Statistical Machine Learning Exercise Sheet 9

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Exercise 16

(a)

First we compute the derivative of the loss with respect to a and c.

$$
\begin{aligned}
\frac{\partial}{\partial c}L(f) &= \frac{\partial}{\partial c}\sum_{i=1}^{n}\gamma_i(Y_i - f(X_i))^2 \\
&= \sum_{i=1}^{n}\gamma_i\frac{\partial}{\partial c}(Y_i - f(X_i))^2 \\
&= \sum_{i=1}^{n}\gamma_i 2(Y_i - f(X_i))\frac{\partial}{\partial c}(Y_i - f(X_i)) \\
&= \sum_{i=1}^{n}-\gamma_i 2(Y_i - f(X_i))
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial a}L(f) &= \frac{\partial}{\partial a}\sum_{i=1}^{n}\gamma_i(Y_i - f(X_i))^2 \\
&= \sum_{i=1}^{n}\gamma_i\frac{\partial}{\partial a}(Y_i - f(X_i))^2 \\
&= \sum_{i=1}^{n}\gamma_i 2(Y_i - f(X_i))\frac{\partial}{\partial a}(Y_i - f(X_i)) \\
&= \sum_{i=1}^{n}-\gamma_i 2(Y_i - f(X_i))1_{\langle w,X_i\rangle+b>0}
\end{aligned}
$$

Now we can set the gradient to 0 and solve both equations for a:

$$
\begin{aligned}
0 &= \sum_{i=1}^{n}\gamma_i 2(Y_i - f(X_i))1_{\langle w,X_i\rangle+b>0} \\
\Leftrightarrow a &= \frac{\sum_{i=1}^{n}\gamma_i(Y_i - c)}{\sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}}
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \sum_{i=1}^{n}-\gamma_i 2(Y_i - f(X_i))1_{\langle w,X_i\rangle+b>0} \\
\Leftrightarrow a &= \frac{\sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}(Y_i - c)}{\sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}}
\end{aligned}
$$

Now set a equal and solve for c:

$$
\begin{aligned}
\frac{\sum_{i=1}^{n}\gamma_i(Y_i + c)}{\sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}} &= \frac{\sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}(Y_i + c)}{\sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}} \\
\Longrightarrow \sum_{i=1}^{n}\gamma_i(Y_i + c) &= \sum_{i=1}^{n}\gamma_i 1_{\langle w,X_i\rangle+b>0}(Y_i + c) \\
\Longrightarrow c &= \frac{\sum_{i=1}^{n}\gamma_i Y_i(1 - 1_{\langle w,X_i\rangle+b>0})}{\sum_{i=1}^{n}\gamma_i(1 - 1_{\langle w,X_i\rangle+b>0})}
\end{aligned}
$$

(b)

In [1]:

```
import numpy as np
from tqdm import tqdm

def FitStump(X, Y, w, gamma):
    # first sort everything so we can use cumsum, and set b values
    # Note: we would have to add small epsilon to b values to make prediction >0, but we can also just use >
    # then we have to make no assumption how far datapoints lie away from each other
    prediction = X @ w
    sorting = np.flip(np.argsort(prediction))
    b = - prediction[sorting]
    b = np.append(b[0]+ np.sign(b[0]) * 1, b)
    gamma_sorted = gamma[sorting]
    Y_sorted = Y[sorting]

    # now compute a and c for all values of b
    c_upper =  np.sum(gamma_sorted * Y_sorted) - np.append(0, np.cumsum(gamma_sorted * Y_sorted))
    c_lower = np.sum(gamma_sorted) - np.append(0, np.cumsum(gamma_sorted))
    c = c_upper / np.maximum(c_lower, 1e-5)
    a = np.sum(gamma * (Y - c.T), axis=0) / np.maximum(np.append(0, np.cumsum(gamma_sorted)), 1e-5)

    # compute the resulting errors and take the smallest one
    errors = np.linalg.norm(gamma_sorted * (Y_sorted - (np.add.outer(prediction[sorting] , b.T) >= 0) * a.T
    min_index = np.argmin(errors)
    return a[min_index], b[min_index], c[min_index], errors[min_index]
```

(c)

