Internal absoluteness

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Overview

- We study *internal absoluteness* (IA) between $M[g]$ and V, where $M \prec H_\theta$ and $g \in V$ is generic over M.
- Here we only consider projective absoluteness.
- The story is that these or similar principles were used in proofs of Steel and Woodin (see Steel 2008) for the tree production lemma
- Our motivation:
  We re-discovered such principles for an application (2018) to selectors for ideals. and a connection with universally Baire sets.
- Our aims:
  Fix a class of forcings.
  - (How) can IA be characterized via descriptive set theory?
  - How is IA related to other notions of absoluteness?
  - What is the consistency strength of IA?
- This is recent work in progress with Sandra Müller.

Forcing $\leftrightsquigarrow$ descriptive set theory

Zapletal (2008): TFAE for forcings of the form $\mathbb{P} = Borel/I$, where I is a σ-ideal

Forcing	*Descriptive set theory*
$\mathbb{P}$ is proper	For every countable transitive $M \prec H_\theta$ and every $A \in \mathbb{P} \cap M$, the set of $\mathbb{P}$-generic reals over M in A is I-positive
$\mathbb{P}$ is ω^ω-bounding	Every Borel I-positive set contains a compact I-positive subset, and continuous reading of names

The second equivalence assumes that $\mathbb{P}$ is proper. Zapletal proved a large number of such characterisations.

Forcing $\leftrightsquigarrow$ descriptive set theory

Ikegami (2010): TFAE for proper tree forcings $\mathbb{P}$

Forcing	*Descriptive set theory*
1-step $\mathbf{\Sigma}^1_3$-$\mathbb{P}$-absoluteness	$\mathbb{P}$-measurability of $\mathbf{\Delta}^1_2$ sets
1-step $\mathbf{\Sigma}^1_4$-$\mathbb{P}$-absoluteness	$\mathbb{P}$-measurability of $\mathbf{\Delta}^1_3$ sets

Restrictions:

1. The second equivalence assumes that every real has a sharp, but there's no inner model with a Woodin cardinal.
2. There's no full equivalence in ZFC: projective 1-step Cohen absoluteness doesn't imply that all projective sets have the Baire property.

Forcing classes

We consider the classes:

1. All forcings
2. Proper forcings
3. Simply definable proper forcings on the reals: forcings of the form $Borel/I$ for σ-ideals I; tree forcings
4. Simply definable ccc forcings on the reals

1-step and 2-step absoluteness

Suppose that $\mathcal{F}$ is a class of forcings and Λ a class of formulas $\varphi(., x)$ (where x ranges over certain parameters).

Definition (1-step absoluteness)

1-step $\mathcal{F}$-Λ-absoluteness means: if $\mathbb{P} \in \mathcal{F}$ and G is $\mathbb{P}$-generic over V, then $V \prec_\Lambda V[G]$ holds.

Definition (2-step absoluteness)

2-step $\mathcal{F}$-Λ-absoluteness means: If $\mathbb{P} \in \mathcal{F}$ and G, H are $\mathbb{P}$-generic filters over V with $V[G] \subseteq V[H]$, then $V[G] \prec_\Lambda V[H]$ holds.

If $\mathcal{F}$ contains the trivial forcing, then 2-step implies 1-step absoluteness.

Why 2-step absoluteness?

Consider the class of all forcings.

Theorem (Martin, Solovay, Woodin)

2-step Σ^1_3-absoluteness $\iff$ "$\forall X\ X^\#$ exists", where X denotes sets of ordinals.

Theorem (Feng, Magidor, Woodin 1992)

1-step Σ^1_3-absoluteness is equiconsistent with the existence of a regular cardinal κ with $H_\kappa \prec_{\Sigma_2} V$ (weaker than a Mahlo cardinal).

Here 2-step-absoluteness is strictly stronger.

