The complexity of club filters

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Introduction

The fact that closed unbounded subsets generate a proper normal filter

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The study of the structural properties of these filters and their dual ideals

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In particular, questions about the complexity of these filters motivated much of the development of generalized descriptive set theory.

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- Through the complexity of the formulas and parameters defining these sets in the structure $\langle \mathrm{V}, \in \rangle$.
- Through the topological complexity of these sets viewed as subsets of the generalized Baire space ${}^\kappa\kappa$ of the corresponding cardinal κ.

The Levy Hierarchy

A formula in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Δ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under negations, conjunctions and bounded quantification.

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Π_1-*formulas* are negations of Σ_1-formulas.

Definition

An $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_n)$ and sets $y_0, \ldots, y_{n-1}$ *define* a class X if

$$X = \{x \mid \varphi(x, y_0, \ldots, y_{n-1})\}.$$

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Definition

Given a set P, a class X is $\mathbf{\Delta}_1(P)$-definable if it is definable by both a Σ_1- and a Π_1-formula with parameters in P.

Generalized Baire spaces

Given an infinite regular cardinal κ, the *generalized Baire space* of κ consists of the set ${}^{\kappa}\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^{\kappa}\kappa \mid s \subseteq x\}$$

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Corollary

Given an uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, a subset of ${}^{\kappa}\kappa$ is a $\mathbf{\Delta}^1_1$-subset if and only if it is $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

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These results motivate the task to answer the above question in different models of set theory.

Positive consistency results

In the following, we present several different examples of models of set theory in which the restrictions

$$NS \restriction S \; = \; NS_\kappa \cap \mathcal{P}(\kappa)$$

of non-stationary ideals on uncountable regular cardinals κ to stationary subsets S of κ are $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

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- *The set* $NS \restriction S^{\mu^+}_\mu$ *is* $\mathbf{\Delta}_1(\mathrm{H}(\mu^{++}))$*-definable.*

The proof of this result makes use of the notion of *Canary trees*.

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Assume that $\mathrm{V} = \mathrm{L}$ *holds. Given an infinite cardinal* μ*, the following statements hold in a cofinality-preserving forcing extension of the ground model:*

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- *The set* NS_{μ^+} *is* $\mathbf{\Delta}_1(\{\mu^+\})$*-definable.*

Dense ideals

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Given a cardinal κ, an ideal $\mathcal{I}$ on a set X is κ-*dense* if the partial order $\mathcal{P}(X)/\mathcal{I}$ has a dense subset of cardinality at most κ.

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The theory $\mathbf{ZFC}$ + "NS_{ω_1} *is* ω_1*-dense" is equiconsistent with the theory* $\mathbf{ZF} + \mathbf{AD}$.

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The theory $\mathbf{ZFC}$ + "NS_{ω_1} is ω_1-dense" is equiconsistent with the theory $\mathbf{ZF} + \mathbf{AD}$.

Proposition

If S is a stationary subset of an uncountable regular cardinal κ with the property that $NS \restriction S$ is κ-dense, then $NS \restriction S$ is $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

Stationary reflection

A crucial ingredient in the proofs of the new results presented in this talk is the observation that the $\mathbf{\Delta}_1$-definability of non-stationary ideals can also be obtained through strong principles of stationary reflection.

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Let S be stationary subsets of an uncountable regular cardinal δ and let $\mathcal{E}$ be a set of stationary subsets of δ.

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Assume that for every stationary subset A of S, there exists $E \in \mathcal{E}$ such that A reflects at every element of E.

If $\mathcal{E}$ is definable by a Σ_1-formula with parameter p, then the set $NS \restriction S$ is definable by a Π_1-formula with parameters p, S and $\mathrm{H}(\delta)$.

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Note that a classical result of Magidor shows that, starting with a weakly compact cardinal, it is possible to construct a model of set theory in which every stationary subset of S^2_0 reflects almost everywhere in S^2_1.

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Note that a classical result of Magidor shows that, starting with a weakly compact cardinal, it is possible to construct a model of set theory in which every stationary subset of S^2_0 reflects almost everywhere in S^2_1.

