GSERM - Oslo 2018
Generalized Estimating Equations

June 6, 2018 (morning session)

Linear-normal model is:

$$Y_i = \mu_i + u_i$$

with:

$$\mu_i = \mathbf{X}_i \boldsymbol{\beta}.$$

Generalize:

$$g(\mu_i) = \mathbf{X}_i \boldsymbol{\beta}$$

and:

$$Y_i \sim \text{i.i.d.}\, F[\mu_i, \mathbf{V}_i].$$

GLM Estimation

"Score" equations:

$$\mathbf{U}(\beta) = \sum_{i=1}^{N} \mathbf{D}_i' \mathbf{V}_i^{-1} [Y_i - \mu_i] = \mathbf{0}.$$

with:

- $\mathbf{D}_i = \frac{\partial \mu_i}{\partial \beta}$,
- $\mathbf{V}_i = \frac{h(\mu_i)}{\phi}$, and
- $(Y_i - \mu_i) \approx$ a "residual."
- Known as "quasi-likelihood" (e.g. Wedderburn 1974 *Biometrika*).

Now suppose:

$$Y_{it} = \mu_{it} + u_{it}$$

where

- $i \in \{1, ...N\}$ are i.i.d. "units,"
- $t \in \{1, ...T\}$, $T > 1$ are "time points,"
- we want $g(\mu_{it}) = \mathbf{X}_{it}\boldsymbol{\beta}$.

Key issue: Accounting for (conditional) dependence in Y over time.

Full joint distributions over T are hard. But...

Define:

$$\underset{T \times T}{\mathbf{R}_i(\boldsymbol{\alpha})} = \begin{pmatrix} 1.0 & \alpha_{12} & \cdots & \alpha_{1,T} \\ \alpha_{21} & 1.0 & \cdots & \alpha_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{T,1} & \cdots & \alpha_{T,T-1} & 1.0 \end{pmatrix},$$

$\rightarrow$ "working correlation" matrix.

- Completely defined by $\boldsymbol{\alpha}$,
- Structure specified by the analyst.

GEE Origins

Liang and Zeger (1986): We can decompose the variance of Y_{it} as:

$$\mathbf{V}_i = \text{diag}(\mathbf{V}_i^{\frac{1}{2}})\,\mathbf{R}_i(\boldsymbol{\alpha})\,\text{diag}(\mathbf{V}_i^{\frac{1}{2}})$$

With a standard GLM assumption about the mean and variance, this is:

$$\mathbf{V}_i = \frac{(\mathbf{A}_i^{\frac{1}{2}})\,\mathbf{R}_i(\boldsymbol{\alpha})\,(\mathbf{A}_i^{\frac{1}{2}})}{\phi}$$

where

$$\underset{T \times T}{\mathbf{A}_i} = \begin{pmatrix} h(\mu_{i1}) & 0 & \cdots & 0 \\ 0 & h(\mu_{i2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & h(\mu_{iT}) \end{pmatrix}$$

What does that mean?

$\mathbf{V}_i = \text{Var}(Y_{it}|\mathbf{X}_{it}, \beta)$ has two parts:

- $\mathbf{A}_i =$ *unit-level* variation,
- $\mathbf{R}_i(\alpha) =$ within-unit *temporal* variation.

Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

Independent:

$$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & 0 & \cdots & 0 \\ 0 & 1.0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1.0 \end{pmatrix}$$

- Assumes no within-unit temporal correlation.
- Equivalent to GLM on pooled data.

Exchangeable:

$$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha & \cdots & \alpha \\ \alpha & 1.0 & \cdots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \cdots & \alpha & 1.0 \end{pmatrix}$$

- One free parameter in $\mathbf{R}_i(\boldsymbol{\alpha})$ ($\alpha_{ts} = \alpha \; \forall \; t \neq s$)
- Temporal correlation within units is constant across time points.
- Akin (in some respects) to a random-effects model...

Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

AR(p) (e.g., AR(1)):

$$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha & \alpha^2 & \cdots & \alpha^{T-1} \\ \alpha & 1.0 & \alpha & \cdots & \alpha^{T-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{T-1} & \cdots & \alpha^2 & \alpha & 1.0 \end{pmatrix}$$

- One free parameter in $\mathbf{R}_i(\boldsymbol{\alpha})$ ($\alpha_{ts} = \alpha^{|t-s|} \; \forall \; t \neq s$).
- Conditional within-unit correlation an exponential function of the lag.

Stationary(p):

$$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha_1 & \cdots & \alpha_p & 0 & \cdots & 0 \\ \alpha_1 & 1.0 & \alpha_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_p & \cdots & \alpha_1 & 1.0 \end{pmatrix}$$

- AKA "banded," or "p-dependent."
- $p \leq T - 1$ free parameters in $\mathbf{R}_i(\boldsymbol{\alpha})$.
- Conditional within-unit correlation an exponential function of the lag, up to lag p, and zero thereafter.

Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

Unstructured: $$\mathbf{R}_i(\boldsymbol{\alpha}) = \begin{pmatrix} 1.0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1,T-1} \\ \alpha_{12} & 1.0 & \alpha_{23} & \cdots & \alpha_{2,T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{1,T-1} & \alpha_{2,T-1} & \cdots & \alpha_{T-1,T-1} & 1.0 \end{pmatrix}$$

- $\frac{T(T-1)}{2}$ free parameters in $\mathbf{R}_i(\boldsymbol{\alpha})$.
- Conditional within-unit correlation is completely data-dependent.

Score equations:

$$\boldsymbol{U}_{GEE}(\boldsymbol{\beta}_{GEE}) = \sum_{i=1}^{N} \mathbf{D}_i' \left[\frac{(\mathbf{A}_i^{\frac{1}{2}})\, \mathbf{R}_i(\boldsymbol{\alpha})\, (\mathbf{A}_i^{\frac{1}{2}})}{\phi} \right]^{-1} [Y_i - \mu_i] = \mathbf{0}$$

Two-step estimation:

- For fixed values of $\boldsymbol{\alpha}_s$ and ϕ_s at iteration s, use Newton scoring to estimate $\hat{\boldsymbol{\beta}}_s$,
- Use $\hat{\boldsymbol{\beta}}_s$ to calculate standardized residuals $(Y_i - \hat{\mu}_i)_s$, from which consistent estimates of $\boldsymbol{\alpha}_{s+1}$ and ϕ_{s+1} can be estimated.

Liang & Zeger (1986):

$$\hat{\boldsymbol{\beta}}_{GEE} \underset{N\to\infty}{\sim} \boldsymbol{N}(\boldsymbol{\beta}, \boldsymbol{\Sigma}).$$

For $\hat{\boldsymbol{\Sigma}}$, two options:

$$\hat{\boldsymbol{\Sigma}}_{\text{Model}} = N\left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)$$

$$\hat{\boldsymbol{\Sigma}}_{\text{Robust}} = N\left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)^{-1} \left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{S}}_i \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right) \left(\sum_{i=1}^{N} \hat{\boldsymbol{D}}_i' \hat{\boldsymbol{V}}_i^{-1} \hat{\boldsymbol{D}}_i\right)^{-1}$$

where $\hat{\boldsymbol{S}}_i = (Y_i - \hat{\mu}_i)(Y_i - \hat{\mu}_i)'$.

Inference (aka, magic!)

- $\hat{\Sigma}_{\text{Model}}$
  - Requires that $\mathbf{R}_i(\boldsymbol{\alpha})$ be "correct" for consistency.
  - Is slightly more efficient than $\hat{\boldsymbol{\Sigma}}_{\text{Robust}}$ if so.

- $\hat{\Sigma}_{\text{Robust}}$
  - Is consistent *even if* $\mathbf{R}_i(\boldsymbol{\alpha})$ *is misspecified*.
  - Is slightly less efficient than $\hat{\boldsymbol{\Sigma}}_{\text{Model}}$ if $\mathbf{R}_i(\boldsymbol{\alpha})$ is correct.

