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Analysis of Dirichlet and Generalized "Hamming" window functions in the fractional Fourier transform domains

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ABSTRACT

A new mathematical model for obtaining the fractional Fourier transforms of Dirichlet and Generalized "Hamming" window functions is presented. The different parameters for the window functions are also obtained with the help of simulations. The fractional Fourier transformation contains an adjustable parameter with which the main lobe width and correspondingly, the minimum stop band attenuation of the resulting window function can be controlled.

1. Introduction

Window functions are used in harmonic analysis to reduce the undesirable effects related to the spectral leakage. They impact on many attributes of a harmonic processor which include detestability, resolution, dynamic range, confidence and ease of implementation [1]. Several standard windows are also used to optimize the requirements of a particular application in signal processing.

Window functions have been successfully used in various areas of signal processing and communications such as, spectrum estimation, speech processing, digital filter design, and in other related fields such as, beamforming. A complete review of many window functions and their properties was presented by Harris [2]. All window functions are designed to reduce the side lobes of the spectral output of Fast Fourier transform (FFT) routines. Whilst applying the window function reduces the side lobe leakage, it causes the main lobe to broaden thus, reducing the resolution. This is a trade-off that has to be made, one should choose the window function, which best suits the application.

Recently, the fractional Fourier transform (FrFT) has been developed and utilized by a number of researchers, and being used in almost all applications where Fourier transforms were used. For example, the FrFT has been applied to optimal Wiener filtering and matched filtering [3]. Applications of FrFT have also been described by Bailey and Swartztrauber et al. [4]. Stankovic et al. [5] have used windowed FrFT for the analysis of non-stationary signals. Also, Sharma et al. [6] have carried out Kaiser and Parzen-cos$^6(\pi t)$ (PC6) window function analysis in fractional Fourier domain to show the dependence of window main-lobe width on the order of FrFT and also an alternate methodology is described to tune FIR filter transition bandwidth based on FrFT. In this paper, the FrFT analysis of Dirichlet and Generalized "Hamming" window functions has been carried out for different values of the FrFT angle α or FrFT order a, both of which are related by $\alpha = a\pi/2$. An attempt has also been made to study the variations of the parameters Half Main Lobe Width (HMLW), Maximum Side Lobe Level (MSLL)

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and Side-lobe fall-off rate (SLFOR) of these window functions with the variation of the parameter a. It is found that with the adjustment of parameter a to different values, these window functions can attain a maximum main lobe width and SLFOR.

The rest of the paper is organized as follows. Section 2 gives an overview of the fractional Fourier transform. An overview of the window functions and its parameters are discussed in Section 3. In Sections 4 and 5, the mathematical model of Dirichlet and Generalized "Hamming" window functions has been derived using fractional Fourier transformation technique. Experimental results are presented in Section 6. The conclusive remarks are presented in Section 7.

2. The fractional Fourier transform

The Fourier transform (FT) is undoubtedly one of the most valuable and frequently used tools in signal processing and analysis [7–9]. Little need be said of the importance and ubiquity of the ordinary FT in many areas of science and engineering. The fractional Fourier transform (FrFT) has been found to have several applications in the areas of optics and signal processing [10–13]. It also leads to the generalization of notion of space (or time) and frequency domains, which are central concepts of signal processing [14–16].

The FrFT is a generalization of the conventional FT, which is richer in theory, flexible in application, and implementation cost is at par with FT. With the advent of FrFT and the related concept, it is seen that the properties and applications of the conventional FT are special cases of those of the FrFT. However, in every area where FT and frequency domain concepts are used, there exists the potential for generalization and implementation by using FrFT.

The continuous-time fractional Fourier transform of a signal $x(t)$ is defined via an integral [10]

$$X_\alpha(u) = \int_{-\infty}^{\infty} x(t) K_\alpha(t,u) dt \tag{1}$$

where the transformation kernel $K_\alpha(t,u)$ of the FrFT is given by

$$K_\alpha(t,u) = \begin{cases} \sqrt{\dfrac{1-i\cot(\alpha)}{2\pi}} \exp\left[i\left(\dfrac{t^2+u^2}{2}\right)\cot(\alpha) - iut\,\mathrm{cosec}(\alpha)\right] & \text{if } \alpha \text{ is not a multiple of } \pi, \\ \delta(t-u) & \text{if } \alpha \text{ is a multiple of } 2\pi, \\ \delta(t+u) & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi. \end{cases} \tag{2}$$

where α indicates the rotation angle of the transformed signal for the FrFT.

