THE VERY BASICS OF DESCRIPTIVE SET THEORY

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Abstract. This note presents elementary proofs that any two uncountable Polish metric spaces are Borel isomorphic, a function is Borel measurable if and only if its graph is a Borel set, and the auxiliary result that disjoint analytic sets can be separated by Borel sets. The proofs are much shorter than those in the literature.

1. Standard Borel spaces

Definition 1.1. (a) A metric space (X, d) is called *Polish* if it is countably based and complete.
(b) A topological space (X, τ) is called *Polish* if there is a Polish metric d on X that induces τ.
(c) A pair (X, σ) is called a *standard Borel space* if σ is the Borel σ-algebra generated by a Polish topology on X.

Our notation is (X, d), (Y, d), (Z, d) for Polish metric spaces and the metric d is often omitted. We further write $d(A)$ for the *diameter* $\sup_{x,y \in A} d(x, y)$ of a subset A of X, and $B_\epsilon(x)$ for the open ball of radius ϵ around x.

Example 1.2. $\mathbb{R}^n$, Baire space $\mathbb{N}^\mathbb{N}$, Cantor space $2^\mathbb{N}$, compact metric spaces, countably based locally compact metric spaces.

Here $\mathbb{N}^\mathbb{N}$ caries the product topology. The standard metric on $\mathbb{N}^\mathbb{N}$ is defined by $d(x, y) = \frac{1}{2^n}$ for the least n with $x(n) \neq y(n)$, where $x \neq y$. The basic open subsets of $\mathbb{N}^\mathbb{N}$ (clopen balls) are denoted $N_t = \{x \in \mathbb{N}^\mathbb{N} \mid t \subseteq x\}$ for $t \in \mathbb{N}^{<\mathbb{N}}$. The following combinatorial presentation of $\mathbb{N}^\mathbb{N}$ via trees is useful for constructions of continuous maps. A *tree* on $\mathbb{N}$ is a subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under initial segments. It is easy to see that the closed subsets of $\mathbb{N}^\mathbb{N}$ are exactly the sets of the form $[T] = \{x \in \mathbb{N}^\mathbb{N} \mid \forall n\ x{\upharpoonright}n \in T\}$ for trees T on $\mathbb{N}$. Let $|t|$ denote the length of a sequence $t \in \mathbb{N}^{<\mathbb{N}}$. Let $x{\upharpoonright}n$ denote the restriction of a sequence $x \in \mathbb{N}^\mathbb{N}$ to $\{0, \ldots, n-1\}$. A tree is called *pruned* if it has no end nodes. Let $\mathrm{Lev}_n(T)$ denote the n-th level of a tree T, so the elements of $\mathrm{Lev}_n(T)$ are sequences of length n.

Note that Polish spaces are closed under countable products. To see this, take a sequence $\vec{X} = \langle X_n \mid n \in \mathbb{N}\rangle$ of Polish spaces and assume that each metric d is bounded by replacing d with $\max\{d, 1\}$. Then form the weighted sum over all coordinates with factors 2^{-n}. It is easy to show that this metric on the product is complete and induces the product topology.

We next show that any two uncountable Polish spaces are *Borel isomorphic* in the sense of a bijection that preserves Borel sets both ways.

Proposition 1.3. *Any two uncountable Polish spaces are Borel isomorphic.*

This will follow from the next two lemmas. To state them, we fix some notation. An *ϵ-cover* of a subset A of X is a sequence $\vec{A} = \langle A_i \mid i < N\rangle$ with $N \in \mathbb{N} \cup \{\mathbb{N}\}$ whose union equals A with $d(A_i) < \epsilon$ and $\overline{A_i} \subseteq A$ for all $i \in \mathbb{N}$. An *F_σ-set* is a countable union of closed sets. Note that all open sets are F_σ.

