The κ-Strongly Proper Forcing Axiom

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Properness and strong properness

(Shelah) A partial order $\mathcal{P}$ is *proper* iff for every large enough cardinal θ (i.e., such that $\mathcal{P} \in H(\theta)$), every countable $N \preccurlyeq H(\theta)$ such that $\mathcal{P} \in N$ and every $p \in \mathcal{P} \cap N$ there is some $q \leq_{\mathcal{P}} p$ which is $(N, \mathcal{P})$-generic, i.e., for every $q' \leq_{\mathcal{P}} q$ and every dense set $D \subseteq \mathcal{P}$ such that $D \in N$, q' is compatible with some condition in $D \cap N$.

(Mitchell) A partial order $\mathcal{P}$ is *strongly proper* iff for every large enough cardinal θ, every countable $N \preccurlyeq H(\theta)$ such that $\mathcal{P} \in N$ and every $p \in \mathcal{P} \cap N$ there is some $q \leq_{\mathcal{P}} p$ which is *strongly* $(N, \mathcal{P})$*-generic*, i.e., for every $q' \leq_{\mathcal{P}} q$ there is some $\pi_N(q') \in \mathcal{P} \cap N$ weaker than q' and such that every $r \in \mathcal{P} \cap N$ such that $r \leq_{\mathcal{P}} \pi_N(q')$ is compatible with q'.

Examples of strongly proper partial orders:

- Cohen forcing
- Baumgartner's forcing for adding a club of ω_1 with finite conditions.
- Given a cardinal $\lambda \geq \omega_2$, the forcing of finite $\in$-chains of countable $N \preccurlyeq H(\lambda)$.

Caution: ccc does not imply strongly proper. In fact, most ccc forcings are not strongly proper.

Some basic facts

Fact

If $\mathcal{P}$ is strongly proper, $N \preccurlyeq H(\theta)$ is countable, $\mathcal{P} \in N$, q is strongly $(N, \mathcal{P})$-generic, $G \subseteq \mathcal{P}$ is generic over V, and $q \in G$, then $G \cap N$ is $\mathcal{P} \cap N$-generic over V.

Fact

Every ω-sequence of ordinals added by a strongly proper forcing notion is in a generic extension of V by Cohen forcing.

Proof.

Let $\mathcal{P}$ be strongly proper, $\dot{r}$ a $\mathcal{P}$-name for an ω-sequence of ordinals, $p \in \mathcal{P}$, and $N \preccurlyeq H(\theta)$ countable and such that $\mathcal{P}$, p, $\dot{r} \in N$.

Let $q \leq_{\mathcal{P}} p$ be strongly $(N, \mathcal{P})$-generic. Then, if G is $\mathcal{P}$-generic over V and $q \in G$, $H = G \cap N$ is $\mathcal{P} \cap N$-generic over V.

But $\mathcal{P} \cap N$ is countable and non-atomic, and therefore forcing-equivalent to Cohen forcing.

And of course $\dot{r}_G = \dot{r}_H$. $\square$

Lemma

(Neeman) Suppose $\mathcal{P}$ is strongly proper, $\dot{f}$ is a $\mathcal{P}$-name for a function with $\mathrm{dom}(\dot{f}) = \alpha \in Ord$. Let $N \preccurlyeq H(\theta)$ countable and such that $\mathcal{P}, \dot{f} \in N$. Let q be strongly $(N, \mathcal{P})$-generic, let G be $\mathcal{P}$-generic over V such that $q \in G$, and suppose $\dot{f}_G \restriction M \in V$. Then $\dot{f}_G \in V$.

Corollary

(Neeman) Suppose $\mathcal{P}$ is strongly proper. Then $\mathcal{P}$ does not add new branches through trees T such that $\mathrm{cf}(ht(T)) \geq \omega_1$.

