Axiomatizing Preferences over Varying Time Horizons*

Philipp Peitler†

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Abstract

We consider the preferences of a decision maker that lives for finitely many periods and hence faces a diminishing number of future periods as time passes. We identify axioms that connect preferences across horizons and lead to exponential and quasi hyperbolic discounting. Existing axiomatizations for an infinite horizon ignore the problem of changing horizons and are not applicable. Existing axiomatizations for a finite horizon do not ensure identical discounting across preference relations and are therefore insufficient. We also extend the environment to allow for an uncertain time horizon.

Keywords: Finite horizon, indefinite horizon, exponential discounting, quasi hyperbolic discounting, expected utility.

1 Introduction

In this paper we consider a decision maker, henceforth DM, who lives for a finite number of time periods. In each period, the DM has preferences over the outcomes she will receive in the remaining future periods. At time t the DM decides over sequences of length l and at time $t+1$ over sequences of length $l-1$. Hence, the time horizon varies across periods. This is different to an infinite horizon environment, where the horizon remains infinite across periods, in two important ways. First, we cannot simply assume that preferences are identical across periods, since the DM has

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†University of Vienna; philipp.peitler@univie.ac.at

preferences over different domains at different points in time. Instead, we have to identify axioms that connect preferences across horizons. Second, there exists a final period, where the DM decides over single outcomes, in absence of any inter-temporal considerations. This allows us to axiomatize a utility representation for sequences, where the per period valuation u is the utility of the agent over single outcomes, justifying the common interpretation that u measures the well-being experienced in that period.

We connect preferences through two axioms, *Constancy* and *Consistency*. Constancy says that preferences are independent, both of the history leading up to the decision and the number of future time periods that are unaffected by the DM's decision. Hence, preferences over the outcomes for the next k periods in period t, when a total of $k+l$ periods lie ahead, are the same as preferences over the outcomes for the next k periods in period $t+1$, when a total of $k+l-1$ periods lie ahead. Consistency says that the preferences between two sequences are the same before and after some outcome is received. Hence, preferences over sequences of length l are the same as preferences over sequences of length $l+1$ where the initial outcome is fixed. We show that these two axioms, together with a non-triviality assumption, lead to an exponentially discounted utility representation for every preference relation, where the discount factor and the per period valuation are the same across preference relations. We then relax Consistency and identify weaker conditions that lead to quasi hyperbolic discounting. These conditions identify all situations where a quasi hyperbolic discounter still behaves consistently. Finally, we extend the environment and require the DM to compare sequences of different lengths. This allows us to derive a utility representation where a positive per period valuation of an outcome informs us that the agent prefers to receive an additional period containing that outcome and where a negative valuation informs us that the agent prefers not to receive an additional period containing that outcome.

Our framework applies to a broad range of models, as we impose no assumptions on the outcome space. For instance, one could model the preferences of a player in a finitely repeated finite stage game or the preferences of a worker choosing consumption levels in $\mathbb{R}_+$ in life cycle model. Our framework allows us to formulate axioms that have straight forward interpretations in terms of how the DM plans for the future, instead of imposing mathematical separability conditions on each preference relation individually.

2 Preliminaries

We begin by outlining the primitives of our framework. Let $\mathbb{T} := \{1, 2, ..., T\}$ denote the set of time periods where $T \in \mathbb{N}$ is the final period. In each period the DM receives an outcome from the set of outcomes A. The set A contains at least two elements, but otherwise we impose no restrictions. For instance, A could be finite or it could be uncountably infinite and unbounded. In any period, the preferences of the DM depend on the history of previously received outcomes. Let $H := \bigcup_{k\in\mathbb{T}} A^{k-1}$ denote the set of all possible histories. We denote the length of history h by $|h|$. After history h, the DM faces decisions that result in sequences of outcomes with length $T - |h|$. We allow for uncertainty and hence require the DM to have preferences over distributions of sequences. Formally, for any $h \in H$, $\succsim_h$ is a preferences relation on $\triangle(A^{T-|h|})$. The family of all preference relations is denoted by $\Phi := \{\succsim_h\}_{h\in H}$.

Next we explain the notation that is used throughout the paper. a and b denote generic outcomes, h and g denote generic histories of outcomes and X and Y denote generic lotteries over sequences of outcomes. When we use generic outcomes, histories or lotteries in an axiom, we mean that the condition holds for all outcomes, histories or lotteries. For simplicity of exposition, we restrict our analysis to *simple* lotteries, which put positive probability on finitely many sequences. Note that this assumption is of no significance for our results and the extension to continuous distributions is straightforward. For some lottery X let $S_X = \{s_1, ..., s_n\}$ denote the finite set of sequences that X can realize. Then X can be written as $((p_1, ..., p_n), (s_1, ..., s_n))$, where p_i denotes the probability which X puts on sequence s_i such that $\sum_{i=1}^{n} p_i = 1$ and $p_i > 0$ for all $i \in \{1, ..., n\}$. We then denote by (X, a) the lottery $((p_1, ..., p_n), ((s_1, a), ..., (s_n, a)))$, where (s_i, a) is the sequence that results from adding $a \in A$ to the end of s_i. Similarly, (a, X) denotes a lottery that results from adding a to the beginning of every sequence in X.

Finally, we describe how the DM deals with uncertainty. We assume that any $\succsim_h \in \Phi$ can be represented by an expected utility representation U_h, such that $U_h(X) = \sum_{i=1}^{n} p_i U_h(s_i)$. Note the this reduces the problem of evaluating lotteries, to the problem of evaluating sequences, but imposes no restrictions on how sequences are evaluated. We refrain from explicitly stating axioms for expected utility, since they are well known (von Neumann and Morgenstern, 1944; Anscombe and Aumann, 1963; Savage, 1972). All of our results can be extended to non-expected utility representations, as long as these representations are unique up to a positive affine transformation.

3 Exponentially Discounted Utility

The standard model of evaluating sequences of outcomes is exponentially discounted utility, henceforth EDU.

Definition 1. $\succcurlyeq_h$ has an EDU representation if there exists a discount factor $\delta_h \in \mathbb{R}_+$ and a per period valuation $u_h : A \mapsto \mathbb{R}$ such that $U_h(a_t, ..., a_T) = \sum_{k=t}^{T} \delta_h^{k-t} u_h(a_k)$.

Even if each $\succcurlyeq_h \in \Phi$ has an EDU representation, δ_h and u_h might differ across histories. Since we want to ensure that the DM has the same EDU representation at all times, it is not sufficient to impose axioms on each preference relation separately, but in addition we have to identify axioms that connect preferences across histories. In the following we explore two notions of connecting preferences. The first is *Constancy*, meaning that preference are the "same" at all times. The second notion is *Consistency*, meaning that the decision of future self's are considered optimal by the current DM. We show that these two conditions, together with a non-triviality assumption, are sufficient for EDU and hence one does not have to impose axioms on each preference relation separately.

The most obvious way to connect preferences across histories is to assume that preferences simply do not depend on the history. In an infinite horizon environment, where all decisions are between sequences of infinite length, this would reduce Φ to a single preference relation. However, in our framework the DM has to evaluate sequences of different lengths at different points in time. *History independence* therefore can only ensures that preferences are the same at time t for all histories leading up to t.

