LECTURE NOTES: AN INTRODUCTION TO COUNTABLE BOREL EQUIVALENCE RELATIONS AND ACTIONS OF COUNTABLE GROUPS

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1. Overview

Borel measurable actions of countable groups have been intensively studied in the last decades from the viewpoints of descriptive set theory, dynamics and measured group theory [Gab10, Eps08, CM17, GJ15, Kec10, KS, Ioa]. Still there are many long-standing open questions in the focus of current research.

We want to study some of the fundamental results in this area, both from the descriptive and measure-theoretic viewpoints. For instance, we want to study universal equivalence relations for actions of countable groups, Dye's theorem from 1959 that any two ergodic probability measure preserving actions of $\mathbb{Z}$ are orbit equivalent, Harrington, Kechris and Louveau's result from 1990 that there exists a least non-smooth Borel equivalence relation, and connections with logic and model theory related to the complexity of isomorphism problems for countable structures. The book [Kec] gives an overview of the theory of countable Borel equivalence relations.

The main reference for the first part of the course is the book *Topics in orbit equivalence* by Alexander S. Kechris and Benjamin D. Miller [KM04]. Most of the background in descriptive set theory can be found in [Gao09, Kec95].

2. Introduction

A Borel equivalence relation on a Polish space is called *countable* if all its equivalence classes are countable. The standard example is an orbit equivalence relation of an action of a countable group by Borel automorphisms.

Example. (a) The rotation $\sigma_\alpha \colon \mathbb{Z} \to S^1$, $\sigma_\alpha(n) = \exp(\pi i n \alpha)$ by α on the unit sphere in $\mathbb{C}$.
  (b) The (left) shift action $\Gamma_2 \curvearrowright 2^{\Gamma_2}$, $(\gamma \cdot x)(\delta) = x(\gamma^{-1}(\delta))$ for $\Gamma = \mathbb{F}_2$, the free group on 2 generators.

The basic problem is to describe the complexity of equivalence relations in comparison with each other. Suppose that $E_\Gamma = E_{\Gamma \curvearrowright X}$ and $E_\Delta = E_{\Delta \curvearrowright Y}$ are the orbit equivalence relations of group actions $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$. The coarsest way to compare these two relations is the following.

Definition. E_Γ is *Borel reducible* to E_Δ ($E_\Gamma \leq E_\Delta$) if there is a Borel measurable *reduction* $f \colon X \to Y$ of E_Γ to E_Δ, i.e. for all $x_0, x_1 \in X$, we have $(x_0, x_1) \in E_\Gamma \iff (f(x_0), f(x_1)) \in E_\Delta$.

Since all reductions will be assumed to be Borel measurable, we will simply speak of reductions. An equivalence relation is called *smooth* if it is reducible to equality. Since these relations are arguably trivial, the interesting stuff begins beyond them. We will see that σ_α in the example above is non-smooth if α is irrational. But this is only the simplest type of non-smooth equivalence relation. Just beyond smoothness, it is interesting to study finer notions of equivalence:

Definition. (a) $\Gamma \curvearrowright X$, $\Delta \curvearrowright Y$ are *conjugate* if there is a group isomorphism $\pi \colon \Gamma \to \Delta$ and a Borel isomorphism $f \colon X \to Y$ with $f(\gamma x) = \pi(\gamma) f(x)$ for all $x \in X$ and $\gamma \in \Gamma$.
  (b) E_Γ, E_Δ are *orbit equivalent*[1] (OE) if there is a Borel isomorphism $f \colon X \to Y$ with $f(\Gamma x) = \Delta f(x)$ for all $x \in X$.
  (c) E_Γ, E_Δ are *Borel bi-reducible* ($E_\Gamma \equiv E_\Delta$) if $E_\Gamma \leq E_\Delta$ and $E_\Delta \leq E_\Gamma$.