The above corollary shows that the set $NS \restriction S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable in Magidor's model.

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In particular, it is possible to show NS_κ can be $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable for a greatly Mahlo cardinal κ.

Negative consistency results

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The following results show that adding κ^+-many Cohen subsets to an uncountable cardinal κ satisfying $\kappa^{<\kappa} = \kappa$ produces a model in which no $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable subset of $\mathcal{P}(\kappa)$ separates $Club_\kappa$ from NS_κ, i.e. there is no set A definable in this way with $Club_\kappa \subseteq A$ and $A \cap NS_\kappa = \emptyset$.

Definition

Given an infinite regular cardinal κ, a subset A of ${}^{\kappa}\kappa$ has the κ-*Baire property* if there exists an open subset U of ${}^{\kappa}\kappa$ and a sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of closed nowhere dense subsets of ${}^{\kappa}\kappa$ satisfying $U_\Delta X \subseteq \bigcup_{\alpha<\kappa} A_\alpha$.

Definition

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Theorem

If κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and G is $\mathrm{Add}(\kappa, \kappa^+)$-generic over V, then all $\mathbf{\Delta}^1_1$-subsets of ${}^{\kappa}\kappa$ have the κ-Baire property in $\mathrm{V}[G]$.

Definition (L.–Schlicht)

Given an infinite regular cardinal κ, a subset X of ${}^\kappa\kappa$ *super-dense* if

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Lemma

The subsets $Club_\kappa$ and NS_κ of ${}^{\kappa}\kappa$ are super-dense.

Weakly compact cardinals

In contrast to the consistency results about greatly Mahlo cardinals presented earlier, the following theorem shows that stronger large cardinal properties outright imply that the corresponding non-stationary ideal is not $\mathbf{\Delta}_1$-definable.

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Sketch of the proof.

Given a Σ_1-formula $\varphi(v_0, \dots, v_{n-1})$ and $A_0, \dots, A_{n-1} \subseteq \mathrm{V}_\kappa$, we have

$$\varphi(A_0, \dots, A_{n-1}) \iff \{\alpha < \kappa \mid \varphi(A_0 \cap \mathrm{V}_\alpha, \dots, A_{n-1} \cap \mathrm{V}_\alpha)\} \in Club_\kappa.$$

Hence the $\mathbf{\Delta}_1$-definability of $Club_\kappa$ implies that every Σ_1-formula with parameters in $\mathrm{H}(\kappa^+)$ is equivalent to a Π_1-formula with parameters in $\mathrm{H}(\kappa^+)$,

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Hence the $\mathbf{\Delta}_1$-definability of $Club_\kappa$ implies that every Σ_1-formula with parameters in $\mathrm{H}(\kappa^+)$ is equivalent to a Π_1-formula with parameters in $\mathrm{H}(\kappa^+)$, which is impossible by the existence of universal Σ_1-formulas. $\square$

The constructible universe

The following result shows that, consistently, no non-stationary ideal is $\mathbf{\Delta}_1$-definable.

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Sketch of the proof.

Given a Σ_1-formula $\varphi(v_0, \dots, v_{n-1})$ and $A_0, \dots, A_{n-1} \subseteq \kappa$, the statement $\varphi(A_0, \dots, A_{n-1})$ holds if and only if the set of all $\alpha < \kappa$ with the property that there exists $\alpha < \beta < \kappa$ with

$$\begin{aligned} \mathrm{L}_\beta \models \mathbf{ZFC}^- \ &+ \ \text{“}\alpha \textit{ is regular}\text{''} \ + \ \text{“}S \cap \alpha \textit{ is stationary}\text{''} \\ &+ \ \varphi(A_0 \cap \kappa, \dots, A_{n-1} \cap \kappa) \end{aligned}$$

has a subset of the form $C \cap S$ for some club C in κ. □

Lightface definability

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Assume that one of the following statements holds:

- *There is a Woodin cardinal and a measurable cardinal.*
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Then no subset of $\mathcal{P}(\omega_1)$ *that separates the club filter from the nonstationary ideal is* $\mathbf{\Delta}_1(\mathrm{H}(\omega_1) \cup \{\omega_1\})$*-definable.*

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Assume that one of the above assumptions holds. Then the following statements hold for every Σ_1-formula $\varphi(v_0, v_1, v_2)$ and all $z \in \mathrm{H}(\omega_1)$:

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Theorem (Larson–Schindler–Wu)

Assume that Woodin's Axiom $(*)$ holds. Then NS_{ω_1} is not $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$-definable.