The FrFT with $\alpha=\pi/2$ corresponds to the conventional Fourier transform, and the one with $\alpha=0$ corresponds to the identity operator. Also, two successive FrFT's with angles α and β are equivalent to a single FrFT with an angle $(\alpha+\beta)$. Hence, the properties of the conventional Fourier transform can be obtained by substituting $\alpha=\pi/2$ in the properties of FrFT.

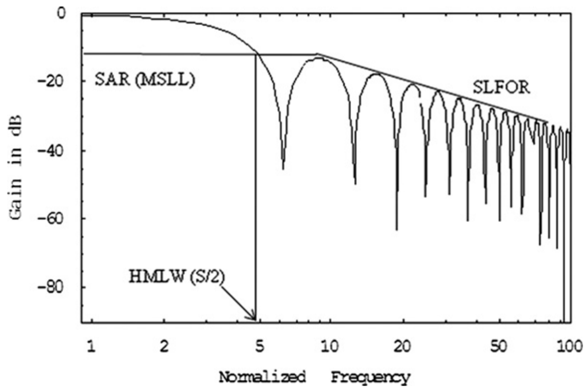

Fig. 1. Log magnitude plot of Dirichlet window to illustrate the definition of the parameters HMLW (S/2), MSLL (SAR) and SLFOR.

3. Window function and its parameters

Window functions are widely used in digital signal processing for the applications in signal analysis and estimation, digital filter design and speech processing [7]. In literature many windows have been proposed [8,9]. The common properties of the window functions can be summarized as follows.

- They are real, even, nonnegative and time-limited.
- Their Fourier transforms have main lobe at the origin and side lobes at both sides. These side lobes are decaying with asymptotic attenuation of f^{-n} as $f \to \infty$ where n is an integer [1].

The parameters of window functions which are generally used for its evaluation are [17–21]:

- **Maximum Side Lobe Level (MSLL):** This is the peak ripple value of the side lobes and it is evaluated from the log magnitude plot of transformed window. This is also known as selectivity amplitude ratio (SAR).
- **Selectivity (S/2) or Half Main Lobe Width (HMLW):** This is the frequency at which the Main Lobe drops to the peak ripple value of the side lobes. For convenience half main lobe width (HMLW) or S/2 is computed.
- **Side-lobe fall-off rate (SLFOR):** This is the asymptotic decay rate of the side lobe level. This is also called asymptotic attenuation. Other parameters associated with the window functions are Equivalent Noise Bandwidth (ENBW) and Scalloping Losses (SL), which are out of scope of this paper.

Fig. 1 presents a plot of the aforementioned parameters for the Dirichlet window function.

4. Dirichlet window function

The mathematical analyses of Dirichlet window function in the fractional Fourier domain is carried in the following section. Without loss of generality, let $w(t)$ be unity at the origin, and time-limited to the interval $|t| \le 1/2$, i.e.,

$$w(t)=\begin{cases}1 & |t|\le 1/2\\ 0 & elsewhere\end{cases} \tag{3}$$

Therefore, the FrFT of $w(t)$ can be written as

$$W_\alpha(u)=\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}1\exp\left[i\frac{t^2}{2}\cot(\alpha)-iut\,\mathrm{cosec}(\alpha)\right]dt \tag{4}$$

Now, rewriting the integral of (4),

$$W_\alpha(u)=\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}\exp\left[\frac{i}{2}\cot(\alpha)\left\{(t-u\sec(\alpha))^2-(u\sec(\alpha))^2\right\}\right]dt \tag{5}$$

By substituting $(t-u\sec(\alpha))=R$ and changing the limits of the integration in (5),

$$W_\alpha(u)=\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(-\frac{i}{2}u^2\tan(\alpha)\right)\times\int_{-(1/2)-u\sec(\alpha)}^{(1/2)-u\sec(\alpha)}\exp\left[\left(\frac{i}{2}\cot(\alpha)\right)R^2\right]dR \tag{6}$$

Now, solving the integral $\int_{-(1/2)-u\sec(\alpha)}^{(1/2)-u\sec(\alpha)}\exp\left[\left(\frac{i}{2}\cot(\alpha)\right)R^2\right]dR$ the following expression results [22],

$$\int_{-(1/2)-u\sec(\alpha)}^{(1/2)-u\sec(\alpha)}\exp\left[\left(\frac{i}{2}\cot(\alpha)\right)R^2\right]dR=-\frac{\sqrt{\pi}}{2}\left\{erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2}-u\sec(\alpha)\right)\right]-erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2}-u\sec(\alpha)\right)\right]\right\} \tag{7}$$

where *erfi* (z) is an entire analytical function of z which is defined in the whole complex z-plane.

