Reinforcement learning

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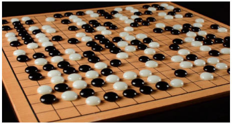
Scool

This presentation:
`https://philippe-preux.github.io/talks/AISS-Insa-Rouen/AISS.pdf`

Based on my *Reinforcement Learning lecture notes*, in French only:
`https://philippe-preux.github.io/Documents/digest-ar.pdf`.

Reinforcement learning



The roots of reinforcement learning

Roots of RL in psychology: the "scientific study of behavior":

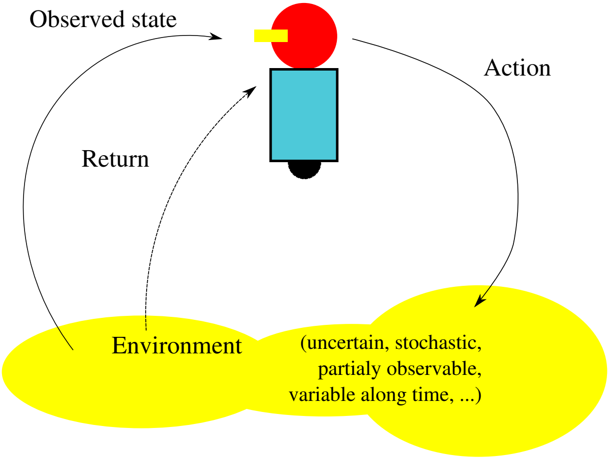
- Law of effect: Thorndike, 1898.
- Classical conditioning: Pavlov, 1903, 1927
- Operant conditioning: Skinner *et al.* from 1931 on.
  Key idea: behaviors are selected by their consequences.
  (selection of behavior akin selection of species.)
- Rescorla-Wagner law: 1972.

- Sutton's Ph.D. defended in Feb. 1984.

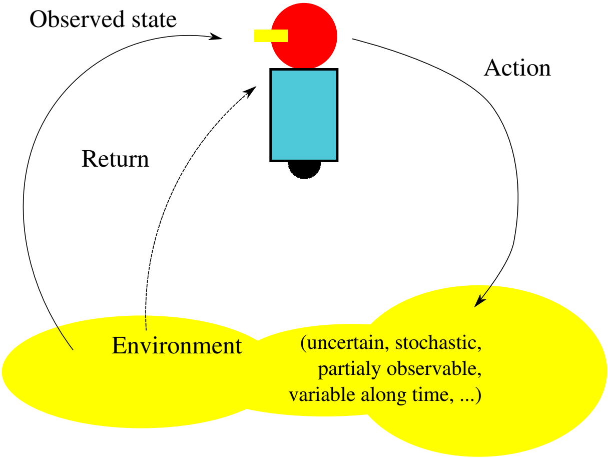
Outline

- Introduction
- Markov decision processes and Markov decision problems
- Reinforcement learning: definition and algorithms
- RL in practice

Markov decision problems



Markov decision problems



Learn an optimal behavior.

Markov decision process

Markov decision process

describes a dynamical decision system

Definition

A Markov decision process is defined by the tuple $(\mathcal{T}, \mathcal{X}, \mathcal{X}_0, \mathcal{X}_f, \mathcal{A}, \mathcal{P}, \mathcal{R})$ where:

- $\mathcal{T}$ is the set of instants of decision, $t \in \mathcal{T}$.
  For the sake of simplicity, $\mathcal{T}$ is usually the sequence of positive integers: $0, 1, 2, \ldots$

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- $\mathcal{P}$ is the transition function.

  Assume at time t, the environment is in state $x_t = x$ and performs action $a_t = a$, then $\mathcal{P}(x, a, x')$ is the probability that the environment will be in state x' at time of decision $t + 1$.

  That is: $\mathcal{P}(x, a, x') = Pr[x_{t+1} = x' | x_t = x, a_t = a]$.

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- $\mathcal{P}$ is the transition function.
- $\mathcal{R}$ is the return function, $r \in \mathbb{R}$.
  With the same assumption as for $\mathcal{P}$: $\mathcal{R}(x, a, x')$ is the expected return for the transition from state x to x' following action a.
  That is: $\mathcal{R}(x, a, x') = \mathbb{E}[r_t | x_{t+1} = x', x_t = x, a_t = a]$.

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1. items depend on t and do not depend on $t-1$, $t-2$, ...: Markov property.

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1. items depend on t and do not depend on $t-1$, $t-2$, ...: Markov property.
2. None of these items depend on $\mathcal{T}$: stationary system ($=$ non autonomous).

Markov decision process

The decision loop

$t \leftarrow 0$

loop

observe state x_t

take action a_t

observe the immediate return r_t

$t \leftarrow t + 1$

end loop

Why would we take an action or an other? What's the point?

Markov decision process

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An MD process only specifies the dynamics of a system that takes decision: it describes the "how", not the "why".

Markov decision process

Example

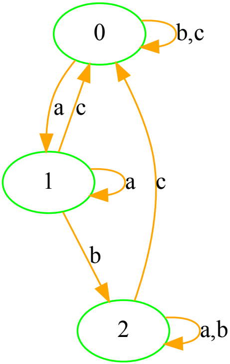
Let's play "21 with a dice".

Rules:

- you need 1 "standard" dice with 6 faces numbered from 1 to 6.
- You roll the dice, and note the value on its upper face: that's your initial score.
- Now repeatedly, you decide whether you roll it again or you stop the game.
- If you roll the dice, you add the value on its upper face to your current score.

Define a Markov decision process that models this dynamical system.

Running example: my little MDP



a, b, and c are actions.

Transitions are deterministic.

All transitions return 0 except the transition from 2 to 0 that returns 1.

Markov decision problem

The "why".

Markov decision problem

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- Solution of an MD problem: a policy π that specifies the probability to perform any action a in any state x in order to optimize ζ.
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  $\pi(x, a) = Pr[a_t = a | x_t = x], \forall a \in \mathcal{A}$.
- Blackwell's theorem tells us that an optimal policy (for this ζ) is determinitic: $\pi(x)$.
  More than 1 action may be optimal is a state.

Markov decision problem

Remarks about the definition

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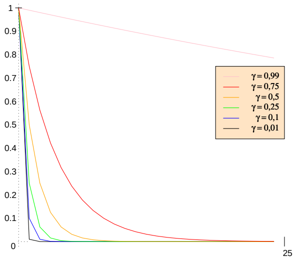

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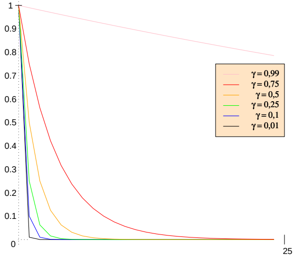

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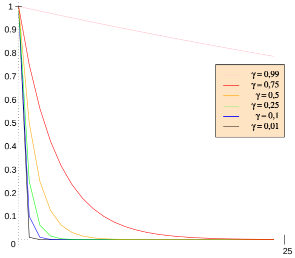

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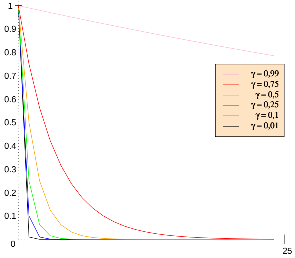

- Nice property: If we assume $\mathcal{R}$ is bounded, then ζ converges.
- This definition makes the maths easier.
- Short term consequences vs. longer consequences of actions.

