Homework Nr.1

Philipp Stassen

September 24, 2018

Remark. In this solution proposals a ring is always meant to be a commutative ring with 1.

7.1

Exercise 26

Let K be a field and $\nu : K^\times \to \mathbb{Z}$ be a discrete valuation on K. Let $R = \{x \in K^\times | v(x) \geq 0\}$.

a) <u>*Claim:*</u> R is a subring of K which contains the identity.

Proof. It suffices to show that R is closed under *addition*, *multiplikation* and *additive inverses* as well as that it contains 0 and 1. Properties as commutativity of $+$ and $\cdot$, associativity or distributivity are all valid as $R \subset K$ and K is a field.

1. R is closed under *addition*: Let $x, y \in R$.
   Case 1: $x + y \neq 0$. Then $\nu(x) \geq 0$ and $\nu(y) \geq 0$. Hence, $\nu(x + y) \geq \min\{\nu(x), \nu(y)\} \geq 0$. Therefore, $x + y \in R$.
   Case 2: $x + y = 0$. This is trivial as $0 \in R$.

2. R is closed unter *multiplikation*:
   Let $x, y \in R$ then $\nu(x) \geq 0$ and $\nu(y) \geq 0$. We have that $\nu(xy) = \nu(x) + \nu(y) \geq 0$. Therefore, $xy \in R$.

3. R contains 0.

4. R contains 1: $\nu(a) = \nu(a1) = \nu(a) + \nu(1)$. Therefore, $\nu(1) = 0$ and $1 \in R$.

5. R is closed under building *additive inverses*:
   Observe that we have for any $x \in R$

$$0 = x \cdot 0 = x \cdot (1 + (-1)) = x \cdot 1 + x \cdot (-1) \tag{1}$$

$$\iff -x = (-1) \cdot x \tag{2}$$

   and by in particular $1 = -(-1) = (-1) \cdot (-1)$. Furthermore, it is

$$0 = \nu(1) = \nu((-1) \cdot (-1)) = 2\nu(-1) \tag{3}$$

Therefore, $\nu(-1) = 0$. Now we can conclude that for any $x \in K$ holds $\nu(-x) = \nu(x \cdot (-1)) = \nu(x) + \nu(-1) = \nu(x)$. Hence, if $x \in R$ then also $-x \in R$.

□

b) _Claim:_ For every nonzero element $x \in K$ either x or x^{-1} is in R.

Proof. Let $x \in K^\times$, as K is a field every nonzero element is a unit. As ν is a discrete valuation we have

$$0 = \nu(1) = \nu(x \cdot x^{-1}) = \nu(x) + \nu(x^{-1}) \tag{4}$$

Hence, $\nu(x) = -\nu(x^{-1})$ and $\nu(x) \geq 0$ or $\nu(x^{-1}) \geq 0$. This implies that either $x \in R$ or $x^{-1} \in R$. □

c) $x \in R$ is a unit iff $\nu(x) = 0$.

Proof. "$\Longrightarrow$" Let $x \in R$ be a unit. Then there is $b \in R$ such that $xb = 1$. We have that $\nu(x) \geq 0$ and $\nu(b) \geq 0$ as $x, b \in R$.

Furthermore,

$$0 = \nu(1) = \nu(xb) = \nu(x) + \nu(b). \tag{5}$$

This implies that $\nu(x) = \nu(b) = 0$.

"$\Longleftarrow$" Let $\nu(x) = 0$. We have that

$$0 = \nu(1) = \nu(x \cdot x^{-1}) = \nu(x) + \nu(x^{-1}) = \nu(x^{-1}). \tag{6}$$

Hence, $\nu(x^{-1}) \geq 0$ and $x^{-1} \in R$. This makes x a unit of R. □

Chapter 7.4

Exercise 37

Let R be a local ring. Let $\mathfrak{m}$ be the maximal ideal of R.

Claim: Every element of $R - \mathfrak{m} = \{r \in R | r \notin \mathfrak{m}\}$ is a unit.

Proof. Assume $r \in R$ is a nonunit and $r \notin \mathfrak{m}$. Then the ideal that is generated from $\mathfrak{m}$ and r - lets call it M - is strict larger than $\mathfrak{m}$ (as it contains r) but also $M \subsetneq R$ as $1 \notin M$. This contradicts the maximality of $\mathfrak{m}$. □

Let $M = \{r \in R | r \text{ is not a unit}\}$ be the ideal of nonunit forms. _Claim:_ M is the unique maximal ideal in R.