By rearranging (6) and (7),

$$W_\alpha(u)=-\sqrt{\frac{1-i\cot(\alpha)}{8}}\exp\left(-\frac{i}{2}u^2\tan(\alpha)\right)\times\left\{erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2}-u\sec(\alpha)\right)\right]-erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2}-u\sec(\alpha)\right)\right]\right\} \tag{8}$$

Thus, from (8), it can be seen that the FrFT of Dirichlet window function is directly dependent on the FrFT angle α.

5. Generalized "Hamming" window function

The expression for the Generalized "Hamming" window function in time domain is given as [18]

$$w(t)=\begin{cases}\beta+(1-\beta)\cos(2\pi t) & |t|\le 1/2\\ 0 & elsewhere\end{cases} \tag{9}$$

For β=0.50, Hanning window results and for β=0.54, Hamming window results.

Rewriting (9) in Euler's form, one gets

$$w(t)=\beta+(1-\beta)\left(\frac{e^{i2\pi t}+e^{-i2\pi t}}{2}\right)$$

or,

$$w(t)=\beta+\left(\frac{1-\beta}{2}\right)(e^{i2\pi t}+e^{-i2\pi t}) \tag{10}$$

Now, because $w(t)\overset{\alpha}{\leftrightarrow}W_\alpha(u)$

Therefore,

$$W_\alpha(u)=\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}w(t)\exp\left[i\frac{t^2}{2}\cot(\alpha)-iut\,\mathrm{cosec}(\alpha)\right]dt \tag{11}$$

$$\text{i.e.,}W_\alpha(u)=\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}\left[\beta+\left(\frac{1-\beta}{2}\right)(e^{i2\pi t}+e^{-i2\pi t})\right]\times exp\left[i\frac{t^2}{2}\cot(\alpha)-iut\,\mathrm{cosec}(\alpha)\right]dt \tag{12}$$

Rewriting the integral in (12),

$$\begin{aligned}W_\alpha(u)=&\beta\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}\exp\left[i\frac{t^2}{2}\cot(\alpha)-iut\,\mathrm{cosec}(\alpha)\right]dt\\&+\frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}\exp\left[i\frac{t^2}{2}\cot(\alpha)+it(2\pi-u\,\mathrm{cosec}(\alpha))\right]dt\\&+\frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}\exp\left[i\frac{t^2}{2}\cot(\alpha)-it(2\pi+u\,\mathrm{cosec}(\alpha))\right]dt\end{aligned} \tag{13}$$

Eq. (13) can be rewritten as

$$W_\alpha(u)=I_1+I_2+I_3 \tag{14}$$

Here

$$I_1=\beta\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)\times\int_{-1/2}^{1/2}\exp\left[i\frac{t^2}{2}\cot(\alpha)-iut\,\mathrm{cosec}(\alpha)\right]dt \tag{15}$$

$$I_2=\frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)$$

$$\times \int_{-1/2}^{1/2} \exp\left[i\frac{t^2}{2}\cot(\alpha) + it(2\pi - u\,\mathrm{cosec}(\alpha))\right] dt \tag{16}$$

$$I_3 = \frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)$$
$$\times \int_{-1/2}^{1/2} \exp\left[i\frac{t^2}{2}\cot(\alpha) - it(2\pi + u\,\mathrm{cosec}(\alpha))\right] dt \tag{17}$$

Now, solving (15) for I_1,

$$I_1 = \beta\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)$$
$$\times \int_{-1/2}^{1/2} \exp\left[i\frac{t^2}{2}\cot(\alpha) - iut\,\mathrm{cosec}(\alpha)\right] dt \tag{18}$$

Solving (18) in the same manner as (4),

$$I_1 = -\beta\sqrt{\frac{1-i\cot(\alpha)}{8}}e^{-\frac{i}{2}u^2\tan(\alpha)}$$
$$\times \left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} - u\sec(\alpha)\right)\right]\right\} \tag{19}$$

