# UNIFORMIZATION AND INTERNAL ABSOLUTENESS

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ABSTRACT. Measurability with respect to ideals is tightly connected with absoluteness principles for certain forcing notions. We study a uniformization principle that postulates the existence of a uniformizing function on a large set, relative to a given ideal. We prove that for all $\sigma$-ideals $I$ such that the ideal forcing $\mathbb{P}_I$ of Borel sets modulo $I$ is proper, this uniformization principle is equivalent to an absoluteness principle for projective formulas with respect to $\mathbb{P}_I$ that we call *internal absoluteness*. In addition, we show that it is equivalent to measurability with respect to $I$ together with 1-step absoluteness for the poset $\mathbb{P}_I$. These equivalences are new even for Cohen and random forcing and they are, to the best of our knowledge, the first precise equivalences between regularity and absoluteness beyond the second level of the projective hierarchy.

## 1. INTRODUCTION

The study of the connections between generic absoluteness and regularity properties has been a central theme in set theory in the past decades. For example many results have appeared in work of Bagaria, Brendle, Halbeisen, Ikegami, Judah and others on regularity properties at the first levels of the projective hierarchy [2–5, 8, 11, 12, 14], see [1] for an overview of these results with background on generic absoluteness. Moreover, this is closely related to the work of Feng, Magidor and Woodin on universally Baire sets [7]. For example, Bagaria showed that $\Sigma^1_3$ absoluteness for Cohen forcing is equivalent to the Baire property for all $\mathbf{\Delta}^1_2$ sets of reals [2].[1] Analogously, $\Sigma^1_3$ absoluteness for Random forcing is equivalent to the statement that all $\mathbf{\Delta}^1_2$ sets of reals are Lebesgue measurable [2].

The regularity property we study here is uniformization up to a small set, where "small" refers to being in some fixed ideal $I$. This has for example already been considered by Solovay in [27, Theorem 1] for the ideal of meager sets and the ideal of null sets. Moreover, Shelah showed in [25] that projective uniformization up to a meager set holds in his model for "all sets of reals have the Baire property and CH fails" (that is obtained as a forcing extension of a ZFC model), see also [15]. Zapletal studied a more general notion for ideals $I$ in the context of idealized forcing [32, Proposition 2.3.4]. Uniformization up to Ramsey-small sets also plays a crucial role in Schrittesser's and Törnquist's celebrated result on infinite maximal almost disjoint families [24].

[1]See also [3, Section 5] and [12, Theorem 1.1].

In general, we define projective uniformization up to a small set as follows. It is the special case of the next definition where $\Gamma$ denotes the class of projective sets. We let $I$ always denotes a $\sigma$-ideal on the Borel subsets of a Polish space $X$ that contains all singletons. By $\mathbb{P}_I$ we denote the preorder *mod I* on the $I$-positive Borel sets, i.e. those not in $I$. Moreover, let $p[R] = \{x \in X \mid \exists y \ (x, y) \in R\}$ denote the projection of a relation $R \subseteq X \times X$.

**Definition 1.1.** For any $\sigma$-ideal $I$ on a Polish space $X$ and any class $\Gamma$ of subsets of $X$, $\Gamma$-*uniformization up to* $I$ denotes the statement:

For any binary relation $R \in \Gamma$ on $X$ and for any Borel set $A \notin I$, there is a Borel subset $B \notin I$ of $A$ such that either:

(1) $B$ is disjoint from $p[R]$, or otherwise
(2) $B$ is a subset of $p[R]$ and there is a Borel measurable function $f \colon B \to X$ whose graph is a subset of $R$.

The main result of this paper is that projective uniformization up to $I$ is equivalent to an absoluteness principle for the forcing $\mathbb{P}_I$ for all $\sigma$-ideals $I$ such that $\mathbb{P}_I$ is proper. More precisely, we consider principles of *internal projective absoluteness* that postulate projective absoluteness between the universe and generic extensions of countable elementary submodels of some $H_\theta$. This is defined more formally in Definition 3.1 below. Variants of this notion for the class of all forcings were used in proofs in inner model theory, see [28, §5], and more recently for the class of all proper forcings in Neeman's and Norwood's triangular embedding theorem [20, Theorem 22]. We rediscovered this principle for an application to selectors for ideals [19]. Moreover, it turns out that it is equivalent to a variant of universal Baireness for formulas (see Theorem 3.6).

We prove that in addition to internal absoluteness, also 1-step absoluteness together with $I$-measurability is equivalent to uniformization up to $I$.

**Definition 1.2** (Khomskii [17]). Let $I$ be a $\sigma$-ideal on a Polish space $X$ and let $A \subseteq X$. We say $A$ is $I$-*measurable*[2] if for every Borel set $B \notin I$, there is some Borel set $C \notin I$ with $C \subseteq B$ and either $C \cap A = \emptyset$ or $C \subseteq A$.

To see that projective uniformization up to $I$ implies $I$-measurability for all projective sets, take a projective set $A$ and an arbitrary Borel set $B \notin I$ and apply uniformization up to $I$ to the characteristic function of $A$ restricted to $B$. For more details, see the proof of Theorem 4.2.

We can now state the main result of this article.

**Theorem 1.3.** *The following statements are equivalent for all proper forcings* $\mathbb{P}_I$:

(1) *Internal projective* $\mathbb{P}_I$-*absoluteness holds.*
(2) *Projective uniformization holds up to sets in* $I$.
(3) *1-step* $\mathbb{P}_I$-*absoluteness holds and all projective sets are* $I$-*measurable.*

We will in fact prove a level-by-level version of this theorem, i.e., internal $\Sigma^1_n$ absoluteness is equivalent to uniformization of $\Sigma^1_n$ relations up to $I$ for all $n \geq 1$.

In the final part of this paper, we look at examples of this equivalence for Cohen and random forcing that lead to consistency strength results for internal projective absoluteness for these forcing notions. Moreover, we briefly discuss the strength of internal projective absoluteness for the class of all forcings.

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[2]This is often called $I$-*regular* in the literature.

## 2. Uniformization

For the class of projective sets, the uniformization principle in Definition 1.1 coincides with the notion studied by Solovay [27, Theorem 1] and Shelah [25] for the ideal $I$ of meager sets. The latter states that any projective relation $R$ on $X$ with $p[R] = X$ can be uniformized by a Borel measurable function on a comeager set. More generally, we call a class $\Gamma$ of subsets of $X$ and its finite products *sufficiently closed* if $\Gamma$ is closed under projections, complements, finite unions, products and contains all singletons.

