The complexity of club filters

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Introduction

The fact that closed unbounded subsets generate a proper normal filter

$$Club_\kappa \;=\; \{A \subseteq \kappa \mid \exists C \subseteq A \textit{ closed and unbounded in } \kappa\}$$

is one of the most important combinatorial properties of uncountable regular cardinals κ.

The study of the structural properties of these filters and their dual ideals

$$NS_\kappa \;=\; \{A \subseteq \kappa \mid \exists C \textit{ closed and unbounded in } \kappa \textit{ with } A \cap C = \emptyset\}$$

plays a central role in modern set theory.

In particular, questions about the complexity of these filters motivated much of the development of generalized descriptive set theory.

There are two canonical approaches to measure the complexity of sets of the form $Club_\kappa$ and NS_κ for uncountable regular cardinals κ:

- Through the complexity of the formulas and parameters defining these sets in the structure $\langle \mathrm{V}, \in \rangle$.
- Through the topological complexity of these sets viewed as subsets of the generalized Baire space ${}^\kappa\kappa$ of the corresponding cardinal κ.

The Levy Hierarchy

A formula in the language $\mathcal{L}_\in = \{\in\}$ of set theory is a Δ_0-*formula* if it is contained in the smallest collection of $\mathcal{L}_\in$-formulas that contains all atomic formulas and is closed under negations, conjunctions and bounded quantification.

An $\mathcal{L}_\in$-formula is a Σ_1-*formula* if it is of the form $\exists x \ \varphi$ for some Δ_0-formula φ.

Π_1-*formulas* are negations of Σ_1-formulas.

Definition

An $\mathcal{L}_\in$-formula $\varphi(v_0, \ldots, v_n)$ and sets $y_0, \ldots, y_{n-1}$ *define* a class X if

$$X \ = \ \{x \mid \varphi(x, y_0, \ldots, y_{n-1})\}.$$

It is easy to see that, given an uncountable regular cardinal κ, the sets $Club_\kappa$ and NS_κ can both be defined by a Σ_1-formula with parameter κ.

Definition

Given a set P, a class X is $\mathbf{\Delta}_1(P)$-definable if it is definable by both a Σ_1- and a Π_1-formula with parameters in P.

Generalized Baire spaces

Given an infinite regular cardinal κ, the *generalized Baire space* of κ consists of the set ${}^{\kappa}\kappa$ of all functions from κ to κ equipped with the topology whose basic open sets are of the form

$$N_s \ = \ \{x \in {}^{\kappa}\kappa \mid s \subseteq x\}$$

for functions $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$.

Definition

Let κ be an infinite regular cardinal and let X be a subset of ${}^{\kappa}\kappa$.

- X is a $\mathbf{\Sigma}^1_1$-subset if it is the projection of a closed subset of ${}^{\kappa}\kappa \times {}^{\kappa}\kappa$.
- X is a $\mathbf{\Pi}^1_1$-subset if ${}^{\kappa}\kappa \setminus X$ is a $\mathbf{\Sigma}^1_1$-subset.
- X is a $\mathbf{\Delta}^1_1$-subset if it is both a $\mathbf{\Sigma}^1_1$- and a $\mathbf{\Pi}^1_1$-subset.

It is easy to see that the sets of characteristic functions of elements of $Club_\kappa$ and NS_κ are disjoint $\mathbf{\Sigma}^1_1$-subsets.

The above notions of complexity are connected in the following way:

Lemma

Let κ be an uncountable regular cardinal and let X be a subset of ${}^{\kappa}\kappa$.

- *If X is definable by a Σ_1-formula with parameters in $\mathrm{H}(\kappa^+)$, then X is a $\mathbf{\Sigma}^1_1$-subset.*
- *If X is a $\mathbf{\Sigma}^1_1$-subset, then X is definable by a Σ_1-formula with parameters in $\mathrm{H}((2^{<\kappa})^+)$.*

Corollary

Given an uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, a subset of ${}^{\kappa}\kappa$ is a $\mathbf{\Delta}^1_1$-subset if and only if it is $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

Several results now show that an answer to the following question has several interesting consequences in different branches of mathematical logic:

Question

Given an uncountable regular cardinal κ, are the sets $Club_\kappa$ and NS_κ $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable?