Now, solving (16) for I_2,

$$I_2 = \frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)$$
$$\times \int_{-1/2}^{1/2} \exp\left[i\frac{t^2}{2}\cot(\alpha) + it(2\pi - u\,\mathrm{cosec}(\alpha))\right] dt$$

$$I_2 = \frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}e^{i/2\cot(\alpha)[u^2 - \{2\pi\tan(\alpha) - u\sec(\alpha)\}^2]}$$
$$\times \int_{-1/2}^{1/2} e^{i/2\cot(\alpha)[t + \{2\pi\tan(\alpha) - u\sec(\alpha)\}]^2} dt \tag{20}$$

Solving the integral $\int_{-1/2}^{1/2} e^{\frac{i}{2}\cot(\alpha)[t + \{2\pi\tan(\alpha) - u\sec(\alpha)\}]^2} dt$, following expression results

$$\int_{-1/2}^{1/2} e^{i/2\cot(\alpha)[t + \{2\pi\tan(\alpha) - u\sec(\alpha)\}]^2} dt$$
$$= -\frac{\sqrt{\pi}}{2}\left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} + 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} + 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right]\right\} \tag{21}$$

Now, rearranging (20) and (21),

$$I_2 = -(1-\beta)\sqrt{\frac{1-i\cot(\alpha)}{32}}e^{i/2\cot(\alpha)[u^2 - \{2\pi\tan(\alpha) - u\sec(\alpha)\}^2]}$$
$$\times \left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} + 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} + 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right]\right\} \tag{22}$$

Now, solving (17) for I_3,

$$I_3 = \frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}\exp\left(i\frac{u^2}{2}\cot(\alpha)\right)$$
$$\times \int_{-1/2}^{1/2} \exp\left[i\frac{t^2}{2}\cot(\alpha) - it(2\pi + u\,\mathrm{cosec}(\alpha))\right] dt \tag{23}$$

$$I_3 = \frac{(1-\beta)}{2}\sqrt{\frac{1-i\cot(\alpha)}{2\pi}}e^{i/2\cot(\alpha)[u^2 - \{2\pi\tan(\alpha) + u\sec(\alpha)\}^2]}$$
$$\times \int_{-1/2}^{1/2} e^{i/2\cot(\alpha)[t - \{2\pi\tan(\alpha) + u\sec(\alpha)\}]^2} dt \tag{24}$$

Solving the integral $\int_{-1/2}^{1/2} e^{\frac{i}{2}\cot(\alpha)[t - \{2\pi\tan(\alpha) + u\sec(\alpha)\}]^2} dt$, following expression results:

$$\int_{-1/2}^{1/2} e^{\frac{i}{2}\cot(\alpha)[t - \{2\pi\tan(\alpha) - u\sec(\alpha)\}]^2} dt$$
$$= -\frac{\sqrt{\pi}}{2}\left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} - 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} - 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right]\right\} \tag{25}$$

Now, rearranging (24) and (25),

$$I_3 = -(1-\beta)\sqrt{\frac{1-i\cot(\alpha)}{32}}e^{\frac{i}{2}\cot(\alpha)\left[u^2 - \{2\pi\tan(\alpha) + u\sec(\alpha)\}^2\right]}$$
$$\times \left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} - 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} - 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right]\right\} \tag{26}$$

Thus, the FrFT of the Generalized "Hamming" window function can be obtained by summing (19), (22), and (26) as

$$W_\alpha(u) = -\beta\sqrt{\frac{1-i\cot(\alpha)}{8}}e^{-i/2u^2\tan(\alpha)}$$
$$\times \left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} - u\sec(\alpha)\right)\right]\right\}$$
$$-(1-\beta)\sqrt{\frac{1-i\cot(\alpha)}{32}}e^{i/2\cot(\alpha)[u^2 - \{2\pi\tan(\alpha) - u\sec(\alpha)\}^2]}$$
$$\times \left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} + 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} + 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right]\right\}$$
$$-(1-\beta)\sqrt{\frac{1-i\cot(\alpha)}{32}}e^{i/2\cot(\alpha)[u^2 - \{2\pi\tan(\alpha) + u\sec(\alpha)\}^2]}$$
$$\times \left\{ erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(\frac{1}{2} - 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right] - erfi\left[\frac{(1+i)}{2}\sqrt{\cot(\alpha)}\left(-\frac{1}{2} - 2\pi\tan(\alpha) - u\sec(\alpha)\right)\right]\right\} \tag{27}$$

Thus, the FrFT of the Generalized "Hamming" window function, as given by (27) is dependent on the FrFT angle α. Also, for β=0.54 and 0.50, one obtains the FrFT of Hamming window function and FrFT of Hanning window function, respectively.