Markov decision problem

A Markov decision process defines the “how”: how does the system evolve in time?

A Markov decision problem defines the “why”: why would the agent choose one or an other action?
Answer: to maximize ζ.

Pending questions:

- Does this problem have a solution?
- Can we compute it? In practice?
- How?

Markov decision problem

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- A "trajectory" is the set of (x_t, a_t, r_t, x_{t+1}) for all t during an episode.
- Let $x_0 \in \mathcal{X}_0$.
- Two episodes starting in x_0 will usually follow different trajectories.
- $\zeta(x), x \in \mathcal{X}_0$ is a random variable.
- We define $V^\pi(x)$, the value of a state x according to a policy π by:

  $$V^\pi(x) \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x) | x_0 = x, a_{t \geq 0} \sim \pi]$$

  the actions being chosen according to policy π.

Markov decision problem

The value function: intuition

$$V^{\pi}(x) \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_{t \geq 0} \sim \pi]$$

$V^{\pi}(x)$ simply quantifies how good it is to be in state x while behaving according to policy π in order to optimize ζ.

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We will soon see that given V^π, we can improve the policy and obtain a policy which value is better.

Markov decision problem

Evaluation of the value of a policy

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express the expectation
and you get:

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$$V^\pi(x) = \sum_{a \in \mathcal{A}} \pi(x, a) \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x') [\mathcal{R}(x, a, x') + \gamma V^\pi(x')]$$

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This may look complicated but all terms are known except the vector V.

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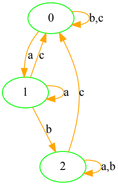
Development $\rightarrow$ a system of N linear equations with N unknowns, the $V^\pi(x), \forall x \in \mathcal{X}$.



For this MDP , compute the value function of the uniformly random policy, $\pi(x, a) = 1/3$, for $\gamma = 0.9$.

System of linear equations?

Reminder:

$V^\pi(x) = \sum_{a \in \mathcal{A}} \pi(x, a) \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^\pi(x')]$

Markov decision problem

Example (continued)

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$$
\begin{aligned}
V^\pi(0) = \quad & \pi(0, a) \sum_{x'} \mathcal{P}(0, a, x')[\mathcal{R}(0, a, x') + \gamma V^\pi(x')] + \\
& \pi(0, b) \sum_{x'} \mathcal{P}(0, b, x')[\mathcal{R}(0, b, x') + \gamma V^\pi(x')] + \\
& \pi(0, c) \sum_{x'} \mathcal{P}(0, c, x')[\mathcal{R}(0, c, x') + \gamma V^\pi(x')] \\
V^\pi(1) = \quad & \pi(1, a) \sum_{x'} \mathcal{P}(1, a, x')[\mathcal{R}(1, a, x') + \gamma V^\pi(x')] + \dots \\
V^\pi(2) = \quad & \pi(2, a) \sum_{x'} \mathcal{P}(2, a, x')[\mathcal{R}(2, a, x') + \gamma V^\pi(x')] + \dots
\end{aligned}
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Markov decision problem

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& \mathcal{P}(0,a,1)[\mathcal{R}(0,a,1) + \gamma V^\pi(1)] + \\
& \mathcal{P}(0,a,2)[\mathcal{R}(0,a,2) + \gamma V^\pi(2)]) + \\
& \pi(0,b)(\mathcal{P}(0,b,0)[\mathcal{R}(0,b,0) + \gamma V^\pi(0)] + \\
& \mathcal{P}(0,b,1)[\mathcal{R}(0,b,1) + \gamma V^\pi(1)] + \\
& \mathcal{P}(0,b,2)[\mathcal{R}(0,b,2) + \gamma V^\pi(2)]) + \\
& \pi(0,c)(\mathcal{P}(0,c,0)[\mathcal{R}(0,c,0) + \gamma V^\pi(0)] + \\
& \mathcal{P}(0,c,1)[\mathcal{R}(0,c,1) + \gamma V^\pi(1)] + \\
& \mathcal{P}(0,c,2)[\mathcal{R}(0,c,2) + \gamma V^\pi(2)]) \\
V^\pi(1) = \quad & \ldots \\
V^\pi(2) = \quad & \ldots
\end{aligned}
$$

$\rightarrow$

$$
\begin{aligned}
V^\pi(0) = \quad & 1/3(0 \times [\ldots] + \\
& 1 \times [0 + \gamma V^\pi(1)] + \\
& 0 \times [\ldots]) + \\
& 1/3(1 \times [0 + \gamma V^\pi(0)] + \\
& 0 \times [\ldots] + \\
& 0 \times [\ldots]) + \\
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& 0 \times [\ldots] + \\
& 0 \times [\ldots]) \\
V^\pi(1) = \quad & \ldots \\
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\end{aligned}
$$

Markov decision problem

Example (continued)

$$
\begin{aligned}
V^\pi(0) = \quad & 1/3(0 \times [\ldots] + \\
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& 0 \times [\ldots] + \\
& 0 \times [\ldots])
\end{aligned}
$$

$$
\begin{aligned}
V^\pi(1) = \quad & \ldots \\
V^\pi(2) = \quad & \ldots
\end{aligned}
$$

$$
\rightarrow \quad
\begin{aligned}
V^\pi(0) = \quad & 1/3(\gamma V^\pi(1) + \gamma V^\pi(0) + \gamma V^\pi(0)) \\
V^\pi(0) = \quad & 1/3(0.9 V^\pi(1) + 0.9 V^\pi(0) + 0.9 V^\pi(0)) \\
\rightarrow \quad & 0.4 V^\pi(0) - 0.3 V^\pi(1) = 0
\end{aligned}
$$

$$
\begin{aligned}
V^\pi(1) = \quad & \ldots \\
V^\pi(2) = \quad & \ldots
\end{aligned}
$$

Markov decision problem
Example (continued)

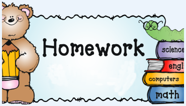
We obtain the linear system of equations:

$$\begin{pmatrix} 0.4 & -0.3 & 0 \\ -0.3 & 0.7 & -0.3 \\ -0.3 & 0 & 0.4 \end{pmatrix} V^\pi = \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix}$$

$\rightarrow$

$$V^\pi = \begin{pmatrix} 0{,}61 \\ 0{,}82 \\ 1{,}29 \end{pmatrix}$$

Markov decision problem

Evaluation of the value of a policy: 

Implement this approach in a generic way to solve any MDP.

Check your implementation on the little example.

Apply it to the “21 with a dice” problem.

