Precision-based sampling with missing observations

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This presentation

Mash-up of two papers in my dissertation!

Method:

Hauber, P and C. Schumacher (2021). *Precision-based sampling with missing observations: A factor model application*, **Bundesbank Discussion Paper 11/2021**.

Application:

Hauber, P. (2021) *How useful is external information from professional forecasters? Conditional forecasts in large factor models*

Precision-based sampling

Motivation

Essential task in the Bayesian estimation of state space models: drawing from $p(\boldsymbol{\eta}|\mathbf{y}, \Theta)$ where $\boldsymbol{\eta}$ is an unobserved component, $\mathbf{y}$ is data and Θ parameters

Precision-based samplers (Chan and Jeliazkov 2009, **IJMMNO**; McCausland 2012, **JEcmtrics**) exploit the fact the precision matrix of $\boldsymbol{\eta}$ is banded in many macroeconomic application $\rightarrow$ alternative to simulation smoothers that rely on the Kalman filter

Applications in macroeconomics (with complete data) include models of trend inflation (Chan et al. 2013, **JBES**), time-varying Bayesian vector autoregressions (Chan 2020, **JBES**) and factor models (Kaufmann and Schumacher 2017, **JAE**)

Missing observations arise frequently in macroeconomic applications/datasets: different starting dates, different release patterns ("ragged edge"), outliers or mixed frequencies

In our paper, we propose a precision-sampler that can handle (most of) these applications!

Precision-based sampling

Simple example

AR(1) process: $\eta_t = \phi\eta_{t-1} + u_t$; $u_t \sim \mathcal{N}(0, \sigma^2)$

Stacking the observations over $t = 1, \ldots, T$ yields

$$\mathbf{H}\boldsymbol{\eta} = \mathbf{u}, \text{ where } \mathbf{u} \sim \mathcal{N}(0, \mathbf{I}_T\sigma^2) \text{ and } \mathbf{H} = \begin{bmatrix} 1 & & & & \\ -\phi & 1 & & & \\ & -\phi & 1 & & \\ & & \ddots & \ddots & \\ & & & -\phi & 1 \end{bmatrix}$$

$\boldsymbol{\eta}$ is Normal with mean $\mathbf{0}_T$ and covariance matrix $\Sigma = \mathbf{H}^{-1}\,\mathbf{I}_T\sigma^2\,\mathbf{H}^{-1^{\mathsf{T}}}$

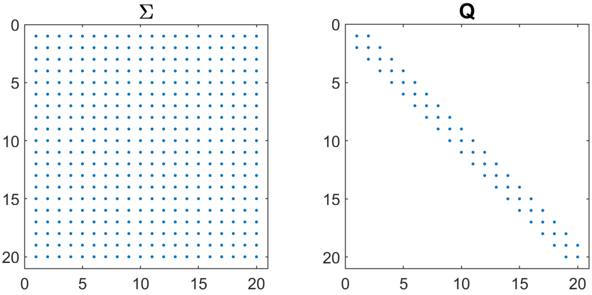
corresponding *precision matrix* is given by $\mathbf{Q} = \Sigma^{-1} = \mathbf{H}^{\mathsf{T}}\,\mathbf{I}_T\sigma^{-2}\,\mathbf{H}$

Precision-based sampling with missing observations

Covariance and precision matrix of η

Properties of the multivariate $\mathcal{N}$:

- $\Sigma_{ij} = 0 \implies$ independence of η_i and η_j (in the example: $Cov(\eta_i, \eta_j) \propto \phi^{|i-j|}$)
- $\mathbf{Q}_{ij} = 0 \implies$ **conditional** independence of η_i and η_j



Notes: The blue dots indicate the non-zero entries in the covariance matrix Σ and precision matrix $\mathbf{Q}$ of an AR(1) process for $T = 20$ observations. The former is a dense matrix while the latter is sparse and banded with lower and upper bandwidth equal to 1.