Using a recent result of Asperó and Schindler that shows that $\mathbf{MM}^{++}$ implies $(*)$, we obtain the following corollary:

Corollary

$\mathbf{MM}^{++}$ *implies that* NS_{ω_1} *is not* $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$*-definable.*

Forcing axioms and the complexity of NS_{ω_2}

Motivated by the above result, Sean Cox and I studied the complexity of NS_{ω_2} and its restrictions in the presence of forcing axioms.

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Assume that $\mathbf{MM}^{++}$ holds. If θ is a cardinal with $\theta^{\omega_2} = \theta$, then there is a $<\omega_2$-directed closed partial order that forces the following statements to hold in the corresponding generic extension of the ground model V:

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Corollary

If $\mathbf{ZFC} + \mathbf{MM}^{++}$ is consistent, then the statement

"No $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable subset of $\mathcal{P}(\omega_2)$ separates $Club_{\omega_2}$ from NS_{ω_2}"

is independent of this theory.

Definition (Shelah)

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- Given a sequence $\vec{z} = \langle z_i \mid i < \kappa^+ \rangle$ of elements of $[\kappa^+]^{<\kappa}$, a limit ordinal $\gamma < \kappa^+$ is *approachable with respect to* $\vec{z}$ if and only if there exists a sequence
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- The *Approachability ideal* $I[\kappa^+]$ *on* κ^+ is the (possibly non-proper) *normal* ideal generated by sets of the form
$$A_{\vec{z}} = \{\gamma < \kappa^+ \mid \gamma \text{ is approachable with respect to } \vec{z}\}$$
for some $\vec{z} \in {}^{\kappa^+}([\kappa^+]^{<\kappa})$.

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Let κ be an infinite regular cardinal with $\kappa^{<\kappa} \leq \kappa^+$ and let $\langle z_i \mid i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. Then the following statements hold:

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Theorem

- $\mathbf{PFA}$ *implies that* $I[\omega_2]$ *is a proper ideal.*
- $\mathbf{MM}$ *implies that if* M *is a maximum element of* $\mathcal{P}(S^2_1) \cap I[\omega_2]$ *mod* NS*, then every stationary subset of* S^2_0 *reflects stationary often in* M.

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- *If $2^\kappa = \kappa^+$ holds, then $\mathbb{P}$ satisfies the κ^{++}-chain condition.*

Corollary

Let κ be an infinite regular cardinal satisfying $\kappa^{<\kappa} \leq \kappa^+$, let $\mathbb{P}$ be the partial order given by the above theorem and let M be the maximum set in $I[\kappa^+] \cap \mathcal{P}(S^{\kappa^+}_{\kappa})$ mod NS. If G is $\mathbb{P}$-generic over V, then $NS \restriction M$ is $\mathbf{\Delta}_1(\mathrm{H}((2^\kappa)^+))$-definable in $\mathrm{V}[G]$.

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Sketch of the proof.

Work in $\mathrm{V}[G]$, let T be the subtree of ${}^{<\kappa^+}\kappa^+$ given by our theorem and define $\mathcal{S}$ to be the collection of all subsets A of M such that either there exists a closed unbounded subset C of κ^+ with $C \cap M \subseteq A$ or there exists an order-preserving embedding of the tree $T(S^{\kappa^+}_{\kappa} \setminus A)$ into the tree T.

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Then the set $\mathcal{S}$ is definable by a Σ_1-formula with parameters M, T and ${}^{<\kappa^+}\kappa^+$, and it is possible to show that $\mathcal{S}$ is equal to the collection of all subsets of M that are stationary in κ^+. □

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Thank you for listening!