Lemma 1.4. *Let $\epsilon > 0$.*

(1) *Any open set has an ϵ-cover consisting of open sets.*
(2) *Any F_σ-set has a disjoint ϵ-cover consisting of F_σ-sets.*

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Proof. (1) Suppose that A is an open subset of X. Fix a countable dense subset Q of X and let $\vec{A} = \langle A_i \mid i \in N \rangle$ list all open balls $B_q(x)$ with $x \in Q$, $q \in (0, \frac{\epsilon}{2}) \cap \mathbb{Q}$ and $\overline{B_q(x)} \subseteq A$. It suffices to show that $A = \bigcup_{i \in N} A_i$. To see this, take any $x \in A$ and find some $q \in \mathbb{Q}$ with $B_{2q}(x) \subseteq A$. Since Q is dense, there is some $y \in B_q(x) \cap Q$. We then also have $x \in B_q(y)$ and thus it suffices to show that $\overline{B_q(y)} \subseteq A$. This holds since $\overline{B_q(y)} \subseteq B_{2q}(x) \subseteq A$.

(2) Suppose that A is an F_σ subset of X. Take a sequence $\vec{A} = \langle A_i \mid i \in N \rangle$ whose union equals A with A_i closed. Then A is the disjoint union of the sets $B_i = A_i \setminus (A_0 \cup \cdots \cup A_{i-1})$. Since B_i is of the form $C \cap D$ with C open and D closed, it suffices to prove the claim for sets of this form. To see this, note that by (a), there is an ϵ-cover $\vec{C} = \langle C_i \mid i \in K \rangle$ of C consisting of open sets. Let $\vec{D} = \langle D_i \mid i \in K \rangle$ with $D_i = (C_i \setminus (C_0 \cup \cdots \cup C_{i-1})) \cap D$. Then $\vec{D}$ is a disjoint ϵ-cover of $C \cap D$ consisting of F_σ-sets. ☐

The existence of Borel isomorphisms between uncountable Polish spaces follows from the second part of the next lemma by noting that countable sets can be discarded. We call a subset of X *co-countable* if its complement is countable.

Lemma 1.5.

(1) *There is a closed subset A of $\mathbb{N}^\mathbb{N}$ and a continuous Borel isomorphism $f : A \to X$.*[1]

(2) *If X is uncountable, then there is a continuous Borel isomorphism $f : \mathbb{N}^\mathbb{N} \to X$ onto a co-countable subset of X.*

Proof. (1) Let $\vec{\epsilon} = \langle \epsilon_i \mid i \in \mathbb{N} \rangle$ be a sequence in $\mathbb{R}^+$ converging to 0. By iterative applications of Lemma 1.4, we obtain a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ without end nodes and a family $\vec{A} = \langle A_t \mid t \in T \rangle$ of nonempty F_σ-subsets of X with $A_0 = X$ that satisfies the following properties for all $t \in T$: (a) $A_t = \bigcup_{ti \in T} A_{ti}$ is a disjoint union, (b) $\overline{A_{ti}} \subseteq A_t$ for $ti \in T$, and (c) $d(A_t) < \epsilon_n$ if $|t| = n$.

We then define $f : [T] \to X$ by choosing $f(x)$ as the unique element of $\bigcap_{n \in \mathbb{N}} A_{x \upharpoonright n}$. The latter is nonempty by (b), (c) and completeness of X, and has a unique element by (c).

Claim. *f is bijective.*

Proof. It follows from (a) that f is injective. To see that f is surjective, take any $x \in X$. By (a), there is for each $n \in \mathbb{N}$ a unique $t_n \in T$ of length n with $x \in A_{t_n}$. By (b), $t_i \subseteq t_j$ for $i \leqslant j$. Hence $f(y) = x$ for $y = \bigcup_{i \in \mathbb{N}} t_i$. ☐

Claim. *f is continuous.*

Proof. If U is open and $f(x) \in U$, then $A_{x \upharpoonright n} \subseteq U$ for sufficiently large n by (c). ☐

Claim. *f is a Borel isomorphism.*

Proof. Since f is continuous, it suffices that the images of Borel sets are Borel. This holds since f maps each basic open set N_t onto A_t and f is bijective. ☐

Thus f is as required.

