Precision-based sampling with missing observations

Philipp Hauber[1,2]

[1]JMU Würzburg
[2]Kiel Institute for the World Economy (IfW)

Kiel Institute Research Seminar

March 22nd, 2022

This presentation

Mash-up of two papers in my dissertation!

Method:

Hauber, P and C. Schumacher (2021). *Precision-based sampling with missing observations: A factor model application*, **Bundesbank Discussion Paper 11/2021**.

Application:

Hauber, P. (2021) *How useful is external information from professional forecasters? Conditional forecasts in large factor models*

Precision-based sampling

Motivation

Essential task in the Bayesian estimation of state space models: drawing from $p(\boldsymbol{\eta}|\mathbf{y}, \Theta)$ where $\boldsymbol{\eta}$ is an unobserved component, $\mathbf{y}$ is data and Θ parameters

Precision-based samplers (Chan and Jeliazkov 2009, **IJMMNO**; McCausland 2012, **JEcmtrics**) exploit the fact the precision matrix of $\boldsymbol{\eta}$ is banded in many macroeconomic application $\rightarrow$ alternative to simulation smoothers that rely on the Kalman filter

Applications in macroeconomics (with complete data) include models of trend inflation (Chan et al. 2013, **JBES**), time-varying Bayesian vector autoregressions (Chan 2020, **JBES**) and factor models (Kaufmann and Schumacher 2017, **JAE**)

Missing observations arise frequently in macroeconomic applications/datasets: different starting dates, different release patterns ("ragged edge"), outliers or mixed frequencies

In our paper, we propose a precision-sampler that can handle (most of these) applications!

Precision-based sampling

Simple example

AR(1) process: $\eta_t = \phi\eta_{t-1} + u_t$; $u_t \sim \mathcal{N}(0, \sigma^2)$

Stacking the observations over $t = 1, \ldots, T$ yields

$$\mathbf{H}\boldsymbol{\eta} = \mathbf{u}, \text{ where } \mathbf{u} \sim \mathcal{N}(0, \mathbf{I}_T\sigma^2) \text{ and } \mathbf{H} = \begin{bmatrix} 1 & & & & \\ -\phi & 1 & & & \\ & -\phi & 1 & & \\ & & \ddots & \ddots & \\ & & & -\phi & 1 \end{bmatrix}$$

$\boldsymbol{\eta}$ is Normal with mean $\mathbf{0}_T$ and covariance matrix $\Sigma = \mathbf{H}^{-1}\,\mathbf{I}_T\sigma^2\,{\mathbf{H}^{-1}}^{\mathsf{T}}$

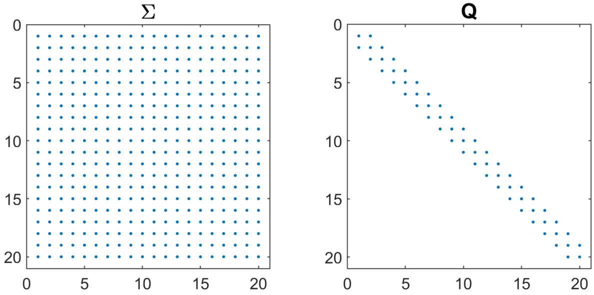
corresponding *precision matrix* is given by $\mathbf{Q} = \Sigma^{-1} = \mathbf{H}^{\mathsf{T}}\,\mathbf{I}_T\sigma^{-2}\,\mathbf{H}$

Precision-based sampling with missing observations

Covariance and precision matrix of η

Properties of the multivariate $\mathcal{N}$:

- $\Sigma_{ij} = 0 \implies$ independence of η_i and η_j $\Rightarrow$ $Cov(\eta_i, \eta_j) \propto \phi^{|i-j|}$
- $\mathbf{Q}_{ij} = 0 \implies$ **conditional** independence of η_i and η_j



Notes: The blue dots indicate the non-zero entries in the covariance matrix Σ and precision matrix $\mathbf{Q}$ of an AR(2) process for $T = 20$ observations. The former is a dense matrix while the latter is sparse and banded with lower and upper bandwidth equal to 1.

