Forcing over choiceless models (3/4)

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Outline

0. Introduction
1. Adding Cohen subsets by $\mathrm{Add}(A, 1)$
   - Preliminaries
   - Cohen's first model and Dedekind finite sets A
   - Properties of $\mathrm{Add}(\kappa, 1)$ and fragments of DC
   - Adding Cohen subsets over $L(\mathbb{R})$
2. Chain conditions and cardinal preservation
   - Variants of the ccc
   - An iteration theorem
   - A ccc_2 forcing that collapses ω_1
3. Generic absoluteness principles inconsistent with choice
   - Hartog numbers
   - Very strong absoluteness and consequences
   - Gitik's model
4. Random algebras without choice
   - Completeness
   - ccc_2^*

Generic diversity

Sentences that can be forced true or false over any model of ZFC:

- CH
- $b \geq \omega_2$
- A Suslin tree exists

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Sentences that can be forced true and remain true in further extensions:

- There exists a non-constructible real
- ω_4^L is countable

Sentences that hold in all generic extension assuming large cardinals:

- Any sentence in $L(\mathbb{R})$
- Any sentence in the Chang model $L(\mathrm{Ord}^\omega)$

Generic diversity

Given at least one regular uncountable cardinal κ, one can force some non-trivial statements.

- ω_1 is regular
- $b = d = \kappa$
- Fragments of Martin's axiom

But there might be no uncountable regular cardinals.

Generic diversity

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In particular, Cohen and random extensions are different. Truss proved the following stronger statement: Cohen and random forcing don't commute.

Theorem (Truss 1983)

A $\mathbb{R} * \dot{\mathbb{C}}$-extension of V is not a $\mathbb{C} * \dot{\mathbb{R}}$-extension of V.

*"Every uncountable subset B of ω_1 contains an infinite subset $A \in V$"
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holds in the former, but not the latter.

We use the special case:

Fact

If y is random over $V[x]$, then x is not Cohen over $V[y]$.

Proof (Glazer, ?). Otherwise $x + y$ is both random over $V[x]$ and Cohen over $V[y]$. Then $x + y$ is both random and Cohen over V, contradiction. □

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$\mathbb{R}_\alpha$ denotes the random algebra on α. It consists of all Borel codes for subsets of 2^α. The quasi-order on $\mathbb{R}_\alpha$ is given by inclusion.

- $\mathbb{R}_\alpha$ is not equivalent to the finite support product of random forcings.
- We will see that $\mathbb{R}_\alpha$ preserves all cardinals.

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Definition

- A Cohen model is a $\mathbb{C}^\kappa$-extension over V for some $\kappa \geq \omega_2$.
- A random model is a $\mathbb{R}_\kappa$-extension over V for some $\kappa \geq \omega_2$.

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Proposition (Woodin)

Cohen and random models over V have different theories.

Proof. In a Cohen model, for any subset A of ω_1 there is a Cohen real over $V[A]$ and hence over $L[A]$, since A is constructed from an ω_1 size piece of the generic.

In the random model, let B be a piece of the random generic of size ω_1. Then there is no Cohen real x over $L[B]$.

To see this, note that for any real x, B adds a random real y over $V[x]$ and hence over $L[x]$, since x is constructed from a countable piece of B. So x is not Cohen over $L[y]$. $\square$

Generic diversity

The next step is to separate the theories of other extensions.

Definition

- A Hechler model is an extension of V by an iteration of Hechler forcing of length $\kappa \geq \omega_2$.

Problem

Do Cohen and Hechler models have different theories?

Generic diversity

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The Chang model cannot have generic absoluteness for its Σ_2 theory in ZF, even in the presence of large cardinals.

Proof sketch. Suppose that V is a model of ZFC with large cardinals.

- Form a symmetric extension M of V such that $M \models \mathsf{cof}(\omega_2) = \omega_1$ via $\mathrm{Col}(\omega_1, {<}\aleph_{\omega_1})$. Then M has the same Chang model $L(\mathrm{Ord}^\omega)$ as V.
- Let G be $\mathrm{Col}(\omega, \omega_1)$-generic over M. $M[G]$ collapses ω_1^M and $M[G] \models \mathsf{cof}(\omega_1) = \omega$.

But $\mathsf{cof}(\omega_1) = \omega$ is a Σ_2 statement over the Chang model. $\square$

Generic diversity

If κ is supercompact in V, then κ is supercompact in M in the following sense for all α:

Definition

κ is V_α-supercompact if for every α, there exists some $\beta > \alpha$ and an elementary embedding $j\colon V_\beta \to N$ with $\alpha < \mathrm{crit}(j) = \kappa$ such that N is a transitive set with $N^{V_\alpha} \subseteq N$.

