On Average Distortion of Non-Convex Quantization Sets and UAV deployments

Jun Guo, Philipp Walk, and Hamid Jafarkhani
University of California Irvine, Department of Electrical Engineering and Compute Science
Center for Pervasive Communications and Computing, Electrical Engineering Hall
Irvine, CA 92697-2625, USA
{guoj4,pwalk,hamidj}@uci.edu

1 Introduction

In this report, we study the optimal deployment of multiple Unmanned Aerial Vehicles (UAVs) with directional antennas. Similar to [1], we consider a wireless network where several ground terminals (GTs) using variable transmission power and fixed data rate. In the recent decade, UAV with direction antenna has been widely studied in literature [2]–[7]. In [2]–[7], the antenna gain within 3dB beamwidth is approximated by a constant. The authors claimed that this approximation comes from the book [8]. Unfortunately, I cannot get this reference (fourth edition). But, I have the second edition of this book [9]. In [9], the maximum antenna (the gain at 0^o) is approximated by the constant that is mentioned in [2]–[7]. Moreover, a cosine-shaped antenna gain is proposed in [9]. Even if antenna gain is approximated as a constant in [8], cosine function seems be a better approximation than a constant. Therefore, I formulate UAV's antenna gain as a cosine function. On the other hand, the existing studies [1] made a common assumption that all UAVs have the identical height. In this report, we will study the UAVs with variant heights which is more reasonable in practice.

Notation We denote the positive integers by $\mathbb{N} = \{1, 2, 3, \dots\}$ and for $N \in \mathbb{N}$ the first N positive integers by $[N] = \{1, 2, \dots, N\}$. We will denote row vectors by bold letters $\mathbf{x} = (x_1, \dots, x_d)$ and matrices by bold capital letters $\mathbf{X} = \mathbb{R}^{d\times N}$. We denote by $(\cdot)^T$ the transpose of a vector or matrix. For convenience, let $B(\mathbf{c}, r) = \{\boldsymbol{\omega} \mid \|\boldsymbol{\omega} - \mathbf{c}\|^2 \leq r\}$ be the open ball in $\mathbb{R}^d$ centered at $\mathbf{c} \in \mathbb{R}^d$ with radius $r \geq 0$. Note that $B(\mathbf{c}, 0)$ is an empty set. We denote by ∂V the boundary of the set $V \subset \mathbb{R}^d$ and by V^c its complement. In particular, $\partial B(\mathbf{c}, r)$ is the sphere with center at $\mathbf{c}$ and radius r. We will denote the positive numbers by $\mathbb{R}_+ := \{a \in \mathbb{R} \mid a > 0\}$. Moreover, for two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, we denote the generated half space between them, which contains $\mathbf{a} \in \mathbb{R}^d$ by $HS(\mathbf{a}, \mathbf{b})$.

2 System model

We consider a network of several GTs at zero elevation on the target region $\Omega \subset \mathbb{R}^2$ and N low-altitude UAVs in the sky. Let $\mathbf{P} = (\mathbf{p}_1^T, \dots, \mathbf{p}_N^T)$ be the UAV (ground) horizontal deployment, where $\mathbf{p}_n = (x_n, y_n) \in \mathbb{R}^2$ is the projected location on the ground Ω, and $\mathbf{h} = (h_1, \ldots, h_N)$ be the UAV vertical (height) deployment, where $h_n \in \mathbb{R}_+$ is UAV n's height. In particular, UAV n's 3-dimensional location is defined as $\mathbf{q}_n = (\mathbf{p}_n, h_n) = (x_n, y_n, h_n)$. The elevation angle, $\theta \in [0, \frac{\pi}{2}]$, from a GT at $\boldsymbol{\omega}$ to UAV n is defined as the angle between the ground distance $\overline{\boldsymbol{\omega}\mathbf{p}_n}$ and flight height h_n.

As Koyuncu et.al. [1], we consider a fixed-rate variable-power transmission scenario where a GT at location ω wishes to communicate with bit-rate R_b in bits/s, and transmits with power P_{TX} in J/s.

To capture the power falloff versus distance along with the random attenuation about the path loss from shadowing, we adopt the propagation model [10, (2.51)] as:

$$PL_{dB} = 10\log_{10} K - \underbrace{10\alpha \log_{10} \frac{d}{d_0}}_{\text{terrestrial path-loss}} - \psi_{dB}, \tag{1}$$

where K is a unitless constant depending on the antenna characteristics, d_0 is a reference distance, $\alpha \geq 1$ is the terrestrial path loss exponent, and ψ_{dB} is a Gaussian random variable following $\mathcal{N}\left(0, \sigma^2_{\psi_{dB}}\right)$. This air-to-ground or terrestrial path loss model is widely used for UAV basestation path-loss models [11]. Practical values are between 2 and 6 and depends on the distance d and the shadowing due to low altitude. For common practical measurements see for example [12]. Typically maximal altitudes for UAV are < 1000m, due to flight zone restrictions of aircrafts. Therefore, the received power at UAV n can be represented as

$$P_{RX} = P_{TX} G_{TX} G_{RX} PL = \frac{P_{TX} G_{TX} G_{RX} K d_0^{\alpha}}{d^{\alpha} 10^{\frac{\psi_{dB}}{10}}}, \tag{2}$$

where

$$d((\mathbf{p}, h), (\boldsymbol{\omega}, 0)) = \sqrt{\|\mathbf{p} - \boldsymbol{\omega}\|^2 + h^2} = \sqrt{(p_x - \omega_x)^2 + (p_y - \omega_y)^2 + h^2} \tag{3}$$

is the Euclidean distance between the UAV at $\mathbf{q} = (\mathbf{p}, h)$ and the ground terminal at $\boldsymbol{\omega} = (\omega_x, \omega_y)$ and G_{TX} and G_{RX} are the antenna gains of the transmitter and the receiver, respectively. In this report, we focus on omnidirectional transmitter (GT) antennas and directional receiver (UAV) antennas. The antenna gains are formulated as the positive real numbers

$$G_{\text{GT}} > 0 \quad , \quad G_{\text{UAV}} = \cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{h}{d((p,h),(\omega,0))}. \tag{4}$$

see [13, pp.52]. We model here the GT antennas as perfect omni-directional, with an isotropic gain. The combined antenna intensity is then proportional to

$$G = G_{\text{UAV}} G_{\text{GT}} K = \frac{h G_{\text{GT}} K}{d((p,h),(\omega,0))} \tag{5}$$

see Figure 1. Accordingly, the received power (2) can be rewritten as

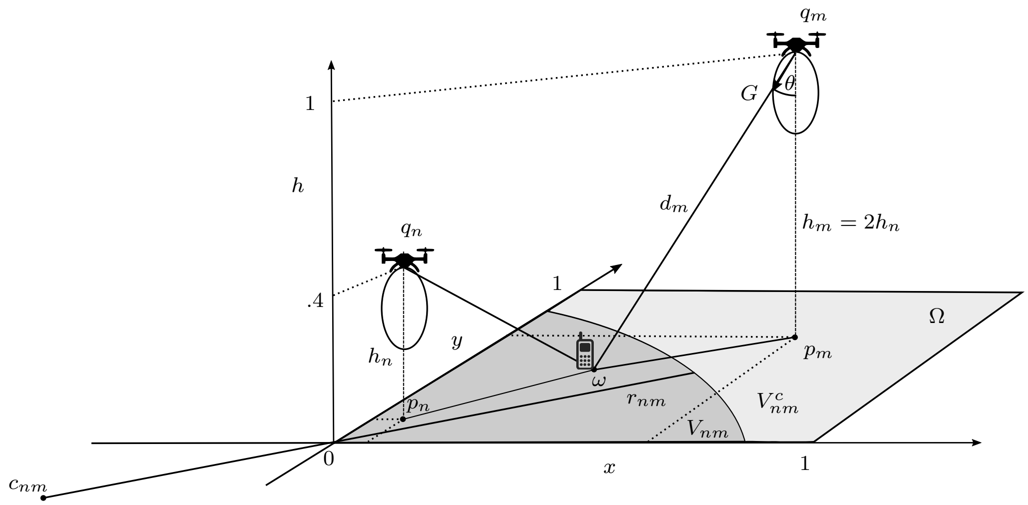

Figure 1: UAV deployment with directed antenna beam for $\alpha = 2$ and $N = 2$ for a uniform GT distribution.

$$P_{RX} = \frac{P_{TX} h G_{\mathrm{GT}} K d_0^{\alpha}}{d((p,h),(\omega,0))^{\alpha+1} 10^{\frac{\psi_{dB}}{10}}}. \tag{6}$$

To achieve the reliable communication between GT and UAV with capacity (bit-data-rate) at least R_b, we have

$$B \log_2 \left(1 + \frac{P_{RX}}{N_0}\right) \geq R_b, \tag{7}$$

where B is the channel bandwidth and N_0 is the noise power. The minimum transmission power is then given by

$$P_{TX} = \frac{\left(2^{\frac{R_b}{B}} - 1\right) N_0 d((p,h),(\omega,0))^{\alpha+1} 10^{\frac{\psi_{dB}}{10}}}{h G_{\mathrm{GT}} K d_0^{\alpha}}. \tag{8}$$

The expectation of the minimum transmitter power of the nth UAV at $\mathbf{q}_n = (p_n, h_n)$ to a ground user in the nth cell at $(\boldsymbol{\omega}, 0)$ is then

$$\begin{aligned} \mathbb{E}[P_{TX,n}] &= \frac{(2^{\frac{R_b}{B}} - 1)N_0}{h_n G_{\mathrm{GT}} K d_0^{\alpha}} \frac{d(\mathbf{q}_n, (\boldsymbol{\omega}, 0))^{\alpha+1}}{\sqrt{2\pi}\sigma_{\psi_{dB}}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\psi_{dB}^2}{2\sigma_{\psi_{dB}}^2} + \ln(10)\frac{\psi_{dB}}{10}\right) d\psi_{dB} \\ &= \beta \cdot \frac{d(\mathbf{q}_n, (\boldsymbol{\omega}, 0))^{2\gamma}}{h_n} =: P(\mathbf{q}_n, \boldsymbol{\omega}) \end{aligned} \tag{9}$$

where the fixed parameters

$$\beta = \frac{(2^{\frac{R_b}{B}} - 1)N_0 \exp\left(-\frac{\sigma_{\psi_{dB}}^2 (\ln 10)^2}{200}\right)}{G_{\mathrm{GT}} K d_0^{\alpha}} \quad , \quad \gamma = \frac{\alpha + 1}{2} \tag{10}$$

are independent of the UAVs and will be chosen by design.

