A SURVEY ON FORCING OVER CHOICELESS MODELS

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Abstract. We describe known results about properties of forcings such as chain conditions and their effect on generic extensions, in particular over models where DC fails. Some of this is taken from joint work with Daisuke Ikegami.

Contents

1. Introduction

We survey known results, some folklore and some recent, about forcing over arbitrary models of ZF, in particular about adding Cohen subsets, variants of the countable chain condition and generic absoluteness. Some of them address a question asked by Asaf Karagila on mathoverflow (2012): "I am looking for theorems such as c.c.c. forcing does not collapse cardinals and similar theorems extended to the choiceless context if possible, or the strength of choice needed for these theorems to hold."

1. **Cohen subsets.** We show that forcing with $\mathrm{Add}(A,1)$ depends heavily on A and fragments of choice. We look at $\mathrm{Add}(A,1)$ for a Dedekind finite set A in Cohen's first model. We characterise when $\mathrm{Add}(\lambda^+,1)$ preserves λ^+ by fragments of DC. Since $\mathrm{Add}(\lambda^+,1)$ collapses λ^+ over $L(\mathbb{R})$, we force with $\mathrm{Add}(\kappa,1)^{\mathrm{HOD}}$.
2. **Chain conditions.** We discuss variants of the ccc and whether they preserve cardinals. We prove an iteration theorem for a variant of the ccc. We discuss Karagila and Schweber's result that ccc_2 forcing can collapse ω_1.
3. **Generic absoluteness.** We study very strong generic absoluteness principles that are inconsistent with choice and their consequences. We study Gitik's model where all infinite cardinals have countable cofinality.
4. **Random algebras.** We prove that random algebras with κ many generators are complete and satisfy a version of the ccc. We show that several results above can be applied to them.

This includes work of Woodin, Cunningham [Cun23], Ikegami, Trang [IT23], Karagila and Schweber [KS22]. Besides the work mentioned here, there has been recent work on properties of forcings over arbitrary models of ZF by Karagila and Schilhan [KS23].

Forcing over models of ZF up to the forcing theorem is the same for ZFC. This can be found in [Kun14] and the Boolean-valued approach in [Jec03, HS12]. Symmetric models and Cohen's first model (the Halpern-Levy model) are studied in [Jec08].

2. Preliminaries

All models are models of ZF. A forcing is a set quasi-order $\mathbb{P}$ with $\leq$ (partial order without reflexivity) with a maximal element $1_{\mathbb{P}}$. We write $p \parallel q$ if p and q are compatible, i.e., there exists some $r \leq p, q$, and $p \perp q$ if they are incompatible. If $\mathbb{P}$ is separative, i.e., $p \nleq q$ implies $\exists r \leq p\ r \perp q$, then $\mathbb{P}$ is dense in its Boolean completion $\mathbb{B}(\mathbb{P})$, the set of regular open subsets of $\mathbb{P}$. If $\mathbb{P}$ is not separative, we form its separative quotient $\mathbb{P}_{\mathrm{sep}}$ by letting $p \sim q$ if $\forall r\ (p \parallel r \Leftrightarrow q \parallel r)$ and work with $\mathbb{B}(\mathbb{P}_{\mathrm{sep}})$. If $\langle x_i \mid i \in I \rangle$ is a family of $\mathbb{P}$-names, $\{x_i \mid i \in I\}^\bullet := \{(\mathbb{1}_{\mathbb{P}}, \dot{x}_i) \mid i \in I\}$ is a name for $\{\dot{x}_i \mid i \in I\}$.

For sets A and B, $A \leq B$ means there exists an injective function $f\colon A \to B$. $A \leq^* B$ means there exists a surjective function $g\colon B \to A$. The partition principle states that $A \leq^* B$ implies $A \leq B$ for all sets A and B. One of the oldest open problems in set theory asks whether the partition principle implies the axiom of choice.

A tree T on a set A consists of sequences of elements of A ordered by end extension.

Definition 2.1. Suppose that $\kappa \in \mathrm{Card}$ and $\delta \in \mathrm{Ord}$.

