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Assignment Nr. 4

4.1 Radiometry

a)

As the point light source emits Φ_S Watts in every direction, its radiance is $L_o(d, \omega) = \frac{\Phi_S}{4\pi}$ in every direction. Because it is the only light source in the scene which emits light to x from the direction ω_i, the integral for the irradiance of x $I_x = \int_\omega L(x, w)cos(\theta)d\omega$ collapses to $I_x = L_o(d, \omega_i)cos(\theta)$. This would assume however that the light source sits on the unit circle around x. Therefore, we have to normalize by the distance between d and x, resulting in $I_x = \frac{L_o(d,\omega_i)cos(\theta)}{|x-d|} = \frac{L_o(d,\omega_i)cos(\theta)}{r^2+d^2} = \frac{\Phi_S cos(\theta)}{4\pi(r^2+d^2)}$

b)

$$
\begin{aligned}
\Phi_E &= \int_0^{2\pi} \int_0^{\infty} I(r)r \ dr \ d\phi \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{\Phi_S \cos(\theta)}{4\pi(r^2+d^2)} r \ dr \ d\phi \\
&= \int_0^{2\pi} \int_0^{\infty} \frac{\Phi_S \frac{d}{\sqrt{d^2+r^2}}}{4\pi(r^2+d^2)} r \ dr \ d\phi \\
&= \Phi_S \int_0^{2\pi} \int_0^{\infty} \frac{\frac{d}{\sqrt{d^2+r^2}}}{4\pi(r^2+d^2)} r \ dr \ d\phi \\
&= \Phi_S \int_0^{2\pi} \int_0^{\infty} \frac{d\sqrt{d^2+r^2}}{4\pi\left(r^2+d^2\right)^2} r \ dr \ d\phi \\
&= \Phi_S \int_0^{2\pi} \frac{d}{4\pi\sqrt{d^2}} \ d\phi \\
&= \Phi_S \int_0^{2\pi} \frac{1}{4\pi} \ d\phi \\
&= \Phi_S \frac{1}{2} \\
&= \frac{1}{2}\Phi_S \\
&= 50W
\end{aligned}
$$

So the total radiant power Φ_E received by E is half of the power of the point light source. Intuitively this could seem much, but it is actually quite reasonable, since exactly half of the light sources power is radiated up while the other half is radiated down. For an infinite plane E all of the power that is emitted downwards is received by the plane, which again is half of the light sources radiant power.

4.2 Analytical solution of the rendering equation in 2D

a)

$$
\begin{aligned}
L(x, \omega_o) &= L_e(x, \omega_o) + \int_0^{\pi} f_r(\omega, x, \omega_o) L(x, \omega) cos(\phi) d\omega \\
&= 0 + \int_0^{\pi} \frac{1}{2} L(x, \omega) cos(\phi) d\omega \\
&= \frac{1}{2} \int_0^{\pi} L(x, \omega) cos(\phi) d\omega \\
&= \frac{1}{2} \int_0^{\pi} L(x, \omega) cos(\phi) d\omega
\end{aligned}
$$

We want to find the first angle, that hits the area light and the last angle that hits the area light. Trigonometric examination shows that those angles can be calculated with

$$
\begin{aligned}
\phi_{lower} &= \frac{\pi}{2} - tan^{-1}(p+1) \\
\phi_{upper} &= \frac{\pi}{2} + tan^{-1}(1-p)
\end{aligned}
$$

