exercise_04_LauraHaegePhilippVonBachmann

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1 Statistical Machine Learning Exercise Sheet 4

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1.1 Exercise 7

1.1.1 (a)

1. Suppose there exist y_1, y_2 such that $P_C(x) = \{y_1, y_2\}$ with $\|x - y_1\|_2^2 = \|x - y_2\|_2^2 = a$. Now construct $y* = p \cdot y_1 + (1-p) \cdot y_2 \in P_C(x)$

$$
\begin{aligned}
&\|x - p \cdot y_1 + (1-p) \cdot y_2\|_2^2 \\
&= \|p \cdot x + (1-p) \cdot x - (p \cdot y_1 + (1-p) \cdot y_2)\|_2^2 \\
&= \|p \cdot x - p \cdot y_1 + (1-p) \cdot x - (1-p) \cdot y_2\|_2^2 \\
&\leq \|p \cdot x - p \cdot y_1\|_2^2 + \|(1-p) \cdot x - (1-p) \cdot y_2\|_2^2 \\
&= p^2 \cdot \|x - y_1\|_2^2 + (1-p)^2 \cdot \|x - y_2\|_2^2 \\
&< p \cdot \|x - y_1\|_2^2 + (1-p) \cdot \|x - y_2\|_2^2 \\
&= p \cdot a + (1-p) \cdot a \\
&= a
\end{aligned}
$$

   Therefore the distance to $y*$ is smaller than to y_1, y_2 and they can not be the minimum. Therefore the minimum has to be unique

2. As we measure the distance in L_2, the distance within one component doesn't affect the distance within the other, and we can optimize each component seperatly. If $x_i < 0$, then the closest point in C is given by 0. Otherwise, we can just stay at x_i, which is the trivial minimal distance. This leads to $P_C(X)_i = max(x_i, 0)$. In total we get $P_C(X) = max(x, 0)$, where max is the element wise max.

2 (b)

1.

[1]:
```
from sys import prefix
import numpy as np
import matplotlib.pyplot as plt
```

```python
def LassoObjective(wplus, wminus, Phi, Y, lmbd):
    ''' evaluates the objective function at (wplus, wminus)
    L2 loss + L1 regularization
    '''
    w = wplus - wminus
    return ((Phi @ w - Y) ** 2).mean(
        ) + lmbd * np.abs(w).sum()


def GradLassoObjective(wplus, wminus, Phi, Y, lmbd):
    ''' computes the gradients of the objective function
    at (wplus, wminus)
    gradwplus: gradient wrt wplus
    gradwminus: gradient wrt minus

    FILL IN
    '''
    prediction_loss = 2* (Y - Phi @ wplus + Phi @ wminus)
    gradwplus =  np.expand_dims(- np.sum(np.multiply(prediction_loss, Phi),␣
 ↪axis=0) / Phi.shape[0] + lmbd, axis=1)
    gradwminus =   np.expand_dims(np.sum(np.multiply(prediction_loss, Phi),␣
 ↪axis=0) / Phi.shape[0] + lmbd, axis=1)
    return gradwplus, gradwminus


def ProjectionPositiveOrthant(x):
    ''' returns the projection of x onto the positive orthant

    FILL IN
    '''
    return np.maximum(0, x)


def getStepSize(wplus, wminus, Phi, Y, lmbd, gradwplus,
                gradwminus, loss):
    ''' performs one step of projected gradient descent (i.e.
    compute next iterate) with step size selection via
    backtracking line search

    input
    loss: objective function at current iterate (wplus, wminus)

    output
    wplusnew, wminusnew: next iterates wplus_{t+1}, wminus_{t+1}
    lossnew: objective function at the new iterate
```

```python
    FILL IN
    '''
    alpha, beta, sigma = 1., .1, .1
    wplusnew, wminusnew = wplus.copy(), wminus.copy()
    lossnew = np.float('Inf') # make sure to enter the loop

    # choose the step size alpha with backtracking line search
    while lossnew > loss + sigma * ((gradwplus * (
        wplusnew - wplus)).sum() + (gradwminus * (
        wminusnew - wminus)).sum()):
        # get new step size to test
        alpha *= beta

        # projected gradient step for wplus and wminus with step size alpha
        # i.e. compute x_{t+1} as in the text
        # FILL IN
        # print('fill in with projected gradient step')
        wplusnew = ProjectionPositiveOrthant(wplus - alpha * gradwplus)
        wminusnew = ProjectionPositiveOrthant(wminus - alpha * gradwminus)

        # compute new value of the objective
        lossnew = LassoObjective(wplusnew, wminusnew, Phi, Y, lmbd)

    return wplusnew, wminusnew, lossnew


def Lasso(Phi, Y, lmbd):
    ''' compute weight of linear regression with Lasso

    Phi: deisgn matrix n x d
    Y: true values n x 1
    lmbd: weight of regularization

    output: weights of linear regression d x 1
    '''
    # initialize wplus, wminus
    wplus = np.random.rand(Phi.shape[1], 1)
    wminus = np.random.rand(*wplus.shape)
    loss = LassoObjective(wplus, wminus, Phi, Y, lmbd)

    counter = 1
    while counter > 0:
        # compute gradients wrt wplus and wminus
        gradwplus, gradwminus = GradLassoObjective(
            wplus, wminus, Phi, Y, lmbd)

        # compute new iterates
```

```
        wplus, wminus, loss = getStepSize(wplus,
            wminus, Phi, Y, lmbd, gradwplus, gradwminus, loss)

        if (counter % 100) == 0:
            # check if stopping criterion is met
            wnew = wplus - wminus
            ind = wnew != 0.
            indz = wnew == 0.
            r = 2 / Phi.shape[0] * (Phi.T @ (Phi @ wnew - Y))
            stop = np.abs(r[ind] + lmbd * np.sign(wnew[ind]
                )).sum() + (np.abs(r[indz]) - lmbd * np.ones_like(
                r[indz])).clip(0.).sum()
            print('iter={} current objective={:.3f} nonzero weights={}'.format(
                counter, loss, ind.sum()) +\
                ' stop={:.5f}'.format(stop / Phi.shape[0]))
            if np.abs(stop) / Phi.shape[0] < 1e-5:
                break
        counter += 1

    return wplus - wminus
```

