Reinforcement learning

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Scool

This presentation:

`https://philippe-preux.github.io/talks/AISS-Insa-Rouen/AISS.pdf`

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Reinforcement learning in 2023

$1/2/++$ slides de propagande pour mettre en appétit.

The roots of reinforcement learning

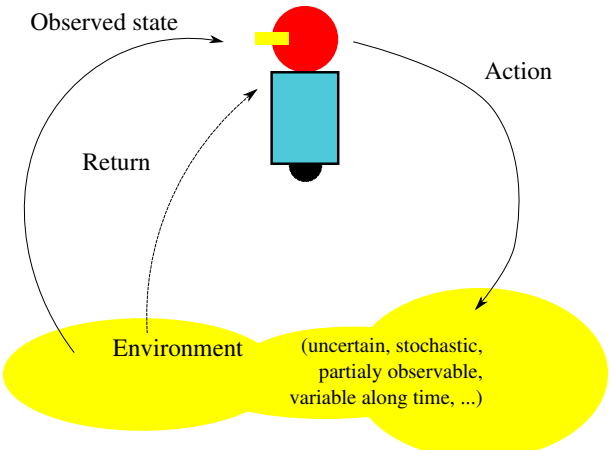
- Law of effect: Thorndike, 1896.
- Classical conditioning: Pavlov, 19..
- Operant conditioning: Skinner *et al.* from 19.. on.
- Rescorla-Wagner law: 1972.
-
- Sutton's Ph.D. defended in Feb. 1984.

frise avec dates importantes

Outline

- Introduction
- Markov decision processes and Markov decision problems
- Reinforcement learning: definition and algorithms
- RL in practice
- How could you contribute to RL?

Markov decision problems



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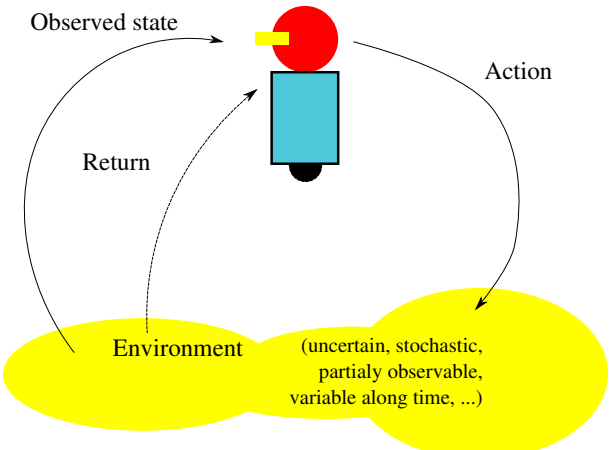

Learn an optimal behavior.

Markov decision process

Markov decision process

describes a dynamical decision system

Definition

A Markov decision process is defined by the tuple $(\mathcal{T}, \mathcal{X}, \mathcal{X}_0, \mathcal{X}_f, \mathcal{A}, \mathcal{P}, \mathcal{R})$ where:

- $\mathcal{T}$ is the set of instants of decision, $t \in \mathcal{T}$.
  For the sake of simplicity, $\mathcal{T}$ is usually the sequence of positive integers: $0, 1, 2, ...$

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- $\mathcal{A}$ is a finite set of actions, $a \in \mathcal{A}$.
- $\mathcal{P}$ is the transition function.
  Assume at time t, the environment is in state $x_t = x$ and performs action $a_t = a$, then $\mathcal{P}(x, a, x')$ is the probability that the environment will be in state x' at time of decision $t + 1$.
  That is: $\mathcal{P}(x, a, x') = Pr[x_{t+1} = x' | x_t = x, a_t = a]$.

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- $\mathcal{P}$ is the transition function.
- $\mathcal{R}$ is the return function, $r \in \mathbb{R}$.
  With the same assumption as for $\mathcal{P}$: $\mathcal{R}(x, a, x')$ is the expected return for the transition from state x to x' following action a.
  That is: $\mathcal{R}(x, a, x') = \mathbb{E}[r_t | x_{t+1} = x', x_t = x, a_t = a]$.