Markov decision problem

Evaluation of the value of a policy: the dynamic programming approach

$$V^\pi(x) = \sum_{a \in \mathcal{A}} \pi(x, a) \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^\pi(x')]$$

Development $\rightarrow$ a system of N linear equations with N unknowns, the $V^\pi(x), \forall x \in \mathcal{X}$.

In principle, an easy problem.
In practice, when N is large (*e.g.* 10^9), this is a challenging problem.

$V^\pi = \mathcal{M} V^\pi + Q$, where $\mathcal{M} \in \mathbb{R}^{N \times N}$ is an $N \times N$ matrix, and $Q \in \mathbb{R}^N$.
$\rightarrow (Id - \mathcal{M}) V^\pi = Q$

Thanks to contraction properties of $\mathcal{M}$, this can be solved iteratively.

express $\mathcal{M}$ and $\mathcal{Q}$ in terms of $\pi, \mathcal{P}, \mathcal{R}, \gamma$.

Markov decision problem

Evaluation of the value of a policy: the dynamic programming approach

$$V^\pi(x) = \sum_{a\in\mathcal{A}} \pi(x,a) \sum_{x'\in\mathcal{X}} \mathcal{P}(x,a,x')[\mathcal{R}(x,a,x') + \gamma V^\pi(x')]$$

Thanks to contraction properties of $\mathcal{M}$, this can be solved iteratively.

$k \leftarrow 0$
$V_k \leftarrow 0, \forall x \in \mathcal{X}$
$\epsilon \leftarrow 10^{-6}$ // some small value
repeat
    $V_{k+1} \leftarrow \dots$
    $\Delta \leftarrow ||V_k - V_{k+1}||_\infty$
    $k \leftarrow k+1$
until $\Delta < \epsilon \frac{1-\gamma}{2\gamma}$

This algorithm provides an estimation of V^π at most ϵ away from its true value.

It is very easy to implement and very fast, even when $N = 10^9$.

Markov decision problem

Evaluation of the value of a policy: the dynamic programming approach

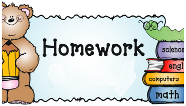
$$V^\pi(x) = \sum_{a \in \mathcal{A}} \pi(x, a) \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^\pi(x')]$$

is known as a "Bellman equation".

[Bellman, *Dynamic Programming*, Princeton U. Press, 1957]

Markov decision problem

Example (continued): the dynamic programming approach



Implement this dynamic programming approach for policy value evaluation and run it on the two examples.

Check that both methods give the same result.

Markov decision problem

Evaluation of the value of a policy: the Monte Carlo approach

Very simple approach: consists in simulating the Markov chain and estimating V^π by computing ζ.

To estimate $V^\pi(x)$:

$cr \leftarrow 0$
$cnt \leftarrow 0$
for $k \in \{0, ..., K-1\}$ **do**
  $t \leftarrow 0$
  $x_t \leftarrow x$
  repeat
    sample $a_t \sim \pi(x_t, .)$
    sample the next state $x_{t+1} \sim \mathcal{P}(x_t, a_t, .)$
    sample $r_t \sim \mathcal{R}(x_t, a_t, .)$
    $cr \leftarrow cr + \gamma^t r_t$
    $cnt \leftarrow cnt + 1$
  until γ^t is very small
end for $V^\pi(x) \leftarrow \frac{cr}{cnt}$

Markov decision problem

Evaluation of the value of a policy: the Monte Carlo approach

When the MDP is deterministic, the double loop becomes a single loop. Wrt. traditionnal tree search algorithms, it is competitive in some cases: very wide tree, or very deep tree, or stochastic dynamics.

The drawback is that the convergence is slow: $\mathcal{O}(\sqrt{K})$.

However, as this is a very simple algorithm, its inner loop is fast.

One run (inner loop) is named a *rollout*.

Monte Carlo tree search is a more sophisticated version of this basic algorithm to deal with large trees, and stochastic dynamics.

Implement a Monte Carlo algorithm for policy evaluation. Apply it to the same examples as above and check that the 3 approaches provide the same results.

Markov decision problem

Policy improvement

- Once the value of a policy has been estimated, the policy can be improved.

Markov decision problem

Policy improvement

- Once the value of a policy has been estimated, the policy can be improved.
- Let us assume we estimated V^π. Then, we compute:

$$\pi'(x) \leftarrow \text{arg max}_{a \in \mathcal{A}} \sum_{x'} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^\pi(x')]$$

for each state $x \in \mathcal{X}$.

Markov decision problem

Policy improvement

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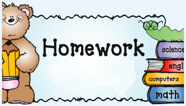
  for each state $x \in \mathcal{X}$.
- Then π' is either better than π, or they are equivalent, that is: $V^{\pi'} \geq V^\pi$.

Markov decision problem

Example (continued) 

Implement the policy improvement algorithm.

Apply it on the little MDP. How is the uniformly random policy improved?

Markov decision problem

Policy iteration

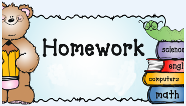
- Start with a random policy.
- Then, alternate:
  - estimate the value of the current policy
  - improve the current policy

  until the value of the policy no longer improves.

This is the "policy iteration" algorithm [Howard, 1958].

Markov decision problem

Example (continued)



Implement policy iteration and test it on the little MDP and on the "21 with a dice" problem.

Markov decision problem
Value iteration

- The value V^* of the optimal policy π^* is:
  $V^*(x) \stackrel{\text{def}}{=} \max_\pi V^\pi(x), \forall x \in \mathcal{X}$

Markov decision problem

Value iteration

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  $V^*(x) = \max_{a \in \mathcal{A}} \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^*(x')]$
  The "Bellman optimality equation".

Markov decision problem

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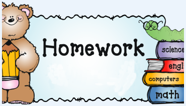
  The "Bellman optimality equation".
- and an algorithm to obtain V^* directly:
  - $k \leftarrow 0$
  - initialize V_k
    - compute $V_{k+1} \leftarrow \max_{a \in \mathcal{A}} \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^*(x')]$ for all x
    - $k \leftarrow k + 1$
  - repeat
  - until $||V_k - V_{k-1}||_\infty \leq \frac{\epsilon(1-\gamma)}{2\gamma}$

named "value iteration".

Markov decision problem

Example (continued) 

Implement value iteration and test it on the little MDP and on the "21 with a dice" problem.

Markov decision problem

As a linear program

We can frame a Markov decision problem as a linear program:

$$
\begin{array}{l}
min \sum_x V[x] \\
\text{s.t. } V[x] - \sum_{x'} \mathcal{P}(x,a,x')(\mathcal{R}(x,a,x')) + \gamma V[x']) \geq 0 \;\forall (x,a) \in \mathcal{X} \times \mathcal{A}
\end{array}
$$

There one constraint for each (state, action) pair and one variable per state.
The dual is:

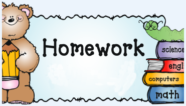
$$
\begin{array}{l}
max \sum_{(x,a)} \sum_{x'} \mathcal{P}(x,a,x')\mathcal{R}(x,a,x')\xi(x,a) \\
\text{s.t. } \sum_a \xi(x',a) - \sum_{x,a} \gamma \mathcal{P}(x,a,x')\xi(x,a) \leq 1, \forall x' \in \mathcal{X} \\
\text{and } \xi(x,a) \geq 0, \forall (x,a) \in \mathcal{X} \times \mathcal{A}
\end{array}
$$

Once solved, $\xi(x,a)$ is non zero if action a is optimal in x.