3 Optimal Static UAV deployment

To define an optimization problem over the UAVs we will need a *single-objective function*, depending on the N UAVs ground positions $\mathbf{P} = (\mathbf{p}_1^T, \mathbf{p}_2^T, \dots, \mathbf{p}_N^T)$ and flight heights $\mathbf{h} = (h_1, h_2, \dots, h_N)$. We will define the average power over all ground terminals (GTs) distributed by λ in Ω as

$$\bar{P}(\mathbf{P}, \mathbf{h}) = \int_\Omega \min_n \{P(\boldsymbol{\omega}, \mathbf{p}_n, h_n)\} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}, \tag{11}$$

where we assume that each GT at ω will chose the UAV which requires the lowest transmit power. Here we assume a *continuous distribution* of GTs, given by $\lambda : \Omega \to [0, 1]$ with $\int_\Omega \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} = 1$, as used in [14]–[17]. The power function P can be interpret as a "distance" or *distortion* function, which measures the required power (distortion) between user at $\boldsymbol{\omega}$ to the nth UAV, which we seek to minimize. We also assume that the communication between all users and UAVs is orthogonal, i.e., separated in frequency or time (slotted protocols). An *optimal static deployment* is then the minimization of the *average power* $\bar{P}$ over all UAVs allowed locations and associated parameters

$$\min_{\mathbf{P} \in \Omega^N, \mathbf{h} \in \mathbb{R}_+^N} \bar{P}(\mathbf{P}, \mathbf{h}), \tag{12}$$

which defines a *locational-parameter optimization problem*. Here we associate to the locations $\mathbf{p}_n$ of the facilities (UAV ground positions) an additional parameter, given by the flight height h_n. Although h_n is a location parameter in space, the power is not a monotone function of the height, due to the directional antenna effect.

To find local extrema of (12) analytically, we will need that the single-objective function $\bar{P}$ is continuously differentiable at any point in $\mathcal{Q}^N = \Omega^N \times \mathbb{R}_+^N$, i.e., the gradient exists and is a continuous function, as was shown for piecewise continuous non-decreasing distortion functions in the Euclidean metric over Ω^N [18, Thm.2.2] and [19]. We could extend this result in [20] to continuous distortion functions on $\Omega \times \mathcal{Q}$ for arbitrary $d-$dimensional parameter sets $\mathcal{Q} \subset \mathbb{R}^d$, which are multivariate polynomials of finite degree if restricted to Ω. Then the necessary condition for a local extrema is the vanishing of the gradient at a critical point[1]. To derive the partial derivatives we will need to rewrite the integral kernel, which is the minimum of N continuous functions, as a sum of N integrals by using generalized Voronoi tessellations of Ω.

First we will derive the tessellation of Ω by generalized Voronoi diagrams in d dimensions for any height parameters $h_n \in \mathbb{R}_+$ associate to the generating (ground) points p_n, which are special cases of *Möbius diagrams*, introduced in [21] and [22].

Lemma 1. *Let* $\mathbf{P} = (\mathbf{p}_1^T, \mathbf{p}_2^T, \dots, \mathbf{p}_N^T) \subset \Omega^N \subset (\mathbb{R}^d)^N$ *for* $d \in \{1, 2\}$ *be the ground positions and* $\mathbf{h} = (h_1, \dots, h_N) \in \mathbb{R}_+^N$ *the associated flight heights of the* N *UAVs.*

[1] Note, if $\nabla \bar{P}$ is not continuous in $\mathcal{Q}^N$ than any jump-point is a potential critical point and has to be checked individually.

Then the total average transmit power to serve any ground terminal in Ω distributed by λ for $\beta = 1$ is given by

$$\bar{P}\left(\mathbf{P},\mathbf{h}\right) = \int_{\Omega} \min_{n} \left\{ \frac{(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma}}{h_n} \right\} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} = \sum_{n=1}^{N} \int_{\mathcal{V}_n} \frac{(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma}}{h_n} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{13}$$

where $\gamma = (1+\alpha)/2$ is given by the path-loss exponent $\alpha \geq 1$. The generalized Voronoi regions for the ground terminals are given by

$$\mathcal{V}_n = \mathcal{V}_n(\mathbf{P},\mathbf{h}) = \bigcap_{m \neq n} \mathcal{V}_{nm}(\mathbf{P},\mathbf{h}) \tag{14}$$

with the dominance regions of $\mathbf{p}_n$ over $\mathbf{p}_m$ are given by

$$\mathcal{V}_{nm} = \Omega \cap \begin{cases} HS(\mathbf{p}_n, \mathbf{p}_m) & , h_m = h_n \\ B(\mathbf{c}_{nm}, r_{nm}) & , h_n < h_m \\ B^c(\mathbf{c}_{nm}, r_{nm}) & , h_n > h_m \end{cases} \tag{15}$$

where the center and radii of the balls are given by

$$\mathbf{c}_{nm} = \frac{\mathbf{p}_n - h_{nm}\mathbf{p}_m}{1 - h_{nm}} \quad , \quad r_{nm} = \left(\frac{h_{nm}}{(1 - h_{nm})^2} \|\mathbf{p}_n - \mathbf{p}_m\|^2 + h_n^2 \frac{h_{nm}^{-\alpha} - 1}{1 - h_{nm}} \right)^{1/2} \tag{16}$$

with the relative flight heights between UAV n and UAV m given by

$$h_{nm} = \left(\frac{h_n}{h_m} \right)^{\frac{2}{1+\alpha}} . \tag{17}$$

Remark. If two UAVs have the same ground position, i.e., $\mathbf{p}_n = \mathbf{p}_m$, then both of them are active, if the relative height difference is not to large. In all other cases, each UAV will be active, as long as it is not to high or to low. Let us note, that the individual transmit powers of each UAV are only bounded by $P(\mathbf{P},\mathbf{h})$. To bound the individual average transmit powers of the nth UAV, one needs to bound the maximal and minimal flight heights, since it holds

$$\bar{P}_n(\mathbf{P},\mathbf{h}) = \int_{V_n} (h_n^{-1/\gamma} \|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^{2-\frac{1}{\gamma}})^{\gamma} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{18}$$

where h_n^{α} will dominate the power for large flight heights. It can be also seen that the radius of the nth GT cells will increase if all relative flight heights increase.

Proof. The minimization of the performance functions over Ω defines an assignment rule for a generalized Voronoi diagram $\mathcal{V}(\mathbf{P},\mathbf{h}) = \{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_N\}$ where

$$\begin{aligned} \mathcal{V}_n = \mathcal{V}_n(\mathbf{P},\mathbf{h}) &:= \left\{ \boldsymbol{\omega} \in \Omega \;\middle|\; \frac{(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma}}{h_n} \leq \frac{(\|\mathbf{p}_m - \boldsymbol{\omega}\|^2 + h_m^2)^{\gamma}}{h_m}, m \neq n \right\} \\ &= \left\{ \boldsymbol{\omega} \in \Omega \;\middle|\; a_n \|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + b_n \leq a_m \|\mathbf{p}_m - \boldsymbol{\omega}\|^2 + b_m, m \neq n \right\} \end{aligned} \tag{19}$$

is the nth Voronoi region. Here we denoted the weights by the positive numbers

$$a_n = h_n^{-\frac{1}{\gamma}}, \quad b_n = h_n^{2-\frac{1}{\gamma}} \tag{20}$$

which define a *compoundly weighted power distance*, see [23, (3.1.12)]. The resulting diagram is also called a *Möbius diagram* and its bisectors are circles or lines in $\mathbb{R}^2$ as we will show below, see also [21], [22]. The nth Voronoi region is defined by $N-1$ inequalities, which can be written as the intersection of the $N-1$ *dominance regions* of $\mathbf{p}_n$ over $\mathbf{p}_m$, given by

$$\mathcal{V}_{nm} = \left\{ \boldsymbol{\omega} \in \Omega \;\middle|\; a_n \left\|\mathbf{p}_n - \boldsymbol{\omega}\right\|^2 + b_n \leq a_m \left\|\mathbf{p}_m - \boldsymbol{\omega}\right\|^2 + b_m \right\}. \tag{21}$$