6. Simulation results

Dirichlet and Hanning window functions are shown in Fig. 2. The continuum of Dirichlet and Hanning window function to sinc pulse, as the parameter a is varied from 0 to 1 is shown in Figs. 3 and 4, respectively. The plots for

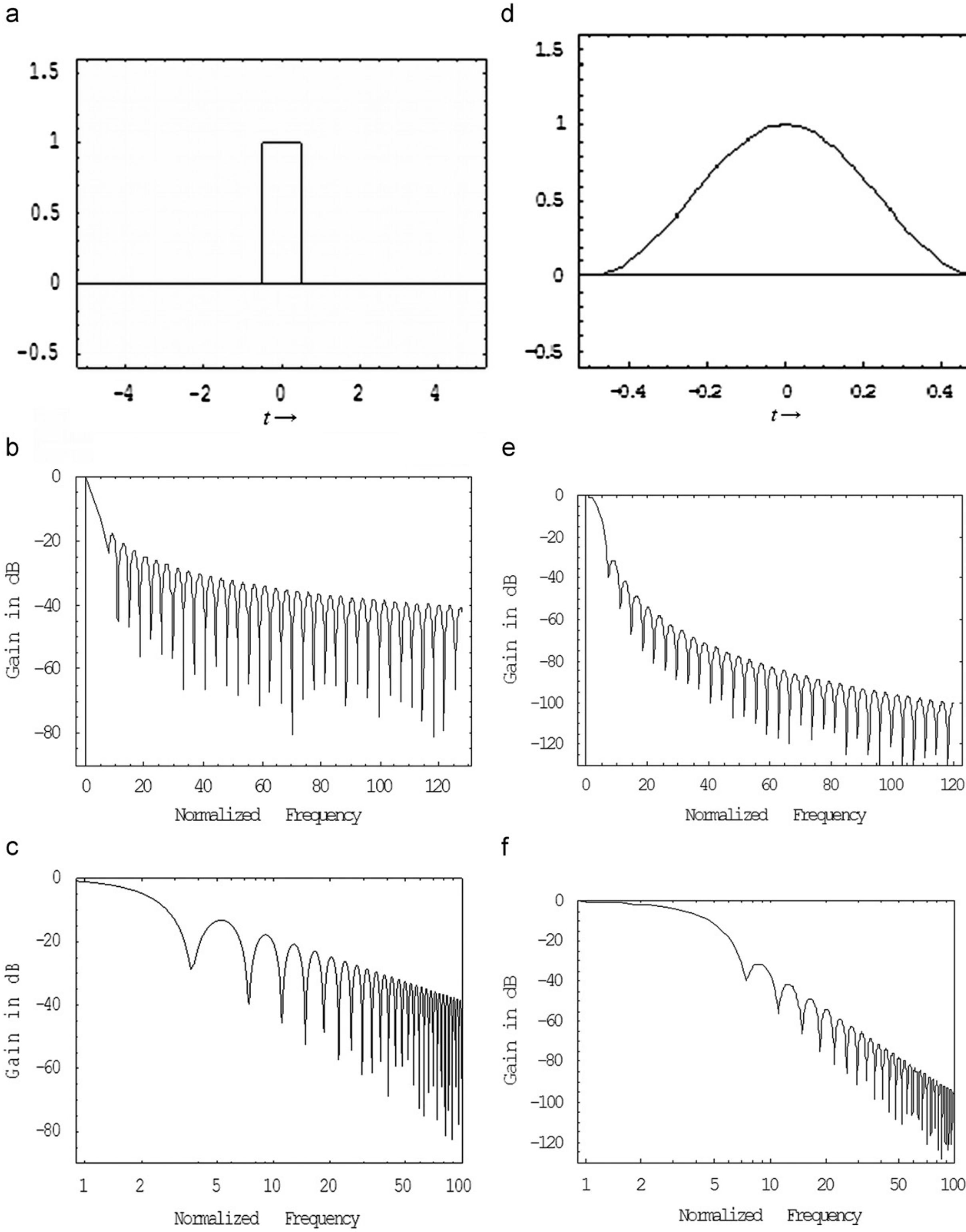

Fig. 2. MSLL and SLFOR plots for Dirichlet and Hanning window functions (for β=0.5) at a=0.4.