Proof. Assume there is an ideal $\mathfrak{m} \supsetneq M$ then there exists $r \in \mathfrak{m}$ such that $r \notin M$. Hence, r is a unit and $1 \in \mathfrak{m}$. Therefore, $\mathfrak{m} = R$ and $\mathfrak{m}$ is not a proper ideal. This implies that M is a maximal ideal.

M is also unique, as any proper ideal $I \subsetneq R$ is also contained in M. If there was an Ideal $I \nsubseteq M$ then by the same argument as before we already have $I = R$. □

Exercise 40

Lemma. Let $\mathfrak{p} \subsetneq R$ be a prime ideal then $\mathfrak{N}(R) \subseteq \mathfrak{p}$.

Proof. Let $r \in \mathfrak{N}(R)$ and $n \in \mathbb{N}$ such that $0 = r^n = r^{n-1}r$, we want to show that $r \in \mathfrak{p}$. As $r^n = 0 \in \mathfrak{p}$ we have that either $r \in \mathfrak{p}$ or $r^{n-1} \in \mathfrak{p}$. Wlog $r^{n-1} \in \mathfrak{p}$ (otherwise we are done). Now we can induct on $n \in \mathbb{N}$ eventually showing that $r \in \mathfrak{p}$. □

<u>*Claim:*</u> Let R be a commutative Ring and let $\mathfrak{N}(R)$ be the nilradical of R. The following are equivalent:

1. R has exactly one prime ideal
2. every element of R is either nilpotent or a unit.
3. $R/\mathfrak{N}(R)$ is a field

Proof. "$(i) \implies (ii)$" Let $\mathfrak{p} \subsetneq R$ be the only prime ideal.

We know that $\mathfrak{N}(R) = \bigcap_{\mathfrak{q} \text{ prime}} \mathfrak{q}$[1]. Hence, $\mathfrak{N}(R) = \mathfrak{p}$. Furthermore, every proper ideal is contained in a maximal ideal[2]. Hence, we have $\mathfrak{p} \subseteq \mathfrak{m}$. As every maximal ideal in a commutative ring is prime[3] we must have $\mathfrak{p} = \mathfrak{m}$. Now we can conclude that either $r \in \mathfrak{N}(R)$, i.e. r is nilpotent, or $r \in R - \mathfrak{N}(R)$, i.e. r is a unit, (by exercise 37).

"$(ii) \implies (iii)$" Let every element of R be either nilpotent or a unit. $\mathfrak{N}(R)$ is an ideal. It is also maximal as every proper superset of $\mathfrak{N}(R)$ contains a unit. Hence, $R/\mathfrak{N}(R)$ is a field[4].

"$(iii) \implies (i)$" Let $R/\mathfrak{N}(R)$ be a field. Then $\mathfrak{N}(R)$ is a maximal ideal[5]. Let $\mathfrak{p}$ be a prime ideal then $\mathfrak{N}(R) \subseteq \mathfrak{p}$ and as $\mathfrak{N}(R)$ is maximal we have $\mathfrak{p} = \mathfrak{N}(R)$. Therefore $\mathfrak{N}(R)$ is the only prime ideal of R. □

[1] DF §15.2 Proposition 12
[2] DF §7.4 Proposition 11
[3] DF §7.4 Corollary 14
[4] DF §7.4 Proposition 12
[5] DF §7.4 Proposition 12

Chapter 15.1

Exercise 2

a) Let R denote the Ring of continuous, real-valued functions on $[0,1]$. *Claim:* R is not noetherian.

Proof. We define the sequence of ideals

$$I_n := \{f \in R | f(x) = 0, \forall x \leq \frac{1}{n} : \} \tag{7}$$

Claim(1): Every I_n is an ideal.
We need to show that

1. I_n is nonempty and $\forall x, y \in I : x - y \in I$

   proof. Take $f, g \in I_n$ then $\forall x \leq \frac{1}{n} : (f-g)(x) = 0$. Hence, $f - g \in I_n$.