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Problem (Aspero, Karagila 2020)

- *Can generic absoluteness for $L(\mathbb{R})$ fail in the presence of large cardinals?*
- *Is it possible that $\mathbb{R}^\#$ exists and ω_1 is singular in $L(\mathbb{R})$?*

$\square$

Definition

Suppose that $\lambda < \kappa$ are cardinals.

- κ is a λ-strong limit if for all $\nu < \kappa$, $\kappa \not\leq^* \nu^\lambda$.
- κ is called λ-inaccessible if it is a λ-strong limit and $\mathrm{cof}(\kappa) > \lambda$.

Hartog numbers

Let $\aleph(x)^-$ denote $\aleph(x)$ if this is a limit cardinal and its cardinal predecessor otherwise.

We write

$$\aleph := \aleph(2^\omega) = \sup\{\alpha \in \mathrm{Ord} \mid \alpha \leq 2^\omega\},$$
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Then

$$\aleph^- = \sup\{\lambda \in \mathrm{Card} \mid \alpha \leq 2^\omega\}.$$

Case

$\aleph = \kappa^+$. Then $\kappa = \sup\{\lambda \in \mathrm{Card} \mid \lambda \leq 2^\omega\}$.

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$\aleph$ is a limit. Then $\aleph = \sup\{\lambda \in \mathrm{Card} \mid \lambda \leq 2^\omega\}$.

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Lemma

$\aleph(\kappa^\omega) = \aleph^{V[G]}$ *for any infinite cardinal* κ *and any* $\mathbb{C}^\kappa$*-generic filter* G *over* V.

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Proof. $\leq$: It suffices to show $\kappa^\omega \leq (2^\omega)^{V[G]}$.

- Map κ^ω injectively to a subset of κ^ω of functions with almost disjoint ranges.
- For each range, glue the list of Cohen reals into a single real. The reals are pairwise different.

$\geq$: Suppose $1 \Vdash \vec{x} = \langle \dot{x}_\alpha \mid \alpha < \gamma \rangle$ is injective. Working in $\mathrm{HOD}_{\vec{x},\Vdash}$, we can replace each $\dot{x}_\alpha$ by a nice name coded by an element of κ^ω. $\square$

Hartog numbers

Lemma

$1_{\mathbb{C}^\kappa} \Vdash \aleph = \kappa^+$ *for any ω-inaccessible cardinal κ.*

Proof. The claim is equivalent to $\aleph(\kappa^\omega) = \kappa^+$ by the previous lemma.

Otherwise there exists an injective function $f\colon \kappa^+ \to \kappa^\omega$.

- $\kappa^\omega = \bigcup_{\alpha<\kappa} \alpha^\omega$, since $\mathsf{cof}(\kappa) > \omega$.
- $|f^{-1}[\alpha^\omega]| \geq \kappa$ for some $\alpha < \kappa$.

This contradicts that κ is an ω-strong limit. $\square$

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This contradicts that κ is an ω-strong limit. □

Corollary

Suppose there exist two uncountable regular cardinals $\kappa < \lambda$. Then we can force two different theories.

Proof. Suppose that $\kappa < \lambda$ are least. Pick ω-inaccessibles ν_κ and ν_λ with cofinalites κ and λ. Then

- $1_{\mathbb{C}^{\nu_\kappa}} \Vdash \mathsf{cof}(\aleph^-) = \kappa$.
- $1_{\mathbb{C}^{\nu_\lambda}} \Vdash \mathsf{cof}(\aleph^-) = \lambda$. □

Hartog numbers

Suppose there exists only a single uncountable regular cardinal κ.

Woodin proved that one can still force two different theories via $\mathbb{C}^\lambda$ for different λ. Next is a version of this argument.

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Definition

Suppose that I and J are subsets of ν^ω.

1. J covers I if for each $f \in I$, there exists some $g \in J$ with $\mathrm{ran}(f) \subseteq \mathrm{ran}(g)$.
2. For any cardinal ν, a subset J of ν^ω of size $\aleph^-$ is called minimal if it is not covered by any subset I of ν^ω of size $<\aleph^-$.
3. m denotes the least cardinal ν such that there exists a minimal subset of ν^ω, if there exists such a ν.

The idea is to find different values of m in $\mathbb{C}^\lambda$-extensions.

Hartog numbers

Lemma

$1_{\mathbb{C}^\kappa} \Vdash \mathrm{m} \geq \kappa$ *for any* ω*-strong limit cardinal* κ.

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Proof. Work in a $\mathbb{C}^\kappa$-generic extension of V. We work in $V[G]$.

Suppose that $\nu < \kappa = \aleph$ and B is a subset of ν^ω of size κ. We claim that B is not minimal.

It suffices to find a wellorderable subset $A \in V$ of ν^ω that covers B. Since κ is an ω-strong limit in V, $|A| < \kappa$ follows.

