Small ultrafilter number and some compactness principles

Šárka Stejskalová

Department of Logic, Charles University
Institute of Mathematics, Czech Academy of Sciences

logika.ff.cuni.cz/sarka

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We will show that the following three properties are consistent together for a strong limit singular κ of countable or uncountable cofinality:

- $2^\kappa > \kappa^+$,
- $\mathfrak{u}(\kappa) = \kappa^+$,
- κ^{++} is compact in a prescribed sense.[1]

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- κ^{++} is compact in a prescribed sense.[1]

This result extends the existing results which study the interplay between compactness and the continuum function by including one more cardinal invariant: the ultrafilter number $\mathfrak{u}(\kappa)$.

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Definition

$\mathfrak{u}(\kappa)$ is the least cardinal α such that there exists a base B of a uniform ultrafilter U on κ of size α, where B is a base of U if for every $X \in U$ there is $Y \in B$ with $Y \subseteq X$.

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For all infinite κ, $\mathfrak{u}(\kappa)$ is always at least κ^+.

Ultrafilter number at ω

- Let us note that for $\kappa = \omega$, it is known that $\mathfrak{u}(\omega) = \omega_1$ is consistent with 2^ω large: for instance the product or iteration of the Sacks forcing of length ω_2, or an iteration of length ω_1 over a model of $\neg$CH of Mathias forcing, achieves this configuration.

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- The iteration of Sacks forcing also forces TP(ω_2) and SR(ω_2) if ω_2 used to be weakly compact in the ground model.
- The situation for $\kappa > \omega$ is less understood.

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- Some techniques are available for *weakly or strongly inaccessible* cardinals.

The following, however, is still open:

Question

Is $\mathfrak{u}(\kappa) < 2^\kappa$ consistent for a successor of a regular cardinal?

"Compactness at κ^{++}" can mean many things, we will in the talk focus on three:

- the tree property,
- stationary reflection,
- the failure of approachability.

The tree property

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Note that with inaccessibility of λ, $\mathsf{TP}(\lambda)$ is equivalent to λ being weakly compact. Over L, $\mathsf{TP}(\lambda)$ is equivalent to λ being weakly compact.

The tree property

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- (Specker) If $\kappa^{<\kappa} = \kappa$ then there exists a κ^+-Aronszajn tree. Therefore $\neg\mathsf{TP}(\kappa^+)$.
  - If GCH then $\neg\mathsf{TP}(\kappa^{++})$ for all $\kappa \geq \omega$.
  - $\mathsf{TP}(\kappa^{++})$ then $2^\kappa > \kappa^+$.

  It follows that the tree property has a non-trivial effect on the continuum function. It is of interest to study the extent of this effect to cardinal invariants.

Stationary reflection

Definition

Let λ be a cardinal of the form $\lambda = \nu^+$ for some regular cardinal ν. We say that the *stationary reflection* holds at λ, and write $\mathsf{SR}(\lambda)$, if every stationary subset $S \subseteq \lambda \cap \mathrm{cof}(< \nu)$ reflects at a point of cofinality ν; i.e. there is $\alpha < \lambda$ of cofinality ν such that $\alpha \cap S$ is stationary in α.

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- Stationary subsets of $\lambda \cap \mathrm{cof}(\nu)$ never reflect. Also note that over L, if every stationary subset of λ reflects, then λ is weakly compact.
- Stationary reflection is consistent with GCH.

The failure of approachability

For a cardinal λ and sequence $\bar{a} = \langle a_\alpha \mid \alpha < \lambda \rangle$ of bounded subsets of λ, we say that an ordinal $\gamma < \lambda$ is approachable with respect to $\bar{a}$ if there is an unbounded subset $A \subseteq \gamma$ of order type $\mathrm{cf}(\gamma)$ and for all $\beta < \gamma$ there is $\alpha < \gamma$ such that $A \cap \beta = a_\alpha$.

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Let us define the *ideal* $I[\lambda]$ *of approachable subsets of* λ:

Definition

$S \in I[\lambda]$ if and only if there are a sequence $\bar{a} = \langle a_\alpha \,|\, \alpha < \lambda \rangle$ and a club $C \subseteq \lambda$ such that every $\gamma \in S \cap C$ is approachable with respect to $\bar{a}$.

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- It is known that $\neg\mathsf{AP}(\lambda)$, for $\lambda = \nu^+$, implies the failure of $\square^*_\nu$, and in this sense $\neg\mathsf{AP}(\lambda)$ can be considered a compactness principle.

