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Bachelor's Thesis: Static and Dynamic Vehicle Routing under Uncertainty

1 SPATIAL RANDOM FIELD GENERATION

1.1 Covariance Models

- Gaussian Kernel:

$$\gamma(r) = \sigma^2 \left(1 - \exp\left(-\left(s \cdot \frac{r}{\ell}\right)^2\right)\right) + n, \qquad s = \frac{\sqrt{\pi}}{2}.$$

- Exponential Kernel:
- Matern Kernel:
- Stable Kernel:
- Rational Kernel:

1.2 Fast Fourier Transformation Generation

LIST OF SYMBOLS AND ABBREVIATIONS

2 ROBUST STATIC NETWORK-FLOW MODEL UNDER BUDGET UNCERTAINTY

This section introduces the mathematical formulation of the robust shortest-path model under budgeted uncertainty, following the approach of *Bertsimas & Sim (2003).* The model determines a cost-minimal path between a start and goal node in a directed network while accounting for uncertain edge costs within a predefined uncertainty budget Γ. The formulation extends the standard single-commodity network-flow model by integrating a robust optimization component, allowing a controlled level of conservatism against cost deviations.

2.1 Mathematical Formulation

$$\min_{x\in\mathcal{X}} \max_{c\in\mathcal{U}} \sum_{e\in E} c_e x_e \tag{4.8}$$

$$= \min \sum_{e\in E} \hat{c}_e x_e + \Gamma\pi + \sum_{e\in E} \rho_e \tag{4.9}$$

$$\text{s.t. } \pi + \rho_e \geq d_e x_e, \quad \forall e \in E \tag{4.10}$$

$$\sum_{e\in\delta^+(v)} x_e - \sum_{e\in\delta^-(v)} x_e = b_v, \quad \forall v \in V \tag{4.11}$$

$$x_e \in \{0,1\}, \quad \pi \geq 0, \quad \rho_e \geq 0, \quad \forall e \in E \tag{4.12)–(4.14}$$

2.2 Model Description

Objective Function (4.8) Equation (4.8) defines the robust shortest-path problem as a nested minmax optimization problem. The outer minimization selects a feasible path x from the feasible set $\mathcal{X}$, while the inner maximization represents natures adversarial choice of edge cost realizations c_e within the uncertainty set $\mathcal{U}$.

Deterministic Equivalent (4.9) Using the *BertsimasSim* budgeted uncertainty model, the inner maximization can be reformulated as a linear deterministic equivalent. Here, $\hat{c}_e$ denotes the nominal cost of edge e, d_e its maximum deviation, and Γ the uncertainty budget defining how many edge costs may simultaneously deviate to their worst case. The auxiliary variables π and ρ_e represent the global and edge-specific deviation protections, respectively.

Key Robust Constraints (4.10) This constraint ensures that each selected edges deviation d_e is adequately covered by the global uncertainty variable π or by its corresponding slack variable ρ_e. This limits the models overall conservatism while ensuring robustness against up to Γ worst-case edge deviations.

Flow Balance Constraints (4.11) Constraint (4.11) enforces flow conservation in the directed graph $G = (V, E)$. For each node $v \in V$, the difference between outgoing and incoming flows equals the node balance parameter b_v, which takes the value 1 for the source node, -1 for the sink node, and 0 for all intermediate nodes. This guarantees that exactly one continuous path connects the start and goal nodes.

Feasibility and Nonnegativity (4.124.14) The binary variable x_e indicates whether edge e is included in the selected path ($x_e = 1$) or not ($x_e = 0$). The continuous nonnegative variables π and ρ_e ensure valid uncertainty protections, maintaining feasibility and nonnegativity in the robust formulation.

2.3 Symbols and Parameters

Symbol	Description	Type / Domain
$\mathcal{X}$	Feasible set of flow-conserving paths	Set
$\mathcal{U}$	Uncertainty set for edge costs	Set
E	Set of edges in the directed network	Set
V	Set of nodes in the directed network	Set
$e \in E$	Edge index, $e = (u, v)$ connecting nodes u and v	Index
$\delta^+(v)$, $\delta^-(v)$	Sets of outgoing and incoming edges for node v	Sets
b_v	Node supply/demand balance (1, -1, or 0)	Parameter
$\hat{c}_e$	Nominal cost of edge e	Parameter
d_e	Maximum cost deviation of edge e	Parameter
Γ	Uncertainty budget controlling robustness	Parameter
x_e	Binary decision variable for edge e	Decision variable ($x_e \in \{0, 1\}$)
π	Global deviation threshold variable	Decision variable ($\pi \geq 0$)
ρ_e	Local slack variable for edge e	Decision variable ($\rho_e \geq 0$)
c_e	Realized edge cost ($c_e \in \mathcal{U}$)	Derived quantity
$\sum_{e \in E} c_e x_e$	Total path cost	Expression

Table 1: Symbols and parameters used in the robust static network-flow model.

2.4 Implementation Correspondence

The model is implemented in Python using the Gurobi optimization solver. Each binary variable x_e corresponds to an edge $e = (u, v)$ in the directed graph. Flow balance constraints (4.11) are constructed for each node $v \in V$ using the outgoing and incoming edge sets $\delta^+(v)$ and $\delta^-(v)$. The robust budget constraints (4.10) link the binary edge decisions x_e with the uncertainty control variables π and ρ_e, allowing a trade-off between nominal efficiency and robustness. The objective function (4.9) minimizes the sum of the nominal edge costs and the uncertainty surcharge term $\Gamma\pi + \sum_{e \in E} \rho_e$. This linearized deterministic equivalent enables the model to be solved efficiently as a mixed-integer linear program (MILP), providing the optimal robust path for a given uncertainty budget Γ.

3 DYNAMIC PATH REPLANNING