Equidistribution of Roots of Unity and the Mahler Measure

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Diophantine Problems, Determinism and Randomness
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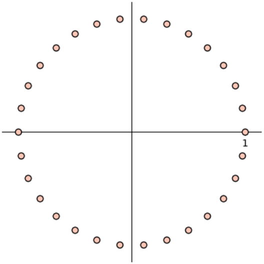
Joint work with Vesselin Dimitrov

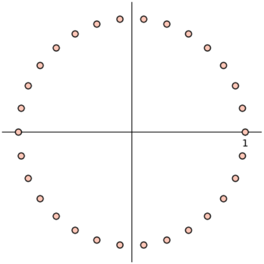
Equidistribution of Roots of Unity



Order dividing 30

$$\left\{ e^{2\pi ik/N} : k \in \mathbb{Z} \right\} \text{ become equdistributed as } N \to \infty.$$

Equidistribution of Roots of Unity



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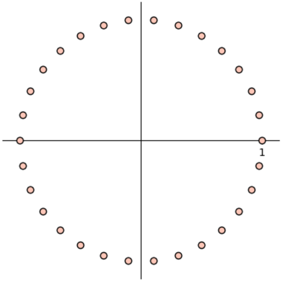
If $f\colon S^1 = \{z \in \mathbb{C} : |z| = 1\} \to \mathbb{C}$ is continuous, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} f(e^{2\pi ik/N}) = \int_0^1 f(e^{2\pi it})\mathrm{d}t.$$

Equidistribution of Roots of Unity



Order dividing 30

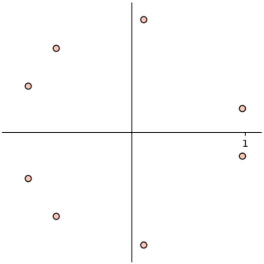
For the test function $f(z) = z^l$ with $l \in \mathbb{Z}$ we have

$$\frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N}) =$$

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{2\pi ikl/N} = \begin{cases} 0 & : N \nmid l, \\ 1 & : N \mid l. \end{cases}$$

$$f(z) = \sum_{l=-L}^{L} a_l z^l \Rightarrow \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N}) = a_0 = \int_0^1 f(e^{2\pi it})\mathrm{d}t.$$

Equidistribution of Galois Conjugates

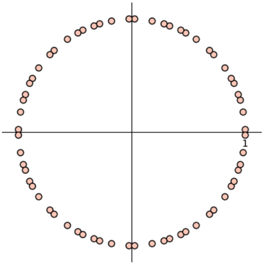


Order (exactly) $N = 30$

Number of roots of unity in $\left\{e^{2\pi ik/N} : k \in \mathbb{Z} \text{ coprime to } N\right\}$ is $\varphi(N)$.
It is the set of $\mathbb{Q}$-Galois conjugates of $e^{2\pi i/N}$. Equidistribution follows from

$$\frac{1}{\varphi(N)} \sum_{\substack{k=1 \\ \gcd(k,N)=1}}^{N} e^{2\pi ikl/N} = \pm\phi\left(\frac{N}{\gcd(N,l)}\right)^{-1}.$$

Equidistribution of Galois Conjugates



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Logarithmic Singularities

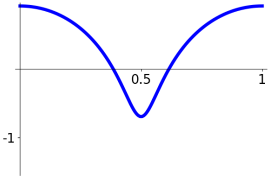
We check equdistribution with **continuous** test functions. What about a weaker hypothesis?

Theorem (M. Baker, Ih, Rumely 2008)

Let $\alpha \in \mathbb{C}$ be algebraic, then

$$\frac{1}{\varphi(N)} \sum_{\substack{k=1 \\ \gcd(k,N)=1}}^{N} \log\left|e^{2\pi i k/N} - \alpha\right| \to \int_0^1 \log\left|e^{2\pi i t} - \alpha\right| \mathrm{d}t$$

as $N = \operatorname{ord}\zeta \to \infty$.



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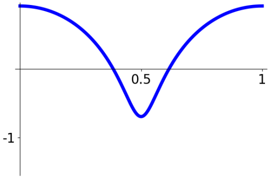

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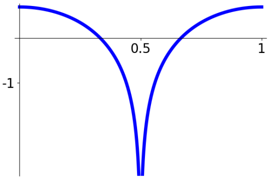

$\alpha = -1$

Logarithmic Singularities

We check equdistribution with **continuous** test functions. What about a weaker hypothesis?