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2. None of these items depend on $\mathcal{T}$: stationary system (= non autonomous).

Why would we take an action or an other? What's the point?

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An MD process only specifies the dynamics of a system that takes decision: it describes the "how", not the "why".

Markov decision process

Example

Let's play "21 with a dice".

Rules:

- you need 1 "standard" dice with 6 faces numbered from 1 to 6.
- You roll the dice, and note the value on its upper face: that's your initial score.
- Now repeatedly, you decide whether you roll it again or you stop the game.
- If you roll the dice, you add the value on its upper face to your current score.

Define a Markov decision process that models this dynamical system.

Markov decision problem

The “why”.

Markov decision problem

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- Solution of an MD problem: a policy π that specifies the probability to perform any action a in any state x in order to optimize ζ.
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  $\pi(x, a) = Pr[a_t = a | x_t = x], \forall a \in \mathcal{A}$.
- Blackwell's theorem tells us that an optimal policy (for this ζ) is determinitic: $\pi(x)$.
  More than 1 action may be optimal is a state.

Markov decision problem

Remarks about the definition

γ?

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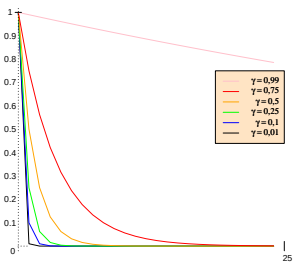

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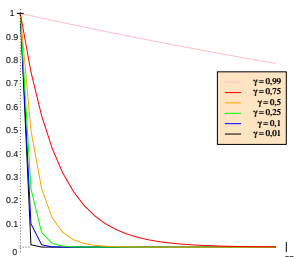

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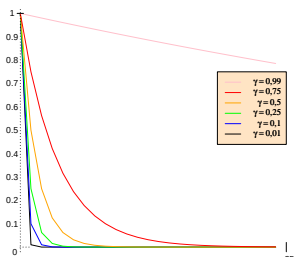

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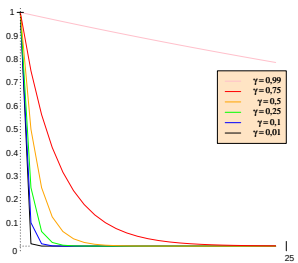

- Nice property: If we assume $\mathcal{R}$ is bounded, then ζ converges.
- This definition makes the maths easier.
- Short term consequences vs. longer consequences of actions.

Markov decision problem

A Markov decision process defines the "how": how does the system evolve in time?

A Markov decision problem defines the "why": why would the agent choose one or an other action?
Answer: to maximize ζ.

Pending questions:

- Does this problem have a solution?
- Can we compute it? In practice?
- How?

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Markov decision problem

The value function

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- Two episodes starting in x_0 will usually follow different trajectories.
- $\zeta(x), x \in \mathcal{X}_0$ is a random variable.
- We define $V^\pi(x)$, the value of a state x according to a policy π by:

$$V^\pi(x) \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x) | x_0 = x, a_{t \geq 0} \sim \pi]$$

the actions being chosen according to policy π.

Markov decision problem

The value function: intuition

$$V^\pi(x) \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_{t\geq 0} \sim \pi]$$

$V^\pi(x)$ simply quantifies how good it is to be in state x while behaving according to policy π in order to optimize ζ.

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We will soon see that given V^π, we can improve the policy and obtain a policy which value is better.

Markov decision problem

Evaluation of the value of a policy

$$V^\pi(x) \stackrel{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_{t\geq 0} \sim \pi]$$

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$$V^\pi(x) = \sum_{a\in\mathcal{A}} \pi(x,a) \sum_{x'\in\mathcal{X}} \mathcal{P}(x,a,x')[\mathcal{R}(x,a,x') + \gamma V^\pi(x')]$$

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This may look complicated but all terms are known except the vector V.

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Development $\rightarrow$ a system of N linear equations with N unknowns, the $V^\pi(x), \forall x \in \mathcal{X}$.