In [2]:

```
def GentleBoost(X, Y, k):
    dim = X.shape[1]
    # init gamma uniformly, prediction with zeros
    gamma = np.ones((X.shape[0], 1))/X.shape[0]
    predictions = np.zeros_like(Y, dtype=float)
    a_s = np.zeros(k)
    b_s = np.zeros(k)
    c_s = np.zeros(k)
    w_s = np.zeros((dim, k))
    training_errors = []
    for episode in tqdm(range(k)):
        # choose random w and calculate a,b,c
        w = np.random.randn(dim)
        w = w/np.linalg.norm(w)
        a,b,c,error = FitStump(X, Y, w, gamma)
        a_s[episode] = a
        b_s[episode] = b
        c_s[episode] = c
        w_s[:, episode] = w
        current_prediction =  a *(((X @ w) + b) >= 0) + c
        # recalculate gamma for next episode
        gamma = gamma * np.expand_dims(np.exp(- np.squeeze(Y) * current_prediction), axis=1)
        gamma = gamma/np.sum(gamma)
        # calculate combined prediction and print their error
        predictions += np.expand_dims(current_prediction, axis=1)
        current_training_error = np.linalg.norm(gamma * (Y - predictions/(episode+1)))
        training_errors.append(current_training_error)
        if (episode % 100) ==0:
            print(f"The training error is:{current_training_error}")

    return w_s, a_s, b_s, c_s, training_errors
```

(d)

In [3]:

```
data = np.load("USPS_data.npy", allow_pickle=True).item()
X_train = data["Xtrain"]
Y_train = data["Ytrain"]
X_train = X_train[np.where(Y_train < 2)[0], :]
Y_train = Y_train[np.where(Y_train < 2)[0]]
X_test = data["Xtest"]
Y_test = data["Ytest"]
X_test = X_test[np.where(Y_train < 2)[0], :]
Y_test = Y_test[np.where(Y_train < 2)[0]]
```

In [4]:

```
w, a, b, c, training_errors = GentleBoost(X_train, Y_train, 1000)
```

```
  1%|          | 7/1000 [00:00<00:34, 28.87it/s]
The training error is:0.030399633413629733
 10%|█         | 105/1000 [00:03<00:31, 28.67it/s]
The training error is:0.00030098646944210297
 21%|██        | 207/1000 [00:06<00:17, 45.53it/s]
The training error is:0.00015124195728188623
 31%|███       | 307/1000 [00:08<00:15, 44.47it/s]
The training error is:0.00010099545984604252
 41%|████      | 407/1000 [00:10<00:13, 42.81it/s]
The training error is:7.580955963505505e-05
 50%|█████     | 505/1000 [00:13<00:11, 43.63it/s]
The training error is:6.0677911005302833e-05
 60%|██████    | 605/1000 [00:15<00:09, 43.63it/s]
The training error is:5.058175276814892e-05
 71%|███████   | 710/1000 [00:18<00:06, 43.17it/s]
The training error is:4.336609616784288e-05
 81%|████████  | 807/1000 [00:20<00:04, 41.89it/s]
The training error is:3.795210164002435e-05
 91%|█████████ | 908/1000 [00:22<00:02, 45.59it/s]
The training error is:3.373988170217523e-05
100%|██████████| 1000/1000 [00:25<00:00, 39.69it/s]
```

In [5]:

