Structural results about projective sets

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Enumerations of Π^1_1 sets

- A Σ^1_1 set is a projection $p[T]$ of a closed subset $[T]$ of $\omega^\omega \times \omega^\omega$, where T is a computable tree on $\omega \times \omega$.

  Equivalently, it is defined by a Σ^1_1-formula $\exists y\ \varphi(x, y)$, where φ is Σ_0.
- A Π^1_1 set is a complement of a Σ^1_1 set.
- A Σ^1_2 set is a projection of a Π^1_1 set, etc.

There is a strong parallel between c.e. sets and Π^1_1 sets.

Reduction property: If A, B are Π^1_1 sets, then there are disjoint Π^1_1 sets $A' \subseteq A$ and $B' \subseteq B$ with $A' \cup B' = A \cup B$.

This is explained by the fact that any Π^1_1 set can be enumerated as a c.e. set, but in ordinal stages.

Enumerations of Π^1_1 sets

A wellorder is a linear order without infinite decreasing sequences.

Example

Let WO denote the Π^1_1 set of wellorders on ω.
Let $\mathrm{WO}_{\leq\alpha}$ denote the Borel subset of wellorders of order type $\leq\alpha$.

WO can be enumerated in ω_1 stages by checking if an input has order type ω, $\omega+1$ etc.

Formally, one inputs a real R into a machine and runs a computation with ordinal stages. (This is explained on a later slide.)

All wellorders $R \in \mathrm{WO}_{\leq\alpha}$ are found by a fixed countable stage.

Enumerations of Π^1_1 sets

A set is Π^1_1 iff it can be enumerated by an algorithm p such that $p(x)$ halts before $\omega_1^{\mathrm{ck},x}$ or diverges (Spector).

A set is Σ^1_2 iff it can be enumerated by an unrestricted algorithm.

These representations play a major role, from classical results about Π^1_1 and Σ^1_2 sets to numerous recent results.

For an introduction, see:

- Greg Hjorth

  Vienna notes on effective descriptive set theory and admissible sets
  http://www.math.uni-bonn.de/ag/logik/events/young-set-theory-2010/Hjorth.pdf

- Chi Tat Chong and Liang Yu

  Recursion Theory: Computational Aspects of Definability

  De Gruyter Series in Logic and Its Applications 8, 2015

Ranks

A *rank* is a notion that abstracts the halting times of infinite processes.

Consider the relation $x \leq y \Leftrightarrow p(x)$ *halts before or at the same time as* $p(y)$.

A Π^1_1-rank on a Π^1_1 set A is a prewellorder on A such that

- comparison is both Π^1_1 and Σ^1_1 on A, and
- A is downwards closed in both relations.

Thus A is written as an increasing union of Borel subsets.

Ranks also arise in other ways, for instance from transfinite iterations of derivation processes such as the Cantor-Bendixson derivative.

Most of the following results hold for both enumerations and ranks.

What was known

Fact

TFAE for a Π^1_1 set A:

- A is Borel.
- Every Π^1_1-rank on A is countable.
- A admits a countable Π^1_1-rank.

The first implication follows from the Kunen-Martin theorem: Every wellfounded $\boldsymbol{\Sigma}^1_1$ relation has countable rank.

Problem

What is the length of countable enumerations of Π^1_1 *sets?*
How long can countable Π^1_1*-ranks be?*

Both implications fail for Σ^1_2 sets.

Problem

What is the length of countable enumerations of Σ^1_2 sets?
How long can countable Σ^1_2-ranks be?

Problem

Which Σ^1_2 sets admit a countable Σ^1_2-rank?

What we showed

τ is defined as the supremum of Σ_2-definable ordinals in $L_{\omega_1^V}$.

Theorem (Welch, Carl, S.)

Each of the following sets of ordinals has supremum τ:

1. a. *Lengths of countable enumerations of Π^1_1 sets*
   b. *Lengths of countable Π^1_1 ranks*
   c. *Countable ranks of wellfounded Π^1_1 relations.*
2. a. *Lengths of countable enumerations of Σ^1_2 sets*
   b. *Lengths of countable Σ^1_2 ranks*
   c. *Countable ranks of wellfounded Σ^1_2 relations.*
3. *Borel ranks of Π^1_1 Borel sets.*

The value in 3. was computed by Kechris, Marker and Sami (JSL 1989) as γ^1_2. Thus $\gamma^1_2 = \tau$.

Lengths of ranks

L

The L-hierarchy is a transfinite extension of the arithmetical hierarchy.

