Connections between generalised Baire spaces and model theory

Miguel Moreno
Kurt Gödel Research Center for Mathematical Logic
University of Vienna

Fifth Workshop on Generalised Baire Spaces

3 February 2020

Outline

1. Borel* sets
2. The isomorphism relation
3. Classifiable theories in the Borel hierarchy
4. The division line Classifiable vs Unclassifiable

Outline

Borel and Δ_1^1 Sets

Definition

The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Borel and Δ_1^1 Sets

Definition

The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Definition

$A \subseteq \kappa^\kappa$ is an analytic if there is a closed subset F of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $pr(F) = \{\eta \in \kappa^\kappa \mid \exists \xi \in \kappa^\kappa \ (\eta, \xi) \in F\}$ is equal to A.

Borel and Δ^1_1 Sets

Definition

The collection of Borel subsets of κ^κ is the smallest set which contains the basic open sets and is closed under unions and intersections, both of length κ.

Definition

$A \subseteq \kappa^\kappa$ is an analytic if there is a closed subset F of the product space $\kappa^\kappa \times \kappa^\kappa$ such that its projection $pr(F) = \{\eta \in \kappa^\kappa \mid \exists \xi \in \kappa^\kappa\ (\eta, \xi) \in F\}$ is equal to A.

Definition

$A \subseteq \kappa^\kappa$ is a Δ^1_1 set if A and $\kappa^\kappa \backslash A$ are analytic sets.

Borel*-code

- A tree T is a κ^+, λ-*tree* if does not contain chains of length λ and its cardinality is less than κ^+. It is *closed* if every chain has a unique supremum.

Borel*-code

- A tree T is a κ^+, λ*-tree* if does not contain chains of length λ and its cardinality is less than κ^+. It is *closed* if every chain has a unique supremum.

- A pair (T, h) is a *Borel*-code* if T is a closed κ^+, κ-tree and h is a function with domain T such that if $x \in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x) \in \{\cup, \cap\}$.

Borel*-game

For an element $\eta \in \kappa^\kappa$ and a Borel*-code (T, h), the *Borel*-game* $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T.

Borel*-game

For an element $\eta \in \kappa^\kappa$ and a Borel*-code (T, h), the *Borel*-game* $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T.
At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice.

Borel*-game

For an element $\eta \in \kappa^\kappa$ and a Borel*-code (T, h), the *Borel*-game* $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T.

At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice.

At limits the game continues from the (unique) supremum of the previous moves by player **I**.

Borel*-game

For an element $\eta \in \kappa^\kappa$ and a Borel*-code (T, h), the *Borel*-game* $B^*(T, h, \eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T.

At each move, if the game is at node $x \in T$ and $h(x) = \cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x) = \cup$, then **II** makes the choice.

At limits the game continues from the (unique) supremum of the previous moves by player **I**.

Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta \in h(x)$.

Borel* sets

Definition (Borel*)

A set $X \subseteq \kappa^\kappa$ is a Borel*-set if there is a Borel*-code (T, h) such that for all $\eta \in \kappa^\kappa$, $\eta \in X$ if and only if **II** *has a winning strategy in the game $B^*(T, h, \eta)$.*

Borel* sets

Definition (Borel*)

A set $X \subseteq \kappa^\kappa$ is a Borel*-set if there is a Borel*-code (T, h) such that for all $\eta \in \kappa^\kappa$, $\eta \in X$ if and only if **II** *has a winning strategy in the game $B^*(T, h, \eta)$.*

We will write $\mathbf{II} \uparrow B^*(T, h, \eta)$ when **II** has a winning strategy in the game $B^*(T, h, \eta)$ and $\mathbf{I} \uparrow B^*(T, h, \eta)$ when **I** has a winning strategy in the game $B^*(T, h, \eta)$.

Borel* sets

Definition (Borel*)

A set $X \subseteq \kappa^\kappa$ is a Borel-set if there is a Borel*-code (T, h) such that for all $\eta \in \kappa^\kappa$, $\eta \in X$ if and only if $\mathbf{II}$ has a winning strategy in the game $B^*(T, h, \eta)$.*

We will write $\mathbf{II} \uparrow B^*(T, h, \eta)$ when $\mathbf{II}$ has a winning strategy in the game $B^*(T, h, \eta)$ and $\mathbf{I} \uparrow B^*(T, h, \eta)$ when $\mathbf{I}$ has a winning strategy in the game $B^*(T, h, \eta)$.