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[1]or *Borel isomorphic*

We have the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c). *Rigidity* is the phenomenon that these implications can be reversed in some cases. The converse phenomenon is *elasticity*: for instance, $\mathbb{Z}$ and $\oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ have actions with the same orbit equivalence relation, so the implication (b) $\Rightarrow$ (c) cannot be reversed. This shows that orbit equivalence is much finer than conjugacy. This phenomenon may depend on the group: for instance, $\mathbb{Z}$ has many non-equivalent actions, but for some other groups there is only one ergodic measure preserving free action up to orbit equivalence. It is therefore important to study connections between groups and orbit equivalence relations of their actions.

3. Some background in descriptive set theory

In section, we review some material on standard Borel spaces, Borel sets, analytic sets and Borel measurable functions, in particular the fact that any two uncountable standard Borel spaces are isomorphic. We give very direct and concise proofs from scratch, in contrast to most proofs in the literature, which refer to other theorems.

3.1. Standard Borel spaces.

Definition 3.1.1. (a) A metric space (X, d) is called *Polish* if it is countably based and complete.
(b) A topological space (X, τ) is called *Polish* if there's a metric d on X that induces τ.
(c) A pair (X, σ) is called a *standard Borel space* if σ is the Borel σ-algebra generated by a Polish topology on X.

It is an easy exercise to show that Polish metric spaces are closed under countable products by taking a weighted sum over the coordinates.

We always write (X, d), (Y, d), (Z, d) for Polish metric spaces and often omit the metric d. We further write $d(A)$ for the *diameter* $\sup_{x,y \in A} d(x, y)$ of a subset A of X. Moreover, $B_\epsilon(x)$ denotes the open ball of radius ϵ around x.

Example 3.1.2. $\mathbb{R}^n$, Baire space $\mathbb{N}^{\mathbb{N}}$, Cantor space $2^{\mathbb{N}}$, compact metric spaces.

Here $\mathbb{N}^{\mathbb{N}}$ caries the product topology. The standard metric on $\mathbb{N}^{\mathbb{N}}$ is defined by $d(x, y) = \frac{1}{2^n}$ for the least n with $x(n) \neq y(n)$, where $x \neq y$. The basic open subsets of $\mathbb{N}^{\mathbb{N}}$ (clopen balls) are denoted $N_t = \{x \in \mathbb{N}^{\mathbb{N}} \mid t \subseteq x\}$ for $t \in \mathbb{N}^{<\mathbb{N}}$.

A *tree* on $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments. It is easy to see that the closed subsets of $\mathbb{N}^{\mathbb{N}}$ are exactly the sets of the form $[T] = \{x \in \mathbb{N}^{\mathbb{N}} \mid \forall n \ x{\restriction}n \in T\}$ for trees T on $\mathbb{N}$. Let $|t|$ denote the length of a sequence $t \in \mathbb{N}^{<\mathbb{N}}$. Let $x{\restriction}n$ denote the restriction of a sequence $x \in \mathbb{N}^{\mathbb{N}}$ to $\{0, \ldots, n-1\}$. A tree is called *pruned* if it has no end nodes. Let $\mathrm{Lev}_n(T)$ denote the n-th level of a tree T.

We next show that any two uncountable Polish spaces are *Borel isomorphic* in the following sense.

Definition 3.1.3. A *Borel isomorphism* $f \colon A \to B$ between Borel subsets A and B of standard Borel spaces is a bijection that preserves Borel sets in both directions.

Proposition 3.1.4. *Any two uncountable Polish spaces are Borel isomorphic.*

This will follow from the next four lemmas. The first three lemmas show that any uncountable Polish space X is Borel isomorphic to a Borel subset of $\mathbb{N}^{\mathbb{N}}$ and vice versa. The last lemma show that this is sufficient to obtain Proposition 3.1.4.

An ϵ-*cover* of a subset A of X is a sequence $\vec{A} = \langle A_i \mid i < N \rangle$ with $N \in \mathbb{N} \cup \{\mathbb{N}\}$ whose union equals A with $d(A_i) < \epsilon$ and $\overline{A_i} \subseteq A$ for all $i \in \mathbb{N}$. An F_σ-*set* is a countable union of closed sets.

