LECTURE NOTES: AN INTRODUCTION TO COUNTABLE BOREL EQUIVALENCE RELATIONS AND COUNTABLE GROUP ACTIONS

PHILIPP SCHLICHT

Contents

1. Introduction

A Borel equivalence relation on a Polish space is called *finite* or *countable*, respectively, if all its equivalence classes are finite or countable. For instance, the orbit equivalence relation of any action of a countable group by Borel automorphisms is a countable Borel equivalence relations.

Example. (a) The rotation $\sigma_\alpha\colon \mathbb{Z} \to S^1$, $\sigma_\alpha(n) = \exp(\pi i n\alpha)$ by α on the unit sphere in $\mathbb{C}$.
(b) The (left) shift action $\Gamma_2 \curvearrowright 2^{\Gamma_2}$, $(\gamma \cdot x)(\delta) = x(\gamma^{-1}(\delta))$ for $\Gamma = \mathbb{F}_2$, the free group on 2 generators.

We study connections between properties of equivalence relations and the groups whose actions generate them.

Suppose that $E_{\Gamma\curvearrowright X}$, $E_{\Delta\curvearrowright Y}$ are the orbit equivalence relations of group actions $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$. Next is the standard way to compare these two relations.

Definition. $E_{\Gamma\curvearrowright X}$ is *(Borel) reducible* to $E_{\Delta\curvearrowright Y}$ ($E_{\Gamma\curvearrowright X} \leq E_{\Delta\curvearrowright Y}$) if there is a Borel measurable *reduction* $f\colon X \to Y$ of $E_{\Gamma\curvearrowright X}$ to $E_{\Delta\curvearrowright Y}$, i.e. for all $xx_0, x_1 \in X$, $(x_0, x_1) \in E_{\Gamma\curvearrowright X} \iff (f(x_0), f(x_1)) \in E_{\Delta\curvearrowright Y}$.

An equivalence relation is called *smooth* if it is reducible to equality. Moreover, *least* and *universal* is meant with respect to the quasi-ordering of Borel reducibility. We study the following topics in reducibility:

- Universal equivalence relations among those induced by actions of a fixed countable group, universal countable Borel equivalence relations
- Harrington, Kechris, Louveau (1990): there is a least non-smooth Borel equivalence relation
- Isomorphism problems for countable structures

There are also finer modes of comparison for group actions. Any group action will be assumed to be Borel measurable. We have three notions of equivalence:

Definition. Let $E = E_{\Gamma\curvearrowright X}$ and $F = E_{\Delta\curvearrowright Y}$.

(a) E, F are *Borel bi-reducible* ($E \equiv F$) if $E \leq F$ and $F \leq E$.
(b) E, F are *orbit equivalent*[1] (OE) if there is a Borel isomorphism $f\colon X \to Y$ with $f(\Gamma x) = \Delta f(x)$ for all $x \in X$.
(c) E, F are *conjugate* if there is a group isomorphism $\pi\colon \Gamma \to \Delta$ and a Borel isomorphism $f\colon X \to Y$ with $f(\gamma x) = \pi(\gamma) f(x)$ for all $x \in X$ and $\gamma \in \Gamma$.

Date: October 17, 2019.
[1]or *Borel isomorphic*

We have the implications

$$(\Gamma \curvearrowright X \sim^{\mathrm{Conj}} \Delta \curvearrowright Y) \Longrightarrow (\Gamma \curvearrowright X \sim^{\mathrm{OE}} \Delta \curvearrowright Y) \Longrightarrow (\Gamma \curvearrowright X \equiv \Delta \curvearrowright Y)$$

In general, orbit equivalence is much finer than conjugacy, as we will see in several examples and results below. *Rigidity* is the phenomenon that these implications can be reversed in some cases. The converse phenomenon is *elasticity.* For example, $\mathbb{Z}$ and $\oplus_{n\in\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$ have actions with the same orbit equivalence relation. Topics in orbit equivalence:

- Dye (1959): any two ergodic p.m.p. transformations are orbit equivalent.