Definition (Dual sets)

We say that X and Y are duals if there is a Borel-code (T, h) such that:*

$$\eta \in X \Leftrightarrow \mathbf{II} \uparrow B^*(T, h, \eta),$$

$$\eta \in Y \Leftrightarrow \mathbf{I} \uparrow B^*(T, h, \eta),$$

Separation Theorem

Theorem (Mekler-Väänänen)

Suppose A and B are disjoint analytic sets. There are Borel sets C_0 and C_1 such that $A \subseteq C_0$ and $B \subseteq C_1$, and C_1 and C_0 are duals.*

Separation Theorem

Theorem (Mekler-Väänänen)

Suppose A and B are disjoint analytic sets. There are Borel* sets C_0 and C_1 such that $A \subseteq C_0$ and $B \subseteq C_1$, and C_1 and C_0 are duals.

Fact

X is a Borel set if and only if there is a Borel*-code (T, h) coding X with T a κ^+, ω-tree.

Separation Theorem

Theorem (Mekler-Väänänen)

Suppose A and B are disjoint analytic sets. There are Borel sets C_0 and C_1 such that $A \subseteq C_0$ and $B \subseteq C_1$, and C_1 and C_0 are duals.*

Fact

X is a Borel set if and only if there is a Borel-code (T, h) coding X with T a κ^+, ω-tree.*

Corollary

- *X is Δ_1^1 if and only if there is a Borel*-code (T, h) coding X such that for all η*
$$\mathbf{II} \uparrow B^*(T, h, \eta) \Leftrightarrow \mathbf{I} \not\uparrow B^*(T, h, \eta).$$
- *Borel $\subseteq \Delta_1^1 \subseteq$ Borel*.*

Outline

The isomorphism relation

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$

The isomorphism relation

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

The isomorphism relation

Fix a relational language $\mathcal{L} = \{P_n | n < \omega\}$

Definition

Let π be a bijection between $\kappa^{<\omega}$ and κ. For every $\eta \in \kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ and for every tuple $(a_1, a_2, \ldots, a_n)$ in κ^n

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \Leftrightarrow \eta(\pi(m, a_1, a_2, \ldots, a_n)) > 0$$

Definition

Given T a first-order complete countable theory in a countable vocabulary, we say that $\eta, \xi \in \kappa^\kappa$ are $\cong_T$ equivalent if

- $\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi$
  or
- $\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T$

Some Division Lines

- Unstable theories

Some Division Lines

- Unstable theories
- Stable unsuperstable theories

Some Division Lines

- Unstable theories
- Stable unsuperstable theories
- Superstable theories with DOP

Some Division Lines

- Unstable theories
- Stable unsuperstable theories
- Superstable theories with DOP
- Superstable theories with OTOP

Some Division Lines

- Unstable theories
- Stable unsuperstable theories
- Superstable theories with DOP
- Superstable theories with OTOP
- Superstable theories with no DOP nor OTOP

Some Division Lines

- Unstable theories
- Stable unsuperstable theories
- Superstable theories with DOP
- Superstable theories with OTOP
- Superstable theories with no DOP nor OTOP

The tools used to construct models of a theory T, strongly depends on what kind of theory is T.

Ehrenfeucht-Fraïssé Game

Let $\mathcal{A}$ and $\mathcal{B}$ be structures with domain κ, and $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration for all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. The game $EF^\kappa_\omega(\mathcal{A},\mathcal{B})$ is played by **I** and **II** as follows.

Ehrenfeucht-Fraïssé Game

Let $\mathcal{A}$ and $\mathcal{B}$ be structures with domain κ, and $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration for all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. The game $EF_\omega^\kappa(\mathcal{A},\mathcal{B})$ is played by **I** and **II** as follows.