1. The axiom of choice AC_κ for families of size κ states that for any $F\colon \kappa \to V$ with $f(\alpha) \neq \varnothing$ for all $\alpha < \kappa$, there exists a choice function $f\colon \kappa \to V$ with $f(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.
2. Suppose that A is a class. The axiom of dependent choice $\mathsf{DC}_\delta(A)$ for trees on A states that any $<\kappa$-closed tree T on A[1] (i.e., such that every strictly increasing sequence in T of length $\alpha < \delta$ has an upper bound) has a branch of length δ. DC_δ denotes $\mathsf{DC}_\delta(V)$. Finally, $\mathsf{DC}_{<\delta}(A)$ and $\mathsf{DC}_{<\delta}$ are defined in the obvious way.
3. The axiom of determinacy AD states that any two-player game with perfect information of length ω with moves in ω is determined, i.e., one of the players has a winning strategy.

Exercise 2.2. Suppose that $\kappa \in \mathrm{Card}$, $\gamma \in \mathrm{Ord}$, $\delta < \gamma^+$ and A, B are classes.

1. DC_κ implies AC_κ.
2. $\mathsf{DC}_\lambda(A)$ implies $\mathsf{DC}_\kappa(A)$ for $\kappa \leq \lambda$.
3. $\mathsf{DC}_\gamma(A)$ for all sets A implies DC_γ.
4. If $A \leq^* B$, then $\mathsf{DC}_\gamma(B) \Rightarrow \mathsf{DC}_\gamma(A)$.
5. If $A^\gamma \leq^* A$, then $\mathsf{DC}_\gamma(A) \Rightarrow \mathsf{DC}_\delta(A)$.

Every cardinal is an ordinal.

Exercise 2.3. DC suffices to show that every σ-closed forcing preserves ω_1.

Exercise 2.4. DC holds if and only if for every infinite cardinal θ, there exists a countable elementary substructure $M \prec H_\theta$.

This shows that properness may become vacuous without DC. Aspero and Karagila showed that DC suffices to show that proper forcings have the usual properties [AK21].

A set X is *Dedekind finite* if it is infinite and there exists no injective function $f\colon \mathbb{N} \to X$. As usual, a set is finite if is has size n for some $n \in \mathbb{N}$.

Exercise 2.5. DC implies that there exist no Dedekind finite sets.

Remark 2.6.

1. Dedekind finite sets of reals can exist in models of ZF. If A is such as set, then linearity of $\leq$ breaks down immediately above the finite sets, since A and ω are incomparable.
2. ω_1 may be measurable, in fact AD implies the club filter on ω_1 is an $<\omega_1$-complete ultrafilter on ω_1 [Kan08].
3. ω_1 may be singular. This holds in the $L(\mathbb{R})$ of a $\mathrm{Col}(\omega, {<}\aleph_\omega)$-generic extension.
4. The set $\mathbb{R}$ of reals may be a countable union of countable sets. Then every set of reals is a Borel set according to the usual definition of Borel sets in ZFC [Jec08].

Exercise 2.7. Suppose that ω_1 is singular. Show that $\mathrm{Add}(\omega_1, 1) = \{p\colon \alpha \to 2 \mid \alpha < \kappa\}$ collapses ω_1.

[1]The height of T is arbitrary. If we consider only trees of height δ, the axiom is weaker.

Even a single regular cardinal κ allows one to do some interesting forcings. For instance, one can force the dominating number to be κ [IS22]. Models without any uncountable regular cardinals seem particularly hard to force over. Gitik constructed such a model [Git80].

Here is a bit of information about forcing choice. One cannot force choice over Gitik's model, since an end segment of cardinals in the extension has countable cofinality. Blass and Usuba characterised the possibility of forcing choice as follows.

Theorem 2.8 (Blass [Bla79]). *The following statements are equivalent:*

(a) $\exists S\ \forall X\ \exists g{:}\, S \times \mathrm{Ord} \to X$ *is surjective. This principle is called SVC (small violations of choice).*
(b) $\exists S\ \forall X\ \exists f{:}\, X \to S \times \mathrm{Ord}$ *is injective.*
(c) *There exists a forcing $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}}$ forces choice.*