This can be seen when examining that the triangle that is formed from the top left point of the lightsource (-1, 1) with (p, 0) and (-1, 0) has an angle α' at (-1,1) with $\frac{p-(-1)}{1} = tan(\alpha')$. Since a tringle has a total inner angle of π and it has a right angle at (-1, 0), the angle α at (p, 0) with $\alpha = \pi - \frac{\pi}{2} - \alpha' = \frac{\pi}{2} - tan^{-1}(p+1)$. This α is the desired angle ϕ_{lower}. Note that this even holds for $p < -1$. In a similar way ϕ_{upper} is found. Examine the triangle ((p,0), (1,0), (1,1)). The angle α_2' at (1,1) has $tan(\alpha_2') = \frac{1-p}{1}$. Therefore the angle α_2 at (p, 0) is $\alpha_2 = \pi - \frac{\pi}{2} - tan^{-1}(1-p) = \frac{\pi}{2} - tan^{-1}(1-p)$ The desired angle ϕ_{upper} is the remaining part of the half circle with $\phi_{upper} = \pi - (\frac{\pi}{2} - tan^{-1}(1-p)) = \frac{\pi}{2} + tan^{-1}(1-p)$.
$\frac{1}{2} \int_0^{\pi} L(x, \omega) cos(\phi) d\omega$ therefore can be split into

$$
= \frac{1}{2}(\int_0^{\phi_{lower}} L(x, \omega) cos(\phi) d\omega + \int_{\phi_{lower}}^{\phi_{upper}} L(x, \omega) cos(\phi) d\omega + \int_{\phi_{upper}}^{\pi} L(x, \omega) cos(\phi) d\omega)
$$

The first and the third integral refer to incident angles that point past the light source, therefore they are 0. The second integral refers to incident angles pointing to the light-source, where $L(x, \omega) = 1$

$$
\begin{aligned}
&= \frac{1}{2}\int_{\phi_{lower}}^{\phi_{upper}} 1 \cdot cos(\phi) d\omega \\
&= \frac{1}{2}\int_{\phi_{lower}}^{\phi_{upper}} cos(|\omega - \frac{\pi}{2}|) d\omega \\
&= \frac{1}{2}\left(\int_{\frac{\pi}{2}-tan^{-1}(p+1)}^{\frac{\pi}{2}} cos(\frac{\pi}{2}-\omega)d\omega + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+tan^{-1}(1-p)} cos(\omega - \frac{\pi}{2})d\omega\right) \\
&= \frac{1}{2}\left(\cos\left(\frac{\pi}{2} - \tan^{-1}(p+1)\right) - \cos\left(\tan^{-1}(-p+1) + \frac{\pi}{2}\right)\right) \\
&= \frac{1}{2}\left(\frac{1-p}{\sqrt{(1-p)^2+1}} + \frac{p+1}{\sqrt{(p+1)^2+1}}\right) \\
&= \frac{1}{2}\left(\frac{1-p}{\sqrt{p^2-2p+2}} + \frac{1+p}{\sqrt{p^2+2p+2}}\right) \\
&= L((p,0), w_o)
\end{aligned}
$$

b)

$$
\begin{aligned}
L(x,\omega_o) &= L_e(x,\omega_o) + \int_{y\in S} f_r(\omega_i, x, \omega_o)\cdot L(y, -\omega_i(x,y)) \cdot \frac{\cos\phi_i \cos\phi_y}{|x-y|}\cdot dy_x \\
&= L_e(x,\omega_o) + \int_{-1}^{1} f_r(\omega_i, x, \omega_o)\cdot L(y, -\omega_i(x,y)) \cdot \frac{\cos\phi_i \cos\phi_y}{|x-y|}\cdot dy_x \\
&\text{If we are not in a light source this is:} \\
&= \int_{-1}^{1} f_r(\omega_i, x, \omega_o)\cdot L(y, -\omega_i(x,y)) \cdot \frac{\cos\phi_i \cos\phi_y}{|x-y|}\cdot dy_x \\
&= \int_{-1}^{1} f_r(\omega_i, x, \omega_o)\cdot L(y, -\omega_i(x,y)) \cdot \frac{\cos\phi_i \cos\phi_y}{\sqrt{(x_x-y_x)^2+(x_y-y_y)^2}}\cdot dy_x
\end{aligned}
$$

