OPEN QUESTIONS ON GENERALISED BAIRE SPACES

Abstract. Open questions collected at the seventh workshop on generalised Baire spaces that took place at the University of Bristol in February 2024.

See https://www.bristol.ac.uk/maths/events/2024/philip-welch-event-.html.

Please add (some) open questions from your talk (and additional problems) below and edit any time (participants of the workshop). Please copy in your changes all at the same time to avoid conflicts. If you send an email (see end of the file) after you've made edits, I'll update the file on the webpage.

1. Borel reducibility and model theory

(1) (Miguel) Do all analytic strong measure 0 subsets of ${}^{\kappa}2$ have size $\leq\kappa$?
(2) (Miguel) Is it consistent that there is a stable unsuperstable theory T such that $\cong_T$ is Δ^1_1?
(3) (Miguel) Are filter reflection and $=^{\kappa}_{X}\hookrightarrow_B=^{2}_{Y}$ equivalent?
(4) (Miguel) Define $=^{\theta}_{S}\subseteq \theta^{\kappa}\times\theta^{\kappa}$, $2\leq\theta\leq\kappa$, as $\eta =^{\theta}_{S}\xi$ if $\{\alpha<\kappa \mid \eta(\alpha)\neq\xi(\alpha)\}\cap S$ is not stationary. For which values of θ and $S\subseteq\kappa$ stationary, does $=^{\theta}_{X}\hookrightarrow=^{2}_{S}$ hold?
(5) (Miguel) Can the cardinal arithmetic assumptions of the Borel reducibility main gap be relaxed?
(6) (Miguel) Is isomorphism of graphs of size κ Σ^1_1-complete?
(7) (Miguel) Is it consistent that the definable sets in $\mathcal{M}_{\kappa^+,\kappa}$ are precisely the Borel* sets under isomorphism?

2. Dichotomies

(1) (Dorottya) Does the κ-perfect set property for κ-analytic sets imply the open graph dichotomy for κ-analytic sets (equivalently for closed sets)?
(2) (Dorottya) Does the PSP for closed sets already imply the statements in problem (1)?
(3) (Dorottya) $\mathrm{CCP}_{\kappa}(X)$ states that for any κ-ideal $\mathcal{I}$ on X generated by a family of closed sets, either $X\in\mathcal{I}$ or there exists a continuous function $f\colon {}^{\kappa}\kappa\to X$ such that $f(N_t)\in\mathcal{I}^+$ for all $t\in {}^{<\kappa}\kappa$. $\mathrm{CCP}_{\kappa}(\mathsf{D}_{\kappa})$ states that $\mathrm{CCP}_{\kappa}(X)$ holds for all subsets X of ${}^{\kappa}\kappa$ definable from a κ-sequence of ordinals.

    Do any of the previous statements in problems (1) and (2) imply $\mathrm{CCP}_{\kappa}(\Sigma^1_1,\mathsf{D}_{\alpha})$, i.e. the version for definable families of closed sets.
(4) (Dorottya) Does $\mathrm{CCP}_{\kappa}(\mathsf{D}_{\kappa})$ have at least the consistency strength of a Mahlo cardinal?
(5) (Philipp S.) Is the consistency strength of (any version of) the Hurewicz dichotomy for Π^1_1 or projective subset of ${}^{\omega}2$ or ${}^{\kappa}2$ an inaccessible or less? (A flawed proof by Stern claims that ZFC suffices to force this for ${}^{\omega}2$.)

3. Uniformisation

(1) (Philipp S.) Does the version of the Lusin-Novikov uniformisation theorem for ${}^{\kappa}\kappa$ always fail, i.e., can one prove that exists a κ-Borel relation with sections of size $\leq\kappa$ that does not admit a κ-Borel measurable uniformisation?

4. Kurepa trees

(1) (Claudio) Is there a κ-Kurepa subtree T of ${}^{<\kappa}\kappa$ such that the κ^+-Borel hierarchy for the space $[T]$ collapses? (This means there exists some $\alpha<\kappa^+$ such that every κ-Borel set is $\mathbf{\Sigma}^0_{\alpha}$.)

5. Forcing

(1) (Yurii) Is there a $<\kappa$-closed or $<\kappa$-distributive forcing adding a dominating κ-real without adding κ-Cohen reals?

6. Combinatorics

(1) (Nick) Is the Borel conjecture consistent for κ?
(2) (Miguel) Is the an analytic κ-MAD family in ${}^{\kappa}2$?

7. Löwenheim-Skolem numbers

(1) (Christopher) Is it consistent that the LST number is singular?

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