Set theory in second-order

Victoria Gitman

vgitman@nylogic.org
http://boolesrings.org/victoriagitman

Set Theory in UK 2

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Classes in naive set theory

The absence of a formal distinction between sets and classes in naive set theory led to supposed paradoxes.

Russell's Paradox

Let X be the set of all sets that are not members of themselves. Assuming $X \in X$ implies $X \notin X$ and assuming $X \notin X$ implies $X \in X$.

Burali-Forti's Paradox

The set of all ordinals is itself an ordinal larger than all ordinals and therefore cannot be an ordinal.

Classes in first-order set theory

The objects of first-order set theory are sets.

Definition: A class is a first-order definable (with parameters) collection of sets.

Classes play an important role in modern set theory.

- Inner models
- Ord-length products and iterations of forcing notions
- Elementary embeddings of the universe

But first-order set theory does not provide a framework for understanding the structure of classes.

In first-order set theory we cannot study:

- non-definable collections of sets,
- properties which quantify over classes.

Second-order set theory is a formal framework in which both sets and classes are objects of the set-theoretic universe. In this framework, we can study:

- non-definable classes
- general properties of classes

Kunen's Inconsistency

How do we formalize the statement of Kunen's Inconsistency that there is no non-trivial elementary embedding $j : V \to V$?

The following result is nearly trivial to prove.

Theorem: If $V \models \mathrm{ZF}$, then there is no definable non-trivial elementary embedding $j : V \to V$.

A non-trivial formulation must involve the existence and formal properties of non-definable collections.

Kunen's Inconsistency: A model of Kelley-Morse second-order set theory cannot have a non-trivial elementary embedding $j : V \to V$.

Properties of class forcing

Question: Does every class forcing notion satisfy the Forcing Theorem?

Fails in weak systems, holds in stronger systems.

Question: Does every class forcing notion have a Boolean completion?

Depends on how you define Boolean completion, with the right definition holds in strong systems.

Question: If two class forcing notions densely embed must they be forcing equivalent?

No.

Question: Does the Intermediate Model Theorem (all intermediate models between a universe and its forcing extension are forcing extensions) hold for class forcing?

Fails in weak systems, holds partially in stronger systems.

Inner model reflection principles

Inner model reflection principle (Barton, Caicedo, Fuchs, Hamkins, Reitz)

Whenever a first-order formula $\varphi(a)$ holds in V, then it holds in some inner model $W \subsetneq V$.

Inner model hypothesis (Friedman)

Whenever a first-order sentence φ holds in some inner model of a universe $V^* \supseteq V$, then it already holds in some inner model of V.

Truth predicates

Informal definition: A truth predicate is a collection of Gödel codes

$$T = \{\ulcorner \varphi(\bar{a}) \urcorner \mid V \models \varphi(\bar{a})\},$$

where $\varphi(x)$ are first-order formulas.

Theorem: (Tarski) A truth predicate is never definable.

Models of sufficiently strong second-order set theories have truth predicates, as well as iterated truth predicates.

Second-order set theory

Second-order set theory has two sorts of objects: sets and classes.

Syntax: Two-sorted logic

- Separate variables for sets and classes
- Separate quantifiers for sets and classes
- Convention: upper-case letters for classes, lower-case letters for sets
- Notation:
  - Σ^0_n - first-order Σ_n-formula
  - Σ^1_n - n-alternations of class quantifiers followed by a first-order formula

Semantics: A model is a triple $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$.

- V consists of the sets.
- $\mathcal{C}$ consists of the classes.
- Every set is a class: $V \subseteq \mathcal{C}$.
- $C \subseteq V$ for every $C \in \mathcal{C}$.

Alternatively, we can formalize second-order set theory with classes as the only objects and define that a set is a class that is an element of some class.