Theorem 2.9 (Usuba [Usu18]). *The following statements are equivalent:*

(a) *There exists an inner (i.e., transitive class) model M of ZFC and a set X such that $V = M(X)$, where $M(X)$ denotes the least transitive model N of ZF with $M \subseteq N$ and $X \in N$.*
(b) *V is a symmetric extension of some inner model M of ZFC.*
(c) *There exists a forcing $\mathbb{P}$ such that $\mathbb{1}_{\mathbb{P}}$ forces choice.*

Theorem 2.10 (Karagila [Kar18]). *If x is a Cohen real over L, then there is an intermediate model $L \subseteq M \subseteq L[x]$, the* Bristol model*, that is not of the form $L(X)$ for a set X.*

It follows from Usuba's result that choice cannot be forced over M. [cite Karagila]

3. Cohen subsets

3.1. **Cohen subsets of Dedekind finite sets.** We sketch how symmetric models are used to construct models of ZF without choice from models of ZFC. Let $\mathbb{P}$ be a notion of forcing and π be an automorphism of $\mathbb{P}$. Then π acts on $\mathbb{P}$-names via

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

Suppose that $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$. A *filter* of subgroups over $\mathscr{G}$ is a nonempty family $\mathscr{F}$ of subgroups of $\mathscr{G}$ closed under finite intersections and supergroups. $\mathscr{F}$ is *normal* if whenever $H \in \mathscr{F}$ and $\pi \in \mathscr{G}$, $\pi H \pi^{-1} \in \mathscr{F}$ as well.

We call $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ a *symmetric system* if $\mathbb{P}$ is a notion of forcing, $\mathscr{G}$ is a subgroup of $\mathrm{Aut}(\mathbb{P})$, and $\mathscr{F}$ is a normal filter of subgroups over $\mathscr{G}$. $\mathrm{sym}_{\mathscr{G}}(\dot{x})$ denotes the group $\{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\}$, the stabiliser of $\dot{x}$. $\dot{x}$ is called *$\mathscr{F}$-symmetric* if $\mathrm{sym}_{\mathscr{G}}(\dot{x}) \in \mathscr{F}$. If is called *hereditarily* $\mathscr{F}$-symmetric if this holds hereditarily for the names in $\dot{x}$. The class $\mathsf{HS}_{\mathscr{F}}$ denotes the class of all hereditarily $\mathscr{F}$-symmetric names. We usually omit the subscript $\mathscr{F}$.

Lemma 3.1 (Symmetry lemma [Jec03, Lemma 14.37])**.** *Suppose that $p \in \mathbb{P}$, $\pi \in \mathrm{Aut}(\mathbb{P})$ and $\dot{x}$ is a $\mathbb{P}$-name. Then*

$$p \Vdash \varphi(\dot{x}) \iff \pi p \Vdash \varphi(\pi\dot{x}).$$

Theorem 3.2 ([Jec03, Lemma 14.37])**.** *Suppose that $G \subseteq \mathbb{P}$ is a V-generic filter and $M := \mathsf{HS}^G := \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}\}$. Then M is a transitive class model of ZF in $V[G]$ such that $V \subseteq M$.*

HS^G is called a *symmetric extension* (of V).

We briefly describe Cohen's first model as an example of a model of set theory where the axiom of choice fails. It is described in detail in [Jec08, Section 5.3]. Suppose that V is a model of ZFC and $\mathbb{P}$ is $\mathrm{Add}(\omega,\omega)$. The group $\mathscr{G}$ consists of all finitary permutations of ω acting on the first coordinate of $\mathbb{P}$ via

$$\pi p(\pi n, m) = p(n, m).$$

Moreover, $\mathscr{F}$ is the filter of subgroups generated by $\{\mathrm{fix}(E) \mid E \in [\omega]^{<\omega}\}$, where $\mathrm{fix}(E) := \{\pi \in \mathscr{G} \mid \pi \upharpoonright E = \mathrm{id}\}$. If $\mathrm{fix}(E) \subseteq \mathrm{sym}(\dot{x})$, we say that E is a *support* for $\dot{x}$.

For each $n \in \omega$, $\dot{a}_n := \{\langle p, \check{m}\rangle \mid p(n,m) = 1\}$ is a name for the nth Cohen real and $\dot{A} := \{\dot{a}_n \mid n \in \omega\}^\bullet$ is a name for the set of them. We have $\pi\dot{A} = \dot{A}$, since $\pi\dot{a}_n = \dot{a}_{\pi^{-1}n}$ for all $\pi \in \mathscr{G}$. Hence $\dot{A} \in \mathsf{HS}$.

