Statistical Machine Learning Exercise Sheet 7

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Exercise 11

(a)

$$\begin{aligned}
\langle\Phi_m(x),\Phi_m(y)\rangle &= \sum_{m\in M}\Phi_m(x),\Phi_m(y)\\
&= \sum_{m\in M}\sqrt{\frac{d!}{\prod_i m_i!}}\prod_i x_i^{m_i}\cdot\sqrt{\frac{d!}{\prod_i m_i!}}\prod_i y_i^{m_i}\\
&= \sum_{m\in M}\frac{d!}{\prod_i m_i!}(\prod_i x_i^{m_i}\prod_i y_i^{m_i})\\
&= \sum_{m\in M}\frac{d!}{\prod_i m_i!}\prod_i x_i^{m_i}y_i^{m_i}\\
&= \sum_{m\in M}\frac{d!}{\prod_i m_i!}\prod_i (x_i\cdot y_i)^{m_i}\\
&= (x_1\cdot y_1+\dots+x_n\cdot y_n)^d\\
&= \langle x,y\rangle^d
\end{aligned}$$

(b)

assume a Loss function $\sum_{i=1}^n L(y_i,\phi(x_i))+\lambda||\phi||_H^2$ minimized by

$\phi^*(x)=\sum_{i=1}^n \alpha_i K(x,x_i)$

equivalent (due to representer theorem) to:
$\sum_{i=1}^n L(y_i,\alpha_i K)+\lambda\sum_{j,i=1}^n \alpha_i\alpha_j K$

for L_2 - Loss there exists a unique solution: $\alpha^*=(K+\lambda I)^{-1}y$ With
$\Omega=R^d$
we get
$K(x,y)=(1+\sum_{i=1}^n x_iy_i)^d=(1+\sum_{i=0}^{n-1}x_iy_i)^d$

$=\binom{d}{k}1^{d-k}*\sum_{i=0}^{n-1}x_iy_i^d$

Therefore the dimension of H is $\binom{(d+n-1)}{d}$

(c)

We have $d=5$ and $n=16*16=256$. Therefore we get a dimension of H of $\binom{5+256+1}{5}=\binom{262}{5}$. This means we have too mayn features to explicitly compute/store. However if we use the kernel $\langle x,y\rangle^5$, we first have to take a product over 256 dimensions and then a sum over 256 dimensions and finally just compute the power of one number, which is much cheaper to compute and store.

Exercise 12

We can reformulate the problem as:

$$\begin{aligned}
R(f') &= \inf_{f\in F}R(f)+R(f')-\inf_{f\in F}R(f)\\
&\le \inf_{f\in F}R(f)+|R(f')-\inf_{f\in F}R(f)|\\
&\le \inf_{f\in F}R(f)+2\sup_{f\in F}|\hat{R}_m(f)-R(f)|
\end{aligned}$$

Therefore it is sufficient to show that $P(2\sup_{f\in F}|\hat{R}_m(f)-R(f)|\ge\epsilon)\le\delta\Rightarrow\epsilon=\sqrt{\frac{2}{m}\log\frac{2N}{\delta}}$

Reformulating gives

$$\begin{aligned}
P(2\sup_{f\in F}|\hat{R}_m(f)-R(f)|\ge\epsilon) &= P(\sup_{f\in F}|\hat{R}_m(f)-R(f)|\ge\frac{\epsilon}{2})\\
&\le P(\cup_{f\in F}(|\hat{R}_m(f)-R(f)|\ge\frac{\epsilon}{2})\\
&= \sum_{f\in F}P(|\hat{R}_m(f)-R(f)|\ge\frac{\epsilon}{2})\\
&= \sum_{f\in F}2exp(-2m(\frac{\epsilon}{2})^2)\\
&= 2Nexp(-\frac{1}{2}m\epsilon^2)\\
&\Rightarrow \delta=2Nexp(-\frac{1}{2}m\epsilon^2)
\end{aligned}$$

Now solve for epsilon

$$\begin{aligned}
\delta &= 2N\exp(-\frac{1}{2}m\epsilon^2)\\
\frac{\delta}{2N} &= \exp(-\frac{1}{2}m\epsilon^2)\\
\frac{2N}{\delta} &= \exp(\frac{1}{2}m\epsilon^2)\\
\log\frac{2N}{\delta} &= \frac{1}{2}m\epsilon^2\\
\frac{2}{m}\log\frac{2N}{\delta} &= \epsilon^2\\
\sqrt{\frac{2}{m}\log\frac{2N}{\delta}} &= \epsilon
\end{aligned}$$

Exercise 13

In [1]:
```python
import numpy as np
from numpy.linalg import norm
from scipy.linalg import cho_factor, cho_solve
from scipy.spatial.distance import cdist
```

In [2]:
```python
data = np.load("diabetes_data.npy", allow_pickle=True).item()
X_train = data["Xtrain"]
Y_train = data["Ytrain"]
X_test = data["Xtest"]
Y_test = data["Ytest"]
```