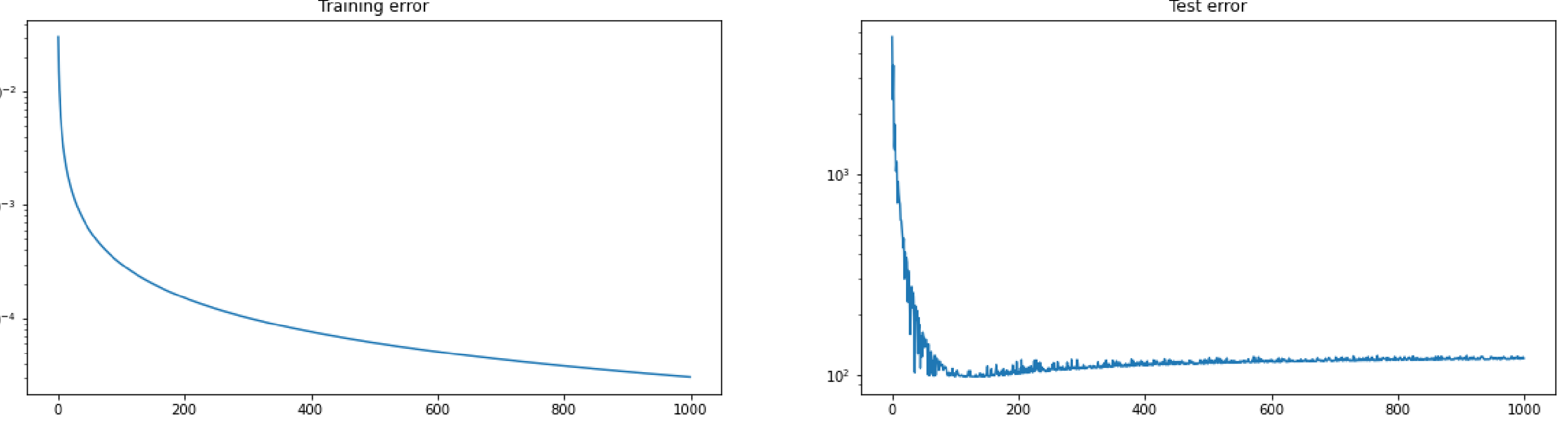
```
def compute_test_errors(X, Y, w, a, b, c):
    predictions = a *(((X @ w) + b) >= 0) + c
    predictions = np.cumsum(predictions, axis=0)/np.arange(1, predictions.shape[1] + 1, 1).T
    test_errors = np.linalg.norm(Y - predictions, axis=0)
    return test_errors
```

In [6]:

```
test_errors = compute_test_errors(X_test, Y_test, w, a, b, c)
```

In [7]:

```
from matplotlib import pyplot as plt
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(20,5))
ax1.plot(training_errors)
ax1.set_title("Training error")
ax1.set_yscale("log")
ax2.plot(test_errors)
ax2.set_yscale("log")
ax2.set_title("Test error");
```



We can see that the test error only increases moderatly again which means the generalization error only increases moderatly while still improving the training error.

(e)

In [8]:

```
data = np.load("USPS_data.npy", allow_pickle=True).item()
prediction_class = 1
X_train = data["Xtrain"]
Y_train = data["Ytrain"]
Y_train[np.where(Y_train != prediction_class)[0]] = 0
X_test = data["Xtest"]
Y_test = data["Ytest"]
Y_test[np.where(Y_test != prediction_class)[0]] = 0
```

In [9]:

```
w, a, b, c, training_errors = GentleBoost(X_train, Y_train, 1000)
```

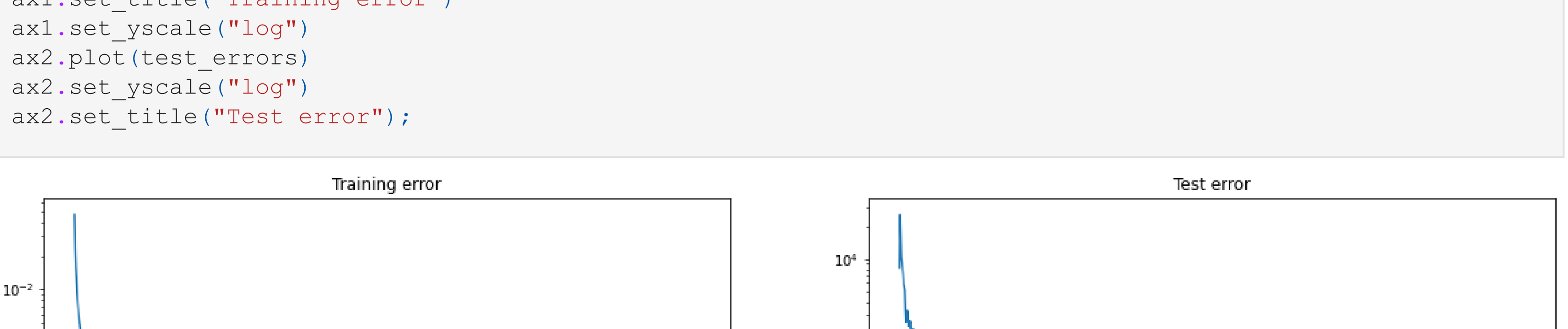
```
  0%|          | 2/1000 [00:00<03:02,  5.47it/s]
The training error is:0.0464860117107709
 10%|█         | 102/1000 [00:16<02:17,  6.54it/s]
The training error is:0.00046025754169097994
 20%|██        | 202/1000 [00:32<02:08,  6.23it/s]
The training error is:0.0002312736901034793
 30%|███       | 302/1000 [00:47<01:46,  6.55it/s]
The training error is:0.00015443857711228934
 40%|████      | 402/1000 [01:06<02:02,  4.89it/s]
The training error is:0.00011592521623640656
 50%|█████     | 502/1000 [01:26<01:47,  4.64it/s]
The training error is:9.278645052055533e-05
 60%|██████    | 601/1000 [01:48<01:19,  5.03it/s]
The training error is:7.734777322928993e-05
 70%|███████   | 702/1000 [02:09<01:00,  4.89it/s]
The training error is:6.631385408103062e-05
 80%|████████  | 802/1000 [02:33<00:45,  4.35it/s]
The training error is:5.803497092485048e-05
 90%|█████████ | 901/1000 [03:00<00:23,  4.17it/s]
The training error is:5.159379768124634e-05
100%|██████████| 1000/1000 [03:20<00:00,  4.98it/s]
```