Lemma 3.1.5. *Let $\epsilon > 0$.*
(a) *Any open set has an ϵ-cover consisting of open sets.*
(b) *Any F_σ-set has a disjoint ϵ-cover consisting of F_σ-sets.*

Proof. (a) Suppose that A is an open subset of X. Let $\vec{A} = \langle A_i \mid i \in N \rangle$ list all open balls $B_q(x)$ with $x \in Q$, $q \in (0, \frac{\epsilon}{2}) \cap \mathbb{Q}$ and $\overline{B_q(x)} \subseteq A$, where Q is a countable dense subset of X.

We claim that $A = \bigcup_{i \in N} A_i$. To see this, take any $x \in A$. Since A is open, there is some $q \in \mathbb{Q}$ with $B_{2q}(x) \subseteq A$. Fix any $y \in B_q(x) \cap Q$. Then $x \in B_q(y)$ and $\overline{B_q(y)} \subseteq A$. By the latter, $B_q(y)$ appears in $\vec{A}$.

(b) Suppose that A is an F_σ subset of X. Take a sequence $\vec{A} = \langle A_i \mid i \in N \rangle$ whose union equals A with A_i closed. Then A is the disjoint union of the sets $B_i = A_i \setminus (A_0 \cup \cdots \cup A_{i-1})$. Since B_i is of the form $C \cap D$ with C open and D closed, it suffices to prove the claim for sets of this form. By (a), there is an ϵ-cover $\vec{C} = \langle C_i \mid i \in K \rangle$ of C consisting of open sets. Let $\vec{D} = \langle D_i \mid i \in K \rangle$ with $D_i = (C_i \setminus (C_0 \cup \cdots \cup C_{i-1})) \cap D$. Then $\vec{D}$ is a disjoint ϵ-cover of $C \cap D$ consisting of F_σ-sets. $\square$

Lemma 3.1.6. *Any Polish space X is Borel isomorphic to a Borel subset of $\mathbb{N}^\mathbb{N}$. In fact, there is a closed subset A of $\mathbb{N}^\mathbb{N}$ and a continuous Borel isomorphism $f \colon A \to X$.*[2]

Proof. Let $\vec{\epsilon} = \langle \epsilon_i \mid i \in \mathbb{N} \rangle$ be a sequence in $\mathbb{R}^+$ converging to 0. By iterative applications of Lemma 3.1.5, we obtain a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ without end nodes and a family $\vec{A} = \langle A_t \mid t \in T \rangle$ of nonempty F_σ-subsets of X with $A_0 = X$ that satisfies the following properties for all $t \in T$:

(a) $A_t = \bigcup_{ti \in T} A_{ti}$ (disjoint), (b) $\overline{A_{ti}} \subseteq A_t$ if $ti \in T$, and (c) $d(A_t) < \epsilon_n$ if $|t| = n$.

We then define $f \colon [T] \to X$ by choosing $f(x)$ as the unique element of $\bigcap_{n \in \mathbb{N}} A_{x \restriction n}$. The latter is nonempty by (b), (c) and completeness of X, and has a unique element by (c).

Claim. *f is bijective.*

Proof. It follows from (a) that f is injective. To see that f is surjective, take any $x \in X$. By (a), there is for each $n \in \mathbb{N}$ a unique $t_n \in T$ of length n with $x \in A_{t_n}$. By (b), $t_i \subseteq t_j$ for $i \leq j$. Hence $f(y) = x$ for $y = \bigcup_{i \in \mathbb{N}} t_i$. $\square$

Claim. *f is continuous.*

Proof. If U is open and $f(x) \in U$, then $A_{x \restriction n} \subseteq U$ for sufficiently large n by (c). $\square$

Claim. *f is a Borel isomorphism.*

Proof. It suffices that the images of Borel sets are Borel. This holds since f maps each basic open set N_t onto A_t and f is bijective. $\square$

Thus f is as required. $\square$

Using the previous lemma, one can see that any nonempty Polish space X is the image of $\mathbb{N}^\mathbb{N}$ under a continuous function. By the lemma, it suffices to show this in the case that $X = [T]$ is a closed subset of $\mathbb{N}^\mathbb{N}$. In this case, map each $x \in \mathbb{N}^\mathbb{N}$ to the leftmost $y \in [T]$ with $x \restriction n \subseteq y$, where n is maximal with $x \restriction n \in T$.

