A descriptive main gap theorem

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Fifth Workshop on Generalised Baire Spaces
University of Bristol, 3–4.2.2020

Joint work with Francesco Mangraviti

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Main question

Given T, can one provide non-trivial lower/upper bounds for the spectrum function $I(\kappa, T)$?

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This gave birth to a beautiful branch of model theory, later called

stability theory.

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So if the function $I(\kappa, T)$ is to have a non-trivial upper bound, then T must be (stable) superstable, NDOP and NOTOP: such theories are briefly called **classifiable**.

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Remark. The upper bound in 1 may trivialize (e.g. when κ is a fixed point of the $\aleph$-function), but in general it is easy to find cardinals for which this is not the case: for example, under GCH there are unboundedly many κ for which such upper bound is $< 2^\kappa$, or even $\leq \kappa$.

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Here is where generalized Descriptive Set Theory enters the scene...

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Within this framework, we can say that

$$\cong_T^\kappa \text{ is "simple" if it is a } (\kappa^+\text{-})\text{Borel subset of } (\mathrm{Mod}_T^\kappa)^2,$$

because in this case we have a "Borel" procedure to decide whether two models are isomorphic or not.

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Question (F-H-K)

If T is classifiable shallow, what is the Borel rank of $\cong^\kappa_T$? Is it related to the depth of T?

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2. If T is not classifiable shallow, then $B(\kappa, T) = \infty$.

Thus in the "good" case the function $B(\kappa, T)$ is almost everywhere dominated by a constant function which (unlike Shelah's upper bound) depends only on $\mathrm{dp}(T)$ and not on the cardinal κ.

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Let T be a countable complete first-order theory, and κ be such that $\kappa^{<\kappa} = \kappa > 2^{\aleph_0}$.

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Remark. This gap theorem, unlike Shelah's, is never trivial for the relevant κ's: in particular, under GCH the descriptive gap is non-trivial for every successor cardinal $\kappa \geq \aleph_2$.

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Let κ be an infinite cardinal. Given $M, N \in \mathrm{Mod}^{\kappa}_{T}$ and an ordinal β, set $M \equiv_{\beta} N$ if and only if M and N satisfy the same $\mathcal{L}_{\infty\kappa}$-formulæ with quantifier rank $< \beta$.

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Remark. $M \equiv_{\beta} N \iff \mathbf{II} \uparrow \mathrm{EF}^{\kappa}_{\mathfrak{T}_{\alpha}}(M, N)$.

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In particular, there is no complete non-κ-categorical first-order theory T for which $\cong^\kappa_T$ is true open.

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(*) Recall that $\alpha = \mathrm{dp}(T) < \aleph_1$.

T countable complete first-order theory; $\kappa = \aleph_\gamma$ with $\gamma \geq 1$.

Uncountably categorical	**Classifiable shallow** $\mathrm{dp}(T) = \alpha$	**Not classifiable shallow**
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Question

How large is the $\leq^\kappa_B$-gap between $\cong^\kappa_T$ and $\cong^\kappa_{T'}$?

Forbidden complexities for $\cong^{\kappa}_{T}$

Proposition (Mangraviti-M.)

Let $\kappa^{<\kappa} = \kappa$ and $\mathcal{L}$ a signature of size $< \kappa$.

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This also show that both the Scott height and the Borel complexity may vary with the size of the models.

Back to countable structures

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Let T be a complete first-order theory in a countable language and $M \in \mathrm{Mod}_T^\omega$. Then $[M]_\cong$ is dense in Mod_T^ω (unconditionally!).

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In particular, other complexities are forbidden: there is no complete first-order theory T such that $\cong^\omega_T$ is true open or true closed.

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What about *topological* smoothness?

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Remark. The result is actually stronger. If $f \colon \mathrm{Mod}^\omega_T \to Y$ is continuous and invariant with Y Hausdoff, then the quotient map $\hat{f} \colon \mathrm{Mod}^\omega_T/{\cong}$ is nowhere injective: for every $M \in \mathrm{Mod}^\omega_T$ we can always find $N \in \mathrm{Mod}^\omega_T$ such that $N \ncong M$, yet $f(N) = f(M)$.

The end

Thank you for your attention!