In the *n*-th turn **I** chooses an ordinal $\beta_n<\kappa$ such that $X_{\beta_{n-1}}\subset X_{\beta_n}$, and **II** an ordinal $\theta_n<\kappa$ such that $X_{\beta_n}\subseteq dom(f_{\theta_n})\cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}}\subset f_{\theta_n}$, the game starts with X_{β_0} and f_{θ_0} as empty sets. The game finish after ω moves.

Ehrenfeucht-Fraïssé Game

Let $\mathcal{A}$ and $\mathcal{B}$ be structures with domain κ, and $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration for all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. The game $EF^\kappa_\omega(\mathcal{A},\mathcal{B})$ is played by **I** and **II** as follows.

In the *n*-th turn **I** chooses an ordinal $\beta_n < \kappa$ such that $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \kappa$ such that $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subset f_{\theta_n}$, the game starts with X_{β_0} and f_{θ_0} as empty sets. The game finish after ω moves.

The player **II** wins if $\bigcup_{i<\omega} f_{\theta_i} : A \to B$ is a partial isomorphism, otherwise the player **I** wins.

Classifiable theories

A first-order complete countable theory in a countable vocabulary is classifiable if it is a superstable theory with no DOP nor OTOP.

Classifiable theories

A first-order complete countable theory in a countable vocabulary is classifiable if it is a superstable theory with no DOP nor OTOP. We will write $\mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B})$ when $\mathbf{II}$ has a winning strategy in the game $EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B})$.

Classifiable theories

A first-order complete countable theory in a countable vocabulary is classifiable if it is a superstable theory with no DOP nor OTOP. We will write $\mathbf{II} \uparrow EF^\kappa_\omega(\mathcal{A}, \mathcal{B})$ when $\mathbf{II}$ has a winning strategy in the game $EF^\kappa_\omega(\mathcal{A}, \mathcal{B})$.

Fact

If T is a classifiable theory and $\mathcal{A}$, $\mathcal{B}$ are models of T, then

$$\mathbf{II} \uparrow EF^\kappa_\omega(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{A} \cong \mathcal{B}.$$

Classifiable theories

A first-order complete countable theory in a countable vocabulary is classifiable if it is a superstable theory with no DOP nor OTOP. We will write $\mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B})$ when $\mathbf{II}$ has a winning strategy in the game $EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B})$.

Fact

If T is a classifiable theory and $\mathcal{A}$, $\mathcal{B}$ are models of T, then

$$\mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{A} \cong \mathcal{B}.$$

Theorem (Friedman-Hyttinen-Kulikov)

If T is a classifiable theory, then $\cong_T$ is Δ^1_1. Moreover, if T is classifiable not shallow, then $\cong_T$ is not Borel.

Classifiable theories

A first-order complete countable theory in a countable vocabulary is classifiable if it is a superstable theory with no DOP nor OTOP. We will write $\mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B})$ when $\mathbf{II}$ has a winning strategy in the game $EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B})$.

Fact

If T is a classifiable theory and $\mathcal{A}$, $\mathcal{B}$ are models of T, then

$$\mathbf{II} \uparrow EF^{\kappa}_{\omega}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{A} \cong \mathcal{B}.$$

Theorem (Friedman-Hyttinen-Kulikov)

If T is a classifiable theory, then $\cong_T$ is Δ^1_1. Moreover, if T is classifiable not shallow, then $\cong_T$ is not Borel.

Corollary

Borel $\neq \Delta^1_1$.

Outline

Reductions

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is *Borel*, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of κ^κ.

Reductions

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is *Borel*, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of κ^κ.

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

Reductions

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is *Borel*, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a Borel subset of κ^κ.

Let E_1 and E_2 be equivalence relations on κ^κ. We say that E_1 is *Borel reducible* to E_2, if there is a Borel function $f\colon \kappa^\kappa \to \kappa^\kappa$ that satisfies $(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$.

We write $E_1 \hookrightarrow_B E_2$ and we say that E_1 is as most as complex as E_2.

The Equivalence Modulo Non-stationary Ideals in GBS

Let $\lambda < \kappa$ be a regular cardinal. We say that $\eta, \xi \in \kappa^\kappa$ are $=_\lambda$ equivalent if the set $\{\alpha < \kappa | cof(\alpha) = \lambda \ \& \ \eta(\alpha) \neq \xi(\alpha)\}$ is not stationary.