This is not always the case:

Theorem (Steel, Woodin)

The first-order theory $\mathrm{Th}(L(\mathbb{R}))$ cannot be changed by forcing $\iff$ $\mathsf{AD}^{L(\mathbb{R})}$ holds in all generic extensions, assuming Ord *is measurable in an outer model.*

The proof shows that 1-step and 2-step absoluteness are equivalent for formulas of the form $\varphi^{L(\mathbb{R})}$ with real parameters.

Internal absoluteness

Definition (Internal absoluteness)

Let $\mathcal{F}$ be a class of forcings, Λ a class of formulas and θ an uncountable regular cardinal.

1. For any countable $M \prec H_\theta$, $\mathsf{IA}^M_{\mathcal{F},\Lambda}$ denotes the statement:

$$M[g] \prec_\Lambda H_\theta$$

   holds for all $\mathbb{P} \in \mathcal{F} \cap \mathrm{Def}(M)$ and all $\mathbb{P}$-generic filters $g \in V$ over M.
2. $\mathsf{IA}^\theta_{\mathcal{F},\Lambda} :\Longleftrightarrow \mathsf{IA}^M_{\mathcal{F},\Lambda}$ holds for club many $M \in [H_\theta]^\omega$.

We write IA if Λ is the set of projective formulas with real parameters, $\mathcal{F}$ is the class of all forcings, and $\mathsf{IA}^\theta_{\mathcal{F},\Lambda}$ holds for all regular θ with H_θ sufficiently elementary in V.

IA implies 1-step and 2-step absoluteness:

$$\begin{tikzcd}
 & M[g] \arrow[dr, "\prec_\Lambda"] \arrow[dd, "\subseteq"] & \\
M \arrow[ur, "\subseteq"] \arrow[dr, "\subseteq"'] & & V \\
 & M[h] \arrow[ur, "\prec_\Lambda"'] &
\end{tikzcd}$$

Separating internal from 1-step and 2-step absoluteness

Fact

$\mathsf{IA}_{\mathbb{C}}$ *implies the Baire property for all projective sets.*

Proof.

Take a countable $M \prec H_\theta$. Note that the set of Cohen reals x over M is comeager.

Let $\varphi(x)$ define a projective set.

The set of Cohen conditions p with $p \Vdash^M_{\mathbb{C}} \varphi(\dot{x})$, where $\dot{x}$ is a name for the Cohen real, defines an open set witnessing the Baire property. $\square$

Projective 1-step and 2-step absoluteness hold in $L^{Add(\omega,\omega_1)}$.

But not all projective sets have the Baire property in this model, so internal Cohen absoluteness $\mathsf{IA}_{\mathbb{C}}$ fails there.

Uniformization up to meager

We now consider Cohen forcing.

Proposition (Müller, S.)

TFAE:

1. $\mathsf{IA}_{\mathbb{C}}$
2. *Every projective relation can be uniformized on a comeager set.*

Shelah (1984) proved the consistency of 2. relative to ZFC.

Proof sketch.

$1 \Rightarrow 2$: Assume $\mathsf{IA}_{\mathbb{C}}$. Let $R = \{(x, y) \mid \varphi(x, y, z)\}$.

Let $M \prec H_\theta$ with $z \in M$ and $\sigma \in M$ be a name for the Cohen real (all names are nice).

Let $\tau \in M$ with $1 \Vdash \exists y\ \varphi(\sigma, y, z) \Rightarrow \varphi(\sigma, \tau, z)$.

$\mathsf{IA}_{\mathbb{C}}$ implies $M[x] \models \varphi(x, \tau^x, z) \Longleftrightarrow V \models \varphi(x, \tau^x, z)$ for Cohen reals $x \in V$ over M.

The continuous function $x \mapsto \tau^x$ uniformizes R on the set of Cohen reals over M. $\square$

Uniformization up to meager

Proposition (Müller, S.)

TFAE:

1. $\mathsf{IA}_{\mathbb{C}}$
2. *Every projective relation can be uniformized on a comeager set.*

Proof sketch.