Very interesting theoretically speaking.
In practice, policy iteration is (usually) the best way to go.

Markov decision problem

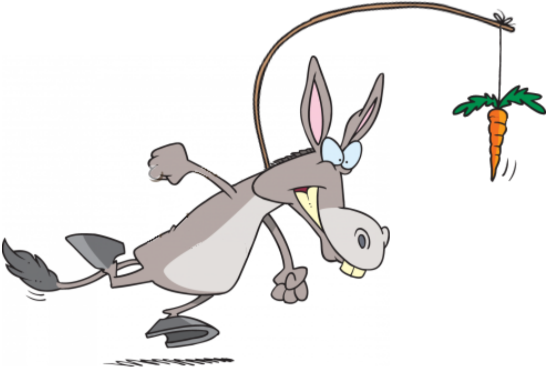
Example (continued)



Implement the LP approach on the little MDP and on the “21 with a dice” problem.

Hint: in Python, use the PULP package to solve an LP.

From MDPs to Reinforcement Learning



From MDPs to Reinforcement Learning

What if $\mathcal{P}$ and $\mathcal{R}$ are unknown?

- Reminder: $V^\pi \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_{t>0} \sim \pi]$
  Let us define $Q^\pi(x, a) \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_0 = a, a_{t>0} \sim \pi]$

  named the "quality" of the (state, action) pair (x, a).

From MDPs to Reinforcement Learning

The quality function

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  $Q^\pi(x, a) =$
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- Bellman optimality equation for Q^*:
  $Q^*(x, a) = \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma \max_{a' \in \mathcal{A}} Q^*(x', a')]$.

Reinforcement Learning

Bellman equation and the TD error [Sutton, 1988]

Bellman approach:

- $\zeta = \sum_{t \geq 0} \gamma^t r_t, \gamma \in [0, 1[$

Reinforcement Learning

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  Reminder: This quantifies what will happen to the agent in its future if it behaves according to π.

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- re-written as: $V(x_t|\pi) = \mathbb{E}(r_t) + \gamma \mathbb{E}(V(x_{t+1}|\pi))$

  sum of what will happen immediately $+ \gamma \times$ what will happen then.

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  sum of what will happen immediately $+ \gamma \times$ what will happen then.
- at time t, $r_t + \gamma(V(x_{t+1}|\pi)) - V(x_t|\pi)$

  is an estimation of the error of estimation of V: TD-error

  This TD-error may be used to learn the optimal behavior.

- computing V by gradient descent:

  $V(x_{t+1}) \leftarrow V(x_t) - \eta[r_t + \gamma(V_t(x_{t+1})) - V(x_t)]$
  where η is a learning rate, adequately decreasing along time.

- computing V by gradient descent:

  $V(x_{t+1}) \leftarrow V(x_t) - \eta[r_t + \gamma(V_t(x_{t+1})) - V(x_t)]$
  where η is a learning rate, adequately decreasing along time.
- We may also consider the quality of an (x, a) pair:
  $Q(x_t, a_t) \stackrel{\text{def}}{=} \mathbb{E}(\zeta(x_t | a_t = a, \pi))$

Reinforcement Learning
The temporal difference

- computing V by gradient descent:

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  $Q(x_t, a_t) \stackrel{\text{def}}{=} \mathbb{E}(\zeta(x_t | a_t = a, \pi))$
- At t, we may consider $r_t + \gamma \max_{a'} Q(x_{t+1}, a') - Q(x_t, a_t)$ as a correction to $Q(x_t, a_t)$.

  $\leadsto$

  $Q(x_t, a_t) \leftarrow Q(x_t, a_t) + \eta[r_t + \gamma \max_b Q(x_{t+1}, b) - Q(x_t, a_t)]$

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  $\rightsquigarrow$
  $Q(x_t, a_t) \leftarrow Q(x_t, a_t) + \eta[r_t + \gamma \max_b Q(x_{t+1}, b) - Q(x_t, a_t)]$

- $\rightsquigarrow$ learning π^* algorithm.

The goal of this algorithm is to estimate Q^* by interacting with the environment and correcting its estimate of Q^*.

1. initialize Q.
2. set the agent in some random initial state $\in \mathcal{X}_0$.
3. run the agent in the environment:
   at each step, record the state x_t, the action performed a_t, the reward collected r_t, and the next state x_{t+1}.
4. correct Q.
5. continue until a terminal state is reached.
6. do it again and again (re-starting at step 2).

Reinforcement Learning

Q-Learning [Watkins, 1989]

$Q(x,a) \leftarrow$ some value (0, random, ...)
repeat
  $t \leftarrow 0$
  Initialize the state of the agent x_t
  while episode not completed **do**
    choose an action to perform in state x_t: a_t
    perform this action and observe r_t and x_{t+1}
    update: $Q(x_t, a_t) \leftarrow Q(x_t, a_t) + \alpha[r_t + \max Q(x_{t+1}, b_b - Q(x_t, a_t)]$
    $t++$
  end while
until some condition is met.

At completion (if you looped enough): $\pi^*(x) = \arg\max_a Q(x,a), \forall x$

At completion (if you looped enough): $\pi^*(x) = \arg\max_a Q(x, a), \forall x$

- Remark: you never know if you looped enough!
- Asymptotic convergence to Q^*.

- α depends on x_t and a_t.
- for each (x, a), we should have:
  $\sum_{\{t \text{ at which } (x,a) \text{ is visited}\}} \alpha_t(s, n(s)) = +\infty$
  and
  $\sum_{\{t \text{ at which } (x,a) \text{ is visited}\}} \alpha_t^2(s, n(s)) < +\infty$
  e.g. $\alpha(x, a) \equiv \frac{1}{n(x,a)+1}$

- "choose an action to perform in state x_t: a_t"
  How do we do that?

Reinforcement Learning
About Q-Learning

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Reinforcement Learning

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Reinforcement Learning

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  - step by step, we acquire some knowledge about which actions are good, and which are bad. We can exploit this knowledge.
    Select the "best" action: $\arg\max_a Q(x_t, a)$.

Reinforcement Learning

About Q-Learning

- “choose an action to perform in state x_t: a_t”
  How do we do that?
- No perfect solution.
- Intuition:
  - initially, we do not know anything about Q^*. We have to test the effect of the actions: we have to explore.
    Select action at random
  - step by step, we acquire some knowledge about which actions are good, and which are bad. We can exploit this knowledge.
    Select the “best” action: $\arg\max_a Q(x_t, a)$.
- Exploration and exploitation should be carefully balanced.