The Equivalence Modulo Non-stationary Ideals in GBS

Let $\lambda < \kappa$ be a regular cardinal. We say that $\eta, \xi \in \kappa^\kappa$ are $=_\lambda$ equivalent if the set $\{\alpha < \kappa | cof(\alpha) = \lambda \ \& \ \eta(\alpha) \neq \xi(\alpha)\}$ is not stationary.

Theorem (Hyttinen-M.)

Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. Then $\cong_T \ \hookrightarrow_B \ =_\lambda$.

Proof

For every $\alpha < \kappa$, structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the game $\mathsf{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha}, \mathcal{B}\restriction_{\alpha})$ is played by **I** and **II** as follows.

Proof

For every $\alpha < \kappa$, structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the game $\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha}, \mathcal{B}\restriction_{\alpha})$ is played by **I** and **II** as follows.

In the *n*-th turn **I** chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$.

Proof

For every $\alpha < \kappa$, structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the game $\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha}, \mathcal{B}\restriction_{\alpha})$ is played by **I** and **II** as follows.

In the *n*-th turn **I** chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$.

The game starts with X_{β_0} and f_{θ_0} as empty sets, and finishes when one of the players cannot choose or after ω moves.

Proof

For every $\alpha < \kappa$, structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the game $\mathrm{EF}^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha}, \mathcal{B}\restriction_{\alpha})$ is played by **I** and **II** as follows.

In the *n*-th turn **I** chooses an ordinal $\beta_n < \alpha$ such that $X_{\beta_n} \subset \alpha$, $X_{\beta_{n-1}} \subset X_{\beta_n}$, and **II** an ordinal $\theta_n < \alpha$ such that $dom(f_{\theta_n}), rang(f_{\theta_n}) \subset \alpha$, $X_{\beta_n} \subseteq dom(f_{\theta_n}) \cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}} \subseteq f_{\theta_n}$.

The game starts with X_{β_0} and f_{θ_0} as empty sets, and finishes when one of the players cannot choose or after ω moves.

The player **II** wins if $\cup_{i<\omega} f_{\theta_i} : A\restriction_{\alpha} \to B\restriction_{\alpha}$ is a partial isomorphism, otherwise the player **I** wins.

Proof

Claim

For every pair of structures, $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the following holds:

- $\mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.
- $\mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.

Proof

Claim

For every pair of structures, $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the following holds:

- $\mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.
- $\mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.

Definition

Given T a first-order complete countable theory in a countable vocabulary and $\alpha \leqslant \kappa$, define the relation $R^{\alpha}_{EF} \subseteq \kappa^{\kappa}\times\kappa^{\kappa}$ as $\eta\ R^{\alpha}_{EF}\ \xi$:

Proof

Claim

For every pair of structures, $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the following holds:

- $\mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \Longleftrightarrow \mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.
- $\mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \Longleftrightarrow \mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.

Definition

Given T a first-order complete countable theory in a countable vocabulary and $\alpha \leqslant \kappa$, define the relation $R^{\alpha}_{EF} \subseteq \kappa^{\kappa} \times \kappa^{\kappa}$ as $\eta\ R^{\alpha}_{EF}\ \xi$:

- $\mathcal{A}_{\eta}\restriction_{\alpha} \not\models T$ *and* $\mathcal{A}_{\xi}\restriction_{\alpha} \not\models T$, *or*

Proof

Claim

For every pair of structures, $\mathcal{A}$ and $\mathcal{B}$ with domain κ, the following holds:

- $\mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{II}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.
- $\mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A},\mathcal{B}) \iff \mathbf{I}\uparrow EF^{\kappa}_{\omega}(\mathcal{A}\restriction_{\alpha},\mathcal{B}\restriction_{\alpha})$ *for club-many* α.