**Lemma 2.1.** *Suppose that $\Gamma$ is sufficiently closed. Then $\Gamma$-uniformization up to $I$ is equivalent to its restriction to relations $R$ with $p[R] = X$.*

*Proof.* Suppose that $R$ is a relation in $\Gamma$. Fix $x_0 \in X$ and assume $x_0 \notin p[R]$. Let $S = R \cup (\{x_0\} \times (X \setminus p[R]))$ and pick a uniformization $f \colon A \to X$ of $S$ up to $I$ for some Borel set $A \notin I$. Now split $A$ into the Borel sets $A_0 = f^{-1}[\{x_0\}]$ and $A_1 = f^{-1}[X \setminus \{x_0\}]$. Since $A \notin I$ and $I$ is an ideal, one of $A_0$, $A_1$ is not in $I$. This shows uniformization of $R$ up to $I$. $\square$

**Lemma 2.2.** *Projective uniformization up to meager sets coincides with Solovay's notion from [27, Theorem 1]. More generally, suppose that $\Gamma$ is sufficiently closed and $Borel/I$ is c.c.c. Then $\Gamma$-uniformization up to $I$ is equivalent to the statement:*

*Any relation $R$ in $\Gamma$ with $p[R] = X$ can be uniformized by a Borel measurable function $f \colon A \to X$, where $A$ is a Borel set with $X \setminus A \in I$.*

*Proof.* Suppose that $R$ is a relation in $\Gamma$. Construct a sequence $\langle X_\alpha, A_\alpha, f_\alpha \mid \alpha < \lambda \rangle$ for some countable ordinal $\lambda$ such that $A_\alpha \notin I$ is a Borel set, $X_\alpha = X \setminus \bigcup_{\beta < \alpha} A_\beta$ the graph of $f_\alpha \colon A_\alpha \to X$ is a subset of $R$ for all $\alpha < \lambda$, and $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha < \beta < \lambda$. In step $\alpha$, define $f_\alpha$ by applying uniformisation up to $I$ to $R$ and $X_\alpha$. Since $Borel/I$ is c.c.c., we have $X_\lambda \in I$ for some countable ordinal $\lambda$. Then $f = \bigcup_{\alpha < \lambda} f_\alpha$ unformizes $R$ on a set with complement in $I$. The converse follows from Lemma 2.1. $\square$

Uniformization up to small sets also plays a crucial role in Schrittesser's and Törnquist's recent result on maximal almost disjoint families [24, Theorem 1.1]. For the ideal $I_R$ of Ramsey null sets, one can see that uniformization up to $I_R$ coincides with the conjunction of $I_R$-measurability and the principle $R$-Unif defined in [24, Section 2.3]. Their main result can be thus restated as follows: Assuming $\mathsf{ZF} + \mathsf{DC}$ and uniformization up to $I_R$ for all sets of reals, there is no infinite maximal almost disjoint family.

In general, the uniformization principle in Definition 1.1 can be equivalently formulated as the conjunction of the conditions:

(1) All sets of the form $p[A]$ for $A \in \Gamma$ are $I$-measurable.
(2) If $R \in \Gamma$ is a relation and $A \notin I$ is a Borel subset of $p[R]$, then there is a Borel subset $B \notin I$ of $A$ and a Borel measurable function $f \colon B \to X$ whose graph is contained in $R$.

## 3. Internal Absoluteness

In this section we define and discuss internal absoluteness, the absoluteness principle that characterizes uniformization up to a small set. We will consider absoluteness for projective formulas with real parameters. Similar notions have been defined in [28] and independently in [19].

**Definition 3.1** (*Internal absoluteness*)**.** Let $\mathbb{F}$ be a class of forcing notions. We say *internal projective $\mathbb{F}$-absoluteness* holds if and only if for any sufficiently large regular

cardinal $\theta$ and a club of countable models $M \prec H_\theta$ with transitive collapse $\bar{M}$, for all $\mathbb{P} \in \mathbb{F} \cap \bar{M}$ and all $\mathbb{P}$-generic filters $g \in V$ over $\bar{M}$,

$$\bar{M}[g] \models \varphi(x) \Longleftrightarrow H_\theta \models \varphi(x),$$

for every projective formula[3] $\varphi(v_0)$ and every real $x$ in $\bar{M}[g]$.

We analogously define internal $\Sigma_n^1$ absoluteness for $n \geq 1$. For a notion of forcing $\mathbb{P}$, we say that *internal projective $\mathbb{P}$-absoluteness* holds if $\mathbb{F} = \{\mathbb{P}\}$ in the previous definition, and *internal projective absoluteness* if $\mathbb{F}$ equals the class of all forcings.

The rest of this section is devoted to general properties of the internal absoluteness principle. Before we argue that internal absoluteness is consistent from large cardinals, we want to remark that internal absoluteness for all forcings $\mathbb{P}$ is related to the notion of being universally Baire. Before we give some details on this, we recall its definition.

**Definition 3.2** (Feng, Magidor, Woodin [7])**.** A subset $A$ of a topological space $Y$ is *universally Baire* if $f^{-1}(A)$ has the property of Baire in any topological space $X$, where $f: X \to Y$ is continuous.[4]

The following well-known equivalence due to Feng, Magidor and Woodin provides a formulation of being universally Baire that has proven to be very useful in set theory, see also [13, Definition 32.21] or [23, Definition 8.6]. To state the equivalence, we need another definition.

**Definition 3.3.** Let $S$ and $T$ be trees on $\omega \times \kappa$ for some ordinal $\kappa$ and let $\lambda$ be an ordinal. We say $(S, T)$ *is $\lambda$-absolutely complementing* if

$$p[S] = {}^\omega\omega \setminus p[T]$$

in every $\mathrm{Col}(\omega, \lambda)$-generic extension of $V$.

For a proof of the following lemma see [7, Theorem 2.1].[5]

**Lemma 3.4** (Feng, Magidor, Woodin [7])**.** *A set of reals $A$ is universally Baire if and only if for every ordinal $\lambda$, there are $\lambda$-absolutely complementing trees $(S, T)$ with $p[S] = A$.*

The above notions of internal absoluteness are closely connected to the following strengthening of being universally Baire.