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- $\square^*_\nu$ is equivalent to the existence of ν^+-special Aronszajn tree; therefore by Specker result, if $\neg$AP(κ^{++}) then $2^\kappa > \kappa^+$.

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- It is known that $\neg$AP(λ), for $\lambda = \nu^+$, implies the failure of $\square^*_\nu$, and in this sense $\neg$AP(λ) can be considered a compactness principle.
- $\square^*_\nu$ is equivalent to the existence of ν^+-special Aronszajn tree; therefore by Specker result, if $\neg$AP(κ^{++}) then $2^\kappa > \kappa^+$.
- Note that AP(λ) does not imply $\square^*_\nu$, so $\neg$AP(λ) is strictly stronger than the fact that there are no special λ-Aronszajn trees.

Compactness at small cardinals

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If $\kappa < \lambda$ are regular cardinals, $\kappa^{<\kappa} = \kappa$, and λ is weakly compact, then the Mitchell forcing turns λ to κ^{++}, adds λ-many Cohen subsets to κ ($2^\kappa = \kappa^{++}$) and the tree property, stationary reflection and the failure of the approachability hold at κ^{++}.

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- To obtain stationary reflection at κ^{++}, the Levy collapse of a weakly compact cardinal is enough. (Recall that we do not need to violate GCH at κ. Moreover, the consistency strength of SR(κ^{++}) is only a Mahlo cardinal. However to achieve SR(κ^{++}) from a Mahlo cardinal, we need to use an additional iteration after turning the Mahlo cardinal to κ^{++} (either by Levy collapse or by Mitchell forcing).

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- The assumption that λ is a Mahlo cardinal is enough for obtaining the failure of approachability at κ^{++} in the Mitchell model.

Main theorem

We sketch the main steps for proving the following theorem:

Theorem (Honzik, S. 2019)

Suppose κ is Laver-indestructible supercompact and $\lambda > \kappa$ is weakly compact. Then there is a forcing extension where the following hold:

- *κ is singular strong limit with countable or uncountable cofinality.*
- $2^\kappa = \lambda = \kappa^{++}$.
- $\mathsf{TP}(\kappa^{++}), \mathsf{SR}(\kappa^{++})$ *and* $\neg\mathsf{AP}(\kappa^{++})$.
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The leitmotif is to give arguments for compactness which are based on "indestructibility" of these principles, avoiding ad hoc arguments for specific forcings. We modify an argument of Garti and Shelah to obtain small $\mathfrak{u}(\kappa)$.

Theorem (Garti,Shelah, [1])

Assume that:

1. κ *is a singular strong limit cardinal with countable cofinality.*
2. E *is a uniform ultrafilter on* ω *and* E^* *its dual.*
3. $\bar{\kappa} = \langle \kappa_n \mid n < \omega \rangle$ *is a sequence of regular cardinals converging to* κ.
4. U_n *is a uniform ultrafiter on* κ_n *for each* $n < \omega$.
5. *For every* $n < \omega$ *there is a* $\subseteq^*$*-decreasing sequence* $\langle A_{n,\alpha} \mid \alpha < \theta_n \rangle$ *for some* θ_n *which generates* U_n *(let* $\bar{\theta} = \langle \theta_n \mid n < \omega \rangle$*).*
6. $\chi_{\bar{\kappa}} = \mathrm{tcf}(\prod_{n<\omega} \kappa_n, <_{E^*})$, $\chi_{\bar{\theta}} = \mathrm{tcf}(\prod_{n<\omega} \theta_n, <_{E^*})$.

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Then $\mathfrak{u}(\kappa) \leq \chi_{\bar{\kappa}} \cdot \chi_{\bar{\theta}}$.

Note that if $\chi_{\bar{\kappa}} = \chi_{\bar{\theta}} = \kappa^+$ then $\mathfrak{u}(\kappa) = \kappa^+$.

The basic strategy of Garti and Shelah to obtain $\mathfrak{u}(\kappa) = \kappa^+$ with $2^\kappa > \kappa^+$ is a follows:

- Iterate up to a singular length $\delta > \kappa$ of cofinality κ^+ a κ^+-cc κ-strategically-closed forcing which will add a scale of size κ^+ to products on a cofinal sequence of

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  κ remains measurable after this stage.

Definition

Let us call this forcing $\mathbb{P}_\delta$.