Definition

Given T a first-order complete countable theory in a countable vocabulary and $\alpha \leqslant \kappa$, define the relation $R^{\alpha}_{EF} \subseteq \kappa^{\kappa}\times\kappa^{\kappa}$ as $\eta\ R^{\alpha}_{EF}\ \xi$:

- $\mathcal{A}_{\eta}\restriction_{\alpha}\not\models T$ *and* $\mathcal{A}_{\xi}\restriction_{\alpha}\not\models T$, *or*
- $\mathcal{A}_{\eta}\restriction_{\alpha}\models T$, $\mathcal{A}_{\xi}\restriction_{\alpha}\models T$ *and the player* $\mathbf{II}$ *has a winning strategy for the restricted game* $EF^{\kappa}_{\omega}(\mathcal{A}_{\eta}\restriction_{\alpha},\mathcal{A}_{\xi}\restriction_{\alpha})$.

Proof

Claim

For every T first-order complete countable theory in a countable vocabulary, there are club many α such that R^{α}_{EF} is an equivalence relation.

Proof

Claim

For every T first-order complete countable theory in a countable vocabulary, there are club many α such that R^{α}_{EF} is an equivalence relation.

Define the reduction as follows.
For every $\eta \in \kappa^\kappa$ define the function f_η, as:

- $f_\eta(\alpha)$ is a code in $\kappa\backslash\{0\}$ for the R^{α}_{EF} equivalence class for $\mathcal{A}_\eta\restriction_\alpha$, when $cf(\alpha)=\lambda$, $\mathcal{A}_\eta\restriction_\alpha\models T$, and R^{α}_{EF} is an equivalence relation;

Proof

Claim

For every T first-order complete countable theory in a countable vocabulary, there are club many α such that R^{α}_{EF} is an equivalence relation.

Define the reduction as follows.
For every $\eta \in \kappa^\kappa$ define the function f_η, as:

- $f_\eta(\alpha)$ is a code in $\kappa \backslash \{0\}$ for the R^{α}_{EF} equivalence class for $\mathcal{A}_\eta \restriction_\alpha$, when $cf(\alpha) = \lambda$, $\mathcal{A}_\eta \restriction_\alpha \models T$, and R^{α}_{EF} is an equivalence relation;
- $f_\eta(\alpha) = 0$ in other case.

The Cantor Space

The isomorphism relation and the equivalence modulo non-stationary ideals can be easily define in the generalised Cantor space 2^κ.

The Cantor Space

The isomorphism relation and the equivalence modulo non-stationary ideals can be easily define in the generalised Cantor space 2^κ.

$$\cong^2_T \text{ is } \cong_T \cap\, (2^\kappa \times 2^\kappa)$$
$$=^2_\lambda \text{ is } =_\lambda \cap\, (2^\kappa \times 2^\kappa)$$

The Cantor Space

The isomorphism relation and the equivalence modulo non-stationary ideals can be easily define in the generalised Cantor space 2^κ.

$$\cong^2_T \text{ is } \cong_T \cap (2^\kappa \times 2^\kappa)$$
$$=^2_\lambda \text{ is } =_\lambda \cap (2^\kappa \times 2^\kappa)$$

Theorem (Hyttinen-Kulikov-M.)

Denote by S^κ_λ the set $\{\alpha < \kappa | cf(\alpha) = \lambda\}$.
Suppose T is a classifiable theory and $\lambda < \kappa$ is a regular cardinal. If $\diamondsuit(S^\kappa_\lambda)$ holds, then $\cong^2_T \hookrightarrow_B =^2_\lambda$.

Outline

Unstable

Theorem (Friedman-Hyttinen-Kulikov)

If T is unstable, then $\cong_T$ is not Δ_1^1.

Unstable

Theorem (Friedman-Hyttinen-Kulikov)

If T is unstable, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. If T is an unstable, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Unstable

Theorem (Friedman-Hyttinen-Kulikov)

If T is unstable, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. If T is an unstable, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(S^\kappa_\lambda)$ holds. If T is a classifiable theory and T' is an unstable, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda \ \hookrightarrow_B \ \cong^2_{T'}$.

Stable unsuperstable

Question

Can it be proved in ZFC that if T is stable unsuperstable, then $\cong_T$ is not Δ_1^1?

Stable unsuperstable

Question

Can it be proved in ZFC that if T is stable unsuperstable, then $\cong_T$ is not Δ_1^1?