Reinforcement Learning
About Q-Learning

- "choose an action to perform in state x_t: a_t".

Reinforcement Learning

About Q-Learning

- "choose an action to perform in state x_t: a_t".
- explore and progressively exploit more and more.

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Reinforcement Learning

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Reinforcement Learning

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  - Each time you need to select an action:
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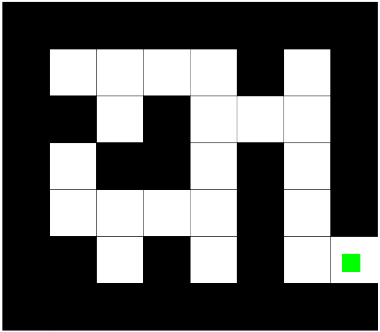
    - draw an action at random according to p.
  - Slowly decrease τ along the episodes.
  - Rationale: when τ is large, uniformly random selection. τ close to 0, greedy selection.



Goal: reach the green cell from any location.

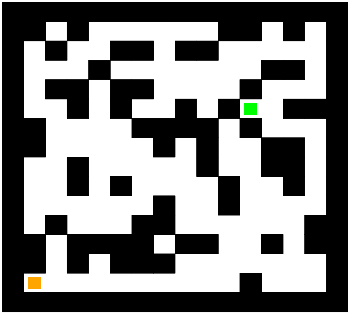
Question 1: propose a Markov decision process: what are $(\mathcal{T}, \mathcal{X}, \mathcal{X}_0, \mathcal{X}_f, \mathcal{A}, \mathcal{P}, \mathcal{R})$?

Question 2: model this task as a Markov decision problem: what is ζ?

Reinforcement Learning

Q-Learning in action: escaping a maze

We use an extremely basic Q-Learning.
Has a very local perception: sees only the 4 neighboring cells.



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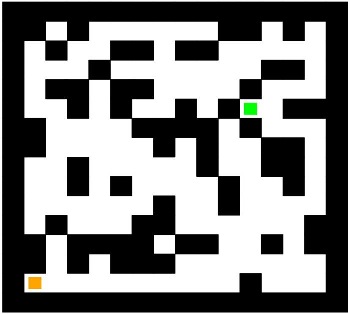
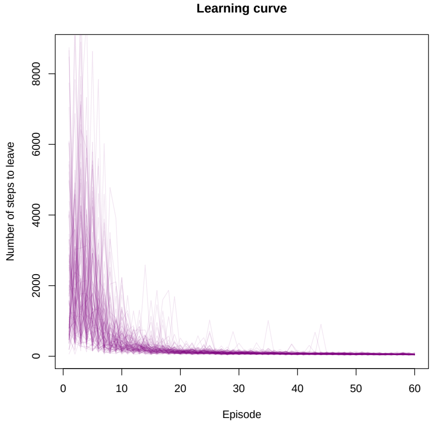
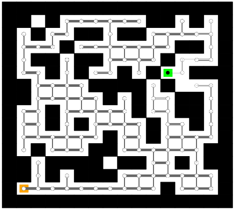




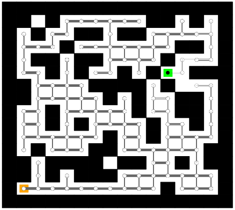
1st reach



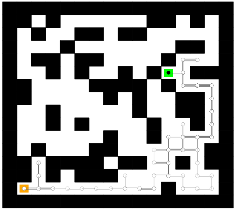
Reinforcement Learning

Q-Learning in action

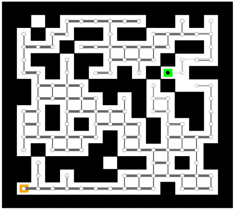
1st reach



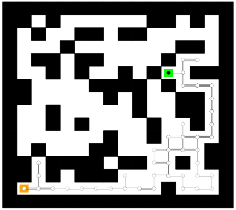
10th reach



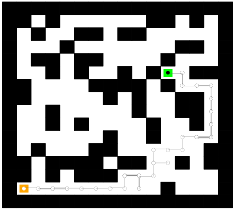
1st reach



10th reach



60th reach



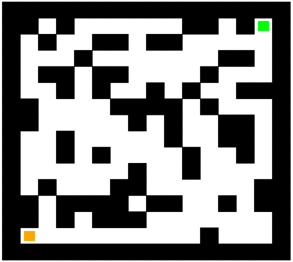
Reinforcement Learning

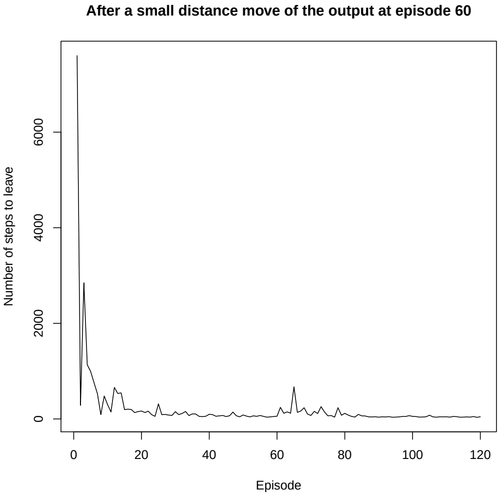
eligibility traces

Reinforcement Learning

Q-Learning continuously adapts to its environment

The goal state moves nearby:

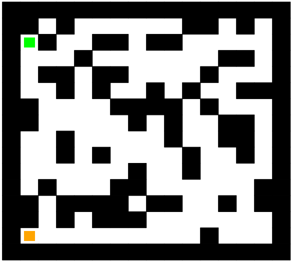


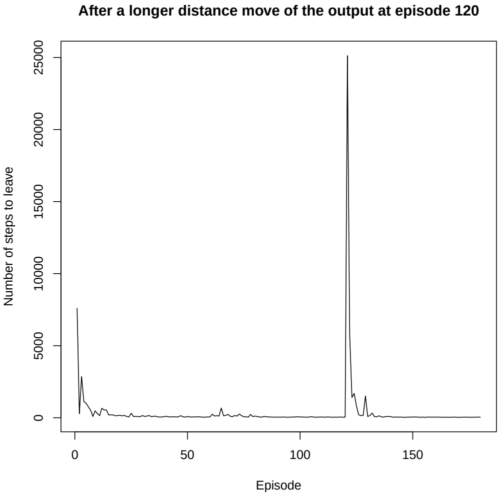


Reinforcement Learning

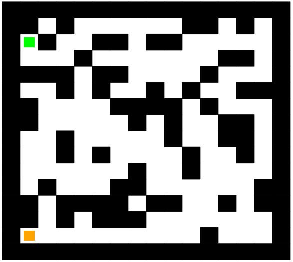
Q-Learning continuously adapts to its environment