Lemma

(Neeman) Suppose $\mathcal{P}$, $\mathcal{Q}$ *are forcing notions,* $N \preccurlyeq H(\theta)$ *is countable and such that* $\mathcal{P}$, $\mathcal{Q} \in N$, p *is strongly* $(N, \mathcal{P})$*-generic, and* q *is* $(N, \mathcal{Q})$*-generic. Then* (p, q) *is* $(N, \mathcal{P} \times \mathcal{Q})$*-generic.*

Corollary

If $\mathcal{P}$ *is strongly proper and* $\mathcal{Q}$ *is proper, then* $\Vdash_{\mathcal{P}} \mathcal{Q}$ *is proper.*

Extending to $\kappa > \omega$

This part is joint work with Sean Cox, Asaf Karagila, and Christoph Weiss.

The notion of strong properness can be naturally extended to higher cardinals:

Suppose κ is an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. A partial order $\mathcal{P}$ is κ-*strongly proper* iff for every $N \preccurlyeq H(\theta)$ such that $\mathcal{P} \in N$ and such that

- $|N| = \kappa$, and
- ${}^{<\kappa}N \subseteq N$,

every $\mathcal{P}$-condition in N can be extended to a strongly $(N, \mathcal{P})$-generic condition.

We will need the following closure property:

Given an infinite regular cardinal κ, a partial order $\mathcal{P}$ is $<\kappa$*-directed closed with greatest lower bounds* in case every directed subset X of $\mathcal{P}$ (i.e., every finite subset of X has a lower bound in $\mathcal{P}$) such that $|X| < \kappa$ has a greatest lower bound in $\mathcal{P}$.

We will also say that $\mathcal{P}$ *is* κ*-lattice*.

All fact about strongly proper (i.e., ω-strongly proper) forcing we have seen extend naturally to κ-strongly proper forcing notion which are κ-lattice (always assuming $\kappa^{<\kappa} = \kappa$).

For example, every κ-sequence of ordinals added by a forcing in this class belongs to a generic extension by adding a Cohen subset of κ.

Lemma

(Reflection Lemma) Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Suppose $\mathcal{P}$ is a κ-lattice and κ-strongly proper forcing. If θ is large enough and $(Q_i)_{i<\kappa^+}$ is a $\subseteq$-continuous $\in$-chain of elementary submodels of $H(\theta)$ such that $\mathcal{P} \in Q_i$, $|Q_i| = \kappa$, and ${}^{<\kappa}Q_i \subseteq Q_i$ for all $i \in S^{\kappa^+}_{\kappa}$, then $\mathcal{P} \cap Q$ is κ-lattice and κ-strongly proper, for $Q = \bigcup_{i<\kappa^+} Q_i$.

Proof.

Given large enough cardinal χ and $N \preccurlyeq H(\chi)$ such that $\mathcal{P}$, $(Q_i)_{i<\kappa^+} \in N$, $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$, $N \cap Q = Q_\delta \in Q$ for $\delta = N \cap \kappa^+$. But any strongly $(Q_\delta, \mathcal{P})$-generic $q \in Q$ is $(N, \mathcal{P} \cap Q)$-generic. $\square$

Compare the above reflection property with the reflection of κ-c.c. forcing to substructures Q such that ${}^{<\kappa}Q \subseteq Q$.

Theorem

Assume GCH, and let $\kappa < \kappa^+ < \theta$ be infinite regular cardinals. Then there is a κ-lattice and κ-strongly proper forcing $\mathcal{P}$ which forces $2^\kappa = \kappa^{++} = \theta$ together with the κ-Strongly Proper Forcing Axiom.

Proof sketch: By first forcing with $\mathrm{Coll}(\kappa^+, {<}\theta)$, we may assume that $\theta = \kappa^{++}$ and that $\diamondsuit(S^\theta_{\kappa^+})$ holds. Hence there is a 'diamond sequence' $\vec{A} = (A_\alpha)_{\alpha \in S^\theta_{\kappa^+}}$, where $A_\alpha \subseteq H(\theta)$ for all α.