**Definition 3.5.** For a forcing notion $\mathbb{P}$ we say a formula $\varphi(v_0, v_1)$ is *treeable with respect to $\mathbb{P}$* if and only if for every parameter $a \in V$ there are trees $S$ and $T$ such that in every $\mathbb{P}$-generic extension $V[G]$ of $V$,

$$p[S] = \{x \in {}^\omega\omega \mid \varphi(x, a)\} \text{ and } p[T] = \{x \in {}^\omega\omega \mid \neg\varphi(x, a)\}.$$

Moreover, we say $\varphi(v_0, v_1)$ is *treeable* if it is treeable with respect to $\mathbb{P}$ for all forcing notions $\mathbb{P}$.

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[3]A formula is projective if its quantifiers range over reals.

[4]In a discussion in 2019, Menachem Magidor attributed this definition to Schilling and Vaught [21]. However, the first explicit definition in the literature is to our knowledge due to Feng, Magidor and Woodin.

[5]The following step is missing in [7]: if a set of reals is universally Baire with respect to preimages in compact Hausdorff spaces, then it is universally Baire with respect to all topological spaces. An argument for this was communicated to us by Menachem Magidor. The claim follows from the universal property of the canonical map $i: X \to \beta X$ of the Stone-Čech compactification $\beta X$ of a topological space $X$: (i) any continuous map $f: X \to 2^\omega$ factors as $f = g \circ i$ with a unique continuous map $g: \beta X \to 2^\omega$ and (ii) the range $i[X]$ is dense in $\beta X$.

It is easy to see that if all projective formulas are treeable, then all projective sets are universally Baire. The converse is a well-known open question, see [7, Question 1, Section 6]. [7, Question 1, Section 6] also asks whether projective absoluteness follows from the statement that all projective sets are universally Baire. The converse was asked in [29, Introduction]. Internal absoluteness is a natural absoluteness principle that is equivalent to the statement that all projective formulas are treeable. See [28, Lemma 5.1] for a proof of the following theorem due to Steel and Woodin. Their proof in fact also shows the "local version" of this theorem, i.e. that internal projective $\mathbb{P}$-absoluteness holds if and only if all projective formuals are treeable with respect to $\mathbb{P}$.

**Theorem 3.6.** *Internal projective absoluteness holds for $\mathbb{P}$ if and only if all projective formulas are treeable with respect to $\mathbb{P}$.*

Many standard proofs for forcing generic absoluteness from large cardinals as for example in [30], [22] or [20] also show that internal absoluteness is consistent from large cardinals. The reason is that they prove that the following property holds which, as we argue below, in turn implies internal absoluteness.

**Definition 3.7.** Let $W$ be an inner model and $\kappa$ a cardinal. The *$<\kappa$-generics property over $W$ in $V$* of a forcing $\mathbb{P}$ is the following statement: For any $\mathbb{P}$-generic extension $V[G]$ of $V$ and any real $x \in V[G]$, there exist

(1) a forcing $\mathbb{Q} \in W$ with $|\mathbb{Q}|^W < \kappa$ and
(2) a $\mathbb{Q}$-generic filter $H \in V[G]$ over $W$

such that $x \in W[H]$.

When $\kappa$ is clear from the context, we will also refer to the $<\kappa$-generics property as the *small generics property*.

**Lemma 3.8.** *Suppose that $\kappa$ is inaccessible and $G$ is $\mathrm{Col}(\omega, <\kappa)$-generic over $V$. If $\mathbb{P}$ has the $<\kappa$-generics property over $V$ in $V[G]$, then every projective formula $\varphi(v_0, v_1)$ is treeable in $V[G]$ with respect to $\mathbb{P}$.*

*Proof.* Suppose that $\kappa$ is inaccessible and $G$ is $\mathrm{Col}(\omega, <\kappa)$-generic over $V$. We are going to use the following standard claim about $\mathrm{Col}(\omega, <\kappa)$-generic extensions of $V$.

**Claim 1.** Suppose that $\mathbb{P}$ is a forcing in $V$ of size $<\kappa$ in $V$, and $H$ is $\mathbb{P}$-generic over $V[G]$. Moreover, suppose that $\mathbb{Q}$ is a forcing in $V$ of size $<\kappa$ in $V$ and $G_0$ is $\mathbb{Q}$-generic over $V$. Then there is a $\mathrm{Col}(\omega, <\kappa)$-generic filter $G^*$ over $V$ with $G_0 \subseteq G^*$ and $\mathbb{R}^{V[G^*]} = \mathbb{R}^{V[G][H]}$.

Now suppose $\mathbb{P}$ has the $<\kappa$-generics property over $V$ and let $\varphi(v_0, v_1)$ be a projective formula. Let $a \in V$ be a parameter. We will construct a tree $S$ such that

$$p[S] = \{x \in {}^\omega\omega \mid \varphi(x, a)\}$$

in every $\mathbb{P}$-generic extension of $V[G]$. The tree $T$ for $\neg\varphi(x, a)$ can then be defined analogously. We obtain the tree $S$ as a union of trees $S_{\mathbb{Q},\sigma}$ for all forcings $\mathbb{Q} \in V$ of size $<\kappa$ in $V$ and all $\mathbb{Q}$-names $\sigma$ for reals. Here the tree $S_{\mathbb{Q},\sigma}$ is the canonical tree searching for a pair $(G_0, x)$, such that

(1) $x$ is a real,
(2) $G_0$ is a $\mathbb{Q}$-generic filter over $V$,
(3) $\sigma^{G_0} = x$, and
(4) $\mathbb{Q} * \mathrm{Col}(\omega, <\kappa)$ forces $\varphi(\sigma, \check{a})$.

See for example the proof of [28, Lemma 5.1] for a formal definition of a similar search tree.

**Claim 2.** $p[S] = \{x \in {}^\omega\omega \mid \varphi(x, a)\}$ holds in all $\mathbb{P}$-generic extensions of $V[G]$.

*Proof.* Let $H$ be $\mathbb{P}$-generic over $V[G]$. Suppose that $(G_0, x)$ is a branch through the tree $S$, say it is a branch through $S_{\mathbb{Q},\sigma}$. Then there is some $G^*$ as in Claim 1. Since $(G_0, x)$ is a branch through $S_{\mathbb{Q},\sigma}$, we have $V[G^*] \models \varphi(x, a)$. As $\mathbb{R}^{V[G^*]} = \mathbb{R}^{V[G][H]}$, this implies $V[G][H] \models \varphi(x, a)$.