$2 \Rightarrow 1$: Take a Σ^1_{n+1}-formula $\exists y\ \varphi(x, y)$, where $n \geq 1$.

For a nice name σ and any $i \in \omega$, let $A_\sigma = \{(x, y) \mid \varphi(\sigma^x, y)\}$.

Let $M \prec H_\theta$.

Claim: For any Cohen real $x \in V$ over M, $\exists y\ \varphi(\sigma^x, y)$ is downwards absolute from V to $M[x]$.

To see this, let f_σ be a uniformization of A_σ up to meager. Since the existence of f_σ is projective, we can take f_σ to be projectively defined in M.

Suppose that $\exists y\ \varphi(u, y)$ holds in V.

Recall that x is Cohen generic over M iff $x \in A$ for every comeager Borel set with a code in M. So $x \in \mathsf{dom}(f_\sigma)$ and $f_\sigma(x) \in M[x]$. $\square$

Forcings of the form $Borel/I$

Let I be a σ-ideal on a Polish space X. A binary relation R on X is called *total* if $\mathrm{proj}(R) = X$.

Definition (Uniformization and regularity)

1. $\mathsf{U}_I :\Longleftrightarrow$ If R is a projective total relation on X, then for any Borel set $A \notin I$, $R{\restriction}A$ has a projective subfunction with domain $B \notin I$.
2. $\mathsf{R}_I :\Longleftrightarrow$ If A is a projective set and $B \notin I$ is a Borel set, then there is some Borel set $C \notin I$ with $C \subseteq B$ and either $C \cap A = \emptyset$ or $C \subseteq A$.

Clearly $\mathsf{U}_I \Longrightarrow \mathsf{R}_I$.

Why don't we uniformize everywhere except for a set in I? This would be much stronger.

Forcings of the form $Borel/I$

Theorem (Müller, S.)

The following statements are equivalent for proper forcings of the form $\mathbb{P} = Borel/I$:

1. $\mathsf{IA}_{\mathbb{P}}$
2. 1-*step* $\mathbb{P}$-*absoluteness holds and* R_I *holds for all projective sets.*
3. U_I

Forcing				*Descriptive set theory*
$\mathsf{IA}_{\mathbb{P}}$				U_I for projective sets
1-step $\mathbb{P}$-absoluteness	$\not\Longrightarrow$		$\Longleftarrow$?	R_I for projective sets

We have seen 1. $\Leftrightarrow$ 3. for Cohen forcing. 1. $\Rightarrow$ 2. is clear.

Consequences of internal Cohen absoluteness

Fact (Müller, S.)

$\mathsf{IA}_{\mathbb{C}}$ *implies:*

1. 1*-step and* 2*-step Cohen absoluteness*
2. *The Baire property for projective sets*
3. *For every real x, the set of Cohen reals over $L[x]$ is comeager*
4. *For every real x, there is a real dominating $L[x]$*
5. *Projective uniformization up to meager*

Obtaining internal absoluteness

$\mathsf{IA}_{\mathbb{C}}$ is consistent relative to ZFC, by the consistency of projective uniformization up to meager (Shelah 1984).

In many cases, $\mathsf{IA}_{\mathbb{P}}$ follows directly from large cardinals:

Proposition (Müller, S.)

PD *implies that any sufficiently absolute Axiom A forcing of the form* $\mathbb{P} = Borel/I$ *satisfies* $\mathsf{IA}_{\mathbb{P}}$.

This is based on work of Fabiana Castiblanco.

The connection with universally Baire sets

Let $A \subseteq \omega^\omega$.

Definition (Feng, Magidor, Woodin)

A is *universally Baire* if for every continuous $f\colon X \to \omega^\omega$, $f^{-1}(A)$ has the Baire property.

A is *absolutely complemented* if there are trees S, T with $A = p[S]$ such that

$$\omega^\omega = p[S] \cup p[T]$$

holds in all generic extensions.