Let

$$E = \{\alpha \in S^\theta_{\kappa^+} \,:\, (A_\alpha; \in, \vec{A} \restriction \alpha) \preccurlyeq (H(\theta); \in, \vec{A})\},$$

$$\mathcal{T} = \{A_\alpha \,:\, \alpha \in E\},$$

and

$$\mathcal{S} = \{N \preccurlyeq H(\theta) \,:\, |N| = \kappa,\ {}^{<\kappa}N \subseteq N\}$$

Our forcing $\mathcal{P}$ is $\mathcal{P}_\theta$, where $(\mathcal{P}_\alpha \in E \cup \{\theta\})$ is a ${<}\kappa$-support iteration à la Neeman with models from $\mathcal{S} \cup \mathcal{T}$ as side conditions.

More specifically, given $\beta \in E \cup \{\theta\}$, $\mathcal{P}_\beta$ is the set of all pairs $\langle p, s\rangle$ such that:

(1) $s \in [\mathcal{S} \cup \mathcal{T}]^{<\kappa}$ and $\in$ is a weak total order on s.

(2) p is a function with $\mathrm{dom}(p) \in [E \cap \beta]^{<\kappa}$ such that for each $\alpha \in \mathrm{dom}(p)$,

 (a) A_α is a $\mathcal{P}_\alpha$-name for a κ-lattice κ-strongly proper forcing notion whose conditions are ordinals,

 (b) $H(\alpha) \in s$, and

 (c) $p(\alpha)$ is a nice $\mathcal{P}_\alpha$-name such that $\Vdash_\alpha p(\alpha) \in A_\alpha$.

(3) For every $\alpha \in \mathrm{dom}(p)$ and every $N \in s \cap \mathcal{S}$ such that $\alpha \in N$, $\langle p \restriction \alpha, s \cap H(\alpha)\rangle$ is a condition in $\mathcal{P}_\alpha$ which forces in $\mathcal{P}_\alpha$ that $p(\alpha)$ is a strongly $(N[\dot{G}_\alpha], A_\alpha)$-generic condition.

Extension relation: $\langle p_1, s_1\rangle \leq_\beta \langle p_0, s_0\rangle$ iff

(i) $s_0 \subseteq s_1$,

(ii) $\mathrm{dom}(p_0) \subseteq \mathrm{dom}(p_1)$, and

(ii) for all $\alpha \in \mathrm{dom}(p_0)$, $\langle p_1 \restriction \alpha, s_1 \cap H(\alpha)\rangle \Vdash_\alpha p_1(\alpha) \leq_{A_\alpha} p_0(\alpha)$.

The Reflection Property is used to show that our construction captures strongly κ-proper forcings of arbitrary size.

Also: The proof crucially uses the fact that our forcings are κ-lattice (it would not work if we just assume $<\kappa$-directed closedness). $\square$

The κ-Strongly Proper Forcing Axiom does not decide 2^κ. In fact:

Theorem

Assume GCH, and let $\kappa < \kappa^+ < \kappa^{++} \leq \theta$ be infinite regular cardinals. Suppose $\diamondsuit(S^{\kappa^{++}}_{\kappa})$ holds. Then there is a κ-lattice and κ-strongly proper forcing $\mathcal{P}$ which forces $2^\kappa = \theta$ together with the κ-Strongly Proper Forcing Axiom.

Proof sketch: We fix 'diamond sequence' $\vec{A} = \langle A_\alpha \, : \, \alpha \in S^{\kappa^{++}}_{\kappa^+} \rangle$, where $A_\alpha \subseteq H(\kappa^{++})$ for all α, and build an iteration $(\mathcal{P}_\alpha \in \alpha \in E \cup \{\kappa^{++}\})$ as before, except that at each stage $\alpha \in E$ now we look at whether A_α is a $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$-name for a κ-lattice and strongly κ-proper poset (and if so we force with it).

