The Diamond Principles

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1 Introduction

The diamond principle is a tool in set theory introduced by Jensen [2], as a part of the proof of the existence of Suslin trees assuming the axiom of constructibility. Together with its many variants, the diamond principle serves as the key to the proofs of many consistency or independence results.

In this article, we give a brief introduction to the diamond principle and one of its equivalent forms in §2. After that, in §3, we present the proof of two consistency results using the diamond principle. The proofs mainly follow [3].

Convention. As is usual in forcing, we assume the existence of a countable transitive model $\mathbf{M}$ of ZFC.

2 The diamond principles

2.1 Diamond

Definition 2.1. Let κ be an infinite cardinal. A subset $C \subseteq \kappa$ is called a *club* set, if the following holds:

(i) C is *closed* in κ under the order topology. Equivalently, for any non-empty subset $C' \subseteq C$, if $\sup C' < \kappa$, then $\sup C' \in C$.

(ii) C is *unbounded* in κ. That is, for any $\alpha < \kappa$, there exists $\beta \in C$ such that $\alpha < \beta$.

A subset $S \subseteq \kappa$ is called a *stationary* set, if it intersects every club set in κ. $\lhd$

A stationary set is a set that is "not too small" in some sense. For example, a stationary set must be unbounded.

Definition 2.2. The *diamond principle* $\diamond$ states that there exists a transfinite sequence of functions

$$\langle\, h_\alpha \colon \alpha \to 2 \mid \alpha \in \omega_1 \,\rangle$$

such that for any function $f \colon \omega_1 \to 2$, the set $\{\alpha \in \omega_1 \mid h_\alpha = f|_\alpha\}$ is stationary.

More generally, for a given stationary set $S \subseteq \omega_1$, the *diamond principle* $\diamond_S$ states that there exists a transfinite sequence of subsets

$$\langle\, h_\alpha \colon \alpha \to 2 \mid \alpha \in S \,\rangle$$

such that for any function $f \colon \omega_1 \to 2$, the set $\{\alpha \in S \mid h_\alpha = f|_\alpha\}eq$ is stationary in ω_1. In particular, the principle $\diamond_{\omega_1}$ is the same as $\diamond$.

The axiom $\forall S \diamond_S$ states that $\diamond_S$ holds for every stationary subset $S \subseteq \omega_1$. $\lhd$

Theorem 2.3. *There exists a generic extension* $\mathbf{M}[G]$ *in which* $\diamond$ *holds.*

Proof. Consider the following forcing notion. Let $\mathbb{P}$ be the poset whose elements are of the form $p = \langle h_\beta \colon \beta \to 2 \mid \beta < \alpha\rangle$, where $\alpha < \omega_1$ is called the *height* of p, and the partial order is defined by $p \le q$ if and only if $p \supseteq q$.

Let $G \subseteq \mathbb{P}$ be a generic filter. Then it is not hard to see that the union of G is a sequence $h = \langle h_\alpha \colon \alpha \to 2 \mid \alpha \in \omega_1\rangle$. Let $\dot{h}$ be a $\mathbb{P}$-name of this sequence.

Since $\mathbb{P}$ is ω-closed, by [3, Lemma 12.2], $\mathbf{M}[G]$ preserves ω_1.

Let $f \colon \omega_1 \to 2$ be a function in $\mathbf{M}[G]$, and let $C \subseteq \omega_1$ be a club set in $\mathbf{M}[G]$. We have to prove that there exists $\alpha \in C$ such that $h_\alpha = f|_\alpha$.

Let $\dot{f}$ and $\dot{C}$ be $\mathbb{P}$-names for f and C. Let $p \in G$ be an element which forces the statement "$\dot{f}$ is a function from ω_1 to 2 and $\dot{C}$ is a club subset of ω_1".

We claim that for any $q \le p$ of height α, there exist $\beta > \alpha$, a function $g \colon \alpha \to 2$ in $\mathbf{M}$, and an element $q' \le q$ whose height is $\ge \beta$ and which forces $\check{\beta} \in \dot{C}$ and $\check{g} = \dot{f}|_{\check{\alpha}}$. To see this, let G' be a generic filter containing q, and let $g = \dot{f}^{G'}|_\alpha$. Then $g \in \mathbf{M}$ by [3, Lemma 12.2], so that some $q'' \le q$ forces $\check{g} = \dot{f}|_{\check{\alpha}}$. On the other hand, since p forces $\dot{C}$ to be unbounded, there exists $\beta > \alpha$ and some $q' \le q''$ which forces $\check{\beta} \in \dot{C}$. We can choose q' so that its height is $\ge \beta$.