Theorem (M. Baker, Ih, Rumely 2008)

Let $\alpha \in \mathbb{C}$ *be* *algebraic, then*

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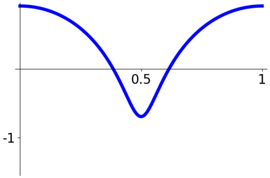

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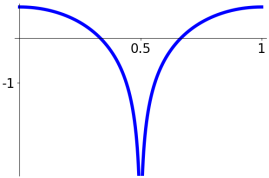

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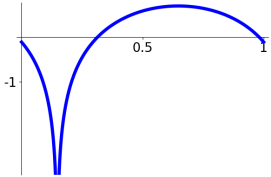


$\alpha = \frac{3+4i}{5}$

- $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ with its Haar measure $\mathrm{d}\lambda$
- $\mu_\infty = \{\zeta \in \mathbb{C}$ a root of unity$\}$
- σ denotes an element of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/\operatorname{ord}(\zeta)\mathbb{Z})^\times$

Theorem (M. Baker, Ih, Rumely 2008)

Suppose $P \in \overline{\mathbb{Q}}[T] \smallsetminus \{0\}$, then

$$\frac{1}{[\mathbb{Q}(\zeta):\mathbb{Q}]} \sum_\sigma \log|P(\zeta^\sigma)| \to \int_0^1 \log\left|P(e^{2\pi i t})\right| \mathrm{d}t = \int_{S^1} \log|P| \mathrm{d}\lambda$$

as $\zeta \in \mu_\infty$ and $\operatorname{ord}\zeta \to \infty$.

The <u>Mahler measure</u> of $P = p_d(T-\alpha_1)\cdots(T-\alpha_d) \in \mathbb{C}[T] \smallsetminus \{0\}$ is

$$m(P) \overset{\mathrm{Def}}{=} \int_{S^1} \log|P| \mathrm{d}\lambda = \log|p_d| + \sum_{k=1}^{d} \log\max\{1, |\alpha_k|\}.$$

What about Higher Dimension?

The $\mathbb{Q}$-Galois conjugates of

$$\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d), \qquad \zeta_j = e^{2\pi i a_j / N} \text{ with } a_j \in \mathbb{Z}, \quad \gcd(a_1, \ldots, a_d, N) = 1.$$

are

$$\boldsymbol{\zeta}^k \text{ with } k \in \mathbb{Z} \text{ and } \gcd(k, N) = 1.$$

For $d > 1$ equdistribution does <u>not</u> follow from $N \to \infty$.

Definition

For $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d) \in \mu_\infty^d$ we define

$$\delta(\boldsymbol{\zeta}) = \min\{|\boldsymbol{b}| : \boldsymbol{b} = (b_1, \ldots, b_d) \in \mathbb{Z}^d \smallsetminus \{0\} \text{ with } \boldsymbol{\zeta}^{\boldsymbol{b}} = \zeta_1^{b_1} \cdots \zeta_d^{b_d} = 1\}.$$

Equidistribution of Galois Conjugates in Dimension d

Fact

Let $f : (S^1)^d \to \mathbb{C}$ be continuous. Then

$$\frac{1}{[\mathbb{Q}(\zeta) : \mathbb{Q}]} \sum_{\sigma} f(\zeta^{\sigma}) \to \int_{(S^1)^d} f \mathrm{d}\boldsymbol{\lambda} \quad \textit{as} \quad \delta(\zeta) \to \infty.$$

Conjecture

Suppose $P \in \overline{\mathbb{Q}}[T_1, \ldots, T_d] \setminus \{0\}$, then

$$\frac{1}{[\mathbb{Q}(\zeta) : \mathbb{Q}]} \sum_{\sigma} \log |P(\zeta^{\sigma})| \to \int_{(S^1)^d} \log |P| \mathrm{d}\boldsymbol{\lambda} \overset{\text{Def}}{=} m(P) \quad \textit{as} \quad \delta(\zeta) \to \infty.$$

Why is the average on the left well-defined for large $\delta(\zeta)$? Laurent's Theorem / the Manin–Mumford Conjecture.

Evidence in Dimension > 1

Theorem (Myerson 1980, Duke 2007)

Let $p \equiv 1 \pmod 3$ be a prime and $G = \{1, a, b\} \subset \mathbb{F}_p^\times$ the subgroup of order 3. Then

$$\frac{1}{p-1}\sum_{\sigma} \log|\zeta^{\sigma} + \zeta^{a\sigma} + \zeta^{b\sigma}| = m(T_1 + T_2 + T_3) + O\left(\frac{\log p}{\sqrt{p}}\right)$$

where $\zeta \in \mu_\infty$ has order p.