2.

[2]:
```
# load data
data = np.load("multidim_data_trainval.npy", allow_pickle=True).item()
x_train = data["Xtrain"]
y_train = data["Ytrain"]
x_val = data["Xval"]
y_val = data["Yval"]

# normalize
x_train -= np.mean(x_train, axis=0)
x_train /= np.std(x_train, axis=0)
x_val -= np.mean(x_val, axis=0)
x_val /= np.std(x_val, axis=0)


# add one vector for offset
x_train = np.hstack((np.ones((x_train.shape[0], 1)), x_train))
x_val = np.hstack((np.ones((x_val.shape[0], 1)), x_val))
```

[3]:
```
lmbd = 10

w = Lasso(x_train, y_train, lmbd=lmbd)
y_train_predict = x_train @ w
y_val_predict = x_val @ w
train_loss = np.linalg.norm(y_train - y_train_predict)**2/x_train.shape[0]
```

```
test_loss = np.linalg.norm(y_val - y_val_predict)**2/x_val.shape[0]
```

```
iter=100 current objective=139277.680 nonzero weights=63 stop=0.11879
iter=200 current objective=138925.125 nonzero weights=45 stop=0.03480
iter=300 current objective=138832.709 nonzero weights=44 stop=0.00569
iter=400 current objective=138786.650 nonzero weights=43 stop=0.00400
iter=500 current objective=138758.119 nonzero weights=43 stop=0.00305
iter=600 current objective=138738.583 nonzero weights=42 stop=0.00240
iter=700 current objective=138724.429 nonzero weights=42 stop=0.00194
iter=800 current objective=138716.244 nonzero weights=41 stop=0.00125
iter=900 current objective=138714.441 nonzero weights=41 stop=0.00078
iter=1000 current objective=138713.814 nonzero weights=40 stop=0.00048
iter=1100 current objective=138713.589 nonzero weights=40 stop=0.00030
iter=1200 current objective=138713.506 nonzero weights=40 stop=0.00018
iter=1300 current objective=138713.476 nonzero weights=40 stop=0.00011
iter=1400 current objective=138713.464 nonzero weights=40 stop=0.00007
iter=1500 current objective=138713.460 nonzero weights=40 stop=0.00004
iter=1600 current objective=138713.458 nonzero weights=40 stop=0.00003
iter=1700 current objective=138713.458 nonzero weights=40 stop=0.00002
iter=1800 current objective=138713.457 nonzero weights=40 stop=0.00001
iter=1900 current objective=138713.457 nonzero weights=40 stop=0.00001
```

i.

