Forcing over choiceless models (2/4)

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Variants of the ccc

We aim for:

- A variant of the ccc the preserves cardinals and cofinalities
- An iteration theorem this variant

Variants of the ccc

The ZFC argument why a ccc (ω_1-cc) forcing preserves ω_1 uses both regularity of ω_1 and the existence of maximal antichains:

- Suppose 1 forces that $\dot{f}\colon \omega \to \omega_1^V$ is surjective.
- Pick a maximal antichain of p_n^i for $i \in \omega$ such that $p_n^i \Vdash \dot{f}(n) = \alpha_n^i$.
- Then $\mathrm{ran}(\dot{f})$ is bounded by $\sup_{n,i\in\omega} \alpha_n^i < \omega_1$.

In ZFC, a forcing has the κ-cc if there exist no antichains of size κ. However, there are other equivalent formulations.

Variants of the ccc

Definition (Karagila, Schweber)

- ccc_1: Every maximal antichain in $\mathbb{P}$ is countable.
- ccc_2: Every antichain in $\mathbb{P}$ is countable.
- ccc_3: Every predense subset of $\mathbb{P}$ contains a countable predense subset.

Moreover, ccc_i^* means ccc_i restricted to wellordered antichains, or predense subsets, of $\mathbb{P}$.

- These notions are equivalent for well-orderable forcings.

Karagila and Schweber showed that the implications

$$\mathrm{ccc}_3 \Rightarrow \mathrm{ccc}_2 \Rightarrow \mathrm{ccc}_1$$

are provable in ZF, but none of these implications can be reversed in $\mathrm{ZF} + \mathrm{DC}$. Moreover, ccc_2 forcings can collapse ω_1.

Exercise

There exists a ccc_2^* forcing collapsing ω_1 if there is no ω_1-sequence of distinct reals.

Variants of the ccc

The following theorem of Bukovsky gives us a new variant of the ccc.

Theorem (Bukovsky)

Suppose that $V \subseteq W$ are models of ZFC. *Then W is a generic extension of V by a ccc forcing if and only if for every $x \in V$ and $f\colon x \to V$ in W, there exists a function $g\colon x \to V$ such that*

1. $V \models |g(u)| < \omega_1$ *for all $u \in x$, and*
2. $W \models f(u) \in g(u)$ *for all $u \in x$.*

Their theorem holds for the κ-cc for other regular κ as well.

Lemma (Karagila, Schweber)

ccc_3 *implies Bukowsky's condition.*

Problem (Karagila, Schweber)

Does Bukowsky's condition imply ccc_3*?*

CHECK ccc_2 for $\mathbb{B}(\mathbb{P})$ implies that ω_1 is preserved (Schilhan, Karagila, S.), but the argument does not generalise to higher chain conditions.

Variants of the ccc

Proposition (Karagila, Schweber)

If $\mathbb{P}$ satisfies Bukovský's condition, then $\mathbb{P}$ preserves any cardinal $\kappa > \omega_1$. If ω_1 is regular, then it is not collapsed.

Proof sketch. Suppose that $\kappa < \lambda$ are cardinals and $f\colon \kappa \to \lambda$ is a surjective function in $V[G]$.

Pick some $F\colon \kappa \to [\lambda]^{<\omega_1}$ such that $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$. Since f is surjective, $\bigcup_{\alpha<\kappa} F(\alpha) = \lambda$.

But $\bigcup_{\alpha<\kappa} F(\alpha)$ has size at most $\kappa \cdot \omega_1 = \kappa$.

If ω_1 is regular, $\kappa = \omega$ and $\lambda = \omega_1$, then $\bigcup_{n<\omega} F(n)$ is countable. $\square$

Problem (Karagila, Schweber)

Is it consistent that a ccc_3 forcing collapses ω_1?

The above variants of the ccc do not seem to suffice.

Linked forcings

We'd like to isolate a variant of the ccc that includes all σ-linked forcings.

Exercise

σ-linked forcings preserve all cardinals.