Precision-based sampling

Computational advantages of banded precision matrices

Solving linear systems of the form $Ux = b$ where U is an $n \times n$ upper-triangular matrix takes n^2 flops (left); when U has bandwidth p the solution can be obtained in $2np$ flops (right):

```
% solution to Ux = b
% U has maximal bandwidth
for i = n:-1:1
    x(i) = b(i)/U(i,i)
    for j = 1:i-1
        b(j) = b(j) - U(j,i)x(i)
    end
end
```

```
% solution to Ux = b
% U has bandwidth p
for i = n:-1:1
    x(i) = b(i)/U(i,i)
    for j = max{1,i-p}:i-1
        b(j) = b(j) - U(j,i)x(i)
    end
end
```

Even larger gains for matrix factorisations, e.g. Cholesky ($Q = LL^\mathsf{T}$) $\implies$ linear instead of cubic costs!

L "inherits" the bandwidth of Q (Golub and Van Loan 2013, Theorem 4.3.1)

Precision-based sampling

Factor model

To fix ideas, consider the following factor model:

$$\mathbf{y}_t = \lambda \boldsymbol{\eta}_t + \mathbf{e}_t;\ \mathbf{e}_t \sim \mathcal{N}(0, \Sigma_e)$$
$$\boldsymbol{\eta}_t = \phi^{\eta} \boldsymbol{\eta}_{t-1} + \mathbf{u}_t;\ \mathbf{u}_t \sim \mathcal{N}(0, \Sigma_u)$$

where $\mathbf{y}_t$ is an $N \times 1$ vector of data, $\boldsymbol{\eta}_t$ is an $R \times 1$ vector of unobserved factors and $\Sigma_e = diag([\sigma_1^2, \cdots, \sigma_N^2]))$ a diagonal matrix

Bayesian estimation of the model is done via a Gibbs Sampler which sequentially draws from

- the conditional distribution of factors given data and parameters: $p(\boldsymbol{\eta}|\mathbf{y}, \Theta)$
- the conditional distribution of parameters given data and factors: $p(\Theta|\boldsymbol{\eta}, \mathbf{y})$

Precision-based sampling

Drawing from $p(\boldsymbol{\eta}|\mathbf{y},\Theta)$

Joint distribution of factors $\boldsymbol{\eta} = [{\boldsymbol{\eta}_1}^\mathsf{T}, \cdots, {\boldsymbol{\eta}_T}^\mathsf{T}]^\mathsf{T}$ and data $\mathbf{y} = [\mathbf{y}_1^\mathsf{T}, \cdots, \mathbf{y}_T^\mathsf{T}]^\mathsf{T}$ given parameters:

$$\mathbf{z} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{-1});\ \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_\eta & \mathbf{Q}_{\eta,y} \\ \mathbf{Q}_{\eta,y}^\mathsf{T} & \mathbf{Q}_y \end{bmatrix}$$

Mapping from Θ to $\mathbf{Q}$

Standard result for the multivariate $\mathcal{N}$: $p(\boldsymbol{\eta}|\mathbf{y},\Theta) = \mathcal{N}(-\mathbf{Q}_\eta^{-1}\mathbf{Q}_{\eta y}\mathbf{y}, \mathbf{Q}_\eta^{-1})$

Sampling from this distribution does **not** require the inversion of (the potentially very large matrix) $\mathbf{Q}_\eta$ and because it is banded

- the mean $-\mathbf{Q}_\eta^{-1}\mathbf{Q}_{\eta y}\mathbf{y}$
- and a random draw given mean and precision matrix

can be obtained efficiently! Rue and Held (2005, Algorithms 2.1, 2.4)

Precision-based sampling with missing observations

Drawing from $p(\eta, \mathbf{y}^m | \mathbf{y}^o, \Theta)$: 3 block ordering

What to do when some observations in $\mathbf{y}$ are missing?

E.g. a model with one factor, two variables and $t = 1:6$ with

$$\mathbf{y} = [y_{11}, y_{12}, \text{NaN}, y_{22}, y_{31}, \text{NaN}, \text{NaN}, \text{NaN}, y_{51}, \text{NaN}, \text{NaN}, \text{NaN}]^{\mathsf{T}}$$

Reorder the model in 3 blocks as:

$$\mathbf{z}_{3b} = \begin{bmatrix} \eta \\ \mathbf{y^m} \\ \mathbf{y^o} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{z}^{3b}_{\eta y^m} \\ \mathbf{z}^{3b}_{y^o} \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{-1}_{3b});\ \mathbf{Q}_{3b} = \begin{bmatrix} \mathbf{Q}^{3b}_{\eta y^m} & \mathbf{Q}^{3b}_{\eta y^m, y^o} \\ \mathbf{Q}^{3b,\mathsf{T}}_{\eta y^m, y^o} & \mathbf{Q}^{3b}_{y^o} \end{bmatrix}$$