Axiom 3.1 (History independence)**.** If $|h| = |g|$ then $\succcurlyeq_h = \succcurlyeq_g$.

Preferences at time t and $t+1$ on the other hand cannot formally be the same, since the DM decides between lotteries over sequences of length $T-t+1$ and $T-t$ respectively. In order to make these decisions comparable, consider a decision at time t where the outcome of the final period T is fixed, such that the decision only affects the outcomes of the periods t to $T-1$. Even though $T-t+1$ periods lie ahead, the DM only decides over the outcomes for the next $T-t$ periods. *Ignorance of common futures* says that in these situations the DM ignores the final period and decides as if she was in period $t+1$, facing sequences of length $T-t$.

Axiom 3.2 (Ignorance of common futures)**.** For every $h \in H$ there exists a $g \in A^{|h|+1}$ such that $(X, a) \succcurlyeq_h (Y, a)$ if and only if $X \succcurlyeq_g Y$.

Even though we formulate the axiom such that it can stand on its own, we want the reader to interpret Axiom 3.2 in conjunction with Axiom 3.1, as a single condition. We refer to this condition as *Constancy*.

Definition 2. Φ satisfies Constancy if Φ satisfies Axioms 3.1 and 3.2.

Constancy ensures that DM makes the same decisions at all times, independent of the history and the number of unaffected future periods. Importantly, it implies that in situations where a decision only affects the outcome of the current period, the DM decides as if she was in the final period, in absence of any inter-temporal considerations.

The second way of connecting preferences is *Consistency*, an axiom that is standard in the literature on dynamic choice.[1]

Axiom 3.3 (Consistency)**.** $(a, X) \succcurlyeq_h (a, Y)$ if and only if $X \succcurlyeq_{(h,a)} Y$.

Consistency says that the DM's preferences between two lotteries are the same before and after receiving some outcome a. This ensures that future selves do not make decisions that are sub-optimal from the current self's perspective. Note that our framework is agnostic about the beliefs of the DM regarding her preferences in the future. The axiom ensures that the DM decides optimally at all times, independent of her beliefs.

Our axioms, besides connecting preference relations, also impose strong separability conditions on each utility representation. As it turns out, Constancy and Consistency together are sufficient for EDU, as long as the DM cares about the order in which outcomes are received. With the following axiom we rule out the knife-edge case where the agent is indifferent between any sequence and its permutation.

Axiom 3.4 (Non-trivial time preference)**.** There exists a history $f \in H$, a sequence $s \in A^{T-|f|}$ and a permutation of s denoted by $\tilde{s}$ such that $s \succ_f \tilde{s}$.

Note that the axiom only requires one preference relation in Φ to care about the order of outcomes. Constancy and Consistency ensure that the same time preference holds for all preference relations.

We now state the main result of this paper.

Theorem 1. Let $T > 2$. Φ satisfies Axioms 3.1 to 3.4 if and only if there exists a $\delta \in \mathbb{R}_+ \setminus \{1\}$ and a $u : A \mapsto \mathbb{R}$ such that for all $h \in H$

$$U_h(a_t, ..., a_T) = \sum_{k=t}^{T} \delta^{k-t} u(a_k),$$

[1] See for instance Kreps and Porteus (1978).

where $t = |h| + 1$. Furthermore, δ is unique and u is unique up to a positive affine transformation.

We prove Theorem 1 in Appendix A. The proof is simple and self contained. First, note that we require $T > 2$.[2] A utility representation for $T = 2$ is provided by Equation 5 in Appendix A. Second, note that u is a utility representation of $\succcurlyeq_h$ for all h such that $|h| = T - 1$. This means that the per period valuation u is the utility function of the DM in the final period, where only one outcome lies ahead. This is unique to our framework and supports the common interpretation that u indicates the well-being experienced in a single period. Finally, note that due to Axiom 3.4, δ cannot be 1. The following proposition provides a representation for the case where Axiom 3.4 is violated.

Proposition 1. Let $T > 2$. Φ satisfies Axioms 3.1 to 3.3 and violates Axiom 3.4 if and only if there exists a $u : A \mapsto \mathbb{R}$ such that for all $h \in H$, either

$$U_h(a_t, ..., a_T) = \sum_{k=t}^{T} u(a_k)$$

or

$$U_h(a_t, ..., a_T) = \sigma^{T-t} \prod_{k=t}^{T} u(a_k),$$

where $t = |h| + 1$ and $\sigma \in \{-1, 1\}$.

In Appendix B we prove Proposition 1 and discuss the case in more detail.

4 Quasi hyperbolic discounting

There is evidence that some decision makers violate Consistency (Thaler, 1981; Kirby and Herrnstein, 1995). In an effort to account for such behaviour, some models assume that a sequence $(a_1, a_2, a_3, ...)$ is evaluated by $u(a_1) + \beta\delta u(a_2) + \beta\delta^2 u(a_3) + ...$, where β is an additional discount factor that applies to periods after the first one. Since more weight is put on the immediate period, the DM's preferences might reverse, once the consequences of a decision become imminent. However, as we will show next, preferences can only reverse when consequences are imminent, while preferences are consistent under any other circumstances. Hence, quasi hyperbolic discounting is not defined by total inconsistency, but instead by consistency in many situations. The following axioms identify these situations.

[2] It is well known that separability conditions that are sufficient for sequences of length greater than 2 are insufficient sequences of length 2. See for instance Karni and Safra (1998).

Axiom 4.1 (Quasi consistency). $(a,b,X) \succsim_h (a,b,Y)$ if and only if $(b,X) \succsim_{(h,a)} (b,Y)$.

Axiom 4.2 (Two period consistency). Let $|h| = T-2$. $(a,X) \succsim_h (a,Y)$ if and only if $X \succsim_{(h,a)} Y$.

Axiom 4.3 (Path independent consistency). If $(a,X) \succsim_h (a,Y)$, $X \succsim_{(h,a)} Y$ and $X \succsim_{(h,b)} Y$ then $(b,X) \succsim_h (b,Y)$.

Axiom 4.1 says that preferences are consistent, as long as the consequences of a decision are not imminent. Axiom 4.2 says that preferences are consistent between the final period and the second to last period. This is due to the fact that preference reversals under hyperbolic discounting occur solely due to trade-offs between the present and future periods. If there is only one period ahead, no such trade-offs can be made. Axiom 4.3 says that if preferences are consistent for one path leading up to the decision, then they are consistent for any path. Hence, preference reversals occur independent of the path leading up the the decision. Again, this is due to the fact that under hyperbolic discounting reversals occur exclusively due to the trade-off between the present and future periods.

Relaxing Axiom 3.3 to Axioms 4.1, 4.2 and 4.3 in Theorem 1 leads to quasi hyperbolic discounting.

Theorem 2. Let $T > 3$. Φ satisfies Axioms 3.1, 3.2, 3.4, 4.1, 4.2 and 4.3 if and only if there exists a $\delta \in \mathbb{R}_+ \setminus \{1\}$, $\beta \in \mathbb{R}_+$ and a $u : A \mapsto \mathbb{R}$ such that for all $h \in H$

$$U_h(a_t, ..., a_T) = u(a_t) + \beta \sum_{k=t+1}^{T} \delta^{k-t} u(a_k),$$

where $t = |h| + 1$. Furthermore, δ and β are unique and u is unique up to a positive affine transformation.