If $h_n = h_m$ then $a_n = a_m$ and $b_n = b_m$, such that $\mathcal{V}_{nm} = HS(\mathbf{p}_n, \mathbf{p}_m)$, the left half-space between $\mathbf{p}_n$ and $\mathbf{p}_m$. For $a_n > a_m$ we can rewrite the inequality as

$$\begin{aligned}
a_n \left\|\mathbf{p}_n\right\|^2 + a_n \left\|\boldsymbol{\omega}\right\|^2 - 2a_n \langle \mathbf{p}_n, \omega\rangle - a_m \left\|\mathbf{p}_m\right\|^2 - a_m \left\|\boldsymbol{\omega}\right\|^2 + 2a_m \langle \mathbf{p}_m, \boldsymbol{\omega}\rangle + (b_n - b_m) &\leq 0 \\
(a_n - a_m) \left\|\boldsymbol{\omega}\right\|^2 - 2\langle a_n\mathbf{p}_n - a_m\mathbf{p}_m, \boldsymbol{\omega}\rangle + a_n \left\|\mathbf{p}_n\right\|^2 - a_m \left\|\mathbf{p}_m\right\|^2 + (b_n - b_m) &\leq 0 \\
\left\|\boldsymbol{\omega}\right\|^2 - 2\langle \mathbf{c}_{nm}, \boldsymbol{\omega}\rangle + \frac{a_n^2 \left\|\mathbf{p}_n\right\|^2 + a_m^2 \left\|\mathbf{p}_m\right\|^2 - a_n a_m(\left\|\mathbf{p}_n\right\|^2 + \left\|\mathbf{p}_m\right\|^2)}{(a_n - a_m)^2} + \frac{b_n - b_m}{a_n - a_m} &\leq 0
\end{aligned}$$

where the center point is given by

$$\mathbf{c}_{nm} = \frac{a_n\mathbf{p}_n - a_m\mathbf{p}_m}{a_n - a_m} = a_n \frac{\mathbf{p}_n - h_{nm}\mathbf{p}_m}{a_n - a_m} = \frac{\mathbf{p}_n - h_{nm}\mathbf{p}_m}{1 - h_{nm}} \tag{22}$$

where we introduced the *relative flight height* between the nth and mth UAV

$$h_{nm} := \frac{a_m}{a_n} = \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}} > 0. \tag{23}$$

If $0 < a_n - a_m$, which is equivalent to $h_n < h_m$, then this defines a ball (disc) and for $h_n > h_m$ its complement. Hence we get

$$\mathcal{V}_{nm} = \begin{cases} B(\mathbf{c}_{nm}, r_{nm}) = \{\boldsymbol{\omega} \in \Omega \mid \left\|\boldsymbol{\omega} - \mathbf{c}_{nm}\right\| < r_{nm}\}, & h_n < h_m \\ HS(\mathbf{p}_n, \mathbf{p}_m) = \{\boldsymbol{\omega} \in \Omega \mid \left\|\boldsymbol{\omega} - \mathbf{p}_n\right\| \leq \left\|\boldsymbol{\omega} - \mathbf{p}_m\right\|\}, & h_n = h_m \\ B^c(\mathbf{c}_{nm}, r_{nm}) = \{\boldsymbol{\omega} \in \Omega \mid \left\|\boldsymbol{\omega} - \mathbf{c}_{nm}\right\| > r_{nm}\}, & h_n > h_m \end{cases} \tag{24}$$

where the radius square is given by

$$r_{nm}^2 = \frac{-2a_n a_m \langle \mathbf{p}_n, \mathbf{p}_m\rangle + a_n a_m(\left\|\mathbf{p}_n\right\|^2 + \left\|\mathbf{p}_m\right\|^2)}{(a_n - a_m)^2} - \frac{b_n - b_m}{a_n - a_m} \tag{25}$$

$$= a_n a_m \frac{\left\|\mathbf{p}_n - \mathbf{p}_m\right\|^2}{(a_n - a_m)^2} + \frac{b_m - b_n}{a_n - a_m} = \frac{a_n}{a_m} \frac{\left\|\mathbf{p}_n - \mathbf{p}_m\right\|^2}{(1 - \frac{a_n}{a_m})^2} + \frac{b_m - b_n}{a_n - a_m} \tag{26}$$

The second summand can be written as

$$\frac{b_m - b_n}{a_n - a_m} = \frac{h_m^{2-\frac{2}{1+\alpha}} - h_n^{2-\frac{2}{1+\alpha}}}{h_n^{-\frac{2}{1+\alpha}} - h_m^{-\frac{2}{1+\alpha}}} = \frac{h_m^{2-\frac{2}{1+\alpha}} - h_n^{2-\frac{2}{1+\alpha}}}{h_n^{-\frac{2}{1+\alpha}}\left(1 - \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}}\right)} \tag{27}$$

$$= \frac{h_m^2 \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}} - h_n^2}{1 - \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}}} = \frac{h_n^2 \left(\left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}-2} - 1\right)}{1 - \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}}} = h_n^2 \frac{h_{nm}^{-\alpha} - 1}{1 - h_{nm}}. \tag{28}$$

For any $\alpha \geq 1$, we have $h_{nm} = (h_n/h_m)^{2/(1+\alpha)} < 1$ if $h_n < h_m$ and $h_{nm} > 1$ if $h_m > h_n$. In both cases (28) is positive, which implies a radius $r_{nm} > 0$ whenever $\mathbf{p}_n \neq \mathbf{p}_m$. Hence, in this case the radius-square is

$$r_{nm}^2 = \frac{h_{nm}}{(1 - h_{nm})^2} \|\mathbf{p}_n - \mathbf{p}_m\|^2 + h_n^2 \frac{h_{nm}^{-\alpha} - 1}{1 - h_{nm}} \tag{29}$$

■

Example 1. We plotted in Figure 1 for $N = 2$ and $\Omega = [0, 1]^2$ the GT cells for a uniform distribution with UAVs placed on

$$\mathbf{p}_1 = (0.1, 0.2), h_1 = 0.5, \quad \text{and} \quad \mathbf{p}_2 = (0.6, 0.6), h_2 = 1 \tag{30}$$

If the second UAV reaches an altitude of $h_2 \geq 2.3$ its Voroni cell $\mathcal{V}_2 = \mathcal{V}_{2,1}$ will be empty and hence become inactive.

3.1 Local optimality conditions for arbitrary heights

To find the global optimal deployment for N static UAVs we have to minimize the average power consumption for all GTs in Ω over its locational-parameter set $\mathcal{Q}^N$, i.e., we have to solve the following non-convex $N-$*facility locational-parameter optimization problem* [20]

$$\min_{\mathbf{Q} \in \mathcal{Q}^N} \bar{P}(\mathbf{Q}) = \min_{\mathbf{P} \in \Omega^N, \mathbf{h} \in \mathbb{R}_+^N} \sum_{n=1}^{N} \int_{\mathcal{V}_n(\mathbf{P},\mathbf{h})} h_n^{-1} (\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^\gamma \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{31}$$

where $\mathcal{V}_n(\mathbf{p}, h)$ are the Möbius regions given in (15) for each fixed $\mathbf{q} = (\mathbf{p}, h)$.

Proposition 1. *Let* $\alpha \geq 1$*. The point* $\mathbf{Q}^* = (\mathbf{P}^*, \mathbf{h}^*)$ *with generalized Voronoi tessellation* $\mathcal{V}^* = \mathcal{V}(\mathbf{p}^*, \mathbf{h}^*) = (\mathcal{V}_1^*, \dots, \mathcal{V}_N^*)$ *given by Lemma 1 is a critical point of* (31) *if and only if for each* $n \in [N]$ *it holds*

$$0 = \int_{\mathcal{V}_n^*} (\mathbf{p}_n^* - \boldsymbol{\omega})(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\frac{\alpha-1}{2}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{32}$$

$$0 = \int_{\mathcal{V}_n^*} (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\frac{\alpha-1}{2}} \cdot (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 - \alpha h_n^{*2}) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{33}$$

Proof. Since the power function

$$P(\boldsymbol{\omega}, \mathbf{p}, h) = (h^{-\frac{2}{1+\alpha}} \|\mathbf{p} - \boldsymbol{\omega}\|^2 + h^{\frac{2\alpha}{1+\alpha}})^{\frac{\alpha+1}{2}} \tag{34}$$

is a polynomial in $\boldsymbol{\omega}$ of degree less than $1+\alpha$ for each fixed $\mathbf{q} = (\mathbf{p}, h)$, the average distortion function is continuous differentiable, and we obtain by [20, Thm.1] for the partial derivatives

$$\frac{\partial \bar{P}(\mathbf{Q})}{\partial q_{n,i}} = \int_{\mathcal{V}_n(\mathbf{Q})} \frac{\partial P(\boldsymbol{\omega}, \mathbf{q})}{\partial q_{n,i}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \quad , \quad i \in \{1,2,3\}, n \in \{1,2,\dots,N\}. \tag{35}$$

Hence, $\mathbf{Q}^* = (\mathbf{P}^*, \mathbf{h}^*)$ is a critical point if and only if all partial derivatives vanish

$$\begin{aligned} 0 \stackrel{!}{=} \nabla_n \bar{P}(\mathbf{Q}^*) &= \begin{pmatrix} \int_{\mathcal{V}_n} h_n^{*-1} \gamma 2(\mathbf{p}_n^* - \boldsymbol{\omega})(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\gamma-1} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \int_{\mathcal{V}_n} \left[-h_n^{*-2}(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\gamma} + 2\gamma(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\gamma-1}\right] \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{pmatrix} \\ \Leftrightarrow 0 &= \begin{pmatrix} \int_{\mathcal{V}_n} (\mathbf{p}_n^* - \boldsymbol{\omega})(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\gamma-1} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \int_{\mathcal{V}_n} (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\gamma-1} \cdot \left(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2} - 2\gamma h_n^{*2}\right) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{pmatrix} \\ \Leftrightarrow 0 &= \begin{pmatrix} \int_{\mathcal{V}_n} (\mathbf{p}_n^* - \boldsymbol{\omega})(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\frac{\alpha-1}{2}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \int_{\mathcal{V}_n} (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\frac{\alpha-1}{2}} \cdot (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 - \alpha h_n^{*2}) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{pmatrix} \end{aligned} \tag{36}$$

■

Remark. For $\alpha = 1$, the optimal ground user tessellation of the UAVs is by (32) centroidal. However, the shape of the user cells depend on the heights, which are different and therefore the cells are spherical and not polyhedral.