The forcing witnessing the theorem is

$$\mathcal{P} = \mathcal{P}_{\kappa^{++}} \times \mathsf{Add}(\kappa, \theta)$$

To see this, take a κ-lattice κ-strongly proper forcing in the

extension via $\mathcal{P}$. By the Reflection Property it reflects to a forcing of size κ^+. Let $\dot{Q}$ be a $\mathcal{P}$-name for the corresponding forcing.

By κ^{++}-c.c. of $\mathcal{P}$ we may identify $\dot{Q}$ with a $\mathcal{P}_{\kappa^{++}} \times \mathsf{Add}(\kappa, \kappa^+)$-name, which of course we may assume is a subset of $H(\kappa^{++})$. Now we use our diamond $\vec{A}$ to capture $\dot{Q}$ by some A_α as in the proof of the previous theorem.

The final point is that A_α will be a $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$-name for a κ-lattice κ-strongly proper forcing. This uses the fact that every κ-sequence of ordinals is in a κ-Cohen extension since $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$ is κ-lattice and κ-strongly proper (which enables A_α to have enough access to arbitrary $\mathcal{P}_\alpha \times \mathrm{Add}(\kappa, \kappa^+)$-names for κ-sized elementary submodels N). $\square$

As far as I know this is the first example of a forcing axiom $\mathrm{FA}_{\kappa^+}(\Gamma)$ such that $\mathrm{FA}_{\kappa^{++}}(\Gamma)$ is false but nevertheless $\mathrm{FA}_{\kappa^+}(\Gamma)$ is compatible with 2^κ arbitrarily large.

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Some applications of the κ-Strongly Proper Forcing Axiom

- $\mathfrak{d}(\kappa) > \kappa^+$
- The covering number of natural meagre ideals is $> \kappa^+$.
- Weak failures of Club-Guessing at κ.
- Suppose $(C_\alpha \in S^{\kappa^+}_\kappa)$ is a club sequence of $S^{\kappa^+}_\kappa$, and let $\vec{F} = (f_\alpha \ , \alpha \in S^{\kappa^+}_\kappa)$ be a colouring, i.e., for each α, $f_\alpha : C_\alpha \longrightarrow \{0, 1\}$. There there is $G : \kappa^+ \to \{0, 1\}$, and clubs $D_\alpha \subseteq C_\alpha$, for $S^{\kappa^+}_\kappa$, such that $G(\beta) = f_\alpha(\beta)$ for all α and all $\beta \in D_\alpha$.

Getting rid of strongness?

No:

Theorem

(Veličković) Suppose $\kappa \geq \omega_1$ is a regular cardinal and $\kappa^{<\kappa} = \kappa$. Then $FA_{\kappa^+}(\{\mathcal{P} : \mathcal{P}\ \kappa\text{-lattice and } \kappa\text{-proper}\})$ is false.

Proof.

Let $(C_\alpha \in S^{\kappa^+}_\kappa)$ be a club sequence of $S^{\kappa^+}_\kappa$, and let $\vec{F} = (f_\alpha\ , \alpha \in S^{\kappa^+}_\kappa)$ be a colouring which cannot be uniformized, i.e., there is no $G : \kappa^+ \to \{0, 1\}$ such that for every $\alpha \in S^{\kappa^+}_\kappa$, $G(\beta) = f_\alpha(\beta)$ for all β on a tail of C_α (by a result of Shelah, there is always such an $\vec{F}$). But the natural forcing $\mathcal{P}$ for adding a uniformizing function G by approximations of size less than κ and using an $\in$-chain, of length less than κ, of κ-sized models as side conditions is κ-lattice and κ-proper and $\mathrm{FA}_{\kappa^+}(\{\mathcal{P}\})$ would give rise to a uniformizing function for $\vec{F}$. $\square$

Getting rid of g.l.b.'s?

No:

Theorem

(Shelah) Suppose $\kappa \geq \omega_1$ is a regular cardinal and $\kappa^{<\kappa} = \kappa$. Then $FA_{\kappa^+}(\{\mathcal{P} : \mathcal{P}$ $<\kappa$-directed closed and κ-proper$\})$ is false.