Conversely, suppose that $V[G][H] \models \varphi(x, a)$. Using the $<\kappa$-generics property for $\mathbb{P}$ over $V$ in $V[G]$, we find some $\mathbb{Q} \in V$ of size $<\kappa$ in $V$ and some $\mathbb{Q}$-generic filter $G_0 \in V[G][H]$ over $V$ with $x \in V[G_0]$. Let $\sigma$ be a $\mathbb{Q}$-name for a real such that $\sigma^{G_0} = x$. Then $(G_0, x)$ is a branch through $S_{\mathbb{Q},\sigma}$. $\qquad\square$

We can now finish the proof of Lemma 3.8 by analogously defining a tree $T$ such that

$$p[T] = \{x \in {}^{\omega}\omega \mid \neg\varphi(x, a)\}$$

holds in all $\mathbb{P}$-generic extensions of $V[G]$. $\qquad\square$

## 4. PROOF OF THE MAIN THEOREM

This section is devoted to the proof of Theorem 1.3. We first introduce some terminology. Fix a $\sigma$-ideal $I$ on the Borel subsets of an uncountable Polish space $X$. We can assume that $X$ equals the Cantor space $2^\omega$, since all uncountable Polish spaces are Borel isomorphic [16, Theorem 15.6]. Elements of $2^\omega$ can be identified with sets of natural numbers and are called *reals*. A generic filter $G$ over $V$ for the forcing $\mathbb{P}_I$ adds a real $x \in X$ such that for every Borel set $B \subseteq X$ coded in the ground model, $B \in G$ if and only if $x \in B$ by [32, Proposition 2.1.2], and in particular $M[x] = M[G]$. We therefore call $x$ a $\mathbb{P}_I$-*generic real*.

Suppose that $M$ is an $\omega$-model[6] of $\mathsf{ZFC}^-$ with $\mathbb{P}_I \in M$ and

$$\tau = \{(\check{n}, p) \mid p \in A_n)\} \in H^M_{\omega_1}$$

is a nice $\mathbb{P}_I$-name for a real, where each $A_n$ is a countable antichain in $\mathbb{P}$. If $g$ is a $\mathbb{P}_I$-generic filter over $M$ that induces the $\mathbb{P}_I$-generic real $x$, we write $\tau^x = \tau^g$. One can compute $\tau^x$ as follows. Define

$$\tau^{(y)} = \{n \in \omega \mid \exists p\ (\check{n}, p) \in \tau \wedge y \in p\}$$

for arbitrary reals $y$. By the definition of $x$ from $g$, $\tau^x = \tau^{(x)}$. Moreover, the map sending any real $y$ to $\tau^{(y)}$ is Borel measurable, since $n \in \tau^{(y)} \iff y \in \bigcup A_n$ for all $n \in \omega$.

If $M$ is an $\omega$-model of $\mathsf{ZFC}^-$ and $\mathbb{P}_I \in M$, we will write $A_{\mathbb{P}_I, M}$ for the set of $M$-generic reals over $\mathbb{P}$. Note that

$$A_{\mathbb{P}_I, M} = \bigcap\{\bigcup D \cap M \mid M \models \text{``$D$ is a dense subset of $\mathbb{P}$''}\}$$

is Borel. From now on, assume that $\mathbb{P}_I$ is proper. Following standard terminology, if $\theta > 2^{|\mathbb{P}_I|}$ is regular, $M \prec H_\theta$ and $\mathbb{P}_I \in M$, then a condition $B \in \mathbb{P}_I$ is called $M$-*generic* if for every maximal antichain $A \in M$ in $\mathbb{P}_I$, $A \cap M$ is predense below $B$ in $\mathbb{P}_I \cap M$. $M$-generic conditions below $B \in \mathbb{P}_I \cap M$ exist if and only if $A_{\mathbb{P}_I, M} \cap B \notin I$ by [32, Proposition 2.2.2] and in this case $A_{\mathbb{P}_I, M} \cap B$ is $M$-generic.

From now on, we will assume that $\mathbb{P}_I$ is a forcing on the reals by working with Borel codes instead of Borel sets. Thus $\mathbb{P}_I$ is the set of Borel codes for Borel sets $A \notin I$. However, we still use the notation "$x \in A$" when $x$ is a real and $A \in \mathbb{P}_I$ (thus $A$ is a Borel code) to mean that $x$ is an element of the set coded by $A$. Note that in this set-up, conditions $A \in \mathbb{P}_I$ and nice $\mathbb{P}_I$-names in $H_{\omega_1}$ will not be moved by transitive collapses.

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[6]We only consider wellfounded models.

We shall use the following standard fact about proper forcings on the reals: for any $\mathbb{P}_I$-name $\sigma$ for a real and any $p \in \mathbb{P}_I$, there is a nice $\mathbb{P}_I$-name $\tau \in H_{\omega_1}$ and some $q \leq p$ with $q \Vdash \sigma = \tau$ (see e.g. [6, Proposition 2.11]).

We now proceed with the proofs. The following level-by-level equivalence strengthens the equivalence of (1) and (2) in Theorem 1.3.

**Lemma 4.1.** *Let $n \geq 1$. The following statements are equivalent for all proper forcings $\mathbb{P}_I$:*

(1) *Internal $\Sigma_n^1$ $\mathbb{P}_I$-absoluteness.*
(2) $\mathbf{\Sigma}_n^1$ *uniformization up to $I$.*
(3) $\mathbf{\Pi}_{n-1}^1$ *uniformization up to $I$.*

*Proof.* Write $\mathbb{P} = \mathbb{P}_I$.

(1) $\Rightarrow$ (2): Suppose that $R = \{(x,y) \mid \varphi(x,y)\}$ is a relation on $2^\omega$, where $\varphi(x,y)$ is $\Sigma_n^1$ with an additional parameter $x_0$. As we mentioned above, we shall identify elements of $2^\omega$ with sets of natural numbers. Fix some $A \in \mathbb{P}$ and take a countable $M \prec H_\theta$ as in (1) that witnesses properness of $\mathbb{P}$ with $A, x_0 \in M$. Let $\bar{M}$ denote its transitive collapse and let $\bar{\mathbb{P}}$ denote the image of $\mathbb{P}$. In $\bar{M}$, let $\sigma$ be a name for the $\bar{\mathbb{P}}$-generic real. We have two cases:

*Case* 1. There exists some $B \leq A$ with $B \Vdash \nexists y \, \varphi(\sigma, y)$. In this case, $C := A_{\bar{\mathbb{P}},\bar{M}} \cap B \subseteq \{x \mid \nexists y \, \varphi(x,y)\}$, so $C$ is disjoint from $p[R]$. Since $C \notin I$ by properness, $C$ is as required.

