OPEN PROBLEMS ON GENERALISED BAIRE SPACES 2020

The *Fifth Workshop on Generalised Baire Spaces* took place at the School of Mathematics of the University of Bristol on 3-4 February, 2020, see `<https://philippschlicht.github.io/meetings/generalizedbairespaces2020>`. This is an incomplete list of some open problems that were discussed at the workshop. [1]

1. Generalized Baire Spaces

1.1. **Claudio Agostini: Winning tactics.** A *tactic* in a game of length ω is a strategy that depends only on the previous move of the opponent. For separable regular Hausdorff spaces, the following conditions are equivalent:

(1) There is a compatible complete metric.
(2) There is a winning tactic for II in the strong Choquet game.
(3) There is a winning strategy for II in the strong Choquet game.

Notice that tactics do not make sense for longer games, since moves at limit times don't have a predecessor. But one can define the following natural variant for the strong Choquet game $G_\kappa(X)$ of length κ for X. A tactic for II in $G_\kappa(X)$ is defined as a strategy with the following properties:

(1) For successor times, the reply of II depends only on the preceding move.
(2) For a play of limit length consisting of $\vec{U} = \langle U_i \mid i < \alpha \rangle$ and $\vec{x} = \langle x_i \mid i < \alpha \rangle$, the reply of II depends only on $\bigcap_{i<\alpha} U_i$ and (possibly) α.

Question 1.1. Suppose that X is a regular Hausdorff space of weight $\leq \kappa$. Does the existence of a winning strategy for player II in $G_\kappa(X)$ imply the existence of a winning tactic?

1.2. **Dorottya Sziraki: The open colouring axiom.** The *open colouring axiom* for X, $\mathsf{OCA}_\kappa(X)$, states that any open graph G on X either has a κ-coloring or else contains a complete subgraph of size κ^+. OCA_κ states that $\mathsf{OCA}_\kappa(X)$ holds for all $X \subseteq {}^\kappa\kappa$.

Question 1.2. Is OCA_κ consistent with ZFC?

The *open graph dichotomy* $\mathsf{OGD}_\kappa(X)$ for X states that any open graph on X either has a κ-colouring or else contains a κ-perfect complete subgraph. The *perfect set property* $\mathsf{PSP}_\kappa(X)$ for $X \subseteq {}^\kappa\kappa$ states that $|X| \leq \kappa$ or X has a κ-perfect subsets.

Question 1.3. Does PSP_κ(closed) imply OGD_κ(closed)?

Fact. $\mathsf{OGD}_\kappa(\text{closed}) \Rightarrow \mathsf{OGD}_\kappa(\Sigma^1_1(\kappa)) \Rightarrow \mathsf{PSP}_\kappa(\Sigma^1_1(\kappa))$.

Fact. *After forcing with* $\mathrm{Col}(\kappa, {<}\lambda)$*, where* $\lambda > \kappa$ *is inaccessible,* OGD_κ *holds for all sets definable from* κ*-sequences of ordinals.*

Question 1.4. Can we force PSP_κ with some forcing other than $\mathrm{Col}(\kappa, {<}\lambda)$?

[1]If you have an answer, comment or further question, please email Philipp Schlicht at philipp.schlicht@bristol.ac.uk.

2. Connections with model theory

2.1. Jan Dobrowolski: Polish groups.

Fact. *Suppose that X is Polish, E is an equivalence relation on X such that*

(1) *the equivalence class $[x]_E$ of any $x \in X$ is closed and*
(2) *the saturation $[U]_E$ of any open subset U of X is Borel.*

Then X/E, with the Effros Borel structure, is standard Borel, and the natural projection $\pi\colon X \to X/E$ has a Borel measurable right inverse f with $\pi \circ f = \mathrm{id}$.

In particular, this holds for any Polish group H and the equivalence relation on H induced by a closed subgroup $G \leq H$.

Question 2.1. Does the analogue of the previous statement about Polish groups hold for κ^κ?

The product of two closed subgroups G, H of $\mathrm{Sym}(\omega)$ has the property of Baire.

Question 2.2. Does the analogue of the previous statement hold for $\mathrm{Sym}(\kappa)$ with the bounded topology?

2.2. Rosario Mennuni: Omitting types.

Disclaimer: The answer to this question might be easy or already known.

The most basic version of the Omitting Type Theorem states:

Theorem. *Let T be a consistent first-order theory in a countable language, $x = (x_0, \dots, x_{n-1})$ an n-tuple of variables, and $\pi(x)$ a partial n-type over $\emptyset$ such that for no formula $\phi(x)$ consistent with T we have $T \cup \{\phi(x)\} \vdash \pi(x)$. Then there is a countable $M \models T$ such that no element of M^n realises $\pi(x)$.*

A better version states (see [2, Theorem 10.3]):

Theorem. *Let T be a consistent first-order theory in a countable language, and for every $n \in \omega$ let A_n be a meagre subset of the space $S_n(T)$ of n-types over $\emptyset$. Then there is a countable $M \models T$ such that, for every $n \in \omega$ and $p(x) \in A_n$, no element of M^n realises $p(x)$.*

The basic version can be generalised to what is called the *κ-omitting types theorem* (see [1, Theorem 2.2.19]).

Theorem. *Let T be a consistent first-order theory in a language of size κ, $x = (x_0, \dots, x_{n-1})$ an n-tuple of variables, and $\pi(x)$ a partial n-type over $\emptyset$ such that for no set of formulas $\Phi(x)$ of size $< \kappa$ consistent with T we have $T \cup \Phi(x) \vdash \pi(x)$. Then there is $M \models T$ with $|M| \leq \kappa$ and such that no element of M^n realises $\pi(x)$.*

Question 2.3. Let L be a language of size κ, and let T be a consistent L-theory. Equip each $S_n(T)$ with the topology generated by declaring partial types of size $< \kappa$ to induce open sets, and replace “meagre” with “κ-meagre”. For which cardinals κ does Theorem 2.2 generalise in the fashion of Theorem 2.2?

References

[1] C.C. Chang, H.J. Keisler. *Model Theory*, Third edition, Studies in Logic and the Foundations of Mathematics 73. North-Holland Publishing Co., Amsterdam, (1990).

[2] B. Poizat. *A Course in Model Theory.* Universitext. Springer (2000).

3. Combinatorics of κ^κ

3.1. Adrian Mathias, Vera Fischer.

Question 3.1. Do analogues to some classical results for mad families on ω hold for mad families on κ?

3.2. Johannes Schürz.

Question 3.2. Is there a forcing that adds a κ-dominating real, but no κ-Cohen real?

3.3. Sarka Stejskalova.

Question 3.3. Is $\mathfrak{u}_\kappa < 2^\kappa$ consistent for the successor κ of a regular cardinal?