Precision-based sampling

Computational advantages of banded precision matrices

Solving linear systems of the form $Ux = b$ where U is an $n \times n$ upper-triangular matrix takes n^2 flops (left); when U has bandwidth p the solution can be obtained in $2np$ flops (right):

```
% solution to Ux = b
% U has maximal bandwidth
for i = n:-1:1
    x(i) = b(i)/U(i,i)
    for j = 1:i-1
        b(j) = b(j) - U(j,i)x(i)
    end
end
```

```
% solution to Ux = b
% U has bandwidth p
for i = n:-1:1
    x(i) = b(i)/U(i,i)
    for j = max{1,i-p}:i-1
        b(j) = b(j) - U(j,i)x(i)
    end
end
```

Even larger gains for matrix factorisations, e.g. Cholesky ($Q = LL^{\mathsf{T}}$) $\implies$ linear instead of cubic costs!

L "inherits" the bandwidth of Q (Golub and Van Loan 2013, Theorem 4.3.1)

Precision-based sampling

Factor model

To fix ideas, consider the following factor model:

$$\mathbf{y}_t = \lambda \boldsymbol{\eta}_t + \mathbf{e}_t;\ \mathbf{e}_t \sim \mathcal{N}(0, \Sigma_e)$$
$$\boldsymbol{\eta}_t = \phi^{\eta} \boldsymbol{\eta}_{t-1} + \mathbf{u}_t;\ \mathbf{u}_t \sim \mathcal{N}(0, \Sigma_u)$$

where $\mathbf{y}_t$ is an $N \times 1$ vector of data, $\boldsymbol{\eta}_t$ is an $R \times 1$ vector of unobserved factors and $\Sigma_e = diag([\sigma_1^2, \cdots, \sigma_N^2]))$ a diagonal matrix

Bayesian estimation of the model is done via a Gibbs Sampler which sequentially draws from

- the conditional distribution of factors given data and parameters: $p(\boldsymbol{\eta}|\mathbf{y}, \Theta)$
- the conditional distribution of parameters given data and factors: $p(\Theta|\boldsymbol{\eta}, \mathbf{y})$

Precision-based sampling

Drawing from $p(\boldsymbol{\eta}|\mathbf{y},\Theta)$

Joint distribution of factors $\boldsymbol{\eta} = [\boldsymbol{\eta}_1{}^\mathsf{T}, \cdots, \boldsymbol{\eta}_T{}^\mathsf{T}]^\mathsf{T}$ and data $\mathbf{y} = [\mathbf{y}_1^\mathsf{T}, \cdots, \mathbf{y}_T^\mathsf{T}]^\mathsf{T}$ given parameters:

$$\mathbf{z} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{-1});\ \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_\eta & \mathbf{Q}_{\eta,y} \\ \mathbf{Q}_{\eta,y}^\mathsf{T} & \mathbf{Q}_y \end{bmatrix}$$

Mapping from Θ to $\mathbf{Q}$

Standard result for the multivariate $\mathcal{N}$: $p(\boldsymbol{\eta}|\mathbf{y},\Theta) = \mathcal{N}(-\mathbf{Q}_\eta^{-1}\mathbf{Q}_{\eta y}\mathbf{y}, \mathbf{Q}_\eta^{-1})$

Sampling from this distribution does **not** require the inversion of (the potentially very large matrix) $\mathbf{Q}_\eta$ and because it is banded

- the mean $-\mathbf{Q}_\eta^{-1}\mathbf{Q}_{\eta y}\mathbf{y}$
- and a random draw given mean and precision matrix

can be obtained efficiently! Rue and Held (2005, Algorithms 2.1, 2.4)

Precision-based sampling with missing observations

Drawing from $p(\eta, \mathbf{y}^m | \mathbf{y}^o, \Theta)$: 3 block ordering

What to do when some observations in **y** are missing?