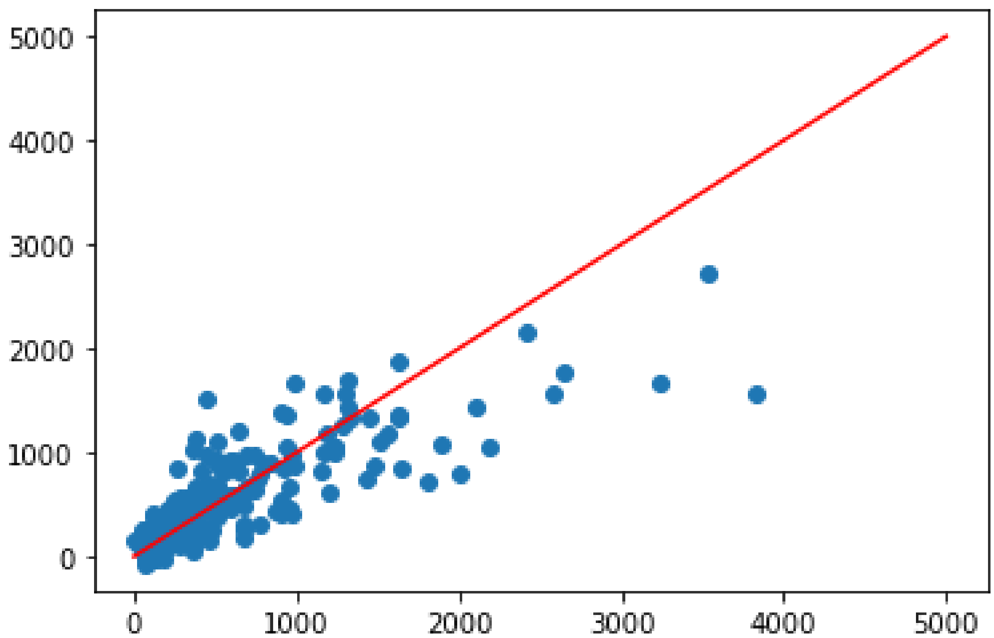
[4]:
```
print(f"The training loss is {train_loss} and the test loss is {test_loss}")
```

```
The training loss is 121332.09617441503 and the test loss is 141717.19168760636
```

ii.

[5]:
```
perfect_fit = np.linspace(0, 5000)
plt.scatter(y_val, y_val_predict)
plt.plot(perfect_fit, perfect_fit, color="red")
```

[5]:
```
[<matplotlib.lines.Line2D at 0x1d693d40708>]
```



We can see that for small values, the predictions are very good. However for larger values, the predictions get more inaccurate

[6]:
```
def basis(X):
    return np.hstack([X, np.sin(np.pi * X), np.cos(np.pi * X)])
```

[7]:
```
def train(X, Y):
    X = basis(X)
    w = Lasso(X, Y, lmbd=5)
    return w
```