3.2 Optimality conditions for common heights

If $h_n = h > 0$ for all n, then the locational-parameter problem becomes

$$\min_{\mathbf{P} \in \Omega^N, h \in \mathbb{R}_+} \bar{P}(\mathbf{P}, h) \tag{37}$$

Since the minimization of the distortion functions in (13) does not depend on a **common** flight height, the Voronoi diagram will be independent of h, in which case the Möbius diagram becomes the ordinary Euclidean Voronoi diagram by Lemma 1. Hence, for all h the Voronoi diagram is $\mathcal{V}(\mathbf{P}) = \mathcal{V}(\mathbf{P}, h)$, i.e., independent of h. This allows to separate the optimization in (37) in an optimization over the ground positions and an optimization over the common flight height $h > 0$. An optimal height is then given if the gradient vanishes

$$\begin{aligned} \frac{\partial \bar{P}(\mathbf{P}, h)}{\partial h} &= \sum_n \int_{\mathcal{V}_n} \left[-h^{-2}(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h^2)^{\gamma} + 2\gamma(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h^2)^{\gamma-1}\right] \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \Leftrightarrow \quad 0 &= \sum_n \int_{\mathcal{V}_n(\mathbf{P})} (\|\boldsymbol{\omega} - \mathbf{p}_n\|^2 + h^2)^{\gamma-1} \cdot ((2\gamma - 1)h^2 - \|\boldsymbol{\omega} - \mathbf{p}_n\|^2) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{aligned} \tag{38}$$

Hence we have shown

Corollary 1. *The optimization problem*

$$\min_{h \in \mathbb{R}_+} \bar{P}(\mathbf{P}, h) \tag{39}$$

has for each fixed ground positions $\mathbf{P} = (\mathbf{p}_1^T, \dots, \mathbf{p}_N^T) \in \Omega^N$ *and* $\alpha \geq 1$ *a local optimal common flight height* h^**, if it holds*

$$h^* = \sqrt{\frac{\sum_n \int_{\mathcal{V}_n} \|\boldsymbol{\omega} - \mathbf{p}_n\|^2 \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}}{\alpha \sum_n \int_{\mathcal{V}_n} (\|\boldsymbol{\omega} - \mathbf{p}_n\|^2 + h^{*2})^{\frac{\alpha-1}{2}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}}}. \tag{40}$$

If $\alpha = 1$ *then this reduces to the global optimal flight height*

$$h^* = \sqrt{\sum_n \int_{\mathcal{V}_n} \|\boldsymbol{\omega} - \mathbf{p}_n\|^2 \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}}. \tag{41}$$

Remark. To find the optimal ground positions for a common flight height $h_n = h$ and $\alpha > 1$ was investigated in [24, Sec.III]. The authors could show asymptotic results for $N \to \infty$ and the one-dimensional case, which reduces to the uniform quantizer $\mathbf{P}^* = (1/2N, 3/2N, \dots, 2N - 1/2N)$. However, they did not considered the optimal flight height for a given ground position.

4 The Optimal UAV deployment over one-dimensional ground

In this section, we discuss the optimal UAV deployment when users are placed on a one-dimensional ground and restrict them to a convex target area, i.e., to an interval $\Omega = [s, t]$. Under such circumstances, the UAVs ground positions are degenerated to scalars, i.e., $p_n \in [s, t]$, $\forall\ n \in \{1, \dots, N\}$. If we shift the target area Ω by an arbitrary number $a \in \mathbb{R}$, then the power consumption, i.e., the objective function, will not change if we shift all UAV ground positions by the same number a. Hence, if we set $a = -s$ we can shift any deployment $[s, t]$ to $[0, A]$ where $A = t - s$ without changing the average power consumption. Hence we will only consider here target areas of the form $[0, A]$ with $A > 0$.

4.1 Optimal UAV deployment for common height and uniform user density

If $d = 1$ the ground users will be located on a line and we have $\mathbf{p}_n = x_n \in \mathbb{R}$ for each n. Let us assume a uniform distribution on Ω, i.e. $\lambda(\omega) = 1/\mu(\Omega)$ where μ is the Lebesgue measure and $0 < A = \mu(\Omega) < \infty$. To show the existence of a unique global deployment and to derive the optimal height we need to derive the minimum of a integral function.

Lemma 2. *The function*

$$F(h, \gamma) = \int_0^1 f(h, \omega, \gamma) d\omega \quad \textit{with} \quad f(h, \omega, \gamma) = \frac{(\omega^2 + h^2)^\gamma}{h} \tag{42}$$

is for any fixed $\gamma \geq 1$ continuous and convex over $\mathbb{R}_+$ and obtains for

$$g(\gamma) = \arg\min_{h>0} F(h,\gamma), \tag{43}$$

the global minimum, which can be derived in closed form for integer valued γ as

$$g(1) = \frac{1}{\sqrt{3}}, \quad g(3) = \frac{\sqrt{\sqrt{32/5}-1}}{3}, \quad g(5) = \sqrt{\frac{(\frac{32}{7})^{1/3}-1}{5}}. \tag{44}$$

Proof. Surprisingly, it can be shown that for each $\gamma \geq 1$ the integral kernel f is a convex function in $\mathbf{x} = (h,\omega)$ over $\mathbb{R}^2_+$. Let us rewrite f as

$$f(h,\omega,\gamma) = \frac{\|(\omega,h)\|_2^{2\gamma}}{h}. \tag{45}$$

Clearly, $\|\mathbf{x}\|_2$ is a convex and continuous function in $\mathbf{x}$ over $\mathbb{R}^2$ and since $(\cdot)^{2\gamma}$ with $2\gamma \geq 2$ is a strictly increasing continuous function the concatenation $f(\mathbf{x},\gamma)$ is a strict convex and continuous function over $\mathbb{R}^2_+$. Hence, for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2$ we have

$$\|\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2\|_2^{2\gamma} < \lambda \|\mathbf{x}_1\|_2^{2\gamma} + (1-\lambda)\|\mathbf{x}_2\|_2^{2\gamma} \tag{46}$$

for all $\lambda \in (0,1)$. But then we have also for any $h_1, h_2 \in \mathbb{R}^2_+$ and $\omega \geq 0$

$$f(\lambda h_1 + (1-\lambda)h_2, \omega, \gamma) < \frac{\lambda\|(\omega,h_1)\|_2^{2\gamma} + (1-\lambda)\|(\omega,h_2)\|_2^{2\gamma}}{\lambda h_1 + (1-\lambda)h_2}. \tag{47}$$

Since it holds

$$\frac{1}{h_1}+\frac{1}{h_2} = \left(\frac{1}{h_1}+\frac{1}{h_2}\right)\frac{\lambda h_1+(1-\lambda)h_2}{\lambda h_1+(1-\lambda)h_2} = \frac{\left(\lambda+\frac{(1-\lambda)h_2}{h_1}+(1-\lambda)+\frac{\lambda h_1}{h_2}\right)}{\lambda h_1+(1-\lambda)h_2} > \frac{1}{\lambda h_1+(1-\lambda)h_2}$$

we get with (47)

$$f(\lambda h_1 + (1-\lambda)h_2, \omega, \gamma) < \lambda f(h_1,\omega,\gamma) + (1-\lambda)f(h_2,\omega,\gamma) \tag{48}$$

for each $\lambda \in (0,1)$. Hence the integral kernel is strict convex for every $\omega \geq 0, \gamma \geq 1$, and since the infinite sum (integral) of convex functions is again a convex function for $h > 0$, we have shown convexity of $F(h,\gamma)$. Note, $f(h,\omega,\gamma)$ is continuous in $\mathbb{R}^2_+$ since it is a product of the continuous functions $\|(h,\omega)\|_2^{2\gamma}$ and $1/(h + 0\cdot\omega)$, and so is $F(h,\gamma)$. Therefore, the only critical point of $F(\cdot,\gamma)$ will be the unique global minimizer, which is defined by the vanishing of the first derivative:

$$\begin{aligned} F'(h) &= \int_0^1 2\gamma(\omega^2+h^2)^{\gamma-1} - h^{-2}(\omega^2+h^2)^\gamma d\omega d\omega \qquad (49) \\ &= \int_0^1 (\omega^2+h^2)^{\gamma-1}((2\gamma-1) - h^{-2}\omega^2)d\omega = \frac{1}{h^\gamma}\int_0^{\frac{1}{h}} (\omega^2+1)^{\gamma-1}(2\gamma-1-\omega^2). \end{aligned}$$

