Perfect Set Games and Colorings on Generalized Baire Spaces

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Perfectness for the κ-Baire space

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Example

A subset of a topological space is perfect in the usual sense iff it is closed and contains no isolated points.

$X_\omega = \{x \in {}^{\kappa}2 : |\{\alpha < \kappa : x(\alpha) = 0\}| < \omega\}$ is perfect in this usual sense, but $|X_\omega| = \kappa$.

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Definition

A subtree T of ${}^{<\kappa}\kappa$ is a strongly κ-perfect tree if T is $<\kappa$-closed and every node of T extends to a splitting node.

A set $X \subseteq {}^{\kappa}\kappa$ is a strongly κ-perfect set if $X = [T]$ for a strongly κ-perfect tree T.

Väänänen's perfect set game

Let $X \subseteq {}^\kappa\kappa$, let $x_0 \in {}^\kappa\kappa$ and let $\omega \leq \gamma \leq \kappa$.

Definition (Väänänen, 1991)

The game $\mathcal{V}_\gamma(X, x_0)$ has length γ and is played as follows:

$$\begin{array}{lccccccccc} \mathbf{I} & & U_1 & & \ldots & & U_\alpha & & \ldots & \\ \mathbf{II} & x_0 & & x_1 & & \ldots & & x_\alpha & & \ldots \end{array}$$

$\mathbf{II}$ first plays x_0. In each round $0 < \alpha < \gamma$, $\mathbf{I}$ plays a basic open subset U_α of X, and then $\mathbf{II}$ chooses

$$x_\alpha \in U_\alpha \text{ with } x_\alpha \neq x_\beta \text{ for all } \beta < \alpha.$$

$\mathbf{I}$ has to play so that $U_{\beta+1} \ni x_\beta$ in each successor round $\beta + 1 < \gamma$ and $U_\alpha = \bigcap_{\beta<\alpha} U_\beta$ in each limit round $\alpha < \gamma$.

$\mathbf{II}$ wins a given run of the game if $\mathbf{II}$ can play legally in all rounds $\alpha < \gamma$.

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Perfect and scattered subsets of the κ-Baire space

Let $X \subseteq {}^\kappa\kappa$, and suppose $\omega \leq \gamma \leq \kappa$.

Definition (Väänänen, 1991)

X is a γ-scattered set if $\mathbf{I}$ wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

X is a γ-perfect set if X is closed and $\mathbf{II}$ wins $\mathcal{V}_\gamma(X, x_0)$ for all $x_0 \in X$.

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- X is ω-perfect iff X is perfect in the usual sense (i.e., iff X closed and has no isolated points).
- X is ω-scattered iff X is scattered in the usual sense (i.e., each nonempty subspace contains an isolated point).
- $\mathcal{V}_\gamma(X, x_0)$ may not be determined when $\gamma > \omega$.

κ-perfect sets vs. strongly κ-perfect sets

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Example (Huuskonen)

The following set is κ-perfect but is not strongly κ-perfect:

$$Y_\omega = \{x \in {}^\kappa 3 : |\{\alpha < \kappa : x(\alpha) = 2\}| < \omega\}.$$

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Proposition

Let X be a closed subset of ${}^\kappa\kappa$.

$$X \text{ is } \kappa\text{-perfect} \iff X = \bigcup_{i \in I} X_i \text{ for strongly } \kappa\text{-perfect sets } X_i.$$

Väänänen's generalized Cantor-Bendixson theorem

Theorem (Väänänen, 1991)

The following Cantor-Bendixson theorem for ${}^{\kappa}\kappa$ is consistent relative to the existence of a measurable cardinal $\lambda > \kappa$:

Every closed subset of ${}^{\kappa}\kappa$ is the (disjoint) union of

a κ-perfect set and a κ-scattered set, which is of size $\leq \kappa$.

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Theorem (Galgon, 2016)

Väänänen's generalized Cantor-Bendixson theorem is consistent relative to the existence of an inaccessible cardinal $\lambda > \kappa$.