In principle, an easy problem.
In practice, when N is large (*e.g.* 10^9), this is a challenging problem.

$V^\pi = \mathcal{M} V^\pi + Q$, where $\mathcal{M} \in \mathbb{R}^{N \times N}$ is an $N \times N$ matrix, and $Q \in \mathbb{R}^N$.
$\rightarrow (Id - \mathcal{M}) V^\pi = Q$

Thanks to contraction properties of $\mathcal{M}$, this can be solved iteratively.

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Thanks to contraction properties of $\mathcal{M}$, this can be solved iteratively.

This algorithm provides an estimation of V^π at most ϵ away from its true value.

It is very easy to implement and very fast, even when $N = 10^9$.

Markov decision problem

Evaluation of the value of a policy

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is known as a "Bellman equation".

[Bellman, *Dynamic Programming*, Princeton U. Press, 1957]

Markov decision problem

Example (continued)

- Once the value of a policy has been estimated, the policy can be improved.

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- Let us assume we estimated V^π. Then, we compute:

$$\pi'(x) \leftarrow \arg\max_{a \in \mathcal{A}} \sum_{x'} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^\pi(x')]$$

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  for each state $x \in \mathcal{X}$.
- Then π' is either better than π, or they are equivalent, that is: $V^{\pi'} \geq V^\pi$.

- Start with a random policy.
- Then, alternate:
  - estimate the value of the current policy
  - improve the current policy

  until the value of the policy no longer improves.

This is the "policy iteration" algorithm [Howard, 1958].

Markov decision problem

Value iteration

- The value V^* of the optimal policy π^* is:
  $V^*(x) \stackrel{\text{def}}{=} \max_\pi V^\pi(x), \forall x \in \mathcal{X}$

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  The "Bellman optimality equation".
- and an algorithm to obtain V^* directly:
  - $k \leftarrow 0$
  - initialize V_k
    - compute $V_{k+1} \leftarrow \max_{a \in \mathcal{A}} \sum_{x' \in \mathcal{X}} \mathcal{P}(x, a, x')[\mathcal{R}(x, a, x') + \gamma V^*(x')]$ for all x
    - $k \leftarrow k + 1$
  - repeat
  - until $||V_k - V_{k-1}||_\infty \leq \frac{\epsilon(1-\gamma)}{2\gamma}$

  named "value iteration".

Markov decision problem

As a linear program

We can frame a Markov decision problem as a linear program:

$$
\begin{aligned}
&min \sum_x V[x] \\
&\text{s.t. } V[x] - \sum_{x'} \mathcal{P}(x, a, x')(\mathcal{R}(x, a, x')) + \gamma V[x']) \geq 0 \ \forall (x, a) \in \mathcal{X} \times \mathcal{A}
\end{aligned}
$$

There one constraint for each (state, action) pair and one variable per state.
The dual is:

$$
\begin{aligned}
&max \sum_{(x,a)} \sum_{x'} \mathcal{P}(x, a, x')\mathcal{R}(x, a, x')\xi(x, a) \\
&\text{s.t. } \sum_a \xi(x', a) - \sum_{x,a} \gamma \mathcal{P}(x, a, x')\xi(x, a) \leq 1, \forall x' \in \mathcal{X} \\
&\text{and } \xi(x, a) \geq 0, \forall (x, a) \in \mathcal{X} \times \mathcal{A}
\end{aligned}
$$

Once solved, $\xi(x, a)$ is non zero if action a is optimal in x.

Very interesting theoretically speaking.
In practice, policy iteration is (usually) the best way to go.

Markov decision problem

Example (continued)

From MDPs to Reinforcement Learning

What if $\mathcal{P}$ and $\mathcal{R}$ are unknown?

- Reminder: $V^\pi \overset{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_{t>0} \sim \pi]$

  Let us define $Q^\pi(x, a) \overset{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_0 = a, a_{t>0} \sim \pi]$

  named the "quality" of the (state, action) pair (x, a).

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  named the "quality" of the (state, action) pair (x,a).
- We have: $V^\pi(x) = \sum_{a \in \mathcal{A}} \pi(x,a) Q^\pi(x,a)$.