Hence, $F'(h)$ can only vanish if $h < 1/\sqrt{2\gamma - 1}$, which is an upper bound on $g(\gamma)$. Let us set $0 < x = h^2$ in (49), then we get for $\gamma \in \mathbb{N}$ a polynomial in ω of degree 2γ and in x of degree γ. The integral does not have a closed form for arbitrary γ. However, we can calculate by hand the integral if $\alpha = 2\gamma - 1$ is integer valued, since the kernel becomes a polynomial. For $\gamma \in \{1, 2, 3\}$ we get

$$(\omega^2 - 1x)(\omega^2 + x)^0 = \omega^2 - x \tag{50}$$
$$(\omega^2 - 3x)(\omega^2 + x)^1 = \omega^4 - 2\omega^2 x - 3x^2 \tag{51}$$
$$(\omega^2 - 5x)(\omega^2 + x)^2 = \omega^6 - 3\omega^4 x - 9\omega^2 x^2 - 5x^3 \tag{52}$$

which yield with the definite integrals to

$$0 = \omega(\frac{\omega^2}{3} - x)\Big|_{\omega=1} \tag{53}$$
$$0 = \omega(\frac{\omega^4}{5} - \frac{2\omega^2 x}{3} - 3x^2)\Big|_{\omega=1} \tag{54}$$
$$0 = \omega(\frac{\omega^6}{7} - \frac{3\omega^4 x}{5} - 3\omega^2 x^2 - 5x^3)\Big|_{\omega=1} \tag{55}$$

Solving the first for x yields to the only feasible solution

$$x = \frac{1}{3} \quad \Rightarrow \quad g(1) = \frac{1}{\sqrt{3}} \approx 0.577. \tag{56}$$

The second yields for x to

$$x_{1,2} = -\frac{1}{9} \pm \sqrt{\frac{1}{81} + \frac{1}{15}} = \frac{\pm\sqrt{32/5} - 1}{9} \tag{57}$$

Since only positive roots are allowed we get as only feasible solution

$$g(3) = \frac{\sqrt{\sqrt{32/5} - 1}}{3} \approx 0.412. \tag{58}$$

Finally, for $\alpha = 5$ we get the cubic equation (55)

$$5x^3 + 3x^2 + \frac{3}{5}x - \frac{1}{7} = 0 \tag{59}$$

Then the discriminant is given by, see for example [25, p. 2.3.2],

$$\Delta = q^2 + 4p^3 \quad \text{with} \quad q = \frac{2b^3 - 9abc + 27a^2 d}{27a^3}, p = \frac{3ac - b^2}{9a^2} \tag{60}$$

Let us identify $a = 5, b = 3, c = 3/5$ and $d = -1/7$, then we get

$$q = \frac{6 \cdot 9 - 9 \cdot 9 - 27 \cdot 5^2 \cdot 1/7}{27 \cdot 5^3} = -\frac{3}{3 \cdot 5 \cdot 25} - \frac{1}{5 \cdot 7} = -\frac{32}{25 \cdot 35} \tag{61}$$
$$\Delta = q^2 + 4\left(\frac{3 \cdot 3 - 9}{9 \cdot 5^2}\right)^3 = q^2 > 0 \tag{62}$$

which indicates only one real-valued root, given by

$$x = \alpha_+^{1/3} + \alpha_-^{1/3} - \frac{b}{3a} \quad \text{with} \quad \alpha_\pm = \frac{-q \pm \sqrt{\Delta}}{2} = \{0, \frac{32}{25 \cdot 35}\} \tag{63}$$

which computes to

$$x = \left(\frac{32}{5^3 \cdot 7}\right)^{1/3} - \frac{1}{5} = \frac{(\frac{32}{7})^{1/3} - 1}{5} \Rightarrow g(5) = \sqrt{\frac{(\frac{32}{7})^{1/3} - 1}{5}} \approx 0.363. \tag{64}$$

■

Remark. The convexity of F can be also shown by using extensions of the Hermite-Hadamard inequality, as for example in [26], which allows to show convexity over any interval. This suggest, that as long as the generalized Voronoi cells remain convex, the objective function might be still convex.

Theorem 1. *Let $N \in \mathbb{N}$ and $\Omega = [0, A]$ for some $A > 0$. The* unique global optimal deployment *for N UAVs located at $\mathbf{q}_n^* = (x_n^*, h^*) \in \Omega \times \mathbb{R}_+$ with path loss $\alpha \geq 1$, for a common flight height and a uniform ground user density is given by*

$$h^* = \frac{A}{2N} g((1+\alpha)/2) \quad , \quad \forall\, n\colon x_n^* = \frac{A}{2N}(2n-1) \tag{65}$$

with minimal average power consumption

$$\bar{P}(\mathbf{x}^*, h^*) = \left(\frac{A}{2N}\right)^\alpha \int_0^1 \frac{(\omega^2 + g^2((1+\alpha)/2))^{\frac{1+\alpha}{2}}}{g((1+\alpha)/2)} d\omega. \tag{66}$$

For $\alpha \in \{1, 3, 5\}$ we can derive $g((1+\alpha)/2)$ in closed form (44).

Proof. The average-communication power is given for a common height $h = h_n$ by

$$\bar{P}(\mathbf{Q}) = \frac{1}{hA} \sum_n \int_{\mathcal{V}_n(\mathbf{x})} ((x_n - \omega)^2 + h^2)^\gamma d\omega \tag{67}$$

Then the optimal ground positions $x_1, \ldots, x_n^*$ need to satisfy by Proposition 1

$$0 = \int_{\mathcal{V}_n(\mathbf{x}^*)} (x_n^* - \omega)((x_n^* - \omega)^2 + h^2)^{\gamma - 1} d\omega \quad , \quad 1 \leq n \leq N \tag{68}$$

Since the Voronoi regions are intervals given by Lemma 1 as

$$\mathcal{V}_n(\mathbf{x}^*) = [b_{n-1}, b_n] \quad \text{with} \quad b_n = \begin{cases} 0, & n = 0 \\ A, & n = N \\ \frac{x_{n+1}^* + x_n^*}{2}, & \text{else} \end{cases} \tag{69}$$

we get by substituting $\tilde{\omega} = x_n^* - \omega$ for each n

$$0 = -\int_{x_n - b_{n-1}}^{x_n - b_n} (\tilde{\omega}^2 + h^2)^{\gamma-1} \tilde{\omega} d\tilde{\omega} = \int_{x_n - b_n}^{x_n - b_{n-1}} (\tilde{\omega}^2 + h^2)^{\gamma-1} \tilde{\omega} d\tilde{\omega}. \tag{70}$$

Since the integral kernel $f(\omega) = (\omega^2 + h^2)^{\gamma-1}\omega$ is anti-symmetric in ω it can only vanish if the integral boundaries have different signs, i.e., since $b_{n-1} \leq b_n$ we have

$$0 = \int_{x_n^* - b_n}^{0} f(\omega) d\omega + \int_{0}^{x_n^* - b_{n-1}} f(\omega) d\omega \tag{71}$$

Moreover, we get by the anti-symmetry of f and a substituting by $\tilde{\omega} = -\omega$

$$\int_{x_n^* - b_n}^{0} f(\omega) d\omega = -\int_{x_n^* - b_n}^{0} f(-\omega) d\omega = \int_{0}^{x_n^* - b_n} f(-\omega) d\omega = -\int_{0}^{b_n - x_n^*} f(\tilde{\omega}) d\tilde{\omega} \tag{72}$$

Hence, we get by inserting in (71)

$$\int_{0}^{b_n - x_n^*} f(\omega) d\omega = \int_{0}^{x_n^* - b_{n-1}} f(\omega) d\omega \tag{73}$$

which implies with (69) for $2 \leq n \leq N-1$

$$\frac{d_n^*}{2} = \frac{x_{n+1}^* - x_n^*}{2} b_n - x_n^* = x_n^* - b_{n-1} = \frac{x_n^* - x_{n-1}^*}{2} = \frac{d_{n-1}^*}{2} = \frac{d^*}{2} \tag{74}$$

and for $n = 1$ and $n = N$

$$\frac{d^*}{2} = \frac{d_1^*}{2} = \frac{x_2^* - x_1^*}{2} = b_1 - x_1 * = x_1^* - b_0 = x_1^* = d_0^* \tag{75}$$

$$\frac{d^*}{2} = \frac{d_{N-1}}{2} = \frac{x_N^* - x_{N-1}^*}{2} = x_N^* - b_{N-1} = b_N - x_N^* = A - x_N^* = d_N^* \tag{76}$$