In [3]:
```python
def gaussian_kernel(x, y, mu):
    return np.exp(- mu * cdist(x,y)**2)

def Loss(Y, Y_prediction):
    return 0.5* norm(Y == np.sign(Y_prediction))
```

In [4]:
```python
def train(X, Y, lambda_reg, mu):
    n = X.shape[0]
    K = gaussian_kernel(X, X, mu)
    alpha = cho_solve(cho_factor(K + n * lambda_reg * np.eye(n)), Y)
    train_loss = Loss(Y, predict(X, X, alpha, mu))
    return alpha, train_loss

def predict(X_pred, X_t, alpha, mu):
    return np.sum(np.multiply(np.squeeze(alpha), gaussian_kernel(X_pred, X_t, mu)), axis=1)

def test(X_test, Y_test, X, alpha, mu):
    return Loss(Y_test, predict(X_test, X, alpha, mu))
```

In [5]:
```python
def CrossValidation(X, Y, lambdas, mus, k=5):
    X_split = np.array(np.array_split(X, k))
    Y_split = np.array(np.array_split(Y, k))
    min_loss=np.inf
    min_mu = None
    min_lambda = None
    CVErrors = np.zeros((len(lambdas), len(mus)))

    for mu_index, mu in enumerate(mus):
        for l_index, l in enumerate(lambdas):
            # compute k-fold cross validation
            validation_loss = []
            for i in range(k):
                x_train = np.concatenate(np.delete(X_split, i, axis=0))
                y_train = np.concatenate(np.delete(Y_split, i, axis=0))
                x_val = X_split[i]
                y_val = Y_split[i]
                alphas, train_loss = train(x_train, y_train, l, mu)
                validation_loss.append(test(x_val, y_val, x_train, alphas, mu))
            # average over all losses and store
            validation_loss = np.mean(validation_loss)
            CVErrors[l_index, mu_index] = validation_loss
            # check if smaller
            if validation_loss < min_loss:
                min_loss = validation_loss
                min_mu = mu
                min_lambda = l
    print(f"The pest parameters are given as mu={min_mu}, lambda={min_lambda}")
    # finally train on the whole set again
    return train(X, Y, min_lambda, min_mu), CVErrors
```

Cross Validation Training

In [6]:
```python
mus = [1e-4, 1e-3, 1e-2, 1e-1, 1]
lambdas = [1e-4, 1e-3, 1e-2, 1e-1, 1]
(best_alphas, final_train_loss), validation_loss = CrossValidation(X_train, Y_train, lambdas, mus, 5)
print("The resulting train loss is: ", final_train_loss)
```

```
The pest parameters are given as mu=0.1, lambda=0.0001
The resulting train loss is:  95.33362470817943
```

Evaluating of the final parameters on the test set

In [7]:
```python
test_loss = test(X_test, Y_test, X_train, alpha=best_alphas, mu=0.1)
print("The resulting test loss is: ", test_loss)
```

```
The resulting test loss is:  193.4644670217247
```

Plots

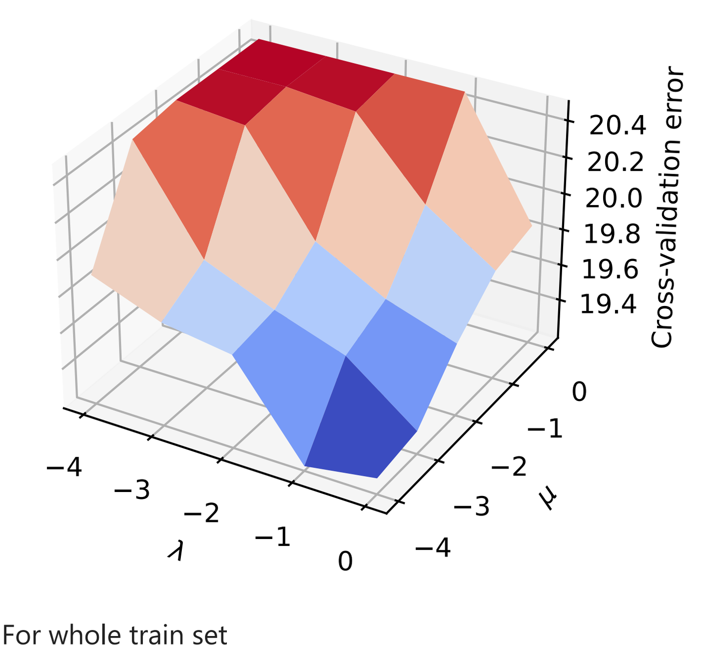
In [8]:
```python
def train_test_plot(x_train, y_train, x_test, y_test, lambdas, mus):
    TrainErrors = np.zeros((len(lambdas), len(mus)))
    TestErrors = np.zeros((len(lambdas), len(mus)))
    for mu_index, mu in enumerate(mus):
        for l_index, l in enumerate(lambdas):
            # compute k-fold cross validation
            alphas, train_loss = train(x_train, y_train, l, mu)
            TrainErrors[l_index, mu_index] = train_loss
            TestErrors[l_index, mu_index] = test(x_test, y_test, x_train, alpha=alphas, mu=mu)
    return TrainErrors, TestErrors
```