[8]:
```
w_predict = train(x_train, y_train)
```

```
iter=100 current objective=121815.273 nonzero weights=229 stop=0.28289
iter=200 current objective=120535.502 nonzero weights=191 stop=0.20205
iter=300 current objective=120065.688 nonzero weights=195 stop=0.12329
iter=400 current objective=119775.663 nonzero weights=184 stop=0.13487
iter=500 current objective=119604.110 nonzero weights=185 stop=0.06124
iter=600 current objective=119506.657 nonzero weights=180 stop=0.04667
iter=700 current objective=119453.600 nonzero weights=180 stop=0.02954
iter=800 current objective=119413.800 nonzero weights=177 stop=0.04661
iter=900 current objective=119381.764 nonzero weights=179 stop=0.03385
iter=1000 current objective=119354.426 nonzero weights=179 stop=0.03198
iter=1100 current objective=119330.297 nonzero weights=179 stop=0.02725
iter=1200 current objective=119308.427 nonzero weights=179 stop=0.02730
iter=1300 current objective=119288.177 nonzero weights=176 stop=0.03379
```

```
iter=1400 current objective=119269.033 nonzero weights=177 stop=0.01911
iter=1500 current objective=119250.651 nonzero weights=177 stop=0.03064
iter=1600 current objective=119232.918 nonzero weights=179 stop=0.02207
iter=1700 current objective=119215.704 nonzero weights=177 stop=0.01735
iter=1800 current objective=119198.842 nonzero weights=178 stop=0.03668
iter=1900 current objective=119182.199 nonzero weights=176 stop=0.03053
iter=2000 current objective=119165.837 nonzero weights=177 stop=0.02698
iter=2100 current objective=119149.694 nonzero weights=180 stop=0.02393
iter=2200 current objective=119137.323 nonzero weights=176 stop=0.01101
iter=2300 current objective=119130.836 nonzero weights=176 stop=0.01331
iter=2400 current objective=119124.776 nonzero weights=176 stop=0.01838
iter=2500 current objective=119118.961 nonzero weights=177 stop=0.01049
iter=2600 current objective=119113.316 nonzero weights=175 stop=0.01290
iter=2700 current objective=119107.859 nonzero weights=175 stop=0.01615
iter=2800 current objective=119102.526 nonzero weights=175 stop=0.01995
iter=2900 current objective=119097.244 nonzero weights=175 stop=0.00973
iter=3000 current objective=119092.038 nonzero weights=176 stop=0.01127
iter=3100 current objective=119086.903 nonzero weights=175 stop=0.01256
iter=3200 current objective=119081.836 nonzero weights=178 stop=0.01422
iter=3300 current objective=119076.828 nonzero weights=174 stop=0.01559
iter=3400 current objective=119071.889 nonzero weights=174 stop=0.01704
iter=3500 current objective=119067.196 nonzero weights=173 stop=0.01763
iter=3600 current objective=119062.598 nonzero weights=174 stop=0.00761
iter=3700 current objective=119058.036 nonzero weights=174 stop=0.00804
iter=3800 current objective=119053.512 nonzero weights=174 stop=0.00838
iter=3900 current objective=119049.018 nonzero weights=174 stop=0.00883
iter=4000 current objective=119044.552 nonzero weights=174 stop=0.00928
iter=4100 current objective=119040.110 nonzero weights=174 stop=0.00975
iter=4200 current objective=119035.691 nonzero weights=174 stop=0.01032
iter=4300 current objective=119032.790 nonzero weights=172 stop=0.00415
iter=4400 current objective=119032.324 nonzero weights=172 stop=0.00458
iter=4500 current objective=119032.011 nonzero weights=172 stop=0.00219
iter=4600 current objective=119031.769 nonzero weights=172 stop=0.00227
iter=4700 current objective=119031.567 nonzero weights=172 stop=0.00245
iter=4800 current objective=119031.392 nonzero weights=171 stop=0.00251
iter=4900 current objective=119031.235 nonzero weights=171 stop=0.00263
iter=5000 current objective=119031.094 nonzero weights=171 stop=0.00276
iter=5100 current objective=119030.964 nonzero weights=171 stop=0.00123
iter=5200 current objective=119030.845 nonzero weights=171 stop=0.00128
iter=5300 current objective=119030.735 nonzero weights=171 stop=0.00134
iter=5400 current objective=119030.633 nonzero weights=170 stop=0.00140
iter=5500 current objective=119030.538 nonzero weights=170 stop=0.00147
iter=5600 current objective=119030.451 nonzero weights=170 stop=0.00154
iter=5700 current objective=119030.369 nonzero weights=170 stop=0.00162
iter=5800 current objective=119030.294 nonzero weights=170 stop=0.00170
iter=5900 current objective=119030.224 nonzero weights=170 stop=0.00180
iter=6000 current objective=119030.159 nonzero weights=170 stop=0.00189
iter=6100 current objective=119030.099 nonzero weights=170 stop=0.00083
```

```
iter=6200 current objective=119030.043 nonzero weights=170 stop=0.00088
iter=6300 current objective=119029.990 nonzero weights=170 stop=0.00091
iter=6400 current objective=119029.942 nonzero weights=170 stop=0.00096
iter=6500 current objective=119029.919 nonzero weights=169 stop=0.00069
iter=6600 current objective=119029.908 nonzero weights=169 stop=0.00043
iter=6700 current objective=119029.900 nonzero weights=169 stop=0.00054
iter=6800 current objective=119029.895 nonzero weights=169 stop=0.00032
iter=6900 current objective=119029.891 nonzero weights=169 stop=0.00020
iter=7000 current objective=119029.889 nonzero weights=169 stop=0.00028
iter=7100 current objective=119029.887 nonzero weights=169 stop=0.00017
iter=7200 current objective=119029.886 nonzero weights=169 stop=0.00023
iter=7300 current objective=119029.885 nonzero weights=169 stop=0.00014
iter=7400 current objective=119029.885 nonzero weights=169 stop=0.00009
iter=7500 current objective=119029.884 nonzero weights=169 stop=0.00012
iter=7600 current objective=119029.884 nonzero weights=169 stop=0.00007
iter=7700 current objective=119029.884 nonzero weights=169 stop=0.00010
iter=7800 current objective=119029.884 nonzero weights=169 stop=0.00006
iter=7900 current objective=119029.883 nonzero weights=169 stop=0.00008
iter=8000 current objective=119029.883 nonzero weights=169 stop=0.00005
iter=8100 current objective=119029.883 nonzero weights=169 stop=0.00003
iter=8200 current objective=119029.883 nonzero weights=169 stop=0.00004
iter=8300 current objective=119029.883 nonzero weights=169 stop=0.00003
iter=8400 current objective=119029.883 nonzero weights=169 stop=0.00003
iter=8500 current objective=119029.883 nonzero weights=169 stop=0.00002
iter=8600 current objective=119029.883 nonzero weights=169 stop=0.00003
iter=8700 current objective=119029.883 nonzero weights=169 stop=0.00002
iter=8800 current objective=119029.883 nonzero weights=169 stop=0.00001
iter=8900 current objective=119029.883 nonzero weights=169 stop=0.00001
iter=9000 current objective=119029.883 nonzero weights=169 stop=0.00001
```