Gödel-Bernays set theory GBC

Axioms

- Sets: ZFC
- Classes:
  - Extensionality
  - Replacement: If F is a function and a is a set, then $F \upharpoonright a$ is a set.
  - Global well-order: There is a class well-order of sets.
  - Comprehension scheme for first-order formulas:

    If $\varphi(x, A)$ is a first-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Models

- L together with its definable collections is a model of GBC.
- Suppose $V \models \mathrm{ZFC}$ has a definable global well-order. Then V together with its definable collections is a model of GBC.
- **Theorem**: (folklore) Every model of ZFC has a forcing extension with the same sets and a global well-order. Force to add a Cohen sub-class to Ord.

Strength

- GBC is equiconsistent with ZFC.
- GBC has the same first-order consequences as ZFC.

$\mathrm{GBC} + \Sigma^1_1$-Comprehension

Axioms

- GBC
- Comprehension for Σ^1_1-formulas:
  If $\varphi(x, A) := \exists X \psi(x, X, A)$ with ψ first-order, then $\{x \mid \varphi(x, A)\}$ is a class.

Note: Σ^1_1-Comprehension is equivalent to Π^1_1-Comprehension.

Question: How strong is $\mathrm{GBC} + \Sigma^1_1$-Comprehension?

Meta-ordinals

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: A meta-ordinal is a well-order $(\Gamma, \leq) \in \mathcal{C}$.

- Examples: Ord, $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$.
- Notation: For $a \in \Gamma$, $\Gamma \upharpoonright a$ is the restriction of the well-order to $\leq$-predecessors of a.

Theorem: $\mathrm{GBC} + \Sigma^1_1$-Comprehension proves that any two meta-ordinals are comparable.

- Suppose $(\Gamma, \leq)$ and $(\Delta, \leq)$ are meta-ordinals.
- Let $A = \{a \in \Gamma \mid \Gamma \upharpoonright a \text{ is isomorphic to an initial segment of } (\Delta, \leq)\}$ (exists by Σ^1_1-Comprehension)
- If $A = \Gamma$, then $(\Gamma, \leq)$ embeds into $(\Delta, \leq)$.
- Otherwise, let a be $\leq$-least not in A. Then $\Gamma \upharpoonright b$ is isomorphic to $(\Delta, \leq)$, where b is immediate $\leq$-predecessor of a.

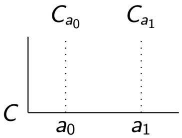
Question: Does GBC imply that any two meta-ordinals are comparable?

Problem: Meta-ordinals don't need to have unique representations! Unless...

Elementary Transfinite Recursion ETR

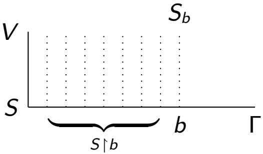
Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: Suppose $A \in \mathcal{C}$ is a class. A sequence of classes $\langle C_a \mid a \in A\rangle$ is a single class C such that $C_a = \{x \mid \langle a, x\rangle \in C\}$.



Definition: Suppose $\Gamma \in \mathcal{C}$ is a meta-ordinal. A solution along Γ to a first-order recursion rule $\varphi(x, b, F)$ is a sequence of classes S such that for every $b \in \Gamma$, $S_b = \varphi(x, b, S \restriction b)$.

Elementary Transfinite Recursion ETR: For every meta-ordinal Γ, every first-order recursion rule $\varphi(x, b, F)$ has a solution along Γ.



Theorem: $\mathrm{GBC}+\Sigma^1_1$-Comprehension implies ETR. Let $A = \{a \mid \varphi(x, b, F) \text{ has a solution along } \Gamma \restriction a\}$.

ETR_Γ: Elementary transfinite recursion for a fixed Γ.

- $\mathrm{ETR}_{\mathrm{Ord}\cdot\omega}$, $\mathrm{ETR}_{\mathrm{Ord}}$, ETR_ω

Theorem: (Williams) If $\Gamma \geq \omega^\omega$ is a (meta)-ordinal, then $\mathrm{GBC} + \mathrm{ETR}_{\Gamma\cdot\omega}$ implies $\mathrm{Con}(\mathrm{GBC} + \mathrm{ETR}_\Gamma)$.

Truth Predicates

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$.