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It suffices to find a wellorderable subset $A \in V$ of ν^ω that covers B. Since κ is an ω-strong limit in V, $|A| < \kappa$ follows.

- Fix a bijection $f\colon \kappa \to B$ and a name $\dot{f}$ for it. Let $\dot{g}$ be a $\mathbb{C}^\kappa$-name for the function $g\colon \kappa \times \omega \to \nu$ with $g(\alpha, n) = f(\alpha)(n)$. Let p force the above for $\dot{f}$ and $\dot{g}$.
- For each $(\alpha, n) \in \kappa \times \omega$, let $D_{\alpha,n}$ denote the set of all conditions $\leq p$ in $\mathbb{C}^\kappa$ that decide $\dot{g}(\alpha)(n)$. Define $g_{\alpha,n}\colon D_{\alpha,n} \to \nu$ such that $g_{\alpha,n} = \gamma$ if $r \Vdash \dot{g}(\alpha)(\beta) = \gamma$.

Then $\mathrm{ran}(g_{\alpha,n})$ is countable. Working in $\mathsf{HOD}_{\mathbb{C}^\kappa, \Vdash, \dot{f}, \dot{g}}$, we can define $h\colon \kappa \times \omega \to \nu^\omega$ such that $h(\alpha, n)$ is an enumeration of $\mathrm{ran}(g_{\alpha,n})$.

Let $\bar{h}\colon \alpha \to \nu^{\omega\times\omega}$, $\bar{h}(\alpha)(m, n) = h(\alpha, m)(n)$. Then $\bar{h}(\alpha)$ covers $f(\alpha)$. $\square$

Hartog numbers

Lemma

Suppose that $\nu \in \mathrm{Card}$, $p \in \mathbb{P}_\nu$ *forces that* $\aleph$ *is a successor cardinal and* $1_{\mathbb{P}} \Vdash \aleph > (\aleph^+)^V$.

Then $p \Vdash_{\mathbb{C}^\nu} \mathrm{m} \leq \nu$.

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Proof. Let $\lambda := (\aleph^-)^{V[G]} = \aleph(\nu^\omega)^-$. Then $\lambda \leq \nu^\omega$.

We claim that any subset of ν^ω of size λ in V is minimal in $V[G]$.

Fix an injective function $f\colon \lambda \to \nu^\omega$ in V.

- If $\mathrm{ran}(f)$ is not minimal, then there exists some $\mu < \lambda$, a $\mathbb{C}^\nu$-name $\dot{g}$ for a function $\dot{g}\colon \mu \to \nu^\omega$ such that some $q \leq p$ forces that $\mathrm{ran}(\dot{g})$ covers $\mathrm{ran}(f)$.
- Like in the previous proof, replace $\dot{g}$ by a function $h\colon \mu \to \nu^\omega$ in V such that $\mathrm{ran}(h)$ covers $\mathrm{ran}(f)$.

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For each $\alpha < \mu$, let $A_\alpha := \{\gamma < \lambda \mid f(\gamma) \subseteq h(\alpha)\}$.

- Since $h(\alpha)$ is countable, $\mathbf{otp}(A_\alpha) < \aleph^V$ for all $\alpha < \mu$.
- We have $\bigcup_{\alpha<\mu} A_\alpha = \lambda$ since $\mathrm{ran}(h)$ covers $\mathrm{ran}(f)$, contradicting $\lambda \geq (\aleph^+)^V$.

$\square$

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Theorem

If $A_{\mathbb{C}^*}$ holds, then $1_{\mathbb{C}^\kappa} \Vdash \aleph > \kappa^+$ for any ω-strong limit cardinal κ.

Proof. Towards a contradiction, suppose that there exists an ω-strong limit cardinal κ with $p \Vdash_{\mathbb{P}_\kappa} \aleph = \kappa^+$ for some $p \in \mathbb{P}_\kappa$. By the above, $p \Vdash_{\mathbb{C}^\kappa} \mathrm{m} \geq \aleph^-$.

It suffices to show that $\mathrm{m} < \aleph^-$ holds in a $\mathbb{C}^\lambda$-generic extension for some $\lambda \in \mathrm{Card}$.

To see this, pick any successor cardinal $\lambda \geq \aleph^+$. Since $\mathbb{C}^\kappa$ forces that $\aleph$ is the successor of a limit, the same holds for $\mathbb{C}^\lambda$ by $A_{\mathbb{C}^*}$.

Since λ is not a limit cardinal, $1_{\mathbb{P}_\lambda} \Vdash \aleph > \lambda^+$.

Since $1_{\mathbb{P}_\lambda}$ forces that $\aleph$ is a successor, $1_{\mathbb{P}_\lambda}$ forces $\mathrm{m} \leq \lambda < \aleph^-$ by the previous Lemma. $\square$

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Corollary (Woodin)

If there exist a uncountable regular cardinal, then $A_{\mathbb{C}^*}$ fails. Then there exists an ω-inaccessible cardinal κ and we get both $\mathbf{1}_{\mathbb{C}^\kappa} \Vdash \aleph = \kappa^+$ and $\mathbf{1}_{\mathbb{C}^\kappa} \Vdash \aleph > \kappa^+$.

