Forcing over choiceless models (4/4)

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Outline

0. Introduction
1. Adding Cohen subsets by $\mathrm{Add}(A, 1)$
   - Preliminaries
   - Cohen's first model and Dedekind finite sets A
   - Properties of $\mathrm{Add}(\kappa, 1)$ and fragments of DC
   - Adding Cohen subsets over $L(\mathbb{R})$
2. Chain conditions and cardinal preservation
   - Variants of the ccc
   - An iteration theorem
   - A ccc_2 forcing that collapses ω_1
3. Generic absoluteness principles inconsistent with choice
   - Hartog numbers
   - Very strong absoluteness and consequences
   - Gitik's model
4. Random algebras without choice
   - Completeness
   - ccc_2^*

Random algebras

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Fact

Any random real over V is random over every inner model $M \subseteq V$ of ZF.

Random algebras

Any Borel subset A of 2^α with the product topology has a countable support, so its Lebesgue measure $\mu(A)$ is well-defined.

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$\mathbb{R}_{\omega+\omega}$ is

- a "product" of random reals with "random support"
- a 2-step iteration of random forcing

Random algebras

Always suppose that α is multiplicatively closed.

Definition

An α-Borel code (for a subset of 2^{α}) is an element x of 2^{α}.

- If $x_0 x_1 = 00$, then x codes a basic open set in the rest of x
- If $x_0 x_1 = 01$, then x codes the complement of the set in the rest of x
- If $x_0 x_1 = 10$, then x codes the union of the sets listed in the rest of x

A Borel code for a subset of 2^{α} is an α-Borel code with countable support.

- Let B_x denote the Borel (α-Borel) set coded by x.

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Definition

The random algebra on α is the set of Borel codes for subsets of 2^α. It is quasi-ordered by $x \leq y$ if $B_x \subseteq_\mu B_y$.

Fact

- *If 2^ω is a countable union of countable sets, then every subset of 2^ω is Borel*
- *If ω_1 is singular, then there exists a Borel subset of 2^ω without a Borel code.*

Random algebras

Proposition

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- locally complete
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Local completeness is a property reminiscent of the fact that a maximal antichain in random forcing in an inner model is an antichain in V.

It is used to show that random algebras are narrow.

Corollary

- *Random algebras can be iterated without collapsing cardinals.*

All iterations have finite support.

Exercise

1. Show that a countable support iteration of Cohen forcing collapses ω_1, if ω_1 is singular.
2. Find a sufficient condition on a forcing $\mathbb{P}$ that shows this for $\mathbb{P}$.

Corollary

- *$\mathbb{R}_*$-absoluteness implies that all uncountable cardinals are singular.*

$\mathbb{R}_*$-absoluteness states that any $\mathbb{R}_\kappa$-generic extension has the same theory as V.

Random algebras

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Theorem (Lebesgue's density theorem)

Suppose that A is a Lebesuge measurable subset of 2^ω. The set

$$D(A) := \{x \in 2^\omega \mid \lim_{n\to\infty} \frac{\mu(A \cap N_t)}{\mu(N_t)} = 1\}$$

of its density points satisfies $\mu(A \triangle D(A)) = 0$.

Hence A can be reconstructed up to a nullset from relative measures on basic open sets.

Random algebras

We modify this reconstruction to 2^α. Let $2^{(\alpha)}$ denote the set of finite partial functions $f\colon \alpha \rightharpoonup 2$.

Definition

For any $A \in \mathbb{R}_\alpha$, call

$$\mu_A = \langle \mu_{A,t} := \frac{\mu(A \cap N_t)}{\mu(N_t)} \mid t \in 2^{(\alpha)} \rangle$$

its footprint.

We have $A \leq B \Leftrightarrow \mu_{A,t} \leq \mu_{B,t}$ for all $t \in 2^{(\alpha)}$.

Definition

Suppose that $x \in 2^\alpha$ and $\vec{\mu} = \langle \mu_t \mid t \in 2^{(\alpha)} \rangle$ is a sequence in $\mathbb{R}_{\geq 0}$.

1. For any $\epsilon > 0$, x is an ϵ-density point of $\vec{\mu}$ if

$$\exists s \, \forall t \supseteq s \; \mu_t > 1 - \epsilon.$$

2. x is a density point of $\vec{\mu}$ if x is an ϵ-density point of $\vec{\mu}$ for all $\epsilon \in \mathbb{Q}^+$.

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Let $D(\mu)$ denote the α-Borel code induced by 2. Its definition is absolute between transitive models of ZF.

$D(\mu)$ is not a Borel code, but we can reduce it to one.

Random algebras

Any α-Borel code A can be reduced to a Borel code as follows.