Example:

$$\mathbf{z}_{3b} = [\eta, \mathbf{y^m}, \mathbf{y^o}]^{\mathsf{T}} = [\eta_1, \ldots, \eta_6, y_{21}, y_{32}, y_{41}, y_{42}, y_{52}, y_{61}, y_{62}, y_{11}, y_{12}, y_{22}, y_{31}, y_{51}]^{\mathsf{T}}$$

Precision-based sampling with missing observations

Drawing from $p(\eta, \mathbf{y}^m | \mathbf{y}^o, \Theta)$: time-t ordering

However, $\mathbf{Q}^{3b}_{\eta y^m}$ will in general **not** be a banded matrix, so sampling as in the case of complete data not feasible!

Different ordering of $\mathbf{z}$ that groups conditionally dependent components – $\boldsymbol{\eta}_t$ and $\mathbf{y}^m_t$ (*time-t ordering*) – together: $\Longrightarrow \mathbf{z}_\tau = \mathcal{P}_\tau \mathbf{z}$ and corresponding precision matrix $Q_\tau = \mathcal{P}_\tau \mathbf{Q} \mathcal{P}_\tau{}^\intercal$

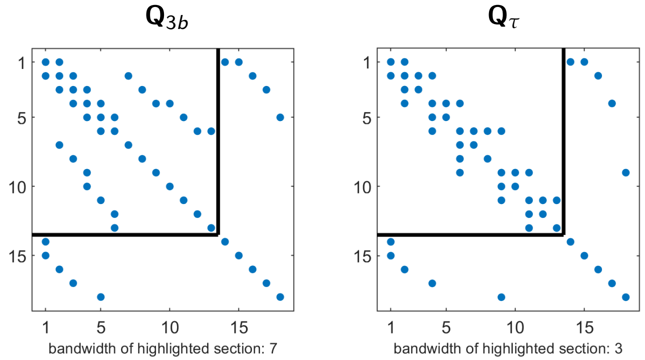
Example: $\mathbf{z}_\tau = [\eta_1, \eta_2, y_{21}, \eta_3, y_{32}, \eta_4, y_{41}, y_{42}, \eta_5, y_{52}, \eta_6, y_{61}, y_{62}, y_{11}, \ldots, y_{51}]^\intercal$

This ensures that the precision matrix of the conditional distribution $Q^\tau_{\eta y^m}$ is banded!

After sampling from $\mathbf{z}^\tau_{\eta y^m | y^o} \sim \mathcal{N}(-\mathbf{Q}^{\tau -1}_{\eta y^m} \mathbf{Q}^\tau_{\eta y^m, y^p} \mathbf{y^o}, \mathbf{Q}^{\tau -1}_{\eta y^m})$, reverse the permutation to back out the draw of $\boldsymbol{\eta}$ and $\mathbf{y}^m$

Precision-based sampling with missing observations

Example: Graphical comparison of Q_{3b} and Q_τ



Notes: The blue dots indicate the non-zero entries in the precision matrix of the 3-block ordering (left) and the time-t permutation (right) of the discussed example with 2 variables and one factor. The highlighted upper left submatrices correspond to the conditional precision matrix of factors and missing values given observations, $\mathbf{Q}^{3b}_{\eta y^m}$ and $\mathbf{Q}^{\tau}_{\eta y^m}$.

Conditional forecasts in large factor models

Motivation

Conditional forecast evaluations in the literature typically condition on realizations (Clark and McCracken 2017, **JAE**; Banbura et al. 2015, **IJoF**) $\Longrightarrow$ useful for model assesement and scenario analysis

Practitioners may also be interested in knowing how accurate forecasts will be when conditioning on forecasts $\Longrightarrow$ compare **information sets** rather than models

I condition on professionals' forecasts for GDP and CPI:

- (conditional) forecasts $\Longrightarrow$ missing values!
- similar approach for the Euro area using BVARs: Ganics and Odendahl (2021, **IJoF**)
- focus of the evaluation on a large cross-section of variables not typically considered in the forecasting literature