Proof.

Similar as previous proof, with slightly different forcing. $\square$

These results are related to:

Theorem

(A.) $FA_{\aleph_2}(\{\mathcal{P} : \mathcal{P}$ proper and $\aleph_2$-c.c.$\})$ is false.

κ-strong semiproperness

Note: Given a forcing notion $\mathcal{P}$, a relevant countable model N and $q \in \mathcal{P}$, q is $(N, \mathcal{P})$-generic iff for every $\mathcal{P}$-generic filter G such that $q \in G$, $N[G] \cap \mathrm{Ord} = N \cap \mathrm{Ord}$.

(Shelah) A forcing notion $\mathcal{P}$ is *semiproper* in case for every relevant countable model N and every $p \in \mathcal{P} \cap N$ there is some $q \leq_{\mathcal{P}} p$ which is $(N, \mathcal{P})$*-semi-generic*, i..e.,
$q \Vdash_{\mathcal{P}} N[\dot{G}] \cap \omega_1^V = N[\dot{G}] \cap \omega_1^V$.

Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$. Let us say that a forcing notion $\mathcal{P}$ is *κ-strongly semiproper* if and only if for every large enough θ and every $N \preccurlyeq H(\theta)$ such that $\mathcal{P} \in N$, $|N| = \kappa$, and ${}^{<\kappa}N \subseteq N$, every $p \in \mathcal{P} \cap N$ can be extended to some $q \in \mathcal{P}$ which is *κ-strongly semiproper*, i.e., the following holds.

(1) q is κ-$(N, \mathcal{P})$-semiproper: $q \Vdash_{\mathcal{P}} N[\dot{G}] \cap (\kappa^+)^V = N \cap (\kappa^+)^V$.

(2) q forces that for every $q' \leq_{\mathcal{P}} q$ there is some $\pi_{N[\dot{G}]}(q') \in \mathcal{P} \cap N[\dot{G}]$ weaker than $1'$ and such that every $r \in \mathcal{P} \cap N[\dot{G}]$ such that $r \leq_{\mathcal{P}} \pi_{N[\dot{G}]}(q')$ is compatible with q'.

Given infinite regular κ, let the κ-Strongly Semiproper Forcing Axiom be

$$\mathrm{FA}_{\kappa^+}(\{\mathcal{P}\,;\,\mathcal{P}\ \kappa\text{-lattice and } \kappa\text{-strongly semiproper}\})$$

A reflection principle

Given an infinite regular κ such that $\kappa^{<\kappa} = \kappa$, let $\mathrm{SRP}(\kappa^+, 1)$ be the following reflection principle: Suppose X is a set and $\mathcal{S} \subseteq [X]^\kappa$. If λ is such that $X \in H(\lambda)$, there is a $\subseteq$-continuous $\in$-chain $(N_i)_{i<\kappa^+}$ such that for each $i < \kappa^+$ such that $\mathrm{cf}(i) = \kappa$:

(1) $N \preccurlyeq H(\lambda)$ and $|N| = \kappa$.

(2) $N_i \cap X \notin \mathcal{S}$ if and only if there is no $x \in X$ such that

 (a) $\mathrm{Sk}_\lambda(N \cup \{x\})$ is a κ^+-end-extension of N (i.e., $\mathrm{Sk}_\lambda(N \cup \{x\}) \cap \kappa^+ = N \cap \kappa^+$), and

 (b) $\mathrm{Sk}_\lambda(N \cup \{x\}) \cap X \in \mathcal{S}$.

Easy: The κ-Strongly Semiproper Forcing Axiom implies $\mathrm{SRP}(\kappa^+, 1)$.

Note: $\mathrm{SRP}(\kappa^+, 1)$ is the simplest application of the κ-Strongly Semiproper Forcing Axiom not covered by the κ-Strongly Proper Forcing Axiom.