Theorem (Friedman-Hyttinen-Kulikov)

Suppose for all $\lambda < \kappa$, $\lambda^\omega < \kappa$. If T is a stable unsuperstable, then $=^2_\omega \ \hookrightarrow_B \ \cong^2_T$.

Stable unsuperstable

Question

Can it be proved in ZFC that if T is stable unsuperstable, then $\cong_T$ is not Δ^1_1?

Theorem (Friedman-Hyttinen-Kulikov)

Suppose for all $\lambda < \kappa$, $\lambda^\omega < \kappa$. If T is a stable unsuperstable, then $=^2_\omega \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+$ and $\lambda^\omega = \lambda$. If T is a classifiable theory and T' is a stable unsuperstable theory, then $\cong^2_T \ \hookrightarrow_B \ =^2_\omega \ \hookrightarrow_B \ \cong^2_{T'}$, and $\cong^2_{T'} \ \not\hookrightarrow_B \ \cong^2_T$.

The Orthogonal Chain Property (OCP)

Definition

Given $p \in S(A)$ and $B \subseteq A$, we say $p \perp B$ if for every $q \in S(A)$ that doesn't fork over B the following holds; for every a, b, and $B' \supseteq A$, if a realizes p, b realizes q, $a \downarrow_A B'$ and $b \downarrow_A B'$, then $a \downarrow_{B'} b$.

The Orthogonal Chain Property (OCP)

Definition

Given $p \in S(A)$ and $B \subseteq A$, we say $p \perp B$ if for every $q \in S(A)$ that doesn't fork over B the following holds; for every a, b, and $B' \supseteq A$, if a realizes p, b realizes q, $a \downarrow_A B'$ and $b \downarrow_A B'$, then $a \downarrow_{B'} b$.

Definition

A stable theory T has the property OCP if there exist $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, and $a \notin \cup_{i<\omega}\mathcal{A}_i$ such that for all $i \leqslant j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$, $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic and for all $j < \omega$, $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_j$.

The Orthogonal Chain Property (OCP)

Definition

Given $p \in S(A)$ and $B \subseteq A$, we say $p \perp B$ if for every $q \in S(A)$ that doesn't fork over B the following holds; for every a, b, and $B' \supseteq A$, if a realizes p, b realizes q, $a \downarrow_A B'$ and $b \downarrow_A B'$, then $a \downarrow_{B'} b$.

Definition

A stable theory T has the property OCP if there exist $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, and $a \notin \cup_{i<\omega}\mathcal{A}_i$ such that for all $i \leqslant j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$, $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic and for all $j < \omega$, $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_j$.

Theorem (Hyttinen-M.)

Suppose T is a classifiable theory, T' is a stable theory with the OCP, and κ is an inaccessible cardinal. Then $\cong_T \ \hookrightarrow_B \ =_\omega \ \hookrightarrow_B \ \cong_{T'}$

Superstable with DOP

Definition (a-isolation)

Denote by F^a_ω the set of pairs (p, A) with $|A| < \omega$, such that for some $B \supseteq A$, $p \in S(B)$, $a \models p$ and $stp(a, A) \vdash p$.

From this isolation the notions of a-saturated, a-primary, and a-minimal are defined.

Superstable with DOP

Definition (a-isolation)

Denote by F^a_ω the set of pairs (p, A) with $|A| < \omega$, such that for some $B \supseteq A$, $p \in S(B)$, $a \models p$ and $stp(a, A) \vdash p$.

From this isolation the notions of a-saturated, a-primary, and a-minimal are defined.

Definition (DOP)

A theory T has the dimensional order property if there are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, $M_1 \downarrow_{M_0} M_2$, and the a-primary model over $M_1 \cup M_2$ is not a-minimal over $M_1 \cup M_2$.

Superstable with DOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is not Δ_1^1.

Superstable with DOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda > 2^\omega$. If T is superstable with DOP, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Superstable with DOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda > 2^\omega$. If T is superstable with DOP, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda > 2^\omega$ and $\diamondsuit(S^\kappa_\lambda)$ holds. If T is a classifiable theory and T' is superstable with DOP, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda \ \hookrightarrow_B \ \cong^2_{T'}$.