Lemma 3.1.7. *If X is an uncountable Polish space, then $\mathbb{N}^\mathbb{N}$ is Borel isomorphic to a Borel subset of X.*

Proof. Fix a sequence $\vec{\epsilon} = \langle \epsilon_i \mid i \in \mathbb{N} \rangle$ in $\mathbb{R}^+$ converging to 0. It is easy to construct a family $\vec{U} = \langle U_t \mid t \in \mathbb{N}^{<\mathbb{N}} \rangle$ of uncountable open subsets of X such that: (a) $\overline{U_t} \subseteq U_s$ for all $s \subseteq t$, (b) $U_s \cap U_t = \emptyset$ for all $s \neq t$ with $|s| = |t|$, and (c) $d(U_t) < \epsilon_n$ if $|t| = n$.

Define $f \colon \mathbb{N}^\mathbb{N} \to X$ by choosing $f(x)$ as the unique element of $\bigcap_{n \in \mathbb{N}} U_{x \restriction n}$. It is then easy to see that f is a homeomorphism onto its image. To see that $f(\mathbb{N}^\mathbb{N})$ is Borel, let $U_n = \bigcup_{|t|=n} U_t$. We then have $f(\mathbb{N}^\mathbb{N}) = \bigcap_{n \in \mathbb{N}} U_n$. $\square$

Lemma 3.1.8. *Suppose that X is Borel isomorphism to a Borel subset of Y and vice versa. Then X and Y are Borel isomorphic.*

Proof. Suppose that $f \colon X \to Y$ and $g \colon Y \to X$ are Borel isomorphisms onto Borel sets. We call a pair (A, B) of Borel subsets of X and Y *nice* if $f \colon A \to B$ and $g \colon B \to A$ are well-defined.[3]

Claim. *For any nice pair (A, B), there is a nice pair (A_*, B_*) such that $A_* \subseteq A \cap g(B)$, $B_* \subseteq B \cap f(A)$ and the Borel sets $A \setminus A_*$, $B \setminus B_*$ are Borel isomorphic.*

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[2] In fact, any continuous bijection is a Borel isomorphism, but we don't prove this here.

[3] This is shorthand for $\mathrm{ran}(f \restriction A) \subseteq B$ and $\mathrm{ran}(g \restriction B) \subseteq A$.

Proof. To see this, let $A^* = A \setminus g(B)$, $B^* = B \setminus f(A)$. Since $A^* \cap g(B^*) = B^* \cap f(A^*) = \emptyset$, the maps $f \colon A^* \to f(A^*)$ and $g \colon B^* \to g(B^*)$ witness that $A^* \cup g(B^*)$ and $B^* \cup f(A^*)$ are Borel isomorphic. Thus $A_* = A \setminus (A^* \cup g(B^*)) = g(B) \setminus g(B^*)$ and $B_* = B \setminus (B^* \cup f(A^*)) = f(A) \setminus f(A^*)$ are as required. $\square$

Let $A_0 = X$ and $B_0 = Y$. By iterative applications of the previous claim, we obtain decreasing sequences $\vec{A} = \langle A_n \mid n \in \mathbb{N} \rangle$ and $\vec{B} = \langle B_n \mid n \in \mathbb{N} \rangle$ of Borel sets such that (A_n, B_n) is nice for all $n \in \mathbb{N}$. Let $A = \bigcap_{n \in \mathbb{N}} A_n$ and $B = \bigcap_{n \in \mathbb{N}} B_n$.