Examples of such consequences:

- *In combinatorial set theory*: Structural properties of the collections of stationary subsets of κ and trees of height and size κ without cofinal branches ("*Canary trees*").
- In model theory: Ehrenfeucht–Fraïssé games ("*Universal non-equivalence trees*").

These results motivate the task to answer the above question in different models of set theory.

Positive consistency results

In the following, we present several different examples of models of set theory in which the restrictions

$$NS \upharpoonright S \;=\; NS_\kappa \cap \mathcal{P}(\kappa)$$

of non-stationary ideals on uncountable regular cardinals κ to stationary subsets S of κ are $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

The case $\mu = \omega$ of the following theorem, first proven by Mekler and Shelah, provided the first example of such a model.

Theorem (Mekler–Shelah, Hyttinen–Rautila)

Assume that GCH *holds. Given an infinite regular cardinal* μ*, the following statements hold in a cofinality-preserving forcing extension of the ground model:*

- GCH.
- *The set* $NS \upharpoonright S^{\mu^+}_\mu$ *is* $\mathbf{\Delta}_1(\mathrm{H}(\mu^{++}))$*-definable.*

The proof of this result makes use of the notion of *Canary trees*.

Using different techniques, Friedman, Wu and Zdomskyy extended the above result to the full non-stationary ideal.

Theorem (Friedman–Wu–Zdomskyy)

Assume that $\mathrm{V} = \mathrm{L}$ holds. Given an infinite cardinal μ, the following statements hold in a cofinality-preserving forcing extension of the ground model:

- GCH.
- *The set NS_{μ^+} is $\mathbf{\Delta}_1(\{\mu^+\})$-definable.*

Dense ideals

In another direction, it turns out that strong forms of saturation of the non-stationary ideal imply its $\mathbf{\Delta}_1$-definability.

Definition

Given a cardinal κ, an ideal $\mathcal{I}$ on a set X is *κ-dense* if the partial order $\mathcal{P}(X)/\mathcal{I}$ has a dense subset of cardinality at most κ.

Theorem (Woodin)

The theory $\mathbf{ZFC}$ + "NS_{ω_1} is ω_1-dense" is equiconsistent with the theory $\mathbf{ZF} + \mathbf{AD}$.

Proposition

If S is a stationary subset of an uncountable regular cardinal κ with the property that $NS \restriction S$ is κ-dense, then $NS \restriction S$ is $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

Stationary reflection

A crucial ingredient in the proofs of the new results presented in this talk is the observation that the $\mathbf{\Delta}_1$-definability of non-stationary ideals can also be obtained through strong principles of stationary reflection.

Proposition (Cox–L.)

Let S be stationary subsets of an uncountable regular cardinal δ and let $\mathcal{E}$ be a set of stationary subsets of δ.

Assume that for every stationary subset A of S, there exists $E \in \mathcal{E}$ such that A reflects at every element of E.

If $\mathcal{E}$ is definable by a Σ_1-formula with parameter p, then the set $NS \restriction S$ is definable by a Π_1-formula with parameters p, S and $\mathrm{H}(\delta)$.

The next corollary provides an easy application of the above observation.

Corollary

Let E and S be stationary subsets of an uncountable regular cardinal δ such that every stationary subset of S reflects almost everywhere in E.

Then the set $NS \restriction S$ is definable by a Π_1-formula with parameters E, S and $\mathrm{H}(\delta)$.

Note that a classical result of Magidor shows that, starting with a weakly compact cardinal, it is possible to construct a model of set theory in which every stationary subset of S^2_0 reflects almost everywhere in S^2_1.

The above corollary shows that the set $NS \restriction S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable in Magidor's model.

The above ideas can be extended to inaccessible cardinals, using the notion of *full reflection* introduced by Jech and Shelah.

In particular, it is possible to show NS_κ can be $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable for a greatly Mahlo cardinal κ.

Negative consistency results

In the following, we present several scenarios in which the non-stationary ideal is not $\mathbf{\Delta}_1$-definable.

We start by showing how generalizations of classical concepts from *descriptive set theory* can be used to achieve this goal.