Since the distances must sum to A we get

$$2\frac{d^*}{2} + (N-1)d^* = A \quad \Leftrightarrow \quad d = \frac{A}{N} \tag{77}$$

and hence we obtain the uniform scalar quantizer, given for $n \in [N]$ by

$$x_n^* = \frac{A}{N}(n - 1/2). \tag{78}$$

To obtain the optimal height we need to minimize (67) over $h > 0$ with the uniform quantizer. In fact, for each ground cell we get for the average power by substituting with $\tilde{\omega} = \omega - x_n^*$

$$\bar{P}_n(x_n^*, h) = \frac{1}{hA} \int_{b_{n-1}}^{b_n} (\omega - x_n^*)^2 + h^2)^{\gamma} d\omega = \frac{1}{hA} \int_{b_{n-1} - x_n^*}^{b_n - x_n^*} (\tilde{\omega}^2 + h^2)^{\gamma} d\tilde{\omega} \tag{79}$$

where $b_n = (x^*_{n+1} + x^*_n)/2$ and $b_{n-1} - x^*_n = (x^*_{n-1} - x^*_n)/2 = -(x_{n+1} - x^*_n)/2 = -(b_n - x^*_n)$ which is $A/2N$ by (78) for ever $n \in [N]$ s.t.

$$= \frac{1}{hA} \int_{-A/2N}^{A/2N} (\omega^2 + h^2)^\gamma d\omega = \frac{2}{A} \int_0^{A/2N} \frac{(\omega^2 + h^2)^\gamma}{h} d\omega. \tag{80}$$

By substituting $\tilde{\omega} = N2\omega/A$ and $\tilde{h} = N2h/A$ we can separate A, N from the integral

$$= \int_0^1 \frac{(A^2\tilde{\omega}^2/4N^2 + A^2\tilde{h}^2/4N^2)^{\frac{1+\alpha}{2}}}{A\tilde{h}/2N} \frac{A}{2N} d\tilde{\omega} = \left(\frac{A}{2N}\right)^{1+\alpha} \int_0^1 \frac{(\omega^2 + \tilde{h}^2)^{\frac{1+\alpha}{2}}}{\tilde{h}^2} d\omega.$$

Hence, we get with $g(\alpha)$

$$h^* = \frac{A}{2N} g(\alpha) \quad \text{with} \quad g(\alpha) := \arg\min_{h>0} \int_0^1 \frac{(\omega^2 + h^2)^{\frac{1+\alpha}{2}}}{h} d\omega. \tag{81}$$

■

4.2 Optimal UAV deployment for different heights and uniform user density

To extend the previous results to different heights we need first to verify that the optimal deployment of $N-$UAVs involves only active UAVs.

Lemma 3. *Let $\Omega = [0, A]$ for some $A > 0$. The optimal deployment $\mathbf{Q}^* \in (\Omega \times \mathbb{R}_+)^N$ of N UAVs for a uniform ground user density has optimal ground cells $\mathcal{V}_n(\mathbf{Q}^*) = [b_{n-1}, b_n]$ with $0 \leq b_{n-1} < b_n \leq A$ for $n \in [N]$, i.e., each optimal ground cell has positive measure and is served by exactly one UAV.*

Remark. For any optimal deployment all UAVs are active, which is intuitively, since each UAV antenna should reduces the power if it covers a non-zero user area.

Proof. Although, this statement seems to be trivial, it is not straight forward to show, since the power function P in (9) is not strictly monotone increasing in $\mathbf{q}_n = (p_n, h_n)$ for any user $\omega \in \Omega$ and UAV $n \in [N]$. To bypass this non-monotonicity property of $P(\mathbf{Q}, \omega)$ we will use the quantization relaxation for the average power consumption $\bar{P}$ in (11) to solve the optimal deployment problem (12). We define, as in quantization theory, see for example [27], an $N-$*point quantizer* for Ω, by a (disjoint) partition $\mathcal{R} = \{\mathcal{R}_n\}_{n=1}^N \subset \Omega$ of Ω and assign to each partition cell $\mathcal{R}_n$ a reproduction point $\mathbf{q}_n \in \mathcal{Q} = \Omega \times \mathbb{R}_+$. The assignment rule or *quantization rule* can be anything and does not have to depend on the performance function P or whatsoever. Minimizing over all quantizer, that is, over all partitions and possible reproduction points will yield to the optimal quantizer, which is by definition the optimal deployment (reproduction points) which generate the generalized Voronoi diagram as the optimal partition (tesselation[2]). This holds for any density function

[2]Since we take here the continuous case, the integral will not distinguish between open or closed sets.

$\lambda(\omega)$ and target area Ω. To see this[3] let us start with any quantizer $(\mathbf{Q}, \mathcal{R})$ for Ω yielding to the average power

$$\bar{P}(\mathbf{Q}, \mathcal{R}) = \sum_{n=1}^{N} \int_{\mathcal{R}_n} P(\mathbf{q}_n, \omega)\lambda(\omega)d\omega \geq \sum_{n=1}^{N} \int_{\mathcal{R}_n} \left(\min_{m\in[N]} P(\mathbf{q}_m, \omega) \right) \lambda(\omega)d\omega \qquad (82)$$
$$= \int_{\Omega} \min_{m\in[N]} P(\mathbf{q}_m, \omega)\lambda(\omega)d\omega = \sum_{n} \int_{\mathcal{V}_n(\mathbf{Q})} P(\mathbf{q}_n, \omega)\lambda(\omega)d\omega$$

where the first inequality is only achieved if for any $\omega \in \mathcal{R}_n$ we have chosen $\mathbf{q}_n$ to be the optimal quantization point with respect to P, or vice versa, if every $\mathbf{q}_n$ is optimal for every $\omega \in \mathcal{R}_n$, which is the definition of the generalized Voronoi cell $\mathcal{V}(\mathbf{Q})$. Therefore, minimizing over all partitions gives equality, i.e.

$$\min_{\mathcal{R}} \bar{P}(\mathbf{Q}, \mathcal{R}) = \bar{P}(\mathbf{Q}, \mathcal{V}(\mathbf{Q})) \qquad (83)$$

for any reproduction points $\mathbf{Q} \in \mathcal{Q}^N$. Hence, we have shown that the optimal quantizer problem is equivalent to the optimal deployment problem

$$\min_{\mathbf{Q}\in\mathcal{Q}^N} \min_{\mathcal{R}\in\Omega^N} \bar{P}(\mathbf{Q}, \mathcal{R}) = \min_{\mathbf{Q}\in\mathcal{Q}^N} \bar{P}(\mathbf{Q}, \mathcal{V}(\mathbf{Q})) = \bar{P}(\mathbf{Q}^*, \mathcal{V}(\mathbf{Q}^*)). \qquad (84)$$

We need to show, that for the optimal $N-$UAV deployment $\mathbf{Q}^*$ we have $\mu(\mathcal{V}_n(\mathbf{Q}^*)) > 0$ for all $n \in [N]$. Let us first show that each cell is indeed a closed interval, i.e., $\mathcal{V}_n(\mathbf{Q}^*) = [b_{n-1}, b_n]$ with $0 \leq b_{n-1} \leq b_n \leq A$.

By the definition of the Moebius diagrams in Lemma 1, each dominance region is either one interval (if it is a ball not contained in the target region or a halfspace) or two intervals (if its a ball contained in the target region), we can not have more than $K_n \leq 2N - 2$ disjoint closed intervals for each Moebius (Voronoi) region. Therefore, the nth optimal Voronoi regions is given as $\mathcal{V}_n^* = \mathcal{V}_n(p^*, h^*) = \bigcup_{k=1}^{K_n} v_{n,k}$, where $v_{n,k} = [a_{n,k-1}, a_{n,k}]$ is a continuous interval and $2N - 2 \geq K_n \geq 1$.

Let us assume at least one UAV has a disconnected region, i.e. $K_n > 1$ for some $n \in [N]$. Then we will re-arrange the partition $\mathcal{V}(\mathbf{p}^*, \mathbf{h}^*)$ by concatenating the K_n intervals $v_{n,k}$ to $\mathcal{R}_n = [b_{n-1}, b_n]$ such that for all $n \in [N]$ it holds $\mu(\mathcal{R}_n) = \mu(\mathcal{V}_n^*) = b_n - b_{n-1}$ where we ordered them such that $b_{n-1} \leq b_n$ for all $n \in [N]$ and set $b_0 = 0$ and $b_N = A$. Then we move each p_n^* to the center of the new arranged cells, i.e., $\tilde{p}_n = \frac{b_n + b_{n-1}}{2}$ without changing the associated heights. This defines a new partition $\mathcal{R} = \{\mathcal{R}_n(\tilde{\mathbf{p}}, \mathbf{h}^*)\}$ for the ground positions $\tilde{\mathbf{p}} = (\tilde{p}_1, \dots, \tilde{p}_N)$. The average power consumption is then

$$\bar{P}(\tilde{\mathbf{p}}^*, \mathbf{h}^*, \mathcal{R}) = \sum_{n=1}^{N} \int_{b_{n-1}}^{b_n} \frac{((\tilde{p}_n - \omega)^2 + h_n^{*2})^\gamma}{h_n^*} d\omega = 2 \sum_{n=1}^{N} \int_0^{\frac{b_n - b_{n-1}}{2}} \frac{(\omega^2 + h_n^{*2})^\gamma}{h_n^*} d\omega \quad (85)$$

where we substituted ω by $\tilde{p}_n - \omega$. Since the height is not changing and the function $(\omega^2 + h_n^{*2})^\gamma$ is strictly monotone increasing in ω for each $\gamma > 0$, it holds for the average

[3] We use the same argumentation as in the prove of [28, Prop.1].

power of the nth cell

$$2\int_0^{\frac{b_n-b_{n-1}}{2}} \frac{(\omega^2+h_n^{*2})^\gamma}{h_n^*}d\omega < \sum_{k=1}^{K_n}\int_{a_{n,k}-p_n^*}^{a_{n,k-1}-p_n^*} \frac{(\omega^2+h_n^{*2})^\gamma}{h_n^*}d\omega \tag{86}$$

since the non-zero gaps in $\mathcal{V}_n^*-p_n^*$ will lead to larger ω and therefore to a strictly larger average power consumption, whereas the other cells with $K_n=1$ will not change the average power. Therefore the deployment $(\tilde{\mathbf{p}},\mathbf{h}^*)$ with convex partition sets $\mathcal{R}(\tilde{\mathbf{p}},\mathbf{h}^*)$ has a smaller power consumption which contradicts the assumption that $(\mathbf{p}^*,\mathbf{h}^*)$ is the optimal deployment (84). Hence $K_n=1$ for each $n\in[N]$ and every $\gamma\geq 1$.