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  Let us define $Q^\pi(x,a) \overset{\text{def}}{=} \mathbb{E}[\zeta(x)|x_0 = x, a_0 = a, a_{t>0} \sim \pi]$

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Reinforcement Learning

Bellman equation and the TD error [Sutton, 1988]

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  sum of what will happen immediately $+ \gamma \times$ what will happen then.
- at time t, $r_t + \gamma(V(x_{t+1}|\pi)) - V(x_t|\pi)$

  is an estimation of the error of estimation of V: TD-error

  This TD-error may be used to learn the optimal behavior.

- computing V by gradient descent:

  $V(x_{t+1}) \leftarrow V(x_t) - \eta[r_t + \gamma(V_t(x_{t+1})) - V(x_t)]$
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- $\rightsquigarrow$ learning π^* algorithm.

The goal of this algorithm is to estimate Q^* by interacting with the environment and correcting its estimate of Q^*.

1. initialize Q.
2. set the agent in some random initial state $\in \mathcal{X}_0$.
3. run the agent in the environment:
   at each step, record the state x_t, the action performed a_t, the reward collected r_t, and the next state x_{t+1}.
4. correct Q.
5. continue until a terminal state is reached.
6. do it again and again (re-starting at step 2).

Reinforcement Learning

Q-Learning [Watkins, 1989]

- $Q(x, a) \leftarrow$ some value (0, random, ...)
- **Repeat**
  - $t \leftarrow 0$
  - Initialize the state of the agent x_t
  - **while** episode not completed, **do:**
    - choose an action to perform in state x_t: a_t
    - perform this action and observe r_t and x_{t+1}
    - update $Q(x, a)$:
      $Q(x_t, a_t) \leftarrow Q(x_t, a_t) + \alpha[r_t + \max Q(x_{t+1}, b_b - Q(x_t, a_t)]$
    - $t++$
- **Until** some condition is met.

At the completion of this algorithm (if you looped enough):
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Remark: you never know if you looped enough!

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    Select the "best" action: $\arg\max_a Q(x_t, a)$.
- Exploration and exploitation should be carefully balanced.

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Reinforcement Learning

About Q-Learning

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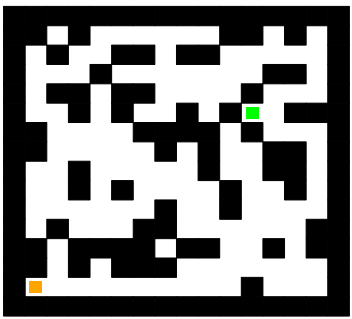
  - Slowly decrease τ along the episodes.
  - Rationale: when τ is large, uniformly random selection. τ close to 0, greedy selection.

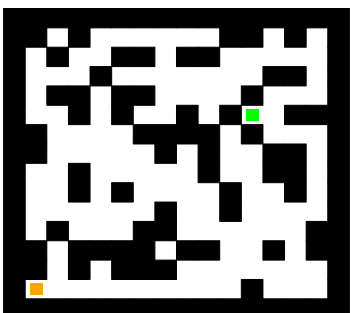


Question 1: propose a Markov decision process: what are ($\mathcal{T}$, $\mathcal{X}$, $\mathcal{X}_0$, $\mathcal{X}_f$, $\mathcal{A}$, $\mathcal{P}$, $\mathcal{R}$)?
Question 2: model this task as a Markov decision problem: what is ζ?

Reinforcement Learning

Q-Learning in action: escaping a maze

We use an extremely basic Q-Learning.
Has a very local perception: sees only the 4 neighboring cells.