The following results show that adding κ^+-many Cohen subsets to an uncountable cardinal κ satisfying $\kappa^{<\kappa} = \kappa$ produces a model in which no $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable subset of $\mathcal{P}(\kappa)$ separates $Club_\kappa$ from NS_κ, i.e. there is no set A definable in this way with $Club_\kappa \subseteq A$ and $A \cap NS_\kappa = \emptyset$.

Definition

Given an infinite regular cardinal κ, a subset A of ${}^{\kappa}\kappa$ has the κ-*Baire property* if there exists an open subset U of ${}^{\kappa}\kappa$ and a sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ of closed nowhere dense subsets of ${}^{\kappa}\kappa$ satisfying $U_\Delta X \subseteq \bigcup_{\alpha<\kappa} A_\alpha$.

Theorem

If κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and G is $\mathrm{Add}(\kappa, \kappa^+)$-generic over V, then all $\mathbf{\Delta}^1_1$-subsets of ${}^{\kappa}\kappa$ have the κ-Baire property in $\mathrm{V}[G]$.

Definition (L.–Schlicht)

Given an infinite regular cardinal κ, a subset X of ${}^{\kappa}\kappa$ *super-dense* if

$$\bigcap\{U_\alpha \cap X \mid \alpha < \kappa\} \neq \emptyset$$

holds for every non-empty open subset U of ${}^{\kappa}\kappa$ and every sequence $\langle U_\alpha \mid \alpha < \kappa\rangle$ of dense open subsets of U.

Proposition

Assume that A and B are disjoint super-dense subsets of ${}^{\kappa}\kappa$.

If $A \subseteq X \subseteq {}^{\kappa}\kappa \setminus B$, then X does not have the κ-Baire property.

Lemma

The subsets $Club_\kappa$ and NS_κ of ${}^{\kappa}\kappa$ are super-dense.

Weakly compact cardinals

In contrast to the consistency results about greatly Mahlo cardinals presented earlier, the following theorem shows that stronger large cardinal properties outright imply that the corresponding non-stationary ideal is not $\mathbf{\Delta}_1$-definable.

Theorem (Friedman–Wu)

If κ is a weakly compact cardinal, then NS_κ is not $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

Sketch of the proof.

Given a Σ_1-formula $\varphi(v_0, \dots, v_{n-1})$ and $A_0, \dots, A_{n-1} \subseteq \mathrm{V}_\kappa$, we have

$$\varphi(A_0, \dots, A_{n-1}) \iff \{\alpha < \kappa \mid \varphi(A_0 \cap \mathrm{V}_\alpha, \dots, A_{n-1} \cap \mathrm{V}_\alpha)\} \in Club_\kappa.$$

Hence the $\mathbf{\Delta}_1$-definability of $Club_\kappa$ implies that every Σ_1-formula with parameters in $\mathrm{H}(\kappa^+)$ is equivalent to a Π_1-formula with parameters in $\mathrm{H}(\kappa^+)$, which is impossible by the existence of universal Σ_1-formulas. $\square$

The constructible universe

The following result shows that, consistently, no non-stationary ideal is $\mathbf{\Delta}_1$-definable.

Theorem (Friedman–Hyttinen–Kulikov)

Assume that $\mathrm{V} = \mathrm{L}$ *holds. If* S *is a stationary subset of an uncountable regular cardinal* κ, *then* $NS \restriction S$ *is not* $\mathbf{\Delta}^1_1$*-definable.*

Sketch of the proof.

Given a Σ_1-formula $\varphi(v_0, \ldots, v_{n-1})$ and $A_0, \ldots, A_{n-1} \subseteq \kappa$, the statement $\varphi(A_0, \ldots, A_{n-1})$ holds if and only if the set of all $\alpha < \kappa$ with the property that there exists $\alpha < \beta < \kappa$ with

$$\begin{aligned} \mathrm{L}_\beta \models \mathbf{ZFC}^- \ &+ \text{“}\alpha \textit{ is regular}\text{''} + \text{“}S \cap \alpha \textit{ is stationary}\text{''} \\ &+ \ \varphi(A_0 \cap \kappa, \ldots, A_{n-1} \cap \kappa) \end{aligned}$$

has a subset of the form $C \cap S$ for some club C in κ. $\square$

Lightface definability

The next result shows that many canonical extensions of $\mathbf{ZFC}$ imply that NS_{ω_1} cannot be defined by a Π_1-formula with *simple* parameters.

