# OPEN PROBLEMS ON GENERALISED BAIRE SPACES 2020

The *Fifth Workshop on Generalised Baire Spaces* took place at the School of Mathematics of the University of Bristol on 3-4 February, 2020, see `<https://philippschlicht.github.io/meetings/generalizedbairespaces2020>`. This is an incomplete list of some open problems that were discussed at the workshop. [1]

## 1. Generalized Baire Spaces

1.1. **Claudio Agostini: Winning tactics.** A *tactic* in a game of length $\omega$ is a strategy that depends only on the previous move of the opponent. For separable regular Hausdorff spaces, the following conditions are equivalent:

(1) There is a compatible complete metric.
(2) There is a winning tactic for II in the strong Choquet game.
(3) There is a winning strategy for II in the strong Choquet game.

Notice that tactics do not make sense for longer games, since moves at limit times don't have a predecessor. But one can define the following natural variant for the strong Choquet game $G_\kappa(X)$ of length $\kappa$ for $X$. A tactic for II in $G_\kappa(X)$ is defined as a strategy with the following properties:

(1) For successor times, the reply of II depends only on the preceding move.
(2) For a play of limit length consisting of $\vec{U} = \langle U_i \mid i < \alpha\rangle$ and $\vec{x} = \langle x_i \mid i < \alpha\rangle$, the reply of II depends only on $\bigcap_{i<\alpha} U_i$ and (possibly) $\alpha$.

**Question 1.1.** Suppose that $X$ is a regular Hausdorff space of weight $\leq \kappa$. Does the existence of a winning strategy for player II in $G_\kappa(X)$ imply the existence of a winning tactic?

1.2. **Dorottya Sziraki: The open colouring axiom.** The *open colouring axiom* for $X$, $\mathsf{OCA}_\kappa(X)$, states that any open graph $G$ on $X$ either has a $\kappa$-coloring or else contains a complete subgraph of size $\kappa^+$. $\mathsf{OCA}_\kappa$ states that $\mathsf{OCA}_\kappa(X)$ holds for all $X \subseteq {}^\kappa\kappa$.

**Question 1.2.** Is $\mathsf{OCA}_\kappa$ consistent with $\mathsf{ZFC}$?

The *open graph dichotomy* $\mathsf{OGD}_\kappa(X)$ for $X$ states that any open graph on $X$ either has a $\kappa$-colouring or else contains a $\kappa$-perfect complete subgraph. The *perfect set property* $\mathsf{PSP}_\kappa(X)$ for $X \subseteq {}^\kappa\kappa$ states that $|X| \leq \kappa$ or $X$ has a $\kappa$-perfect subsets.

**Question 1.3.** Does $\mathsf{PSP}_\kappa(\text{closed})$ imply $\mathsf{OGD}_\kappa(\text{closed})$?

**Fact.** $\mathsf{OGD}_\kappa(\text{closed}) \Rightarrow \mathsf{OGD}_\kappa(\Sigma^1_1(\kappa)) \Rightarrow \mathsf{PSP}_\kappa(\Sigma^1_1(\kappa))$.

**Fact.** *After forcing with* $\mathrm{Col}(\kappa, {<}\lambda)$*, where* $\lambda > \kappa$ *is inaccessible,* $\mathsf{OGD}_\kappa$ *holds for all sets definable from* $\kappa$*-sequences of ordinals.*

**Question 1.4.** Can we force $\mathsf{PSP}_\kappa$ with some forcing other than $\mathrm{Col}(\kappa, {<}\lambda)$?

[1] If you have an answer, comment or further question, please email Philipp Schlicht at philipp.schlicht@bristol.ac.uk.

## 2. Connections with model theory

### 2.1. Jan Dobrowolski: Polish groups.

**Fact.** *Suppose that $X$ is Polish, $E$ is an equivalence relation on $X$ such that*

(1) *the equivalence class $[x]_E$ of any $x \in X$ is closed and*
(2) *the saturation $[U]_E$ of any open subset $U$ of $X$ is Borel.*

*Then $X/E$, with the Effros Borel structure, is standard Borel, and the natural projection $\pi\colon X \to X/E$ has a Borel measurable right inverse $f$ with $\pi \circ f = \mathrm{id}$.*

In particular, this holds for any Polish group $H$ and the equivalence relation on $H$ induced by a closed subgroup $G \leq H$.

**Question 2.1.** Does the analogue of the previous statement about Polish groups hold for $\kappa^\kappa$?

The product of two closed subgroups $G$, $H$ of $\mathrm{Sym}(\omega)$ is again closed.

**Question 2.2.** Does the analogue of the previous statement hold for $\mathrm{Sym}(\kappa)$ with the bounded topology?

### 2.2. Rosario Mennuni: Omitting types.

**Disclaimer:** The answer to this question might be easy or already known.

The most basic version of the Omitting Type Theorem states:

**Theorem.** *Let $T$ be a consistent first-order theory in a countable language, $x = (x_0, \ldots, x_{n-1})$ an $n$-tuple of variables, and $\pi(x)$ a partial $n$-type over $\emptyset$ such that for no formula $\phi(x)$ consistent with $T$ we have $T \cup \{\phi(x)\} \vdash \pi(x)$. Then there is a countable $M \models T$ such that no element of $M^n$ realises $\pi(x)$.*

A better version states (see [2, Theorem 10.3]):

**Theorem.** *Let $T$ be a consistent first-order theory in a countable language, and for every $n \in \omega$ let $A_n$ be a meagre subset of the space $S_n(T)$ of $n$-types over $\emptyset$. Then there is a countable $M \models T$ such that, for every $n \in \omega$ and $p(x) \in A_n$, no element of $M^n$ realises $p(x)$.*

The basic version can be generalised to what is called the *$\kappa$-omitting types theorem* (see [1, Theorem 2.2.19]).

**Theorem.** *Let $T$ be a consistent first-order theory in a language of size $\kappa$, $x = (x_0, \ldots, x_{n-1})$ an $n$-tuple of variables, and $\pi(x)$ a partial $n$-type over $\emptyset$ such that for no set of formulas $\Phi(x)$ of size $< \kappa$ consistent with $T$ we have $T \cup \Phi(x) \vdash \pi(x)$. Then there is $M \models T$ with $|M| \leq \kappa$ and such that no element of $M^n$ realises $\pi(x)$.*

**Question 2.3.** Let $L$ be a language of size $\kappa$, and let $T$ be a consistent $L$-theory. Equip each $S_n(T)$ with the topology generated by declaring partial types of size $< \kappa$ to induce open sets, and replace "meagre" with "$\kappa$-meagre". For which cardinals $\kappa$ does Theorem 1.2.2 generalise in the fashion of Theorem 1.2.2?

## References


[1] C.C. Chang, H.J. Keisler. *Model Theory*, Third edition, Studies in Logic and the Foundations of Mathematics 73. North-Holland Publishing Co., Amsterdam, (1990).

[2] B. Poizat. *A Course in Model Theory.* Universitext. Springer (2000).

## 3. Combinatorics of $\kappa^\kappa$

### 3.1. **Adrian Mathias, Vera Fischer.**

**Question 3.1.** Do analogues to some classical results for mad families on $\omega$ hold for mad families on $\kappa$?

### 3.2. **Johannes Schürz.**

**Question 3.2.** Is there a forcing that adds a $\kappa$-dominating real, but no $\kappa$-Cohen real?

### 3.3. **Sarka Stejskalova.**

**Question 3.3.** Is $\mathfrak{u}_\kappa < 2^\kappa$ consistent for the successor $\kappa$ of a regular cardinal?