Definition

A forcing $\mathbb{P}$ is σ-linked (ω-linked) if there exists a (linking) function $f\colon \mathbb{P} \to \omega$ such that for all $p, q \in \mathbb{P}$:

$$f(p) = f(q) \Rightarrow p \parallel q.$$

$\mathbb{P}$ is split into countably many pieces, each one consisting of pairwise compatible conditions.

The definition of κ-linked is analogous.

Example

Hechler forcing is σ-linked:

$$\mathbb{H} := \{(s, f) \mid s \in \omega^{<\omega},\ f \in \omega^{\omega},\ s \subseteq f\}$$

where $(t, g) \leq (s, f)$ if $s \subseteq t$ and $f(n) \leq g(n)$ for all $n \in \omega$.

Every σ-linked forcing satisfies ccc_2.

Problem

Does every σ-linked forcing satisfy ccc_3*?*

Linked forcings

The definition of κ-linked could say

$$p \perp q \Rightarrow f(p) \neq f(q).$$

We equip Ord with the discrete partial order $=$. This suggests a generalisation of κ-linked relative to a forcing $\mathbb{Q}$:

Definition

$\mathbb{P}$ is $\mathbb{Q}$-linked if there exists a $\perp$-homomorphism $f\colon \mathbb{P} \to \mathbb{Q}$, i.e., such that for all $p, q \in \mathbb{P}$

$$p \perp q \Rightarrow f(p) \perp f(q).$$

In ZFC, if $\mathbb{P}$ is $\mathbb{Q}$-linked and $\mathbb{Q}$ is ccc, then $\mathbb{P}$ is ccc.

Linked forcings

Exercise

Well-ordered c.c.c. forcings preserve cardinals.
(To see this, work in **HOD** with the relevant parameters.)

$\mathbb{C} := \{p \mid p\colon n \to 2, n \in \omega\}$ denotes Cohen forcing and $\mathbb{C}^\kappa$ the finite support product of κ many copies. They are well-ordered. This goes further:

Lemma

Suppose that $\mathbb{P}$ is $\mathbb{Q}$-linked and $\mathbb{Q}$ is well-ordered and c.c.c. Then $\mathbb{P}$ preserves all cardinals.

Proof sketch. Suppose that $1_\mathbb{P} \Vdash \dot{f}\colon \omega \to \check{\omega}_1$ is surjective.

Let $g\colon \mathbb{P} \to \mathbb{Q}$ be a $\perp$-homomorphism. Define $q \Vdash^* \varphi \Leftrightarrow \exists p \;\; f(p) = q \wedge p \Vdash \varphi$.

- If $q \Vdash^* \varphi$, $q' \Vdash^* \psi$ and φ, ψ are contradictory, then $q \perp q'$, since
$$p \Vdash \varphi \wedge p' \Vdash \psi \Rightarrow p \perp p' \Rightarrow f(p) \perp f(p').$$
- Let A_n be a maximal antichain of $q \in \mathbb{Q}$ with $q \Vdash^*$ "$\dot{f}(n) = \alpha$"
- This can be done in $M := \mathbf{HOD}_{\{\mathbb{P},\mathbb{Q},\dot{f}\}}$, since $\mathbb{Q} \subseteq M$.
- In M, ω_1^V is regular, $\bigcup_{n\in\omega} A_n$ is countable and $\omega_1^V \leq^* \bigcup_{n\in\omega} A_n$. $\square$

Linked forcings

Exercise

Let $\mathbb{P}_\alpha$ denote α with the discrete partial order. Then $\prod_{\alpha<\omega_1} \mathbb{P}_\alpha$ collapses ω_1.

We therefore need a uniformity requirement on an iteration.

A product or iteration of σ-linked forcings is called uniform if it comes with a sequence of names for linking functions.

Theorem

Any uniform finite support iteration of σ-linked forcings of length κ is $\mathbb{C}^\kappa$-linked.

Hence cardinals are preserved.

Problem

Do Cohen and Hechler models over V have different theories?

