Application

Given Equations

Using Algebraic Geometry in F-Theory

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Physica Context

Mathematical

Racio Obioete

Ideal-Varie

Correspond

Dimension

Application Computations

Observe Fibre

$$0 = d_0 c_2^2 + b_0^2 c_1 - b_0 b_1 c_2$$

$$0 = d_1 b_0 c_2 - b_0^2 b_2 - c_2^2 d_2$$

$$0 = d_0 b_2 c_1 - b_0 b_2^2 - c_1^2 d_2$$

$$0 = d_1 c_1^2 - b_1 b_2 c_1 + b_2^2 c_2$$

$$0 = d_0 c_1^3 c_2^2 + b_0^2 c_1^4 - b_0 b_1 c_1^3 c_2 + c_2^3 (b_1 b_2 c_1 - b_2^2 c_2 - c_1^2 d_1)$$

$$0 = d_2 c_1^4 c_2^2 + (b_0 c_1^2 + c_2 (-b_1 c_1 + b_2 c_2)) \times$$

$$\times (b_0 b_2 c_1^2 + c_2 (-b_1 b_2 c_1 + b_2^2 c_2 + c_1^2 d_1))$$

Extract Matter Curves

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Physical Context

Mathematical Tools

Ideal-Variety Correspondent Decompositio

Dimension
Applicatio

Computations
Observe Fibre
Enhancement

```
singular.lib('primdec.lib');
singular.lib('sing.lib');
R = singular.ring(0, '(b0, b1, b2, c1, c2, d0, d1, d2)', '
   dp');
I1 = singular.ideal('d0*c2^2+b0^2*c1-b0*b1*c2','d1*
   b0*c2-b0^2*b2-c2^2*d2').std();
12 = singular.ideal('d0*b2*c1-b0*b2^2-c1^2*d2','d1*
   c1^2-b1*b2*c1+b2^2*c2').std();
13 = singular.ideal('d0*c1^3*c2^2-(-b0^2*c1^4+b0*b1*)
   c1^3*c2+c2^3*(-b1*b2*c1+b2^2*c2+c1^2*d1))','d2*
   c1^4*c2^2+(b0*c1^2+c2*(-b1*c1+b2*c2))*(b0*b2*c1
   ^2+c2*(-b1*b2*c1+b2^2*c2+c1^2*d1))').std();
# Extract the matter curves
I1pr = I1.minAssGTZ();
12pr = 12.minAssGTZ();
13pr = 13.minAssGTZ();
```

Matter Curves

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Physical Context

Mathematical Tools

Basic Objects
Ideal-Variety

Decomposition

Dimension

Application
Computations

Every pair of the above equations defines two matter curves.

```
# I1
Cl1 = I1pr[2].std(); # c2 = 0 = b0
Cl2 = I1pr[1].std(); # complicated

# I2
Cl3 = I2pr[2].std(); # b2 = 0 = c1
Cl4 = I2pr[1].std(); # complicated

# I3
Cl5 = I3pr[2].std(); # c2 = 0 = c1
Cl6 = I3pr[1].std(); # complicated
```

Allowed Yukawa Interactions

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Physica Context

Mathematica Tools

Tools

Ideal-Variety

correspond

Dimension

Application

Computations

$$\begin{array}{c} \mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)} \\ \mathbf{1}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)} \\ \mathbf{1}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}. \end{array}$$

```
inters1 = (CI1+CI4+CI5).std();
inters2 = (CI2+CI3+CI5).std();
inters3 = (CI2+CI4+CI6).std();
inters4 = (CI1+CI2+CI6).std();
inters5 = (CI3+CI4+CI6).std();
singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

Allowed Yukawa Interactions

Using Algebraic Geometry in F-Theory

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Physica Context

Mathematica

Tools

Basic Objects

Decomposit

Dimension

Application

Computations
Observe Fibre

```
inters1 = (CI1+CI4+CI5).std();
inters2 = (CI2+CI3+CI5).std();
inters3 = (CI2+CI4+CI6).std();
inters4 = (CI1+CI2+CI6).std();
inters5 = (CI3+CI4+CI6).std();
singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

Compute Codimension

```
Using
Algebraic
Geometry in
F-Theory
```

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Physical Context

Mathematical Tools

Ideal-Variety Correspondence Decomposition Dimension

Application
Computations
Observe Fibre
Enhancement

```
inters1pr = inters1.minAssGTZ();
inters2pr = inters2.minAssGTZ();
inters3pr = inters3.minAssGTZ();
inters4pr = inters4.minAssGTZ();
inters5pr = inters5.minAssGTZ();
inters6pr = inters6.minAssGTZ();
inters61pr = inters61.minAssGTZ();
inters1pr[1].std().dim() \# -> codim 3
inters2pr[1].std().dim() # -> codim 3
inters3pr[1].std().dim() \# \rightarrow codim 3
inters3pr[2].std().dim() \# -> codim 4 \# wrong codim
inters4pr[1].std().dim() \# \rightarrow codim 3
inters5pr[1].std().dim() \# \rightarrow codim 3
inters6pr[1].std().dim() # -> codim 3
inters61pr[1].std().dim() \# \rightarrow codim 3
inters61pr[2].std().dim() \# -> codim 4 \# wrong codim
```