Literature: Su Gao: Invariant descriptive set theory; Alexander S. Kechris and Benjamin D. Miller: Topics in orbit equivalence, Alexander S. Kechris: Classical descriptive set theory

2. Basic results

2.1. Standard Borel spaces.

Definition 2.1.1. (a) A metric space (X, d) is called *Polish* if it is countably based and complete.
(b) A topological space (X, τ) is called *Polish* if there's a metric d on X that induces τ.
(c) A pair (X, σ) is called a *standard Borel space* if σ is the Borel σ-algebra generated by a Polish topology on X.

It is an easy exercise to show that Polish metric spaces are closed under countable products by taking a weighted sum over the coordinates.

Example 2.1.2. $\mathbb{R}^n$, Baire space $\mathbb{N}^\mathbb{N}$, Cantor space $2^\mathbb{N}$, compact metric spaces.

Here $\mathbb{N}^\mathbb{N}$ caries the product topology. The standard metric on $\mathbb{N}^\mathbb{N}$ is defined by $d(x, y) = \frac{1}{2^n}$ for the least n with $x(n) \neq y(n)$, where $x \neq y$. The basic open subsets of $\mathbb{N}^\mathbb{N}$ (clopen balls) are denoted $N_t = \{x \in \mathbb{N}^\mathbb{N} \mid t \subseteq x\}$ for $t \in \mathbb{N}^{<\mathbb{N}}$.

A *tree* on $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments. It is easy to see that the closed subsets of $\mathbb{N}^\mathbb{N}$ are exactly the sets of the form $[T] = \{x \in \mathbb{N}^\mathbb{N} \mid \forall n\ x{\restriction}n \in T\}$ for trees T on $\mathbb{N}$. Let $|t|$ denote the length of a sequence $t \in \mathbb{N}^{<\mathbb{N}}$. Let $x{\restriction}n$ denote the restriction of a sequence $x \in \mathbb{N}^\mathbb{N}$ to $\{0, \ldots, n-1\}$. A tree is called *pruned* if it has no end nodes. Let $\mathrm{Lev}_n(T)$ denote the n-th level of a tree T.

We always write (X, d), (Y, d), (Z, d) for Polish metric spaces. The metric d is often omitted. We write $d(A)$ for the *diameter* $\sup_{x,y\in A} d(x, y)$ of a subset A of X. Moreover, $B_\epsilon(x)$ denotes the open ball of radius ϵ around x.

We next show that any two uncountable Polish spaces are *Borel isomorphic* in the following sense.

Definition 2.1.3. A *Borel isomorphism* $f : X \to Y$ between standard Borel spaces is a bijection that preserves Borel sets in both directions.

An F_σ*-set* is a countable union of closed sets. An ϵ*-cover* of a subset A of X is a sequence $\vec{A} = \langle A_i \mid i < N\rangle$ with $N \in \mathbb{N} \cup \{\mathbb{N}\}$ whose union equals A with $d(A_i) < \epsilon$ and $\overline{A_i} \subseteq A$ for all $i \in \mathbb{N}$.

Lemma 2.1.4. *Let* $\epsilon > 0$.
(a) *Any open set has an* ϵ*-cover consisting of open sets.*
(b) *Any* F_σ*-set has a disjoint* ϵ*-cover consisting of* F_σ*-sets.*

Proof. (a) Suppose that A is an open subset of X. Let $\vec{A} = \langle A_i \mid i \in N\rangle$ list all open balls $B_q(x)$ with $x \in Q$, $q \in (0, \frac{\epsilon}{2}) \cap \mathbb{Q}$ and $\overline{B_q(x)} \subseteq A$, where Q is a countable dense subset of X.

We claim that $A = \bigcup_{i\in N} A_i$. To see this, take any $x \in A$. Since A is open, there is some $q \in \mathbb{Q}$ with $B_{2q}(x) \subseteq A$. Fix any $y \in B_q(x) \cap Q$. Then $x \in B_q(y)$ and $\overline{B_q(y)} \subseteq A$. By the latter, $B_q(y)$ appears in $\vec{A}$.