- *A Cohen model is a $\mathbb{C}^\kappa$-generic extension for some $\kappa \geq \omega_2$.*
- *A Hechler model is obtained by a finite support iteration of $\mathbb{H}$ of some length $\kappa \geq \omega_2$.*

Linked forcings

Theorem (cont.)

Any uniform finite support iteration of σ-linked forcings of length κ is $\mathbb{C}^\kappa$-linked.

Proof idea. Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\alpha, \dot{f}_\alpha \mid \alpha < \kappa \rangle$ denote such an iteration, where $\dot{f}_\alpha$ is a $\mathbb{P}_\alpha$-name for a σ-linking function for $\dot{\mathbb{P}}_\alpha$.

Show that the set $\tilde{\mathbb{P}}$ of all $p \in \mathbb{P}_\kappa$ such that for all $\alpha \in \mathrm{supp}(p)$, $p{\restriction}\alpha$ decides $\dot{f}_\alpha(p(\alpha))$, is dense.

Use the values of these functions to read off a $\perp$-homomorphism from $\tilde{\mathbb{P}}$ to the set $\mathrm{Fun}_{<\omega}(\kappa, \omega)$ of finite partial functions $p \colon \kappa \to \omega$.

$\mathrm{Fun}_{<\omega}(\kappa, \omega)$ can be densely embedded into $\mathbb{C}^\kappa$. $\square$

Narrow forcings

The following is just the ccc_2^* for $\mathbb{B}(\mathbb{P})$.

Definition

$\mathbb{P}$ is called $(\omega, 1)$-narrow if all partial $\|$-homomorphisms $f\colon \mathbb{P} \rightharpoonup \mathrm{Ord}$ have countable range.

- A partial $\|$-homomorphism f corresponds to a function on the set D all $p \in \mathbb{P}$ deciding a statement, for instance $p \Vdash \dot{g}(n) = \alpha_p$. f sends $p \in D$ to α_p.
- A partial $\|$-homomorphism f can be thought of a generalised antichain consisting of "blocks" $f^{-1}(\alpha)$. Different blocks are incompatible.
- In a complete Boolean algebra, a partial $\|$-homomorphism corresponds to an antichain, since subsets A and B of $\mathbb{P}$ are elementwise incompatible if and only if $\sup(A)$ is incompatible with $\sup(B)$.

However, when trying to prove cardinal preservation via a function $\dot{f}\colon \omega \to \omega_1$, an ω-sequence of such homomorphisms appears.

This is captured by a uniform version of ccc_2^* for many homomorphisms.

Narrow forcings

Definition

1. Suppose that ν is an ordinal.

   $\mathbb{P}$ is called (ω, ν)-narrow if for any sequence $\vec{f} = \langle f_i \mid i < \mu \rangle$ of partial $\|$-homomorphisms $f_i \colon \mathbb{P} \to \mathrm{Ord}$, where $\mu \leq \nu$,

$$|\bigcup_{i<\mu} \mathrm{ran}(f_i)| \leq |\max(\omega, \mu)|.$$

2. $\mathbb{P}$ is called ω-narrow or just narrow if it is (ω, ν)-narrow for all ν.

Exercise

$(\omega, 1)$-narrow implies (ω, ν)-narrow for all $\nu \geq \omega_1$.

Narrow forcings

Lemma

Every $(\omega, 1)$-narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities $\geq \omega_2$.

Proof sketch. Let $\lambda \geq \omega_2$ be a cardinal.

Suppose that $\mu < \lambda$ is a cardinal and $p \Vdash_{\mathbb{P}}$ "$\dot{f} \colon \mu \to \lambda$ is surjective".

- For each $\alpha < \mu$, let D_α be the set of $q \leq p$ deciding $\dot{f}(\alpha)$.
- Let $f_\alpha \colon D_\alpha \to \lambda$ send q to the unique $\beta < \lambda$ with $q \Vdash \dot{f}(\alpha) = \beta$.
- Each f_α is a partial $\|$-homomorphism.