Proposition 3.3. $\mathbb{1} \Vdash \dot{A}$ *is Dedekind finite.*

Proof. Suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash \dot{f}{:}\check{\omega} \to \dot{A}$. Let E be a support for $\dot{f}$, and without loss of generality $\mathrm{supp}(p) \subseteq E$ as well.

We claim that p forces that the range of $\dot{f}$ is a subset of $\{\dot{a}_n \mid n \in E\}$ and hence finite. To see this, pick some $n \notin E$. Suppose towards contradiction that $q \leq p$ is a condition such that $q \Vdash \dot{f}(\check{m}) = \dot{a}_n$ for some $m < \omega$. Let $j \notin E \cup \mathrm{supp}(q)$ and π the 2-cycle $(n\ j)$. Then the following statements hold:

1. $\pi \in \mathrm{fix}(E)$ and therefore $\pi p = p$ and $\pi\dot{f} = \dot{f}$.
2. $\pi\dot{a}_n = \dot{a}_j$.
3. $\pi q \Vdash \pi\dot{f}(\pi\check{m}) = \pi\dot{a}_n$ and therefore $\pi q \Vdash \dot{f}(\check{m}) = \dot{a}_j$.
4. πq is compatible with q.

The last claim holds since $j \notin \mathrm{supp}(q)$ and π only swaps the coordinates j and n. Thus, $q \cup \pi q \Vdash$ "$\dot{a}_n = \dot{f}(\check{m}) = \dot{a}_j$". This is impossible, since $\mathbb{1}_{\mathbb{P}} \Vdash \dot{a}_n \neq \dot{a}_j$. $\square$

We fix a $\mathbb{P}$-generic filter G over V and write $M := \mathsf{HS}^G$ for the Cohen model. We write a_n for $\dot{a}_n^G$ and A for $\dot{A}^G$. One can show as in [Jec08, Lemma 5.25 & Lemma 5.26] that $M = V(A)$, the smallest transitive subclass of $V[G]$ that is a model of ZF, contains V and has A as an element.

For a Dedekind finite set A, let $\mathrm{Add}(A,1) := \{p{:}\, F \to 2 \mid F \subseteq A \text{ is finite }\}$ ordered by reverse inclusion. We identify a subset of A with its characteristic function. So $\mathrm{Add}(A,1)$ adds a new subset of A.

Let $f{:}\,\omega \to 2$ be a finite partial function, and let $\dot{q}_f$ denote the following name:

$$\dot{q}_f = \left\{\langle \dot{a}_n, \check{f(n)}\rangle^\bullet \mid \alpha \in \operatorname{dom} f\right\}^\bullet.$$

Then $\dot{\mathbb{Q}} \coloneqq \{\dot{q}_f \mid f{:}\,\omega \to 2 \text{ is a finite partial function}\}^\bullet$ is a name for $\operatorname{Add}(\kappa,1)^M$. In particular, if $q \in \operatorname{Add}(A,1)$, then there exists a finite partial function $f{:}\,\omega \to 2$ in V such that $q = \dot{q}_f^G$.

Theorem 3.4. *Suppose that G is an $\operatorname{Add}(A,1)$-generic filter over M. Then M and $M[G]$ have the same sets of ordinals.*

Proof. Let $\dot{X} \in \mathsf{HS}$ be a $\mathbb{P}$-name for an $\operatorname{Add}(A,1)$-name for a set of ordinals. Any $\pi \in \mathscr{G}$ acts on $\mathbb{P} * \operatorname{Add}(\dot{A},1)^\bullet$ via

$$\pi\langle p, \dot{q}_f\rangle = \langle \pi p, \pi \dot{q}_f\rangle = \langle \pi p, \dot{q}_{f\circ\pi}\rangle.$$

We write $\langle p, \dot{q}_f\rangle \Vdash^{\mathsf{HS}} \varphi$ to mean that p forces that $\dot{q}_f \Vdash \varphi$ holds in $V(A)$.

