Higher independence

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Independence Number

A family $\mathscr{A} \subseteq [\omega]^\omega$ is said to be independent for any two non-empty finite disjoint subfamilies $\mathscr{A}_0$ and $\mathscr{A}_1$ the set

$$\bigcap \mathscr{A}_0 \setminus \bigcup \mathscr{A}_1$$

is infinite. It is a maximal independent family if it is maximal under inclusion and

$$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \text{ is a m.i.f.}\}$$

Boolean combinations

Let $\mathscr{A}$ be a independent and let $\mathrm{FF}(\mathscr{A})$ be the set of all finite partial functions from $\mathscr{A}$ to 2. For $h \in \mathrm{FF}(\mathscr{A})$ define

$$\mathscr{A}^h = \bigcap \{A^{h(A)} : A \in \mathrm{dom}(h)\},$$

where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \omega \backslash A$ if $h(A) = 1$.

Remark

We refer to $\{\mathscr{A}^h : h \in \mathrm{FF}(\mathscr{A})\}$ as a set of boolean combinations.

- There is a maximal independent family of cardinality $2^{\aleph_0}$.
- $\aleph_0 < \mathfrak{i} \leq 2^{\aleph_0}$.
- If $\mathscr{A}$ is a maximal independent family then $\{\mathscr{A}^h : h \in \mathrm{FF}(\mathscr{A})\}$ is an un-reaped family. Thus $\mathfrak{r} \leq \mathfrak{i}$.
- (Shelah) $\mathfrak{d} \leq \mathfrak{i}$.

$\mathfrak{i}$ vs. $\mathfrak{u}$

In the Miller model $\mathfrak{u} < \mathfrak{i}$, while Shelah devised a special ${}^{\omega}\omega$-bounding poset the countable support iteration of which produces a model of $\mathfrak{i} = \aleph_1 < \mathfrak{u} = \aleph_2$.

$\mathfrak{a}$ vs. $\mathfrak{u}$

In the Cohen model $\mathfrak{a} < \mathfrak{u}$, while assuming the existence of a measurable one can show the consistency of $\mathfrak{u} < \mathfrak{a}$. The use of a measurable has been eliminated by Guzman and Kalajdzievski.

$\mathfrak{a}$ vs $\mathfrak{i}$

In the Cohen model $\mathfrak{a} < \mathfrak{i} = \mathfrak{c}$.

Question:

Is it consistent that $\mathfrak{i} < \mathfrak{a}$?

Diagonalization

$\mathscr{A}$-diagonalization filters (F., Shelah)

Let $\mathscr{A}$ be an independent family. A filter $\mathscr{U}$ is said to be an $\mathscr{A}$-diagonalization filter if

$$\forall F \in \mathscr{U} \forall B \in \mathrm{BC}(\mathscr{A})(|F \cap B| = \omega)$$

and is maximal with respect to the above property.

Diagonalization

Lemma (F., Shelah)

If $\mathscr{U}$ is a $\mathscr{A}$-diagonalization filter and G is $\mathbb{M}(\mathscr{U})$-generic and $x_G = \bigcup\{s : \exists F(s,F) \in G\}$, then:

1. $\mathscr{A} \cup \{x_G\}$ is independent
2. If $y \in ([\omega]^\omega \backslash \mathscr{A}) \cap V$ is such that $\mathscr{A} \cup \{y\}$ is independent, then $\mathscr{A} \cup \{x_G, y\}$ is not independent.

Diagonalization

Corollary (F., Shelah)

Let κ be of uncountable cofinality. Then it is relatively consistent that $\aleph_1 < \mathfrak{i} = \kappa < \mathfrak{c}$.

Theorem (F., Shelah)

Assume GCH. Let $\kappa_1 < \cdots < \kappa_n$ be regular uncountable cardinals. Then it is consistent that $\{\kappa_i\}_{i=1}^n \subseteq \mathrm{Sp}(\mathfrak{i})$.

Theorem (F., Shelah)

Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ be measurable cardinals witnessed by κ_i-complete ultrafilters $\mathscr{D}_i \subseteq \mathscr{P}(\kappa_i)$. Then there is a ccc generic extension in which $\{\kappa_i\}_{i=1}^n = \mathfrak{sp}(\mathfrak{i}) = \{|\mathscr{A}| : \mathscr{A} \text{ m.i.f.}\}$.