2. $\forall x \in I, \forall r \in R : r \cdot x \in I$

   proof. Take $f \in I_n$ and $r \in R$. Then $(f \cdot r)(x) = f(x)r(x)$. As $f(x) = 0$ for all $x \leq \frac{1}{n}$ we also have that $f(x)r(x) = 0$ for all $x \leq \frac{1}{n}$. Hence, $(f \cdot r) \in I_n$. $\square(1)$

Furthermore, for all $m, n \in \mathbb{N}$ with $m < n$ we have that $I_m \subsetneq I_n$. Hence, $I_0 \subsetneq I_1 \subsetneq ... \subsetneq$. is an infinite ascendig chain of ideals that is not eventually constant. $\square$

b) Let X be any infinite set and let $R = \{f : X \to \mathbb{Z}/2\mathbb{Z}\}$ denote the Ring of all functions from X to $\mathbb{Z}/2\mathbb{Z}$. *Claim:* R is not noetherian.

Proof. As X is infinite we have an injection $\iota : \mathbb{N} \hookrightarrow X$. Now we can define the ideals

$$I_n := \{f \in R | f(\iota(m)) = 0, \forall m \geq n : \} \tag{8}$$

Claim(1): Every I_n is an ideal.
We need to show that

1. I_n is nonempty and $\forall x, y \in I : x - y \in I$

   proof. Take $f, g \in I_n$ then $\forall m \geq n : (f-g)(\iota(m)) = 0$. Hence, $f - g \in I_n$.

2. $\forall x \in I, \forall r \in R : r \cdot x \in I$

   proof. Take $f \in I_n$ and $r \in R$. Then $(f \cdot r)(\iota(m)) = f(\iota(m))r(\iota(m))$. As $f(\iota(m)) = 0$ for all $m \geq n$ we also have that $f(\iota(m))r(\iota(m)) = 0$ for all $m \geq n$. Hence, $(f \cdot r) \in I_n$. qed(1)

we have that each I_n is an ideal and that $A_0 \subsetneq ... \subsetneq ...$ is an ascending chain of ideals that is not eventually constant. $\square$

Exercise 3

Claim: Let K be a field. $K(x)$ is not a finitely generated k-algebra.

Proof. Let $g_1, ..., g_n \in K(x)$ with $g_i = \frac{P_i}{Q_i}$ and $P_i, Q_i \in K[X]$ be a finite set of generators. Define $d := \prod_{i \le n} Q_i$. Then we have for each g_i that $g_i \in K[X, d^{-1}]$ as

$$g_i = \frac{P_i \cdot \prod_{j \neq i} Q_i}{d^{-1}} \tag{9}$$

with $P_i \cdot \prod_{j \neq i} Q_i \in K[X]$. Hence, $K[g_1, ..., g_n] \subset K[X, d^{-1}]$.

Notice, that $K[X]$ contains infinitely many irreducible polynoms. This can be easily concluded from the fact that $K[X]$ contains infinitely many primes[6]

Hence, we can take an irreducible polynomial $R \in K[X]$ such that $R \nmid d$.

Claim(1): $\frac{1}{R} \notin K[X, d]$.

Note that - by taking the greatest common divisor - each $p \in K[X, d^{-1}]$ is of the form

$$p = P + p_n d^{-n} + ... + p_0 = PQd^{-n} \tag{10}$$

with $PQ \in K[X]$. Assume we had

$$\frac{1}{R} = \frac{PQ}{d^n}. \tag{11}$$

Hence, $1 = PQ\frac{R}{d^n}$. However, since R is irreducible and $R \nmid d$ this is not possible. □

Exercise 4

Claim: If R is noetherian then so is $R[[X]]$. In addition to that we define

$$\deg\left(\sum_{i=0}^{\infty} r_i X^i\right) := \min\{i \in \mathbb{N} | r_i \neq 0\}. \tag{12}$$

Definition. For any power series we define $\text{lead}(a_k + a_{k+1}X + ...) := a_k$ to be the minimal nonzero coefficient. Furthermore for any ideal $I \subseteq R[[X]]$ let

$$L := \{\text{lead}(P) | P \in I\} \tag{13}$$
$$L_d := \{\text{lead}(P) | P \in I \wedge \deg(P) = d\} \cup \{0\} \quad \forall d \geq 1 \tag{14}$$