Theorem (L.–Schindler–Schlicht)

Assume that one of the following statements holds:

- *There is a Woodin cardinal and a measurable cardinal.*
- *Bounded Martin's Maximum* $\mathbf{BMM}$ *holds and* NS_{ω_1} *is precipitous.*
- *There is a measurable cardinal and a precipitous ideal on* ω_1*.*
- *Woodin's Axiom* $(*)$ *holds.*

Then no subset of $\mathcal{P}(\omega_1)$ *that separates the club filter from the nonstationary ideal is* $\mathbf{\Delta}_1(\mathrm{H}(\omega_1) \cup \{\omega_1\})$*-definable.*

The above theorem is a consequence of the following lemma, whose proof makes use of *generic iterations* of countable models and *Woodin's countable stationary tower forcing*.

Lemma

Assume that one of the above assumptions holds. Then the following statements hold for every Σ_1-formula $\varphi(v_0, v_1, v_2)$ and all $z \in \mathrm{H}(\omega_1)$:

- *If there is a stationary subset A of ω_1 such that $\varphi(A, \omega_1, z)$ holds, then there is an element B of the club filter on ω_1 such that $\varphi(B, \omega_1, z)$ holds.*
- *If there is a costationary subset A of ω_1 such that $\varphi(A, \omega_1, z)$ holds, then there is an element B of the non-stationary ideal on ω_1 such that $\varphi(B, \omega_1, z)$ holds.*

Forcing axioms

Recently, Schindler initiated the study of the complexity of NS_{ω_1} in the presence of strong forcing axioms.

Theorem (Larson–Schindler–Wu)

Assume that Woodin's Axiom $(*)$ holds. Then NS_{ω_1} is not $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$-definable.

Using a recent result of Asperó and Schindler that shows that $\mathbf{MM}^{++}$ implies $(*)$, we obtain the following corollary:

Corollary

$\mathbf{MM}^{++}$ *implies that* NS_{ω_1} *is not* $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$*-definable.*

Forcing axioms and the complexity of NS_{ω_2}

Motivated by the above result, Sean Cox and I studied the complexity of NS_{ω_2} and its restrictions in the presence of forcing axioms.

Theorem (Cox–L.)

Assume that $\mathbf{MM}^{++}$ holds. If θ is a cardinal with $\theta^{\omega_2} = \theta$, then there is a $<\omega_2$-directed closed partial order that forces the following statements to hold in the corresponding generic extension of the ground model V:

- $2^{\omega_2} = \theta$.
- *The set $NS \restriction S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable.*

Corollary

If $\mathbf{ZFC} + \mathbf{MM}^{++}$ is consistent, then the statement

"No $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable subset of $\mathcal{P}(\omega_2)$ separates $Club_{\omega_2}$ from NS_{ω_2}"

is independent of this theory.

Definition (Shelah)

Let κ be an infinite regular cardinal.

- Given a sequence $\vec{z} = \langle z_i \mid i < \kappa^+ \rangle$ of elements of $[\kappa^+]^{<\kappa}$, a limit ordinal $\gamma < \kappa^+$ is *approachable with respect to* $\vec{z}$ if and only if there exists a sequence
$$\vec{\alpha} = \langle \alpha_\xi \mid \xi < \mathrm{cof}(\gamma) \rangle$$
cofinal in γ such that every proper initial segment of $\vec{\alpha}$ is equal to z_i for some $i < \gamma$.
- The *Approachability ideal* $I[\kappa^+]$ *on* κ^+ is the (possibly non-proper) *normal* ideal generated by sets of the form
$$A_{\vec{z}} = \{\gamma < \kappa^+ \mid \gamma \text{ is approachable with respect to } \vec{z}\}$$
for some $\vec{z} \in {}^{\kappa^+}([\kappa^+]^{<\kappa})$.