- $L_0 = \emptyset$
- $L_{\alpha+1} = \{X \subseteq L_\alpha \mid \exists\varphi(., u)\ X = \{x \in L_\alpha \mid (L_\alpha, \in) \models \varphi(x, u)\}\}$
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for limits λ
- $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$

L equals the class of sets written by a transfinite process (Koepke).

The fine structure of L was analysed by Jensen.

σ

A Σ_1-formula is of the form $\exists x\ \varphi(x, y)$, where φ contains only bounded quantifiers.

As L grows, more Σ_1-statements become true.

α is called Σ_1-definable if it is unique with $\varphi(\alpha)$, for some Σ_1-formula φ.

Definition

σ is defined as the supremum of ordinals which are Σ_1-definable in $L_{\omega_1^V}$.

Fact

1. *σ is least with $L_\sigma \prec_{\Sigma_1} L$.*
2. *σ is least such that L_σ contains all Π^1_1-singletons.*
3. *σ equals δ^1_2, the supremum lengths of Δ^1_2-wellorders on ω.*

τ

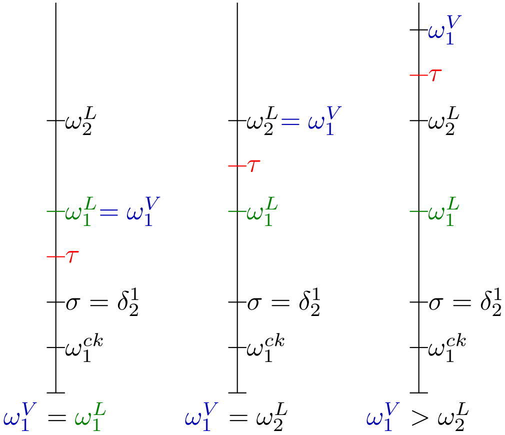
Definition

τ is defined as the supremum of ordinals which are Σ_2-definable in $L_{\omega_1^V}$.

Lemma (Welch, Carl, S.)

τ equals the supremum of ordinals which are Π_1-definable in $L_{\omega_1^V}$.

τ


ω_2^L
$\omega_1^L = \omega_1^V$
τ
$\sigma = \delta_2^1$
ω_1^{ck}
$\omega_1^V = \omega_1^L$
$\omega_2^L = \omega_1^V$
τ
ω_1^L
$\sigma = \delta_2^1$
ω_1^{ck}
$\omega_1^V = \omega_2^L$
ω_1^V
τ
ω_2^L
ω_1^L
$\sigma = \delta_2^1$
ω_1^{ck}
$\omega_1^V > \omega_2^L$

τ

Let τ_* be least such that L_{τ_*} and $L_{\omega_1^V}$ agree on Σ_2-truth. Let τ^* be least with $L_{\tau^*} \prec_{\Sigma_2} L_{\omega_1^V}$.

Then $\tau_* \leq \tau \leq \tau^*$.

Lemma (Welch, Carl, S.)

1. *If* $\omega_1^L = \omega_1^V$, *then* $\tau_* = \tau = \tau^*$.
2. *If* $\omega_1^L < \omega_1^V$, *then* $\tau_* < \omega_1^L < \tau < \tau^*$.

Lengths of ranks

Theorem (Welch, Carl, S.)

The supremum of lengths of countable ranks in the following classes equals τ:

1. Π^1_1-*ranks*
2. Σ^1_2-*ranks*

Lower bound for Π^1_1-ranks

For $x \in \mathrm{WO}$, let α_x denote the ordinal coded by x.

We call an ordinal β an α-index if $\beta > \alpha$ and some $\Sigma_1^{L_{\omega_1}}$ fact with parameters in $\alpha \cup \{\alpha\}$ first becomes true in L_β.

σ_α is defined as the supremum of α-indices.

Suppose that ν is $\Pi_1^{L_{\omega_1}}$-definable by $\varphi(u)$.

We will define a Π^1_1 subset A of WO. A will be bounded in WO, since for all $x \in A$, α_x will be a $\bar{\nu}$-index for some $\bar{\nu} \leq \nu$ and hence $\alpha_x < \sigma_\nu$.

For each $x \in \mathrm{WO}$, let ν_x denote the least ordinal $\bar{\nu} < \alpha_x$ with $L_{\alpha_x} \models \varphi(\bar{\nu})$, if this exists. Let

$$A = \{x \in \mathrm{WO} \mid \nu_x \text{ exists and } \alpha_x \text{ is a } \nu_x\text{-index}\}.$$

Clearly A is Π^1_1.

Lower bound for Π^1_1-ranks

One can show that A admits a countable Π^1_1-rank, and any Π^1_1-rank on A has length at least σ_ν.