Now, we have the optimal deployment $\mathbf{Q}^*$ with $\mathcal{V}_n^*=\{[b_{n-1},b_n]\}_{n=1}^n$. Let us show that for each $n\in[N]$ it also holds $b_{n-1}<b_n$. By (84) we get

$$\min_{\mathcal{R}} \bar{P}(\mathbf{Q}^*,\mathcal{R}) = \min_{\mathcal{R}}\sum_n\int_{\mathcal{R}_n} P(\mathbf{q}_n^*,\boldsymbol{\omega})d\boldsymbol{\omega} = \bar{P}(\mathbf{Q}^*,\mathcal{V}(\mathbf{Q}^*)). \tag{87}$$

If $N=1$ there is nothing to optimize and we have $\mathcal{R}=\mathcal{R}_1=\Omega=[a,b]$. Furthermore, we know by Theorem 1, that the optimal $\mathbf{q}^{(1)*}=(p_1^{(1)*},h_1^{(1)*})$ for $N=1$ is the uniform quantizer given by $p_1^{(1)*}=a+A/2$ with $A=b-a$ and $h^*=h_1^{(1)*}=\operatorname{argmin}_{h>0}\int_0^{A/2}\frac{(\omega^2+h^2)^\gamma}{h}d\omega$. Let us note, that the average power $\bar{P}(p_1^{(1)*},\cdot)$ is convex (second-derivative is always positive) in h and hence there is only one critical point h^* in $\mathbb{R}_+$, which is the global minimum. Now assume, we add one more quantizer $\mathbf{q}_2^{(2)}=(p_2^{(2)},h^*)$ with same height and partitioning Ω by $\mathcal{R}_\epsilon$ given by $R_1=[a,a+\epsilon]$ and $R_2=[a+\epsilon,b]$ for some $\epsilon\in(0,A)$. If we minimize only over the ground positions, we get

$$\bar{P}(\mathbf{p}^{(2)*},h^*,\mathcal{R}_\epsilon) := \min_{p_1}\int_a^{a+\epsilon} P(p_1,h^*,\omega)d\omega + \min_{p_2}\int_{a+\epsilon}^b P(p_2,h^*,\omega)d\omega \tag{88}$$

By Theorem 1 we get for the optimal uniform ground positions for each **independent** integral

$$p_1^{(2)*}=\frac{2a+\epsilon}{2} \quad \text{and} \quad p_2^{(2)*}=\frac{a+b+\epsilon}{2} \tag{89}$$

inserting and substituting with $\tilde{\omega}=\omega-p_i^{(2)*}$ we get with $h^*=\sqrt{c}$

$$\bar{P}(\mathbf{p}^{(2)*},h^*,\mathcal{R}_\epsilon) = \frac{2}{\sqrt{c}}\left(\int_0^{\epsilon/2}(\omega^2+c)^\gamma d\omega + \int_0^{\frac{b-a}{2}-\frac{\epsilon}{2}}(\omega^2+c)^\gamma d\omega\right). \tag{90}$$

Since $(\omega^2+c)^\gamma < ((\omega+\epsilon)^2+c)^\gamma$ for each $\epsilon>0$ and $\gamma>0$ we have

$$< \frac{2}{\sqrt{c}}\left(\int_{\frac{b-a}{2}-\frac{\epsilon}{2}}^{\frac{b-a}{2}}(\omega^2+c)^\gamma d\omega + \int_0^{\frac{b-a}{2}-\frac{\epsilon}{2}}(\omega^2+c)^\gamma d\omega\right) = \bar{P}(\mathbf{q}^{(1)*},[a,b])$$

for any $b-a>\epsilon>0$. By minimizing over the heights, we only obtain smaller values

$$\min_{\mathbf{p}\in\Omega^2,\mathbf{h}\in\mathbb{R}_+^2}\ \min_{\mathcal{R}_\epsilon=\{\mathcal{R}_{\epsilon,n}\}_{n=1}^2}\sum_{n=1}^{2}\int_{R_n}P(p_n,h_n,\omega)d\omega\leq\bar{P}(\mathbf{p}^{(2)*},h^*,\mathcal{R}_\epsilon)<\min_{p\in\Omega,h\in\mathbb{R}+}\bar{P}(p,h,\omega)d\omega$$

Hence, if we add a UAV, we can split an arbitrary ground cell $[a,b]$ with $b>a$ in two cells with positive measure and obtain a strictly smaller power consumption. Therefore, the global minimum can only be attained if all UAVs are active, i.e., all optimal ground cells have positive measure $\mu(\mathcal{R}_n^*)>0$. Since the optimal partition is by (84) a Möbius diagram, each optimal ground cell is given as $\mathcal{R}_n^*=\mathcal{V}_n(\mathbf{q}^*)=[b_{n-1},b_n]$ with $0\leq b_{n-1}<b_n\leq A$. ∎

Now we are now ready to extend Theorem 1 to arbitrary heights.

Theorem 2. *Over the uniform distributed one-dimensional ground $\Omega=[0,A]$ for some $A>0$, the minimum average total power of N UAVs is given by (66), which is only attained for uniform ground positions and common flight heights, given by (65).*

Proof. We can generalize the prove of Theorem 1 to optimal UAV ground positions at different heights by only using the all active property of the optimal UAV deployment from Lemma 3. We also know, that the optimal ground cells are closed non-vanishing intervals, given by $\mathcal{V}_n(\mathbf{q}^*)=[b_{n-1}^*,b_n^*]$ with $b_{n-1}^*<b_n^*$. Hence, for the optimal ground positions x_n^* at the optimal height h_n^* it must hold

$$0=\int_{b_{n-1}^*}^{b_n^*}(x_n^*-\omega)((x_n^*-\omega)^2+{h_n^*}^2)^{\gamma-1}d\omega\quad,\quad 1\leq n\leq N \tag{91}$$

where we get by substituting $\tilde{\omega}=x_n^*-\omega$ for each n

$$0=-\int_{x_n^*-b_{n-1}^*}^{x_n^*-b_n^*}(\tilde{\omega}^2+{h_n^*}^2)^{\gamma-1}\tilde{\omega}d\tilde{\omega}=\int_{x_n^*-b_n^*}^{x_n^*-b_{n-1}^*}(\omega^2+{h_n^*}^2)^{\gamma-1}\omega d\omega \tag{92}$$

Since the integral kernel $f(\omega)=(\omega^2+{h_n^*}^2)^{\gamma-1}\omega$ is anti-symmetric in ω, (92) can only vanish if the integral boundaries have different signs, i.e.,

$$0=\int_{x_n^*-b_n^*}^{0}f(\omega)d\omega+\int_0^{x_n^*-b_{n-1}^*}f(\omega)d\omega\quad\Leftrightarrow\quad\int_0^{b_n^*-x_n^*}f(\omega)d\omega=\int_0^{x_n^*-b_{n-1}^*}f(\omega)d\omega$$

Hence it must hold for $1\leq n\leq N$

$$b_n^*-x_n^*=x_n^*-b_{n-1}^*\Leftrightarrow x_n^*=\frac{b_{n-1}^*+b_n^*}{2} \tag{93}$$

where for $n=1$ and $n=N$ we get $b_0=0$ and $b_N=A$, since $\{[b_{n-1}^*,b_n^*]\}$ is a partition of $[0,A]$. Hence the optimal ground positions are the centroids of the cells. Let us set

$\mu_n = b_n^* - b_{n-1}^*$ for $n \in [N]$, then we get by substituting $\frac{2(x_n^* - \omega)}{\mu_n} = \tilde{\omega}$ and $h_n^* = \tilde{h}_n \mu_n / 2$ in the optimal average power consumption

$$
\begin{aligned}
\bar{P}(\mathbf{p}^*, \mathbf{h}^*) &= \sum_{n=1}^{N} \int_{b_{n-1}^*}^{b_n^*} \frac{((x_n^* - \omega)^2 + h_n^{*2})^\gamma}{h_n^*} \frac{d\omega}{A} = \sum_n \int_1^{-1} - \frac{(\mu_n^2 \tilde{\omega}^2/4 + \tilde{h}_n^2 \mu_n^2/4)^\gamma}{g(\alpha)\mu_n/2} \frac{\mu_n}{2A} \\
&= \frac{2}{A} \sum_n \int_0^1 \frac{(\omega^2 + \tilde{h}_n^2)^\gamma}{\tilde{h}} d\omega \cdot \mu_n^{1+\alpha} 2^{-1-\alpha}
\end{aligned} \tag{94}
$$

where we inserted (93) to get $2(x_n^* - b_{n-1}^*)/\mu_n = 1 = -2(x_n^* - b_n^*)/\mu_n$. Since we do not know what $\tilde{h}_n$ is, we will lower bound over all possible optimal $\tilde{h}_n$, which is certainly the case if we minimize each nth integral over $\tilde{h}_n$

$$
\geq \frac{1}{2^\alpha A} \sum_n \left(\min_{\tilde{h}_n > 0} \int_0^1 \frac{(\omega^2 + \tilde{h}_n^2)^\gamma}{\tilde{h}_n} d\omega \right) \mu_n^{1+\alpha}. \tag{95}
$$