For each $q \le p$, we construct a sequence $q = p_0 \ge p_1 \ge \cdots$ in $\mathbb{P}$, as follows. Let $p_0 = q$; given p_n with height α_n, use the claim to find $p_{n+1} \le p_n$, $\beta_n > \alpha_n$ and $g_n \colon \alpha_n \to 2$, such that p_{n+1} forces $\check{\beta}_n \in \dot{C}$ and $\check{g}_n = \dot{f}|_{\check{\alpha}_n}$. Then the element $q^* = \bigcup_n p_n$ has height $\alpha^* = \sup \alpha_n = \sup \beta_n$. Since $p \Vdash$ "$\dot{C}$ is closed", we see that $q^* \Vdash (\check{\alpha}^* \in \dot{C})$. Let g^* be the union of the functions g_n, so that $q^* \Vdash (\dot{f}|_{\check{\alpha}^*} = \check{g}^*)$. Let $q' \in \mathbb{P}$ be the element of height $\alpha^* + 1$ obtained by extending q^* by the additional element $g^* \colon \alpha^* \to 2$. Then for any generic filter G' containing q', the union of G' is a sequence $\langle h'_\beta \colon \beta \to 2 \mid \beta < \omega_1 \rangle$ such that $h'_{\alpha^*} = g^*$. In other words, q' forces the statement "there exists $\dot{\alpha} \in \dot{C}$ such that $\dot{f}|_{\dot{\alpha}} = \dot{h}_{\dot{\alpha}}$".

We have shown that any $q \leq p$ has an extension $q' \leq q$ with this property, so G must contain such an element since it is generic. Therefore, there exists $\alpha \in C$ such that $f|_\alpha = h_\alpha$. □

Remark 2.4. In standard texts such as [1], the compatibility of $\diamond$ with ZFC is often proved as a result of the fact that $\diamond$ holds in the constructible universe, and that the axiom of constructibility is compatible with ZFC. Our proof, which is an alteration of the proof of [3, Theorem 17.2], is an alternative way to see this. ◁

Theorem 2.5. *There exists a generic extension* $\mathbf{M}[G]$ *in which* $\forall S \diamond_S$ *holds.*

Proof. See [3, Ch. 20]. □

Theorem 2.6. $\diamond$ *implies the continuum hypothesis.*

Proof. For each $f \in 2^\omega$, regarded as a subset of ω, the set $\{\alpha \in \omega_1 \mid f|_\alpha = h_\alpha\}$ is stationary. Let $\alpha(f)$ be the least $\alpha \geq \omega$ such that $f|_\alpha = h_\alpha$. Note that if $f \neq f'$ in 2^ω, then $\alpha(f) \neq \alpha(f')$. This gives an injection $\alpha \colon 2^\omega \to \omega_1$. □

Corollary 2.7. $\diamond$ *is independent of ZFC if ZFC is consistent.*

Proof. By Theorem 2.3, $\diamond$ is consistent with ZFC. By Theorem 2.6, $\neg\diamond$ follows from $\neg$CH, which is also consistent with ZFC. □

2.2 An equivalent form

We discuss a variant of $\diamond$ that looks stronger than $\diamond$, but is actually equivalent to it.

Definition 2.8. The principle $\diamond'$ states that there exists a transfinite sequence of functions

$$\langle\, h_\alpha \colon \alpha \to \omega_1 \mid \alpha \in \omega_1 \,\rangle$$

such that for any function $f \colon \omega_1 \to \omega_1$, the set $\{\alpha \in \omega_1 \mid h_\alpha = f|_\alpha\}$ is stationary.

More generally, for a given stationary set $S \subseteq \omega_1$, the principle $\diamond'_S$ states that there exists a transfinite sequence of subsets

$$\langle\, h_\alpha \colon \alpha \to \omega_1 \mid \alpha \in S \,\rangle$$

such that for any function $f \colon \omega_1 \to \omega_1$, the set $\{\alpha \in S \mid h_\alpha = f|_\alpha\}$ is stationary in ω_1. In particular, the principle $\diamond'_{\omega_1}$ is the same as $\diamond'$.