Let $\langle p, \dot{q}_f\rangle$ be a condition which forces that $\dot{X}$ is a name for a set of ordinals. Let E be a support for $\dot{X}$. Then E is a finite subset of ω with $\operatorname{fix}(E) \subseteq \operatorname{sym}(\dot{X})$. We can assume that $\operatorname{supp}(p) = E = \operatorname{dom} f$.

Suppose that $\langle p_0, \dot{q}_{f_0}\rangle$ and $\langle p_1, \dot{q}_{f_1}\rangle$ are two extensions of $\langle p, \dot{q}_f\rangle$. Again, we can assume that $\operatorname{supp}(p_i) = \operatorname{dom} f_i$ for $i < 2$.

We claim that $\dot{X}$ is a name for a set in M. It suffices to show that if $p_1 \upharpoonright E = p_2 \upharpoonright E$, then p_0 and p_1 must agree on any statement of the form $\check{\alpha} \in \dot{X}$. This is because there is an automorphism in $\operatorname{fix}(E)$ moving $\operatorname{supp}(p_0) \setminus E$ to be disjoint of $\operatorname{supp}(p_1)$, which means that $\langle \pi p_0, \pi \dot{q}_{f_0}\rangle$ is compatible with $\langle p_1, \dot{q}_{f_1}\rangle$ while $\pi\check{\alpha} = \check{\alpha}$ and $\pi\dot{X} = \dot{X}$. Here we use the fact that $\operatorname{dom} f = E$ and $\operatorname{dom} f_i = E_i$ for $i < 2$. It follows that if $\langle p_i, \dot{q}_{f_i}\rangle \Vdash \check{\alpha} \in \dot{X}$, then $\langle p_i \upharpoonright E, \dot{q}_{f_i \upharpoonright E}\rangle = \langle p_i \upharpoonright E, \dot{q}_f\rangle$ already forces this statement. $\square$

In particular, forcing with $\operatorname{Add}(A,1)$ over M preserves all cardinals and cofinalities.

Remark 3.5. In Cohen's first model, every set is linearly ordered by results of Halpern and Levy. Cohen's second model N witnesses a failure of AC_ω by a sequence $\langle F_n \mid n \in \omega\rangle$ of pairwise disjoint finite sets. In particular, the union A of these sets cannot be linearly ordered. We now force with $\operatorname{Add}(A,1)$ over the model to add a function $g{:}\,A \to 2$. A density argument shows that $\{n \in \omega \mid g[F_n] = \{0\}\}$ is a Cohen real over N. So in contrast to the situation in Cohen's first model, $\operatorname{Add}(A,1)$ adds new reals. A result characterising when this happens for an arbitrary Dedekind finite set A can be found in [KS20, Section 6].

3.2. **Cohen subsets of cardinals.** The forcing $\operatorname{Add}(\kappa,1) = \{p{:}\,\alpha \to 2,\ \alpha < \kappa\}$ ordered by reverse inclusion is not $<\kappa$-closed unless κ is regular. However, studying $\operatorname{Add}(\kappa,1)$ for successors κ tells us much about $<\kappa$-closed forcings, since $\operatorname{Add}(\kappa,1)$ is forcing equivalent (i.e., the two forcings have the same generic extensions) to

$$\operatorname{Add}_*(\kappa,1) \coloneqq \{(f,g) \mid f \in \operatorname{Add}(\kappa,1),\ g{:}\operatorname{dom}(f) \to |\operatorname{dom}(f)| \text{ is bijective}\},$$

ordered by reverse inclusion in the first coordinate. (The second coordinate is not used.) $\operatorname{Add}_*(\lambda^+,1)$ is $<\lambda^+$-closed for any cardinal λ.

Exercise 3.6. Show that $\operatorname{Add}(\kappa,1)$ wellorders $2^{<\kappa}$. In particular, $\operatorname{Add}(\omega_1,1)$ wellorders the reals.

By a $<\lambda$-distributive forcing $\mathbb{P}$, we mean one such that for any sequence $\langle U_i \mid i < \alpha\rangle$ of dense open subsets of $\mathbb{P}$ of length $\alpha < \lambda$, $\bigcap_{i<\alpha} U_i \neq \varnothing$. A λ-distributive forcing does not add element of V^λ. (The converse implication may fail by [KS23].)