Smyth computed $m(T_1 + T_2 + T_3) = L'(-1, \chi_3) = 0.323\ldots$.

Atoral Polynomials

A following definition is similar as in work of Agler–McCarthy–Stankus (2006) and Lind–Schmidt-Verbitskiy (2013).

Say $P \in \mathbb{C}[T_1, \ldots, T_d] \setminus \{0\}$. Consider the **real-algebraic** set

$$A = \left\{ (z_1, \ldots, z_d) \in (S^1)^d : P(z_1, \ldots, z_d) = 0 \right\}.$$

If $(z_1, \ldots, z_d) \in A$, complex conjugation gives 2 relations in Laurent polynomials. They may or may not be coprime.

$$\begin{cases} P(z_1, \ldots, z_d) & = 0 \\ \overline{P}(z_1^{-1}, \ldots, z_d^{-1}) & = 0 \end{cases}$$

Definition

We call P <u>atoral</u> if there exist coprime polynomials R and S such that

$$A \subset R^{-1}(\{0\}) \cap S^{-1}(\{0\}).$$

- $d = 1$: atoral amounts to $P^{-1}(\{0\}) \cap S^1 = \emptyset$.
- $d = 2$: atoral amounts to "$\{(z_1, z_2) \in S^1 \times S^1 : P(z_1, z_2) = 0\}$ finite"
- $T_1 + T_2 + \cdots + T_d$ is atoral
- **Not all polynomials are atoral.** e.g. Blaschke products

Theorem (Lind–Schmidt–Verbitskiy (2013))

Let $P \in \mathbb{Z}[T_1, \ldots, T_d] \smallsetminus \{0\}$ be atoral, then

$$\frac{1}{\# G} \sum_{\substack{\zeta \in G \\ P(\zeta) \neq 0}} \log |P(\zeta)| = m(P) + o(1)$$

as G ranges over finite subgroups of $(\mathbb{C}^{\times})^d$ with $\delta(G) \to \infty$.

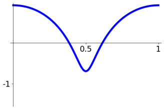
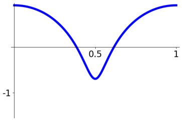
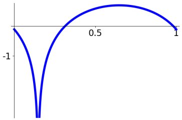
- If $P^{-1}(\{0\}) \cap (S^1)^d$ is empty, then convergences follows from classical equidistribution of tuples of roots of unity.
- Lind–Schmidt–Verbitskiy (2010): convergence if $P^{-1}(\{0\}) \cap (S^1)^d$ is finite
- Dimitrov (2017) dropped the hypothesis on P when averaging over subgroups $G = \{\boldsymbol{\zeta} \in \mu_\infty^d : \boldsymbol{\zeta}^N = 1\}$.

Theorem (Dimitrov–H.)

Suppose $P \in \overline{\mathbb{Q}}[T_1, \ldots, T_d] \setminus \{0\}$ *is* <u>*atoral*</u>, *then*

$$\frac{1}{[\mathbb{Q}(\zeta) : \mathbb{Q}]} \sum_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \log |P(\zeta^\sigma)| \to m(P) \quad \textit{as} \quad \delta(\zeta) \to \infty.$$

- The convergence rate is $O_P(\delta(\zeta)^{-\epsilon_P})$
- We can generalize to **essentially atoral** P
- $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ can be replaced by $\mathrm{Gal}(K(\zeta)/K)$ with K a fixed number field

$d = 1$			
$d \geq 2$	$\alpha = -\frac{3}{2}$ **atoral**	$\alpha = -1$ **essentially atoral**	$\alpha = \frac{3+4i}{5}$ **(conjectural)**

An Example

$$\boxed{\begin{array}{c} T_1 + T_2 + T_3 + T_4, \\ T_1^{-1} + T_2^{-1} + T_3^{-1} + T_4^{-1} \\ \text{are coprime in } \mathbb{C}[T_1^{\pm 1}, \ldots, T_4^{\pm 1}] \end{array}} \Rightarrow P = T_1 + T_2 + T_3 + T_4 \text{ is atoral}$$

Boyd (1981): $m(P) = \frac{4\zeta(3)}{2\pi^2} = 0.243587656467\ldots > 0.$

Let $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4) \in \mu_\infty^4$ be a quadruple with