E.g. a model with one factor, two variables and $t = 1:6$ with

$$\mathbf{y} = [y_{11}, y_{12}, \text{NaN}, y_{22}, y_{31}, \text{NaN}, \text{NaN}, \text{NaN}, y_{51}, \text{NaN}, \text{NaN}, \text{NaN}]^{\mathsf{T}}$$

Reorder the model in 3 blocks as:

$$\mathbf{z}_{3b} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{y^m} \\ \mathbf{y^o} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{z}^{3b}_{\eta y^m} \\ \mathbf{z}^{3b}_{y^o} \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{-1}_{3b}); \; \mathbf{Q}_{3b} = \begin{bmatrix} \mathbf{Q}^{3b}_{\eta y^m} & \mathbf{Q}^{3b}_{\eta y^m, y^o} \\ \mathbf{Q}^{3b,\mathsf{T}}_{\eta y^m, y^o} & \mathbf{Q}^{3b}_{y^o} \end{bmatrix}$$

Example:

$$\mathbf{z}_{3b} = [\eta, \mathbf{y^m}, \mathbf{y^o}]^{\mathsf{T}} = [\eta_1, \ldots, \eta_6, y_{21}, y_{32}, y_{41}, y_{42}, y_{52}, y_{61}, y_{62}, y_{11}, y_{12}, y_{22}, y_{31}, y_{51}]^{\mathsf{T}}$$

Precision-based sampling with missing observations

Drawing from $p(\eta, \mathbf{y}^m | \mathbf{y}^o, \Theta)$: time-t ordering

However, $\mathbf{Q}^{3b}_{\eta y^m}$ will in general **not** be a banded matrix, so sampling as in the case of complete data not feasible!

Different ordering of $\mathbf{z}$ that groups conditionally dependent components – $\boldsymbol{\eta}_t$ and $\mathbf{y}^m_t$ (*time-t ordering*) – together: $\implies \mathbf{z}_\tau = \mathcal{P}_\tau \mathbf{z}$ and corresponding precision matrix $Q_\tau = \mathcal{P}_\tau \mathbf{Q} \mathcal{P}_\tau{}^\mathsf{T}$

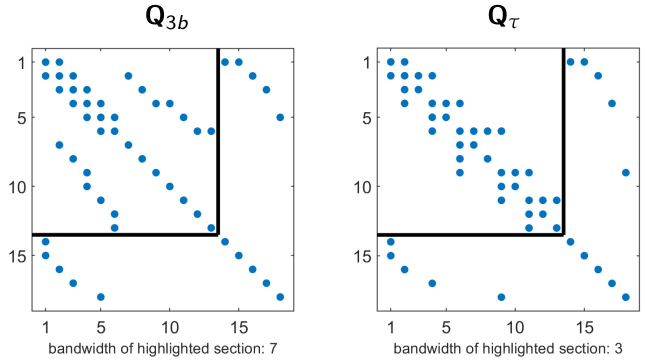
Example: $\mathbf{z}_\tau = [\eta_1, \eta_2, y_{21}, \eta_3, y_{32}, \eta_4, y_{41}, y_{42}, \eta_5, y_{52}, \eta_6, y_{61}, y_{62}, y_{11}, \dots, y_{51}]^\mathsf{T}$

This ensures that the precision matrix of the conditional distribution $Q^\tau_{\eta y^m}$ is banded!

After sampling from $\mathbf{z}^\tau_{\eta y^m | y^o} \sim \mathcal{N}(-\mathbf{Q}^{\tau -1}_{\eta y^m} \mathbf{Q}^\tau_{\eta y^m, y^p} \mathbf{y^o}, \mathbf{Q}^{\tau -1}_{\eta y^m})$, reverse the permutation to back out the draw of $\boldsymbol{\eta}$ and $\mathbf{y}^m$

Precision-based sampling with missing observations

Example: Graphical comparison of $\mathbf{Q}_{3b}$ and $\mathbf{Q}_{\tau}$



Notes: The blue dots indicate the non-zero entries in the precision matrix of the 3-block ordering (left) and the time-t permutation (right) of the discussed example with 2 variables and one factor. The highlighted upper left submatrices correspond to the conditional precision matrix of factors and missing values given observations, $\mathbf{Q}^{3b}_{\eta y^m}$ and $\mathbf{Q}^{\tau}_{\eta y^m}$.