We prove Theorem 2 in Appendix C. Other axiomatizations of quasi hyperbolic discounting have been provided by Hayashi (2003) and Montiel Olea and Strzalecki (2014) for an infinite horizon and by Anchugina (2017) for a finite horizon. The axiomatization of Fishburn (1970) for EDU under a finite horizon could be easily extended to quasi hyperbolic discounting. As discussed in the previous section, axioms for a finite horizon are insufficient in our framework as one has to impose additional axioms to connect preferences across horizons. We will show in Section 6 that axiomatizations for an infinite horizon are insufficient for our framework as well.

5 Undetermined horizon

We now extend the framework of Section 2, by requiring the DM to compare sequences of different lengths. One might think of a model where the DM's decisions influence the length of her life time. For instance, the DM might have to decide whether to partake in risky activities like smoking or extreme sports. Alternatively, one might think of an indefinitely repeated game, where the time horizon is uncertain, as the game can end in any period with some exogenous probability. Even though in an indefinitely repeated game the DM's decisions do not affect the length of the time horizon, the DM has to compare sequences of different lengths in order to evaluate a lottery over sequences of different lengths.

We extend the set of time periods to $\mathbb{N}_1$ and let $H = \bigcup_{k\in\mathbb{N}_1} A^{k-1}$ be the set of all possible histories. For any $h \in H$, $\succcurlyeq_h$ is a preferences relation on $\triangle(\bigcup_{k\in\mathbb{N}_1} A^k)$, the set of lotteries over sequences of all finite lengths. While there are infinitely many time periods, the time horizon is not infinite, as the DM is never asked to evaluate sequences of infinite length. Any period $t \in \mathbb{N}_1$ could potentially be reached by the DM, depending on choice or chance, but outcomes will ultimately be receive only up to some finite period. As before, the family of preference relations is denoted by Φ.

We want to build on our previous insights and hence impose the axioms of Section 3 with a few adjustments. First, note that in Section 3 the lotteries X and Y in Axioms 3.2 and 3.3 necessarily refer to lotteries over sequences of the same length. In this section we require Axioms 3.2 and 3.3 to hold for any two lotteries X and Y, even when they realize sequences of different lengths. Second, note that now every preference relation is over the same domain and hence we can impose an axiom stronger than Axiom 3.1, which equates all preference relations.

Axiom 5.1 (History independence)**.** $\succcurlyeq_h = \succcurlyeq_g$.

Axiom 5.1 reduces Φ to a single preference relation, which we denote by $\succcurlyeq$.

Let U_m be a utility representation of $\succcurlyeq$, where sequences of length m are evaluated according to $\sum_{k=1}^{m} \delta^{k-1} u(a_k)$. Moreover, let U_{m+1} be a utility representation of $\succcurlyeq$, where sequences of length $m+1$ are evaluated according to $\sum_{k=1}^{m+1} \delta^{k-1} u(a_k)$. We know from Theorem 1 that U_m and U_{m+1} must exist. Hence, as in Theorem 1, the DM compares sequences of the same length by the exponentially discounted sum of per period valuations and discounts identically across different horizons.

Next we explore how the DM compares sequences of different lengths. Since both U_m and U_{m+1} are representations of the same preference relation, U_m must be a positive affine transformation of U_{m+1}. Hence, there exists an $\alpha \in \mathbb{R}_+$ and $\mu \in \mathbb{R}$ such that $U_m = \alpha U_{m+1} + \mu$. For instance, when the DM compares the sequence (a, b) to the outcome a, a sequence of length 1, then $(a, b) \succcurlyeq a$ if and only if $\alpha\,[u(a) + \delta u(b)] + \mu \geq u(a)$. Note that u is unique only up to a positive affine transformation. This means that $u(b) > 0$ alone cannot tell us anything about whether the DM prefers to receive an additional period containing b over time ending after a. Note further that α and μ depend on the u that is chosen for the representation. This raises the question whether there exists a $\tilde{u}$ such that $\alpha = 1$ and $\mu = 0$, implying that $(a, b) \succcurlyeq a$ if and only if $\tilde{u}(b) \geq 0$. We find that an additional axiom is required to ensure the existence of such a $\tilde{u}$.

Axiom 5.2 (Horizon separability)**.** *There exist outcomes $c, d, e \in A$ such that $\frac{1}{2}[c] + \frac{1}{2}[(d, e)] \sim \frac{1}{2}[d] + \frac{1}{2}[(c, e)]$ and $c \succ d$.*

The probabilities to receive either c or d in the first period are the same for both lotteries. Likewise the probability to receive an addition period containing e is the same for both lotteries. Axiom 5.2 merely states that the DM does not care whether the additional period is received after c or after d. Note that since we are building on the axioms of Section 3, e is already being evaluated independently from the outcome of the first period.

Adding Axiom 5.2 to Theorem 1 and assuming a stronger notion of History Independence yields the following result.

Theorem 3. *Φ satisfies Axioms 3.2, 3.3, 3.4, 5.1 and 5.2 if and only if there exists a $\delta \in \mathbb{R}_+ \setminus \{1\}$ and a $\tilde{u} : A \mapsto \mathbb{R}$ such that*

$$U(a_1, ..., a_m) = \sum_{k=1}^{m} \delta^{k-1} \tilde{u}(a_k),$$

for all $m \in \mathbb{N}_1$. Furthermore, δ is unique and $\tilde{u}$ is unique up to a positive linear transformation.

We prove Theorem 3 in Appendix D. As mentioned earlier, the per period valuation of Theorem 1 and 2 is in fact the utility function of the DM when choosing over single outcomes, allowing for the interpretation that u measures the well-being experienced in a single period. On top of that, $\tilde{u}$ has the property that desirable outcomes induce positive values and undesirable outcomes induce negative values. $\tilde{u}(a) > 0$ informs us that the DM is better off when a sequence is extended by the outcome a and $\tilde{u}(a) < 0$ informs us that the agent is worse off. Note that we allow for the cases where either all outcomes in A are desirable or no outcome is desirable.

6 Discussion

In this section we discuss in detail how our framework and axioms relate to the existing literature. We restrict our attention to axiomatizations where the DM faces uncertainty, since a utility representation for preferences over deterministic sequences cannot simply be extended to risky decisions by imposing axioms for expected utility ex post.[3] First, we discuss EDU for lotteries over finite sequences. The only axiomatization for this case is provided by Fishburn (1970). A similar axiomatization by Anchugina (2017) considers the more restrictive case of sequences of independent lotteries and is therefore not applicable to our framework. Second, we discuss EDU for lotteries over infinite sequences. Here the only axiomatization is provided by Epstein (1983). Finally, our paper is related to axiomatizations of preferences in dynamic decision problems with a finite horizon. To our knowledge, we are the first to axiomatize EDU in such an environment. We compare our approach to Kreps and Porteus (1978), the paper most closely related.