The axiom $\forall S \diamond'_S$ states that $\diamond'_S$ holds for every stationary subset $S \subseteq \omega_1$. ◁

Theorem 2.9. *$\diamond$ is equivalent to $\diamond'$.*

Proof. Since every function $\alpha \to 2$ is also a function $\alpha \to \omega_1$, we see that $\diamond'$ implies $\diamond$.

To see the reverse implication, assume that $\diamond$ holds. Let $\langle h_\alpha \colon \alpha \to 2 \mid \alpha \in \omega_1 \rangle$ be a sequence with the property specified by $\diamond$.

Let $\langle \beta_\alpha \mid \alpha < \omega_1 \rangle$ be an increasing sequence that enumerates all non-successor ordinals below ω_1, so that $\beta_0 = 0$, $\beta_1 = \omega$, $\beta_2 = 2\omega$, and so on. Fix an injective function $j \colon \omega_1 \to 2^\omega$. By induction, for $\alpha \leq \omega_1$, we construct an injective function

$$i_\alpha \colon \omega_1^\alpha \longrightarrow 2^{\beta_\alpha},$$

as follows. Let i_0 be the unique function from 1 to 1. If $\alpha = \alpha' + 1$, then let $i_\alpha(g) = (i_{\alpha'}(g|_{\alpha'}),\ j(g(\alpha')))$, where we have identified 2^{β_α} with $2^{\beta_{\alpha'}} \times 2^\omega$. If α is a limit, then let $i_\alpha(g) = \bigcup_{\alpha' < \alpha} i_{\alpha'}(g|_{\alpha'})$. Then we obtain a function

$$i = i_{\omega_1} \colon \omega_1^{\omega_1} \longrightarrow 2^{\omega_1},$$

with the property that $i(g)|_{\beta_\alpha} = i_\alpha(g|_\alpha)$ for all $\alpha < \omega_1$ and $g \colon \omega_1 \to \omega_1$.

For each $\alpha < \omega_1$, define $h'_\alpha \colon \alpha \to \omega_1$ by

$$h'_\alpha = \begin{cases} i_\alpha^{-1}(h_{\beta_\alpha}), & \text{if this preimage exists,} \\ 0, & \text{otherwise,} \end{cases}$$

where 0 denotes the constant function.

Now, let $f \colon \omega_1 \to \omega_1$ be a function. Then, by $\diamond$, the set of $\alpha < \omega_1$ with $i(f)|_\alpha = h_\alpha$ is stationary. Since the set of limit ordinals is club in ω_1, the set of all $\alpha < \omega_1$ with $i(f)|_{\beta_\alpha} = h_{\beta_\alpha}$ is stationary. Since $i(f)|_{\beta_\alpha} = i_\alpha(f|_\alpha)$, we see that the sequence $\langle h'_\alpha \mid \alpha < \omega_1 \rangle$ has the property specified by $\diamond'$. $\square$

By a similar process, one can also prove the following.

Theorem 2.10. *$\forall S \diamond_S$ is equivalent to $\forall S \diamond'_S$.* $\square$

3 Applications

3.1 Whitehead's problem

The *Whitehead problem* is the following question.

Question 3.1. *Is every abelian group A with $\mathrm{Ext}^1(A, \mathbb{Z}) = 0$ a free abelian group?*

It is known that when A is countable, the answer to this problem is positive. However, the general case turns out to be independent of ZFC.

The stronger diamond principle $\forall S \diamond_S$ implies a positive answer to Whitehead's problem for groups of cardinality $\aleph_1$. Therefore, we see that such a positive answer is consistent with ZFC. We will not go through the details for the other direction, i.e. the consistency of a negative answer of Whitehead's problem with ZFC. Our argument follows [3, Ch. 21].

Notation 3.2. Let $\langle A_\alpha \mid \alpha < \omega_1\rangle$ be an increasing sequence of sets. We use the notation

$$A_{<\alpha} = \bigcup_{\beta<\alpha} A_\beta$$

for all $\alpha \leq \omega_1$. ◁

Lemma 3.3. *Assume $\forall S \diamond_S$, and let A be an abelian group of cardinality $\aleph_1$, such that $\operatorname{Ext}^1(A,\mathbb{Z}) = 0$. Suppose that $A = \bigcup_{\alpha<\omega_1} A_\alpha$, where $\langle A_\alpha\rangle$ is a strictly increasing sequence of subgroups of A with cardinality $\aleph_0$. Then there exists a club subset $C \subset \omega_1$ such that for all $\alpha \in C$, $A_\alpha/A_{<\alpha}$ is free.*

Proof. For notational reasons, we assume that $A_\alpha \neq A_{<\alpha}$ for all α. Then $A_\alpha \setminus A_{<\alpha}$ must be countably infinite for all α.