In [10]:

```
test_errors = compute_test_errors(X_test, Y_test, w, a, b, c)
```

In [12]:

```
from matplotlib import pyplot as plt
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(20,5))
ax1.plot(training_errors)
ax1.set_title("Training error")
ax1.set_yscale("log")
ax2.plot(test_errors)
ax2.set_yscale("log")
ax2.set_title("Test error");
```



Exercise 17

(a)

We can rewrite $P(X_{n+1} < X_1) = P(X_1 - X_{n+1} > 0)$ and we can write $X_1 - X_{n+1} = A\begin{pmatrix} X_1 \\ X_{n+1}\end{pmatrix} = \begin{pmatrix} 1 & -1\end{pmatrix}\begin{pmatrix} X_1 \\ X_{n+1}\end{pmatrix}$.

We know that $\begin{pmatrix} X_1 \\ X_{n+1}\end{pmatrix} \sim \mathcal{N}(\begin{pmatrix} 0 \\ \mu\end{pmatrix}), \sigma^2 I) \implies A\begin{pmatrix} X_1 \\ X_{n+1}\end{pmatrix} = X_1 - X_{n+1} \sim N(\mu, 2\sigma^2)$

Now it follows that

$$
\begin{aligned}
P(X_1 - X_{n+1} > 0) &= \int_0^\infty \mathcal{N}(x, \mu, 2\sigma^2)dx && (1) \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi 2\sigma^2}}exp(-\frac{(x-\mu)^2}{22\sigma^2}) && (2) \\
&= \int_0^\infty \frac{1}{2\sqrt{\pi}\sigma}exp(-\frac{(x-\mu)^2}{4\sigma^2}) && (3) \\
&= \int_0^\infty \frac{1}{2\sqrt{\pi}\sigma}exp(-\frac{(x-\mu)^2}{(2\sigma)^2}) && (4) \\
&= \frac{1}{2\sqrt{\pi}\sigma}\int_0^\infty exp(-(\frac{(x-\mu)}{(2\sigma)})^2) && (5)
\end{aligned}
$$

(b)

$$
\begin{aligned}
P(\max_i X_i < X_{n+1}) &= P(\bigcap_i X_i < X_{n+1}) && (6) \\
&= 1 - P(\overline{\bigcap_i X_i < X_{n+1}}) && (7) \\
&= 1 - P(\bigcup_i X_{n+1} > X_i) && (8) \\
&\geq 1 - \sum_i P(X_{n+1} > X_i) && (9) \\
&= 1 - nP(X_{n+1} \leq X_1) && (10)
\end{aligned}
$$

Where the last step follows from the X_i beeing iid. Technically, the couter-event of $X_i < X_n$ is $X_n \leq X_i$ but since $P(X_i = X_n)$ is a single point and thus has probability 0 in an continuous space, we can drop the equal.