6.1 Finite Sequences

Fishburn (1970) considers the preference relation of a DM over lotteries over sequences of some fixed length $m \in \mathbb{N}$. He assumes that the preference relation has an expected utility representation and identifies three axioms that lead to exponentially discounted utility. In the following we consider the case where each $\succcurlyeq_h \in \Phi$ satisfies these axioms. Fishburn's central axiom is *Indifference to autocorrelation*, which says that the DM only cares about the marginal distributions of outcomes induced by a lottery over sequences. We denote the marginal distribution of outcomes of period t induced by lottery X by $X_t \in \triangle A$. Note that $t = |h| + 1$.

Axiom 6.1 (Indifference to autocorrelation)**.** If $X_k = Y_k$ for all $k \in \{t, ..., T\}$ then $X \sim_h Y$.

Axiom 6.2 (Monotonicity)**.** $(X, a_{t+1}, ..., a_T) \succcurlyeq_h (Y, a_{t+1}, ..., a_T)$ if and only if $(a_t, ..., a_k, X, a_{k+2}, ..., a_T) \succcurlyeq_h (a_t, ..., a_k, Y, a_{k+2}, ..., a_T)$ for all $k \in \{t, ..., T-1\}$.

Axiom 6.3 (Stationarity)**.** $(a, X) \succcurlyeq_h (a, Y)$ if and only if $(X, a) \succcurlyeq_h (Y, a)$.

These axioms lead to the following result.

[3] This is due to the fact that ordinal representations are unique only up to a positive monotonic transformation. The existence of an additive representation for ordinal preferences does not ensure that the cardinal representation is additive as well, as there are monotonic transformations that are not additive.

Proposition 2. Φ satisfies Axioms 6.1, 6.2 and 6.3 if and only if for each $h \in H$ there exists a $\delta_h \in \mathbb{R}_+$ and a $u_h : A \mapsto \mathbb{R}$ such that

$$U_h(a_t, ..., a_T) = \sum_{k=t}^{T} \delta_h^{k-t} u_h(a_k),$$

where $t = |h| + 1$. Furthermore, each δ_h is unique and each u_h is unique up to a positive affine transformation.

Note that even though the DM is an exponential discounter at every point in time, the DM might have a different discount factor and per period valuation after each history. Hence, Fishburn's axioms are insufficient for ensuring the representation of Theorem 1. In order to ensure that for all $h, g \in H$, $\delta_g = \delta_h$ and that u_h is a positive affine transformation of u_g, we require additional axioms to connect preferences across histories. Note that History independence is not sufficient, as it does not connect preferences across different horizons. Either Constancy or Consistency would suffice.

Of course, one could start from Fishburn's axioms and then impose either Constancy or Consistency to ensure identical discounting. However, we belief that axioms of inter-temporal decision making are naturally interpreted by thinking about the behaviour of the DM at different points in time. For instance, consider the normative appeal of Axiom 6.1.

If $\succcurlyeq_h$ satisfies Axiom 6.1 then there exists $u_{t,h}$ to $u_{T,h}$ such that $U_h(a_t, ..., a_T) = \sum_{k=t}^{T} u_{k,h}(a_k)$, hence the axiom is quite powerful as it immediately ensures that $\succcurlyeq_h$ is additively separable. It is not clear however how one should interpret the axiom. Is it normatively appealing? Should it apply to all situations or only hold in certain environments? To illustrate the point, consider the following example. The DM can consume one of two goods, a or b, in each period $t \in \{1, 2\}$. The goods are such that they are more enjoyable if they are consumed more often. Therefore $\frac{1}{2}[(a, a)] + \frac{1}{2}[(b, b)] \succ_\emptyset \frac{1}{2}[(a, b)] + \frac{1}{2}[(b, a)]$, even thought the two lotteries induce the same marginal distribution of outcomes for both periods, violating Axiom 6.1. Hence Axiom 6.1 seems reasonable only in environments where preferences do not depend on past outcomes. However, Axiom 6.1 does not rule out that $a \succ_a b$ and $b \succ_b a$, allowing for the paradoxical case where the consumer is both indifferent to autocorrelation and influenced by the history of outcomes. We believe that our approach, where Indifference to autocorrelation follows from axioms that connect the DM's preferences at different points in time, is more insightful than assuming Axiom 6.1 directly.

6.2 Infinite Sequences

Epstein (1983) considers a preference relation on $\triangle(A^\infty)$, which we denote by $\succcurlyeq_\infty$, and assumes that $\succsim_\infty$ has an expected utility representation U_∞. In contrast to our framework, Epstein restricts the outcome space to $A = [0, L]$ for $L \in \mathbb{R}_+$ and assumes that U_∞ is not only continuous in probability but also continuous in outcomes. We present Epstein's axioms in terms of our notation.

Axiom 6.4 (Infinite horizon consistency)**.** $(a, X) \succsim_\infty (a, Y)$ if and only if $X \succsim_\infty Y$.

Axiom 6.5 (Independence of common infinite futures)**.** $(X, s) \succsim_\infty (Y, s)$ if and only if $(X, r) \succsim_\infty (Y, r)$, where $X, Y \in \triangle(A^2)$ and $s, r \in A^\infty$.

These axioms lead to the following result.

Proposition 3. $\succsim_\infty$ satisfies Axioms 6.4 and 6.5 if and only if there exists a $\delta \in (0, 1)$ and a $u : A \mapsto \mathbb{R}$ such that

$$U_\infty(a_1, a_2, ...) = \sum_{k=1}^{\infty} \delta^{k-1} u(a_k).$$

Furthermore, δ is unique and u is unique up to a positive affine transformation.

In the following we show that these axioms, when applied to a finite horizon environment, do not ensure exponentially discounted utility. First, note that there is no straight forward way to translate Axiom 6.4 as a condition on a single preference relation $\succsim_m$ over $\triangle(A^m)$ for $m \in \mathbb{N}$. While adding an outcome to the beginning of an infinite sequence results in an infinite sequence, adding an outcome to the beginning of $X \in \triangle(A^m)$ results in a lottery over sequences of length $m + 1$, which cannot be evaluated by $\succsim_m$. Hence, we have to translate Epstein's axioms in terms of our framework, where Time consistency connects preferences at different points in time. We assume that History independence is satisfied, such that Axiom 3.3 is the finite horizon equivalent of Axiom 6.4. For the finite horizon equivalent of Axiom 6.5, we suggest the following condition.

Axiom 6.6 (Independence of common futures)**.** $(X, s) \succsim_h (Y, s)$ if and only if $(X, r) \succsim_h (Y, r)$, where $s, r \in \bigcup_{k=1}^{T-|h|-1} A^k$.

Note that this translation is quite charitable, as we do not restrict X and Y to lotteries over sequences of length 2.

Applying Epstein's axioms to the finite horizon environment means to assume that Φ satisfies History Independence, Axiom 3.3 and Axiom 6.6. This set of axioms is strictly weaker than the axioms assumed in Theorem 1. For instance, Epstein's axioms are consistent with the existence of $u_t : A \mapsto \mathbb{R}$ for all $t \in \mathbb{T}$ such that $U_h(a_t, ..., a_T) = \sum_{k=t}^{T} u_k(a_k)$ for all $h \in H$, where each u_t ranks outcomes in A differently. Hence, Epstein's axioms, translated to the finite horizon environment, are insufficient for ensuring the representation of Theorem 1.