Lemma 3.7 (folklore)**.** *For any infinite cardinal λ, DC_λ holds if and only if every $<\lambda^+$-closed forcing is $<\lambda^+$-distributive.*

Proof. Using DC_λ, we can find a sequence $\langle p_i \mid i < \lambda\rangle$ with $p_i \in U_i$ for all $i < \lambda$. Any lower bound p of this sequence is in $\bigcap_{i<\lambda} U_i$. Conversely, if DC_λ fails then there exists a $<\lambda$-closed tree T with no λ-sequences, so T is $<\lambda^+$-closed. Forcing with $(T,\geq)$ adds a new λ-sequence, so T cannot be $<\lambda$-distributive. $\square$

We aim for a similar result for Cohen subsets.

Lemma 3.8. *Suppose that $\lambda \in \mathrm{Card}$ and $\mathbb{P} = \mathrm{Add}(\lambda^+, 1)$. The following conditions are equivalent:*

(a) $\mathsf{DC}_\lambda(2^\lambda)$.
(b) *$\mathbb{P}$ is λ-distributive.*
(c) *$\mathbb{P}$ does not change V^λ.*

Proof. (a) ⇒ (b) ⇒ (c) are as in the previous lemma. (c) ⇒ (a): $\mathbb{P}$ wellorders $(2^\lambda)^{<\lambda}$. Thus the given tree T has a λ-branch in the generic extension. Since $\mathbb{P}$ does not change V^λ, this branch is in V. □

Proposition 3.9 ([IS22, Section 3.5]). *Suppose that $\lambda \in \mathrm{Card}$ and $\mathbb{P} = \mathrm{Add}(\lambda^+, 1)$. The following conditions are equivalent:*

(a) $\mathsf{DC}_\lambda(2^\lambda)$.
(b) *$\mathbb{P}$ preserves all cardinals $\alpha \leq \lambda^+$ and the cofinality of all ordinals $\alpha \leq \lambda^+$.*
(c) *$\mathbb{P}$ preserves λ^+ as a cardinal.*
(d) *$\mathbb{P}$ forces that λ^+ is regular.*

Proof. (a) ⇒ (b) holds by the previous lemma and (b) ⇒ (c) is clear. (c) ⇒ (d) holds since $\mathbb{P}$ wellorders 2^λ. (d) ⇒ (a): Towards a contradiction, suppose that $\nu \leq \lambda$ is least such that $\mathbb{P}$ adds new elements to V^ν. It suffices to show that $\mathbb{P}$ is ν-distributive. Suppose that $\langle U_i \mid i < \nu\rangle \in V$ is a sequence of dense open subsets of $\mathbb{P}$ and G is a $\mathbb{P}$-generic filter over V. Since $\mathbb{P}$ wellorders $2^{<\lambda^+}$ and does not change $V^{<\nu}$, we can construct a strictly decreasing sequence $\langle p_i \mid i < \nu\rangle$ with $p_i \in U_i \cap G$ in $V[G]$. Since λ^+ is regular in $V[G]$ and $\nu < \lambda^+$, we have $\mu := \sup_{i<\nu} \mathrm{lh}(p_i) < \lambda^+$ and therefore $p := \bigcup_{i<\nu} p_i \in \mathrm{Add}(\lambda^+, 1)$ is the unique condition in G of length μ. In particular, $p \in V$. Hence $p \in \bigcap_{i<\nu} U_i$ as required. □

If $\mathsf{DC}_\nu(2^\nu)$ fails for some $\nu \leq \lambda$, it thus follows that $\mathrm{Add}(\lambda^+, 1)$ collapses λ^+. For example, this holds for all $\lambda \geq \omega_2$ in $L(\mathbb{R})$, assuming there exists no ω_1-sequence of distinct reals in $L(\mathbb{R})$.

Exercise 3.10. Prove that (c) implies (d) in Lemma 3.8.

Problem 3.11. *Which combinations of cardinals $\leq\lambda$ can $\mathrm{Add}(\lambda^+, 1)$ preserve/collapse?*

3.3. **Adding Cohen subsets over $L(\mathbb{R})$.** Forcing over a model V of ZFC does not change the theory of $L(\mathbb{R})$ if there is a proper class of Woodin cardinals in V by a result of Woodin. Then $L(\mathbb{R})$ satisfies the axiom of determinacy AD and in $L(\mathbb{R})$, AD implies DC by a result of Kechris [Kec84]. A generic extension $L(\mathbb{R})[G]$ of $L(\mathbb{R})$ may be different from $L(\mathbb{R})^{V[G]}$ since the set $\mathbb{R}^V$ of ground model reals might not be in $L(\mathbb{R})^{V[G]}$.