[6] similar to euclids proof for the conjecture that there are infinitely many primes in $\mathbb{Z}$

Claim(1): For every $d \geq 1$ L_d is an ideal, i.e. $\forall a, b \in L_d\,, \forall r \in R:\ ar - b \in L_d$.
proof. Let $a, b \in L_d$ and $r \in R$. Wlog $ar - b \neq 0$, otherwise there is nothing to show. Take $P, Q \in I$ such that $\text{lead}(P) = a$ resp. $\text{lead}(Q) = b$. Then we have $\text{lead}(rP - Q) = ar - b$. qed(1)

Claim(2): L is an ideal.
proof. Let $a, b \in L$ and $r \in R$. Again we may assume that wlog $ar - b \neq 0$. As $a, b \in L$ we can take $P, Q \in I$ such that $\text{lead}(P) = a$ respectively $\text{lead}(Q) = b$. Furthermore, we define $d := \deg(P)$ and $e := \deg(Q)$. Therefore, it follows that $\text{lead}(rPx^e - Qx^d) = ar - b$ qed(2)

proof of <u>claim</u>. As R is noetherian we have that L and all the L_d are finitely generated; let $\{a_1, ..., a_n\}$ and $\{a_{d,1}, ..., a_{d,n_d}\}$ be generators for L and L_d. Choose $P_i, Q_{d,i} \in I$ such that $\text{lead}(P_i) = a_i$, respectively $\deg(Q_{d,i}) = d$ and $\text{lead}(Q_{d,i}) = a_{d,i}$. We define $e_i := \deg(P_i)$, $N := \max_{i=1}^n e_i$ and finally

$$I' := \{P_i, |1 \leq i \leq n\} \cup \{Q_{d,i} | 1 \leq d \leq N, 1 \leq i \leq n_d\}. \tag{15}$$

Clearly I' is finitely generated and $I' \subseteq I$. It remains to show that $I' = I$.

Claim(3): $I \subseteq I'$ Assume there was $P \in I$ such that $P \notin I'$; wlog P is of minimal codegree d.
Case 1: $d \neq 0$
Let $a = \text{lead}(P) \in L$ then $a = \sum_{i=k}^{m} a_i r_i$ with $r_i \in R$ and $k \leq m$ being the minimal nonzero coefficient. We define

$$Q = x^{d-e_i} P_i \tag{16}$$

Now clearly □

Exercise 5

Let M be a noetherian R-module and $\varphi : M \to M$ an R-module endomorphism of M.

Lemma. Let M, N be R-modules and $\varphi : M \to N$ a R-module-morphism. Then $\ker(\varphi)$ is a submodule of M.

Proof. We need to verify that $\ker(\varphi)$ is nonempty and $\forall x, y \in \ker(\varphi)\,, \forall r \in R:$ $x + ry \in \ker(\varphi)$. Clearly $\ker(\varphi) \neq \emptyset$ as $0 \in \ker(\varphi)$. Now let $x, y \in \ker(\varphi)$ and $r \in R$. We have that $\varphi(x) = \varphi(y) = 0$. Hence, $\varphi(x + ry) = \varphi(x) + r\varphi(y) = 0 + r0 = 0$. Therefore $x + ry \in \ker(\varphi)$. □

<u>*Claim:*</u> There is an $n \in \mathbb{N}$ such that $\ker(\varphi^n) \cap \text{im}(\varphi^n) = 0$.

Proof. As $\varphi(0) = 0$ and we have that $\ker(\varphi) \subset \ker(\varphi^2) \subset ..$ is an increasing chain of submodules. As M is noetherian it is eventually constant and there exists $N \in \mathbb{N}$ such that $\ker(\varphi^N) = \ker(\varphi^m)$ for all $m \geq N$.

In particular $\ker(\varphi^N) = \ker(\varphi \circ \varphi^N)$. Hence, $\ker(\varphi \upharpoonright_{\mathrm{im}(\varphi^N)}) = 0$ and $\varphi \upharpoonright_{\mathrm{im}(\varphi^N)}$ is injective. Hence, if $x \in \mathrm{im}(\varphi^{N+1})$ and $\varphi(x) = 0$ then $x = 0$. This implies that $\mathrm{im}(\varphi^{N+1}) \cap \ker(\varphi^{N+1}) = 0$. □

<u>*Claim:*</u> If φ is surjective then φ is an isomorphism.