*Case* 2. There exists some $B \leq A$ with $B \Vdash \exists y \, \varphi(\sigma, y)$. By fullness,[7] there is a nice name $\tau \in \bar{M}$ with $B \Vdash \varphi(\sigma, \tau)$. We can assume that $\tau \in H_{\omega_1}$ by strengthening $B$ using the standard fact above (since $\mathbb{P}$ is assumed to be a forcing on the reals). We further have $C := A_{\bar{\mathbb{P}},\bar{M}} \cap B \notin I$, since $\mathbb{P}$ is proper. Now for any $x \in C$, $\bar{M}[x] \models \varphi(x, \tau^{(x)})$ and hence $V \models \varphi(x, \tau^{(x)})$ by (1). Therefore the graph of the function $f \colon C \to 2^\omega$ defined by $f(x) = \tau^{(x)}$ is a subset of $R$. As noted above, $f$ is Borel measurable as required.

The implication (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1): Fix a large regular $\theta$ and a countable $M \prec H_\theta$ with $\mathbb{P} \in M$. Let $\bar{M}$ denote the transitive collapse of $M$ and let $\bar{\mathbb{P}}$ be the image of $\mathbb{P}$ under the collapse. Let $\tau \in H_{\omega_1}^{\bar{M}}$ be a nice name for a real. For (1), it suffices to show the implication

$$V \models \psi(\tau^g) \implies \bar{M}[g] \models \psi(\tau^g)$$

for all $1 \leq i \leq n$, $\bar{\mathbb{P}}$-generic filters $g \in V$ over $\bar{M}$ and formulas $\psi(x) = \exists y \, \varphi(x,y)$, where $\varphi$ is $\Pi_{i-1}^1$, since the converse follows by taking $\neg\varphi$ instead of $\psi$. We show this implication by induction on $i$. The case $i = 1$ follows from $\Sigma_1^1$ absoluteness. Assume $i \geq 2$.

Let $R := \{(x,y) \mid \varphi(\tau^{(x)}, y)\}$. For a formula $\theta(x)$ and a nice name $\sigma$ for a real, write

$$S_{\theta,\sigma} = \{x \in 2^\omega \mid \theta(\sigma^{(x)})\}.$$

There are by (3) densely many $A \in \bar{\mathbb{P}}$ with the properties:

(a) $A \subseteq S_{\psi,\tau}$ or $A \subseteq 2^\omega \setminus S_{\psi,\tau}$, and
(b) there is a Borel measurable function $f \colon A \to X$ whose graph is a subset of $R$.

Now suppose that $x$ is $\bar{\mathbb{P}}$-generic over $\bar{M}$ and $V \models \psi(\tau^x)$. Pick some $A$ with (a) and (b) such that $x \in A$. Since $V \models \psi(\tau^{(x)})$, $x \in S_{\psi,\tau}$ and hence $A \subseteq S_{\psi,\tau}$ by (a). Pick a Borel measurable function $f \colon A \to X$ that uniformizes $R$ by (b). Since $x \in A$, we have $V \models \varphi(\tau^x, f(x))$. By the inductive hypothesis, $\bar{M}[x] \models \varphi(\tau^x, f(x))$. Hence $\bar{M}[x] \models \psi(\tau^x)$ as required. $\square$

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[7]By fullness, we mean that if an existential statement is forced, then this is witnessed by a name.

Now we are ready to prove Theorem 1.3. We will show the stronger level-by-level version:

**Theorem 4.2.** *Let $n \geq 1$. The following statements are equivalent for all proper forcings $\mathbb{P}_I$:*

(1) *Internal $\Sigma_n^1$ $\mathbb{P}_I$-absoluteness.*
(2) *$\mathbf{\Sigma}_n^1$ uniformization up to sets in $I$.*
(3) *$\mathbf{\Pi}_{n-1}^1$ uniformization up to sets in $I$.*
(4) *1-step $\Sigma_{n+1}^1$ $\mathbb{P}_I$-absoluteness and $I$-measurability of all $\mathbf{\Sigma}_n^1$ sets.*

*Proof.* Write $\mathbb{P} = \mathbb{P}_I$. The equivalence of (1), (2) and (3) was shown in Lemma 4.1.

(1) $\Rightarrow$ (4): Take $\theta$ and $M \prec H_\theta$ as in the definition of internal $\Sigma_n^1$ $\mathbb{P}$-absoluteness. Let $\bar{M}$ be the transitive collapse of $M$ and let $\bar{\mathbb{P}}$ be the image of $\mathbb{P}$ under the collapse. For 1-step $\Sigma_{n+1}^1$ $\mathbb{P}$-absoluteness, it suffices to have $\bar{M} \prec_{\Sigma_{n+1}^1} \bar{M}[g]$ for any $\bar{\mathbb{P}}$-generic filter $g \in V$ over $\bar{M}$. So take any $\Pi_n^1$-formula $\varphi(x, y)$ with additional parameters in $M$, let $\psi(x)$ denote the formula $\exists y \, \varphi(x, y)$ and suppose that $M[g] \models \psi(x)$, where $x \in M$ is a real. Then $V \models \psi(x)$ holds by internal $\Sigma_n^1$ $\mathbb{P}$-absoluteness applied to $M$, $\varphi$ and a witness $y$ for $\varphi$ in $M[g]$. Since $M \prec H_\theta$, we have $M \models \psi(x)$ as required.

It remains to show that any $\mathbf{\Sigma}_n^1$ set $A$ is $I$-measurable. Recall that (1) $\Rightarrow$ (2) by Lemma 4.1. By (2) applied to the relation $A \times \{0\}$, we have two cases. In the first case, there exists an $I$-positive set $B$ disjoint from $A$. In the second case, there exists an $I$-positive set $B$ and a Borel measurable subfunction $f \colon B \to 2^\omega$ of $A \times \{0\}$. Then $B \subseteq A$.

(4) $\Rightarrow$ (1): Take some $M \prec H_\theta$ witnessing properness and let $\bar{M}$ denote the transitive collapse of $M$. Let $\bar{\mathbb{P}}$ denote the image of $\mathbb{P}$ under the transitive collapse. Let $\sigma \in M$ be a nice $\mathbb{P}$-name for a real and $\bar{\sigma}$ its image under the collapse. Take any $\Sigma_n^1$ formula $\varphi(x)$ with additional parameters in $M$. Suppose that $x$ is a $\bar{\mathbb{P}}$-generic real over $\bar{M}$. For (1), it suffices to show:

$$\bar{M}[x] \models \varphi(\bar{\sigma}^x) \Longleftrightarrow V \models \varphi(\bar{\sigma}^x).$$

We first prove two general claims. Suppose that $\tau \in H_{\omega_1}$ is a nice name for a real. Recall that $S_{\varphi, \tau} = \{x \in 2^\omega \mid \varphi(\tau^{(x)})\}$ from the proof of Lemma 4.1. Call a set $A \in \mathbb{P}$ $\tau$-*decisive* if $\tau \in H_{\omega_1}$ is a nice name for a real, $A \Vdash \sigma = \tau$ and either $A \subseteq S_{\varphi, \tau}$ or $A \subseteq 2^\omega \setminus S_{\varphi, \tau}$. Call $A$ *decisive* if it is $\tau$-decisive for some $\tau$.