Since $\mathbb{P}$ is $(\omega, 1)$-narrow, $\operatorname{otp}(\operatorname{ran}(f_\alpha)) < \omega_1$ for each $\alpha < \mu$. Hence

$$|\bigcup_{\alpha<\mu} \operatorname{ran}(f_\alpha)| \leq |\max(\omega_1, \mu)| < \lambda.$$

But $\bigcup_{\alpha<\mu} \operatorname{ran}(f_\alpha) = \lambda$.

A similar argument works for cofinalities. $\square$

Lemma

Every narrow forcing $\mathbb{P}$ preserves all cardinals and cofinalities.

Proof. It suffices to show that $\mathbb{P}$ preserves ω_1.

Suppose that $p \Vdash_{\mathbb{P}}$ "$\dot{f}\colon \omega \to \omega_1$ is surjective".

- For each $n < \omega$, let D_n denote the set of $q \leq p$ deciding $\dot{f}(n)$.
- Let $f_n : D_n \to \omega_1$ send q to the unique $\beta < \omega_1$ with $q \Vdash \dot{f}(n) = \beta$.
- Since $\mathbb{P}$ is narrow, we have $|\bigcup_{n<\omega} \mathrm{ran}(f_n)| \leq \omega$. But $\bigcup_{n<\omega} \mathrm{ran}(f_n) = \omega_1$.

A similar argument works for preserving cofinality ω_1. $\square$

Exercise

Every σ-linked forcing is $(\omega, 1)$-narrow. (Uses the next lemma.)

Lemma

If $\mathbb{Q}$ is $(\omega, 1)$-narrow and $f\colon \mathbb{P} \to \mathbb{Q}$ is a $\perp$-homomorphism, then $\mathbb{P}$ is $(\omega, 1)$-narrow.

Proof. Suppose that $g\colon \mathbb{P} \rightharpoonup \mathrm{Ord}$ is a partial $\|$-homomorphism.

Let $D := \mathrm{ran}(f)$ and define $h\colon D \to \mathrm{Ord}$ as follows.

- For all $p, r \in \mathbb{P}$ with $f(p) = f(r)$, we have $g(p) = g(r)$, since f is a $\perp$-homomorphism and g is a $\|$-homomorphism.
- For $f(p) = q \in D$, we can thus define $h(q) := g(p)$.

We claim that h is a partial $\|$-homomorphism.

- Suppose that $q, s \in D$ with $f(p) = q$, $f(r) = s$ and $q \parallel s$.
- Since f is a $\perp$-homomorphism, $p \parallel r$.
- Since g is a $\|$-homomorphism, $h(q) = g(p) \parallel g(r) = h(s)$ as desired.

Since $\mathrm{ran}(g) = \mathrm{ran}(h)$ and $\mathbb{Q}$ is $(\omega, 1)$-narrow, the claim follows. □

Narrow forcings

We need a stronger variant of narrow and a uniformity requirement for an iteration.

Definition

$\mathbb{P}$ is called uniformly narrow if there exists a function G that sends each partial $\|$-homomorphism $f\colon \mathbb{P} \rightharpoonup \mathrm{Ord}$ to an injective function $G(f)\colon \mathrm{ran}(f) \to \omega$.

A uniform iteration comes with a sequence of functions G_α.

Theorem

A uniform iteration of uniformly narrow forcings with finite support is again uniformly narrow.

Example

One can iterate combinations of $\mathbb{C}^\kappa$, σ-linked forcings such as Hechler forcing or eventually different forcing and (as we see later) random algebras, while preserving cardinals and cofinalities.

Theorem (cont.)

A uniform iteration of uniformly narrow forcings with finite support is again uniformly narrow.

Proof. Let $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha, \dot{\mathbb{P}}_\beta, \dot{g}_\beta \mid \alpha \leq \delta, \beta < \delta \rangle$ denote the iteration.

We construct a sequence $\langle G_\gamma \mid \gamma \leq \delta \rangle$ of functions by recursion on $\gamma \leq \delta$ from $\vec{\mathbb{P}}$ and θ, where G_γ witnesses that $\mathbb{P}_\gamma$ is uniformly narrow.

Case. γ is a successor.