(b) Suppose that A is an F_σ subset of X. Take a sequence $\vec{A} = \langle A_i \mid i \in N\rangle$ whose union equals A with A_i closed. Then A is the disjoint union of the sets $B_i = A_i \setminus (A_0 \cup \cdots \cup A_{i-1})$. Since B_i is of the form $C \cap D$ with C open and D closed, it suffices to prove the claim for sets of this form. By (a), there is an ϵ-cover $\vec{C} = \langle C_i \mid i \in K \rangle$ of C consisting of open sets. Let $\vec{D} = \langle D_i \mid i \in K \rangle$ with $D_i = (C_i \setminus (C_0 \cup \cdots \cup C_{i-1})) \cap D$. Then $\vec{D}$ is a disjoint ϵ-cover of $C \cap D$ consisting of F_σ-sets. □

Lemma 2.1.5. *For any Polish space X, there exists a closed subset A of $\mathbb{N}^\mathbb{N}$ and a bijection $f\colon A \to X$ that is both continuous and a Borel isomorphism.*[2]

Proof. Let $\vec{\epsilon} = \langle \epsilon_i \mid i \in \mathbb{N} \rangle$ be a sequence in $\mathbb{R}^+$ converging to 0. By iterative applications of Lemma 2.1.4, we obtain a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ without end nodes and a family $\vec{A} = \langle A_t \mid t \in T \rangle$ of nonempty F_σ-subsets of X with $A_0 = X$ that satisfies the following properties for all $t \in T$:

(a) $A_t = \bigcup_{ti \in T} A_{ti}$ (disjoint),
(b) $\overline{A_{ti}} \subseteq A_t$ and
(c) $d(A_t) < \epsilon_n$ if $|t| = n$.

Note that $[T]$ is a closed subset of $\mathbb{N}^{<\mathbb{N}}$.

Define $f\colon [T] \to X$ by letting $f(x)$ denote the unique element of $\bigcap_{n\in\mathbb{N}} A_{x\restriction n}$. The latter is nonempty by (b), (c) and completeness of X, and unique by (c).

It follows from (a) that f is injective. To see that f is surjective, take any $x \in X$. By (a), there is for each $n \in \mathbb{N}$ a unique $t_n \in T$ of length n with $x \in A_{t_n}$. By (b), $t_i \subseteq t_j$ for $i \leq j$. Hence $f(y) = x$ for $y = \bigcup_{i\in\mathbb{N}} t_i$.

by (c), f is continuous: if $f(x) \in U$ and U is open, then $A_{x\restriction n} \subseteq U$ for sufficiently large n.

To see that f is a Borel isomorphism, it is sufficient that the images of Borel sets are Borel. This holds since f maps each basic open set N_t onto A_t and f is bijective. □

Using Lemma 2.1.5, it is easy to see that any nonempty Polish space X is the image of $\mathbb{N}^\mathbb{N}$ under a continuous function. By the lemma, it suffices to show this in the case that X is a closed subset of $\mathbb{N}^\mathbb{N}$. In this case, map each $x \in \mathbb{N}^\mathbb{N}$ to the closest element of X.

Proposition 2.1.6. *Any two uncountable Polish spaces are Borel isomorphic.*

Proof. The statement follows from the next two lemmas. □

Lemma 2.1.7. *For any uncountable Polish space X, there is a Borel isomorphism $f\colon \mathbb{N}^\mathbb{N} \to X$ onto a Borel subset of X.*

Proof. Fix a sequence $\vec{\epsilon} = \langle \epsilon_i \mid i \in \mathbb{N} \rangle$ in $\mathbb{R}^+$ converging to 0.

It is easy to construct a family $\vec{U} = \langle U_t \mid t \in \mathbb{N}^{<\mathbb{N}} \rangle$ of open subsets of X such that $\overline{U_t} \subseteq U_s$ for all $s \subseteq t$, $U_s \cap U_t = \emptyset$ for all $s \neq t$ with $|s| = |t|$, and $d(U_t) < \epsilon_n$ if $|t| = n$.

Define $f\colon \mathbb{N}^\mathbb{N} \to X$ as follows. Let $f(x)$ be the unique element of $\bigcap_{n\in\mathbb{N}} U_{x\restriction n}$. It is clear that f is a homeomorphism onto its image.