Väänänen's generalized Cantor-Bendixson theorem

Proposition (Sz)

Väänänen's generalized Cantor-Bendixson theorem is equivalent to the κ-perfect set property for closed subsets of ${}^\kappa\kappa$ (i.e, the statement that every closed subset of ${}^\kappa\kappa$ of size $> \kappa$ has a κ-perfect subset).

Remark: The κ-PSP for closed subsets of ${}^\kappa\kappa$ is equiconsistent with the existence of an inaccessible cardinal $\lambda > \kappa$.

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Sketch of the proof.

Let X be a closed subset of ${}^{\kappa}\kappa$. Its set of κ-condensation points is defined to be

$$CP_{\kappa}(X) = \{x \in X : |X \cap N_{x \restriction \alpha}| > \kappa \text{ for all } \alpha < \kappa\}.$$

If the κ-PSP holds for closed subsets of ${}^{\kappa}\kappa$, then $CP_{\kappa}(X)$ is a κ-perfect set and $X - CP_{\kappa}(X)$ is a κ-scattered set of size $\leq \kappa$. $\square$

Perfect and scattered trees

Let T be a subtree of ${}^{<\kappa}2$, let $t \in T$, and let $\omega \leq \gamma \leq \kappa$.

Definition (Galgon, 2016)

The game $\mathcal{G}_\gamma(T,t)$ has length γ and is played as follows:

$$\begin{array}{c|ccccccccc} \mathbf{I} & \delta_0 & & i_0 & & \dots & \delta_\alpha & & i_\alpha & \dots \\ \mathbf{II} & & t_0 & & & \dots & & t_\alpha & & \dots \end{array}$$

In each round $\alpha < \gamma$, player $\mathbf{I}$ first plays $\delta_\alpha < \kappa$. Then $\mathbf{II}$ plays a node $t_\alpha \in T$ of height $\geq \delta_\alpha$, and $\mathbf{I}$ chooses $i_\alpha < 2$.

$\mathbf{II}$ has to play so that $t \subseteq t_0$, and $t_\beta {}^\frown \langle i_\beta \rangle \subseteq t_\alpha$ for all $\beta < \alpha < \gamma$.

$\mathbf{II}$ wins a given run of the game if $\mathbf{II}$ can play legally in all rounds $\alpha < \gamma$.

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Perfect and scattered trees

Definition (Galgon, 2016)

T is a γ-scattered tree if player $\mathbf{I}$ wins $\mathcal{G}_\gamma(T,t)$ for all $t \in T$.

T is a γ-perfect tree if player $\mathbf{II}$ wins $\mathcal{G}_\gamma(T,t)$ for all $t \in T$.

Proposition

Let T be a subtree of ${}^{<\kappa}\kappa$.

1. T *is a* κ*-perfect tree* $\iff$ $[T]$ *is a* κ*-perfect set.*

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Proposition

Let T be a subtree of ${}^{<\kappa}\kappa$.

1. $$T \text{ is a } \kappa\text{-perfect tree} \iff [T] \text{ is a } \kappa\text{-perfect set.}$$
2. *If the κ-PSP holds for closed subsets of ${}^{\kappa}\kappa$, then*
$$T \text{ is a } \kappa\text{-scattered tree} \iff [T] \text{ is a } \kappa\text{-scattered set.}$$

γ-perfect sets and trees when $\gamma \leq \kappa$

Theorem (Sz)

Let T be a subtree of ${}^{<\kappa}\kappa$ and let $\omega \leq \gamma \leq \kappa$.

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2. If T is a γ-scattered tree, then $[T]$ is a γ-scattered set.
3. If κ is weakly compact and $T \subseteq {}^{<\kappa}2$, then

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$$T \text{ is a } \gamma\text{-perfect tree} \quad \Longleftrightarrow \quad [T] \text{ is a } \gamma\text{-perfect set.}$$

More generally: this holds if κ has the tree property and T is a κ-tree.

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Question

Is it consistent that 3 holds for "scattered" instead of "perfect"?

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Analogues of 1–3 hold for levels of "generalized Cantor-Bendixson hierarchies" associated to subsets of ${}^{\kappa}\kappa$ and to subtrees of ${}^{<\kappa}\kappa$ (see next 4 slides).