Reinforcement Learning

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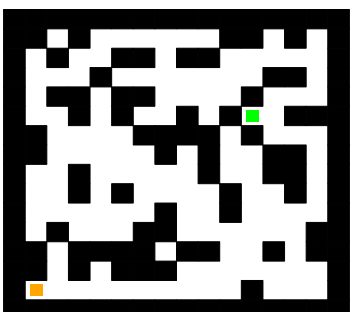
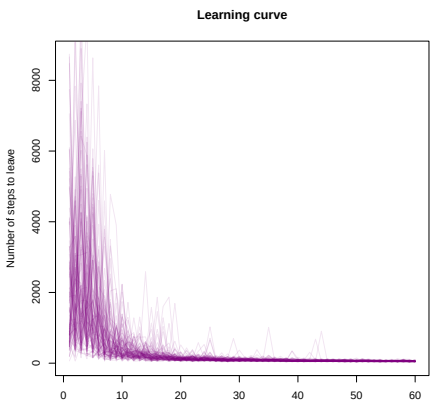
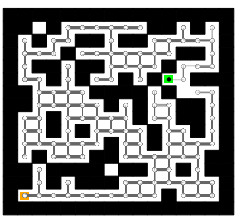




1st reach



Reinforcement Learning

Q-Learning in action

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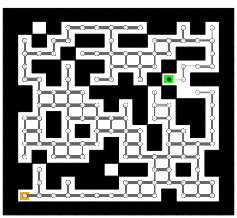
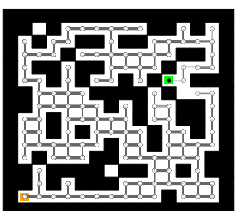


10th reach

1st reach



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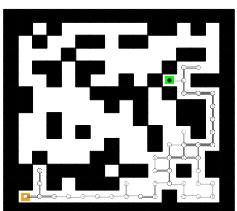
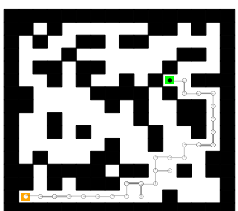
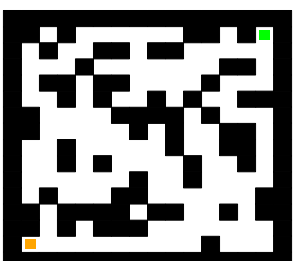
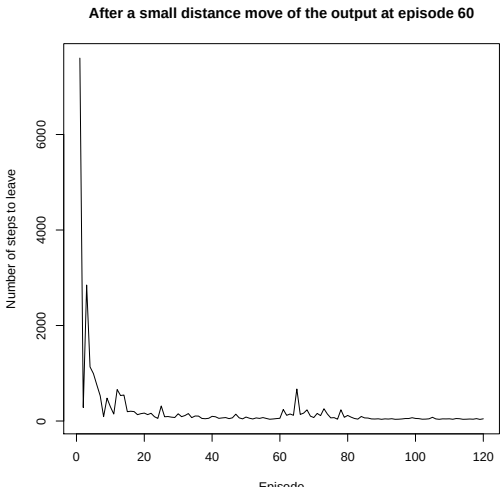


60th reach

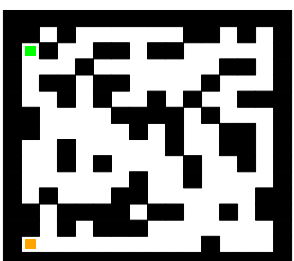


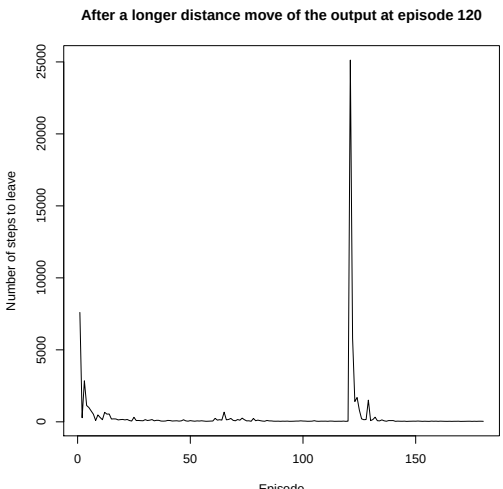
The goal state moves nearby:



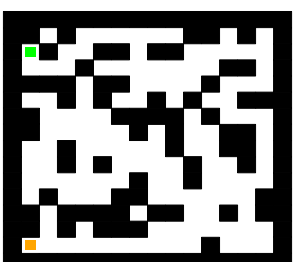


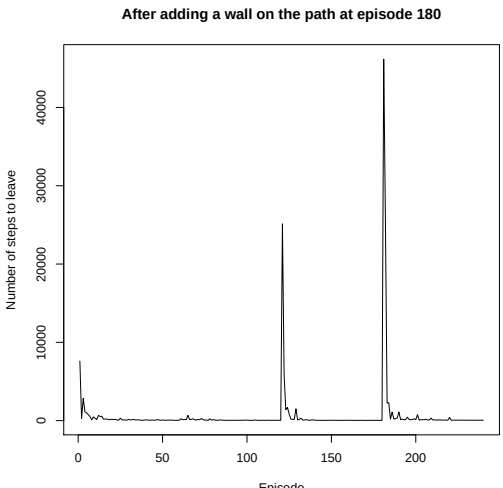
The goal state moves farther away:





Blocking the path:





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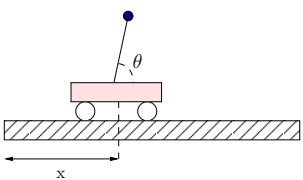
- This estimate may be updated/improved by learning.

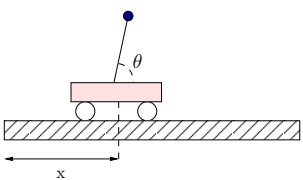


State is $(\theta, \dot{\theta})$

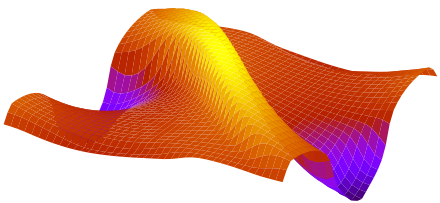
Action is $\ddot{\theta}$

Reinforcement Learning

Value function



State is $(\theta, \dot{\theta})$
Action is $\ddot{\theta}$



$(\theta, \dot{\theta})$ plane
z is $V(x)$
Maximize value $\rightsquigarrow$ reach the top of V

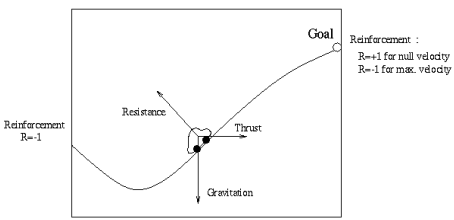
Reinforcement Learning

Handling large $\mathcal{X}$: the function approximator zoo

- neural network [Lin, 1991; Riedmiller, 2005; ...],
- random forest [Geurts *et al.*, 2006],
- SVM and kernels,
- and many other ideas from statistical learning (supervised learning).
- Tabular with progressive and adaptive state partitioning.

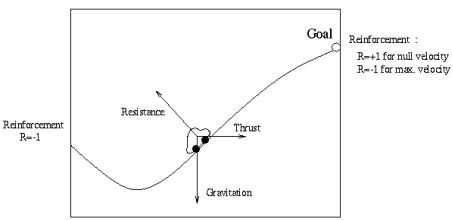
Reinforcement Learning

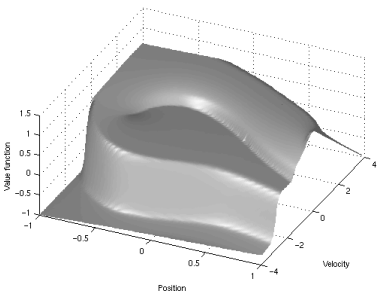
Progressive and adaptive state partitioning [Munos, Moore, MLJ, 2001]



Reinforcement Learning

Progressive and adaptive state partitioning [Munos, Moore, MLJ, 2001]