**Claim 1.** The set of decisive $A \in \mathbb{P}$ is dense.

*Proof.* Since $\mathbb{P}$ is a proper forcing on the reals, the set of $A \in \mathbb{P}$ such that $A \Vdash \sigma = \tau$ holds for some nice name $\tau \in H_{\omega_1}$ is dense. For any such $A$, the set $S_{\varphi, \tau} = \{x \in \omega^\omega \mid \varphi(\tau^{(x)})\}$ is $\mathbf{\Sigma}_n^1$ and hence $I$-measurable by (4). Therefore, there is some $B \leq A$ with either $B \subseteq S_{\varphi, \tau}$ or $B \subseteq \omega^\omega \setminus S_{\varphi, \tau}$ as required. $\square$

**Claim 2.** If $A \in M$ is $\tau$-decisive for some $\tau \in M$, then:

(a) If $A \subseteq S_{\varphi, \tau}$, then $A \Vdash \varphi(\tau)$.
(b) If $A \subseteq \omega^\omega \setminus S_{\varphi, \tau}$, then $A \Vdash \neg\varphi(\tau)$

*Proof.* Recall that the elements of $\mathbb{P}$ are Borel codes. In particular, conditions $A \in \mathbb{P}$ and nice $\mathbb{P}$-names in $H_{\omega_1}$ are not moved in the transitive collapse of $M$.

(a): Towards a contradiction, suppose that $A \not\Vdash \varphi(\tau)$. Since $\bar{M} \cong M \prec H_\theta$, we have $A \not\Vdash^{\bar{M}} \varphi(\tau)$. Hence there is some $B \leq A$ in $\bar{M}$ with $B \Vdash^{\bar{M}} \neg\varphi(\tau)$. Then $\exists x \in B \, (\neg\varphi(\tau^{(x)}))$ holds in any $\bar{\mathbb{P}}$-generic extension of $\bar{M}$. By 1-step $\Sigma_{n+1}^1$ $\mathbb{P}$-absoluteness by (4), $\bar{M} \models \exists x \in B \, (\neg\varphi(\tau^{(x)}))$. Since $\bar{M} \cong M \prec H_\theta$, we have $V \models \exists x \in B \, (\neg\varphi(\tau^{(x)}))$. But this contradicts $B \subseteq A \subseteq S_{\varphi, \tau}$ (here we identify $A$ and $B$ with the Borel sets which they code).

(b): An argument analogous to (a) works. The roles of $\varphi$ and $\neg\varphi$ are switched and 1-step $\Sigma_n^1$ $\mathbb{P}$-absoluteness by (4) is used. $\qquad\square$

We can now prove the theorem with the help of the previous claims. Recall that $x$ is $\bar{\mathbb{P}}$-generic over $\bar{M}$. By Claim 1 applied in $\bar{M}$, there exists some decisive $A \in M$ with $x \in A$. Since $\bar{M} \cong M \prec H_\theta$, there is some $\tau \in M$ such that $A$ is $\tau$-decisive. We first claim that $\bar{\sigma}^x = \tau^x = \tau^{(x)}$. To see this, note that $A \Vdash \sigma = \tau$, since $A$ is $\tau$-decisive. Since $\bar{M} \cong M \prec H_\theta$, we then have $A \Vdash^{\bar{M}} \bar{\sigma} = \tau$. Since $x \in A$, $A$ is in the filter induced by $x$. Hence $\bar{\sigma}^x = \tau^x$. We further have $\tau^x = \tau^{(x)}$, since $x$ is $\bar{\mathbb{P}}$-generic over $\bar{M}$.

It thus suffices to show:

$$\bar{M}[x] \models \varphi(\tau^x) \Longleftrightarrow V \models \varphi(\tau^{(x)}).$$

To see this, consider the two cases in Claim 2. First suppose that $A \subseteq S_{\varphi,\tau}$. We have $\bar{M}[x] \models \varphi(\tau^x)$, since $A \Vdash \varphi(\tau)$ holds by Claim 2 and therefore $A \Vdash^{\bar{M}} \varphi(\tau)$ using $\bar{M} \cong M \prec H_\theta$. Moreover, $V \models \varphi(\tau^{(x)})$ since $x \in A \subseteq S_{\varphi,\tau}$. Finally, suppose that $A \subseteq \omega^\omega \setminus S_{\varphi,\tau}$. Similar to the previous case, we have $\bar{M}[x] \models \neg\varphi(\tau^x)$ and $V \models \neg\varphi(\tau^{(x)})$. $\qquad\square$

## 5. Consistency strength and examples

We round off this paper by some remarks of the consistency strength of the statements shown to be equivalent in Theorem 1.3. More precisely, we consider the strength of internal projective absoluteness, first for the class of all forcings and then for the specific examples of Cohen forcing and random forcing.

5.1. **The strength of internal projective absoluteness.** Woodin proved that 2-step projective absoluteness holds in generic extensions collapsing certain large cardinals. More precisely, he showed that if $\kappa_1 < \cdots < \kappa_n$ are strong cardinals, then 2-step $\boldsymbol{\Sigma}_{n+3}^1$ generic absoluteness holds in $\mathrm{Col}(\omega, 2^{2^{\kappa_n}})$-generic extensions, see [28, Corollary 4.7]. Wilson [30] improved Woodin's result by showing that 2-step $\boldsymbol{\Sigma}_{n+3}^1$ generic absoluteness already holds in $\mathrm{Col}(\omega, 2^{\kappa_n})$-generic extensions of the universe. Woodin's argument shows in fact that internal projective absoluteness can be forced from infinitely many strong cardinals. More precisely, if $\lambda$ is a limit of strong cardinals, then internal projective absoluteness holds in $V[G]$ where $G$ is a $\mathrm{Col}(\omega, \lambda)$-generic filter over $V$. In addition, local versions of this result hold analogous to [28, Corollary 4.7] and [30, Theorem 1.1] for 2-step projective generic absoluteness.