The only material in the scene is the Lambertian meterial at $y = 0$ with $f_r(\omega_i, (p,0), w_o) = \frac{1}{2}$. The light source has uniforma radiance of 1 for each point and direction. So:

$$
= \int_{-1}^{1} \frac{1}{2}\cdot 1 \cdot \frac{\cos\phi_i \cos\phi_y}{\sqrt{(x_x-y_x)^2+(x_y-y_y)^2}}\cdot dy_x
$$

From inspection, for this specific case for the angles ϕ_i and ϕ_y it holds $\phi_i = \phi_y$ (Since the area light and the ground line are parallel). Furthermore it holds: $tan(\phi_i) = \frac{|y_x - x_x|}{1} = |y_x - x_x|$. Therefore:

$$
= \frac{1}{2}\int_{-1}^{1} \frac{cos(\tan^{-1}(|y_x - x_x|))^2}{\sqrt{(x_x-y_x)^2+(x_y-y_y)^2}} dy_x
$$

With $cos(tan^{-1}(x) = \frac{1}{\sqrt{1+x^2}})$ we get:

$$
\begin{aligned}
&= \frac{1}{2}\int_{-1}^{1} \frac{\left(\frac{1}{\sqrt{1+(|y_x-x_x|)^2}}\right)^2}{\sqrt{(x_x-y_x)^2+(x_y-y_y)^2}} dy_x \\
&= \frac{1}{2}\int_{-1}^{1} \frac{\left(\frac{1}{\sqrt{1+(y_x-x_x)^2}}\right)^2}{\sqrt{(x_x-y_x)^2+(x_y-y_y)^2}} dy_x \\
&= \frac{1}{2}\int_{-1}^{1} \frac{\frac{1}{1+(y_x-x_x)^2}}{\sqrt{(x_x-y_x)^2+(x_y-y_y)^2}} dy_x \\
&= \frac{1}{2}\int_{-1}^{1} \frac{1}{(\sqrt{(x_x-y_x)^2+(x_y-y_y)^2})\cdot(1+(y_x-x_x)^2)} dy_x
\end{aligned}
$$

by replacing $x_x := p$, $x_y = 0, y_y = 1$ and $y_y := t$ wet get

$$
\begin{aligned}
&= \frac{1}{2}\int_{-1}^{1} \frac{1}{(\sqrt{(p-y_x)^2+(0-1)^2})\cdot(1+(y_x-p)^2)} dy_x \\
&= \frac{1}{2}\int_{-1}^{1} \frac{1}{(\sqrt{(p-y_x)^2+1})\cdot(1+(y_x-p)^2)} dy_x \\
&= \frac{1}{2}\int_{-1}^{1} \frac{1}{(\sqrt{(p-t)^2+1})\cdot(1+(t-p)^2)} dt
\end{aligned}
$$

Automatic integration shows that

$$
\int \frac{1}{(\sqrt{(p-t)^2+1})\cdot(1+(t-p)^2)} dt = \frac{t-p}{\sqrt{p^2-2pt+t^2+1}} + C
$$

so

$$
\begin{aligned}
\frac{1}{2}\int_{-1}^{1} \frac{1}{(\sqrt{(p-t)^2+1})\cdot(1+(t-p)^2)} dt &= \frac{1}{2}\left[\frac{t-p}{\sqrt{p^2-2pt+t^2+1}}\right]_{-1}^{1} \\
&= \frac{1}{2}\left(\frac{1-p}{\sqrt{p^2-2p+1^2+1}} - \frac{(-1)-p}{\sqrt{p^2-2p(-1)+(-1)^2+1}}\right) \\
&= \frac{1}{2}\left(\frac{1-p}{\sqrt{p^2-2p+1^2+1}} + \frac{1+p}{\sqrt{p^2-2p(-1)+(-1)^2+1}}\right) \\
&= \frac{1}{2}\left(\frac{1-p}{\sqrt{p^2-2p+2}} + \frac{1+p}{\sqrt{p^2+2p+2}}\right) \\
&= L((p,0),w_o)
\end{aligned}
$$

4.3 Simple Path Tracer