6.3 Dynamic Decision under a Finite Horizon

Kreps and Porteus (1978) consider an environment similar to ours, where the DM has to make decisions at multiple points in time and the number of the remaining periods decreases as time passes. Unlike in our framework, the DM has preferences over the outcome in the current period and the continuation decision problem that she will face in the following period. There are two key differences to our approach. First, the timing of the resolution of uncertain might matter to the DM, whereas our framework is agnostic about the time at which uncertainty is resolved. Second, the DM might not simply prefer the continuation decision problem that contains the most preferred outcome path, since she might be aware that her future self's will not choose that outcome path. In their final section, Kreps and Porteus (1978) consider the case where the DM is indifferent towards the resolution of uncertainty and satisfies History independence and Consistency. Note that this is equivalent to our framework, where Φ satisfies Axioms 3.1 and 3.3. They find that, in terms of our notation, $\succcurlyeq_h$ is represented by $U_t(a_t, ..., a_T) = \eta_t(a_t)U_{t+1}(a_{t+1}, ..., a_T) + \theta_t(a_t)$, where $t = |h| + 1$, but do not identify additional axioms that lead to EDU. The addition of Axioms 3.2 and 3.4 is not trivial, which is corroborated by fact that the Kreps & Porteus representation is the starting point of our analysis (Equation 1 in Appendix A).

Appendix A

In this section we prove Theorem 1. History independence allows us to replace the subscript on the utility functions by the time period t, where $t = |h| + 1$. Axiom 3.2 implies $U_t(X, a) \geq U_t(Y, a) \iff U_{t+1}(X) \geq U_{t+1}(Y)$. Axiom 3.3 implies $U_t(a, X) \geq U_t(a, Y) \iff U_{t+1}(X) \geq U_{t+1}(Y)$. Therefore, if we hold a fixed, U_t and U_{t+1} must represent the same preferences over lotteries from $\triangle(A^{T-t+1})$. Since expected utility representations are unique up to a positive affine transformation, for each $t \in \mathbb{T} \setminus \{T\}$ there exists $\gamma_{t+1} : A \mapsto \mathbb{R}_+$, $\iota_{t+1} : A \mapsto \mathbb{R}$, $\eta_{t+1} : A \mapsto \mathbb{R}_+$ and $\theta_{t+1} : A \mapsto \mathbb{R}$ such that the following two equations hold.

$$U_t(a_t, ..., a_T) = \eta_{t+1}(a_t)U_{t+1}(a_{t+1}, ..., a_T) + \theta_{t+1}(a_t). \tag{1}$$

$$U_t(a_t, ..., a_T) = \gamma_{t+1}(a_T)U_{t+1}(a_t, ..., a_{T-1}) + \iota_{t+1}(a_T). \tag{2}$$

We assume w.l.o.g. that there exists $o \in A$ such that for all $t \in \mathbb{T}$, $U_t(o, ..., o) = 0$. This implies $\theta_{t+1}(o) = \iota_{t+1}(o) = 0$. We abbreviate $\eta_{t+1}(o)$ and $\gamma_{t+1}(o)$ by $\bar{\eta}_{t+1}$ and $\bar{\gamma}_{t+1}$.

The proof proceeds as follows. First we consider the relation between U_{T-1} and U_T and show that U_{T-1} has a representation that takes the form of a polynomial. We then consider the relation between U_{T-1} and U_{T-2} and show that U_{T-1} collapses to exponential discounting. Finally, we show that if exponential discounting holds for some U_{t+1} it must also hold for U_t.

T-1 to T

We equate (1) and (2) at $t = T-1$, which gives

$$\eta_T(a)U_T(b) + \theta_T(a) = \gamma_T(b)U_T(a) + \iota_T(b). \tag{3}$$

To make it easier on the eyes, we have replaced a_{T-1} with a and a_T with b. $a = o$ implies $\iota_T(b) = \bar{\eta}_T U_T(b)$ and $b = o$ implies $\theta_T(a) = \bar{\gamma}_T U_T(a)$. Substituting these identities, we find

$$\frac{\gamma_T(b) - \bar{\gamma}_T}{U_T(b)} = \frac{\eta_T(a) - \bar{\eta}_T}{U_T(a)} = \kappa, \tag{4}$$

where $\kappa \in \mathbb{R}$ is some constant, which holds for all a and b such that $U_T(a) \neq 0$ and $U_T(b) \neq 0$. We solve (4) for $\eta_T(a)$ and substitute in (1) at $t = T-1$, which gives

$$U_{T-1}(a,b) = \bar{\gamma}_T U_T(a) + \bar{\eta}_T U_T(b) + \kappa U_T(a)U_T(b). \tag{5}$$

T-2 to T-1

Next we show that $\kappa = 0$. We equate (1) and (2) at $t = T-2$, which gives

$$\eta_{T-1}(a)U_{T-1}(b,c) + \theta_{T-1}(a) = \gamma_{T-1}(c)U_{T-1}(a,b) + \iota_{T-1}(c). \tag{6}$$

$a, b = o$ implies $\iota_{T-1}(c) = \bar{\eta}_{T-1}U_{T-1}(o,c)$ and $b, c = o$ implies $\theta_{T-1}(a) = \bar{\gamma}_{T-1}U_{T-1}(a,o)$. We substitute these identities and the representation from (5) in (6) and receive

$$\begin{aligned} &U_T(b)\left[\eta_{T-1}(a)\left[\bar{\gamma}_T + \kappa U_T(c)\right] - \gamma_{T-1}(c)\left[\bar{\eta}_T + \kappa U_T(a)\right]\right] \\ &\qquad = \bar{\eta}_T U_T(c)[\bar{\eta}_{T-1} - \eta_{T-1}(a)] - \bar{\gamma}_T U_T(a)[\bar{\gamma}_{T-1} - \gamma_{T-1}(c)]. \end{aligned} \tag{7}$$

Note that b only appears in the very first term. Since we can freely vary $U_T(b)$ while keeping the remaining terms constant, (7) can only hold if

$$\eta_{T-1}(a)\left[\bar{\gamma}_T + \kappa U_T(c)\right] - \gamma_{T-1}(c)\left[\bar{\eta}_T + \kappa U_T(a)\right] = 0. \tag{8}$$

From (8) it follows that

$$\frac{\bar{\gamma}_T + \kappa U_T(c)}{\gamma_{T-1}(c)} = \frac{\bar{\eta}_T + \kappa U_T(a)}{\eta_{T-1}(a)} = \frac{\bar{\eta}_T}{\bar{\eta}_{T-1}}. \tag{9}$$

Furthermore, the right hand side of (7) must also be equal to 0. We solve (9) for $\gamma_{T-1}(c)$ and $\eta_{T-1}(a)$ and substitute into the right hand side of (7) to find $0 =$