Use $\hat{\Sigma}_{\textbf{Robust}}$.

Summary

GEEs:

- Are a straightforward variation on GLMs, and so
- Can be applied to a range of data types (continuous, binary, count, proportions, etc.),
- Yield robustly consistent point estimates of βs,
- Account for within-unit correlation in an informed way, but also
- <u>Provide consistent inferences even if that correlation is misspecified</u>.

Practical Issues: Model Interpretation

- In general, GEEs = GLMs.
- GEEs are *marginal* models, so:
  - $\hat{\beta}$s have an interpretation as *average* / total effects.
  - Estimates / effect sizes generally be smaller than conditional (e.g. fixed/random) effects models.
  - E.g., for logit, $\hat{\boldsymbol{\beta}}_M \approx \frac{\hat{\boldsymbol{\beta}}_C}{\sqrt{1+0.35\sigma_\eta^2}}$, where $\sigma_\eta^2 > 0$ is the variance of the unit effects.

Practical Issues: Specifying $\mathbf{R}_i(\boldsymbol{\alpha})$

- Has been called "more art than science."
- Pointers:
  - Choose based on *substance* of the problem.
  - Remember that $\mathbf{R}_i(\boldsymbol{\alpha})$ is conditional on $\mathbf{X}$, $\hat{\boldsymbol{\beta}}$.
  - Consider unstructured when T is small and N large.
  - Try different ones, and compare.
- In general, it shouldn't matter terribly much...

Extensions: GEE2

Substantive interest in $\mathbf{R}_i(\boldsymbol{\alpha})$ (e.g., Prentice 1988)?

Add:

$$\mathbf{U}_{GEE}(\alpha) = \sum_{i=1}^{N} \mathbf{E}_i' \mathbf{W}_i^{-1} (\mathbf{Z}_i - \eta_i)$$

where

- $\mathbf{E}_i = \frac{\partial \eta_i}{\partial \alpha}$,
- $\mathbf{W}_i$ is the "working" VCV matrix for the $\mathbf{Z}$s,
- $\mathbf{Z}_i' = (Z_{i12}, Z_{i13}, ...Z_{iT-1,T-1})$ are the $\frac{T(T-1)}{2}$ observed sample pairwise correlations for i, and
- η_i is a vector of expected values for $\mathbf{Z}_i$ *which may include covariates*.

GEE2: Estimation

Independently from $U_{GEE}(\boldsymbol{\beta})$:

$$\mathbf{U}_{GEE}(\alpha, \boldsymbol{\beta}) = \sum_{i=1}^{N} \begin{pmatrix} \mathbf{D}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{E}_i' \end{pmatrix} \begin{pmatrix} \mathbf{V}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_i^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_i - \mu_i \\ \mathbf{Z}_i - \eta_i \end{pmatrix}$$

or allowing the two to covary:

$$\boldsymbol{U}_{GEE}(\alpha, \boldsymbol{\beta}) = \sum_{i=1}^{N} \begin{pmatrix} \mathbf{D}_i' & \mathbf{0} \\ \mathbf{F}_i' & \mathbf{E}_i' \end{pmatrix} \begin{pmatrix} \mathbf{V}_i^{-1} & \mathrm{Cov}(\mathbf{Y}_i, \mathbf{W}_i) \\ \mathrm{Cov}(\mathbf{W}_i, \mathbf{Y}_i) & \mathbf{W}_i^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_i - \mu_i \\ \mathbf{Z}_i - \eta_i \end{pmatrix}$$

where $\mathbf{F}_i = \frac{\partial \alpha_i}{\partial \boldsymbol{\beta}}$.