To see that $f(\mathbb{N}^\mathbb{N})$ is Borel, let $U_n = \bigcup_{|t|=n} U_t$. We then have $f(\mathbb{N}^\mathbb{N}) = \bigcap_{n\in\mathbb{N}} U_n$. □

Lemma 2.1.8. *Suppose that there is a Borel isomorphism $f\colon X \to Y$ onto a Borel subset of Y, and vice versa. Then X and Y are Borel isomorphic.*

Proof. Suppose that $f\colon X \to Y$ and $g\colon Y \to X$ are as above.

We call a pair (A, B) of Borel subsets of X and Y, respectively, *nice* if $f\colon A \to B$ and $g\colon B \to A$ are well-defined.

We claim that for any nice pair (A, B), there is a nice pair (A', B') with $A' \subseteq A \cap g(B)$ and $B' \subseteq B \cap f(A)$ such that $A \setminus A'$ and $B \setminus B'$ are Borel isomorphic. To see this, let $A^* = A \setminus g(B)$, $B^* = B \setminus f(A)$. Then A^*, $g(B^*)$ are disjoint, B^*, $f(A^*)$ are disjoint and $f\colon A^* \to f(A^*)$, $g\colon B^* \to g(B^*)$ are Borel isomorphisms. Let $A' = A \setminus (A^* \cap g(B^*))$ and $B' = B \setminus (B^* \cap f(A^*))$. Since $A' \subseteq g(B)$ and $B' \subseteq f(A)$, they are as required.

Starting with $A_0 = X$ and $B_0 = Y$ and using the claim iteratively, we obtain decreasing sequences $\vec{A} = \langle A_n \mid n \in \mathbb{N} \rangle$ and $\vec{B} = \langle B_n \mid n \in \mathbb{N} \rangle$ such that (A_n, B_n) is nice for all $n \in \mathbb{N}$.

It suffices to show that $A = \bigcap_{n\in\mathbb{N}} A_n$ and $B = \bigcap_{n\in\mathbb{N}} B_n$ are Borel isomorphic. Since $f\colon A_n \to B_n$ is well-defined for all $n \in \mathbb{N}$, $f\colon A \to B$ is well-defined. Since $B_{n+1} \subseteq f(A_n)$ for all $n \in \mathbb{N}$, $f\colon A \to B$ is surjective. □

[2] In fact, any injective Borel measurable function is a Borel isomorphism onto its image, but we do not prove this here.

2.2. **Analytic sets.** Let $p\colon X \times Y \to X$ denote the first projection. A subset A of X is called *analytic* if $A = p(B)$ for some Borel subset B of $X \times Y$.

By Proposition 2.1.6, these are precisely the images of analytic subsets of $\mathbb{N}^{\mathbb{N}}$ under Borel isomorphisms. Hence we can work in $\mathbb{N}^{\mathbb{N}}$ to study their properties.

We will go through a slight detour. We call a subset A of $\mathbb{N}^{\mathbb{N}}$ **analytic* if $A = p[C]$ for some closed subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, equivalently $A = p[T] = \{x \mid \exists y\ (x, y) \in [T]\}$ for some subtree T of $(\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$.

Problem 2.2.1. (a) *The class of *analytic sets is closed under countable unions and intersections. It follows that every Borel subset of $\mathbb{N}^{\mathbb{N}}$ is *analytic.* (Hint: first show that the intersection of two *analytic sets is again *analytic by merging trees projecting to the sets.)
(b) *Every analytic subsets of $\mathbb{N}^{\mathbb{N}}$ is *analytic.* (Hint: this is an easy consequence of (a).)

The previous result also holds for finite products of $\mathbb{N}^{\mathbb{N}}$, by observing that the proof still works or noting that they are homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Lemma 2.2.2. *Every nonempty analytic subset of X equals $f(\mathbb{N}^{\mathbb{N}})$ for some continuous $f\colon \mathbb{N}^{\mathbb{N}} \to X$.*

Proof. Since every nonempty Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$ by the remark after Lemma 2.1.5, it suffices to find some continuous $f\colon Y \to X$ with $f(Y) = X$.

Take an analytic subset A of X. By Lemma 2.1.5, there is a closed subset Z of $\mathbb{N}^{\mathbb{N}}$ and a continuous Borel isomorphism $g\colon Z \to A$. Since g is continuous, it suffices to find a continuous $f\colon Y \to \mathbb{N}^{\mathbb{N}}$ with $f(Y) = B := g^{-1}(A)$.