Sp($\mathfrak{i}$) can be small

Theorem (F., Shelah)

Assume GCH. Let λ be a cardinal of uncountable cofinality. Let G be $\mathbb{P}$-generic filter, where $\mathbb{P}$ is the countable support product of Sacks forcing of length λ. Then $V[G] \vDash \mathrm{Sp}(\mathfrak{i}) = \{\aleph_1, \lambda\}$.

$\mathfrak{sp}(\mathfrak{i})$ can be large

Lemma

Let $\mathscr{A}$ be an independent family, $\mathscr{U}$ a diagonalization filter for $\mathscr{A}$. For each $i \in n$, let $\mathscr{U}_i = \mathscr{U}$ and let $G = \prod_{i \in n} G_i$ be a $\prod_{i \in n} \mathbb{M}(\mathscr{U}_i)$-generic filter, x_i a $\mathbb{M}(\mathscr{U}_i)$-generic real. Then in $V[G]$:

- $\mathscr{A} \cup \{x_i\}_{i \in n}$ is independent.
- For all $y \in V \cap [\omega]^\omega$ such that $\mathscr{A} \cup \{y\}$ is independent and each $i \in n$, the family $\mathscr{A} \cup \{y, x_i\}$ is not independent.

$\mathfrak{sp}(\mathfrak{i})$ can be large

Definition (F., Shelah)

Let θ be an uncountable cardinal and let $S \subseteq \theta^{<\omega_1}$ be an θ-splitting tree of height ω_1. For each $\alpha \in \omega_1$ let S_α denote the α-th level of S. Recursively, define a finite support iteration

$$\mathbb{P}_S = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha \leq \omega_1, \beta < \omega_1 \rangle$$

as follows:

- Let $\mathbb{P}_0 = \{\emptyset\}$, $\mathbb{Q}_0$ be a $\mathbb{P}_0$-name for the trivial poset.
- Let $\mathscr{A}_0 = \emptyset$ be the empty independent family and let $\mathscr{U}_0$ be an $\mathscr{A}_0$-diagonalizing real.

$\mathfrak{sp}(\mathfrak{i})$ can be large

- For $\eta \in S_1 = \mathrm{succ}_S(\emptyset)$ let $\mathscr{U}_\eta = \mathscr{U}_0$ and let $\mathbb{Q}_1 = \prod^{<\omega}_{\eta \in S_1} \mathbb{M}(\mathscr{U}_\eta)$.
- In $V^{\mathbb{P}_1 * \dot{\mathbb{Q}}_1} = V^{\mathbb{P}_2}$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.
- Let $\alpha \geq 2$ and in $V^{\mathbb{P}_\alpha}$ for each $\eta \in S_\alpha$ let $\mathscr{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}_S(\eta \restriction \xi), \xi < \alpha\}$ be an ind. family.
- For each $\eta \in S_\alpha$, let $\mathscr{U}_\eta$ be an $\mathscr{A}_\eta$-diagonalisation filter and let $\mathbb{Q}_\alpha = \prod^{<\omega}_{\eta \in S_\alpha} \mathbb{M}(\mathscr{U}_\eta)$.
- In $V^{\mathbb{P}_\alpha * \mathbb{Q}_\alpha}$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.

$\mathfrak{sp}(\mathfrak{i})$ can be large

Theorem(F., Shelah)

In $V^{\mathbb{P}_S}$ for each branch η of S the family

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \text{succ}(\eta \restriction \xi), \xi < \sigma\}$$

is a maximal independent family of cardinality θ.

Remark

The idea can be extended to adjoin with a sufficiently homogenous forcing witnesses for many distinct values in $\mathfrak{sp}(\mathfrak{i})$ simultaneously.

$\mathfrak{sp}(\mathfrak{i})$ can be large

Theorem (V.F., Shelah, 2019)

Assume GCH. Let $A \subseteq \{\aleph_n\}_{n\in\omega}$. Then there is a ccc generic extension in which

$$\mathfrak{sp}(\mathfrak{i}) = A.$$

Remark

A more involved argument shows that $\mathfrak{sp}(\mathfrak{i})$ can be quite arbitrary. A remaining open question is the consistency of $\min \mathrm{sp}(\mathfrak{i}) = \mathfrak{i} = \aleph_\omega$.

Definition

Let κ be a regular uncountable cardinal.