Next slide: Lower bound for enumerations

Lower bound for Σ^1_2-enumerations

Lemma (Welch, Carl, S.)

Any Σ^1_2-enumeration of A has length at least σ_ν.

Proof.

- A is unbounded in σ_ν by the definition of A.

Suppose that for some $\gamma < \sigma_\nu$, there is an algorithm p that enumerates A within time γ.

Let g be $\mathrm{Col}(\omega, \gamma)$-generic over L_{σ_ν} and $x_g \in L_{\sigma_\nu}[g]$ a real coding g.

A is $\Sigma^1_1(x_g)$, since $x \in A$ holds if and only if there is a halting run $p(x)$ of length at most γ.

- A is bounded below $\omega_1^{\mathrm{ck}, x_g}$ by the effective boundedness lemma.

Since σ_ν is a limit of admissibles and g is set generic over L_{σ_ν}, σ_ν is a limit of x_g-admissibles. Hence $\omega_1^{\mathrm{ck},y} < \sigma_\nu$.

□

Upper bound for wellfounded Σ^1_2-relations

Lemma (Welch, Carl, S.)

For any wellfounded Σ^1_2-relation R of countable rank, $\mathrm{rank}(R) < \tau$.

This is proved via:

Lemma (Welch, Carl, S.)

Suppose that R is a wellfounded Σ^1_2 relation and M is a Σ_1-correct admissible set.

If $\mathrm{rank}(x) = \alpha < \omega_1^M$, *then there is some $x' \in M$ with* $\mathrm{rank}(x') = \alpha$.

This is applied to a $Col(\omega, \gamma)$-generic extension of L, where $\mathrm{rank}(R) = \gamma$.

Sets with countable ranks

Σ^1_2-ranks

The implications between Borel and admits a countable rank for Π^1_1 sets break at the level of Σ^1_2.

The simplest Π^1_2 sets: Π^1_2-singletons.

Theorem (Silver)

If there exists a Ramsey cardinal, then $0^\#$ is a Π^1_2-singleton that is not in L.

Theorem (Jensen)

By forcing over L, one can add a Π^1_2-singleton that is not in L.

The complements of these singletons do not admit countable Σ^1_2-ranks.

Σ^1_2-ranks

Theorem (Welch, Carl, S.)

The following conditions are equivalent for any Π^1_2-singleton x:

a. $x \in L$.
b. *x is covered by a countable Σ^1_2 set.*
c. *x is covered by a countable Δ^1_2 set.*
d. *The complement of $\{x\}$ admits a countable Σ^1_2-rank.*

This result can be extended to countable Π^1_2 sets.

Σ^1_2-ranks

Proof.

$a \Rightarrow b$:
Suppose that x is defined by a Π_1-formula $\varphi(u)$.

Let A denote the complement of $\{x\}$.

Let B denote the set of y such that for some countable α, $L_\alpha \models$ "y is defined by $\varphi(u)$".

B is a Σ^1_2-set containing x. Moreover, B is countable, since it is contained in L_α, where α is least with $L_\alpha \models \forall y <_L x \neg\varphi(y)$. □

Note that B is in fact Δ^1_2:

$y \notin B$ iff there exists a countable β with $x \in L_\beta$ and either

i. $L_\beta \models \neg\varphi(x)$, or
ii. $L_\beta \models \varphi(x)$ and for all $\alpha \leq \beta$ with $x \in L_\alpha$, $L_\alpha \models \exists y \neq x\ \varphi(y)$.

Borel ranks

Δ^1_1 sets

An ordinal is called computable if it is coded by a computable real. ω_1^{ck} is the supremum of computable ordinals.

Fact

The supremum of Borel ranks of Δ^1_1 sets is ω_1^{ck}.

This uses an effective version of Lusin's separation theorem: Any two disjoint Σ^1_1 sets are separated by a hyperarithmetic set, i.e. a Borel set with a computable code.

$L_{\omega_1^{ck}}$ is the least admissible set. An admissible set is a transitive model of KP: Axioms of set theory with only Σ_1-collection and Δ_0-separation.

Theorem (Louveau TAMS 1980)

Given a Δ^1_1 set that is also $\mathbf{\Sigma}^0_\alpha$, there is a $\mathbf{\Sigma}^0_\alpha$-code in $L_{\omega_1^{ck}}$.

Π^1_1 Borel sets

Assuming Π^1_1-determinacy, all truly Π^1_1 (i.e. non-Borel) sets are Wadge equivalent. It thus remains to understand Π^1_1 Borel sets.