$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$ an **algebraic unit**.

$$\frac{1}{[\mathbb{Q}(\boldsymbol{\zeta}) : \mathbb{Q}]} \sum_\sigma \log|\zeta_1^\sigma + \cdots + \zeta_4^\sigma| \overset{\text{Theorem}}{=} \frac{4\zeta(3)}{2\pi^2} + o(1) \text{ as } \delta(\boldsymbol{\zeta}) \to \infty$$

$$\Rightarrow \delta(\boldsymbol{\zeta}) \text{ is bounded.}$$

Conclusion: There exists a constant $B \geq 1$ such that if $\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4$ is an algebraic unit, then

$$\zeta_1^{a_1}\zeta_2^{a_2}\zeta_3^{a_3}\zeta_4^{a_4} = 1 \text{ for some } (a_1, \ldots, a_4) \in \mathbb{Z}^4 \smallsetminus \{0\}$$
$$\text{with } \max\{|a_1|, \ldots, |a_4|\} \leq B.$$

Step A: Univariable Case

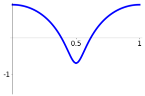
Proposition

Say $Q \in \mathbb{Z}[T] \smallsetminus \{0\}$ has no roots on S^1, $\epsilon > 0$. If $\zeta \in \mu_\infty$ has order N, then

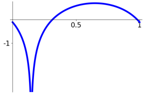
$$\frac{1}{\varphi(N)} \sum_{\sigma} \log|Q(\zeta^\sigma)| = m(Q) + O_\epsilon\left(\frac{(\deg Q)^{1+\epsilon}(1+m(Q))}{N^{1-\epsilon}}\right).$$

We treat first $Q = T - \alpha$. Truncate

$$\frac{1}{\varphi(N)} \sum_{\sigma:|\zeta^\sigma - \alpha| > 1/N^2} \log|\zeta^\sigma - \alpha| + \frac{1}{\varphi(N)} \sum_{\sigma:|\zeta^\sigma - \alpha| \le 1/N^2} \log|\zeta^\sigma - \alpha|$$



average is about $\log \max\{1, |\alpha|\}$



at most one term σ^*

Worst case:

$$\frac{1}{\varphi(N)}\sum_{\sigma}\log|\zeta^{\sigma}-\alpha| = \log\max\{1,|\alpha|\} + \frac{1}{\varphi(N)}\log|\zeta^{\sigma^*}-\alpha| + O(\cdots)$$

with $|\zeta^{\sigma^*}-\alpha|$ very small. NB: $\zeta^{\sigma^*}=e^{2\pi iq}$ with $q\in\mathbb{Q}$.
Temptation: apply Baker's linear forms in logarithms. Dependency on $[\mathbb{Q}(\alpha):\mathbb{Q}]$ not good enough to help, as observed by Duke.

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$$|\zeta^{\sigma}-\alpha| \geq |1-|\alpha|| \gg |\alpha-\overline{\alpha}^{-1}| \text{ for } |\alpha| \text{ close to } 1.$$

α and $\overline{\alpha}^{-1}$ are roots of $Q(T)Q(T^{-1})T^d \in \mathbb{Z}[T]$.

Theorem (Mahler 1964)

If $z, w \in \mathbb{C}$ are distinct roots of $F \in \mathbb{Z}[T] \smallsetminus \{0\}$ and $D = \deg F$, then

$$\log|z-w| \geq -\frac{1}{2}(D+2)\log D - Dm(F).$$

$$\Rightarrow \frac{1}{\varphi(N)}\log|\zeta^{\sigma}-\alpha| \gg -\deg(P)\frac{\log\deg(P)+m(P)}{\varphi(N)}.$$

Mignotte (1995): strengthening of Mahler to several pairs of roots

Step B: Reducing to the Univariate Case

Now $P \in \mathbb{Z}[T_1, \dots, T_d] \smallsetminus \{0\}$ is atoral.
Let $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_d)$ have order N. There is $\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ with

$$\begin{aligned}
&\boldsymbol{\zeta} = (\zeta^{a_1}, \dots, \zeta^{a_d}) = \zeta^{\boldsymbol{a}} && \text{(for some } \zeta \text{ of order } N) \\
\Rightarrow &\boldsymbol{\zeta}^{\tau} = \zeta^{\tau \boldsymbol{a} + N\boldsymbol{b}} && \text{(for all } \boldsymbol{b} \in \mathbb{Z}^d, \tau \in \mathbb{Z} : \gcd(\tau, N) = 1) \\
\Rightarrow &|P(\boldsymbol{\zeta}^{\tau})| = |Q(\zeta)| && \text{(with } Q(T) = T^{?} P(T^{\tau \boldsymbol{a} + N\boldsymbol{b}}) \in \mathbb{Z}[T]) \\
\Rightarrow &\sum_{\sigma} \log |P(\boldsymbol{\zeta}^{\sigma})| = \sum_{\sigma} \log |Q(\zeta^{\sigma})| && \text{(by the univariate case)}
\end{aligned}$$

