Technical Report with Proofs for A Full Picture in Conformance Checking: Efficiently Summarizing All Optimal Alignments

4 Summarizing Alignments with Skip Alignments

4.1 Skip Alignments

Lemma 7 (Finite). *A trace and an SBWF-net have finitely many skip alignments.*

Proof. Assume a trace σ has infinitely many skip alignments on an SBWF-net N. The number of synchronous and log moves is limited by $|\sigma|$. Since N is finite, only finitely many different skip moves exist. Hence, for every $n > 0$ there must exist a skip alignment δ such that $|\delta| > n$. Only loop blocks in N can contribute to an unbounded length of δ. Therefore, from some $m > 0$ on, all skip alignments δ with $|\delta| > n$ and $n \geq m$ execute at least one loop block $N' \in N$ by consecutively skipping over both its children, contradicting the definition of $\mathcal{S}(N')$.

4.2 Transforming Alignments to Skip Alignments

Lemma 9 (Transformation to Skip Alignments). *Let σ be a trace and N an SBWF-net. Let γ be an alignment for σ on N and let γ' be the mix alignment obtained by replacing all model moves in γ with skip moves. Then, the exhaustive application of transformation rules to γ' results in a skip alignment δ for σ on N.*

Proof. Every move in δ is a log, synchronous, or skip move. It holds that $\pi_1(\delta)_{\downharpoonleft \neq \gg} = \pi_1(\gamma)_{\downharpoonleft \neq \gg} = \sigma$ since only skip moves are modified in the transformation. Furthermore, because γ is an alignment on N, $\pi_2(\gamma)_{\downharpoonleft \neq \gg}$ and $\pi_2(\gamma')_{\downharpoonleft \neq \gg}$ are executions of N. Cycling rules trivially maintain that property. Lifting rules replace moves in γ' only at the positions their children where executed at, i.e., the application of any rule ensures that $\pi_2(\delta)_{\downharpoonleft \neq \gg}$ is an execution of N. In δ, every skip move is most general, otherwise one of (L1-L4) could be applied and δ contains no cycles, otherwise (C1-C2) could be applied. Hence, $\pi_2(\delta)_{\downharpoonleft \neq \gg} \in \mathcal{S}(N)$ and δ is a skip alignment for σ on N.

4.4 A Canonical Normal Form for Skip Alignments

We show that reduction rules may be applied in any order resulting in the same unique skip alignment (*canonicity*). To that end, we first prove *termination* and then *local confluency*, i.e., when applying two rules yields two different skip alignments, then there are subsequent reduction rules to make them equal again.

Proposition 1 (Termination of Reduction Rules). *Let δ be a skip alignment. Repeated application of the reduction rules to δ is terminating.*

Proof. We give a weighting function that monotonically decreases with every application of a rule. With every application of (R1) or (R2), the size of the set of pairs $t_1 = \{(i,j) \mid 1 \leq i < j \leq |\delta| \wedge \delta_i = \left(\begin{smallmatrix}\gg\\ a\end{smallmatrix}\right) \wedge \delta_j \in \{\left(\begin{smallmatrix}e\\ \gg\end{smallmatrix}\right), \left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right)\}\}$ decreases. None of (R1-R3) increases the size of this set again. With every application of (R3), the size of the set of pairs $t_2 = \{(i,j) \mid 1 \leq i < j \leq |\delta| \wedge \delta_i = \left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right) \wedge \delta_j = \left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right) \wedge N_2 < N_1\}$ decreases. Only the application of (R2) may increase t_2 by at most $|\delta|$ elements. Hence, $(|\delta|+1) \cdot |t_1| + |t_2|$ is monotonically decreasing with every rule application.

We prove local confluency for every pair of rules where the left sides overlap. We show that after applying one of the rules the resulting skip alignment can be made equal to the one after applying the other rule.