GEE2: Costs and Benefits

- Allows simultaneous modeling of first and second moments.
- Conditional on proper specification, $\hat{\beta}_{GEE2}$s are somewhat more efficient than $\hat{\beta}_{GEE}$s.
- Model (1) requires specification of third and fourth moments.
- Many (e.g. Diggle et al.) suggest using $\boldsymbol{W}_i = \underset{m \times m}{\boldsymbol{I}}$.
- Biggest drawback: *Requires correct specification* of $\boldsymbol{R}_i(\boldsymbol{\alpha})$ for consistent estimates of $\hat{\beta}$.
- Software is somewhat limited (`EE`, `MAREG/WinMAREG`, `geepack`, `orth`, possibly SAS$^{\text{TM}}$).

GEEs: Software

Software	Command(s)/Package(s)
R	`gee` / `geepack` / `multgeeB` / `orth` / `repolr`
Stata	`xtgee` / `xtlogit` / `xtprobit` / `xtpois` / etc.
SAS	`genmod` (w/ `repeated`)

GEEs: Software Tips

- Generally follow GLMs (specify "family" + "link")
- Certain combinations not possible/recommended
- Estimation: Fisher scoring, MLE, etc. (MCMC?)

From the `geepack` manual:

Warning

Use "unstructured" correlation structure only with great care. (It may cause R to crash).

Example: President Bush (41) Approval

```
> url <- getURL("https://raw.githubusercontent.com/PrisonRodeo/GSERM-2018-git/master/Data/
> Bush <- read.csv(text = url)
> summary(Bush)
      idno            year         approval           partyid            perfin
 Min.   :  1.0   Min.   :1990   Min.   :-2.0000   Min.   :-3.0000   Min.   :-2.00000
 1st Qu.:156.8   1st Qu.:1990   1st Qu.:-1.2500   1st Qu.:-2.0000   1st Qu.:-1.00000
 Median :312.5   Median :1991   Median : 1.0000   Median : 1.0000   Median : 0.00000
 Mean   :312.5   Mean   :1991   Mean   : 0.2302   Mean   : 0.3793   Mean   : 0.02724
 3rd Qu.:468.2   3rd Qu.:1992   3rd Qu.: 2.0000   3rd Qu.: 2.0000   3rd Qu.: 1.00000
 Max.   :624.0   Max.   :1992   Max.   : 2.0000   Max.   : 3.0000   Max.   : 2.00000
     nateco             age             educ            class          nonwhite
 Min.   :-2.0000   Min.   :18.00   Min.   :1.000   Min.   :1.000   Min.   :0.0000
 1st Qu.:-2.0000   1st Qu.:32.00   1st Qu.:3.000   1st Qu.:1.000   1st Qu.:0.0000
 Median :-1.0000   Median :41.00   Median :4.000   Median :4.000   Median :0.0000
 Mean   :-0.9797   Mean   :45.34   Mean   :4.048   Mean   :3.002   Mean   :0.1378
 3rd Qu.: 0.0000   3rd Qu.:59.00   3rd Qu.:6.000   3rd Qu.:4.000   3rd Qu.:0.0000
 Max.   : 2.0000   Max.   :85.00   Max.   :7.000   Max.   :6.000   Max.   :1.0000
     female
 Min.   :0.0000
 1st Qu.:0.0000
 Median :1.0000
 Mean   :0.5192
 3rd Qu.:1.0000
 Max.   :1.0000

 > pdim(Bush)
Balanced Panel: n=624, T=3, N=1872
```

GEE: Independence

```
> library(geepack)
> GEE.IND<-geeglm(approval~partyid+perfin+nateco+age+educ+class+nonwhite+female,
  data=Bush,id=idno,family=gaussian,corstr="independence")
> summary(GEE.IND)

 Coefficients:
             Estimate   Std.err    Wald Pr(>|W|)
(Intercept)  1.118752  0.165415  45.742 1.35e-11 ***
partyid     -0.317251  0.017570 326.032  < 2e-16 ***
perfin       0.118223  0.032527  13.211 0.000278 ***
nateco       0.360036  0.039828  81.719  < 2e-16 ***
age         -0.001526  0.002270   0.452 0.501292
educ        -0.048732  0.026603   3.355 0.066982 .
class       -0.035451  0.024571   2.082 0.149078
nonwhite    -0.287660  0.112827   6.500 0.010786 *
female      -0.011875  0.076408   0.024 0.876493
---
Signif. codes:  0  ***  0.001  **  0.01  *  0.05  .  0.1     1

Estimated Scale Parameters:
            Estimate Std.err
(Intercept)    1.839 0.05423

Correlation: Structure = independenceNumber of clusters: 624  Maximum cluster size: 3
```