Definition: A class $T \in \mathcal{C}$ is a truth predicate for $\langle V, \in\rangle$ if it satisfies Tarksi's truth conditions: For every $\ulcorner\varphi\urcorner \in V$ (φ possibly nonstandard),

- if φ is atomic, $V \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$,
- $\ulcorner\neg\varphi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \notin T$,
- $\ulcorner\varphi(\bar{a}) \wedge \psi(\bar{a})\urcorner \in T$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$ and $\ulcorner\psi(\bar{a})\urcorner \in T$,
- $\ulcorner\exists x\varphi(x, \bar{a})\urcorner \in T$ iff $\exists b\, \ulcorner\varphi(b, \bar{a})\urcorner \in T$.

Observation: If T is a truth predicate, then $V \models \varphi(\bar{a})$ iff $\ulcorner\varphi(\bar{a})\urcorner \in T$.

Corollary: GBC cannot imply that there is a truth predicate.

Observation: If T is a truth predicate and $\ulcorner\varphi\urcorner \in \mathrm{ZFC}^{\mathrm{V}}$ (φ possibly nonstandard), then $\varphi \in T$.

Theorem: If there is a truth predicate $T \in \mathcal{C}$, then V is the union of an elementary chain of its rank initial segments V_α:

$$V_{\alpha_0} \prec V_{\alpha_1} \prec \cdots \prec V_{\alpha_\xi} \prec \cdots \prec V,$$

such that V thinks that each $V_{\alpha_\xi} \models \mathrm{ZFC}$.

Let $\langle V_\alpha, \in, T \cap V_\alpha\rangle \prec_{\Sigma_1} \langle V, \in, T\rangle$.

Iterated truth predicates

Theorem: $\mathrm{GBC}+\mathrm{ETR}_\omega$ implies that for every class A, there is a truth predicate for $\langle V,\in,A\rangle$. A truth predicate is defined by a recursion of length ω.

Question: Is $\mathrm{GBC}+\mathrm{ETR}_\omega$ equivalent to GBC plus there are truth predicates for all classes?

Theorem: $\mathrm{GBC}+\mathrm{ETR}_\omega$ (and therefore $\mathrm{GBC}+\Sigma^1_1$-Comprehension) implies $\mathrm{Con}(\mathrm{ZFC})$, $\mathrm{Con}(\mathrm{Con}(\mathrm{ZFC}))$, etc.

Definition: Suppose $\mathscr{V}=\langle V,\in,\mathcal{C}\rangle\models\mathrm{GBC}$ and $(\Gamma,\leq)$ is a meta-ordinal. A sequence of classes $\vec{T}=\langle T_a\mid a\in\Gamma\rangle$ is an iterated truth predicate of length Γ if for every $a\in\Gamma$, T_a is a truth predicate for $\langle V,\in,\vec{T}\upharpoonright a\rangle$.

Theorem: (Fujimoto)

- Suppose $\mathscr{V}=\langle V,\in,\mathcal{C}\rangle\models\mathrm{GBC}$ and $\Gamma\geq\omega^\omega$ is a meta-ordinal. ETR_Γ is equivalent to the existence of an iterated truth predicate of length Γ.
- Over GBC, ETR is equivalent to the existence of an iterated truth predicate of length Γ for every meta-ordinal Γ.

A single recursion, the iterated truth recursion, suffices to give all other recursions.

The second-order constructible universe

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC} + \mathrm{ETR}$.

Theorem: $\mathrm{GBC} + \mathrm{ETR}$ implies that any two meta-ordinals are comparable.

Problem: Meta-ordinals don't need to have unique representations! Unless...

Given a meta-ordinal Γ, we can build a meta-constructible universe L_Γ by a recursion of length Γ.

Definition: A meta-ordinal Γ is constructible if there is another meta-ordinal Δ such that L_Δ has a well-order of Ord isomorphic to Γ. ("$\Gamma \in L_{\mathrm{Ord}^+}$")

Theorem: (Tharp) Constructible meta-ordinals have unique representations.

