OPEN PROBLEMS
STUK 2 BRISTOL, 8 MAY 2019

The second *Set theory in the United Kingdom* workshop took place in Bristol on 8 May, 2019, see `<https://philippschlicht.github.io/meetings>`. This is a collection of open problems discussed by some of the speakers in their talks, the open problems and discussion sessions.[1]

1. Victoria Gitman: Second-order set theory

Classes, from class forcing notions to elementary embeddings of the universe to inner models, play a fundamental role in modern set theory. But within first-order set theory we are limited to studying only definable classes and we cannot even express properties that necessitate quantifying over classes.

Second-order set theory is a formal framework in which a model consists both of a collection of sets and a collection of classes (which are themselves collections of sets). In second-order set theory, we can study classes such as truth predicates, which can never be definable over a model of ZFC, and properties that, for instance, quantify over all inner models. With this formal background we can develop a theory of class forcing that explains why and when class forcing behaves differently from set forcing.

A number of interesting second-order set theoretic principles arose out of recent work in this area, such as, class choice principles, transfinite recursion with classes, determinacy of games on the ordinals, and the class Fodor Principle. The study of where these principles fit in the hierarchy of second-order set theories — starting from the weak Gödel-Bernays set theory GBC and going beyond the relatively strong Kelley-Morse theory KM — serves as the beginning of a reverse mathematics program that I encourage set theorists to take part in.

Question 1.1. Does Gödel-Bernays set theory GBC prove that any two meta-ordinals are comparable?

The current best upper bound is $\mathsf{GBC} + \mathsf{ETR}$.

Question 1.2. Does tame forcing preserve elementary transfinite recursion ETR?

Hamkins and Woodin showed that tame forcing preserves open class determinacy, a principle slighly strengthening ETR, which is implied by Σ^1_1-comprehension.

Question 1.3. Can tame forcing add meta-ordinals?

Tame forcing cannot add meta-ordinals to a model of $\mathsf{GBC} + \Sigma^1_1$-comprehension by a result of Hamkins and Woodin, so the question is only about weak theories.

Question 1.4. Can the Class Fodor Principle hold in a model of ZFC with definable classes?

Question 1.5. Is the Class Fodor Principle preserved by tame forcing?

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References

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[GHK] Victoria Gitman, Joel David Hamkins, and Asaf Karagila. Kelley-Morse set theory does not prove the class Fodor theorem. Submitted.

[HGH+] Peter Holy, Victoria Gitman, Joel David Hamkins, Philipp Schlicht, and Kameryn Williams. The exact strength of the class forcing theorem. Submitted.

[HW] Joel David Hamkins and William Hugh Woodin. Open class determinacy is preserved by forcing. Submitted.

2. Dan Nielsen: Level-by-level virtual large cardinals

The study of generic large cardinals goes back a long way, at least back to the 70's. Back then it seems that the primary interest was the existence of *precipitous* and *saturated* ideals on small cardinals like ω_1 and ω_2. This moved to more general generic embeddings, both defined on V but also on rank-initial segments of V — these were investigated by e.g. Donder and Levinsky (1989) and Ferber and Gitik (2010).

The move to *virtual* large cardinals was probably with the *remarkable cardinals*, introduced in Schindler (2000), and virtual large cardinals were properly investigated in Gitman and Schindler (2018), where the key difference between the generics and the virtuals is that in the virtual case we require that the target model is a subset of the ground model.

These large cardinals are unique in the sense that they allow us to work with embeddings as in the higher reaches of the large cardinal hierarchy, but being consistently below $V = L$, enabling equiconsistencies at these "lower levels".

To take a few examples, Schindler (2000) has shown that the existence of a remarkable cardinal is equiconsistent with thte statement that the theory of $L(\mathbb{R})$ cannot be changed by proper forcing, which was improved to semi-proper forcing in Schindler (2004). Wilson (∞) has shown that the existence of a generic Vopěnka cardinal is equiconsistent with

$$\mathsf{ZF} + \mathbf{\Sigma}^1_2 \text{ is the class of all } \omega_1\text{-Suslin sets} + \Theta = \omega_2,$$

and Schindler and Wilson (∞) has shown that the existence of a virtually Shelah cardinal is equiconsistent with

$$\mathsf{ZF} + \text{every universally Baire set of reals has the perfect set property.}$$

Definition 2.1. Let $\theta > \kappa$ be regular. Then κ is...

- **generically θ-(power)-measurable** if there is a generic (κ-powerset preserving) embedding $\pi : H^V_\theta \to N$ for some transitive N with $\mathrm{crit}\pi = \kappa$;
- **generically θ-prestrong** if it's generically θ-measurable and $H^V_\theta \subset N$;
- **generically θ-strong** if it's generically θ-prestrong and $\pi(\kappa) \geq \theta$;
- **generically θ-supercompact** if it's generically θ-strong and ${}^{<\theta}N \cap V \subset N$.

We further replace "generically" by **virtually** when $N \subset V$. When we don't mention θ we mean that it holds for all $\theta > \kappa$, e.g. a generically measurable cardinal κ is generically θ-measurable for all regular $\theta > \kappa$.

Theorem 2.2 (Gitman)**.** *Every virtually $(2^{<\theta})^+$-strong cardinal is virtually θ-supecompact.*

Question 2.3. Are generically θ-strongs always generically θ-supercompact? Maybe in L?

Let κ be an uncountable cardinal, $\theta > \kappa$ regular and $\gamma < \kappa^+$ an ordinal. Then define the **filter game** $\mathcal{G}^\theta_\gamma(\kappa)$ with $\gamma{+}1$ rounds:

$$\begin{array}{ccccc} \mathrm{I} & M_0 & M_1 & \cdots & M_\gamma \\ \mathrm{II} & \mu_0 & \mu_1 & \cdots & \mu_\gamma \end{array}$$

Here $M_\alpha \prec H_\theta$ is a weak κ-model for every $\alpha \leq \gamma$, μ_α is an M_α-normal M_α-measure on κ with $\mathrm{ult}(M_\alpha, \mu_\alpha)$ being wellfounded for all $\alpha \leq \gamma$, and the M_α's and μ_α's are $\subset$-increasing. For limit ordinals $\alpha \leq \gamma$ we furthermore require that $M_\alpha = \bigcup_{\xi<\alpha} M_\xi$ and $\mu_\alpha = \bigcup_{\xi<\alpha} \mu_\xi$. Player II wins iff they could continue playing throughout all $\gamma{+}1$ rounds.

Theorem 2.4 (Schindler-N.)**.** *Let $\kappa < \theta$ be regular cardinals. If κ is virtually θ-prestrong then player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$, and if player II has a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$ then κ is generically θ-power-measurable. In particular, $\mathcal{G}^\theta_\omega(\kappa)^L \sim \mathcal{C}^\theta_\omega(\kappa)^L$.*

Question 2.5. If κ is generically θ-power-measurable, does player II then have a winning strategy in $\mathcal{G}^\theta_\omega(\kappa)$?

Proposition 2.6. *For any regular cardinal θ, generically θ-measurable cardinals are equivalent to virtually θ-prestrong cardinals in L.*

Question 2.7. What happens to the level-by-level picture in larger core models? E.g., is every generically θ-measurable also virtually θ-prestrong in K below a Woodin? Or just virtually θ-measurable?

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