Since analytic sets are preserved under Borel isomorphisms, B is an analytic subset of Z and thus of $\mathbb{N}^{\mathbb{N}}$.

Let C be a Borel subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $B = p(C)$.

Since Borel subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ are *analytic by Problem 2.2.1 (a), there is a closed subset D of $(\mathbb{N}^{\mathbb{N}})^3$ with $B = p(D)$. Then $p\colon D \to \mathbb{N}^{\mathbb{N}}$ is as required. $\square$

Subsets A, B of X are called *Borel separated* if there's a Borel set C with $A \subseteq C$ and $B \subseteq X \setminus C$.

Proposition 2.2.3. *Any two disjoint analytic subsets A, B of X are Borel separated.*

Proof. It suffices to show this for $X = \mathbb{N}^{\mathbb{N}}$, and by Problem 2.2.1 A and B are *analytic. Let $f\colon [S] \to A$ and $g\colon [T] \to B$ be continuous surjections, where S and T are pruned subtrees of $\mathbb{N}^{<\mathbb{N}}$.

Suppose that A, B are not Borel separated.

We construct increasing sequences $\vec{s} = \langle s_i \mid i \in \mathbb{N}\rangle$ and $\vec{t} = \langle t_i \mid i \in \mathbb{N}\rangle$ in $\mathbb{N}^{<\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $f(N_{s_i})$ and $g(N_{t_i})$ are not Borel separated.

Let $s_0 = t_0 = \emptyset$. Given s_i and t_i, there are $k, m \in \mathbb{N}$ with $s_i k \in S$ and $t_i m \in T$ such that $f(N_{s_i k})$ and $g(N_{t_i m})$ are not Borel separated. If each of these pairs were Borel separated by some $B_{k,m}$, then $C_k = \bigcap_{t_i m \in T} B_{k,m}$ separates $f(N_{s_i k})$ and $g(N_{t_i})$ for each $k \in \mathbb{N}$ with $s_i k \in S$, and hence $C = \bigcup_{s_i k \in S} C_k$ separates $f(N_{s_i})$ and $g(N_{t_i})$, contradicting our assumption. Let $s_{i+1} = s_i k$ and $t_{i+1} = t_i m$.

Let $x = \bigcup_{i \in \mathbb{N}} s_i \in [S]$ and $y = \bigcup_{i \in \mathbb{N}} t_i \in [T]$. Since A and B are disjoint, $f(x) \neq g(y)$. Let U, V be disjoint open subsets of X with $f(x) \in U$ and $g(y) \in V$. By continuity, $f(N_{x\restriction n}) \subseteq U$ and $g(N_{y\restriction n}) \subseteq V$ for sufficiently large n. Then these two sets are Borel separated, contradicting the above facts. $\square$

2.3. **Borel measurable functions.** A function $f\colon X \to Y$ between standard Borel spaces is called *Borel measurable* if f-preimages of Borel sets are Borel.

Lemma 2.3.1. *The following are equivalent for a function $f\colon X \to Y$:*
(a) *f is Borel measurable.*
(b) *The graph $G(f)$ of f is a Borel subset of $X \times Y$.*

Proof. (a) $\Rightarrow$ (b): $f(x) = y \iff x \in \bigcap_{k\in\mathbb{N}} f^{-1}(N_{y\restriction k}) \iff (x, y) \in \bigcap_{k\in\mathbb{N}} \bigcup_{t\in\mathbb{N}^k} f^{-1}(N_t) \times N_t$.

(b) $\Rightarrow$ (a): It suffices to show that $f^{-1}(U)$ is Borel for any open subset of Y. By Proposition 2.2.3, it is sufficient to show that both $f^{-1}(U)$ and its complement in X are analytic. These claims follows from the equivalences $x \in f^{-1}(U) \iff \exists y \in Y\ (x, y) \in G(f) \iff \forall y \in Y\ ((x, y) \in G(f) \Rightarrow y \in U)$. $\square$

Philipp Schlicht, School of Mathematics, University of Bristol, Fry Building, Woodland Road, Bristol, BS8 1UG, UK
Email address: `philipp.schlicht@bristol.ac.uk`