Reinforcement Learning

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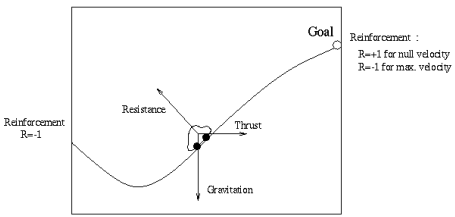
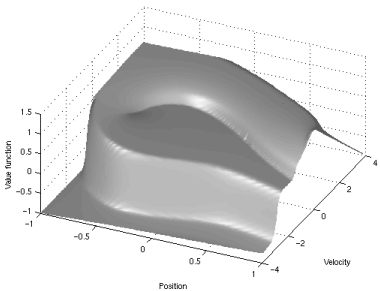
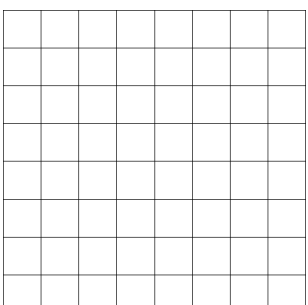
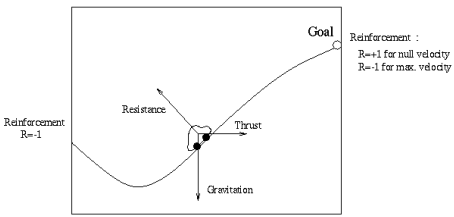


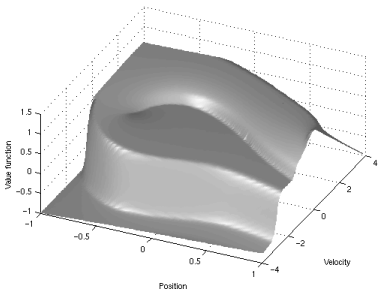


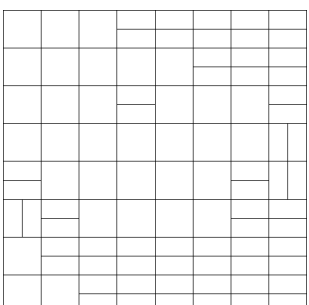


Reinforcement Learning

Progressive and adaptive state partitioning [Munos, Moore, MLJ, 2001]

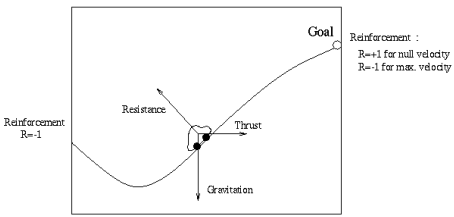


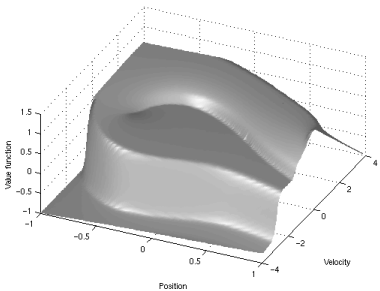


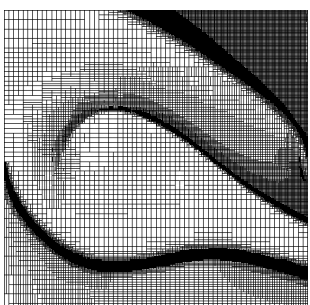


Reinforcement Learning

Progressive and adaptive state partitioning [Munos, Moore, MLJ, 2001]







Trendy people call that "deep RL" though there is only shallow networks most of the times.

- Use of a neural network to represent the Q table.
- Input = the current state
- Output = either an estimate of $Q(x, a), \forall a$, or an estimate of $\pi(x)$.

