Precision-based sampling with missing observations

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This presentation

Mash-up of two papers in my dissertation!

Method:

Hauber, P and C. Schumacher (2021). *Precision-based sampling with missing observations: A factor model application*, **Bundesbank Discussion Paper 11/2021**.

Application:

Hauber, P. (2021) *How useful is external information from professional forecasters? Conditional forecasts in large factor models*

Motivation

Essential task in the Bayesian estimation of state space models: drawing from $p(\boldsymbol{\eta}|\mathbf{y}, \Theta)$ where $\boldsymbol{\eta}$ is an unobserved component, $\mathbf{y}$ is data and Θ parameters

Precision-based samplers (Chan and Jeliazkov 2009, **IJMMNO**; McCausland 2012, **JEcmtrics**) exploit the fact the precision matrix of $\boldsymbol{\eta}$ is banded in many macroeconomic application $\rightarrow$ alternative to simulation smoothers that rely on the Kalman filter

Applications in macroeconomics (with complete data) include models of trend inflation (Chan et al. 2013, **JBES**), time-varying Bayesian vector autoregressions (Chan 2020, **JBES**) and factor models (Kaufmann and Schumacher 2017, **JAE**)

Missing observations arise frequently in macroeconomic applications/datasets: different starting dates, different release patterns (”ragged edge”), outliers or mixed frequencies

In our paper, we propose a precision-sampler that can handle (most of these) applications!

Precision-based sampling

Simple example

AR(2) process: $\eta_t = \phi_1 \eta_{t-1} + \phi_2 \eta_{t-2} + u_t;\ u_t \sim \mathcal{N}(0, \sigma^2)$

Stacking the observations over $t = 1, \ldots, T$ yields

$$\mathbf{H}\boldsymbol{\eta} = \mathbf{u}, \text{ where } \mathbf{u} \sim \mathcal{N}(0, \mathbf{I}_T \sigma^2) \text{ and } \mathbf{H} = \begin{bmatrix} 1 & & & & & \\ -\phi_1 & 1 & & & & \\ -\phi_2 & -\phi_1 & 1 & & & \\ & -\phi_2 & -\phi_1 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\phi_2 & -\phi_1 & 1 \end{bmatrix}$$

$\boldsymbol{\eta}$ is Normal with mean $\mathbf{0}_T$ and covariance matrix $\Sigma = \mathbf{H}^{-1}\, \mathbf{I}_T \sigma^2\, {\mathbf{H}^{-1}}^{\mathsf{T}}$

corresponding *precision matrix* is given by $\mathbf{Q} = \Sigma^{-1} = \mathbf{H}^{\mathsf{T}}\, \mathbf{I}_T \sigma^{-2}\, \mathbf{H}$

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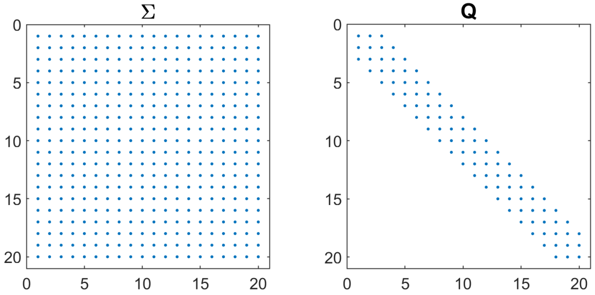
Covariance and precision matrix of η

Properties of the multivariate $\mathcal{N}$:

- $\Sigma_{ij} = 0 \implies$ independence of η_i and η_j
- $\mathbf{Q}_{ij} = 0 \implies$ **conditional** independence of η_i and η_j



Notes: The blue dots indicate the non-zero entries in the covariance matrix Σ and precision matrix $\mathbf{Q}$ of an AR(2) process for $T = 20$ observations. The former is a dense matrix while the latter is sparse and banded with lower and upper bandwidth equal to 2.

Precision-based sampling

Computational advantages of banded precision matrices

Solving linear systems of the form $Ux = b$ where U is an $n \times n$ upper-triangular matrix takes n^2 flops (left); when U has bandwidth p the solution can be obtained in $2np$ flops (right):

```
% solution to Ux = b
% U has maximal bandwidth
for i = n:-1:1
    x(i) = b(i)/U(i,i)
    for j = 1:i-1
        b(j) = b(j) - U(j,i)x(i)
    end
end
```

```
% solution to Ux = b
% U has bandwidth p
for i = n:-1:1
    x(i) = b(i)/U(i,i)
    for j = max{1,i-p}:i-1
        b(j) = b(j) - U(j,i)x(i)
    end
end
```

Even larger gains for matrix factorisations, e.g. Cholesky ($Q = LL^{\mathsf{T}}$) $\Longrightarrow$ linear instead of cubic costs!