Conditional forecasts in large factor models

Motivation

Conditional forecast evaluations in the literature typically condition on realizations (Clark and McCracken 2017, **JAE**; Banbura et al. 2015, **IJoF**) $\Longrightarrow$ useful for model assesement and scenario analysis

Practitioners may also be interested in knowing how accurate forecasts will be when conditioning on external (but quite likely imperfect) information $\Longrightarrow$ compare **information sets** rather than models

I condition on professionals' forecasts for GDP and CPI:

- (conditional) forecasts $\Longrightarrow$ missing values!
- similar approach for the Euro area using BVARs: Ganics and Odendahl (2021, **IJoF**)
- focus of the evaluation on a large cross-section of variables not typically considered in the forecasting literature

Conditional forecasts in large factor models

Data

Quarterly real-time dataset for the German economy (in total **57 series**):

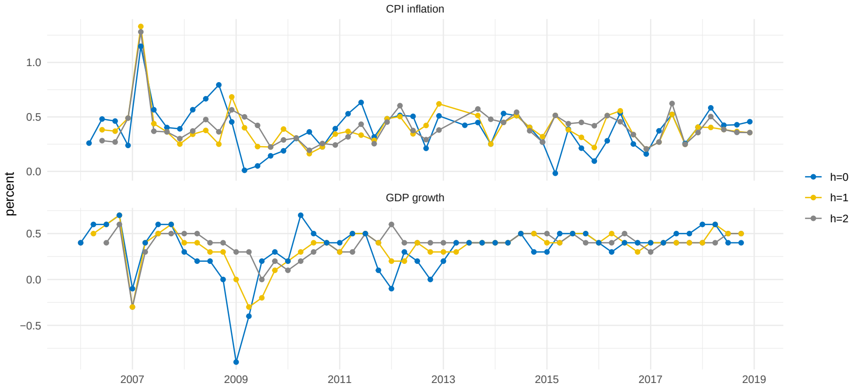
- activity indicators (expenditure and production components, IP, orders, turnovers)
- prices (CPI, PPI, deflators corresponding to chained volume indices from the national accounts)
- labor market (employment, wages, hours worked)
- financial indicators (interest rates, stock market prices, exchange rates)
- survey indicators (sectoral ESI data and employment expectations index)

Reuters Poll of professional forecasters

- $\approx$ 20 different forecasts from private sector and research institutes, up to two quarters ahead ($h = 2$)
- quarterly GDP growth and y/y CPI inflation (transformed to coincide with model definition of q/q change in CPI)
- GDP (inflation) nowcasts, i.e. $h = 0$, considerably (slightly) more accurate than model's unconditional forecast

Conditional forecasts in large factor models

Reuters Poll of professional forecasters



Notes: Median forecast from the Reuters Poll of professional forecasters for quarter-on quarter change in the comsumer price index (top panel) and quarter-on-quarter GDP growth. The forecast horizon *h* is in quarters and relative to the reference period. Source: Thomson Reuters, author's calculations.

Conditional forecasts in large factor models

Predictive density and forecast set-up

Precision-based algorithms to sample from the predictive density:

$$p(\mathbf{y}^{\mathrm{f}}|\mathbf{y}^{\mathrm{o}}, \mathbf{y}^{\mathrm{c}}) \propto \int_{\Theta} p(\mathbf{y}^{\mathrm{f}}|\mathbf{y}^{\mathrm{o}}, \mathbf{y}^{\mathrm{c}}, \Theta) p(\Theta|\mathbf{y}^{\mathrm{o}}, \mathbf{y}^{\mathrm{c}})\, \mathrm{d}\Theta$$

Estimation sample starts in 1996Q1, factor model with $R = 2$, evaluation sample: 2006Q1-2017Q4