Superstable with S-DOP

Definition (S-DOP)

We say that a theory T has the strong dimensional order property if the following holds:
There are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$

Superstable with S-DOP

Definition (S-DOP)

We say that a theory T has the strong dimensional order property if the following holds:
There are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 a-primary model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, that does not fork over $M_1 \cup M_2$.

Superstable with S-DOP

Definition (S-DOP)

We say that a theory T has the strong dimensional order property if the following holds:
There are a-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 a-primary model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, that does not fork over $M_1 \cup M_2$.

Theorem (M.)

Suppose T is a classifiable theory, T' is a superstable theory with the S-DOP, $\lambda = (2^\omega)^+$, and κ an inaccessible cardinal. Then
$\cong_T \quad \hookrightarrow_B \quad =_\lambda \quad \hookrightarrow_B \quad \cong_{T'}$

Superstable with OTOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is stable with OTOP, then $\cong_T$ is not Δ^1_1.

Superstable with OTOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is stable with OTOP, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. If T is superstable with OTOP, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Superstable with OTOP

Theorem (Friedman-Hyttinen-Kulikov)

If T is stable with OTOP, then $\cong_T$ is not Δ_1^1.

Theorem (Friedman-Hyttinen-Kulikov)

Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$. If T is superstable with OTOP, then $=^2_\lambda \ \hookrightarrow_B \ \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+ = 2^\lambda$, $\lambda^{<\lambda} = \lambda$ and $\diamondsuit(S^\kappa_\lambda)$ holds. If T is a classifiable theory and T' is superstable with OTOP, then $\cong^2_T \ \hookrightarrow_B \ =^2_\lambda \ \hookrightarrow_B \ \cong^2_{T'}$.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the Borel reducibility hierarchy.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the Borel reducibility hierarchy.

  Let $H(\kappa)$ be the following property: If T is a classifiable theory and T' is not a classifiable theory, then $\cong^2_T \ \hookrightarrow_B \ \cong^2_{T'}$ and $\cong^2_{T'} \ \not\hookrightarrow_B \ \cong^2_T$.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the Borel reducibility hierarchy.

  Let $H(\kappa)$ be the following property: If T is a classifiable theory and T' is not a classifiable theory, then $\cong^2_T \hookrightarrow_B \cong^2_{T'}$ and $\cong^2_{T'} \not\hookrightarrow_B \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ *and* $\lambda^{<\lambda} = \lambda$.

1. *If* S^κ_ω *and* S^κ_λ *hold, then* $H(\kappa)$ *holds.*

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the Borel reducibility hierarchy.

  Let $H(\kappa)$ be the following property: If T is a classifiable theory and T' is not a classifiable theory, then $\cong^2_T \hookrightarrow_B \cong^2_{T'}$ and $\cong^2_{T'} \not\hookrightarrow_B \cong^2_T$.

Theorem (Hyttinen-Kulikov-M.)

Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ *and* $\lambda^{<\lambda} = \lambda$.

1. *If* S^κ_ω *and* S^κ_λ *hold, then* $H(\kappa)$ *holds.*
2. *It is consistent that* $H(\kappa)$ *holds and there are* 2^κ *equivalence relations strictly between* $\cong^2_T$ *and* $\cong^2_{T'}$.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Theorem (Friedman-Hyttinen-Kulikov)

- *If T is classifiable and Shallow, then $\cong_T$ is Borel.*

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Theorem (Friedman-Hyttinen-Kulikov)

- *If T is classifiable and Shallow, then $\cong_T$ is Borel.*
- *If T is classifiable but not shallow, then $\cong_T$ is Δ_1^1 but not Borel.*

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Theorem (Friedman-Hyttinen-Kulikov)

- *If T is classifiable and Shallow, then $\cong_T$ is Borel.*
- *If T is classifiable but not shallow, then $\cong_T$ is Δ_1^1 but not Borel.*
- *If T is stable unsuperstable, then $\cong_T$ is analytic but not Borel.*

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Theorem (Friedman-Hyttinen-Kulikov)

- *If T is classifiable and Shallow, then $\cong_T$ is Borel.*
- *If T is classifiable but not shallow, then $\cong_T$ is Δ_1^1 but not Borel.*
- *If T is stable unsuperstable, then $\cong_T$ is analytic but not Borel.*
- *If T is unstable or stable with OTOP, then $\cong_T$ is analytic but not Δ_1^1.*

Sum up

Shelah's division lines can be studied in generalised descriptive set theory with two different approaches.

