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Assignment Nr. 1

1 Bayes Optimal Function

$$
\begin{aligned}
&\frac{\partial}{\partial c(x)} \int_y (log(c(x)) + \frac{y}{c(x)})p(y|x)dy \\
&= \int_y (\frac{\partial}{\partial c(x)} log(c(x)) + \frac{\partial}{\partial c(x)} \frac{y}{c(x)})p(y|x)dy \\
&= \int_y (\frac{1}{c(x)} + y \cdot (-\frac{1}{c(x)^2}))p(y|x)dy \\
&= \frac{1}{c(x)} \int_y (1 - y \cdot \frac{1}{c(x)})p(y|x)dy \\
&= \frac{1}{c(x)} (\int_y p(y|x)dy - \int_y y \cdot \frac{1}{c(x)} p(y|x)dy) \\
&= \frac{1}{c(x)} (1 - \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x)dy)
\end{aligned}
$$

Now set to 0:

$$
\begin{aligned}
0 &= \frac{1}{c(x)} (1 - \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x)dy) \\
&\text{because } \frac{1}{c(x)} > 0, x \in \mathbb{R}: \\
\Rightarrow 0 &= (1 - \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x)dy) \\
1 &= \frac{1}{c(x)} \cdot \int_y y \cdot p(y|x)dy) \\
c(x) &= \cdot \int_y y \cdot p(y|x)dy) = E[Y|X]
\end{aligned}
$$

2 Bayes Optimal Function

(a)

$$
\begin{aligned}
E[L(yf(x))|X] &= \sum_y L(yf(x))p(y|X) \\
&= L(f(x))p(y = 1|X) + L(-f(x))p(y = -1|X) \\
&> (1 - kf(x))p(y = 1|X) + (1 - kf(x))p(y = -1|X)
\end{aligned}
$$

$$
\begin{aligned}
E[L(yf(x))|X] &= \sum_y L(yf(x))p(y|X) \\
&= L(f(x))p(y=1|X) + L(-f(x))p(y=-1|X) \\
&< (1-f(x))p(y=1|X) + (1-f(x))p(y=-1|X) \\
0 &= -p(y=1|X) - p(y=-1|X)
\end{aligned}
$$

$$
\frac{\partial}{\partial f(x)}(1-f(x)) = -1 \frac{\partial}{\partial f(x)}(1-kf(x)) = -k
$$

(b)

Because left and right side derivative at x=0 is not the same: left side: -k, right side: -1

3 Bayes error

(a)

$$
\begin{aligned}
R^* &= min(E[\mathbb{1}_{f(x)y\leq 0}|X]) \\
&= min(E_X[\mathbb{1}_{f(x)=-1}P(Y=1|X) + \mathbb{1}_{f(x)=1}P(Y=-1|X)]) \\
&= min(\mathbb{1}_{f(x)=-1}(\int_0^{\frac{1}{8}} P(Y=1,x)dx + \int_{\frac{1}{8}}^{\frac{7}{8}} P(Y=1,x)dx + \int_{\frac{7}{8}}^{1} P(Y=1,x)dx) \\
&+ \mathbb{1}_{f(x)=1}(\int_0^{\frac{1}{8}} P(Y=-1,x)dx + \int_{\frac{1}{8}}^{\frac{7}{8}} P(Y=-1,x)dx + \int_{\frac{7}{8}}^{1} P(Y=-1,x)dx) \\
&= min(\mathbb{1}_{f(x)=-1}(\frac{1}{8}0.1 + \frac{6}{8}0.9 + \frac{1}{8}0.1) \\
&+ \mathbb{1}_{f(x)=-1}(\frac{1}{8}0.9 + \frac{6}{8}0.1 + \frac{1}{8}0.9)) \\
&= min(\mathbb{1}_{f(x)=-1} \cdot 0.7 + \mathbb{1}_{f(x)=1} \cdot 0.3) \\
&\Rightarrow f(x) = 1
\end{aligned}
$$

$$
\begin{aligned}
R^* &= min(E_X[\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)] \\
&= min(\int_x [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)]) \\
&= min(\int_{x\in[0,\frac{1}{8}]} [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)] \\
&+ \int_{x\in[\frac{1}{8},\frac{7}{8}]} [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)] \\
&+ \int_{x\in[\frac{7}{8},1]} [\mathbb{1}_{f(x)=1}P(Y=-1,X) + \mathbb{1}_{f(x)=-1}P(Y=1,X)])
\end{aligned}
$$

all three terms can be minimized on their own

(b)

Probably w = 1, b = 0.1 or multiples of it.