Proof. If $\varphi : M \to M$ is surjective then so is φ^n for any $n \in \mathbb{N}$. As before we have $N \in \mathbb{N}$ such that $\varphi \upharpoonright_{\mathrm{im}(\varphi^N)}$ is injective. Considering that φ^N is surjective we can conclude that $\mathrm{im}(\varphi^N) = M$ and therefore $\varphi \upharpoonright_{\mathrm{im}(\varphi^N} = \varphi$ is both, surjective and injective, and thus an isomorphism. □

Exercise 6

Consider the R-modules M, M', M'' and the following exact sequence; I will refer to it as $\mathcal{S}$ from now on.

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

<u>*Claim:*</u> M is a noetherian R-module iff M' and M'' are noetherian R-modules.

Proof. "$\Longrightarrow$" Let M be a noetherian R-module.

Let $M'_1 \subseteq M'_2 \subseteq ...$ be an increasing sequence of submodules of M'. Then $f(M'_1) \subseteq f(M'_2) \subseteq ...$ is an increasing sequence of submodules of M. As M is noetherian there exists $N \in \mathbb{N}$ such that $f(M'_N) = f(M'_m)$ for all $m \geq n$. As f is injective we also have $M'_N = M'_m$ for all $m \geq N$. Hence, the (arbitrary) sequence of R-submodules in M' is eventually constant and thus M' noetherian.

Let $M''_1 \subseteq M''_2 \subseteq ...$ be an increasing sequence of R-submodules of M''. It follows that $g^{-1}(M''_1) \subseteq g^{-1}(M''_2) \subseteq ...$ is an increasing sequence R-submodules of M. As M is noetherian the latter one is eventually constant, lets say from index N onwards. As g is surjective we have that $g(g^{-1}(M''_i) = M''_i$. Hence, we can conclude that $M''_1 \subseteq M''_2 \subseteq ...$ is constant from N onwards as well.

qed($\Rightarrow$)

"$\Longleftarrow$" Let M' and M'' be noetherian R-modules.

Let $M_1 \subseteq M_2 \subseteq ...$ be an increasing sequence of submodules of M. This implies that $f^{-1}(M_1) \subseteq ...$ is an increasing sequence in M' and $g(M_1) \subseteq$ is an increasing sequence in M''. The latter ones are eventually constant as M' and M'' are noetherian; let us assume from index N' resp. N'' onwards. We define $N := \max(N', N'')$. If we prove that $M_n \subseteq M_N$ for any $n \in \mathbb{N}$ we can conclude that $M_1 \subseteq ...$ is eventually constant and thus M noetherian (as the sequence was arbitrary).

Let $x \in M_n$. As $g \upharpoonright_{M_N} \twoheadrightarrow g(M_n)$ there exists $y \in M_N$ such that $g(y) = g(x)$ (see also fig. 1). It follows that $x - y \in \ker(g) \cap M_n$. As $\mathcal{S}$ is exact we have

$$
\begin{tikzcd}
 & y \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
f^{-1}(M_N) \arrow[r, hook, "f"] \arrow[d, equal] & M_N \arrow[r, two heads, "g"] \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] & g(M_N) \arrow[d, equal] \\
f^{-1}(M_n) \arrow[r, hook, "f"] & M_n \arrow[r, two heads, "g"] & g(M_n) \\
f^{-1}(x-y) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd}
$$

Figure 1:

that $x - y \in \mathrm{im}(f) \cap M_n$ and thus may consider $f^{-1}(x-y) \in M'_n = M'_N$. As f is injective we have that $f(f^{-1}(x-y)) = x - y \in M_N$. But this means that $x = y + (x-y) \in M_N$ and as $x \in M_n$ was arbitrary we can conclude $M_n \subset M_N$. □

Exercise 11

Let R be a commutative ring in which all prime ideals are finitely generated.
a) *Claim:* If the collection of ideals that are not finitely generated is nonempty, then it contains a maximal element I. Then R/I is a noetherian ring.

Proof. Let $L := \{J \subset R | J \text{ is an not f.g. ideal}\}$. We want to use the *Lemma of Zorn* to deduce that L possesses a maximal element. Clearly L is partially ordered by $\subset$. We need to find an upper bound in L for every chain in L. Let $J_1 \subset J_2 \subset ...$ be a chain in L. Clearly $\bigcup_{i>0} J_i$ is an upper bound; it remains to show that $\bigcup_{i>0} J_i \in L$.