Identical to GLM

```
> GLM <- glm(approval~partyid+perfin+nateco+age+educ+class+nonwhite+female,
             data=Bush,family=gaussian)

> # Coefficients:
> cbind(GEE.IND$coefficients,GLM$coefficients)
                 [,1]     [,2]
(Intercept)  1.11875  1.11875
partyid     -0.31725 -0.31725
perfin       0.11822  0.11822
nateco       0.36004  0.36004
age         -0.00153 -0.00153
educ        -0.04873 -0.04873
class       -0.03545 -0.03545
nonwhite    -0.28766 -0.28766
female      -0.01188 -0.01188

> # Standard Errors:
> cbind(sqrt(diag(GEE.IND$geese$vbeta.naiv)),sqrt(diag(vcov(GLM))))
               [,1]    [,2]
(Intercept) 0.13827 0.13861
partyid     0.01615 0.01619
perfin      0.02963 0.02970
nateco      0.03857 0.03866
age         0.00193 0.00194
educ        0.02148 0.02153
class       0.02066 0.02071
nonwhite    0.09477 0.09500
female      0.06356 0.06371
```

GEE: Exchangeable

```
> GEE.EXC<-geeglm(approval~partyid+perfin+nateco+age+educ+class+nonwhite+female,
  data=Bush,id=idno,family=gaussian,corstr="exchangeable")
> summary(GEE.EXC)

 Coefficients:
            Estimate  Std.err   Wald Pr(>|W|)
(Intercept)  1.14375  0.16592  47.52  5.4e-12 ***
partyid     -0.31881  0.01738 336.60  < 2e-16 ***
perfin       0.10193  0.03195  10.18   0.0014 **
nateco       0.32912  0.03964  68.94  < 2e-16 ***
age         -0.00262  0.00228   1.32   0.2512
educ        -0.05096  0.02669   3.65   0.0562 .
class       -0.03311  0.02471   1.80   0.1803
nonwhite    -0.29156  0.11374   6.57   0.0104 *
female      -0.01596  0.07687   0.04   0.8356
---
Signif. codes:  0  ***  0.001  **  0.01  *  0.05  .  0.1      1

Estimated Scale Parameters:
            Estimate Std.err
(Intercept)     1.84  0.0542

Correlation: Structure = exchangeable  Link = identity

Estimated Correlation Parameters:
      Estimate Std.err
alpha    0.232  0.0275
Number of clusters:   624   Maximum cluster size: 3
```

GEE: AR(1)

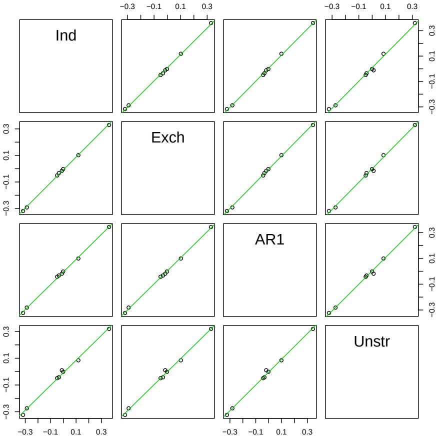
```
> GEE.AR1<-geeglm(approval~partyid+perfin+nateco+age+educ+class+nonwhite+female,
  data=Bush,id=idno,family=gaussian,corstr="ar1")
> summary(GEE.AR1)

 Coefficients:
            Estimate  Std.err   Wald Pr(>|W|)
(Intercept)  1.03609  0.16610  38.91  4.4e-10 ***
partyid     -0.32297  0.01736 346.07  < 2e-16 ***
perfin       0.09890  0.03186   9.64   0.0019 **
nateco       0.34337  0.03967  74.94  < 2e-16 ***
age         -0.00191  0.00229   0.70   0.4038
educ        -0.04255  0.02658   2.56   0.1094
class       -0.03270  0.02488   1.73   0.1888
nonwhite    -0.28120  0.11208   6.29   0.0121 *
female      -0.01873  0.07690   0.06   0.8075
---
Signif. codes:  0  ***  0.001  **  0.01  *  0.05  .  0.1     1

Estimated Scale Parameters:
            Estimate Std.err
(Intercept)     1.84  0.0543

Correlation: Structure = ar1  Link = identity

Estimated Correlation Parameters:
      Estimate Std.err
alpha    0.285  0.0303
Number of clusters:   624   Maximum cluster size: 3
```