$\kappa[\bar{\gamma}_{T-1} - \bar{\eta}_{T-1}]$. We show in Appendix B that $\bar{\gamma}_T = \bar{\eta}_T$ implies indifference towards the order of outcomes for all $\succsim_t$, which is ruled out by Axiom 3.4, and hence $\bar{\gamma}_T \neq \bar{\eta}_T$. (9) further implies $\bar{\gamma}_{T-1} \neq \bar{\eta}_{T-1}$ and therefore $\kappa = 0$. It then follows from (5) that $\succsim_{T-1}$ has an EDU representation with $\delta = \frac{\bar{\eta}_T}{\bar{\gamma}_T}$ and $u = U_T$.

t+1 to t

If we can show that the implication of $U_{t+1}(a_{t+1}, ..., a_T) = \sum_{k=t+1}^{T} \delta^{k-t-1} u(a_k)$ is that U_t also satisfies exponential discounting with the same discount factor and per period valuation, then we have proven Theorem 1 by induction, since we have already established that U_{T-1} is an exponentially discounted utility representations. We equate (1) and (2) and substitute U_{t+1} to receive

$$\left[\sum_{k=t+1}^{T-1} \delta^{k-t-1} u(a_k)\right] [\eta_{t+1}(a_t) - \delta\gamma_{t+1}(a_T)] \\ = \gamma_{t+1}(a_T)u(a_t) - \eta_{t+1}(a_t)\delta^{T-t-1}u(a_T) + \iota_{t+1}(a_T) - \theta_{t+1}(a_t). \quad (10)$$

Since a_{t+1} to a_{T-1} only appear in the very first term, (10) can only hold if $\eta_{t+1}(a_t) - \delta\gamma_{t+1}(a_T) = 0$, implying that both η_{t+1} and γ_{t+1} are constant. Furthermore, the right hand side of (10) must also be equal to 0. We set $a_t = o$ to find $\iota_{t+1}(a_T) = \bar{\eta}_{t+1}\delta^{T-t-1}u(a_T)$ and further $\iota_{t+1}(a_T) = \bar{\gamma}_{t+1}\delta^{T-t}u(a_T)$. Substituting $\iota_{t+1}(a_T)$ in (2) confirms that U_t is an exponentially discounted utility representation with per period utility u and discount factor δ. This concludes the proof.

Appendix B

In this section we discuss the case where the DM is indifferent towards the order of outcomes and show that if Constancy and Consistency are satisfied but Axiom 3.4 is violated, either each $\succsim_h$ can be represented by

$$U_h(a_t, ..., a_T) = \sum_{k=t}^{T} u(a_k), \quad (11)$$

or each $\succsim_h$ can be represented by

$$U_h(a_t, ..., a_T) = \sigma^{T-t} \prod_{k=t}^{T} u(a_k), \quad (12)$$

where $\sigma \in \{-1, 1\}$ and u is unique up to a positive linear transformation. The part *T-1 to T* in Appendix A still goes through and (5) holds. If the DM is indifferent towards the order of outcomes then $\bar{\gamma}_T = \bar{\eta}_T$ and hence the part *T-1 to T* does not go through. If $\kappa = 0$ anyway, then the rest of the proof goes through and preferences can be represented by (11). Hence, we assume $\kappa \neq 0$ for the rest of this section.

Consider two outcomes a and b such that $a \succ_T b$ and consider the preferences of the DM between $\frac{1}{2}[(a,a)] + \frac{1}{2}[(b,b)]$ and (a,b). The deterministic sequence gives the better outcome once with certainty and the lottery either gives the better outcome twice or not at all. Note the similarity to risk attitudes over money, where the DM is risk averse if $\$M$ is preferred to $\frac{1}{2}\$0 + \frac{1}{2}\$2M$ and risk loving if preferences are reversed. Similarly, if $\kappa < 0$ then the DM strictly prefers (a,b) and is in some sense risk averse regarding the total number of preferred outcomes. If $\kappa > 0$ the DM prefers the lottery and is in some sense risk seeking.

First, we consider $\kappa > 0$. $\tilde{U}_{T-1} = \kappa U_{T-1} + \bar{\gamma}^2$ is a positive affine transformation and hence $\tilde{U}_{T-1}$ is a representation of $\succsim_{T-1}$. We define $u := \bar{\gamma}_T + \kappa U_T$, which preserves the property that u is a representation of $\succsim_T$ and find

$$\tilde{U}_{T-1}(a,b) = u(a)u(b). \tag{13}$$

Note that $u(a)$ must be strictly positive for all $a \in A$ to ensure that $\gamma_T(a) > 0$ and hence $\kappa > 0$ is not possible when U_T has no lower bound.

Next we assume that $U_{t+1}(a_{t+1}, ..., a_T) = \prod_{k=t+1}^{T} u(a_k)$ and show that this implies that U_t has the same representation. We equate (1) and (2) and substitute U_{t+1} to receive

$$\left[\prod_{k=t+1}^{T-1} u(a_k)\right] [\eta_{t+1}(a_t)u(a_T) - \gamma_{t+1}(a_T)u(a_t)] = \iota_{t+1}(a_T) - \theta_{t+1}(a_t). \tag{14}$$

Since a_{t+1} to a_{T-1} only appear in the very first term, (14) can only hold if $\eta_{t+1}(a_t)u(a_T) - \gamma_{t+1}(a_T)u(a_t)$, implying that η_{t+1} is a positive linear transformation of u. Furthermore, $0 = \iota_{t+1}(a_T) - \theta_{t+1}(a_t)$ implying that θ_{t+1} is constant. Substituting η_{t+1} in (2) confirms that U_t has the same multiplicative representation. This proves (12) for $\sigma = 1$ by induction.

Second, we consider $\kappa < 0$. The difference to $\kappa > 0$ is that multiplying a representation by κ is no longer a positive affine transformation. Hence, we define $\tilde{U}_{T-1} := -\kappa U_{T-1} - \bar{\gamma}^2$ and $u := -\bar{\gamma}_T - \kappa U_T$ and find

$$\tilde{U}_{T-1}(a,b) = -u(a)u(b). \tag{15}$$

Note that now $u(a)$ must be strictly negative for all $a \in A$ and $\kappa < 0$ is not possible when U_T has no upper bound.

The induction is the same as for $\kappa > 0$, with the only difference that η_{t+1} is now a negative linear transformation of u. This leads to the alternating sign from t to $t+1$. This concludes the proof.

Appendix C

In this section we prove Theorem 2. We advice the reader to get familiar with Appendix A before reading this section. First, we show that Axiom 4.3 implies $(a, X) \succsim_t (a, Y)$ if and only if $(b, Y) \succsim_t (b, X)$.