Claim(1): $\bigcup_{i>0} J_i$ is an ideal.
proof. Take $x, y \in \bigcup_{i>0} J_i$ then there is an $n \in \mathbb{N}$ such that $x, y \in J_n$. Hence, $x - y \in J_n \subset \bigcup_{i>0} J_1$.

Furthermore, let $x \in \bigcup_{i>0} J_i$ and $r \in R$. Again there exists $n \in \mathbb{N}$ such that $x \in J_n$. Hence, $r \cdot x \in J_n \subset \bigcup_{i>0} J_i$. qed(1)

Claim(2): $\bigcup_{i>0} J_i$ is not finitely generated, i.e. $\bigcup_{i>0} J_i \in L$.
proof. Assume $\bigcup_{i>0} J_i$ was finitely generated by the generators $g_1, .., g_m$. Take $n \in \mathbb{N}$ such that $g_1, ..., g_m \in J_n$. But this implies that $J_n \subseteq \bigcup_{i>0} J_i \subseteq J_n$ and J_n would be finitely generated. Hence, $J_n \notin L$. ↯ qed(2)

By *Zorn's Lemma* we can conclude that L possesses a maximal element; let us call it I.

As I is an ideal and R a ring we have a canonical Ringhomorphism $\phi : R \twoheadrightarrow R/I$ that provides a Ring structure on R/I. Now let $J \subseteq R/I$ be a non-trivial ideal. Then $\phi^{-1}(J) \supset I$. As I is maximal in L we have that $\phi^{-1}(J)$ is either finitely generated or equal to I. However, $\phi^{-1}(J)$ is not possible as then $J = (0)$ would be trivial. Hence, $\phi^{-1}(J)$ must be f.g. and therefore also J is. As every ideal of R/I is finitely generated we have that R/I is noetherian. $\square$

b) *Claim:* There are f.g. ideals J_1 and J_2 such that $J_{1/2} \supseteq I$ and $J_1 J_2 \subseteq I$.

Proof. We know that I is not a prime ideal as any prime ideal in R is finitely generated. Hence, we can find $x, y \in R$ such that $x, y \notin I$ but $xy \in I$. Let $\mathfrak{J}_x := (\phi(x))_{R/I}$ denote the R/I-ideal generated by $\phi(x)$. We define $J_1 := \phi^{-1}(\mathfrak{J}_x)$ and $J_2 := \phi^{-1}(\mathfrak{J}_y)$. As $x, y \notin I$ we have that $J_1, J_2 \supseteq I$.

Claim(1): $\mathfrak{J}_x\mathfrak{J}_y = (\phi(xy))_{R/I}$
proof.

$$\begin{aligned}
\mathfrak{J}_x\mathfrak{J}_y &= \{\sum_{i\leq n} a_i b_i | a_i \in \mathfrak{J}_x \wedge b_i \in \mathfrak{J}_y\} && (17)\\
&= \{\sum_{i\leq n} (r_i\phi(x))(s_i\phi(y) | r_i, s_i \in R/I\} && (18)\\
&= \{\phi(x)\phi(y)\sum_{i\leq n} r_i s_i | r_i, s_i \in R/I\} && (19)\\
&= \{\phi(xy)r | r \in R/I\} && (20)\\
&= (\phi(xy))_{R/I} && (21)
\end{aligned}$$

qed(1)

Claim(2): Let $\phi : R_1 \twoheadrightarrow R_2$ be a ringhomorphism and I_1, I_2 ideals of R_2. Then $\phi^{-1}(I_1 I_2) \supseteq \phi^{-1}(I_2)\phi^{-1}(I_1)$.
proof. Let $x \in \phi^{-1}(I_1)\phi^{-1}(I_2)$ such that $x = ab$ with $a \in \phi^{-1}(I_1)$ and $b \in \phi^{-1}(I_2)$. Hence, $\phi(x) = \phi(ab) = \phi(a)\phi(b) \in I_1 I_2$. Therefore, $x \in \phi^{-1}(I_1 I_2)$.
qed(2)