It is easy to see that internal projective absoluteness implies 2-step generic absoluteness.

**Lemma 5.1.** *Let $\mathbb{F}$ be a class of forcing notions. The internal projective $\mathbb{F}$-absoluteness implies 2-step projective absoluteness for forcings in $\mathbb{F}$, i.e., if $\mathbb{P}$ is a forcing in $\mathbb{F}$, $\dot{\mathbb{Q}} \in V$ is a name for a forcing in $\mathbb{F}$, and $g * h$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over $V$, then*

$$V[g] \models \varphi(x) \Longleftrightarrow V[g][h] \models \varphi(x)$$

*for every projective formula $\varphi(v_0)$ and every real $x \in V[g]$.*

*Proof.* Take any countable $\bar{M} \cong M \prec H_\theta$ witnessing internal projective absoluteness. It suffices to show that 2-step projective absoluteness holds for generic extensions of $\bar{M}$ via forcing notions in the transitive collapse of $\mathbb{F} \cap M$. To this end, suppose that $\mathbb{P}, \dot{\mathbb{Q}} \in \bar{M}$ are in the transitive collapse of $\mathbb{F} \cap M$ and let $g * h \in V$ be a $\mathbb{P} * \dot{\mathbb{Q}}$-generic filter over $\bar{M}$. Let $\varphi(v_0)$ be a projective formula and $x \in \bar{M}[g]$ some real. Then

$$\bar{M}[g] \models \varphi(x) \Longleftrightarrow V \models \varphi(x) \Longleftrightarrow \bar{M}[g * h] \models \varphi(x),$$

by internal projective absoluteness applied to $\bar{M}[g]$ and $\bar{M}[g * h]$. $\qquad\square$

Lemma 5.1 together with a result by Hauser, see [9], yields the following fact.

**Fact 5.2.** Internal projective absoluteness implies the existence of an inner model with infinitely many strong cardinals.

Therefore, the consistency strength of internal projective absoluteness is exactly $\omega$ strong cardinals.

5.2. **Cohen forcing.** In this section we look more closely at internal projective absoluteness for Cohen forcing. Is is not hard to see that internal projective $\mathbb{C}$-absoluteness, where $\mathbb{C}$ denotes Cohen forcing, does not follow from projective Cohen 2-step absoluteness.

**Lemma 5.3.** *Internal projective $\mathbb{C}$-absoluteness does not follow from projective 2-step absoluteness for $\mathbb{C}$.*

*Proof.* We will use the following well-known claim. We include a proof for the reader's convenience.

**Claim 3.** Suppose that $M$ is an inner model of $V$ and $R = {}^{\omega}2 \cap M \neq 2^{\omega}$. If $R$ has the property of Baire, then it is meager.

*Proof.* First fix some notation. Write $(x + y)(i) = x(i) + y(i) \bmod 2$ for $x, y \in {}^{\omega}2$ and $x + Y = \{x + y \mid y \in Y\}$ for $x \in {}^{\omega}2$ and $Y \subseteq {}^{\omega}2$. Let $R$ and $M$ be as in the statement of the claim and suppose that $R$ is not meager. Since $R$ has the property of Baire, there is some $t \in 2^{<\omega}$ such that $R \cap N_t$ is comeager in $N_t$, where $N_t = \{s \in {}^{\omega}2 \mid s \text{ extends } t\}$. Let $x \in N_{0^{|t|}} \setminus R$, in particular $x \neq 0$. Then $R \cap N_t$ and $x + (R \cap N_t)$ are disjoint as if $y \in (R \cap N_t) \cap (x + (R \cap N_t))$ then there is some real $z \in R \cap N_t$ such that $y = x + z$. But then $x = y - z \in R$ as $R = {}^{\omega}2 \cap M$ and hence closed under subtraction. This contradicts our choice of $x \notin R$. Note that $x + (R \cap N_t)$ is comeager in $N_t$ as $R \cap N_t$ is comeager in $N_t$ and "$+x$" is a homeomorphism. But this contradicts the fact that $R \cap N_t$ and $x + (R \cap N_t)$ are disjoint subsets of $N_t$. □

It is well-known that the set of ground model reals is not meager in $\mathrm{Add}(\omega, \omega_1)$-generic extensions, see [18]. Therefore, using the claim, the set of ground model reals in $\mathrm{Add}(\omega, \omega_1)$-generic extensions cannot have the property of Baire. Recall that by Theorem 1.3 and the remark before the statement of Theorem 1.3 in the introduction, internal projective $\mathbb{C}$-absoluteness implies that all projective sets have the property of Baire. See also [19, Lemma 5.7] for a direct proof of this fact. So no non-trivial $\mathrm{Add}(\omega, \omega_1)$-generic extension of $L$ satisfies internal projective $\mathbb{C}$-absoluteness as the set of ground model reals is projective if the ground model is $L$. In addition, it is easy to see that projective Cohen 2-step absoluteness holds in these generic extensions. □

Using Claim 3, we can obtain some additional consequences of internal projective $\mathbb{C}$-absoluteness. It for example implies that for any real $x$, $L[x]$ is meager and there is a Cohen real over $L[x]$. Moreover, it implies by [31, Lemma 4] that if $\omega_1 = \omega_1^L$, there is a real $x$ such that there is no random real over $L[x]$.

Nevertheless, internal projective $\mathbb{C}$ can be forced over a ZFC model. For example, it holds by Theorem 1.3 in Shelah's model for uniformization up to a meager set [15, 25].

5.3. **Random forcing.** In this section, we discuss the consistency strength of internal projective $\mathbb{R}$-absoluteness for random forcing $\mathbb{R}$. Recall that 2-step absoluteness for random forcing can be forced over ZFC, it for example holds in a generic extension by the random algebra for uncountably many generators. But by Theorem 1.3, internal projective $\mathbb{R}$-absoluteness implies that all projective sets of reals are Lebesgue measurable. In particular, an inaccessible cardinal is a consistency strength lower bound for internal projective $\mathbb{R}$-absoluteness. Theorem 1.3 combined with results of Solovay show

that this lower bound is optimal: Internal projective $\mathbb{R}$-absoluteness is equivalent to projective uniformization up to a null set. Solovay showed that in his famous model after collapsing an inaccessible cardinal projective uniformization up to a null set holds [26]. We summarize this discussion in the following lemma.