Proposition 2 (Local Confluency of Reduction Rules). *The reduction rules are locally confluent.*

Proof. We only inspect moves where there can be overlapping rules, i.e., (R2,R3) and (R2,R2). Let $\delta = \langle \text{—} \left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right) \rightsquigarrow \left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right), \left(\begin{smallmatrix}\gg\\ s(N_3)\end{smallmatrix}\right) \rightsquigarrow \left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right) \text{—} \rangle$. We show that independent of the order in which we apply (R2,R3) to $\langle \left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right), \left(\begin{smallmatrix}\gg\\ s(N_3)\end{smallmatrix}\right) \rangle$ or (R2,R2) to $\langle \left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right) \rightsquigarrow \left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right) \rangle$, the resulting skip alignments can be made equal again.

- First applying (R2) pushing $\left(\begin{smallmatrix}\gg\\ s(N_3)\end{smallmatrix}\right)$ over $\left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right)$ is equal to swapping $\left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right)$ with $\left(\begin{smallmatrix}\gg\\ s(N_3)\end{smallmatrix}\right)$ by (R3) and shifting back $\left(\begin{smallmatrix}\gg\\ s(N_3)\end{smallmatrix}\right)$ afterwards with (R2).
- First applying (R2) to $\left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right)$ pushing the move after $\left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right)$ and then applying (R2) to $\left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right)$ is equal to applying (R2) first to $\left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right)$ and then to $\left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right)$, followed by an additional application of (R3) to the order $\langle \left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right), \left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right), \left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right) \rangle$ of the first (R2,R2) if the constraint $[N_2 < N_1]$ holds, otherwise we apply (R3) to the result $\langle \left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right), \left(\begin{smallmatrix}\gg\\ s(N_2)\end{smallmatrix}\right), \left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right) \rangle$ of the second (R2,R2).

Canonicity follows from both propositions together with Newman's Lemma [1].

Lemma 13 (Unique Coinciding Normal Form). *Every alignment coincides with a single skip alignment in normal form.*

Proof. We show that for an alignment γ the set of corresponding skip alignments $\mathcal{C}(\gamma)$ coincides with a single normal form. Let δ and $\hat{\delta}$ in $\mathcal{C}(\gamma)$ be corresponding skip alignments of γ. With canonicity, they reduce to coinciding normal forms δ' and $\hat{\delta}'$. Assume $\delta' \neq \hat{\delta}'$. Both normal forms share the same moves, so their order must be different. At the position i of their first difference, one of δ' and $\hat{\delta}'$ has to perform a skip move. W.l.o.g., we assume $\delta'_i = \left(\begin{smallmatrix}\gg\\ s(N_1)\end{smallmatrix}\right)$.

- If $\hat{\delta}'_i = \left(\begin{smallmatrix}e\\ \gg\end{smallmatrix}\right)$ performs a log move, then this move appears at a position δ'_j with $j > i$. Because no synchronous move can take place in between positions i and j in δ', (R1) can be applied contradicting the normal form.
- If $\hat{\delta}'_i = \left(\begin{smallmatrix}e\\ a\end{smallmatrix}\right)$ performs a synchronous move, then the skip move δ'_i is not required before performing the synchronous move in $\hat{\delta}'$, so (R2) can be applied to δ' contradicting the normal form.

- If $\hat{\delta}'_i = \binom{\gg}{s(N_2)}$ performs a skip move, then this move is performed at a position $j > i$ in δ'. If there is a synchronous move in δ' in between i and j, then $\binom{\gg}{s(N_2)}$ is independent from the synchronous move, i.e., (R2) can be applied to $\hat{\delta}'$ contradicting the formal form. If there is a log move in δ' in between i and j, then (R1) can be applied to δ' contradicting to the normal form. Otherwise, there are only skip moves in between positions i and j. Similarly, only skip moves appear in between $\binom{\gg}{s(N_2)}$ and $\binom{\gg}{s(N_1)}$ in $\hat{\delta}'$. As both orders are valid (both are skip alignments for the same trace), N_1 and N_2 must be descendants in different branches of a parallel block. Hence, the order of $\binom{\gg}{s(N_2)}$ and $\binom{\gg}{s(N_1)}$ depends on the total order of blocks, i.e., (R3) can be applied to one of δ' and $\hat{\delta}'$ contradicting the normal form.

References

1. Newman, M.H.A.: On theories with a combinatorial definition of "equivalence". Annals of Mathematics (1942)