- By the different analytic sets.

Theorem (Friedman-Hyttinen-Kulikov)

- *If T is classifiable and Shallow, then $\cong_T$ is Borel.*
- *If T is classifiable but not shallow, then $\cong_T$ is Δ_1^1 but not Borel.*
- *If T is stable unsuperstable, then $\cong_T$ is analytic but not Borel.*
- *If T is unstable or stable with OTOP, then $\cong_T$ is analytic but not Δ_1^1.*
- *If T is superstable with DOP and $\kappa > \omega_1$, then $\cong_T$ is analytic but not Δ_1^1.*

Sum up

By using both approaches at the same time:

Sum up

By using both approaches at the same time:

Let $G(\kappa)$ be the following property: If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:

- $\cong_T$ is Δ_1^1.
- $\cong_T$ is Σ_1^1-complete.

Sum up

By using both approaches at the same time:

Let $G(\kappa)$ be the following property: If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:

- $\cong_T$ is Δ_1^1.
- $\cong_T$ is Σ_1^1-complete.

Theorem (Hyttinen-Kulikov-M.)

Suppose $V = L$. $G(\kappa)$ holds for all κ successor of a regular uncountable cardinal λ.

Sum up

By using both approaches at the same time:

Let $G(\kappa)$ be the following property: If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:

- $\cong_T$ is Δ_1^1.
- $\cong_T$ is Σ_1^1-complete.

Theorem (Hyttinen-Kulikov-M.)

Suppose $V = L$. $G(\kappa)$ holds for all κ successor of a regular uncountable cardinal λ.

Theorem (Fernandes-M.-Rinot)

Suppose $\kappa = \lambda^+$ and $\lambda^{<\lambda} = \lambda > \omega$. There exists a $< \kappa$-closed κ^+-cc forcing extension, in which $G(\kappa)$ holds.

Fake reflection

It is a consequence of Π_1^1-filter reflection with $\diamondsuit$, $DI_S^*(\Pi_1^1)$.

Fake reflection

It is a consequence of Π^1_1-filter reflection with $\diamondsuit$, $DI^*_S(\Pi^1_1)$.

Definition (Filter reflection with $\diamondsuit$)

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

1. *We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is nonstationary;*
2. *We say that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S \rangle$ such that, for every stationary $Y \subset X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \;\&\; Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.*

Fake reflection

It is a consequence of Π^1_1-filter reflection with $\diamondsuit$, $DI^*_S(\Pi^1_1)$.

Definition (Filter reflection with $\diamondsuit$)

Suppose X and S are stationary subsets of κ, and $\vec{\mathcal{F}} = \langle \mathcal{F}_\alpha \mid \alpha \in S \rangle$ is a sequence such that, for each $\alpha \in S$, $\mathcal{F}_\alpha$ is a filter over α.

1. *We say that $\vec{\mathcal{F}}$ captures clubs iff, for every club $C \subseteq \kappa$, the set $\{\alpha \in S \mid C \cap \alpha \notin \mathcal{F}_\alpha\}$ is nonstationary;*
2. *We say that X $\vec{\mathcal{F}}$-reflects with $\diamondsuit$ to S iff $\vec{\mathcal{F}}$ captures clubs and there exists a sequence $\langle Y_\alpha \mid \alpha \in S \rangle$ such that, for every stationary $Y \subset X$, the set $\{\alpha \in S \mid Y_\alpha = Y \cap \alpha \text{ \& } Y \cap \alpha \in \mathcal{F}^+_\alpha\}$ is stationary.*

It can be forced by Sakai's forcing, Friedman-Holy's forcing, and Holy-Wu-Welch's forcing. It can be killed by $\mathrm{Add}(\kappa, \kappa^+)$. It follows from $V = L$ but also from Martin's Maximum.

Thank you