Definable well-orderings with compactness

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We will discuss the following result, and more importantly also consider some open questions with regard to higher cardinals and generalized cardinal invariants:

Theorem (Friedman, H., Stejskalova (2020))

1. *If there is a Mahlo cardinal, then there is a model with* $2^\omega = \omega_2$, $\neg\Box^*_{\omega_1}$ *and with a* Σ^1_3 *well-ordering of the reals.*
2. *If there is a weakly compact cardinal, then there is a model with* $2^\omega = \omega_2$, $\mathsf{TP}(\omega_2)$ *and with a* Σ^1_3 *well-ordering of the reals.*

BPFA *may be false if we wish.*

Let us say a few things about the context and motivations.

- With new methods and forcings available, it is natural to ask not only whether ω_2 can have the tree property, stationary reflection, etc. (compactness properties), but whether the compactness at ω_2 is compatible with other interesting properties.
- One such property is a projective well-ordering.
- Another properties might be a small $\mathfrak{u}_\omega$, etc.

This is interesting both from the point of development of new forcings and from the point of compatibility of compactness principles with other set-theoretical properties.

It appears that at the moment there is virtually only one forcing method which achieves $\mathrm{TP}(\omega_2)$ (the tree property at ω_2) with an iteration which can code some information such as a projective well-order:

Theorem (Friedman, Torres-Perez, [2])

If there is a Σ_1-reflecting cardinal which is also weakly compact, then there is a model with BPFA, $2^\omega = \omega_2$, $\mathrm{TP}(\omega_2)$ *and with a* Σ^1_3 *well-ordering of the reals.*

Let us say a few words about this theorem and how it relates to our result.

Recall that a regular cardinal κ is Σ_1-*reflecting* if for any regular cardinal χ and any formula φ and $a \in H(\kappa)$, $H(\chi) \models \varphi(a)$, implies $(\exists \delta < \kappa)\ a \in H(\delta)$ and $H(\delta) \models \varphi$. By a Löwenheim-Skolem argument, the cardinals κ which are possibly non-regular and satisfy this condition are closed unbounded in ORD. The following therefore follows:

Lemma

(i) *If there are stationarily many Mahlo in* ORD, *then there are stationarily many Mahlo cardinals which are also* Σ_1*-reflecting.*

(ii) *If there are stationarily many weakly compact cardinals in* ORD, *then there are stationarily many weakly compact cardinals which are also* Σ_1*-reflecting.*

Note that if κ is strong, it is also Σ_1-reflecting.

By Goldstern and Shelah [3], BPFA[1] is equiconsistent with a Σ_1-reflecting cardinal.

For our present topic, the following surprising result is important:

Theorem (Caicedo, Friedman [1])

If BPFA *holds and for some real r, $\omega_1 = \omega_1^{L[r]}$, then there is a $\Sigma^1_3(r)$ well-ordering of the reals.*

Thus forcing BPFA over L by the standard proper forcing iteration up to a Σ_1-reflecting cardinal automatically gives a Σ^1_3 well-ordering of the reals (this is the optimal complexity because already MA_{ω_1} implies there are no $\boldsymbol{\Sigma}^1_2$ well-orderings of the reals[2]).

[1] Bounded Proper Forcing Axiom: Pseudo-generics exist which meet ω_1-many maximal antichains each of size at most ω_1 for proper forcing notions.

[2] Some anti-large cardinal assumption is required because if ω_1 is inaccessible to reals, then MA implies there are no $\boldsymbol{\Sigma}^1_3$ well-orderings.

We wish to find a different construction which will:

- Achieve $\mathrm{TP}(\omega_2)$ with a Σ^1_3 well-ordering from the optimal hypothesis of a weakly compact cardinal.
- Remove the dependence of the construction on the proper forcing arguments used to get BPFA to prepare for generalizations to larger cardinals.

Recall we prove:

Theorem (Friedman, H., Stejskalova (2020))

(i) *If there is a Mahlo cardinal, then there is a model with $2^\omega = \omega_2$, $\neg\Box^*_{\omega_1}$ and with a Σ^1_3 well-ordering of the reals.*

(ii) *If there is a weakly compact cardinal, then there is a model with $2^\omega = \omega_2$, $\mathrm{TP}(\omega_2)$ and with a Σ^1_3 well-ordering of the reals.*

BPFA *may be false if we wish.*

Let us say a few words about the proof:

- Let κ be weakly compact. We define an iteration $\langle(\mathbb{P}_\alpha, \dot{Q}_\alpha) \mid \alpha < \kappa\rangle$ with countable support which turns κ to ω_2, ensures a Σ^1_3 well-ordering at successor stages and $\mathrm{TP}(\omega_2)$ at certain limit stages:

In the ground model L fix a definable sequence $\langle S_\alpha \mid \alpha < \kappa \rangle$ where each S_α is stationary/costationary subset of $i_\alpha^+ \cap \mathrm{cof}(\omega)$, where $\langle i_\alpha \mid \alpha < \kappa \rangle$ is an increasing enumeration of inaccessible cardinals below κ.

- **Successor stages α:** $\dot{Q}_\alpha$ codes a given pair of reals (x, y) (so that in the end $x < y$ in the well-ordering) by selectively killing stationarity of certain stationary subsets in an ω-block in $\langle S_\alpha \mid \alpha < \kappa \rangle$, then localizing this information into a subset of ω_1 and then code it further by means of an almost disjoint forcing to a subset of ω.

This forcing is not proper (it is S-proper) but it can be computed it generalizes to larger cardinals.

- **Limit stages $\alpha = i_\alpha$:** If a Π^1_1 diamond sequence guesses at stage α an α-Aronszajn tree $\dot{T}_\alpha$ (note that α has size ω_2 in $V[\mathbb{P}_\alpha]$), then $\dot{Q}_\alpha$ first collapses α to ω_1 and then applies the specialization forcing to $\dot{T}_\alpha$.

This forcing is proper, and most importantly the specialization forcing is ccc.

Intuition. The specialization forcing "seals-off" $\dot{T}_\alpha$: it cannot get a cofinal branch in any outer universe which preserves ω_1. With all other known forcings which force the tree property, $\dot{T}_\alpha$ cannot get a cofinal branch in the further generic extension by a more delicate argument which uses the forcing properties of the iteration (closure, fusion, chain condition): typically, this disqualifies more complex forcing notions which are just distributive.

Open questions and possible generalizations

- It is open how to get the present theorem with $2^\omega > \omega_2$.
- It is open how to get $\mathsf{TP}(\omega_3)$ with a definable well-ordering of $H(\omega_2)$. In fact it is open how to get $\mathsf{TP}(\kappa)$, for $\kappa > \omega_2$, by a forcing which can code some desired information.

- Andrés E. Caicedo and Sy-David Friedman, *BPFA and projective well-orderings of the reals*, The Journal of Symbolic Logic **76** (2011), no. 4, 1126–1136.
- Sy-David Friedman and Victor Torres-Pérez, *The tree property at ω_2 and Bounded Forcing Axioms*, Bul. Pol. Acad. of Sc. Math. **63** (2015), no. 3.
- Martin Goldstern and Saharon Shelah, *The Bounded Proper Forcing Axiom*, The Journal of Symbolic Logic **60** (1995), no. 1, 58–73.