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Physica Context

Tools

Basic Objects

Ideal-Variety

Decompositi

Application

Observe Fibre

Hypersurface equation:

$$P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0.$$

- Consider e.g. $\mathbf{1}^{(1)}$: $c_2 = 0 = b_0$.
- Then: 2-fold factorization

$$P_{T^2}|_{\mathbf{1}^{(1)}} \equiv P_{T^2}|_{b_0=0, c_2=0}$$

= $s_1(d_2s_0^2s_1u^3 + d_0s_0^2u^2v + \ldots)$

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Physica Contex

Tools

Basic Object

Ideal-Variety

Decomposition

Applicatio

Observe Fibre

Hypersurface equation:

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Using Algebraic Geometry in F-Theory

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Physical Context

Mathematica Tools

Tools

Basic Objects

Corresponden

Decompositio

Computations
Observe Fibre

Hypersurface equation:

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Using Algebraic Geometry in F-Theory

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Context

Mathematical Tools

Basic Objects

Corresponder Decomposition

Decompositio Dimension

Computations
Observe Fibre

Hypersurface equation:

$$P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0.$$

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= $s_1(d_2s_0^2s_1u^3 + d_0s_0^2u^2v + \ldots).$

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Physica Contex

Tools

basic Objec

Corresponde

Decomposit Dimension

Application Computations

Consider e.g. $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$: $c_2 = 0 = b_0$ and $d_1 = \frac{b_1 b_2 d_0 + b_1 c_1 d_2 \pm \sqrt{b_1^2 - 4c_1 d_0}(b_2 d_0 - c_1 d_2)}{2c_1 d_0}$.

■ Then: 3-fold factorization

$$P_{T^2}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4c_1d_0} s_1 \left[\left(b_1 \pm \sqrt{b_1^2 - 4c_1d_0} \right) s_0 u + 2c_1 w \right] \times \left[\left(b_1 \mp \sqrt{b_1^2 - 4c_1d_0} \right) d_2 s_0 s_1 u^2 + \dots \right]$$

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Physical Context

Mathematica Tools

Basic Objects

Corresponde Decompositi

Application
Computations
Observe Fibre

■ Consider e.g. $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$: $c_2 = 0 = b_0$ and $d_1 = \frac{b_1b_2d_0 + b_1c_1d_2 \pm \sqrt{b_1^2 - 4c_1d_0}(b_2d_0 - c_1d_2)}{2c_1d_0}$.

■ Then: 3-fold factorization

$$P_{T^{2}}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4c_{1}d_{0}}s_{1}\left[\left(b_{1} \pm \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)s_{0}u + 2c_{1}w\right] \times \left[\left(b_{1} \mp \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)d_{2}s_{0}s_{1}u^{2} + \ldots\right]$$

Using Algebraic Geometry in F-Theory

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Physical Context

Mathematical Tools

Basic Objects Ideal-Variety Correspondence

Decompositio Dimension

Application

Computations

Observe Fibre

Consider e.g. $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$: $c_2 = 0 = b_0$ and $d_1 = \frac{b_1 b_2 d_0 + b_1 c_1 d_2 \pm \sqrt{b_1^2 - 4c_1 d_0}(b_2 d_0 - c_1 d_2)}{2c_1 d_0}$.

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$$P_{T^{2}}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} = \frac{1}{4c_{1}d_{0}}s_{1}\left[\left(b_{1} \pm \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)s_{0}u + 2c_{1}w\right] \times \left[\left(b_{1} \mp \sqrt{b_{1}^{2} - 4c_{1}d_{0}}\right)d_{2}s_{0}s_{1}u^{2} + \ldots\right]$$

Using Algebraic Geometry in F-Theory

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Physica Context

Mathematica Tools

Basic Objects

Corresponder
Decomposition
Dimension

Computations
Observe Fibre
Enhancement

| Locus | Physical content | $\dim_{\mathbb{C}}$ |
|--|--------------------|---------------------|
| Torus degenerates into one \mathbb{P}^1 | 7-branes | 2 |
| 7-branes intersect $/$ degeneration into two \mathbb{P}^1 s | matter curves | 1 |
| matter curves intersect $/$ degeneration into three \mathbb{P}^1 s | intersection locus | 0 |



Using Algebraic Geometry in F-Theory

Philipp Arra

Physica Context

Mathematica Tools

Basic Objects

Corresponder
Decomposition
Dimension

Computations
Observe Fibre
Enhancement

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