Definition:

- A class $A \in \mathcal{C}$ is constructible if there is a constructible meta-ordinal Γ such that $A \in L_\Gamma$.
- The second-order constructible universe is $\mathscr{L} = \langle L, \in, \mathcal{L}\rangle$, where $\mathcal{L}$ consists of the constructible classes.

Theorem: If $\mathscr{V} \models \mathrm{GBC} + \Sigma^1_1$-Comprehension, then $\mathscr{L} \models \mathrm{GBC} + \Sigma^1_1$-Comprehension. It also satisfies a version of the Axiom of Choice for classes. (Coming up!)

The Class Forcing Theorem

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$ and $\mathbb{P} \in \mathcal{C}$ is a class partial order.

Definition:

- A class $\mathbb{P}$-name is a collection of pairs $\langle \sigma, p\rangle$ such that $p \in \mathbb{P}$ and $\sigma \in V^{\mathbb{P}}$.
- G is $\mathscr{V}$-generic for $\mathbb{P}$ if G meets every dense class $D \in \mathcal{C}$ of $\mathbb{P}$.
- The forcing extension $\mathscr{V}[G] = \langle V[G], \in, \mathcal{C}[G]\rangle$.

Observation: $\mathscr{V}[G]$ may not satisfy GBC. Force with $\mathrm{Coll}(\omega, \mathrm{Ord})$.

The Class Forcing Theorem: There is a solution to the recursion defining the forcing relation for atomic formulas.

Observation: Suppose the Class Forcing Theorem holds.

- The forcing relation for all first-order formulas with a fixed class parameter is a class.
- For every second-order formula $\varphi(x, Y)$ the relation $p \Vdash \varphi(\tau, \Gamma)$ is (second-order) definable.

Theorem: (Krapf, Njegomir, Holy, Lücke, Schlicht) GBC does not imply the Class Forcing Theorem.

Theorem: (G., Hamkins, Holy, Schlicht, Williams) Over GBC, $\mathrm{ETR}_{\mathrm{Ord}}$ is equivalent to the Class Forcing Theorem.

Class games and determinacy

Let Ord^ω be the topological space of ω-length sequences of ordinals with the product topology.

Fix a class $A \subseteq \mathrm{Ord}^\omega$.

The Game $\mathcal{G}_A$

- Player I (Alice) and Player II (Bob) alternately play ordinals for ω-many steps.

I	α_0		α_1		α_2		$\ldots$		α_n	
II		β_0		β_1		β_2		$\ldots$		β_n

- Alice wins if $\vec{\alpha} = \langle \alpha_0, \beta_0, \alpha_1, \beta_1 \ldots, \alpha_n, \beta_n \ldots \rangle \in A$. Otherwise, Bob wins.
- The game is determined if one of the players has a winning strategy.

Note: A strategy for a player in the game $\mathcal{G}_A$ is a class.

Open Class Determinacy: $\mathcal{G}_A$ is determined for every open $A \subseteq \mathrm{Ord}^\omega$.

Clopen Class Determinacy: $\mathcal{G}_A$ is determined for every clopen $A \subseteq \mathrm{Ord}^\omega$.

Theorem: (G., Hamkins) $\mathrm{GBC}+\Sigma^1_1$-Comprehension implies Open Class Determinacy.

Theorem: (G., Hamkins) Over GBC, ETR is equivalent to Clopen Class Determinacy.

Beyond ETR

Theorem: (Sato) Over GBC, Open Class Determinacy is stronger than ETR.

Question: Is ETR preserved by tame forcing? Tame forcings preserve GBC.

Theorem: (G., Hamkins) Suppose $\mathscr{V} \models \mathrm{GBC} + \mathrm{ETR}_\Gamma$, $\mathbb{P}$ is a tame forcing notion and $G \subseteq \mathbb{P}$ is $\mathscr{V}$-generic. Then $\mathscr{V}[G] \models \mathrm{GBC} + \mathrm{ETR}_\Gamma$.

Theorem: (Hamkins) Tame countably strategically-closed forcing preserves ETR.

Question: Can forcing add meta-ordinals?

Theorem: (Hamkins, Woodin) Over GBC, Open Class Determinacy

- is preserved by tame forcing,
- implies that tame forcing does not add meta-ordinals.