Proof. We assume $(a, X) \succsim_t (a, Y)$ and consider the two possible cases, $X \succsim_{t+1} Y$ and $Y \succ_{t+1} X$. First, if $X \succsim_{t+1} Y$ then $(b, X) \succsim_t (b, Y)$ follows directly from Path independence. For the second case, $Y \succ_{t+1} X$, we consider again two possible cases, $(a, X) \succ_t (a, Y)$ and $(a, X) \sim_t (a, Y)$. First, if $(a, X) \succ_t (a, Y)$ then not $(b, Y) \succsim_t (b, X)$ since this would imply $(a, Y) \succsim_t (a, X)$ by Path independence. Hence $(b, X) \succ_t (b, Y)$. Second, if $(a, X) \sim_t (a, Y)$ then $(b, Y) \succsim_t (b, X)$ and so we need to show $(b, X) \sim_t (b, Y)$, which we do by contradiction. Assume $(a, X) \sim_t (a, Y)$, $Y \succ_{t+1} X$ and $(b, Y) \succ_t (b, X)$. Then consider any lottery (a, X^*) such that $(a, X^*) \succ_t (a, X)$ and denote $X_\varepsilon := \varepsilon X^* + (1 - \varepsilon)X$. If (a, X^*) does not exist then consider (a, Y^*) instead such that $(a, Y^*) \succ (a, Y)$. As ε goes to 0, $(a, X_\varepsilon) \succ_t (a, Y)$, $Y \succ_{t+1} X_\varepsilon$ and $(b, Y) \succ_t (b, X_\varepsilon)$, which contradicts Path independence. Hence $(b, X) \sim_t (b, Y)$. $\square$

(2) still holds. Furthermore, Axiom 4.2 implies that (1) holds at $t = T - 1$. Finally, Axiom 4.1 implies $(a, b, X) \succsim_t (a, b, Y)$ if and only if $(b, X) \succsim_{t+1} (b, Y)$ which together with Axiom 4.3 implies $(a, b, X) \succsim_t (a, b, Y)$ if and only if $(a, X) \succsim_{t+1} (a, Y)$. Hence for any $t \in \mathbb{T} \setminus \{T, T-1\}$ there exists $\eta_{t+1} : A^2 \mapsto \mathbb{R}_+$ and $\theta_{t+1} : A^2 \mapsto \mathbb{R}$ such that

$$U_t(a_t, ..., a_T) = \eta_{t+1}(a_t, a_{t+1})U_{t+1}(a_t, a_{t+2}, ..., a_T) + \theta_{t+1}(a_t, a_{t+1}). \tag{16}$$

As in the previous section, we assume $U_t(o, ..., o) = 0$ w.l.o.g. and abbreviate $\eta_{t+1}(o, o)$ and $\gamma_{t+1}(o)$ by $\bar{\eta}_{t+1}$ and $\bar{\gamma}_{t+1}$.

Since (2) and (1) hold at $t = T - 1$, the first part of the proof of Theorem 1 goes through and (5) holds. We then consider the relation between U_{T-2} and U_{T-3} and show that U_{T-3} has a representation that takes the form of a polynomial. Then we consider the relation between U_{T-3} and U_{T-4} and show that U_{T-3} collapses to a quasi hyperbolic discounting representation. Finally, we show that if quasi hyperbolic discounting holds for some U_{t+1} it must also hold for U_t

T-2 to T-3

This part is analogous to the part *T-1 to T-2* in Appendix A. We equate (16) and (2) at $t = T - 2$ and $a_t = o$, which gives

$$\eta_{T-1}(o, a)U_{T-1}(o, b) + \theta_{T-1}(o, a) = \gamma_{T-1}(b)U_{T-1}(o, a) + \iota_{T-1}(b). \tag{17}$$

$a = o$ implies $\iota_{T-1}(b) = \bar{\eta}_{T-1}U_{T-1}(o,b)$ and $b = o$ implies $\theta_{T-1}(o,a) = \bar{\gamma}_{T-1}U_{T-1}(o,a)$. Substituting these identities, we find

$$\frac{\gamma_{T-1}\left(b\right)-\bar{\gamma}_{T-1}}{U_{T-1}\left(o,b\right)}=\frac{\eta_{T-1}\left(o,a\right)-\bar{\eta}_{T-1}}{U_{T-1}\left(o,a\right)}=\kappa', \tag{18}$$

where $\kappa' \in \mathbb{R}$ is some constant. We solve (18) for $\eta_{T-1}(o,a)$ and substitute in (16) at $t = T-2$ and for $a_t = o$, which gives

$$U_{T-2}(o,a,b) = \bar{\gamma}_{T-1}U_{T-1}(o,a) + \bar{\eta}_{T-1}U_{T-1}(o,b) + \kappa' U_{T-1}(o,a)U_{T-1}(o,b). \tag{19}$$

T-3 to T-4

Next we equate (16) and (2) at $t = T-3$ and for $a_t = o$, then substitute (19) and find that $\kappa' = 0$. This is analogous to the part *T-2 to T-1* in Appendix A and therefore details are omitted. $\kappa' = 0$ and (18) imply that γ_{T-1} is constant. Hence (2) at $t = T-2$ is

$$U_{T-2}(a,b,c) = \bar{\gamma}_{T-1}U_{T-1}(a,b) + \bar{\eta}_{T-1}U_{T-1}(o,c). \tag{20}$$

$(a,X) \succcurlyeq_t (a,Y)$ if and only if $(b,X) \succcurlyeq_t (b,Y)$, which we have shown to follow from Path independence, implies that there exists $\zeta : A \mapsto \mathbb{R}_+$ and $\lambda : A \mapsto \mathbb{R}$ such that

$$U_{T-2}(a,b,c) = \zeta(a)U_{T-2}(o,b,c) + \lambda(a). \tag{21}$$

We substitute (20) for $a = o$ in (21) and then equate (20) and (21), which gives

$$U_{T-1}(o,c)[\bar{\eta}_{T-1} - \zeta(a)\bar{\eta}_{T-1}] = \zeta(a)\bar{\gamma}_{T-1}U_{T-1}(o,b) + \lambda(a) - \bar{\gamma}_{T-1}U_{T-1}(a,b). \tag{22}$$

Since c only occurs in the very first term, this implies that $\bar{\eta}_{T-1} - \zeta(a)\bar{\eta}_{T-1} = 0$ and hence $\zeta(a) = 1$ and further

$$\bar{\gamma}_{T-1}U_{T-1}(o,b) + \lambda(a) = \bar{\gamma}_{T-1}U_{T-1}(a,b), \tag{23}$$

implying that U_{T-1} is additively separable and hence $\kappa = 0$ in (5). Substituting (5) in (20) shows that U_{T-2} is a quasi hyperbolic discounting representation with $\delta = \frac{\bar{\eta}_{T-1}}{\bar{\gamma}_{T-1}}$ and $\beta = \frac{\bar{\eta}_T\bar{\gamma}_{T-1}}{\bar{\gamma}_T\bar{\eta}_{T-1}}$.

t+1 to t

If we can show that the implication of $U_{t+1}(a_{t+1},...,a_T) = u(a_{t+1}) + \beta\sum_{k=t+2}^{T}\delta^{k-t-1}u(a_k)$ is that U_t also satisfies quasi hyperbolic discounting with the same discount factors and per period valuation, then we have proven Theorem 2 by induction, since we have already established that U_{T-1} and U_{T-2} are hyperbolic discounting representations. We equate (16) and (2) and substitute U_{t+1} to receive

$$\beta\left[\sum_{k=t+2}^{T-1}\delta^{k-t-1}u(a_k)\right]\left[\eta_{t+1}(a_t,a_{t+1})-\delta\gamma_{t+1}(a_T)\right] \\ =\gamma_{t+1}(a_T)\left[u(a_t)+\beta\delta u(a_{t+1})\right]-\eta_{t+1}(a_t,a_{t+1})\left[u(a_t)+\beta\delta^{T-t-1}u(a_T)\right] \\ +\iota_{t+1}(a_T)-\theta_{t+1}(a_t,a_{t+1}). \quad (24)$$

Since a_{t+2} to a_{T-1} only appear in the very first term, (24) can only hold if $\eta_{t+1}(a_t,a_{t+1})-\delta\gamma_{t+1}(a_T)=0$, implying that both η_{t+1} and γ_{t+1} are constant. Furthermore, the right hand side of (24) must also be equal to 0. We set $a_t,a_{t+1}=o$ to find $\iota_{t+1}(a_T)=\bar{\eta}_{t+1}\beta\delta^{T-t-1}u(a_T)$ and further $\iota_{t+1}(a_T)=\bar{\gamma}_{t+1}\beta\delta^{T-t}u(a_T)$. Substituting $\iota_{t+1}(a_T)$ in (2) confirms that U_t is a quasi hyperbolic discounting representation with discount factors δ and β and per period valuation u. This concludes the proof.