GEE: Unstructured

```
> GEE.UNSTR<-geeglm(approval~partyid+perfin+nateco+age+educ+class+nonwhite+female,
  data=Bush,id=idno,family=gaussian,corstr="unstructured")
> summary(GEE.UNSTR)

 Coefficients:
            Estimate  Std.err   Wald Pr(>|W|)
(Intercept)  1.00139  0.16016  39.09    4e-10 ***
partyid     -0.32372  0.01724 352.37   <2e-16 ***
perfin       0.08457  0.03017   7.86   0.0051 **
nateco       0.31947  0.03741  72.94   <2e-16 ***
age         -0.00111  0.00220   0.26   0.6135
educ        -0.04884  0.02586   3.57   0.0589 .
class       -0.04235  0.02421   3.06   0.0803 .
nonwhite    -0.27429  0.11139   6.06   0.0138 *
female       0.01041  0.07479   0.02   0.8893
---
Signif. codes:  0  ***  0.001  **  0.01  *  0.05  .  0.1     1

Estimated Scale Parameters:
            Estimate Std.err
(Intercept)     1.85  0.0542

Correlation: Structure = unstructured  Link = identity

Estimated Correlation Parameters:
          Estimate Std.err
alpha.1:2  0.51573  0.0371
alpha.1:3  0.18614  0.0407
alpha.2:3  0.00277  0.0400
Number of clusters:   624   Maximum cluster size: 3
```

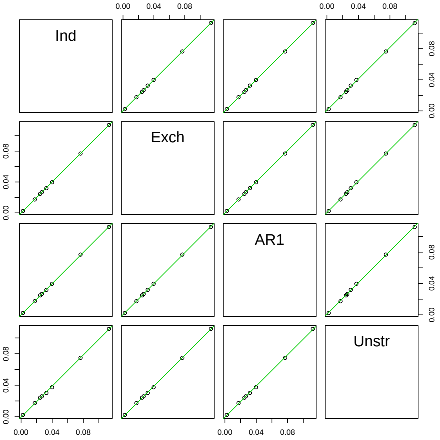
Comparing $\hat{\beta}$s

```
> betas<-cbind(GEE.IND$coefficients,GEE.EXC$coefficients,GEE.AR1$coefficients,
  GEE.UNSTR$coefficients)
> library(car)
> scatterplotMatrix(betas[-1,],smooth=FALSE,var.labels=c("Ind","Exch","AR1","Unstr"),
  diagonal="none")
```



Comparing $\widehat{\text{s.e.}}$s

```
> ses<-cbind(sqrt(diag(GEE.IND$geese$vbeta)),sqrt(diag(GEE.EXC$geese$vbeta)),
  sqrt(diag(GEE.AR1$geese$vbeta)),sqrt(diag(GEE.UNSTR$geese$vbeta)))
> scatterplotMatrix(ses[-1,],smooth=FALSE,var.labels=c("Ind","Exch","AR1","Unstr"),
  diagonal="none")
```



GEEs are:

- Robust
- Flexible
- Extensible beyond panel/TSCS context