- Let $\mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ be the set of all finite partial functions with domain included in $\mathscr{A}$ and range the set $\{0,1\}$.
- For each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ let $\mathscr{A}^h = \bigcap\{A^{h(A)} : A \in \mathrm{dom}(h)\}$ where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \kappa \backslash A$ if $h(A) = 1$.

Definition

1. A family $\mathscr{A} \subseteq [\kappa]^\kappa$ is said to be κ-independent if for each $h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$, $\mathscr{A}^h$ is unbounded. It is maximal κ-independent family if it is κ-independent, maximal under inclusion.
2. The least size of a maximal κ-independent family is denoted $\mathfrak{i}(\kappa)$.

Lemma (V.F., D. Montoya)

Let κ be a regular infinite cardinal.

1. There is a maximal κ-independent family of cardinality 2^κ.
2. $\kappa^+ \leq \mathfrak{i}(\kappa) \leq 2^\kappa$
3. $\mathfrak{r}(\kappa) \leq \mathfrak{i}(\kappa)$
4. $\mathfrak{d}(\kappa) \leq \mathfrak{i}(\kappa)$.

Corollary

If κ is regular uncountable, then if $\mathfrak{i}(\kappa) = \kappa^+$ also $\mathfrak{a}(\kappa) = \kappa^+$.

Definition

A κ-independent family $\mathscr{A}$ is densely maximal if

- for every $X \in [\kappa]^\kappa \backslash \mathscr{A}$ and every $h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ extending h such that either $\mathscr{A}^{h'} \cap X = \emptyset$ or $\mathscr{A}^{h'} \cap (\kappa \backslash X) = \emptyset$.

Definition (V.F., D. Montoya)

Let κ be a measurable cardinal and $\mathscr{U}$ a normal measure on κ. Let $\mathbb{P}_{\mathscr{U}}$ be the poset of all pairs $(\mathscr{A}, A)$ where

- $\mathscr{A}$ is a κ-independent family of cardinality κ,
- $A \in \mathscr{U}$ is such that $\forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$, $\mathscr{A}^h \cap A$ is unbounded.

The extension relation is defined as follows: $(\mathscr{A}_1, A_1) \leq (\mathscr{A}_0, A_0)$ iff $\mathscr{A}_1 \supseteq \mathscr{A}_0$ and $A_1 \subseteq^* A_0$.

Lemma (V.F., D. Montoya)

Assume $2^\kappa = \kappa^+$. Then $\mathbb{P}_{\mathscr{U}}$ is κ^+-closed and κ^{++}-cc and if G is a $\mathbb{P}_{\mathscr{U}}$-generic filter, then

$$\mathscr{A}_G = \bigcup \{\mathscr{A} : \exists A \in \mathscr{U} \text{ with } (\mathscr{A}, A) \in G\}$$

is a densely maximal κ-independent family.

Definition

Let $\mathscr{A}$ be an independent family. The density independence filter $\mathscr{F}_{<\omega,\kappa}(\mathscr{A})$ is the filter of all $X \in \mathscr{U}$, such that $\forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ there is $h' \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A})$ such that $h' \supseteq h$ and $\mathscr{A}^{h'} \subseteq X$.

Definition

We refer to a partition $\mathscr{E}$ of κ into bounded sets as a bounded partition.

1. If $\mathscr{E}$ is a bounded partition of κ, $A \in [\kappa]^\kappa$ is such that for all $E \in \mathscr{E}(|E \cap A| \leq 1)$, we say that A is a semi-selector for $\mathscr{E}$.
2. If $\mathscr{E}$ is a bounded partition of κ and $A \in [\kappa]^\kappa$ is such that for all $E \in \mathscr{E}$, $|E \cap A| \leq 2$, then A is called a 2-semi-selector of $\mathscr{E}$.

Remark

Since $\mathscr{U}$ is a normal measure on κ, for each bounded partition $\mathscr{E}$ of κ there is a semi-selector of $\mathscr{E}$ in $\mathscr{U}$.

Definition

Let $\mathscr{F} \subseteq [\kappa]^\kappa$. We say that:

1. $\mathscr{F}$ is a κ-P-set if every $\mathscr{H} \subseteq \mathscr{F}$ of cardinality $\leq \kappa$ has a pseudo-intersection in $\mathscr{F}$;
2. $\mathscr{F}$ is a κ-Q-set if every bounded partition of κ has a 2-semi-selector in $\mathscr{F}$.