The hierarchy so far

$$\mathrm{GBC} + \Sigma^1_1\text{-Comprehension}$$

$$\downarrow$$

$$\mathrm{GBC} + \text{Open Determinacy}$$

$$\downarrow$$

$$\mathrm{GBC} + \text{For every well-order } \Gamma\text{, there is an iterated truth predicate of length } \Gamma$$

$$\mathrm{GBC} + \text{Clopen Class Determinacy}$$

$$\mathrm{GBC} + \mathrm{ETR}$$

$$\downarrow$$

$$\mathrm{GBC} + \text{There is an iterated truth predicate of length } \mathrm{Ord}$$

$$\mathrm{GBC} + \text{Class Forcing Theorem}$$

$$\mathrm{GBC} + \mathrm{ETR}_{\mathrm{Ord}}$$

$$\downarrow$$

$$\mathrm{GBC} + \mathrm{ETR}_{\omega}$$

$$\downarrow$$

$$\mathrm{GBC} + \text{There exists a truth predicate}$$

$$\downarrow$$

$$\mathrm{GBC}$$

Kelley-Morse set theory KM

Axioms

- GBC
- Full comprehension:
  If $\varphi(x, A)$ is a second-order formula, then $\{x \mid \varphi(x, A)\}$ is a class.

Models

Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}$.

Theorem: (Antos) The theory KM is preserved by tame forcing.

Theorem: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM}$. Then its constructible universe $\mathscr{L} = \langle L, \in, \mathcal{L}\rangle \models \mathrm{KM}$. It also satisfies a choice principle for classes. (Next slide!)

A choice principle for classes

Choice Scheme (CC): Given a second-order formula $\varphi(x, X, A)$, if for every set x, there is a class X witnessing $\varphi(x, X, A)$, then there is a sequence of classes collecting witnesses for every x:

$$\forall x\, \exists X\, \varphi(x, X, A) \to \exists Y\, \forall x\, \varphi(x, Y_x, A).$$

Σ^1_n-Choice Scheme (Σ^1_n-CC): Choice Scheme for Σ^1_n-formulas.

Set Choice Scheme (Set-CC): Given a second-order formula $\varphi(x, X, A)$ and a set a:

$$\forall x \in a\, \exists X\, \varphi(x, X, A) \to \exists Y\, \forall x \in a\, \varphi(x, Y_x, A).$$

Proposition: Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}{+}\mathrm{CC}$.

Theorem: (Marek, Mostowski, Ratajczyk) If $\mathscr{V} \models \mathrm{GBC} + \Sigma^1_n$-Comprehension, then its constructible universe $\mathscr{L} \models \mathrm{GBC} + \Sigma^1_n$-Comprehension$+\Sigma^1_n$-$\mathrm{CC}$.

- If $\mathscr{V} \models \mathrm{KM}$, then its constructible universe $\mathscr{L} \models \mathrm{KM}{+}\mathrm{CC}$.
- The theories KM and $\mathrm{KM}{+}\mathrm{CC}$ are equiconsistent.

Theorem: (G., Hamkins) There is a model of KM in which the Choice Scheme fails for for ω-many choices for a first-order formula.

Theorem: (Antos, Friedman, G.) The theory $\mathrm{KM}{+}\mathrm{CC}$ is preserved by tame forcing.

Fodor's Lemma for classes

The Class Fodor Principle: Every regressive class function $f : S \to \mathrm{Ord}$ from a stationary class S into Ord is constant on a stationary subclass $T \subseteq S$.

Theorem: $\mathrm{GBC} + \Sigma^0_1\text{-}\mathrm{CC}$ implies the Class Fodor Principle.

- Suppose each $A_n = f^{-1}(\{\alpha\})$ is not stationary.
- By $\Sigma^1_0\text{-}\mathrm{CC}$, there is a sequence of clubs $\langle C_n \mid n < \alpha \rangle$ such that A_α misses C_α.
- Let $C = \Delta_{\alpha \in \mathrm{Ord}} C_\alpha$. Then $C \cap S = \emptyset$.

