Baire property and the Ellentuck-Prikry topology

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Joint work with Xianghui Shi

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We say that $\mathrm{I0}(\lambda)$ holds iff there is an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ such that $j \restriction V_{\lambda+1}$ is not the identity

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It is a large cardinal: if I0(λ) holds, then λ is a strong limit cardinal of cofinality ω, limit of cardinals that are n-huge for every $n \in \omega$.

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Theorem (Laver)

Let $\langle \kappa_n : n \in \omega \rangle$ be a cofinal sequence in λ. For every $A \subseteq V_\lambda$:

- A is Σ^1_1-definable in $(V_\lambda, V_{\lambda+1})$ iff there is a tree $T \subseteq \prod_{n\in\omega} V_{\kappa_n} \times \prod_{n\in\omega} V_{\kappa_n}$ whose projection is A;
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This is similar to AD: in fact, under AD every subset of the reals has the Perfect Set Property.

But the proof is completely different: Cramer uses heavily elementary embeddings (inverse limit reflection), while in the classical case involves games.

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Moreover, they are λ-Polish, i.e., completely metrizable and with a dense subset of cardinality λ.

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Theorem (Cramer, to appear)

Suppose I0(λ). Then every $X \subseteq V_{\lambda+1}$, $X \in L(V_{\lambda+1})$ is representable.

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- I0 $\Rightarrow$ every subset of $V_{\lambda+1}$ in $L(V_{\lambda+1})$ is representable
- Every representable set has the λ-Perfect Set Property

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Suppose $\mathrm{I0}(\lambda)$. Then it is consistent that there is κ strong limit of cofinality ω such that all the subsets of ${}^{\omega}\kappa$ in $L(V_{\kappa+1})$ have the κ-PSP, and $\neg\mathrm{I0}(\kappa)$.

The next step would be to analyze the Baire Property.

The most natural thing is to define nowhere dense sets as usual, λ-meager sets as λ-union of nowhere dense sets and λ-comeager sets as complement of λ-meager sets.

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But ${}^{\omega}\lambda = \bigcup_{f\in{}^{\omega}\omega} D_f$, therefore the whole space is λ-meagre (in fact, it is $\mathfrak{c}$-meagre), and the Baire property in this setting is just nonsense.

Or is it?

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- But there are other forcings on λ that are $<\lambda$-distributive, like Prikry forcing
- We can try to define Baire category via Prikry forcing instead of Cohen forcing.

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$\langle\beta_1,\dots,\beta_m,B_{m+1},B_{m+2}\dots\rangle \leq \langle\alpha_1,\dots,\alpha_n,A_{n+1},A_{n+2}\dots\rangle$ iff $m\geq n$ and

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$p \leq^* q$ if $p \leq q$ and they have the same stem.

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- a set $A \subseteq X$ has the λ-Baire property iff there is an open set U such that $A \triangle U$ is λ-meagre

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Let X be a topological space.

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- a set $A \subseteq X$ is λ-comeagre iff it is the complement of a λ-meagre set
- a set $A \subseteq X$ has the λ-Baire property iff there is an open set U such that $A \triangle U$ is λ-meagre
- X is a λ-Baire space iff every nonempty open set in X is not λ-meagre, i.e., the intersection of λ-many open dense sets is dense.

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Coupled with the fact that if $p \in \mathbb{P}_{\vec{\mu}}$ has stem of length n, then the intersection of $< \kappa_n$-many $\leq^*$-extensions of p is still in $\mathbb{P}_{\vec{\mu}}$, we have:

Proposition (D.-Shi)

The space $\prod_{n\in\omega}\kappa_n$ with the EP-topology is λ-Baire.

(Generalized) Mycielski Theorem (D.-Shi)

In $\prod_{n\in\omega} \kappa_n$ with the EP-topology every λ-comeagre set contains a λ-perfect set

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Test case:

Kuratowski-Ulam Theorem

Let X, Y be second-countable spaces, and $A \subseteq X \times Y$ with the Baire property. Then A is meagre iff $\{x \in X : \{y \in Y : (x, y) \in A\}$ is meagre in $Y\}$ is λ-comeagre in X.

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Topologically: Let $U\subseteq\prod_{n\in\omega}\kappa_n$ be an open set. Then for any $p\in\mathbb{P}_{\vec{\mu}}$, there is a $p^A\leq^* p$ such that either $O_{p^A}\subseteq A$, or $O_{p^A}\cap A=\emptyset$.

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So to test that A open is dense, we do not need to test it for all the basic open sets, just for a subfamily of them of size λ!

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Let $A \subseteq \prod_{n\in\omega}\kappa_n \times \prod_{n\in\omega}\kappa_n$ be with the λ-Baire property

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Let $A \subseteq \prod_{n\in\omega}\kappa_n \times \prod_{n\in\omega}\kappa_n$ be with the λ-Baire property. Then A is λ-meagre iff $\{x \in X : A_x \text{ is } \lambda\text{-meagre}\}$ is λ-comeagre.

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What about λ-universally Baire sets?

Thanks for watching