(2) A slight modification of the construction in (a) works. We will ensure that A_t is uncountable for all $t \in T$. The change is that we remove a set A_{ti} if it is countable. Note that one can easily ensure that A_{ti} is uncountable for infinitely many $i \in \mathbb{N}$ by partitioning the sets further. Therefore one can assume that $ti \in T$ for all $i \in \mathbb{N}$ and thus $T = \mathbb{N}^{<\mathbb{N}}$. During the construction only a countable set is removed in total, yielding the claim. ☐

Note that in (2), for any X with isolated points, one cannot replace the co-countable set by X.

2. Analytic sets

Let $p : X \times Y \to X$ denote the first projection.

Definition 2.1. A subset A of X is called *analytic* if $A = p(B)$ for some Borel subset B of $X \times Y$ for some standard Borel space Y.

[1] In fact, any continuous bijection is a Borel isomorphism, but we don't prove this here.

Clearly this class contains all Borel sets.

By Proposition 1.3, one can take Y to be a fixed uncountable Polish space. Moreover, the analytic subsets of an uncountable standard Borel space X are precisely the images of analytic subsets of $\mathbb{N}^{\mathbb{N}}$ under Borel isomorphisms. Hence we can work in $\mathbb{N}^{\mathbb{N}}$ to study their properties. To do this, we will go through a slight detour. We call a subset A of $\mathbb{N}^{\mathbb{N}}$ **analytic* if $A = p[C]$ for some closed subset C of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, equivalently $A = p[T] = \{x \mid \exists y\ (x, y) \in [T]\}$ for some subtree T of $(\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$.

Lemma 2.2. (1) *The class of *analytic sets is closed under countable unions and intersections.*
(2) *Every analytic subset of $\mathbb{N}^{\mathbb{N}}$ is *analytic.*

Proof. (1) Closure under countable unions is easily seen by forming unions of disjointified trees.

Closure under countable intersections is proved by merging trees. We do only the case of two trees, since the general case is similar. So suppose that T_0 and T_1 are subtrees of $(\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$. We identify elements of $(\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$ with pairs of sequences. For $t \in \mathbb{N}^{<\mathbb{N}}$, define $t^0, t^1 \in \mathbb{N}^{<\mathbb{N}}$ by letting $t^0(i) = t(2i)$ if $2i < k$ and $t^1(i) = t(2i + 1)$ if $2i + 1 < k$. Let U be the subtree of $(\mathbb{N} \times \mathbb{N})^{<\mathbb{N}}$ that consists of all $(s, t) \in (\mathbb{N} \times \mathbb{N})^k$ with $(s, t^0) \in S$ and $(s, t^1) \in T_1$. One can now check that $p[S] \cap p[T] = p[U]$.

(2) The collection of subsets generated from basic open sets by countable unions and intersections forms a σ-algebra and thus equals the Borel σ-algebra. Hence all Borel sets are *analytic. The claim follows by composing two projections. □

The previous result also holds for finite products of $\mathbb{N}^{\mathbb{N}}$, by observing that the proof still works or noting that they are homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

In the next proof, subsets A, B of X are called *Borel separated* if there is a Borel set C with $A \subseteq C$ and $B \subseteq X \backslash C$.

Proposition 2.3. *Any two disjoint analytic subsets A, B of X are Borel separated.*

Proof. It suffices to show this for $X = \mathbb{N}^{\mathbb{N}}$, and by Problem 2.2 A and B are then *analytic. It follows that there are continuous surjections $f\colon [S] \to A$ and $g\colon [T] \to B$, where S and T are pruned subtrees of $\mathbb{N}^{<\mathbb{N}}$. It is an easy to see that there is a continuous surjection from $\mathbb{N}^{\mathbb{N}}$ onto $[S]$, in fact one that fixes all elements of $[S]$. We can thus assume that $[S] = [T] = \mathbb{N}^{\mathbb{N}}$.

Suppose that A, B are not Borel separated. We then construct increasing sequences $\vec{s} = \langle s_i \mid i \in \mathbb{N}\rangle$ and $\vec{t} = \langle t_i \mid i \in \mathbb{N}\rangle$ in $\mathbb{N}^{<\mathbb{N}}$ such that for all $i \in \mathbb{N}$, $f(N_{s_i})$ and $g(N_{t_i})$ are not Borel separated as follows.