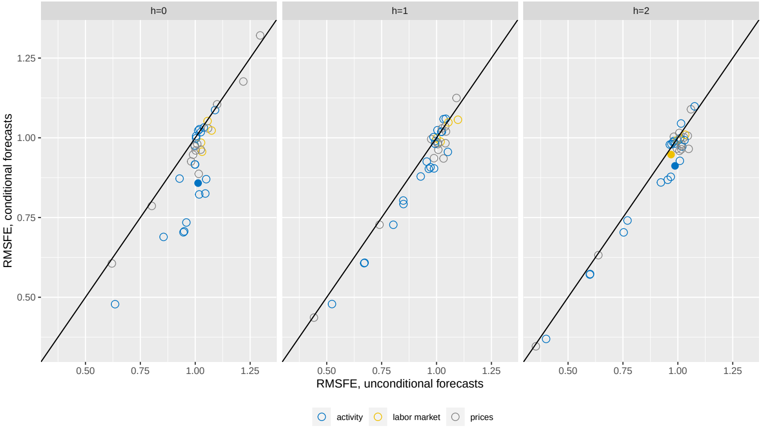
Point (RMSFE) and density forecast (CRPS) accuracy, relative to naïve benchmark

Real-time evaluation $\Longrightarrow$ exactly mimic the information set available to professional forecasters

Diebold-Mariano tests to assess if differences between unconditional and conditional forecast accuracy is significant

Conditional forecasts in large factor models

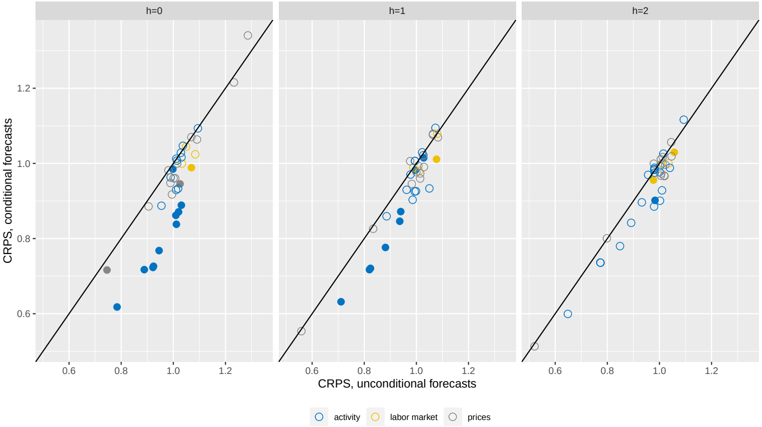
Results: point forecasts



Notes: Root mean squared forecast error (RMSFE) corresponding to unconditional forecasts (x-axis) and forecasts conditional on professional forecasters' view on GDP growth and CPI inflation (y-axis) for different time series. Filled points correspond to those variables for which the null hypothesis of the Diebold-Mariano test can be rejected at the 5 percent level.

Conditional forecasts in large factor models

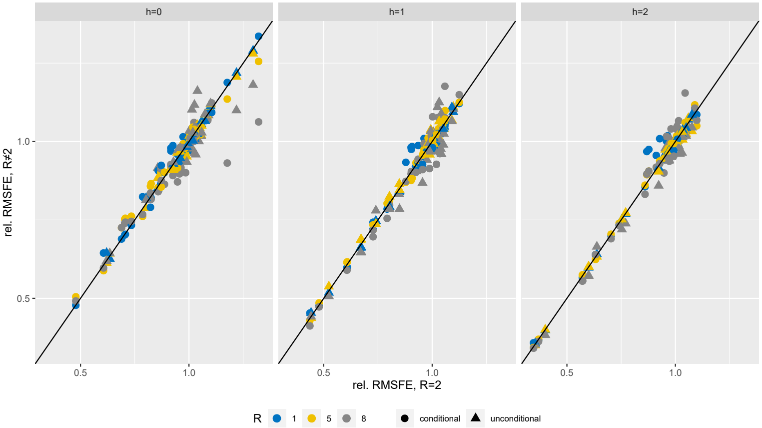
Results: density forecasts



Notes: Average continuous ranked probability score (CRPS) corresponding to unconditional forecasts (x-axis) and forecasts conditional on professional forecasters' view on GDP growth and CPI inflation (y-axis) for different time series. Filled points correspond to those variables for which the null hypothesis of the Diebold-Mariano test can be rejected at the 5 percent level.