Theorem: (G., Hamkins, Karagila) Every model of KM has an extension to a model of KM with the same sets in which the Class Fodor Principle fails.

- Force to add a class Cohen function $f : \mathrm{Ord} \to \omega$.
- Let $A_n = \{\alpha \mid f(\alpha) > n\}$.
- Let $\mathbb{Q} = \Pi_{n<\omega} \mathbb{Q}_n$ be the product forcing with $\mathbb{Q}_n$ shooting a club through A_n.
- Let $G \subseteq \mathbb{Q}$ be $\mathscr{V}$-generic and take only classes added by some finite stage n.

Question: How strong is the Class Fodor Principle? Does it imply Con(ZFC)?

Theorem: (G., Hamkins, Karagila) The Class Fodor Principle is preserved by set forcing.

Question: Is the Class Fodor Principle preserved by tame forcing?

The Łoś Theorem for second-order ultrapowers

Suppose $\mathscr{V} = (V, \in, \mathcal{C}) \models \mathrm{KM}$.

- Suppose U is an ultrafilter on a cardinal κ.
- Define that functions $f : \kappa \to V$ and $g : \kappa \to V$ are equivalent when $\{\xi < \kappa \mid f(\xi) = g(\xi)\} \in U$.
- Let M be the collection of the equivalence classes $[f]_U$.
- Define that $[f]_U \, \mathrm{E} \, [g]_U$ when $\{\xi < \kappa \mid f(\xi) \in g(\xi)\} \in U$.
- Define that class sequences F and G are equivalent when $\{\xi < \kappa \mid F_\xi = G_\xi\} \in U$.
- Let $\mathcal{M}$ be the collection of the equivalence classes $[F]_U$.
- $[f]_U \, \mathrm{E} \, [F]_U$ when $\{\xi < \kappa \mid f(\xi) \in F(\xi)\} \in U$.
- $\langle M, \mathrm{E}, \mathcal{M}\rangle$ is the ultrapower of $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle$ by U.
- Define $j : \langle V, \in, \mathcal{C}\rangle \to \langle M, \mathrm{E}, \mathcal{M}\rangle$ by $j(a) = [c_a]_U$ and $j(A) = [C_A]_U$.

The Łoś Theorem

Problem: For the class existential quantifier, we need

$$\mathscr{V} \models \forall \xi < \kappa \, \exists X \, \varphi(X, f(\xi)) \to \exists Y \, \forall \xi < \kappa \, \varphi(Y_\xi, f(\xi)).$$

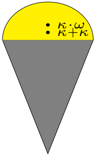
Theorem: (G., Hamkins) Over KM, the Łoś Theorem for second-order ultrapowers is equivalent to Set-CC.

Back to first-order with KM+CC

Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM+CC}$.

- View each extensional well-founded class relation $R \in \mathcal{C}$ as coding a transitive set.
- Define a membership relation E on the collection of all such relations R (modulo isomorphism).
  - $\mathrm{Ord} + \mathrm{Ord}$, $\mathrm{Ord} \cdot \omega$.
  - $V \cup \{V\}$.
- Let $\langle M_{\mathscr{V}}, \mathrm{E}\rangle$, the companion model of $\mathscr{V}$, be the resulting first-order structure.
  - $M_{\mathscr{V}}$ has the largest cardinal $\kappa \cong \mathrm{Ord}^{\mathscr{V}}$.
  - $V_\kappa^{M_{\mathscr{V}}} \cong V$.
  - $\mathcal{P}(V_\kappa)^M_{\mathscr{V}} \cong \mathcal{C}$.
  - $\langle M_{\mathscr{V}}, \mathrm{E}\rangle \models \mathrm{ZFC}^-_{\mathrm{I}}$. (Next slide!)

$V = V_\kappa^{M_{\mathscr{V}}}$



The theory $\mathrm{ZFC}^-_{\mathrm{I}}$

Axioms

- ZFC without powerset (Collection scheme instead of Replacement scheme).
- There is a largest cardinal κ.
- κ is inaccessible. (κ is regular and for all $\alpha < \kappa$, 2^α exists and $2^\alpha < \kappa$.)