L "inherits" the bandwidth of Q (Golub and Van Loan 2013, Theorem 4.3.1)

Precision-based sampling

Factor model

To fix ideas, consider the following factor model:

$$\begin{aligned}
\mathbf{y}_t &= \lambda \boldsymbol{\eta}_t + \mathbf{e}_t \\
\mathbf{e}_t &= \phi^e \mathbf{e}_{t-1} + \boldsymbol{\epsilon}_t;\ \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, diag([\sigma_1^2, \cdots, \sigma_N^2])) \\
\boldsymbol{\eta}_t &= \phi^\eta \boldsymbol{\eta}_{t-1} + \mathbf{u}_t;\ \mathbf{u}_t \sim \mathcal{N}(0, \Sigma_u)
\end{aligned}$$

where $\mathbf{y}_t$ is an $N \times 1$ vector of data and η_t is an $R \times 1$ vector of unobserved factors

Bayesian estimation of the model is done via a Gibbs Sampler which sequentially draws from

- the conditional distribution of factors given data and parameters: $p(\boldsymbol{\eta}|\mathbf{y}, \Theta)$
- the conditional distribution of parameters given data and factors: $p(\Theta|\boldsymbol{\eta}, \mathbf{y})$

Precision-based sampling

Drawing from $p(\boldsymbol{\eta}|\mathbf{y},\Theta)$

Joint distribution of states $\boldsymbol{\eta} = [\boldsymbol{\eta}_1{}^{\mathsf{T}}, \cdots, \boldsymbol{\eta}_T{}^{\mathsf{T}}]^{\mathsf{T}}$ and data $\mathbf{y} = [\mathbf{y}_1^{\mathsf{T}}, \cdots, \mathbf{y}_T^{\mathsf{T}}]^{\mathsf{T}}$ given parameters:

$$\mathbf{z} = \begin{bmatrix} \boldsymbol{\eta} \\ \mathbf{y} \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}^{-1}); \; \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_\eta & \mathbf{Q}_{\eta y} \\ \mathbf{Q}'_{\eta y} & \mathbf{Q}_y \end{bmatrix}$$

Mapping from Θ to $\mathbf{Q}$

Standard result for the multivariate $\mathcal{N}$: $p(\boldsymbol{\eta}|\mathbf{y},\Theta) = \mathcal{N}(-\mathbf{Q}_\eta^{-1}\mathbf{Q}_{\eta y}\mathbf{y}, \mathbf{Q}_\eta^{-1})$

Sampling from this distribution does **not** require the inversion of (the potentially very large matrix) $\mathbf{Q}_\eta$ and because it is banded,

- the mean $-\mathbf{Q}_\eta^{-1}\mathbf{Q}_{\eta y}\mathbf{y}$
- and a random draw given mean and precision matrix

can be obtained efficiently! Rue and Held (2005, Algorithms 2.1, 2.4)

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Drawing from $p(\eta, \mathbf{y}^m | \mathbf{y}^o, \Theta)$

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Appendix

Mapping from Θ to Q

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Appendix

Rue and Held (2005, Algorithm 2.1, 2.4)

Algorithm 2.1 Solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} > 0$

1. Compute the Cholesky factorization $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathsf{T}}$
2. Solve $\mathbf{Lv} = \mathbf{b}$ via forward substitution
3. Solve $\mathbf{L}^{\mathsf{T}}\mathbf{x} = \mathbf{v}$ via backward substitution
4. **return** $\mathbf{x}$

Algorithm 2.4 Sampling $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{Q}^{-1})$

1. Compute the Cholesky factorization $\mathbf{Q} = \mathbf{L}\mathbf{L}^{\mathsf{T}}$
2. Sample $\mathbf{z} \sim \mathcal{N}(\boldsymbol{\mu}, \mathbf{I})$
3. Solve $\mathbf{L}^{\mathsf{T}}\mathbf{v} = \mathbf{z}$
4. Compute $\mathbf{x} = \boldsymbol{\mu} + \mathbf{v}$
5. **return** $\mathbf{x}$

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