In [9]:
```python
TrainErrors, TestErrors = train_test_plot(X_train, Y_train, X_test, Y_test, lambdas, mus)
```

In [10]:
```python
plot_lambda, plot_mu = np.meshgrid(np.log10(lambdas), np.log10(mus))
```

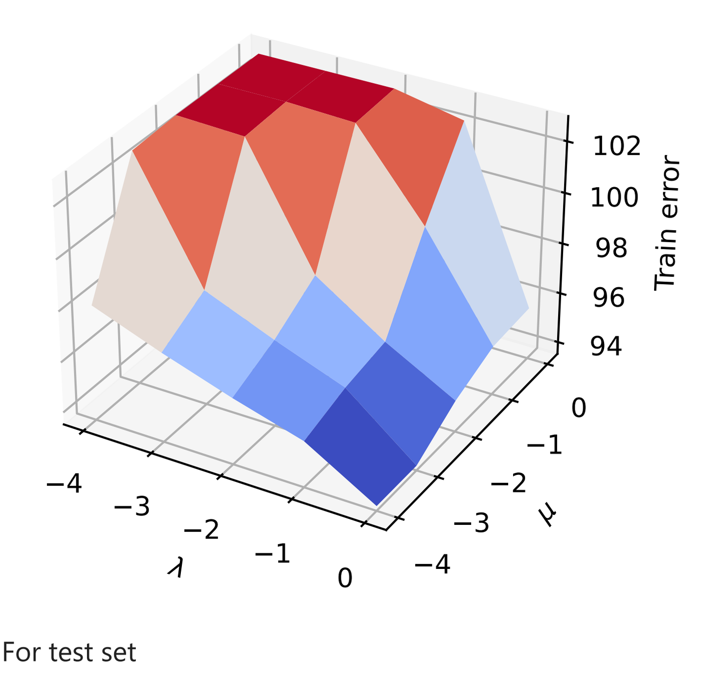
For cross validation

In [11]:
```python
from matplotlib import pyplot as plt
from matplotlib import cm
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(plot_lambda, plot_mu, validation_loss, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False)
ax.set(xlabel="$\lambda$", ylabel="$\mu$", zlabel="Cross-validation error");
```



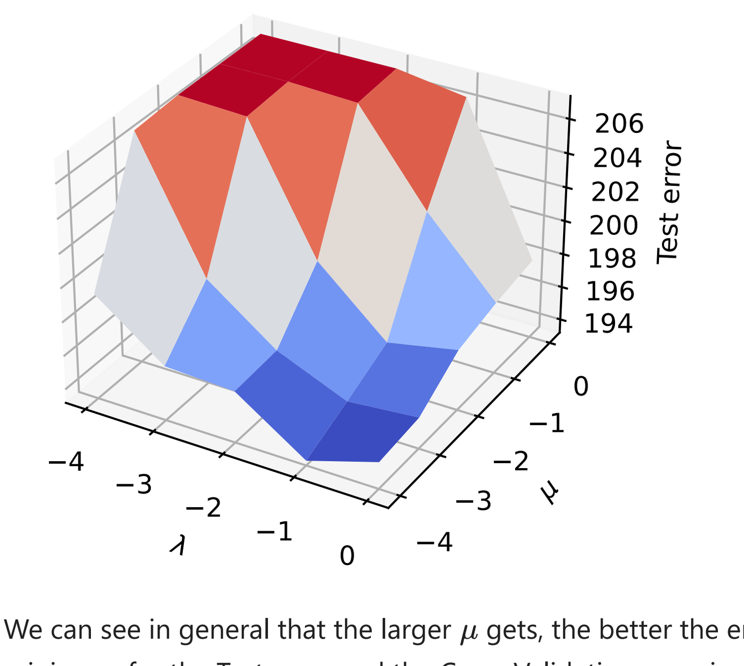
For whole train set

In [12]:
```python
from matplotlib import pyplot as plt
from matplotlib import cm
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(plot_lambda, plot_mu, TrainErrors, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False)
ax.set(xlabel="$\lambda$", ylabel="$\mu$", zlabel="Train error");
```



For test set

In [13]:
```python
from matplotlib import pyplot as plt
from matplotlib import cm
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(plot_lambda, plot_mu, TestErrors, cmap=cm.coolwarm,
                       linewidth=0, antialiased=False)
ax.set(xlabel="$\lambda$", ylabel="$\mu$", zlabel="Test error");
```



We can see in general that the larger μ gets, the better the error becomes. For λ it is the other way around. Note however that the minimum for the Test error and the Cross Validation error is not given for $\mu=1$ (as in the Train set), but for $\mu=0.1$. This shows that Cross validation gives a better estimate of the test error.