As we have $xy \in I$ we can now conclude

$$\begin{aligned}
I = \phi^{-1}((0)_{R/I}) \supseteq \phi^{-1}((\phi(xy))_{R/I}) &= \phi^{-1}(\mathfrak{J}_x\mathfrak{J}_y) && (22)\\
&\supseteq \phi^{-1}(\mathfrak{J}_x)\phi^{-1}(\mathfrak{J}_y) && (23)\\
&= J_1 J_2 && (24)
\end{aligned}$$

Clearly $J_1 J_2 = \{\sum_{i\leq n} a_i b_i | a_i \in J_1 \wedge b_i \in J_2\}$ is finitely generated as J_1 and J_2 are. $\square$

c)

Remark. Let R be a ring and I an ideal. Then I can be naturally viewed as a R-module by simply interpreting the ring multiplication as module product. Indeed, as ideals are abelian groups in $+$ and closed under multiplication they satisfy the R module axioms.

Recall that for ideals I, J of some ring R with $J \supset I$ we have that J/I is an ideal of R/I by the *third isomorphism theorem for rings*[7]. Hence, J/I always forms an abelian group with the induced binary operation $+_{R/I}$.

<u>*Claim:*</u> I/J_1J_2 is a finitely generated R/I-submodule of J_1/J_1J_2.

Proof. It suffices to prove that J_1/J_1J_2 is a f.g. R/I module. Indeed, as R/I is noetherian we can deduce by a previous exercise[8] that $I/J_1J_2 \subset J_1/J_1J_2$ is f.g..

We define an action on J_1/J_1J_2

$$\star : R/I \times J_1/J_1J_2 \to J_1/J_1J_2 \tag{25}$$
$$[r]_I \star [j]_{J_1J_2} := [r \cdot j]_{J_1J_2} \tag{26}$$

where "$\cdot$" denotes the ring-multiplication in R.

Claim(1): $\star$ is well defined.
proof. We have for $i \in I$ and $l \in J_1J_2$

$$[r+i]_I \star [j+l]_{J_1J_2} = [(r+i)\cdot(j+l)]_{J_1J_2} \tag{27}$$
$$= [r \cdot j + r \cdot l + i \cdot j + i \cdot l]_{J_1J_2} \tag{28}$$
$$= [r \cdot j]_{J_1J_2}. \tag{29}$$

The last inequality follows from the fact that $r \cdot l, i \cdot j, i \cdot l \in J_1J_2$ qed(1)

Claim(2): J_1/J_1J_2 is a R/I-module
proof.

1. As J_1 and J_1J_2 are ideals in R we have that $(J_1/J_1J_2, +_{R/I})$ is abelian.
2. for $[r]_I, [s]_I \in R/I$ and $j \in J_1/J_1J_2$ we have

$$([r]_I + [s]_I) \star [j]_{J_1J_2} = [r+s]_I \star [j]_{J_1J_2} \tag{30}$$
$$= [(r+s)\cdot j]_{J_1J_2} \tag{31}$$
$$= [r \cdot j]_{J_1J_2} + [s \cdot j]_{J_1J_2} \tag{32}$$
$$= [r]_I \star [j]_{J_1J_2} + [s]_I \star [j]_{J_1J_2} \tag{33}$$

3. for $[r]_I, [s]_I \in R/I$ and $j \in J_1/J_1J_2$ we have

$$([r]_I \cdot_{R/I} [s]_I) \star [j]_{J_1J_2} = [r \cdot s]_I \star [j]_{J_1J_2} \tag{34}$$
$$= [(r \cdot s)\cdot j]_{J_1J_2} \tag{35}$$
$$= [r \cdot (s \cdot j)]_{J_1J_2} \tag{36}$$
$$= [r]_I \star [s \cdot j]_{J_1J_2} \tag{37}$$
$$= [r]_I \star ([s]_I \star [j]_{J_1J_2}) \tag{38}$$

[7] DF § 7.3 Theorem 8 (2)
[8] DF §15.1 Exercise 8

4. for $[r]_I \in R/I$ and $j, l \in J_1/J_1J_2$ we have

$$\begin{aligned}
[r]_I \star ([j]_{J_1J_2} + [l]_{J_1J_2}) &= [r]_I \star [j+l]_{J_1J_2} && (39)\\
&= [(r)\cdot(j+l)]_{J_1J_2} && (40)\\
&= [r\cdot j]_{J_1J_2} + [r\cdot l]_{J_1J_2} && (41)\\
&= [r]_I \star [j]_{J_1J_2} + [r]_I \star [l]_{J_1J_2} && (42)
\end{aligned}$$