Generalizing the Cantor-Bendixson hierarchy

Let $X \subseteq {}^\kappa\kappa$, let $x_0 \in {}^\kappa\kappa$, and let S be a tree without branches of length $\geq \kappa$.

Definition (Hyttinen; Väänänen)

The S-approximation $\mathcal{V}_S(X, x_0)$ of $\mathcal{V}_\kappa(X, x_0)$ is the following game.

I		s_1, U_1		$\dots$		s_α, U_α	$\dots$
II	x_0		x_1		$\dots$	x_α	$\dots$

In each round $\alpha > 0$, **I** first plays $s_\alpha \in S$ such that $s_\alpha >_S s_\beta$ for all $0 < \beta < \alpha$. Then **I** plays U_α and **II** plays x_α according to the same rules as in $\mathcal{V}_\kappa(X, x_0)$.

The first player who can not move loses, and the other player wins.

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The first player who can not move loses, and the other player wins. Let

$$\mathrm{Sc}_S(X) = \{x \in X : \mathbf{I} \text{ wins } \mathcal{V}_S(X, x)\};$$
$$\mathrm{Ker}_S(X) = \{x \in {}^\kappa\kappa : \mathbf{II} \text{ wins } \mathcal{V}_S(X, x)\}.$$

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Given an ordinal α, let $B_\alpha =$ the tree of descending sequences of elements of α. $X^{(\alpha)}$ denotes the α^{th} Cantor-Bendixson derivative of X.

Observation 1 (Väänänen)

$X^{(\alpha)} = X \cap \mathrm{Ker}_{B_\alpha}(X) = X - \mathrm{Sc}_{B_\alpha}(X).$

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Corollary

$$\mathrm{Ker}_\omega(X) = \bigcap\{\mathrm{Ker}_S(X) : \textit{ S is a tree without infinite branches}\};$$
$$\mathrm{Sc}_\omega(X) = \bigcup\{\mathrm{Sc}_S(X) : \textit{ S is a tree without infinite branches}\}.$$

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Theorem 2 (Hyttinen (1990); Väänänen (1991))

$$\mathrm{Ker}_\kappa(X) = \bigcap\{\mathrm{Ker}_S(X) : S \text{ is a tree without branches of length } \geq \kappa\};$$
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The sets $X \cap \mathrm{Ker}_S(X)$ (resp. $X - \mathrm{Sc}_S(X)$) can be seen as the "levels of a generalized Cantor-Bendixson hierarchy" for the set X associated to $\mathbf{II}$ (resp. $\mathbf{I}$).

Generalizing the Cantor-Bendixson hierarchy for trees

Theorem (Sz, part 1)

There exists a family $\{\mathcal{G}'_\gamma(T,t) : T$ *is a subtree of* ${}^{<\kappa}\kappa$, $t \in T$ *and* $\omega \leq \gamma \leq \kappa\}$ *of games such that the following hold for all such* T, t *and* γ.

- *The games* $\mathcal{G}'_\gamma(T,t)$ *and* $\mathcal{G}_\gamma(T,t)$ *are equivalent whenever* $T \subseteq {}^{<\kappa}2$.
- *Given a tree* S *without branches of length* $\geq \kappa$, *let* $\mathcal{G}'_S(T,t)$ *denote the* S*-approximation of* $\mathcal{G}'_\kappa(T,t)$,[1] *and let*

  $$\mathrm{Sc}_S(T) = \{t \in T : \mathbf{I} \text{ wins } \mathcal{G}'_S(T,t)\}; \quad \mathrm{Ker}_S(T) = \{t \in T : \mathbf{II} \text{ wins } \mathcal{G}'_S(T,t)\}.$$

  Then the analogues of Observation 1 and Theorem 2 hold.[2]

The analogue of Theorem 2 is a special case of a general theorem due to Hyttinen (1990).

[1] This is defined analogously to the S-approximation $\mathcal{V}_S(T,x)$.

[2] We consider the Cantor-Bendixson derivative of subtrees T of ${}^{<\kappa}\kappa$ which was defined in: G. Galgon. *Trees, refining, and combinatorial characteristics.* PhD thesis, University of California, Irvine, 2016.