We assume $V = L(\mathbb{R}) \models \mathsf{AD}$ and ask which Cohen subsets preserve cardinals and AD.

Cohen reals preserve all cardinals, since the usual argument for ccc forcings works for wellordered forcings. $\mathrm{Add}(\lambda^+, 1)$ collapses λ^+ by Proposition 3.9.

Problem 3.12. *(see Problem 3.11) Which cardinals $\leq\lambda$ does $\mathrm{Add}(\lambda^+, 1)$ preserve or collapse over $L(\mathbb{R})$?*

Exercise 3.13. Suppose that $V = L(\mathbb{R})$ and there exists no ω_1-sequence of distinct reals. Show that $L(\mathbb{R})[G] \models \mathrm{cof}(\kappa) \leq |\mathbb{R}|$ for any $\mathrm{Add}(\kappa, 1)$-generic $x \in 2^\kappa$ over $L(\mathbb{R})$ and $\kappa \geq \omega_1$.

Ikegami and Trang observed that a Cohen real destroys AD. This follows from a result of Kunen that $\mathbb{R}^V$ does not have the Baire property adding a Cohen real.

Proposition 3.14 (Chan, Jackson, Goldberg 2021 [CJ21, Fact 3.3]). *Suppose that $V = L(\mathbb{R}) \models \mathsf{AD}$. Then any well-ordered forcing destroys AD.*

Proof sketch. In $V[G]$, take a perfect tree T with $[T] \subseteq (2^\omega)^V$.

- For each $p \in \mathbb{P}$, let A_p be the set of $x \in 2^\omega$ such that $p \Vdash x \in [T]$.
- Some A_p is uncountable, since a wellordered union of meager sets is meager by Kuratowski-Ulam and the Baire property.

- Take a perfect tree T' with $[T'] \subseteq A_p$. Then $p \Vdash [T'] \subseteq [T]$. Since $\mathbb{P}$ adds new reals, it adds a new element of $[T']$.

$\square$

$\mathrm{Add}(\lambda^+, 1)$ destroys AD, since it forces AC.

However, one can add new subsets of regular cardinals using $\mathrm{Add}(\kappa, 1)^{\mathrm{HOD}}$ while preserving AD.

Definition 3.15. Suppose that M is a transitive model and $A \subseteq \kappa$. A is a *fresh* subset of κ over M if $A \notin M$, but $A \cap \alpha \in$ for all $\alpha < \kappa$.

Theorem 3.16 (Cunningham [Cun23, Theorem 2.4])**.** *Suppose that V is a model of ZFC, $L(\mathbb{R}) \vDash \mathsf{ZF} + \mathsf{AD}$ and $\kappa > |\mathbb{R}|$ is a regular cardinal. If A is a fresh subset of κ over $L(\mathbb{R})$, then $L(\mathbb{R})[A]$ does not have new sets of reals and hence it is a model of AD.*

Proof. Suppose that $B \in L(\mathbb{R})[A]$ is a set of reals. B is definable over some $L_\alpha(\mathbb{R})[A]$ from a real and an ordinal and we assume that no parameters are needed.

We claim that $B \in L(\mathbb{R})$. If $\alpha < \kappa$, then $L_\alpha(\mathbb{R})[A] = L_\alpha(\mathbb{R})$ since $A \subseteq \kappa$ is fresh and thus $B \in L(\mathbb{R})$.