5. for $[j]_{J_1J_2} \in J_1/J_1J_2$

$$[1]_I \star [j]_{J_1J_2} = [1\cdot j]_{J_1J_2} = [j]_{J_1J_2}. \tag{43}$$

qed(2)

Claim(3): J_1/J_1J_2 is finitely generated as R/I module.
proof. Let $\{g_1, ..g_n\}$ be a set of generators for J_1 and $[j]_{J_1J_2} \in J_1/J_1J_2$ be arbitrary. It follows that

$$\begin{aligned}
[j]_{J_1J_2} &= [\sum_{i=1}^{n} r_i \cdot g_i]_{J_1J_2} && (44)\\
&= \sum_{i=1}^{n} [r_i \cdot g_i]_{J_1J_2} && (45)\\
&= \sum_{i=1}^{n} [r_i]_I \star [g_i]_{J_1J_2}. && (46)
\end{aligned}$$

Hence, $\{[g_1]_{J_1J_2}, ..., [g_n]_{J_1J_2}\}$ is a generating set for J_1/J_1J_2. Therefore, J_1/J_1J_2 is finitely generated. qed(3)

By the preliminary remarks we can conclude the claim □

d) *Claim:* (c) implies that I is finitely generated over R.

Remark. Recall that any R/I-module M with action $\cdot : R/I \times M \to M$ can be transformed into a R-module in a canoncal way; use the projection $\phi : R \twoheadrightarrow R/I$ to define the action

$$\begin{aligned}
\cdot_R : R \times M &\to M && (47)\\
r \cdot_R m &:= \phi(r) \cdot m && (48)
\end{aligned}$$

Clearly if M is f.g. as R/I module it is also f.g. as R module. In fact the same elements that generate M as R/I-module will also do it as R-module.

Lemma 0.1. *Let M be a R-module and $N \subset M$ be an R-submodule of M. If N and M/N are finitely generated then so is M.*

Proof. Observe that we have the exact sequence in fig. 2.

Let $\{g_1, ..., g_n\}$ be the generators for N and $\{[h_1]_N, ..., [h_m]_N\}$ for M/N. We can use the projection map $\phi : M \twoheadrightarrow M/N$ to get a set $\{\tilde{h}_1, ..., \tilde{h}_m\} \subset M$ such that $\phi(\tilde{h}_1) = [h_1]_N$.

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\phi} M/N \longrightarrow 0$$

Figure 2: Exact sequence

Now take $x \in M$ arbitrary, then $\phi(x) = [x]_N = \sum_{i \leq m} r_i [h_i]_N$. Define $x' := \sum_{i \leq m} r_i \tilde{h}_i \in M$, then clearly $x - x' \in \ker(\phi) = \mathrm{im}(\iota)$. Hence, we can write $x - x' = \sum_{j \leq n} r_i \iota(g_i)$. All together we have $x = x' + x - x' = \sum_{i \leq m} r_i \tilde{h}_i \in M + \sum_{j \leq n} r_i \iota(g_i)$ and as x was arbitrary we can conclude that M is generated by $\{\tilde{h}_1, ..., \tilde{h}_m, \iota(g_1), ..., \iota(g_n)\}$. $\square$

We can now prove that I is finitely generated over R.

proof of claim. We have that I is a R-module and $J_1 J_2$ is a R-submodule of I. As both - $J_1 J_2$ and $I/J_1 J_2$ are finitely generated as R-modules we can conclude by the previous lemma that I is finitely generated. $\square$

Assuming that R is a ring in which all prime ideals are finitely generated we have shown (in **(a)**) that if there was an ideal which is not finitely generated then we would also have an inclusion maximal ideal which is not finitely generated, which we named I. Now in **(d)** we deduced that such an I would in fact be finitely generated(↯). Hence, we must reject the premise of **(a)** and end up with

$$\text{The collections of ideals of } R \text{ that are not finitely generated is empty.} \tag{49}$$

For a ring this is equivalent to being noetherian. $\square$