Since $\diamond$ implies the continuum hypothesis, for each $\alpha < \omega_1$, we may find a bijection $i_\alpha\colon (\mathbb{N}\times(\alpha+1))^{A_\alpha\setminus A_{<\alpha}} \to \omega_1$. Suppose that the lemma fails. Then the set S of $\alpha < \omega_1$ such that $A_\alpha/A_{<\alpha}$ is not free is stationary. Let $\langle h_\alpha\colon \alpha \to \omega_1 \mid \alpha \in S\rangle$ be a sequence with the property specified by $\diamond'_S$. Define $h'_\alpha\colon A_{<\alpha} \to \mathbb{N}\times\alpha$ by taking $i_\beta^{-1}(h_\alpha(\beta))$ on $A_\beta \setminus A_{<\beta}$ for all $\beta < \alpha$.

We construct a sequence $\langle C_\alpha \mid \alpha < \omega_1\rangle$ of abelian groups, together with a bijection $g_\alpha\colon C_\alpha \to \mathbb{N}\times(\alpha+1)$ and a surjective homomorphism $\pi_\alpha\colon C_\alpha \to A_\alpha$ such that $\ker\pi_\alpha \simeq \mathbb{Z}$ and $\pi_\beta = \pi_\alpha|_{C_\beta}$ for all $\beta < \alpha$. Given the part of the sequence below α, consider the map $\pi_{<\alpha}\colon C_{<\alpha} \to A_{<\alpha}$, which is an extension of $A_{<\alpha}$ by $\mathbb{Z}$. If $\alpha \notin S$ or $g_{<\alpha}^{-1}\circ h'_\alpha$ does not split $\pi_{<\alpha}$, then we may extend $\pi_{<\alpha}$ to an extension $\pi_\alpha\colon C_\alpha \to A_\alpha$ of A_α by $\mathbb{Z}$ (which is always possible regardless of our assumption), and then extend $g_{<\alpha}$ to a bijection $g_\alpha\colon C_\alpha \to \mathbb{N}\times(\alpha+1)$ in any way. Otherwise, if $\alpha \in S$ and $g_{<\alpha}^{-1}\circ h'_\alpha$ splits $\pi_{<\alpha}$, we may find a non-trivial extension of $A_\alpha/A_{<\alpha}$ by $\mathbb{Z}$ by the positive answer to Whitehead's problem for countable groups. By a fact in algebra [3, Lemma 21.3], we may extend $\pi_{<\alpha}$ to an extension $\pi_\alpha\colon C_\alpha \to A_\alpha$ of A_α by $\mathbb{Z}$, such that $g_{<\alpha}^{-1}\circ h'_\alpha$ does not embed in a splitting of π_α. Then, we extend $g_{<\alpha}$ to a bijection $g_\alpha\colon C_\alpha \to \mathbb{N}\times(\alpha+1)$ in any way. This completes the construction.

Define $C = C_{<\omega_1}$, and define $\pi = \pi_{<\omega_1}\colon C \to A$. By assumption, π has a splitting $\rho\colon A \to C$. By $\diamond'_S$, there exists $\alpha \in S$ with $\rho|_{A_{<\alpha}} = g_\alpha^{-1}\circ h'_\alpha$. By construction, $\rho|_{A_{<\alpha}}$ does not embed in a splitting of π_α, but $\rho|_{A_\alpha}$ is such an embedding, a contradiction. This proves the lemma. □

Theorem 3.4. *Assume $\forall S\diamond_S$, and let A be an abelian group of cardinality $\aleph_1$, such that $\operatorname{Ext}^1(A,\mathbb{Z}) = 0$. Then A is free.*

Proof. First, we claim that for any countable subset $S \subseteq A$, there exists a countable subgroup $B \subseteq A$ containing S, such that for any countable subgroup $B' \subseteq A$ containing

B, the quotient B'/B is free. To see this, assume that this is not the case. Construct an increasing sequence of countable subgroups $\langle A_\alpha \subseteq A \mid \alpha < \omega_1 \rangle$ by letting A_α be the group B' that falsifies the claim for $B = A_{<\alpha}$. We see that this is impossible if we apply Lemma 3.3 to the group $A_{<\omega_1}$.