Feng, Magidor and Woodin proved the equivalence of these notions:

Forcing	*Descriptive set theory*
A is absolutely complemented	A is universally Baire

The connection with universally Baire sets

Definition

We call a tree T on $\omega \times \lambda$ *absolute* if for all regular θ such that H_θ is sufficiently elementary in V, there is a club of $M \in [H_\theta]^\omega$ such that for all generic extensions $M[g] \subseteq V$:

$$p[T]^{M[g]} = p[T] \cap M[g].$$

Definition

We call a formula $\varphi(x)$ *$\mathbb{P}$-treeable* if there is an absolute tree T on $\omega \times \lambda$, for some λ, such that

$$p[T] = \{x \in {}^\omega\omega \mid \varphi(x)\}$$

holds in every $\mathbb{P}$-generic extension.

Moreover, *treeable* means $\mathbb{P}$-treeable for all forcings $\mathbb{P}$.

Fact

Every treeable formula is internally absolute.

From internally absolute to treeable

Proposition (essentially Steel or Woodin)

If $\varphi(x)$ is internally absolute, then $\varphi(x)$ is treeable.

Proof sketch.

The tree for $\varphi(x)$ searches for

1. A countable transitive model M of ZFC^-
2. An elementary $j\colon M \to H_\theta$
3. A generic filter g over M with $x \in M[g]$ and $M[g] \models \varphi(x)$

$\square$

Internal projective absoluteness $\Longleftrightarrow$ Projective treeability

Internally absolute and universally Baire

The following are equivalent for a formula $\varphi(x)$:

1. φ is internally absolute
2. $\{x \mid \varphi(x)\}$ is universally Baire, with a tree projecting to $\{x \mid \varphi(x)\}$ in all generic extensions.

$$\begin{tikzcd} & \mathsf{IA} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\ \text{All projective sets are universally Baire} & & \text{1-step projective absoluteness} \end{tikzcd}$$

Absoluteness from absorption

How does one prove internal absoluteness?

Next is a well-known property that is used in several proofs of absoluteness.

Definition

Let M be an inner model and κ a cardinal. (M, V) has the *$<\kappa$-absorption* property for a forcing $\mathbb{P}$ if:

For any $\mathbb{P}$-generic extension $V[H]$ of V and any real $x \in V[H]$, there exist:

1. a forcing $\mathbb{Q} \in M$ with $|\mathbb{Q}|^M < \kappa$ and
2. a $\mathbb{Q}$-generic filter $I \in V[H]$ over M

with $x \in M[I]$.

For instance, Schindler (2001) used this property to obtain $L(\mathbb{R})$-absoluteness for proper forcings from a remarkable cardinal.

Internal absoluteness from absorption

Lemma (Müller, S.)

Suppose that κ is inaccessible and G is $\mathrm{Col}(\omega, {<}\kappa)$-generic for V. Suppose that $(V, V[G])$ has the ${<}\kappa$-absorption property for $\mathbb{P}$.

Then every $L(\mathbb{R})$-formula with real and ordinal parameters is $\mathbb{P}$-treeable.

The proof combines previous arguments for absoluteness with the construction of a tree projecting to $\{x \mid \varphi(x)\}$.

Thus one can, in many cases, prove internal absoluteness in a similar way as 1-step absoluteness.

Some open questions

For Cohen forcing, our next goals are:

Question

- *Does* $\mathsf{IA}_{\mathbb{C}}$ *fail in Shelah's "first" model of the Baire property for all projective sets?*
- *For Cohen forcing, does* 2*-step imply* 1*-step projective absoluteness?*

Similar questions appear for forcings of the form $Borel/I$.

For the class of all forcings, the following are long-standing open questions:

Question (Feng, Magidor, Woodin, Wilson)

- *If every projective set is universally Baire, does internal projective absoluteness hold?*
- *If every projective set is universally Baire, does* 1*-step projective absoluteness hold? Does the converse implication hold?*

Literature

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Thank you for listening!