Comparing the Cantor-Bendixson hierarchies

$\mathrm{Sc}_S(T) = \{t \in T : \mathbf{I} \text{ wins } \mathcal{G}'_S(T,t)\}$; $\quad \mathrm{Ker}_S(T) = \{t \in T : \mathbf{II} \text{ wins } \mathcal{G}'_S(T,t)\}$.

$\mathrm{Sc}_S(X) = \{x \in X : \mathbf{I} \text{ wins } \mathcal{V}_S(X,x)\}$; $\quad \mathrm{Ker}_S(X) = \{x \in {}^\kappa\kappa : \mathbf{II} \text{ wins } \mathcal{V}_S(X,x)\}$.

Theorem (Sz, part 2)

- *Let S be a tree without branches of length $\geq \kappa$. Then*
  1. $\mathrm{Ker}_S([T]) \subseteq [\mathrm{Ker}_S(T)]$

     (i.e., if $\mathbf{II}$ wins $\mathcal{V}_S([T],x)$ then $\mathbf{II}$ wins $\mathcal{G}'_S(T,t)$ when $t \subsetneq x \in {}^\kappa\kappa$.)

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  2. $[T] - \mathrm{Sc}_S([T]) \subseteq [T - \mathrm{Sc}_S(T)]$
     (i.e., if $\mathbf{I}$ wins $\mathcal{G}'_S(T,t)$ then $\mathbf{I}$ wins $\mathcal{V}_S([T],x)$ when $t \subsetneq x \in {}^\kappa\kappa$).

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  3. *If κ has the tree property and T is a κ-tree, then*

     $$\mathrm{Ker}_S([T]) = [\mathrm{Ker}_S(T)]$$

     (i.e., $\mathcal{V}_S([T],x)$ and $\mathcal{G}'_S(T,t)$ are equivalent for $\mathbf{II}$ when $t \subsetneq x \in {}^\kappa\kappa$).

Density in itself for the κ-Baire space

Definition

A subset $X \subseteq {}^{\kappa}\kappa$ is κ-dense in itself if $\overline{X}$ is a κ-perfect set.

A subset $X \subseteq {}^{\kappa}\kappa$ is strongly κ-dense in itself if $\overline{X}$ is a strongly κ-perfect set.

Density in itself for the κ-Baire space

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Proposition (Sz)

The following are equivalent for any $X \subseteq {}^{\kappa}\kappa$.

- X *is κ-dense in itself.*
- $X = \bigcup_{i \in I} X_i$ *where each X_i is strongly κ-dense in itself.*
- $X \subseteq \mathrm{Ker}_\kappa(X)$ *(i.e., player $\mathbf{II}$ wins $\mathcal{V}_\kappa(X, x)$ for all $x \in X$.)*

Density in itself for the κ-Baire space

Theorem (Väänänen, 1991)

If $\lambda > \kappa$ is measurable and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic, then in $V[G]$,

$$\text{every subset of } {}^\kappa\kappa \text{ of size } \kappa^+ \text{ contains a } \kappa\text{-dense in itself subset.} \tag{1}$$

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Question

What is the consistency strength of (1)?

A Cantor-Bendixson theorem for G_δ relations

$\mathcal{R}$ is a collection of finitary relations on a set X.

$Y \subseteq X$ is $\mathcal{R}$-*homogeneous* if for all $1 \leq k < \omega$ and k-ary $R \in \mathcal{R}$ we have:

$(x_1, \dots, x_k) \in R$ for all pairwise distinct $x_1, \dots, x_k \in Y$.

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Theorem (Kubiś, 2003; Doležal, Kubiś 2015)

Let $\mathcal{R}$ be a countable set of G_δ relations on a Polish space X (i.e., every $R \in \mathcal{R}$ is an G_δ subset of ${}^k X$ for some $1 \leq k < \omega$).