It suffices to show that A and B are Borel isomorphic via f. To see this, note that $f \colon A \to B$ is well-defined, since $f \colon A_n \to B_n$ is well-defined for all $n \in \mathbb{N}$. Moreover, $f \colon A \to B$ is surjective, since $B_{n+1} \subseteq f(A_n)$ for all $n \in \mathbb{N}$. $\square$

3.2. Analytic sets.
Let $p \colon X \times Y \to X$ denote the first projection.

Definition 3.2.1. A subset A of X is called *analytic* if $A = p(B)$ for some Borel subset B of $X \times Y$.

By Proposition 3.1.4, these are precisely the images of analytic subsets of $\mathbb{N}^{\mathbb{N}}$ under Borel isomorphisms. Hence we can work in $\mathbb{N}^{\mathbb{N}}$ to study their properties.

To do this, we will go through a slight detour. We call a subset A of $\mathbb{N}^{\mathbb{N}}$ **analytic* if $A = p[C]$ for some closed subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, equivalently $A = p[T] = \{x \mid \exists y \ (x, y) \in [T]\}$ for some subtree T of $(\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$.

Problem 3.2.2. (a) *The class of *analytic sets is closed under countable unions and intersections. It follows that every Borel subset of $\mathbb{N}^{\mathbb{N}}$ is *analytic.* (Hint: first show that the intersection of two *analytic sets is again *analytic by merging trees projecting to the sets.)
  (b) *Every analytic subsets of $\mathbb{N}^{\mathbb{N}}$ is *analytic.* (Hint: this is an easy consequence of (a).)

The previous result also holds for finite products of $\mathbb{N}^{\mathbb{N}}$, by observing that the proof still works or noting that they are homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Lemma 3.2.3. *Every nonempty analytic subset of X equals $f(\mathbb{N}^{\mathbb{N}})$ for some continuous $f \colon \mathbb{N}^{\mathbb{N}} \to X$.*

Proof. Since every nonempty Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$ by the remark after Lemma 3.1.6, it suffices to find some continuous $f \colon Y \to X$ with $f(Y) = X$.

Take an analytic subset A of X. By Lemma 3.1.6, there is a closed subset Z of $\mathbb{N}^{\mathbb{N}}$ and a continuous Borel isomorphism $g \colon Z \to A$. Since g is continuous, it suffices to find a continuous $f \colon Y \to \mathbb{N}^{\mathbb{N}}$ with $f(Y) = B := g^{-1}(A)$.

Since analytic sets are preserved under Borel isomorphisms, B is an analytic subset of Z and thus of $\mathbb{N}^{\mathbb{N}}$. Let C be a Borel subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $B = p(C)$. Since Borel subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ are *analytic by Problem 3.2.2 (a), there is a closed subset D of $(\mathbb{N}^{\mathbb{N}})^3$ with $B = p(D)$. Then $p \colon D \to \mathbb{N}^{\mathbb{N}}$ is as required. $\square$

Subsets A, B of X are called *Borel separated* if there is a Borel set C with $A \subseteq C$ and $B \subseteq X \setminus C$.

Proposition 3.2.4. *Any two disjoint analytic subsets A, B of X are Borel separated.*

Proof. It suffices to show this for $X = \mathbb{N}^{\mathbb{N}}$, and by Problem 3.2.2 A and B are then *analytic. Let $f \colon [S] \to A$ and $g \colon [T] \to B$ be continuous surjections, where S and T are pruned subtrees of $\mathbb{N}^{<\mathbb{N}}$.[4]

Suppose that A, B are not Borel separated. We then construct increasing sequences $\vec{s} = \langle s_i \mid i \in \mathbb{N} \rangle$ and $\vec{t} = \langle t_i \mid i \in \mathbb{N} \rangle$ in $\mathbb{N}^{<\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $f(N_{s_i})$ and $g(N_{t_i})$ are not Borel separated as follows.