- Let U be a normal measure on κ in $V[\mathbb{P}_\delta]$ and $\mathbb{Q}_U$ be the Prikry forcing with U. Then Garti and Shelah argue that in $V[\mathbb{P}_\delta][\mathbb{Q}_U]$, enough of pcf-configurations ensured by $\mathbb{P}_\delta$ is preserved to apply the above mentioned theorem with κ which now has cofinality ω. This gives $\mathfrak{u}(\kappa) = \kappa^+$, i.e.

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Theorem (Garti, Shelah)

$V[\mathbb{P}_\delta * \mathbb{Q}_U] \models \mathfrak{u}(\kappa) = \kappa^+$ and $2^\kappa = \lambda$.

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Note that the argument does not prove $\mathfrak{u}(\kappa) = \kappa^+$ already in $V[\mathbb{P}_\delta]$.

We "Mitchell-ize" the forcing $\mathbb{P}_\delta$ as follows. Recall that $\lambda > \kappa$ is weakly compact. Let δ be an ordinal with cofinality κ^+ between λ and λ^+.

Definition

$\mathbb{P}^*_\delta$ is a forcing with conditions $p = (p^0, p^1)$ such that:

- $p^0 \in \mathbb{P}_\delta$,
- p^1 is a function with domain $\mathrm{dom}(p^1)$ of size at most κ such that $\mathrm{dom}(p^1)$ is included in the set of successor cardinals below λ.

The ordering is the usual Mitchell ordering: $(p^0, p^1) \le (p'^0, p'^1)$ iff $p^0 \le_{\mathbb{P}_\delta} p'^0$ and the domain of p^1 extends the domain of p'^1 and for all $\alpha \in \mathrm{dom}(p'^1)$,

$$p^0 \restriction \alpha \Vdash_{\mathbb{P}_\alpha} p^1(\alpha) \le p'^1(\alpha).$$

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This uses the fact that there are natural projections from $\mathbb{P}^*_\delta$ onto $\mathbb{P}_\delta$ and from $\mathbb{T} \times \mathbb{P}_\delta$ onto $\mathbb{P}^*_\delta$ where $\mathbb{T}$ is a κ^+-closed forcing.

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In particular,

$$\mathbb{P}^*_\delta \equiv \mathbb{P}_\delta * \dot{R}$$

for some forcing $\dot{R}$ which is forced κ^+-distributive.

Let U be a normal measure in $V[\mathbb{P}_\delta]$ on κ, and let $\mathbb{Q}_U$ be the Prikry forcing. Note that U is still a normal measure in $V[\mathbb{P}^*_\delta]$ and $\mathbb{Q}_U$ is the Prikry forcing here.

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Lemma

$V[\mathbb{P}^*_\delta * \mathbb{Q}_U] \models \mathfrak{u}(\kappa) = \kappa^+$ *and* $2^\kappa = \kappa^{++} = \lambda$.

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Lemma

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Proof.

Recall $\mathbb{P}^*_\delta \equiv \mathbb{P}_\delta * \dot{R}$. Since $\dot{R}$ is κ^+-distributive, the desired pcf structure of scales on κ is preserved from $V[\mathbb{P}_\delta]$ to $V[\mathbb{P}^*_\delta]$ and the argument follows as in Garti and Shelah. $\square$

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The rest of the proof for stationary reflection and the failure of the approachability follows by "indestructibility" arguments for κ^+-cc forcings and κ-centered forcings, respectively. We briefly discuss this indestructibility on next slides.

Indestructibility of the failure of approachability at κ^{++} by κ-centered forcigs

Theorem (Gitik, Krueger, [2])

Assume $\neg\mathsf{AP}(\kappa^{++})$ *holds and* $\mathbb{Q}$ *is* κ*-centered. Then the forcing* $\mathbb{Q}$ *forces* $\neg\mathsf{AP}(\kappa^{++})$*.*

Indestructibility of stationary reflection at κ^{++} by κ^{+}-cc forcings

Theorem (Honzik, S. (2019))

Suppose λ is a regular cardinal, $\mathrm{SR}(\lambda^+)$ *holds and $\mathbb{Q}$ is λ-cc. Then $\mathbb{Q}$ preserves* $\mathrm{SR}(\lambda^+)$*, i.e. $V[\mathbb{Q}] \models \mathrm{SR}(\lambda^+)$.*

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Suppose λ is a regular cardinal, $\mathrm{SR}(\lambda^+)$ holds and $\mathbb{Q}$ is λ-cc. Then $\mathbb{Q}$ preserves $\mathrm{SR}(\lambda^+)$, i.e. $V[\mathbb{Q}] \models \mathrm{SR}(\lambda^+)$.