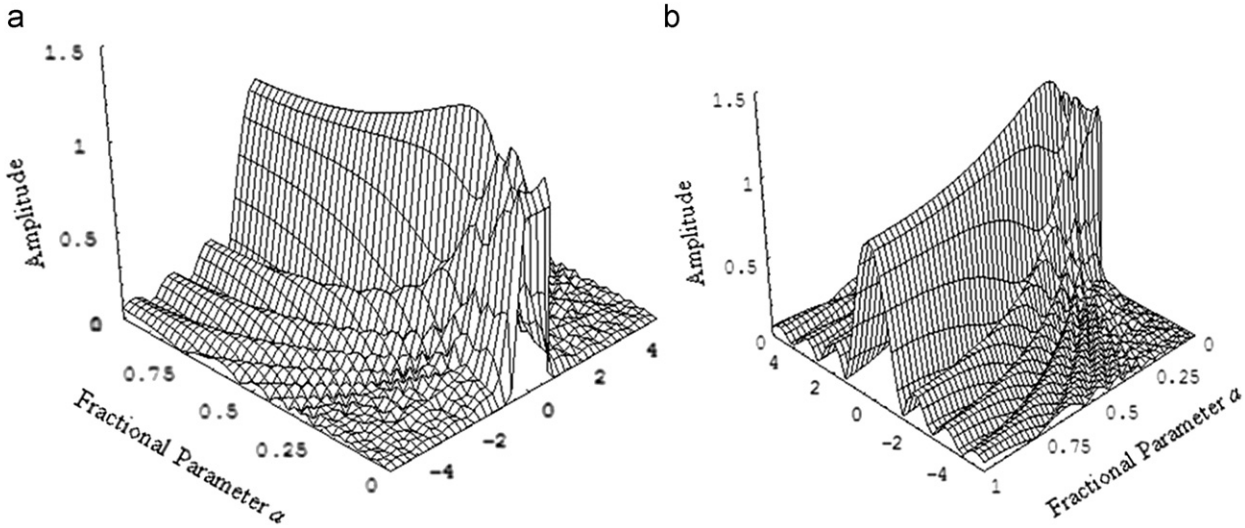

Fig. 3. The continuum of fractional Fourier transform of Dirichlet window function.

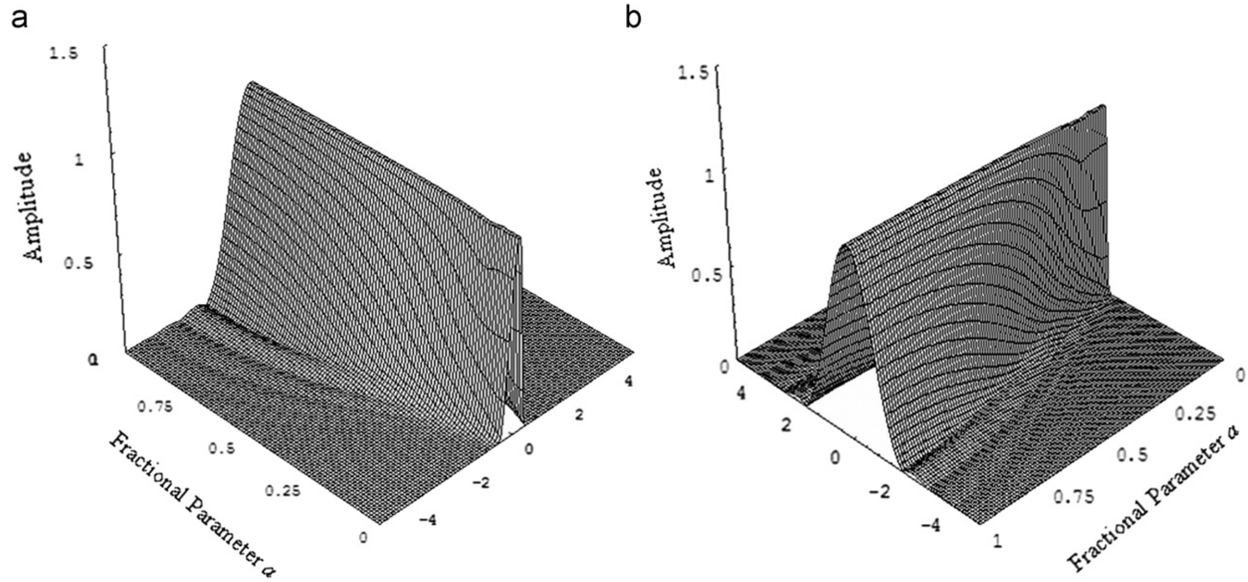

Fig. 4. The continuum of fractional Fourier transform of Hanning window function.