Let $s_0 = t_0 = \emptyset$. Given s_i and t_i, there are $k, m \in \mathbb{N}$ with $s_i k \in S$ and $t_i m \in T$ such that $f(N_{s_i k})$ and $g(N_{t_i m})$ are not Borel separated. If each of these pairs were Borel separated by some $B_{k,m}$, then $C_k = \bigcap_{t_i m \in T} B_{k,m}$ separates $f(N_{s_i k})$ and $g(N_{t_i})$ for each $k \in \mathbb{N}$ with $s_i k \in S$, and hence $C = \bigcup_{s_i k \in S} C_k$ separates $f(N_{s_i})$ and $g(N_{t_i})$, contradicting our assumption. Let $s_{i+1} = s_i k$ and $t_{i+1} = t_i m$.

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[4]One can assume that $[S] = [T] = \mathbb{N}^{\mathbb{N}}$ to simplify the notation at the expense of using Lemma 3.2.3.

Let $x = \bigcup_{i \in \mathbb{N}} s_i \in [S]$ and $y = \bigcup_{i \in \mathbb{N}} t_i \in [T]$. Since A and B are disjoint, $f(x) \neq g(y)$. Let U, V be disjoint open subsets of X with $f(x) \in U$ and $g(y) \in V$. By continuity of f and g, $f(N_{x \restriction n}) \subseteq U$ and $g(N_{y \restriction n}) \subseteq V$ for sufficiently large n, so $f(N_{x \restriction n})$ and $g(N_{y \restriction n})$ are Borel separated. But this contradicts the construction. $\square$

3.3. **Borel measurable functions.** A function $f \colon X \to Y$ between standard Borel spaces is called *Borel measurable* if f-preimages of Borel sets are Borel.

Lemma 3.3.1. *The following are equivalent for a function $f \colon X \to Y$:*

(a) *f is Borel measurable.*
(b) *The graph $G(f)$ of f is a Borel subset of $X \times Y$.*

Proof. (a) $\Rightarrow$ (b): $f(x) = y \iff x \in \bigcap_{k \in \mathbb{N}} f^{-1}(N_{y \restriction k}) \iff (x, y) \in \bigcap_{k \in \mathbb{N}} \bigcup_{t \in \mathbb{N}^k} f^{-1}(N_t) \times N_t$.

(b) $\Rightarrow$ (a): It suffices to show that $f^{-1}(U)$ is Borel for any open subset of Y. By Proposition 3.2.4, it is sufficient to show that both $f^{-1}(U)$ and its complement in X are analytic. These claims follows from the equivalences $x \in f^{-1}(U) \iff \exists y \in Y \ (x, y) \in G(f) \iff \forall y \in Y \ ((x, y) \in G(f) \Rightarrow y \in U)$. $\square$

References

[CM17] Clinton T. Conley and Benjamin D. Miller. Measure reducibility of countable Borel equivalence relations. *Ann. of Math. (2)*, 185(2):347–402, 2017.

[Eps08] Inessa Epstein. *Some results on orbit inequivalent actions of non-amenable groups.* ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–University of California, Los Angeles.

[Gab10] Damien Gaboriau. Orbit equivalence and measured group theory. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1501–1527. Hindustan Book Agency, New Delhi, 2010. Available at https://hal-ens-lyon.archives-ouvertes.fr/ensl-00512729/document.

[Gao09] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[GJ15] Su Gao and Steve Jackson. Countable abelian group actions and hyperfinite equivalence relations. *Invent. Math.*, 201(1):309–383, 2015.

[Ioa] Adrian Ioana. Orbit equivalence of ergodic group actions. Unpublished lecture notes, available at http://www.math.ucsd.edu/.

[Kec] Alexander S. Kechris. The theory of countable Borel equivalence relations. Preprint.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[Kec10] Alexander S. Kechris. *Global aspects of ergodic group actions*, volume 160 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2010.

[KM04] Alexander S. Kechris and Benjamin D. Miller. *Topics in orbit equivalence*, volume 1852 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2004.

[KS] Alexander S. Kechris and Forte Shinko. Realizations of countable Borel equivalence relations. Preprint, 2019.

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