Models

Suppose that $V \models \mathrm{ZFC}$ and κ is inaccessible. Then $H_{\kappa^+} \models \mathrm{ZFC}^-_{\mathrm{I}}$.

Moving to second-order

Suppose $M \models \mathrm{ZFC}^-_{\mathrm{I}}$ with a largest cardinal κ.

- $V = V^M_\kappa$
- $\mathcal{C} = \{X \in M \mid X \subseteq V^M_\kappa\}$
- $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{KM{+}CC}$
- $M_{\mathscr{V}} \cong M$ is the companion model of $\mathscr{V}$.

Theorem: (Marek, Mostowski) The theory $\mathrm{KM{+}CC}$ is bi-interpretable with the theory $\mathrm{ZFC}^-_{\mathrm{I}}$.

The Class Reflection Principle

Reflection Principle: Every formula is reflected by a transitive set:

For every first-order formula $\varphi(x)$, there is a transitive set M such that for all $a \in M$, $\varphi(a)$ holds if and only if $M \models \varphi(a)$.

Theorem: (Lévy) ZFC proves the reflection principle.
Every first-order formula is reflected by some V_α.

Class Reflection Principle: Every formula is reflected by a sequence of classes:

For every second-order formula $\varphi(X)$, there is a sequence of classes $S = \langle S_\xi \mid \xi \in \mathrm{Ord}\rangle$ such that for all $\xi \in \mathrm{Ord}$, $\varphi(S_\xi)$ if and only if $\langle V, \in, S\rangle \models \varphi(S_\xi)$.

Theorem: Suppose $\mathscr{V} \models \mathrm{KM}+\mathrm{CC}$ and $M_{\mathscr{V}} \models \mathrm{ZFC}^-_{\mathrm{I}}$ is its companion model. Then $\mathscr{V}$ satisfies the Class Reflection Principle if and only if $M_{\mathscr{V}}$ satisfies the Reflection Principle.

Another class choice principle

ω-Dependent Choice Scheme (ω-DC): Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is another class Y such that $\varphi(X, Y, A)$ holds, then we can make ω-many dependent choices according to φ:

$$\forall X\, \exists Y\, \varphi(X, Y, A) \rightarrow \exists Y\, \forall n\, \varphi(Y_n, Y_{n+1}, A).$$

Theorem: Over KM+CC, ω-DC is equivalent to the Class Reflection Principle.

Question: Does KM+CC imply ω-DC?

Conjecture: (Friedman, G.) KM+CC does not imply the ω-DC.

Suppose that α is an uncountable regular cardinal or $\alpha = \mathrm{Ord}$.

α-Dependent Choice Scheme (α-DC: Given a second-order formula $\varphi(X, Y, A)$, if for every class X, there is another class Y such that $\varphi(X, Y, A)$ holds, then we can make α-many dependent choices according to φ.

Proposition: If $\mathscr{V} \models \mathrm{KM}+\alpha\text{-DC}$, then its companion model $M_{\mathscr{V}}$ satisfies that every first-order formula reflects to a transitive set that is closed under $<\alpha$-sequences.

Theorem: If $\mathscr{V} \models \mathrm{KM}$, then its constructible universe $\mathscr{L} \models \mathrm{KM}+\mathrm{CC}+\mathrm{Ord}\text{-DC}$.

- The theories KM and KM+CC+Ord-DC are equiconsistent.

Proposition: Suppose $V \models \mathrm{ZFC}$ and κ is an inaccessible cardinal. Then $\langle V_\kappa, \in, V_{\kappa+1}\rangle \models \mathrm{KM}+\mathrm{CC}+\mathrm{Ord}\text{-DC}$.