Suppose that $\gamma = \beta + 1$ and G_β has been constructed. Let $f \colon \mathbb{P}_\beta * \dot{\mathbb{P}}_\beta \rightharpoonup \mathrm{Ord}$ be a partial $\parallel$-homomorphism. and

$$\dot{f} := \{((\dot{q}, \check{\alpha})^\bullet, p) \mid f(p, \dot{q}) = \alpha\}.$$

Claim

$1_{\mathbb{P}_\beta}$ forces that $\dot{f}$ is a partial $\parallel$-homomorphism on $\dot{\mathbb{P}}_\beta$.

$$\dot{f} := \{((\dot{q}, \check{\alpha})^\bullet, p) \mid f(p, \dot{q}) = \alpha\}.$$

Claim (cont.)
$1_{\mathbb{P}_\beta}$ forces that $\dot{f}$ is a partial $\|$-homomorphism on $\dot{\mathbb{P}}_\beta$.

Proof. Suppose that G is $\mathbb{P}_\beta$-generic over V.

In $V[G]$, take $q_0, q_1 \in \dot{\mathbb{P}}_\beta^G$ with $\dot{f}^G(q_i) = \alpha_i$ for $i < 2$. Suppose that $q_0 \| q_1$.

- There exist $\dot{q}_i$ with $\dot{q}_i^G = q_i$ and $p_i \in G$ with $((\dot{q}_i, \check{\alpha}_i)^\bullet, p_i) \in \dot{f}$ for $i < 2$.
- Since $q_0 \| q_1$, some $p \in G$ forces $\dot{q}_0 \| \dot{q}_1$.
- Since we can assume $p \leq p_0, p_1$, we have $(p_0, \dot{q}_0) \| (p_1, \dot{q}_1)$.
- $\alpha_0 = f(p_0, \dot{q}_0) = f(p_1, \dot{q}_1) = \alpha_1$, since f is a $\|$-homomorphism. $\square$

Claim (cont.)

$1_{\mathbb{P}_\beta}$ forces that $\dot{f}$ is a partial $\|$-homomorphism on $\dot{\mathbb{P}}_\beta$.

By the claim,

$$1 \Vdash_{\mathbb{P}_\beta} \dot{g}_\beta(\dot{f}) \colon \operatorname{ran}(\dot{f}) \to \omega \text{ is injective.}$$

We can read off a $\mathbb{P}_\beta$-name $\dot{h}$ for a function extending $\dot{g}_\beta(\dot{f})^{-1}$. Then

$$1 \Vdash_{\mathbb{P}_\beta} \dot{h} \colon \omega \to \operatorname{ran}(\dot{f}) \text{ is surjective.}$$

For each $n < \omega$, let D_n denote the set of all $p \in \mathbb{P}_\beta$ that decide $\dot{h}(n)$.

Let $h_n \colon D_n \to \mathrm{Ord}$, where $h_n(p)$ is the unique δ such that $p \Vdash \dot{h}(n) = \delta$.
h_n is a $\|$-homomorphism.

Since G_β witnesses that $\mathbb{P}_\beta$ is uniformly narrow, $\langle G_\beta(h_n) \mid n < \omega \rangle$ consists of injective functions $G_\beta(h_n) \colon \operatorname{ran}(h_n) \to \omega$.

Glue them to an injective function $i \colon \bigcup_{n<\omega} \operatorname{ran}(h_n) \to \omega$.

Since $1_{\mathbb{P}} \Vdash \operatorname{ran}(\dot{f}) \subseteq \bigcup_{\alpha<\theta} \operatorname{ran}(h_\alpha)$, $\operatorname{ran}(f) \subseteq \bigcup_{\alpha<\theta} \operatorname{ran}(h_\alpha)$ by the definition of $\dot{f}$.

Thus $i \restriction \operatorname{ran}(f) \to \theta$ is injective. Let $G_\gamma(f) := i \restriction \operatorname{ran}(f)$.

Narrow forcings

Case. γ is a limit.

Suppose that $f\colon \mathbb{P}_\gamma \rightharpoonup \mathrm{Ord}$ is a partial $\|$-homomorphism.