The supremum of Borel ranks of Π^1_1 Borel sets was calculated by Kechris, Marker and Sami as γ^1_2 (JSL 1989).

Proposition (Welch, Carl, S.)

The supremum of Borel ranks of Π^1_1 Borel sets equals τ.

Thus $\gamma^1_2 = \tau$.

The lower bound

Lemma (Welch, Carl, S.)

For any $\alpha < \tau$, there is a Π^1_1 Borel set A of Borel rank at least α.

Proof.

Let α_x denote the order type of $x \in \mathrm{WO}$.

Suppose that $\delta > \omega^\alpha$ is a Π_1-singleton defined by $\varphi(x)$. Let

$$A = \{(x, y) \in \mathrm{WO}^2 \mid \alpha_y \text{ is least with } L_{\alpha_y} \models \text{"}\varphi \text{ defines } \alpha_x\text{"}\} \in \Pi^1_1.$$

Let $\eta > \delta$ be least with $L_\eta \models$ "φ defines δ". Note that for any $(x, y) \in A$, we have $\alpha_x \leq \delta$ and $\alpha_y \leq \eta$.

A is a countable union of Borel sets of the form $\mathrm{WO}_\mu \times \mathrm{WO}_\nu$ and thus Borel.

Plug in η on the right to obtain the slice WO_δ. But WO_δ has Borel rank at least α (Stern). $\square$

The Borel ranks of Σ^1_2 Borel sets are not bounded by τ.

Δ^1_2 Borel sets

A Borel code is a subset of ω that codes a tree which describes the way the Borel set is built up from basic open sets.

An ∞-Borel code is a set of ordinals defined similarly, but allowing wellordered unions and intersections.

Do all Δ^1_2 Borel sets have ∞-Borel codes in $L_{\omega_1^V}$?

A set is absolutely Δ^1_2 if it has a uniform Δ^1_2-definition in generic extensions.

Theorem

Suppose that either

a. ω_1^V *is inaccessible in* L *(Stern), or*

b. V *is a generic extension of* L *by proper forcing (Welch, Carl, S.).*

Then any absolutely Δ^1_2 *Borel set has an* ∞*-Borel code of the same rank in* L_τ*.*

There is no such result for Σ^1_2 sets, since Π^1_2 singletons can exist outside of L.

Δ^1_2 Borel sets

Proving this result in ZFC would simultaneously generalise:

- The above result of Kechris, Marker and Sami
- The Mansfield-Solovay theorem: Countable Δ^1_2 sets are contained in L
- Stern's theorem on Δ^1_2 Borel sets that corresponds to the first case.
- Shoenfield absoluteness

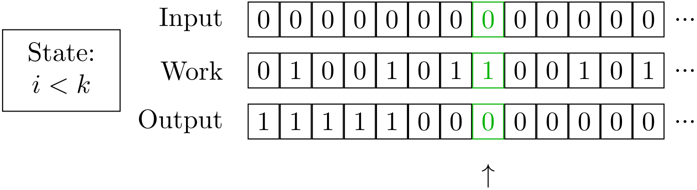
Appendix: infinite time algorithms

Ittm's

We discuss infinite time Turing machines (ittm's, Hamkins, Kidder 2000); unrestricted machines work similarly, but have an ordinal tape.

An ittm is a Turing machine with three tapes whose cells are indexed by natural numbers:

- The input tape
- The output tape
- The working tape



Ittm's

It behaves like a standard Turing machine at successor steps of a computation. At limit steps of computation:

- The head goes back to the first cell.
- The machine goes into a limit state.
- The value of each cell equals the lim inf of the values at previous stages of computation.

State: limit

Input	0	0	0	0	0	0	0	0	0	0	0	0	0	⋯
Work	0	1	0	0	1	0	1	1	0	0	1	0	1	⋯
Output	1	1	1	1	1	0	0	0	0	0	0	0	0	⋯
	$\uparrow$													

Further results for ittm's

Theorem (Welch, Carl, S.)

There is an open ittm-decidable set A that is not ittm-semidecidable in countable time.

Theorem (Welch, Carl, S.)

The suprema of ittm-semidecision times for the following sets equal σ:

1. *Singletons*
2. *Complements of singletons.*

Some open problems

Question

Which Σ^1_2 sets admit countable Σ^1_2-ranks?

The above results only answer this if either the set or its complement is countable. This remaining cases could be related to the next question:

Question

Does every Δ^1_2 Borel set have an ∞-Borel code in $L_{\omega_1^V}$?

Combined with Stern's results, our partial result covers many interesting cases. But a general result seems out of reach. In particular, I was not able to adapt Louveau's method (TAMS 1980).

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