Suppose that $\alpha \geq \kappa$. Fix an operator H in V for Skolem hulls in $L_\alpha(\mathbb{R})$ and let

$$H_\xi := H^{L_\alpha(\mathbb{R})[A]}(\mathbb{R} \cup \{\xi\})$$

for any $\xi \in \mathrm{Ord}$. Let $M_0 := H_0$. Given M_n, let $\xi_n := M_n \cap \kappa$. Given M_n and ξ_n, let $M_{n+1} := H_{\xi_n}$. Then $M := \bigcup_{n \in \omega} M_n \prec L_\alpha(\mathbb{R})$ and $\xi := M \cap \kappa \in \kappa$, since κ is regular. Let N denote the transitive collapse of M. Since $|M| = |\mathbb{R}| < \kappa$, $N = L_\beta(\mathbb{R})[A \cap \xi]$ for some $\beta < \kappa$. Since B is definable over N, we have $B \in L(\mathbb{R})[A]$. $\square$

The extension $L(\mathbb{R})[A]$ constructed in Theorem 3.16 is in fact a model of DC [Cun23, Theorem 3.3].

The next result shows that one can add fresh subsets to some regular cardinals in $L(\mathbb{R})$.

Theorem 3.17 (Cunningham [Cun23, Theorem 3.3])**.** *Suppose that $L(\mathbb{R}) \vDash \mathsf{ZF} + \mathsf{AD}$ and $\kappa > (\aleph^*(\mathbb{R})^+)^{L(\mathbb{R})}$ is regular. Let $\mathbb{P} \in \mathrm{HOD}^{L(\mathbb{R})}$ be a forcing such that*

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“}\kappa \textit{ is a regular cardinal”}$$

holds in $\mathrm{HOD}^{L(\mathbb{R})}$. Then AD holds in any $\mathbb{P}$-generic extension of $L(\mathbb{R})$.

Proof sketch. We will assume that κ is a sufficiently large regular cardinal in $L(\mathbb{R})$.

Suppose that $G \times H$ is $\mathbb{P} \times \mathrm{Col}(\omega_1, \mathbb{R})$-generic over $L(\mathbb{R})$, where $\mathrm{Col}(\lambda, X) := \{p : \alpha \to X \mid \alpha < \lambda\}$. Note that G is $\mathbb{P}$-generic over $\mathrm{HOD}^{L(\mathbb{R})}$ as well. Let A be the subset of κ given by G. Then A is a fresh subset of κ over $L(\mathbb{R})$.

We will assume that $\kappa > |\mathbb{R}|^{L(\mathbb{R})[H]}$ is regular. One can check that κ remains regular in $L(\mathbb{R})[H]$ [Cun23, Theorem 2.7].

It suffices to show that κ is regular in $L(\mathbb{R})[G \times H]$. Since this is a model of choice, we can apply Theorem 3.16 to A in this model and will thus obtain $L(\mathbb{R})[G] \vDash \mathsf{AD}$.

Let $M := \mathrm{HOD}^{L(\mathbb{R})}$. Since $\mathrm{Col}(\omega_1, \mathbb{R})$ is homogeneous, we have $M = \mathrm{HOD}^{L(\mathbb{R})[H]}$. Let $\mathbb{Q}$ denote Vopenka's forcing in M for subsets of ω_1 [Jec03]. Then any subset of ω_1 in $L(\mathbb{R})[H]$ is $\mathbb{Q}$-generic over M. In particular, this holds for the subset of ω_1 given by H. Since this set codes all reals, we have that $L(\mathbb{R})[H] = M[H]$ is a $\mathbb{Q}$-generic extension of M and

$$M[G][H] = M[H][G] = L(\mathbb{R})[G \times H]$$

By the assumption of the theorem, κ remains regular in $M[G]$. We now assume $\kappa > |\mathbb{Q}|^M$ so that κ remains regular in $M[G][H]$ as required.

To see that the lower bound on κ suffices, one needs to check that $\omega_2^{\mathrm{HOD}[H]} = \aleph^*(\mathbb{R})^{L(\mathbb{R})}$ and $|\mathbb{Q}|^M \leq \omega_3^{\mathrm{HOD}[H]} = (\aleph^*(\mathbb{R})^+)^{L(\mathbb{R})}$ [Cun23, Theorem 2.7]. $\square$

Remark 3.18. Ikegami and Trang proved a stronger version of Theorem 3.17 for all $\kappa \geq \aleph^*(\mathbb{R})^{L(\mathbb{R})}$ [IT23, Theorem 5.1]. Chan and Jackson proved that $\aleph^*(\mathbb{R})$ is least, since any forcing over $L(\mathbb{R})$ that is a surjective image of $\mathbb{R}$ destroys AD [CJ21, Theorem 5.6].

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