Conditional forecasts in large factor models

Data

Quarterly real-time dataset for the German economy (in total **57 series**):

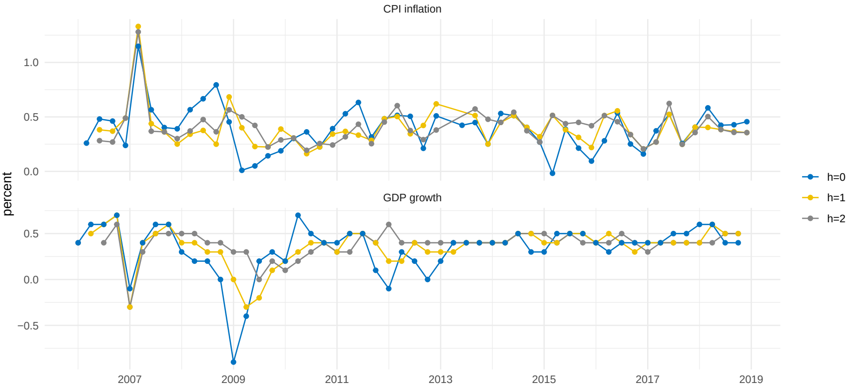
- activity indicators (expenditure and production components, IP, orders, turnovers)
- prices (CPI, PPI, deflators corresponding to chained volume indices from the national accounts)
- labor market (employment, wages, hours worked)
- financial indicators (interest rates, stock market prices, exchange rates)
- survey indicators (sectoral ESI data and employment expectations index)

Reuters Poll of professional forecasters

- $\approx$ 20 different forecasts from private sector and research institutes, up to two quarters ahead ($h = 2$)
- quarterly GDP growth and y/y CPI inflation (transformed to coincide with model definition of q/q change in CPI)
- GDP (inflation) nowcasts, i.e. $h = 0$, considerably (slightly) more accurate than model's unconditional forecast

Conditional forecasts in large factor models

Reuters Poll of professional forecasters



Notes: Median forecast from the Reuters Poll of professional forecasters for quarter-on quarter change in the comsumer price index (top panel) and quarter-on-quarter GDP growth. The forecast horizon h is in quarters and relative to the reference period. Source: Thomson Reuters, author's calculations.

Conditional forecasts in large factor models

Predictive density and forecast set-up

Precision-based algorithms to sample from the predictive density:

$$p(\mathbf{y}^{\mathrm{f}}|\mathbf{y}^{\mathrm{o}},\mathbf{y}^{\mathrm{c}}) \propto \int_{\Theta} p(\mathbf{y}^{\mathrm{f}}|\mathbf{y}^{\mathrm{o}},\mathbf{y}^{\mathrm{c}},\Theta)\,p(\Theta|\mathbf{y}^{\mathrm{o}},\mathbf{y}^{\mathrm{c}})\,\mathrm{d}\Theta$$

Estimation sample starts in 1996Q1, factor model with $R = 2$, evaluation sample: 2006Q1-2017Q4

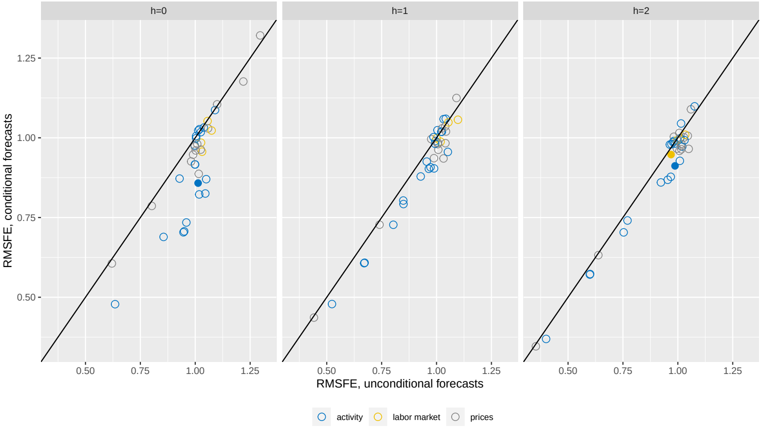
Point (RMSFE) and density forecast (CRPS) accuracy, relative to naïve benchmark

Real-time evaluation $\Longrightarrow$ exactly mimic the information set available to professional forecasters

Diebold-Mariano tests to assess if differences between unconditional and conditional forecast accuracy is significant