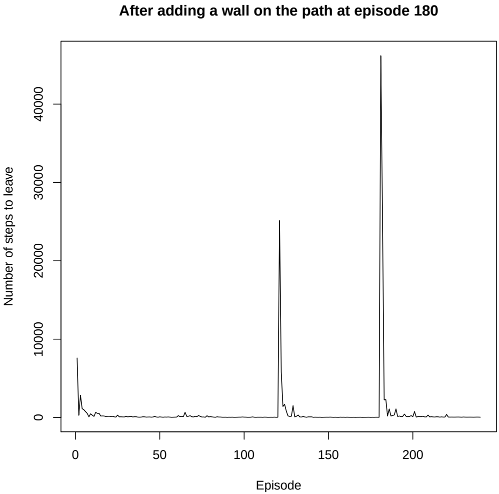
The goal state moves farther away:





Blocking the path:





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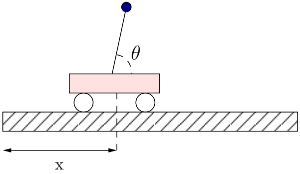
- `Q (x, a)` returns an estimate of $Q(x, a)$.
- This estimate may be updated/improved by learning.

Reinforcement Learning

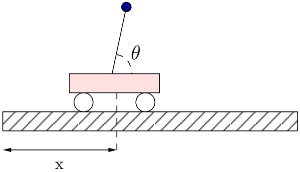
Value function



State is $(\theta, \dot{\theta})$
Action is $\ddot{\theta}$

Reinforcement Learning

Value function



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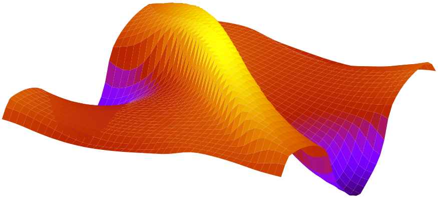


$(\theta, \dot{\theta})$ plane
z is $V(x)$
Maximize value $\rightsquigarrow$ reach the top of V

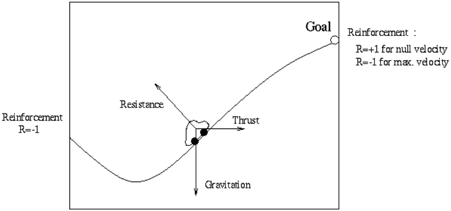
Reinforcement Learning

Handling large $\mathcal{X}$: the function approximator zoo

- neural network [Lin, 1991; Riedmiller, 2005; ...],
- random forest [Geurts *et al.*, 2006],
- SVM and kernels,
- and many other ideas from statistical learning (supervised learning).
- Tabular with progressive and adaptive state partitioning.

Reinforcement Learning

Progressive and adaptive state partitioning [Munos, Moore, MLJ, 2001]



Reinforcement Learning

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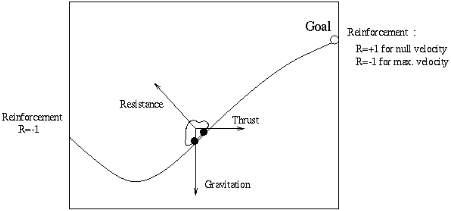

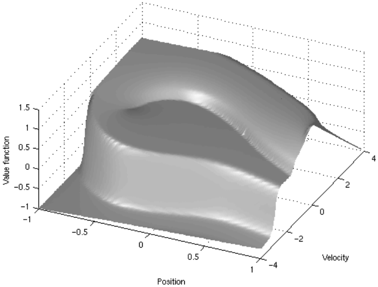

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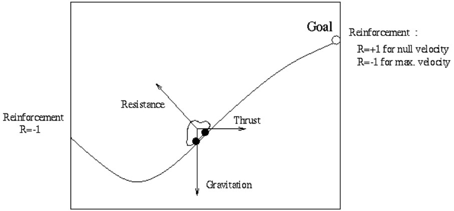

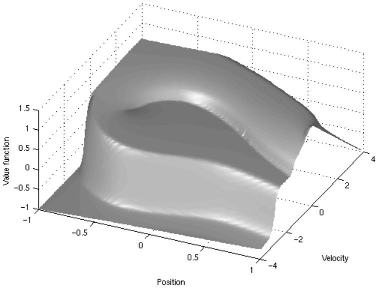

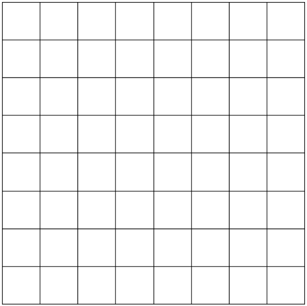

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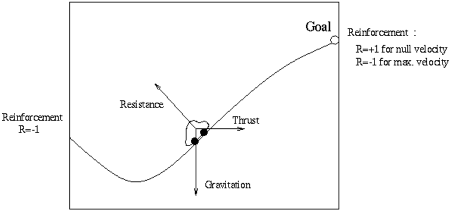

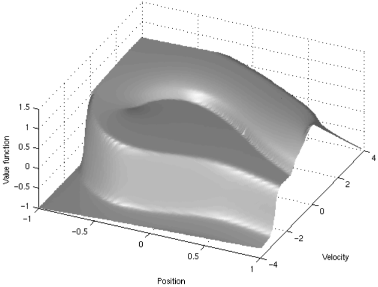

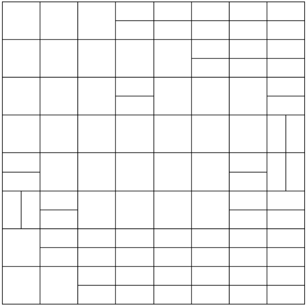

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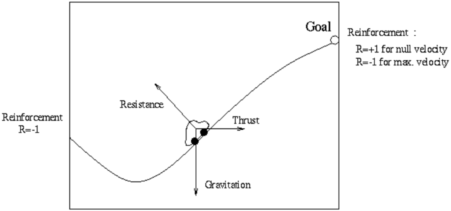

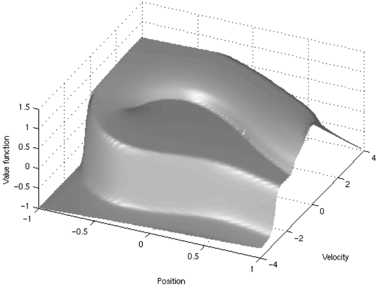

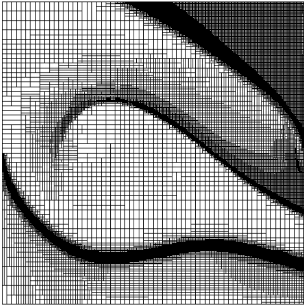

Reinforcement Learning

Neural Q-Learning

Trendy people call that “deep RL” though there is only shallow networks most of the times.

- Use of a neural network to represent Q.
- Input $=$ the current state
- Output $=$ either an estimate of $Q(x, a), \forall a$, or an estimate of $\pi(x)$.