[9]:
```python
def Prediction(X):
    return basis(X) @ w_predict
```

2.1 Exercise 8

2.1.1 (a)

The minimization problem is convex because the primal problem is convex (L_2 norm is convex), and the equality constrain $Xw = b$ is linear.

2.1.2 (b)

First step: Show that the solution satisfies the constrain:

$$\begin{aligned}
X(w_0 - X^T(XX^T)^{-1}(Xw_0 - b)) &= Xw_0 - X(X^T(XX^T)^{-1}(Xw_0 - b)) \\
&= Xw_0 - XX^T(XX^T)^{-1}(Xw_0 - b)) \\
&= Xw_0 - (Xw_0 - b)) \\
&= Xw_0 - Xw_0 + b \\
&= b
\end{aligned}$$

Now setup Lagrange Dual Problem:

$$L(w, v) = \|w - w_0\| + v^T(Xw - b)$$

Compute derivative

$$\begin{aligned}
\nabla_w L(w, v) &= \nabla_w \|w - w_0\| + v^T(Xw - b) \\
&= 2(w - w_0) + v^T X
\end{aligned}$$

Set to 0:

$$\begin{aligned}
0 &= 2(w - w_0) + v^T X \\
w &= -v^T X \frac{1}{2} + w_0
\end{aligned}$$

Plug in L to get g.

$$\begin{aligned}
g(v) = L(-v^T X \frac{1}{2} + w_0, v) &= \| - v^T X \frac{1}{2} + w_0 - w_0\| + v^T(X(-v^T X \frac{1}{2} + w_0) - b) \\
&= \frac{1}{4}\|v^T X\| + v^T(X(-v^T X \frac{1}{2} + w_0) - b)
\end{aligned}$$