By Theorem 1 we know that the minimum of the user integrals is achieved for $\tilde{h}_n = g(\alpha)$, which leaves only an uncertainty for μ_n. Hence we get

$$
= \frac{1}{2^\alpha A} \int_0^1 \frac{(\omega^2 + g^2(\alpha))^\gamma}{g(\alpha)} d\omega \cdot \sum_n \mu_n^{1+\alpha} \cdot \left(\frac{\sum_n 1^q}{N} \right)^{p/q} \tag{96}
$$

which can be lower bounded by Hölder inequality with $p = 1 + \alpha, q = (1+\alpha)/\alpha$ to

$$
\geq \frac{1}{2^\alpha A} \int_0^1 \frac{(\omega^2 + g^2(\alpha))^\gamma}{g(\alpha)} d\omega \cdot \underbrace{(\sum_n \mu_n)^{1+\alpha}}_{=A^{1+\alpha}} N^{-\alpha} = \left(\frac{A}{2N} \right)^\alpha \int_0^1 \frac{(\omega^2 + g^2(\alpha))^\gamma}{g(\alpha)} d\omega
$$

which can be achieved if and only if $\mu_n = A/N$. But this is the average power for the common height case (66), and hence, the only (global) optimal deployment is the common height deployment with uniform ground positions given in Theorem 1. ■

Let us set $\beta = 1 = A$. Then the optimal UAV deployment is pictured in Figure 2 for $N = 2$ and $N = 4$. The maximal evaluation angle $\theta_{\max}$ is hereby constant for each UAV and does not change if the number of UAVs N increase. Moreover, it is also independent of A and β, since with (65) we have $d_n = x_n^* - x_{n-1}^* = A/N$ and

$$
\cos(\theta_{\max}) = \frac{h^*}{d_n^*/2} = \frac{2N}{A} \frac{A}{2N} g(1) = \frac{1}{\sqrt{3}}. \tag{97}
$$

4.3 Deployment for non-uniform user density

Note that a great number of literature assumes the common flight height for convenience when they derive the optimal UAV deployment or trajectory. However, in this section, we show via an example that the common flight height is not a necessary

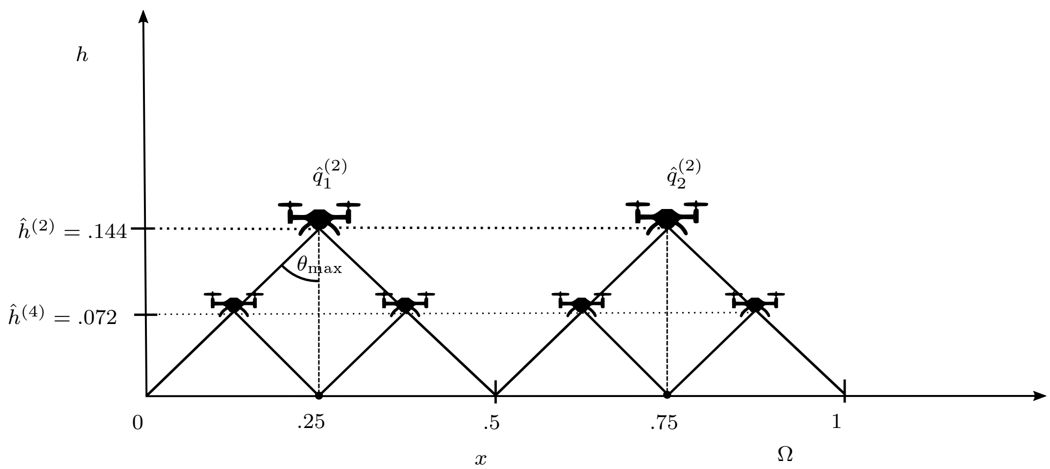

Figure 2: UAV deployment in one dimension for $\Omega = [0, 1]$ for $\alpha = 1$ and $N = 2, 4$ for a uniform GT distribution by (65).

condition for the optimal deployment over the space including but not limited to the non-uniform distributed one-dimensional ground.

In what follows, we provide an example to compare the best deployment with common flight height and an alternative deployment with different flight heights. Consider two UAVs over a one-dimensional ground $[0, 1]$ with path loss parameter $\gamma = 1$ and a non-uniform density function

$$\lambda(\omega) = \begin{cases} 4 & \text{if } 0 \leq \omega \leq 0.2 \\ \frac{1}{4} & \text{if } 0.2 < \omega \leq 1 \\ 0 & \textit{otherwise} \end{cases} \tag{98}$$

Let $q_1 = (p_1, h_1)$ and $q_2 = (p_2, h_2)$ be the deployment of UAV 1 and 2. First, we derive the best deployment with common flight height assumption, i.e., $h_1 = h_2 = h$. Since the common flight height, the generalized Voronoi partition is degenerated to Voronoi partition, $V_n(p.h) = \{\omega | \|\omega - p_n\| \leq \|\omega - p_m\|, \ \forall \ m \neq n\}$. Without loss of generality, we assume $0 < p_1 < p_2 < 1$, and thus the Voronoi partition can be represented as $V_1(p, h) = [0, b]$ and $V_2(p, h) = [b, 1]$, where the boundary is

$$b = \frac{p_1 + p_2}{2}. \tag{99}$$

(i) If $b \leq 0.2$, the distortions can be rewritten as $D(p, h) = 4\int_0^b \frac{(p_1-\omega)^2+h^2}{h} + 4\int_b^{0.2} \frac{(p_2-\omega)^2+h^2}{h} + \frac{1}{4}\int_{0.2}^1 \frac{(p_2-\omega)^2+h^2}{h}$. The best deployment satisfies zero-gradient, i.e.,

$$\frac{\partial D}{\partial p_1} = 4\int_0^b 2\frac{(p_1-\omega)}{h} d\omega = 0 \tag{100}$$

$$\frac{\partial D}{\partial p_2} = 4\int_b^{0.2} 2\frac{(p_2-\omega)}{h} d\omega + \frac{1}{4}\int_{0.2}^1 2\frac{(p_2-\omega)}{h} d\omega = 0 \tag{101}$$

$$\frac{\partial D}{\partial h} = 4\int_0^b \frac{h^2-(p_1-\omega)^2}{h^2} d\omega + 4\int_b^{0.2} \frac{h^2-(p_2-\omega)^2}{h^2} d\omega + \frac{1}{4}\int_{0.2}^1 \frac{h^2-(p_2-\omega)^2}{h^2} d\omega = 0$$

Solving (99), (100), and (101), we get a quadratic function $8b^2 - 3b + 0.4 = 0$ which has no real root. Therefore, $b \le 0.2$ is not feasible.

(ii) If $b > 0.2$, the distortions can be rewritten as $D(p,h) = 4\int_0^{0.2} \frac{(p_1-\omega)^2+h^2}{h} + \frac{1}{4}\int_{0.2}^{b} \frac{(p_1-\omega)^2+h^2}{h} + \frac{1}{4}\int_b^1 \frac{(p_2-\omega)^2+h^2}{h}$. The best deployment satisfies zero-gradient, i.e.,

$$\frac{\partial D}{\partial p_1} = 4\int_0^{0.2} 2\frac{(p_1-\omega)}{h}d\omega \frac{1}{4}\int_{0.2}^{b} 2\frac{(p_1-\omega)}{h}d\omega = 0 \tag{102}$$

$$\frac{\partial D}{\partial p_2} = 4\int_b^1 2\frac{(p_2-\omega)}{h}d\omega = 0 \tag{103}$$

$$\frac{\partial D}{\partial h} = 4\int_0^{0.2} \frac{h^2-(p_1-\omega)^2}{h^2}d\omega + \frac{1}{4}\int_{0.2}^{b} \frac{h^2-(p_1-\omega)^2}{h^2}d\omega + \frac{1}{4}\int_b^1 \frac{h^2-(p_2-\omega)^2}{h^2}d\omega = 0 \tag{104}$$

Solving (99), (102), (103), and (104), we get $p_1^* = \frac{3\sqrt{5.8}-7}{2}$, $p_2^* = \frac{\sqrt{5.8}-1}{2}$, and $h^* = 0.096$. Hence, the minimum distortion with the common flight height is $D(p^*, h^*) = 0.247$. Next, let $\widetilde{p} = p^* = (p_1^*, p_2^*)$ and $\widetilde{h} = (0.05, 0.2)$ be an alternative deployment. By straightforward calculation, we get the corresponding distortion $D(\widetilde{p}, \widetilde{h}) = 0.192$ which is smaller than that of $D(p^*, h^*)$. In sum, the common flight height is not a necessary condition for the optimal deployment. According to our simulations, there is a big performance gap between the "optimal" deployment with common flight height and the real optimal deployment.

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