Reminder: function approximation

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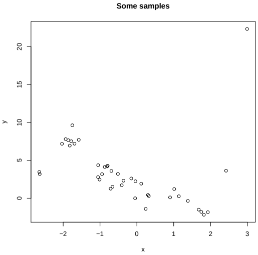
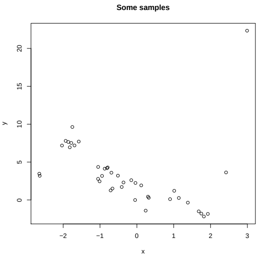
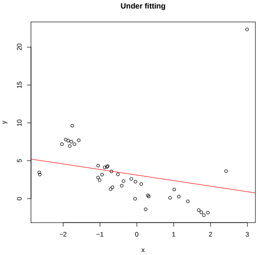
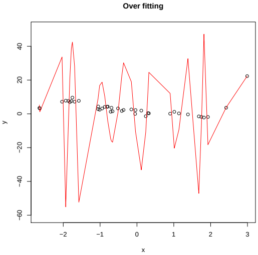


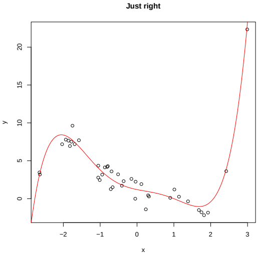
A few samples.

Reminder: function approximation









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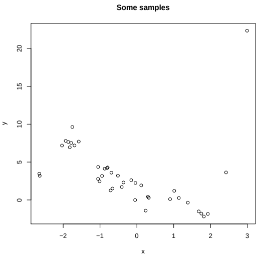

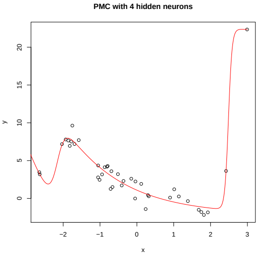

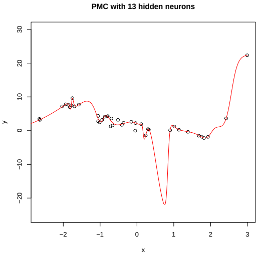

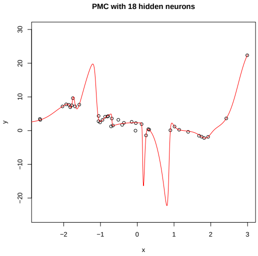

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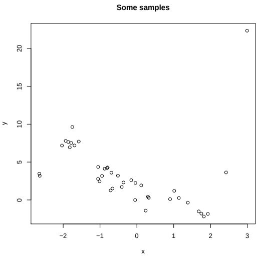

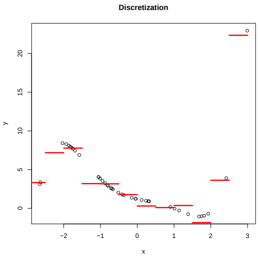

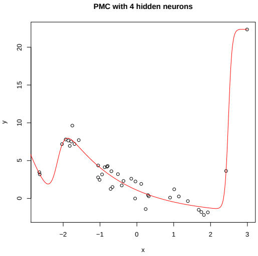

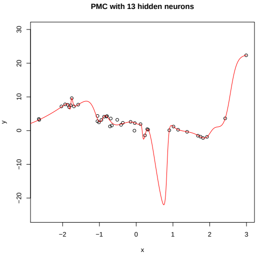

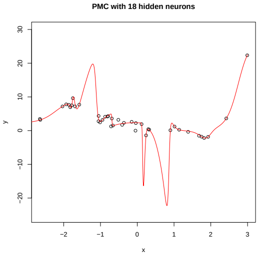

Reinforcement Learning
Neural Q-Learning

Tabular Q-Learning	DQN

Initialize the Q-table Q.
repeat
  set initial state x_0
  $t \leftarrow 0$
  while episode not complete **do**
    • select a_t and emit it
    • observe r_t and x_{t+1}
    • update $Q(x_t, a_t)$
    • $t \leftarrow t + 1$
  end while
until …
	Initialize parameters θ Q
	repeat
	  set initial state x_0
	  $t \leftarrow 0$
	  while episode not complete **do**
	    • select a_t and emit it
	    • observe r_t and x_{t+1}
	    • update θ to minimise the prediction error (temporal difference) for (x_t, a_t)
	    • $t \leftarrow t + 1$
	  end while
	until …

There are many things that can go wrong with this algorithm.
May be painful to debug.

- Design of the state space: the state is a vector; neighboring vectors should be neighbor states.

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  - and more.

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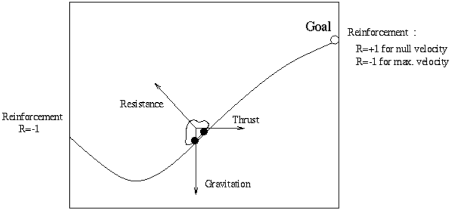
Neural networks may be tricky to train in supervised learning.
In RL, NN are used in a feedback loop: feedback loops are well-known to be very sensible, prone to very quickly (exponential rate) turn tiny errors into catastrophes.

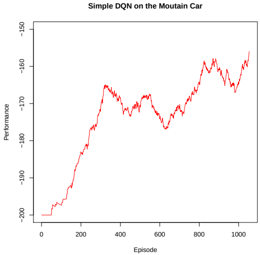
When coding DQN:

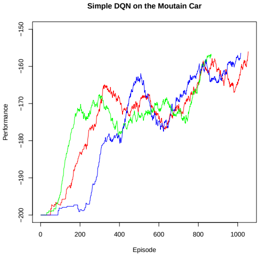
- Each part of the algorithm should be tested.
- One should check that the NN is able to regress the Q function.

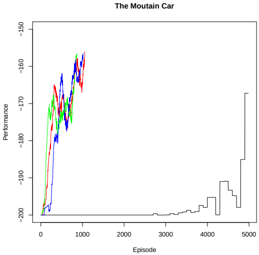


Goal
Reinforcement :
R=+1 for null velocity
R=-1 for max. velocity
Resistance
Reinforcement
R=-1
Thrust
Gravitation

On the mountain-car:







red: 128 neurons

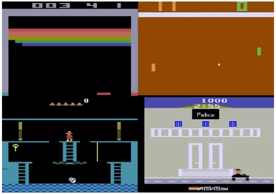
green: 32 neurons,
blue: 64 neurons

+ tabular QL

Reinforcement Learning

Neural Q-Learning/DQN: state = an image [Mnih et al., 2015]

The Atari game suite:



- Input: image
  $\rightarrow$ CNN
  $\rightarrow$ fully connected layers
  $\rightarrow$ P outputs, one per action.
- DQN learns to play 49 games.
- The CNN part learns an embedding of the state (image) that is used to learn the value of each action.

An other family of RL algorithms

- Do we have to learn a value function to learn a policy?

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- Do we have to learn a value function to learn a policy?
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- Idea:
  - Policy represented by a NN with parameters θ.
  - Run one or more trajectories with this policy.
  - We can show that maximizing ζ is equivalent to minimizing $\mathcal{L}(\theta) \equiv -Q \log(\pi^\theta)$.