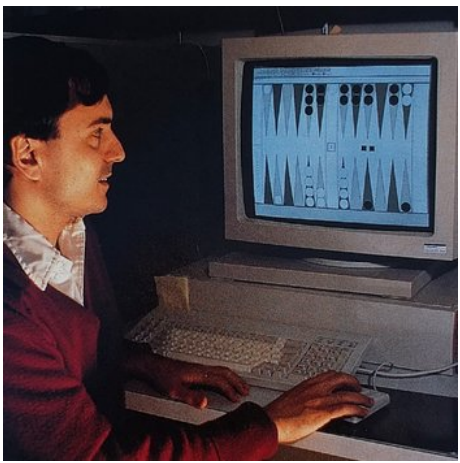
Reinforcement Learning

Neural RL

-
-

Reinforcement Learning

Application: TD-Gammon



Reinforcement Learning

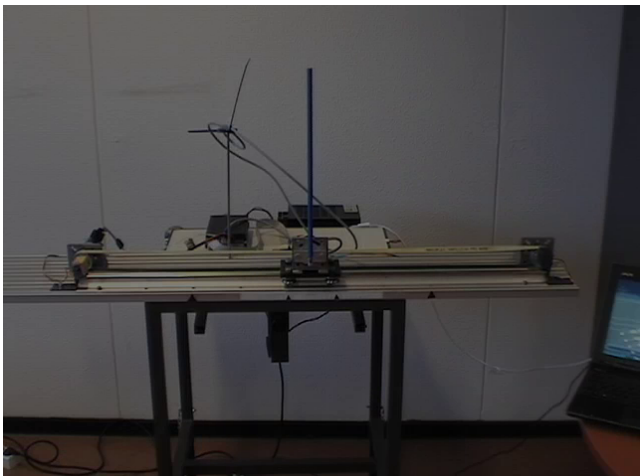
Application: TD-Gammon

- Backgammon is studied at least since 1974
- Branching factor: 800
- TD-Gammon: "successor of Neurogammon, trained by superviser learning. NeuroGammon won the 1st Computer Olympiad in London in 1989, handily defeating all opponents. Its level of play was that of an intermediate-level human player." (Source: wikipedia)
- raw representation of the board position
- trained with TD(λ) algorithm
- no knowledge, self-play
- hand-crafted features
- 3-plies in v3

Tesauro, Temporal Difference Learning and TD-Gammon, *Communications of the ACM*, 1995

Reinforcement Learning

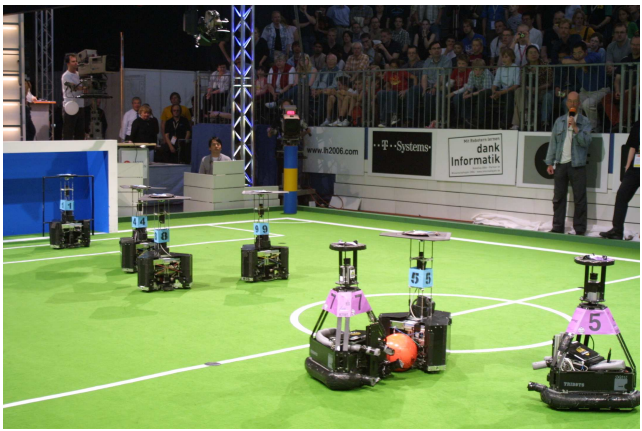
Application to robotics



Riedmiller, Neural reinforcement learning to swing-up and balance a real pole, *Proc. 2005 IEEE International Conference on Systems, Man and Cybernetics*

Reinforcement Learning

Application to robotics



Lauer et al., Cognitive Concepts in Autonomous Soccer Playing Robots, *Cognitive Systems Research*, **11**(3), 287:309, September, 2010
(No Deep Learning! only shallow multi-layer perceptron)

Reinforcement Learning

Application: board games

- Learning to play board games using only the rules of the game
- Alpha Go learned to play Go by using games played by humans
- Alpha Zero learned to play even better by itself by RL.
- then other board games (chess, draughts, reversi, ...)
- then Starcraft II

Reinforcement Learning

Alpha Zero type of algorithms

- RL
- + various tricks to stabilize learning and make it more efficient
- MCTS as a key component
- moderately deep network as function approximator

Outro

- learning options
- learning representation
- generalization in RL
- time varying environments
- transfer learning
- life-long learning
- explaination/accountability of the learned behavior

Take home message

- Many problems can benefit from a sequential decision making point of view.




Not only games.

Take home message

- Many problems can benefit from a sequential decision making point of view.




  Not only games.

- Reinforcement learning outperforms supervised learning.

Bibliography

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