Let $s_0 = t_0 = \varnothing$. Given s_i and t_i, there are $k, m \in \mathbb{N}$ such that $f(N_{s_i k})$ and $g(N_{t_i m})$ are not Borel separated. If each of these pairs were Borel separated by some $B_{k,m}$, then $C_k = \bigcap_{t_i m \in T} B_{k,m}$ separates $f(N_{s_i k})$ and $g(N_{t_i})$ for each $k \in \mathbb{N}$, and hence $C = \bigcup_{s_i k \in S} C_k$ separates $f(N_{s_i})$ and $g(N_{t_i})$, contradicting our assumption. Let $s_{i+1} = s_i k$ and $t_{i+1} = t_i m$.

Let $x = \bigcup_{i \in \mathbb{N}} s_i \in [S]$ and $y = \bigcup_{i \in \mathbb{N}} t_i \in [T]$. Since A and B are disjoint, $f(x) \neq g(y)$. Let further U, V be disjoint open subsets of X with $f(x) \in U$ and $g(y) \in V$. By continuity of f and g, $f(N_{x \restriction n}) \subseteq U$ and $g(N_{y \restriction n}) \subseteq V$ for sufficiently large $n \in \mathbb{N}$, so $f(N_{x \restriction n})$ and $g(N_{y \restriction n})$ are Borel separated. However, they are not Borel separated by the construction. □

Next is a very useful characterization of analytic sets.

Lemma 2.4. *Every nonempty analytic subset of X is the range of some continuous $f\colon \mathbb{N}^{\mathbb{N}} \to X$.*

Proof. Take an analytic subset A of X. By Lemma 1.5, there are a closed subset Y of $\mathbb{N}^{\mathbb{N}}$ and a continuous Borel isomorphism $f\colon Y \to X$. It suffices to find a continuous bijection $g\colon \mathbb{N}^{\mathbb{N}} \to f^{-1}(A)$. Then A equals the range of $fg\colon \mathbb{N}^{\mathbb{N}} \to X$.

Since analytic sets are preserved under Borel isomorphisms, $f^{-1}(A)$ is an analytic subset of Y and thus of $\mathbb{N}^{\mathbb{N}}$. Let B be a Borel subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $f^{-1}(A) = p(B)$. Since Borel subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ are *analytic by Problem 2.2 (1), there is a closed subset C of $(\mathbb{N}^{\mathbb{N}})^3$ with $f^{-1}(A) = p(C)$. Since $(\mathbb{N}^{\mathbb{N}})^3$ and $\mathbb{N}^{\mathbb{N}}$ are homeomorphic, it is easy to see that there is a continuous surjection $h\colon \mathbb{N}^{\mathbb{N}} \to C$. Then $g = fph$ is as required. □

3. Borel measurable functions

A function $f\colon X \to Y$ between standard Borel spaces is called *Borel measurable* if f-preimages of Borel sets are Borel.

Lemma 3.1. *The following are equivalent for a function $f\colon X \to Y$:*

(a) *f is Borel measurable.*
(b) *The graph $G(f)$ of f is a Borel subset of $X \times Y$.*

Proof. (a) $\Rightarrow$ (b): $f(x) = y \iff x \in \bigcap_{k\in\mathbb{N}} f^{-1}(N_{y\restriction k}) \iff (x,y) \in \bigcap_{k\in\mathbb{N}} \bigcup_{t\in\mathbb{N}^k} f^{-1}(N_t) \times N_t$.

(b) $\Rightarrow$ (a): It suffices to show that $f^{-1}(U)$ is Borel for any open subset of Y. By Proposition 2.3, it is sufficient to show that both $f^{-1}(U)$ and its complement in X are analytic. These claims follows from the equivalences $x \in f^{-1}(U) \iff \exists y \in Y\ (x,y) \in G(f) \iff \forall y \in Y\ ((x,y) \in G(f) \Rightarrow y \in U)$. □

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