Conditional forecasts in large factor models

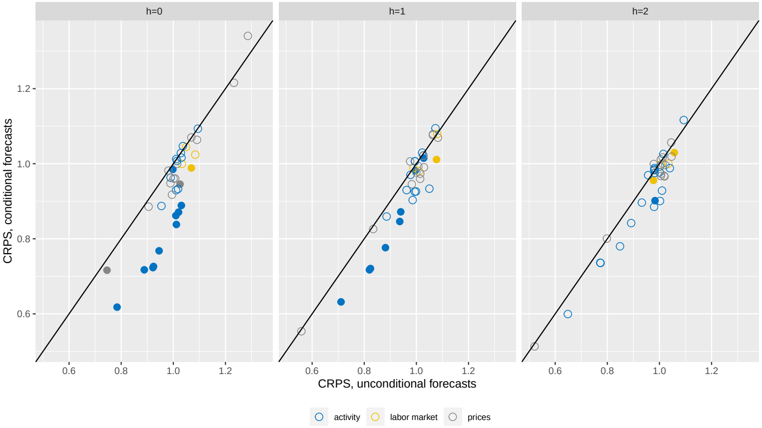
Results: point forecasts



Notes: Root mean squared forecast error (RMSFE) corresponding to unconditionalforecasts (x-axis) and forecasts conditional on professional forecasters' view on GDP growth and CPI inflation (y-axis) for different time series. Filled points correspond to those variables for which the null hypothesis of the Diebold-Mariano test can be rejected at the 5 percent level.

Conditional forecasts in large factor models

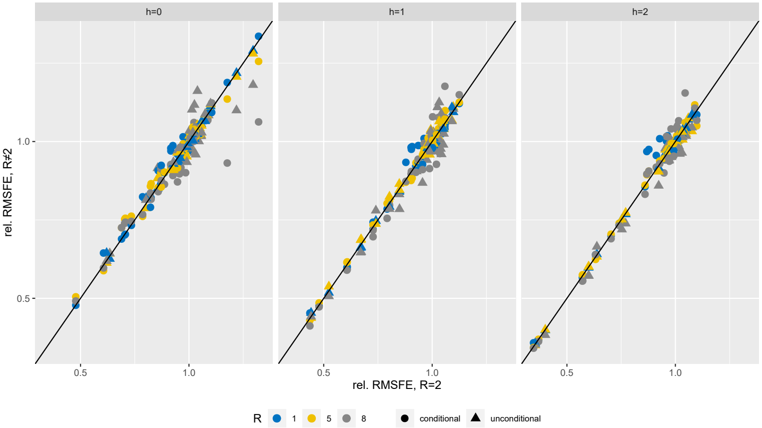
Results: density forecasts



Notes: Average continuous ranked probability score (CRPS) corresponding to unconditionalforecasts (x-axis) and forecasts conditional on professional forecasters' view on GDP growth and CPI inflation (y-axis) for different time series. Filled points correspond to those variables for which the null hypothesis of the Diebold-Mariano test can be rejected at the 5 percent level.

Conditional forecasts in large factor models

Robustness checks



Notes: Root mean squared forecast error (RMSFE) relative to the autoregressive benchmarks for different number of factors. Entries above (below) the 45-degree line indicate that the model with two factors performs better (worse) than the alternative models.

References

Banbura, Marta, Domenico Giannone, and Michele Lenza, "Conditional forecasts and scenario analysis with vector autoregressions for large cross-sections," *International Journal of Forecasting*, 2015, *31* (3), 739–756.

Chan, Joshua C. C., "Large Bayesian VARs: A Flexible Kronecker Error Covariance Structure," *Journal of Business & Economic Statistics*, 2020, *38* (1), 68–79.

__, Gary Koop, and Simon M. Potter, "A New Model of Trend Inflation," *Journal of Business & Economic Statistics*, 2013, *31* (1), 94–106.

Chan, Joshua CC and Ivan Jeliazkov, "Efficient simulation and integrated likelihood estimation in state space models," *International Journal of Mathematical Modelling and Numerical Optimisation*, 2009, *1* (1-2), 101–120.

Clark, Todd E. and Michael W. McCracken, "Tests of Predictive Ability for Vector Autoregressions Used for Conditional Forecasting," *Journal of Applied Econometrics*, 2017, *32* (3), 533–553.