It suffices to show $\mathrm{HOD}_{\vec{\mathbb{P}},f} \models \mathrm{ran}(f) \leq \theta$. Then take the least injective function $G_\gamma(f)\colon \mathrm{ran}(f) \to \theta$ in $\mathrm{HOD}_{\vec{\mathbb{P}},f}$. Work in $\mathrm{HOD}_{\vec{\mathbb{P}},f}$.

Otherwise $\mathrm{ran}(f) > \theta$. We can assume $\mathrm{ran}(f) = \theta^+$ by restricting f.

Let $s_\alpha \in [\gamma]^{<\omega}$ for $\alpha \in \mathrm{ran}(f)$ be least in $[\mathrm{Ord}]^{<\omega}$ such that there exists some $p \in \mathbb{P}_\gamma$ with support s_α and $f(p) = \alpha$. Let $\vec{s} = \langle s_\alpha \mid \alpha \in \mathrm{ran}(f)\rangle$.

We can assume:

- All $p \in \mathbb{P}_\gamma$ with $f(p) = \alpha$ have support s_α.
- $\vec{s}$ forms a Δ-system with root r.

Fix $\gamma' < \gamma$ such that $\alpha + 1 < \gamma_0$ for all $\alpha \in r$. Let $D := \{p{\restriction}\gamma' \mid p \in \mathbf{dom}(f)\}$ be the projection of $\mathbf{dom}(f)$ to $\mathbb{P}_{\gamma'}$.

Let $g\colon D \to \mathrm{Ord}$, where $g(p) := \alpha$ if

$$\exists q \in \mathbf{dom}(f) \;\; (q{\restriction}\gamma' = p \wedge f(q) = \alpha)$$

g well-defined by the next claim.

Narrow forcings

Recall $g\colon D \to \mathrm{Ord}$, $g(p) := \alpha$ if $\exists q \in \mathbf{dom}(f)\ \ (q\restriction\gamma' = p \wedge f(q) = \alpha)$.

Claim

If $u, v \in \mathbf{dom}(f)$ with $u\restriction\gamma' = v\restriction\gamma' = p \in D$, then $f(u) = f(v)$.

Claim

$g\colon \mathbb{P}_\beta \rightharpoonup \mathrm{Ord}$ is a partial $\|$-homomorphism.

Claim

$\mathrm{ran}(f) = \mathrm{ran}(g)$.

The inductive hypothesis for γ' yields an injective function $G_{\gamma'}(g)\colon \mathrm{ran}(g) \to \theta$. Since $G_{\gamma'}, g \in \mathbf{HOD}_{\vec{\mathbb{P}},f}$, we have $\mathbf{HOD}_{\vec{\mathbb{P}},f} \models \mathrm{ran}(f) = \mathrm{ran}(g) \leq \theta$, contradicting the assumption.

We're done! $\square$

A counterexample with CCC_2

The next result uses a standard technique for symmetric models.

Let $\mathcal{L}$ be a first-order language and M an $\mathcal{L}$-structure. Suppose that $\mathscr{G} \subseteq \mathrm{Aut}(M)$ is a group and $\mathscr{I}$ an ideal of subsets of M.

- A subgroup of $\mathscr{G}$ is called large if it contains $\mathrm{fix}(A) = \{\pi \in \mathscr{G} \mid \pi \restriction A = \mathrm{id}\}$ for some $A \in \mathscr{I}$.
- A subset X of M is called stable if there exists a large subgroup $\mathscr{H}$ of $\mathscr{G}$ such that $\pi[X] = X$ for all $\pi \in \mathscr{H}$.

Theorem (Karagila, Schweber)

In a model of ZFC, let $\mathcal{L}$, M, $\mathscr{G}$ and $\mathscr{I}$ be as above. There is a symmetric extension of the universe in which there exists an isomorphic copy N of M such that every subset of N^k in the symmetric extension is a stable isomorphic copy of a subset of M^k.