Enumerate the elements of A as $\{x_\alpha \mid \alpha < \omega_1\}$. Use the claim to find an increasing sequence of countable subgroups $\langle A_\alpha \subseteq A \mid \alpha < \omega_1 \rangle$, such that A_α contains $A_{<\alpha} \cup \{x_\alpha\}$ for each α, and for any countable subgroup $B' \subseteq A$ containing A_α, the quotient B'/A_α is free. By Lemma 3.3, there is a club sequence $\langle \alpha_\beta \rangle_{\beta < \omega_1}$, such that $A_{\alpha_\beta}/A_{<\alpha_\beta}$ is free. Let $A'_\beta = A_{\alpha_\beta}$. Then by our construction, $A'_\beta / A'_{<\beta}$ is free for all β. By induction, one shows that $A'_{<\beta}$ is free for all $\beta \leq \omega_1$, and hence $A = A'_{<\omega_1}$ is free. □

3.2 Suslin's problem

Definition 3.5. A *tree* is a poset T with a least element 0, such that for every $x \in T$, the set $x^< = \{y \in T \mid y < x\}$ is well-ordered.

The *height* of an element $x \in T$ is the ordinal with the order type of $x^<$. The *height* of T is the smallest ordinal greater than the heights of all its elements.

A *branch* of T is a maximal totally ordered set. An *anti-chain* of T is a set of pairwise incomparable elements.

A *Suslin tree* is a tree T with the following properties.

(i) The height of T is ω_1.
(ii) Every anti-chain of T is countable.
(iii) Every branch of T is countable.

A *normal Suslin tree* is a tree T with the following properties.

(i) The height of T is ω_1.
(ii) Every anti-chain of T is countable.
(iii) Every element of T has infinitely many immediate successors.
(iv) Every element of T has extensions at all higher levels.
(v) At limit levels, no two elements have the same set of predecessors. ◁

Note that normal Suslin trees are Suslin trees, since if B is a branch of a normal Suslin tree, then for each vertex $v \in B$, we may choose an immediate successor of v that is not in B, producing an anti-chain, which must be countable.

It turns out that the existence of Suslin trees is independent of ZFC. We demonstrate one part of the proof of this fact, by showing that $\diamond$ implies the existence of normal Suslin trees, so that the existence is compatible with ZFC. The proof follows [3, Ch. 18].

Theorem 3.6. *Assume $\diamond$. Then normal Suslin trees exist.*

Proof. Let $\langle h_\alpha \colon \omega_1 \to 2 \rangle$ be a sequence with the property specified by $\diamond$, and let $A_\alpha \subseteq \omega_1$ be the subset of ω_1 defined by h_α.

We construct a normal Suslin tree whose vertices are precisely the ordinals below ω_1, and we add them in order. Let 0 be the root, and for each successor ordinal $\alpha + 1$, construct the $(\alpha + 1)$-th level of T by adding ω immediate successors to each vertex at level α. For a limit ordinal α, let T_α be the part of T that is already constructed, consisting of all vertices of level $< \alpha$. for each vertex $v \in T_\alpha$, we choose a branch of T_α that contains v and has height α (which is possible by this inductive construction), and we add one vertex at level α to lie above each of these branches. (When some of these branches coincide, no repeated vertices will be added.) Moreover, if A_α is a maximal anti-chain in T_α, then we require that each of these branches contain an element of A_α. This completes the construction of T.

To show that T is indeed a normal Suslin tree, the main difficulty is in showing that every anti-chain is countable. To see this, let $A \subseteq T$ be a maximal anti-chain. Then the set of α for which $A \cap T_\alpha$ is a maximal anti-chain of T_α is club. Also, the set of α satisfying this condition together with $T_\alpha = \alpha$ (where T_α denotes the set of its vertices) is club. By $\diamond$, there exists α such that $A_\alpha = A \cap \alpha = A \cap T_\alpha$ is a maximal anti-chain of T_α. But by our construction at level α, every vertex of T of height at least α lies above some element of $A \cap T_\alpha$, so that $A \cap T_\alpha$ is a maximal anti-chain of T, and hence equals A. This shows that A is countable. $\square$

References

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