Lemma

Let κ be an infinite regular cardinal with $\kappa^{<\kappa} \leq \kappa^+$ and let $\langle z_i \mid i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. Then the following statements hold:

- $M_{\vec{z}} = A_{\vec{z}} \cap S^{\kappa^+}_{\kappa}$ *is a stationary subset of* κ^+.
- $M_{\vec{z}} \in I[\kappa^+]$.
- $M_{\vec{z}}$ *is a "maximal element of* $\mathcal{P}(S^{\kappa^+}_{\kappa}) \cap I[\kappa^+]$ *mod* NS*", i.e. whenever* $S \in I[\kappa^+]$ *is a stationary subset of* $S^{\kappa^+}_{\kappa}$*, then* $S \setminus M_{\vec{z}}$ *is non-stationary.*
- *If* $\kappa^{<\kappa} = \kappa$ *holds, then* $S^{\kappa^+}_{\kappa} \setminus M_{\vec{z}}$ *is non-stationary. In particular,* $S^{\kappa^+}_{\kappa} \in I[\kappa^+]$ *holds in this case.*

Theorem

- **PFA** *implies that* $I[\omega_2]$ *is a proper ideal.*
- **MM** *implies that if* M *is a maximum element of* $\mathcal{P}(S^2_1) \cap I[\omega_2]$ *mod* NS*, then every stationary subset of* S^2_0 *reflects stationary often in* M.

Theorem

Given an infinite regular cardinal κ, there is a partial order $\mathbb{P}$ with the following properties:

- $\mathbb{P}$ *is* $<\kappa^+$*-directed closed.*
- *If G is $\mathbb{P}$-generic over V, then, in $V[G]$, there is a tree T of height κ^+ and size 2^κ, without cofinal branches, such that the following statements hold:*
    - *If $\kappa^{<\kappa} \leq \kappa^+$ holds in V and $M \in \mathrm{V}$ is a maximal set in $I[\kappa^+] \cap \mathcal{P}(S^{\kappa^+}_\kappa)$ mod NS in V, then the following statements hold in $\mathrm{V}[G]$:*
        - *M is a maximal set in $I[\kappa^+] \cap \mathcal{P}(S^{\kappa^+}_\kappa)$ mod NS.*
        - *If S is a bistationary in $S^{\kappa^+}_\kappa$ and $M \setminus S$ is stationary, then there is an order-preserving embedding from $T(S)$ to T.*
- *If $2^\kappa = \kappa^+$ holds, then $\mathbb{P}$ satisfies the κ^{++}-chain condition.*

Corollary

Let κ be an infinite regular cardinal satisfying $\kappa^{<\kappa} \leq \kappa^+$, let $\mathbb{P}$ be the partial order given by the above theorem and let M be the maximum set in $I[\kappa^+] \cap \mathcal{P}(S^{\kappa^+}_{\kappa})$ mod NS. If G is $\mathbb{P}$-generic over V, then $NS \restriction M$ is $\mathbf{\Delta}_1(\mathrm{H}((2^\kappa)^+))$-definable in $\mathrm{V}[G]$.

Sketch of the proof.

Work in $\mathrm{V}[G]$, let T be the subtree of ${}^{<\kappa^+}\kappa^+$ given by our theorem and define $\mathcal{S}$ to be the collection of all subsets A of M such that either there exists a closed unbounded subset C of κ^+ with $C \cap M \subseteq A$ or there exists an order-preserving embedding of the tree $T(S^{\kappa^+}_{\kappa} \setminus A)$ into the tree T.

Then the set $\mathcal{S}$ is definable by a Σ_1-formula with parameters M, T and ${}^{<\kappa^+}\kappa^+$, and it is possible to show that $\mathcal{S}$ is equal to the collection of all subsets of M that are stationary in κ^+. $\square$

The same technical result allows us to prove analogous conclusions for the full non-stationary ideal on ω_2 and forcing axioms compatible with CH.

Theorem (Cox–L.)

Assume that $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$ and either $\mathrm{FA}^{+\omega_1}(\sigma$-closed$)$ or the Subcomplete Forcing Axiom SCFA holds.

If θ is a cardinal with $\theta^{\omega_2} = \theta$, then there is a $<\omega_2$-directed closed partial order that forces the following statements to hold in the corresponding generic extension of the ground model V:

- $2^{\omega_2} = \theta$.
- *The set NS_{ω_2} is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable.*

Thank you for listening!