Gitik's model

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Theorem (Gitik 1980)

Suppose that V is a model of BG *with a global wellorder and a proper class of strongly compact cardinals, but no regular limit of strongly compact cardinals.*

Then there is a symmetric class extension $V(G)$ of V such that:

- $V(G) \models \mathrm{ZF}$.
- *In $V(G)$, every infinite cardinal has countable cofinality.*

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Theorem (Busche, Schindler)

The consistency strength of the theory ZF *and "every infinite cardinal has countable cofinality" is at least* ZFC *with infinitely many Woodin cardinals.*

Gitik's model

Gitik's model is constructed as a symmetric extension $V(G)$ of V.

The forcing $\mathbb{P}$ is constructed from a sequence of interleaved strongly compact Prikry forcings.

- Let $\langle \kappa_i \mid i \in \mathrm{Ord} \rangle$ list all strongly compact cardinals in V. Its closure equals the class of uncountable cardinals in $V(G)$.
- I is a class of finite subsets of Reg with a closure property. Each $s \in I$ induces a subforcing $\mathbb{P}_s$ of $\mathbb{P}$.

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Lemma (Gitik)

For any set of ordinals $X \in V(G)$, there exists some $s \in I$ with $X \in V[G \restriction \mathbb{P}_s]$.

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For any $s \in I$ and any strongly compact $\kappa_i \in s$, $\mathbb{P}_s$ is equivalent to a forcing $\mathbb{P}_{s \cap \kappa_i} * \dot{\mathbb{Q}}$, where $\mathbb{P}_{s \cap \kappa_i}$ forces that $\dot{\mathbb{Q}}$ does not add any bounded subsets of κ_i.

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Proof. Suppose that κ is an infinite cardinal in $V(G)$ and $f\colon \gamma \to {}^\omega\kappa$ is an injective function in $V(G)$. It suffices to show $\gamma < (\kappa^+)^{V(G)}$.

$\kappa = \kappa_\zeta$ and $(\kappa^+)^{V(G)} = \kappa_\xi$ for some $\zeta < \xi$, where κ_i is the ith strongly compact cardinal in V.

By the above properties of Gitik's construction, there exists some $s \in I$ with $f \in V[G{\restriction}\mathbb{P}_s]$. We may assume $\kappa_\zeta, \kappa_\xi \in s$.

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Let λ be inaccessible in V with $\max(s \cap \kappa_\xi) < \lambda < \kappa_\xi$. Then:

- $\mathbb{P}_s$ is equivalent to a forcing of the form $\mathbb{P}_{s \cap \kappa_\xi} * \dot{\mathbb{Q}}$, where $\mathbb{P}_{s \cap \kappa_\xi}$ forces that $\dot{\mathbb{Q}}$ does not add new bounded subsets of κ_ξ.
- Since $|\mathbb{P}_{s \cap \kappa_\xi}| < \lambda$, λ remains inaccessible in $V[G \restriction \mathbb{P}_s]$.
- Since $f \in V[G \restriction \mathbb{P}_s]$ and $\kappa < \lambda$, we have $\gamma < \lambda < \kappa_\xi = (\kappa^+)^{V(G)}$. $\square$

Gitik's model

We have seen that in Gitik's model, $\aleph(2^\kappa) = \kappa^+$ for all infinite cardinals κ. Hence $1_{\mathbb{C}^\kappa} \Vdash \aleph = \kappa^+$.

One can change the theory of Gitik's model by forcing:

- Otherwise for any ω-strong limit cardinal κ, $1_{\mathbb{C}^\kappa} \Vdash \aleph > \kappa^+$.
- Since $\aleph(2^\omega) = \omega_1$, no cardinal characteristics of the reals exist. But $\mathbb{C}^\kappa$ forces $b \geq \omega_1$ by $\mathbb{C}^\kappa$ for any uncountable cardinal κ.

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Remark

One can show $\mathbb{C}^\kappa$ forces $b = \omega_1$ for all uncountable κ. It is open whether one can force $b \geq \omega_2$.

Similarly, one can show $\mathbb{C}^\kappa$ forces "d does not exist". It is open whether one can force "d exists".

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Similarly, one can show $\mathbb{C}^\kappa$ forces "d does not exist". It is open whether one can force "d exists".

Problem

What else can you force over Gitik's model?

Problem
Is the theory of Gitik's model the same when leaving out some strongly compact cardinals?

Problem
Is $\mathbb{C}^{\kappa}$-generic absoluteness consistent?

Very different properties than those of Gitik's model are needed.