1. *Either there exists a perfect $\mathcal{R}$-homogeneous set, or there exists $\alpha < \omega_1$ such that every $\mathcal{R}$-homogeneous set Y has Cantor-Bendixson rank $< \alpha$ (i.e., $Y^{(\alpha)} = \emptyset$).*

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2. *If there exists an uncountable $\mathcal{R}$-homogeneous set, then there exists a perfect $\mathcal{R}$-homogeneous set.*

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Recall that $\quad Y^{(\alpha)} = Y \cap \mathrm{Ker}_{B_\alpha}(Y) = Y - \mathrm{Sc}_{B_\alpha}(Y).$

A dichotomy for infinitely many $\mathbf{\Pi}^0_2(\kappa)$ relations

R is a $\mathbf{\Pi}^0_2(\kappa)$ *relation* on a topological space X iff
R is an intersection of $\leq \kappa$ many open subsets of ${}^k X$ for some $1 \leq k < \omega$.

Theorem (Sz)

Assume $\diamondsuit_\kappa$ or κ is inaccessible.
Let $\mathcal{R}$ be a collection of $\leq \kappa$ many $\mathbf{\Pi}^0_2(\kappa)$ relations on a closed subset X of ${}^\kappa\kappa$. Then either

- *X has a κ-perfect $\mathcal{R}$-homogeneous subset, or*
- *there exists a tree T without κ-branches, $|T| \leq 2^\kappa$, such that for all $\mathcal{R}$-homogeneous $Y \subseteq X$, we have $Y \cap \mathrm{Ker}_T(Y) = \emptyset$ (that is, player $\mathbf{II}$ does not win $\mathcal{V}_T(Y, y)$ for any $y \in Y$).*

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Sketch of the proof

First, show that if X has a κ-dense in itself $\mathcal{R}$-homogeneous subset, then X has a κ-perfect $\mathcal{R}$-homogeneous subset.

Step 2

Let $\mathcal{R}$ be an arbitrary set of finitary relations on ${}^{\kappa}\kappa$.

Lemma

If X does not have a κ-dense in itself $\mathcal{R}$-homogeneous subset, then there exists a tree T without κ-branches, $|T| \leq 2^{\kappa}$, such that $Y \cap \mathrm{Ker}_T(Y) = \emptyset$ holds for all $\mathcal{R}$-homogeneous $Y \subseteq X$.

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Proof.

The assumption holds iff $\mathbf{II}$ does not win $\mathcal{V}_{\kappa}(Y, x)$ for any $\mathcal{R}$-homogeneous $Y \subseteq {}^{\kappa}\kappa$ and $x \in Y$.

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$T_0 =$ the tree of winning strategies τ of $\mathbf{II}$ in short games $\mathcal{V}_{\delta}(X, x)$ (where $\delta < \kappa$ and $x \in X$) such that the set of all possible τ-moves of $\mathbf{II}$ $\mathcal{R}$-homogeneous.

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$T = \sigma T_0$, the tree of ascending chains in T_0. $\square$

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Remark

If all $\mathcal{R}$-homogeneous sets $Y \subseteq X$ are κ-scattered (i.e., $\mathbf{I}$ wins $\mathcal{V}_{\kappa}(Y, y)$ for all $y \in Y$), then there exists a tree S without κ-branches, $|S| \leq 2^{\kappa}$ such that $\mathrm{Sc}_S(Y) = Y$ (i.e., $\mathbf{I}$ wins $\mathcal{V}_S(Y, y)$ for all $y \in Y$) for all $\mathcal{R}$-homogeneous $Y \subseteq {}^{\kappa}\kappa$.

A corollary

Corollary

If $\lambda > \kappa$ is weakly compact, and G is $\mathrm{Col}(\kappa, {<}\lambda)$-generic, then in $V[G]$:

Let X be a $\mathbf{\Sigma}^1_1(\kappa)$ subset of ${}^\kappa\kappa$, let $\mathcal{R}$ a set of $\leq \kappa$ many $\mathbf{\Pi}^0_2(\kappa)$ relations on X.

$$\text{If } X \text{ has an } \mathcal{R}\text{-homogeneous subset of size} > \kappa, \text{ then } X \text{ has a } \kappa\text{-perfect } \mathcal{R}\text{-homogeneous subset.} \tag{2}$$

- This was known for measurable $\lambda > \kappa$ (Sz, Väänänen).

Question

What is the consistency strength of (2)?

Thank you!