The Class Intermediate Model Theorem

Intermediate Model Theorem: (Solovay)

- If $V \models \mathrm{ZFC}$ and $W \models \mathrm{ZFC}$ is an intermediate model between V and its set-forcing extension $V[G]$, then W is a set-forcing extension of V.
- If $V \models \mathrm{ZF}$ and $V[a] \models \mathrm{ZF}$, with $a \subseteq V$, is an intermediate model between V and its set-forcing extension $V[G]$, then $V[a]$ is a set-forcing extension of V.

Definition: Suppose $\mathscr{V} = \langle V, \in, \mathcal{C}\rangle \models \mathrm{GBC}$. Then $\mathscr{W} = \langle W, \in, \mathcal{C}^*\rangle$ is a simple extension of $\mathscr{V}$ if $\mathcal{C}^*$ is generated by $\mathcal{C}$ together with a single new class.

- Forcing extension are simple extensions.

Definition: Suppose T is a second-order set theory.

- The Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is an intermediate model between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G]$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.
- The simple Intermediate Model Theorem holds for T if whenever $\mathscr{V} \models T$ and $\mathscr{W} \models T$ is a simple extension of $\mathscr{V}$ between $\mathscr{V}$ and its class-forcing extension $\mathscr{V}[G]$, then $\mathscr{W}$ is a class-forcing extension of $\mathscr{V}$.

The Class Intermediate Model Theorem (continued)

Theorem:

- (Friedman) The simple Intermediate Model Theorem for GBC fails.
- (Hamkins, Reitz) The simple Intermediate Model Theorem for GBC fails even for Ord-cc forcing.

Theorem: (Antos, Friedman, G.) The simple Intermediate Model Theorem for $\mathrm{KM}+\mathrm{CC}$ holds.

Theorem: (Antos, Friedman, G.) Every model $\mathscr{V} \models \mathrm{KM}+\mathrm{CC}$ has a forcing extension $\mathscr{V}[G]$ with a non-simple intermediate model. Therefore the Intermediate Model Theorem for $\mathrm{KM}+\mathrm{CC}$ fails.

Question: Does the simple Intermediate Model Theorem for KM hold?

Boolean completions of class partial orders

Definition: Suppose $\mathbb{B}$ is a class Boolean algebra.

- $\mathbb{B}$ is set-complete if all its subsets have suprema.
- $\mathbb{B}$ is class-complete if all its subclasses have suprema.

Theorem: (Krapf, Njegomir, Holy, Lücke, Schlicht)

- Suppose $\mathscr{V} \models \mathrm{GBC}$. A class partial order $\mathbb{P}$ has a set-complete Boolean completion if and only if the Class Forcing Theorem holds for $\mathbb{P}$.
- If a Boolean algebra is class-complete, then it has the Ord-cc. Therefore a partial order has a class-complete Boolean completion if and only if it has the Ord-cc.

Definition: In a model of second-order set theory, a hyperclass is a definable collection of classes.

Definition: Suppose $\mathbb{B}$ is a hyperclass Boolean algebra.

- $\mathbb{B}$ is class-complete if all its subclasses have suprema.
- $\mathbb{B}$ is complete if all its sub-hyperclasses have suprema.

Boolean completions of class partial orders (continued)

Theorem: Suppose $\mathscr{V} \models \mathrm{GBC}$. Every class partial order $\mathbb{P}$ has a class-complete hyperclass Boolean completion. Standard regular cuts construction.

Theorem: (Antos, Friedman, G.) Suppose $\mathscr{V} \models \mathrm{KM}$. Then every class partial order $\mathbb{P}$ has a complete hyperclass Boolean completion.

Theorem: (Antos, Friedman, G.) Suppose $\mathscr{V} \models \mathrm{GBC}$. If some non-$\mathrm{Ord}$-cc class partial order $\mathbb{P}$ has a complete hyperclass Boolean completion, then $\mathscr{V} \models \mathrm{KM}$.

Suppose $\mathscr{V} \models \mathrm{KM}+\mathrm{CC}$ and $\mathbb{P}$ is class partial order. In the companion model $M_{\mathscr{V}}$ (with the largest cardinal κ):

- $\mathbb{P}$ is a set partial order.
- $\mathbb{P}$ has a class-complete Boolean completion $\mathbb{B}$.