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Assignment Nr. 1

1 Bayes Optimal Function

$$
\begin{aligned}
&\frac{\partial}{\partial c(x)} \int_y (log(c(x)) + \frac{y}{c(x)}) p(y|x) dy \\
&= \int_y (\frac{\partial}{\partial c(x)} log(c(x)) + \frac{\partial}{\partial c(x)} \frac{y}{c(x)}) p(y|x) dy \\
&= \int_y (\frac{1}{c(x)} + y \cdot (-\frac{1}{c(x)^2})) p(y|x) dy \\
&= \frac{1}{c(x)} \int_y (1 - y \cdot \frac{1}{c(x)}) p(y|x) dy \\
&= \frac{1}{c(x)} (\int_y p(y|x) dy - \int_y y \cdot \frac{1}{c(x)} p(y|x) dy) \\
&= \frac{1}{c(x)} (1 - \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x) dy)
\end{aligned}
$$

Now set to 0:

$$
\begin{aligned}
0 &= \frac{1}{c(x)} (1 - \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x) dy) \\
&\text{because } \frac{1}{c(x)} > 0, x \in \mathbb{R}: \\
\Rightarrow 0 &= (1 - \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x) dy) \\
1 &= \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x) dy) \\
c(x) &= \int_y y \cdot p(y|x) dy) = E[Y|X]
\end{aligned}
$$

2 Bayes Optimal Function

(a)

$$
\begin{aligned}
E[L(Y)|X] &= max(0, 1 - f(x)) P(Y = -1|X) + (1 + kf(x)) P(Y = -1|X) \\
&+ max(0, 1 - f(x)) P(Y = 1|X) + (1 + kf(x)) P(Y = 1|X) \\
&= (max(0, 1 - f(x)) + (1 + kf(x))) \cdot (P(Y = -1|X) + P(Y = 1|X)) \\
&= (max(0, 1 - f(x)) + (1 + kf(x))) \cdot 1
\end{aligned}
$$

We see that this expression gets maximized for $f(x) = 0$, as $k \geq 1$.

(b)

As we have seen in (a), the best function always predicts 0. As 0 is undefined, if we define it one class or another, it will always predict that class and is therefore not classification calibrated. Another way to see this is that the derivative at 0 doesn't exits, as the right side derivative is 1 where the left side derivative is k and k¿1.

3 Bayes error

(a)

$$\begin{aligned}
R^* &= min(E_X[\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)] \\
&= min(\int_x [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)]) \\
&= min(\int_{x\in[0,\frac{1}{8}]} [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)] \\
&+ \int_{x\in[\frac{1}{8},\frac{7}{8}]} [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)] \\
&+ \int_{x\in[\frac{7}{8},1]} [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)])
\end{aligned}$$

all three terms can be minimized on their own, and we get the optimal classifier as:

$$f(x) = \begin{cases} -1 \text{ if } 0 \leq x \leq \frac{1}{8} \\ 1 \text{ if } \frac{1}{8} < x \leq \frac{7}{8} \\ -1 \text{ if } \frac{7}{8} < x \leq 1 \end{cases}$$

This results in the error: $R^* = \frac{1}{8}\cdot 0.1 + \frac{6}{8}\cdot 0.1 + \frac{1}{8}\cdot 0.1 = 0.1$

(b)

If we set $b = -\frac{1}{8}$, we see that we get the best (meaning same as in (a)) predictions for $X \in [0, \frac{7}{8}]$. As we can only linearly seperate the space X with the given classifier, we can not improve the class separation and the Loss upon that. As we have 0-1 Loss, we can now choose $w \in \mathbb{R}_{\geq 0}$ arbitrary.