Reinforcement Learning

Policy gradient approach: REINFORCE [Williams, 1992]

initialize θ
repeat
  perform K episodes and collect transitions $(x_{k,t}, a_{k,t}, r_{k,t}, x_{k,t+1})$
  for for each episode k **do**
    for for each step t in episode k **do**
      $Q_{k,t} \leftarrow \sum_{j=t}^{j=|\tau_k|-1} \gamma^{j-t} r_{k,j}$
      $\theta \leftarrow \theta + \alpha\gamma^t Q_{k,t} \nabla_\theta \log\left(\pi^\theta(s_{k,t}, a_{k,t})\right)$
    end for
  end for
until ...

REINFORCE suffers from important drawbacks:

- REINFORCE needs full episodes.
- REINFORCE is unstable because the update have a large variance.
- there is no explicit exploration.

One may use other types of optimization algorithm to optimize ζ.

Some researchers have investigated the use of evolutionary algorithms.

The combination of the goods of value based and policy gradient approaches.

- $\mathcal{L}(\theta) \equiv -Q \log(\pi^\theta)$
  $\rightarrow$
  $\mathcal{L}(\theta) \equiv -(Q - b) \log(\pi^\theta)$.
  where b is a function of the (current) state.

Reinforcement Learning

Actor-critic

- $\mathcal{L}(\theta) \equiv -Q \log\left(\pi^\theta\right)$
  $\rightarrow$
  $\mathcal{L}(\theta) \equiv -(Q - b) \log\left(\pi^\theta\right)$.
  where b is a function of the (current) state.
- $A(x, a) \equiv Q(x, a) - V(x)$ is the *advantage* function.
  $\rightarrow$
  $\mathcal{L}_{AC}(\theta) \equiv -(Q - V) \log\left(\pi^\theta\right)$.

Reinforcement Learning
Actor-critic

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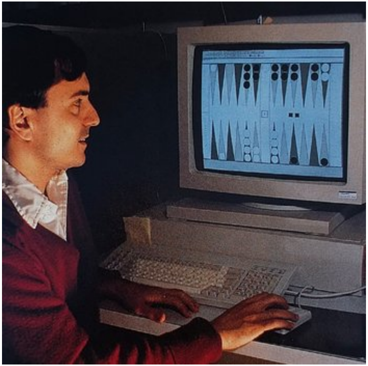
Reinforcement Learning
Application: TD-Gammon



Early 1990's, a real tour de force.

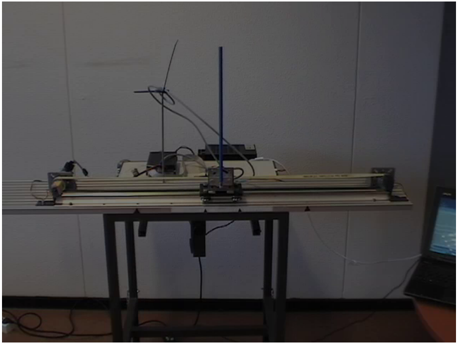
Reinforcement Learning
Application: TD-Gammon

- Backgammon is studied at least since 1974
- Branching factor: 800
- TD-Gammon: “successor of Neurogammon, trained by supervised learning. NeuroGammon won the 1st Computer Olympiad in London in 1989, handily defeating all opponents. Its level of play was that of an intermediate-level human player.” (Source: wikipedia)
- raw representation of the board position
- trained with TD(λ) algorithm
- no knowledge, self-play
- hand-crafted features
- 3-plies in v3

Tesauro, Temporal Difference Learning and TD-Gammon, *Communications of the ACM*, 1995

Reinforcement Learning

Application to robotics: a real cartpole

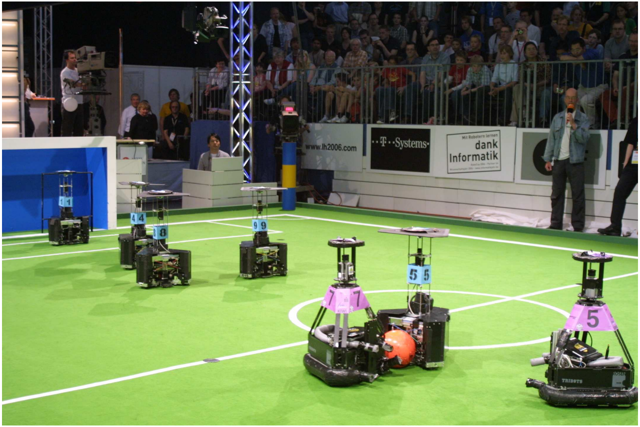


Neural Q-Learning. 2 hidden layers each made of 5 neurons with sigmoid activation function.

Riedmiller, Neural reinforcement learning to swing-up and balance a real pole, *Proc. 2005 IEEE International Conference on Systems, Man and Cybernetics*

Reinforcement Learning

Application to robotics



RL is used only to control the ball and the speed of the robot.

Lauer et al., Cognitive Concepts in Autonomous Soccer Playing Robots, *Cognitive Systems Research*, **11**(3), 287:309, September, 2010

(No Deep Learning! only shallow multi-layer perceptron)

- Learning to play board games using only the rules of the game.
- Alpha Go learned to play Go by using games played by humans.
- Alpha Zero learned to play even better by itself by RL.
- then other board games (chess, draughts, reversi, ...).
- then Starcraft II.

Reinforcement Learning

Alpha Zero type of algorithms

- RL
- $+$ various tricks to stabilize learning and make it more efficient
- MCTS as a key component
- moderately deep network as function approximator

Outro

- learning options
- learning representation
- generalization in RL
- time varying environments
- transfer learning
- life-long learning
- explaination/accountability of the learned behavior

Reinforcement learning resources

The RL community is open source.
Lots of excellent code available online.
Some nice RL libraries:

- rlberry: our home-brewed RL library:
  `https://github.com/rlberry-py/rlberry`
- stable-baselines3:
  `https://stable-baselines3.readthedocs.io/en/master/`
- clean-RL: `https://github.com/vwxyzjn/cleanrl`
- spinning-up RL: `https://spinningup.openai.com/en/latest/`
- mushroom RL:
  `https://mushroomrl.readthedocs.io/en/latest/`
- ...

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- Tesauro, Temporal Difference Learning and TD-Gammon, *Communications of the ACM*, 1995

Credits

All figures are mine, except:

- slide **??**, the backgammon image comes from , the go image comes from , and the chatGPT logo comes from .
- Slide 60, the figures come from Rémi Munos' website `http://researchers.lille.inria.fr/munos/variable/index.html`
- Slide 40, the donkey was designed and realized by Sertan Girgin in 2008 while he was a post-doc in my research group, SequeL.
- Slide **??**,
- Slide **??**,
- Slide **??**,
- Slide **??**,
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image comes from `https://newquayjunior.net/homework/`.