Selective Independence at κ

Theorem (V.F., D. Montoya)

The density independence filter of $\mathscr{F}_{<\omega,\kappa}(\mathscr{A}_G)$ is both a κ-Q-set and a κ-P-set, which is generated by $\{A : \exists \mathscr{A}(\mathscr{A}, A) \in G\}$.

Theorem (V.F., D. Montoya)

The generic maximal independent family $\mathscr{A}_G$ adjoined by $\mathbb{P}_{\mathscr{U}}$ over a model of GCH remains maximal after the κ-support product $\mathbb{S}_\kappa^\lambda$.

Corollary

Let κ be a measurable cardinal. There is a cardinal preserving generic extension in which

$$\mathfrak{a}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{r}(\kappa) = \mathfrak{i}(\kappa) = \kappa^+ < 2^\kappa.$$

Question

Let κ be a regular uncountable cardinal. Is it consistent that

$$\kappa^+ < \mathfrak{i}(\kappa) < 2^\kappa?$$

Definition

Let $\mathscr{A}$ be a κ-independent family. A κ-complete filter $\mathscr{F}$ is said to be an *κ-diagonalization filter for $\mathscr{A}$* if $\forall F \in \mathscr{F} \forall h \in \mathrm{FF}_{<\omega,\kappa}(\mathscr{A}) |F \cap \mathscr{A}^h| = \kappa$ and $\mathscr{F}$ is maximal with respect to the above property.

Question

Given a κ-independent family $\mathscr{A}$ is there a κ-diagonalizazion filter for $\mathscr{A}$? Is there a large cardinal property which guarantees the existence of such maximal filter?

Definition

Let κ be a regular uncountable cardinal, $\mathscr{A} \subseteq [\kappa]^\kappa$ of size at least κ.

1. Let $\mathrm{FF}_{<\kappa,\kappa}(\mathscr{A}) = \{h : \mathscr{A} \to \{0,1\} :$ such that $|\mathrm{dom}(h)| < \kappa\}$.
2. For each $h \in \mathrm{FF}_{<\kappa,\kappa}(\mathscr{A})$ let $\mathscr{A}^h = \bigcap\{A^{h(A)} : A \in \mathrm{dom}(h)\}$ where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \kappa \setminus A$ if $h(A) = 1$.
3. $\mathscr{A}$ is said to be strongly-κ-independent if for each $h \in \mathrm{FF}_{<\kappa,\kappa}(\mathscr{A})$, $\mathscr{A}^h$ is unbounded.
4. $\mathscr{A}$ is maximal strongly-κ-independent family if it is κ-independent, maximal under inclusion.

Lemma (V.F., D. Montoya)

Let κ be a regular infinite cardinal.

1. For κ strongly inaccessible, there is a strongly-κ-independent family of cardinality 2^κ.
2. If $\mathscr{A}$ is strongly-κ-independent and $|\mathscr{A}| < \mathfrak{r}(\kappa)$ then $\mathscr{A}$ is not maximal.
3. Suppose $\mathfrak{d}(\kappa)$ is such that for every $\gamma < \mathfrak{d}(\kappa)$, $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$. If $\mathscr{A}$ is strongly-κ-independent and $|\mathscr{A}| < \mathfrak{d}(\kappa)$ then $\mathscr{A}$ is not maximal.

Corollary

Thus if

$$\mathfrak{i}_s(\kappa) = \min\{|\mathscr{A}| : \mathscr{A} \text{ maximal strongly-}\kappa\text{-independent family}\}$$

is defined, then

- $\kappa^+ \leq \mathfrak{i}_s(\kappa) \leq 2^\kappa$;
- $\mathfrak{r}(\kappa) \leq \mathfrak{i}_s(\kappa)$;
- if for every $\gamma < \mathfrak{d}(\kappa)$, $\gamma^{<\kappa} < \mathfrak{d}(\kappa)$, then $\mathfrak{d}(\kappa) \leq \mathfrak{i}_s(\kappa)$.

Theorem (Kunen, 1983)

1. The existence of a maximal strongly-ω_1-independent family implies CH and the existence of a weakly inaccessible cardinal between ω_1 and 2^{ω_1};
2. The existence of a measurable cardinal is equiconsistent with the existence of a maximal strongly-ω_1-independent family.

Thank you!

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