Use Erdös–Turán–Koksma to find τ and $\boldsymbol{b}$ with

$$\deg Q = O(|\tau \boldsymbol{a} + N\boldsymbol{b}|) = O\left(\frac{N}{\delta(\boldsymbol{\zeta})^{\kappa_d}}\right).$$

If $\delta(\boldsymbol{\zeta})$ grows at least like a small power of N, the proposition gives

$$\frac{1}{\varphi(N)} \sum_{\sigma} \log |P(\boldsymbol{\zeta}^{\sigma})| = m(Q) + O_{\epsilon}\left(\frac{(\deg P)^{1+\epsilon}(1 + m(Q))}{N^{1-\epsilon}}\right) = m(Q) + o(1).$$

$$\delta(\zeta) \text{ grows polynomially in } N \Rightarrow \frac{1}{\varphi(N)} \sum_{\sigma} \log |P(\zeta^{\sigma})| = m(Q) + o(1)$$

Issues:

- If $\delta(\zeta)$ grows slowly: monomial change of coordinates and induction.
- Need to relate $m(Q)$ to $m(P)$. We require a quantitative version of

Theorem (Lawton 1983)

Let $d \geq 2, P \in \mathbb{C}[T_1, \dots, T_d] \setminus \{0\}$ and $\boldsymbol{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$, then

$$m(P(T^{a_1}, \dots, T^{a_d})) = m(P) + o(1)$$

as $\min\{|\boldsymbol{b}| : \boldsymbol{b} \in \mathbb{Z}^d \setminus \{0\}$ and $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = 0\} \to \infty$.

This kind of statement helps in showing $m(Q) = m(P) + o(1)$.

- **Warning:** The proposition only applies if $Q(z) \neq 0$ for all $z \in S^1$.

Warning: The proposition only applies if $Q(z) \neq 0$ for all $z \in S^1$. Up-to a power of T:

$$Q(T) = P(T^{\tau \boldsymbol{a} + N\boldsymbol{b}})$$

If $z \in S^1$, then

$$Q(z) = 0 \Rightarrow P(z^{\tau \boldsymbol{a} + N\boldsymbol{b}}) = 0 \Rightarrow z^{\tau \boldsymbol{a} + N\boldsymbol{b}} \in P^{-1}(\{0\}) \cap (S^1)^d$$

Recall that P is atoral. There are coprime polynomials R and S in d variables, fixed in terms of P, with

$$R(z^{\tau \boldsymbol{a} + N\boldsymbol{b}}) = S(z^{\tau \boldsymbol{a} + N\boldsymbol{b}}) = 0.$$

The point $z^{\tau \boldsymbol{a} + N\boldsymbol{b}}$ lies in a 1-dimensional algebraic subgroup of $(\mathbb{C}^{\times})^d$. It is an unlikely intersection.

Theorem (Bombieri–Masser–Zannier 2007)

In the setup above, there exists $B \geq 1$, depending only on (R, S), with

$$\langle \tau \boldsymbol{a} + N\boldsymbol{b}, \boldsymbol{c} \rangle = 0 \quad \textit{for some} \quad \boldsymbol{c} \in \mathbb{Z}^d \smallsetminus \{0\} \textit{ with } |\boldsymbol{c}| \leq B.$$

By the choice of τ and $\boldsymbol{b}$ we have

$$\begin{aligned}\boldsymbol{\zeta} = \zeta^{\tau \boldsymbol{a} + N\boldsymbol{b}} &\Rightarrow \boldsymbol{\zeta}^{\boldsymbol{c}} = \zeta^{\langle \tau \boldsymbol{a} + N\boldsymbol{b}, \boldsymbol{c}\rangle} = 1 \\ &\Rightarrow \delta(\boldsymbol{\zeta}) \leq |\boldsymbol{c}| \leq B.\end{aligned}$$

Thanks for your attention!