Table 1
Parameters of Dirichlet window function with variations in parameter *a*.

Sl. no.	*a*	MSLL(dB)	HMLW(bins)	SLFOR(dB/octave)
1	1.0	−13.0	0.81	−6.00
2	0.9	−12.9	0.80	−6.20
3	0.8	−12.8	0.78	−6.20
4	0.7	−12.7	0.76	−6.25
5	0.6	−12.5	0.75	−6.29
6	0.5	−11.9	0.74	−6.54
7	0.4	−11.7	0.72	−6.69
8	0.3	−11.3	0.70	−6.75
9	0.2	−11.2	0.65	−10.30

Table 2
Parameters of Hanning window function (for β=0.5) with variations in parameter *a*.

Sl. no.	*a*	MSLL(dB)	HMLW(bins)	SLFOR(dB/octave)
1	1.0	−32.0	1.87	−18.00
2	0.9	−30.7	1.77	−18.00
3	0.8	−30.5	1.71	−18.10
4	0.7	−30.5	1.61	−18.17
5	0.6	−30.4	1.51	−18.20
6	0.5	−30.2	1.50	−18.30
7	0.4	−29.9	1.44	−18.32
8	0.3	−29.6	1.36	−18.35
9	0.2	−40.5	1.22	−17.03

calculating MSLL and SLFOR for Dirichlet and Hanning window functions for particular value of the parameter a is also shown in Fig. 2, respectively. The values of MSLL, SLFOR and HMLW for Dirichlet and Hanning window functions are tabulated in Table 1 and Table 2, respectively, for various values of the parameter a. From these figures and Tables, it is clear that at $a=1$, MSLL of Dirichlet window function is -13 dB down from the main lobe peak and the side lobes fall off rate is -6 dB/octave whereas, for Hanning window function, it is -32 dB down from the main lobe peak and the side lobes fall off rate is -18 dB/octave as in the case of FT.

Thus, it can be observed that for Dirichlet window function, as the value of the parameter a decreases from 1 to 0, the MSLL starts increasing upto -11.2 dB at 0.2, HMLW decreases from 0.81 bins to 0.65 bins and SLFOR decreases from -6 dB/octave to -10.3 dB/octave, whereas, in the case of Hanning window function (for $\beta=0.50$), as the value of the parameter a decreases from 1 to 0, the MSLL starts increasing upto -40.5 dB at 0.2, HMLW decreases from 1.87 bins to 1.22 bins and SLFOR decreases from -18 dB/octave to -17.03 dB/octave.

7. Conclusions

Based on the fractional Fourier transformation techniques, the mathematical analyses of Dirichlet and Generalized "Hamming" window functions have been carried out. It can be shown that the FrFT of Dirichlet and Generalized "Hamming" window functions are directly dependent on the fractional Fourier transform angle α as given by (8) and (27), respectively.

It can also be shown through simulations, that the increasing value of the parameter a reduces the side lobe levels, which in turn broadens the main lobe width, thus reducing resolution. It can further be concluded that for Dirichlet window function, as the parameter a is increased, MSLL, HMLW and SLFOR starts increasing. Similarly, for Hanning window function (for $\beta=0.50$), MSLL, HMLW and SLFOR starts increasing with increasing value of the parameter a. Also, it can be seen that below the value of the parameter $a=0.2$, it is impossible to detect the variations in the window parameters, as evident from Figs. 3 and 4, respectively. This is because of oscillations that occur at around $a=0.2$ and also it can be seen that all the lobes are merging to the main lobe.

Thus, this study reveals that there is a variation in the window function parameters with the variation in the FrFT parameter a and a best optimal solution can be obtained for the variety of practical applications such as, in image compressions. Efforts have also been made to choose the most convenient parameter adjustment to reduce the side lobe effect and to increase the intensity of the main lobe. Also, the results discussed in the above techniques can be beneficial to reduce the undesirable effects of the spectral leakage.

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