Appendix D

In this section we prove Theorem 3. We start out by considering only comparisons where both lotteries realize sequences of the same length, as it was the case in Section 3. We can show that there exists a δ and u such that for each $m\in\mathbb{N}_1$ there exists a U_m representing $\succcurlyeq$ such that sequences of length m are evaluated according to $\sum_{k=1}^{m}\delta^{k-1}u(a_k)$. The proof is identical to the one in Appendix A. We simply replace the notion of utility at time t, where $T-t-1$ periods lie ahead, with utility from sequences of length m, such that $m=T-t-1$.

Each U_m must be a positive affine transformation of U_{m+1}, since they both represent the same preference relation. Hence, for each $m\in\mathbb{N}_1$ there exists $\alpha_{m+1}\in\mathbb{R}_+$ and $\mu_{m+1}\in\mathbb{R}_+$ such that $U_m=\alpha_{m+1}U_{m+1}+\mu_{m+1}$. The following two lemmas identify α_{m+1} and μ_{m+1}.

Lemma 1. $\alpha_{m+1}=1$ for all m.

Proof. Consider the following indifference relation, which must hold according to Axiom 5.2.

$$\frac{1}{2}[c]+\frac{1}{2}[(d,e)]\sim\frac{1}{2}[d]+\frac{1}{2}[(c,e)]. \quad (25)$$

Equating expected utility of both sides according to U_1 gives $(1-\alpha_2)u(c)=(1-\alpha_2)u(d)$ and since $u(c)\neq u(d)$ it must be that $\alpha_2=1$. According to Axiom 3.3, the DM remains indifferent when we add any outcome a to the beginning of each sequence.

$$\frac{1}{2}[(a,c)]+\frac{1}{2}[(a,d,e)]\sim\frac{1}{2}[(a,d)]+\frac{1}{2}[(a,c,e)]. \quad (26)$$

Equating expected utility of both sides according to U_2 shows that $\alpha_3 = 1$. We can add another a to the beginning to show that $\alpha_4 = 1$ and so on. $\square$

Lemma 2. $\mu_{m+1} = \delta\mu_m$ for all m.

Proof. Consider three sequences $q = (q_1, ..., q_m)$, $r = (r_1, ..., r_m)$ and $s = (s_1, ..., s_{m-1})$ such that $q \succ r \succ s$. If such sequences do not exist, since all sequences of length $m-1$ are preferred to sequences of length m, then consider $s \succ r \succ q$ instead. Since $\succcurlyeq$ satisfies expected utility, there must exist a $p \in (0, 1)$ such that

$$p[q] + (1-p)[s] \sim r. \tag{27}$$

Equating expected utility of both sides of (27) according to U_{m-1} gives

$$p\sum_{k=1}^{m} \delta^{k-1}u(q_k) + (1-p)\sum_{k=1}^{m-1} \delta^{k-1}u(s_k) - \sum_{k=1}^{m} \delta^{k-1}u(r_k) = (1-p)\mu_m. \tag{28}$$

According to Axiom 3.3, the DM remains indifferent when we add any outcome a to the beginning of each sequence in (27) and hence

$$p[(a, q)] + (1-p)[(a, s)] \sim (a, r). \tag{29}$$

Equating expected utility of both sides of (29) according to U_m gives

$$p\sum_{k=1}^{m} \delta^{k-1}u(q_k) + (1-p)\sum_{k=1}^{m-1} \delta^{k-1}u(s_k) - \sum_{k=1}^{m} \delta^{k-1}u(r_k) = \frac{1}{\delta}(1-p)\mu_{m+1}. \tag{30}$$

Since the left hand side of (28) is equal to the left hand side of (30) we find that $\mu_{m+1} = \delta\mu_m$. $\square$

Finally, consider the following positive affine transformation of U_1, $U := U_1 + \frac{\mu_2}{\delta}$ and define $\tilde{u} = u + \frac{\mu_2}{\delta}$. We show that U is the representation of Theorem 3. First, note that $U(a) = \tilde{u}(a)$. Furthermore,

$$\begin{aligned} U(a_1, .., a_m) &= U_1(a_1, ..., a_m) + \frac{\mu_2}{\delta} \\ &= \sum_{k=1}^{m} \delta^{k-1}u(a_k) + \mu_2 \sum_{k=2}^{m} \delta^{k-2} + \frac{\mu_2}{\delta} \\ &= \sum_{k=1}^{m} \delta^{k-1}u(a_k) + \frac{\mu_2}{\delta} \sum_{k=1}^{m} \delta^{k-1} \\ &= \sum_{k=1}^{m} \delta^{k-1}\tilde{u}(a_k). \end{aligned} \tag{31}$$

This concludes the proof.

References

Anchugina, N. (2017). A simple framework for the axiomatization of exponential and quasi-hyperbolic discounting. *Theory and Decision*, 82(2):185–210.

Anscombe, F. and Aumann, R. (1963). A definition of subjective probability. *The Annals of Mathematical Statistics*, 34(1):199–205.

Epstein, L. G. (1983). Stationary cardinal utility and optimal growth under uncertainty. *Journal of Economic Theory*, 31(1):133–152.

Fishburn, P. C. (1970). Utility theory for decision making. Technical report, Research analysis corp McLean VA.

Hayashi, T. (2003). Quasi-stationary cardinal utility and present bias. *Journal of Economic Theory*, 112(2):343–352.

Karni, E. and Safra, Z. (1998). The hexagon condition and additive representation for two dimensions: An algebraic approach. *Journal of Mathematical Psychology*, 42(4):393–399.

Kirby, K. N. and Herrnstein, R. J. (1995). Preference reversals due to myopic discounting of delayed reward. *Psychological Science*, 6(2):83–89.

Kreps, D. M. and Porteus, E. L. (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica*, 46(1):185–200.

Montiel Olea, J. L. and Strzalecki, T. (2014). Axiomatization and measurement of quasi-hyperbolic discounting. *The Quarterly Journal of Economics*, 129(3):1449–1499.

Savage, L. J. (1972). *The foundations of statistics.* Courier Corporation.

Thaler, R. (1981). Some empirical evidence on dynamic inconsistency. *Economic letters*, 8(3):201–207.

von Neumann, J. and Morgenstern, O. (1944). *Theory of games and economic behavior.* Princeton university press.