Definition

The reduct $\mathrm{re}(A)$ of A is the following Borel code.

1. If A codes a basic open set, then $\mathrm{re}(A) = A$.
2. If A_0 codes $\neg A_1$, then $\mathrm{re}(A_0)$ codes $\neg\mathrm{re}(A_1)$.
3. If A codes $\bigcup_{i<\alpha} A_i$, then $\mathrm{re}(A)$ codes $\bigcup_{i\in I} \mathrm{re}(A_i)$, where
    - I is the largest subset of α such that for each $j \in I$, A_j adds measure to $\bigcup_{j\in I\cap j} \mathrm{re}(A_i)$.

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Fact

In every outer model M where α is countable,

- $\mathrm{re}(A) =_\mu A$
- $D(\mu_A) =_\mu A$ *by Lebesgue's density theorem in M.*

$\mathrm{re}(A) =_\mu A$ may fail in V, since A may be an ω_1 length union of singletons and CH holds.

Random algebras

By the previous fact, $A^* := \mathrm{re}(D(\mu_A)) =_\mu A$ in V. The map $A \mapsto A^*$ picks a representative in each equivalence class.

- We can replace $\mathbb{R}_\alpha$ by the set of A^*. This definition of $\mathbb{R}_\alpha$ is absolute between transitive models of ZF.

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Given a subset X of $\mathbb{R}_\alpha$, we construct its supremum. Let

$$\mu_{X,t} := \sup_{A \in X} \mu_{A,t}$$

$$\mu_X := \langle \mu_{X,t} \mid t \in 2^{(\alpha)} \rangle$$

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Fact

In any outer model M of V where α is countable, $D(\mu_X)$ is a least upper bound for X.

Proof. If B is an upper bound for X, then $\mu_A \leq \mu_B$ for all $A \in X$ and hence $\mu_X \leq \mu_B$. Then $D(\mu_X) \leq D(\mu_B) =_\mu B$. $\square$

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Fact

$\mathbb{R}_\alpha$ is complete.

Proof. $\mathrm{re}(D(\mu_X))$ is a least upper bound for X in some outer model and hence in V. □

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- It holds for random algebras and all well-ordered forcings.
- It implies that any $\mathbb{P}$-generic filter over V is $(\mathbb{P} \cap \mathrm{HOD}_x)$-generic over HOD_x.
- With locally ccc, it implies uniformly narrow. Hence well-ordered locally ccc forcings and random algebras can be iterated while preserving cardinals

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A forcing $\mathbb{P}$ is locally complete if there exists a finite set containing $\mathbb{P}$ such that: For any nonempty $A \subseteq \mathbb{P}$, there exists some $p \leq \sup(A)$ in $\mathbb{P} \cap \mathsf{HOD}_{x \cup \{A\}}$.

Random algebras

Lemma

Suppose that $\mathbb{P}$ is a locally complete forcing.

1. *If $\mathbb{P}$ is ccc, then it is narrow.*
2. *If $\mathbb{P}$ is locally ccc, then it is uniformly narrow.*

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Proof sketch. We prove 2. The proof of 1 is similar.

Suppose that $f\colon \mathbb{P} \rightharpoonup \mu$ is a partial $\|$-homomorphism (generalised antichain).

Let $A := \mathrm{dom}(f)$, $A_\alpha := f^{-1}(\{\alpha\})$ and $\vec{A} = \langle A_\alpha \mid \alpha \in \mathrm{ran}(f)\rangle$.

Let $H := \mathrm{HOD}_{x\cup\{\vec{A}\}}$, where x witnesses that $\mathbb{P}$ is locally complete.

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For each $\alpha \in \mathrm{ran}(f)$, there exists some $p_\alpha \in \mathbb{P} \cap H$ with $p_\alpha \leq \mathbf{sup}(A_\alpha)$. We can assume that p_α is least such p in the canonical well-order of H.

- Then $p_\alpha \perp p_\beta$ for all $\alpha \neq \beta$.

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Let $\lambda \leq \theta^+$ be the chain condition of $\mathbb{P} \cap H$ in H, i.e., the least ν such that there exists no antichain of size ν. This is always a regular cardinal in models of ZFC.

- Then $|\mathrm{ran}(f)|^H < \lambda \leq \theta^+$.

Let $G(f)$ be the least injective function $F\colon \mathrm{ran}(f) \to \theta$ in H. $\square$

Problem

Is $\mathbb{R}_\kappa$ ccc$_2$ for every infinite cardinal κ?
Can there be Dedekind finite antichains?

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Is $\mathbb{R}_{\omega_1}$ σ-linked in ZFC*?*

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Problem

Over Gitik's model, can you add fresh subsets of some uncountable cardinal?