Proof.

(Sketch). Suppose for contradiction $1_{\mathbb{Q}}$ forces $\dot{S}$ is a non-reflecting stationary set in $\lambda^+ \cap \mathrm{cof}(< \lambda)$. Set

$$U = \{\alpha < \lambda^+ \mid \exists q \; q \Vdash \alpha \in \dot{S}\}.$$

U is stationary and by $\mathrm{SR}(\lambda^+)$ there is α of cof λ such that $(*) U \cap \alpha$ is stationary. But also, by our assumption, $(**) 1 \Vdash \dot{S} \cap \alpha$ is non-stationary. We will argue that $(*)$ and $(**)$ are contradictory which will finish the proof. □

Proof.

- From $(**)$ we get: There is a maximal antichain A in $\mathbb{Q}$ such that for every $q \in A$, there is some club D_q in α with $q \Vdash \dot{S} \cap D_q = \emptyset$. A has size $< \lambda$, and therefore

$$C = \bigcap_{q \in A} D_q \text{ is a club in } \alpha.$$

It is easy to check that $(\dagger) 1 \Vdash \dot{S} \cap C = \emptyset$.

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$$C = \bigcap_{q \in A} D_q \text{ is a club in } \alpha.$$

It is easy to check that $(\dagger) 1 \Vdash \dot{S} \cap C = \emptyset$.

- From $(*)$ we get: There is $\beta < \alpha$ with $\beta \in C \cap U$, i.e. some q such that $q \Vdash \beta \in \dot{S} \cap C$, this contradicts $(\dagger)$.

$\square$

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However, we showed that a reasonably strong indestructibility for the tree property holds in many Mitchell-like models, in particular in our current model for the small $\mathfrak{u}(\kappa)$.

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where $(\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U)$ is κ^+-cc.

Lemma

$V[\mathbb{P}^*_\lambda] \models 2^\kappa = \lambda = \kappa^{++}$ and $\mathrm{TP}(\kappa^{++})$.

The tree property

$\mathbb{P}^*_\delta * \mathbb{Q}_U$ can be written as

$$\mathbb{P}^*_\lambda * (\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U),$$

where $(\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U)$ is κ^+-cc.

Lemma

$V[\mathbb{P}^*_\lambda] \models 2^\kappa = \lambda = \kappa^{++}$ *and* $\mathrm{TP}(\kappa^{++})$.

Proof.

This is standard to check using a Mitchell-style argument. $\square$

Theorem (Honzik, S. (2019))

$\mathsf{TP}(\kappa^{++})$ *is indestructible under* $\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U$ *over* $V[\mathbb{P}^*_\lambda]$.

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Proof.

This follows by a modification of an indestructibility theorem in Honzik and S. [3], crucially using the fact that $\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U$ lives already in $V[\mathbb{P}_\lambda]$: one can argue that there is a projection from

$$j(\mathbb{P}_\lambda * (\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U))/(\mathbb{P}_\lambda * (\mathbb{P}_{[\lambda,\delta)} * \mathbb{Q}_U)) \times \mathrm{Term}(j(\mathbb{P}^*_\lambda)/\mathbb{P}^*_\lambda)$$

onto the quotient of $j(\mathbb{P}^*_\delta * \mathbb{Q}_U)$ over the generic extension by $\mathbb{P}^*_\delta * \mathbb{Q}_U$ (where j is an elementary embedding with critical point λ). □

Note that since the indstructibility arguments use just the κ^+-cc of the relevant forcings (κ-centeredness in the non-approachability case), the argument for small $\mathfrak{u}(\kappa)$ is not limited to countable cofinality and the vanilla Prikry forcing.

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It directly generalizes to the Magidor forcing; we therefore get our result immediately for a singular strong limit κ of a prescribed uncountable cofinality.

Some open questions:

1. Can we obtain a similar model with $\kappa = \aleph_\omega$ or $\kappa = \aleph_{\omega_1}$?

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1. Can we obtain a similar model with $\kappa = \aleph_\omega$ or $\kappa = \aleph_{\omega_1}$?
2. Can we have $\mathsf{SR}(\kappa^+)$ or $\mathsf{TP}(\kappa^+)$ with $\mathfrak{u}(\kappa) = \kappa^+$ and $2^\kappa > \kappa^+$? κ singular or regular.
3. Can the indestructibility theorems be improved? In particular, (1) can we get a general form of indestructibility for the tree property over any model, and (2) can we show indestructibility for stronger forms of stationary reflection, such as the club stationary reflection?

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