In addition, we can require:

- $\mathrm{DC}_{<\kappa}$ *holds in the extension, if $\mathscr{I}$ is $<\kappa$-complete.*
- *The extension has no new λ-sequences for any prescribed cardinal λ.*

A counterexample with ccc_2

Theorem (Karagila, Schweber)
It is consistent with ZF + DC *that there is a* ccc_2 *forcing which collapses* ω_1.

Proof sketch. We construct a symmetric model over a model of ZFC. Let $\mathbb{P}$ denote $\mathrm{Add}(\omega, \omega_1)$ without 1. $\mathbb{P}$ is productively c.c.c.

$\mathbb{P}_\infty := \bigoplus_{\langle n,\alpha\rangle \in \omega\times\omega_1} \mathbb{P}_{n,\alpha}$ is the lottery sum, where each $\mathbb{P}_{n,\alpha} \cong \mathbb{P}$.

Let $\mathscr{G}$ act on each $\mathbb{P}_{n,\alpha}$ individually for countably many $\langle n, \alpha\rangle$ at the same time. Let $\mathscr{I}$ be the ideal of countable subsets of $\mathbb{P}_\infty$.

We get a symmetric extension M of V and working in M, an isomorphic copy of $\mathbb{P}_\infty$, such that M is a model of DC and ω_1 remains uncountable in M.

We use the same notation for the copies.

For any subset A of $\mathbb{P}_\infty^k$, there is a countable $\alpha < \omega_1$ such that if $\alpha \leq \beta$ and $p(i) \in \mathbb{P}_{n,\beta}$ for any $p \in A^k$, $i < k$ and $n \in \omega$, then any condition q obtained by replacing $p(i)$ by an arbitrary condition in $\mathbb{P}_{n,\beta}$ is in A.

A counterexample with CCC_2

In N, $\mathbb{Q}$ consists of pairs $\langle t, \vec{b} \rangle$ such that:

1. $t \in \omega_1^{<\omega}$ and $\mathbf{dom}(t) = n$.
2. $\vec{b} = \langle b_0, \ldots, b_{n-1} \rangle$ and $b_i \in \mathbb{P}_{i,t(i)}$.

Let $\langle t, \vec{b} \rangle \leq \langle t', \vec{b}' \rangle$ if:

1. $t' \subseteq t$.
2. For all $i \in \mathbf{dom}(t')$, $b_i \leq_{n,\alpha} b'_i$.

This two-step iteration first adds a surjection $f \colon \omega \to \omega_1$ and then forces with the product $\prod_{\langle n,\alpha \rangle} \mathbb{P}_{n,\alpha}$. Forcing with $\mathbb{Q}$ collapses ω_1.

A counterexample with ccc_2

To see that every antichain in $\mathbb{Q}$ is countable, let π be the projection of $\mathbb{Q}$ to $\omega_1^{<\omega}$ and $\pi_{n,\alpha}$ the projection to $\mathbb{P}_{n,\alpha}$.

Let D be an uncountable subset of $\mathbb{Q}$.

It suffices to show that $\pi^{-1}(t) \cap D$ is uncountable for some $t \in \omega_1^{<\omega}$, since it is a subset of $\{t\} \times \prod_{i \in \mathbf{dom}(t)} \mathbb{P}_{i,t(i)}$ and $\mathbb{P} = \mathrm{Add}(\omega, \omega_1)$ is productively ccc.

Case

$\pi(D)$ is countable. Then by DC, there exists some $t \in \omega_1^{<\omega}$ such that $\pi^{-1}(t) \cap D$ is uncountable.

Case

$\pi(D)$ is uncountable. We can assume that for some $k \in \omega$, $\mathbf{dom}(t) = k$ for all $t \in \pi(D)$ by shrinking D. We can then identify D with a subset of $\mathbb{P}^k_\infty$.

- Pick $\alpha < \omega_1$ as above by stability of D.
- Since $\pi(D)$ is uncountable, there exists some $t \in \pi(D)$ with $t(i) \geq \alpha$ for some $i < k$. Then $\pi^{-1}(t) \cap D$ is uncountable.