**Lemma 5.4.** *The consistency strength of internal projective $\mathbb{R}$-absoluteness is an inaccessible cardinal. In particular, internal projective $\mathbb{R}$-absoluteness does not follow from projective 2-step absoluteness for $\mathbb{R}$.*

## 6. Open questions

We close this paper with two questions about projective uniformization up to small sets. In Solovay's model projective uniformization holds up to meager sets and up to null sets. This raises the next question.

**Question 6.1.** *In Solovay's model, does projective uniformization up to $I$ hold for every $\sigma$-ideal $I$ such that $\mathbb{P}_I$ is proper?*

As projective uniformization up to $I$ implies $I$-measurability, this would imply a positive answer to the question whether all sets are $I$-measurable in Solovay's model. This was asked by Khomskii, Ikegami and others (see [10, Question 6.3]).[8]

We have seen in Lemma 5.3 that internal projective absoluteness for Cohen forcing does not follow from 2-step projective Cohen absoluteness. In terms of regularity properties, it is natural to conjecture that projective uniformization up to a meager set does not follow from the fact that all projective sets have the property of Baire. Surprisingly, this is open.

**Question 6.2.** *Does the Baire property for all projective sets of reals imply projective uniformization up to meager sets?*

Shelah has shown that both the Baire property for all projective sets and projective uniformization up to meager set are consistent relative to ZFC [15, 25]. He produced two different models for this and it is not clear whether for example projective uniformization up to a meager set fails in his first model for the Baire property.

## References

[1] Joan Bagaria. *Generic Absoluteness*. Book draft.

[2] Joan Bagaria. *Definable forcing and regularity properties of projective sets of reals*. PhD thesis, University of California at Berkeley, 1991.

[3] Joan Bagaria. Axioms of generic absoluteness. In *Logic Colloquium '02*, volume 27 of *Lect. Notes Log.*, pages 28–47. Assoc. Symbol. Logic, La Jolla, CA, 2006.

[4] Tomek Bartoszyński and Haim Judah. *Set Theory: On the Structure of the Real Line*. 1995.

[5] Jörg Brendle. Amoeba-absoluteness and projective measurability. *The Journal of Symbolic Logic*, 58(4):1284–1290, 1993.

[6] Fabiana Castiblanco and Philipp Schlicht. Preserving levels of projective determinacy by tree forcings. *Annals of Pure and Applied Logic*, 172(4):102918, 2021.

[7] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[8] Lorenz Halbeisen and Haim Judah. Mathias absoluteness and the Ramsey property. *The Journal of Symbolic Logic*, 61(1):177–194, 1996.

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[8]An answer is claimed in [17, Proposition 2.2.8], but Ikegami found a gap [10, Section 6].

[9] Kai Hauser. The consistency strength of projective absoluteness. *Annals of Pure and Applied Logic*, 74(3):245 – 295, 1995.

[10] Daisuke Ikegami. I-regularity, determinacy and $\infty$-borel sets of reals. Preprint, 2021.

[11] Daisuke Ikegami. Projective absoluteness for Sacks forcing. *Arch. Math. Logic*, 48:679–690, 2009.

[12] Daisuke Ikegami. Forcing absoluteness and regularity properties. *Ann. Pure Appl. Logic*, 161(7):879–894, 2010.

[13] Thomas Jech. *Set Theory*. Springer Monographs in Mathematics. Springer, 2003.

[14] Haim Judah. Absoluteness for projective sets. In J. Oikkonen and J. Väänänen, editors, *Logic Colloquium '90: ASL Summer Meeting in Helsinki*, volume 2 of *Lecture Notes in Logic*, pages 145–154. Cambridge University Press, 1993.

[15] Haim Judah and Andrzej Roslanowski. On Shelah's amalgamation. In Haim Judah, editor, *Set Theory of the Reals*, Israel Mathematical Conference Proceedings, pages 385–414. American Mathematical Society, 1993.

[16] Alexander Kechris. *Classical descriptive set theory*, volume 156. Springer, 2012.

[17] Yurii Khomskii. *Regularity Properties and Definability in the Real Number Continuum*. PhD thesis, Universiteit van Amsterdam, 2012.

[18] Kenneth Kunen. Random and Cohen reals. In Kenneth Kunen and Jerry E. Vaughan, editors, *Handbook of set-theoretic topology*, pages 887–911. 1984.

[19] Sandra Müller, Philipp Schlicht, David Schrittesser, and Thilo Weinert. Lebesgue's density theorem and definable selectors for ideals. *Israel Journal of Mathematics*. To appear.

[20] Itay Neeman and Zach Norwood. Coding along trees and generic absoluteness. Preprint.

[21] Kenneth Schilling and Robert Vaught. Borel games and the Baire property. *Trans. Amer. Math. Soc.*, 279(1):411–428, 1983.

[22] Ralf Schindler. Proper forcing and remarkable cardinals II. *J. Symbolic Logic*, 66(3):1481–1492, 2001.

[23] Ralf Schindler. *Set Theory*. Universitext. Springer-Verlag, 2014.

[24] David Schrittesser and Asger Törnquist. The Ramsey property implies no mad families. *Proceedings of the National Academy of Sciences*, 116(38):18883–18887, 2019.

[25] Saharon Shelah. Can you take Solovay's inaccessible away? *Israel Journal of Mathematics*, 48(1):1–47, 1984.

[26] Robert M. Solovay. The cardinality of $\Sigma_2^1$ sets of reals. In J. J. Bulloff, T. C. Holyoke, and S. W. Hahn, editors, *Symposium Papers Commemorating the Sixtieth Birthday of Kurt Gödel*, pages 58–73. Springer Berlin Heidelberg, 1969.

[27] Robert M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. of Math. (2)*, 92:1–56, 1970.

[28] John R. Steel. The derived model theorem. In S. Barry Cooper, Herman Geuvers, Anand Pillay, and Jouko Väänänen, editors, *Logic Colloquium 2006*, Lecture Notes in Logic, pages 280–327. Cambridge University Press, 2009.

[29] Trevor Wilson. Universally Baire sets and generic absoluteness. *J. Symb. Log.*, 82(4):1229–1251, 2017.

[30] Trevor Wilson. On forcing projective generic absoluteness from strong cardinals. 2018. https://arxiv.org/abs/1807.02206.

[31] W. Hugh Woodin. On the consistency strength of projective uniformization. In J. Stern, editor, *Proceedings of the Herbrand symposium. Logic Colloquium '81*, pages 365–383. North-Holland, 1982.

[32] Jindřich Zapletal. *Forcing idealized*, volume 174 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2008.

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