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Saturation

Given an infinite regular κ and a stationary $S \subseteq \kappa^+$, $\mathrm{NS}_{\kappa^+} \upharpoonright S$ *is saturated* iff every collection $\mathcal{A}$ of stationary subsets of S such that $S_0 \cap S_1$ is nonstationary for all $S_0 \neq S_1$ in $\mathcal{A}$ is such that $|\mathcal{A}| \leq \kappa^+$.

Fact

(Shelah) If $S \subseteq S^{\kappa^+}_{<\kappa}$ is stationary, then $\mathrm{NS}_{\kappa^+} \restriction S$ is not saturated.

Proof.

If $\mathrm{NS}_{\kappa^+} \restriction S$ is saturated, then $\mathcal{P}(\kappa^+)/(NS_{\kappa^+} \restriction S)$ preserves κ^{++}.

$\mathcal{P}(\kappa^+)/(NS_{\kappa^+} \restriction S)$ forces $\mathrm{cf}((\kappa^+)^V) = \mu < \kappa$ for some $\mu < \kappa$ (as this is true in the corresponding generic ultrapower of V).

Also, $\mathcal{P}(\kappa^+)/(NS_{\kappa^+} \restriction S)$ preserves κ and μ (since the generic embedding has critical point κ^+ and the generic ultrapower is closed under $(\kappa^+)^V$-sequences in the extension by saturation of $NS_{\kappa^+} \restriction S$.

But by a theorem of Shelah, if λ is regular, and $\mathbb{P}$ is a partial order forcing $\mathrm{cf}(\lambda) \neq |\lambda|$, then $\mathbb{P}$ collapses λ^+. Contradiction. $\square$

Fact

If κ is an infinite regular cardinal, SRP$(\kappa^+, 1)$ implies that $\mathrm{NS}_{\kappa^+} \restriction S^{\kappa^+}_{\kappa}$ *is saturated.*

Proof: Let $\mathcal{A}$ be a collection of stationary subsets of $S^{\kappa^+}_{\kappa}$ with pairwise nonstationary intersection. We want to show $|\mathcal{A}| \leq \kappa^+$. Let $X = \mathcal{A} \cup \kappa^+$ and let $\mathcal{S}$ be the collection of $Z \in [X]^{\kappa}$ such that

- $\delta_Z := Z \cap \kappa^+ \in \kappa^+$ and
- $\delta_Z \in S$ for some $S \in \mathcal{A} \cap Z$.

Let $(N_i)_{i<\kappa^+}$ be a reflecting sequence for $\mathcal{S}$ as given by SRP$(\kappa^+, 1)$, and suppose $S \in \mathcal{A} \setminus \bigcup_{i<\kappa^+} N_i$. Let $N_i' = \mathrm{Sk}_\lambda(N_i \cup \{S\})$ for all i and note that

$$\{i < \kappa^+ \ : \ \mathrm{cf}(i) = \kappa \Rightarrow N_1' \cap \kappa^+ = N_i \cap \kappa^+\}$$

contains a club $C \subseteq \kappa^+$.

Hence, for every $i \in C \cap S$ there is some $S(i) \in N_i$ such that $N_i \cap \kappa^+ \in S(i)$. By Fodor's lemma there is some S_0 such that

$$T = \{i \in S \cap C \,:\, S(i) = S_0\}$$

is stationary. But that is a contradiction since $N_i \cap \kappa^+ \in S \cap S_0$ for every $i \in T$ and therefore $S \cap S_0$ is stationary. $\square$

Question: Can there be any regular cardinal $\kappa \geq \omega_1$ such that the κ-Strongly Semiproper Forcing Axiom holds?

Question: Suppose $\kappa \geq \omega_1$ is regular and $\mathrm{NS}_{\kappa^+} \restriction S^{\kappa^+}_{\kappa}$ is saturated. Does it follow that GCH cannot hold below κ?

Thank you!