Ganics, Gergely and Florens Odendahl, "Bayesian VAR forecasts, survey information, and structural change in the euro area," *International Journal of Forecasting*, 2021, *37* (2), 971–999.

Golub, Gene H. and Charles F. Van Loan, *Matrix Computations* Johns Hopkins Studies in the Mathematical Sciences, 4th ed., Johns Hopkins University Press, 2013.

Kaufmann, Sylvia and Christian Schumacher, "Identifying relevant and irrelevant variables in sparse factor models," *Journal of Applied Econometrics*, 2017, *32* (6), 1123–1144.

McCausland, William J., "The HESSIAN method: Highly efficient simulation smoothing, in a nutshell," *Journal of Econometrics*, 2012, *168* (2), 189–206.

Rue, Havard and Leonhard Held, *Gaussian Markov random fields: theory and applications*, CRC press, 2005.

Appendix

Mapping from Θ to $\mathbf{Q}$

Stacked model: $\mathbf{z} = \begin{bmatrix} \eta \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbf{\Lambda} & I \end{bmatrix} \begin{bmatrix} \eta \\ \mathbf{e} \end{bmatrix}$ where $\mathbf{\Lambda} = I_T \otimes \boldsymbol{\lambda}$

Also, let $\mathbf{V}_e = I_T \otimes \Sigma_e = \mathbf{S}_e^{-1}$, $\mathbf{V}_u = I_T \otimes \Sigma_u = \mathbf{S}_u^{-1}$ and $\mathbf{H} = \begin{bmatrix} 1 & & & \\ -\phi & 1 & & \\ & \ddots & \ddots & \\ & & -\phi & 1 \end{bmatrix}$

Then

$$\begin{aligned}
\mathbf{Q}_z = \mathrm{Var}\left(\begin{bmatrix} \eta \\ \mathbf{y} \end{bmatrix}\right)^{-1} &= \left(\begin{bmatrix} I & 0 \\ \mathbf{\Lambda} & I \end{bmatrix} \begin{bmatrix} \mathbf{H}^{-1}\mathbf{V}_u\mathbf{H}^{-1^{\mathsf{T}}} & 0 \\ 0 & \mathbf{V}_e \end{bmatrix} \begin{bmatrix} I & \mathbf{\Lambda}^{\mathsf{T}} \\ 0 & I \end{bmatrix}\right)^{-1} \\
&= \begin{bmatrix} I & -\mathbf{\Lambda}^{\mathsf{T}} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{H}^{\mathsf{T}}\mathbf{S}_u\mathbf{H} & 0 \\ 0 & \mathbf{S}_e \end{bmatrix} \begin{bmatrix} I & 0 \\ -\mathbf{\Lambda} & I \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{H}^{\mathsf{T}}\mathbf{S}_u\mathbf{H} + \mathbf{\Lambda}^{\mathsf{T}}\mathbf{V}_e\mathbf{\Lambda} & -\mathbf{\Lambda}^{\mathsf{T}}\mathbf{S}_e \\ -\mathbf{S}_e\mathbf{\Lambda} & \mathbf{S}_e \end{bmatrix} \equiv \begin{bmatrix} \mathbf{Q}_\eta & \mathbf{Q}_{\eta y} \\ \mathbf{Q}_{\eta y}^{\mathsf{T}} & \mathbf{Q}_y \end{bmatrix}
\end{aligned}$$

Appendix

Rue and Held (2005, Algorithm 2.1, 2.4)

Algorithm 2.1 Solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} > 0$

1: Compute the Cholesky factorization $\mathbf{A} = \mathbf{L}\mathbf{L}^\mathsf{T}$
2: Solve $\mathbf{Lv} = \mathbf{b}$ via forward substitution
3: Solve $\mathbf{L}^\mathsf{T}\mathbf{x} = \mathbf{v}$ via backward substitution
4: **return** $\mathbf{x}$

Algorithm 2.4 Sampling $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{Q}^{-1})$

1: Compute the Cholesky factorization $\mathbf{Q} = \mathbf{L}\mathbf{L}^\mathsf{T}$
2: Sample $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$
3: Solve $\mathbf{L}^\mathsf{T}\mathbf{v} = \mathbf{z}$
4: Compute $\mathbf{x} = \boldsymbol{\mu} + \mathbf{v}$
5: **return** $\mathbf{x}$

Back