Conditional forecasts in large factor models

Robustness checks



Notes: Root mean squared forecast error (RMSFE) relative to the autoregressive benchmarks for different number of factors. Entries above (below) the 45-degree line indicate that the model with two factors performs better (worse) than the alternative models.

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Appendix

Mapping from Θ to $\mathbf{Q}$

Stacked model: $\mathbf{z} = \begin{bmatrix} \eta \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \boldsymbol{\Lambda} & I \end{bmatrix} \begin{bmatrix} \eta \\ \mathbf{e} \end{bmatrix}$ where $\boldsymbol{\Lambda} = I_T \otimes \boldsymbol{\lambda}$

Also, let $\mathbf{V}_e = I_T \otimes \Sigma_e = \mathbf{S}_e^{-1}$, $\mathbf{V}_u = I_T \otimes \Sigma_u = \mathbf{S}_u^{-1}$ and $\mathbf{H} = \begin{bmatrix} 1 & & & \\ -\phi & 1 & & \\ & \ddots & \ddots & \\ & & -\phi & 1 \end{bmatrix}$

Then

$$\mathbf{Q}_z = \mathrm{Var}\left(\begin{bmatrix} \eta \\ \mathbf{y} \end{bmatrix}\right)^{-1} = \left(\begin{bmatrix} I & 0 \\ \boldsymbol{\Lambda} & I \end{bmatrix} \begin{bmatrix} \mathbf{H}^{-1}\mathbf{V}_u\mathbf{H}^{-1^\mathsf{T}} & 0 \\ 0 & \mathbf{V}_e \end{bmatrix} \begin{bmatrix} I & \boldsymbol{\Lambda}^\mathsf{T} \\ 0 & I \end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix} I & -\boldsymbol{\Lambda}^\mathsf{T} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{H}^\mathsf{T}\mathbf{S}_u\mathbf{H} & 0 \\ 0 & \mathbf{S}_e \end{bmatrix} \begin{bmatrix} I & 0 \\ -\boldsymbol{\Lambda} & I \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{H}^\mathsf{T}\mathbf{S}_u\mathbf{H} + \boldsymbol{\Lambda}^\mathsf{T}\mathbf{V}_e\boldsymbol{\Lambda} & -\boldsymbol{\Lambda}^\mathsf{T}\mathbf{S}_e \\ -\mathbf{S}_e\boldsymbol{\Lambda} & \mathbf{S}_e \end{bmatrix} \equiv \begin{bmatrix} \mathbf{Q}_\eta & \mathbf{Q}_{\eta y} \\ \mathbf{Q}_{\eta y}^\mathsf{T} & \mathbf{Q}_y \end{bmatrix}$$

Appendix

Rue and Held (2005, Algorithm 2.1, 2.4)

Algorithm 2.1 Solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A} > 0$

1: Compute the Cholesky factorization $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathsf{T}}$
2: Solve $\mathbf{L}\mathbf{v} = \mathbf{b}$ via forward substitution
3: Solve $\mathbf{L}^{\mathsf{T}}\mathbf{x} = \mathbf{v}$ via backward substitution
4: **return** $\mathbf{x}$

Algorithm 2.4 Sampling $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{Q}^{-1})$

1: Compute the Cholesky factorization $\mathbf{Q} = \mathbf{L}\mathbf{L}^{\mathsf{T}}$
2: Sample $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$
3: Solve $\mathbf{L}^{\mathsf{T}}\mathbf{v} = \mathbf{z}$
4: Compute $\mathbf{x} = \boldsymbol{\mu} + \mathbf{v}$
5: **return** $\mathbf{x}$

Back