Minimal Average Distortion of non-convex distortion functions for optimal UAV deployments

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1 Introduction

In this report, we study the optimal deployment of multiple Unmanned Aerial Vehicles (UAVs) with directional antennas. Similar to [1], we consider a wireless network where several ground terminals (GTs) using variable transmission power and fixed data rate. In the recent decade, UAV with direction antenna has been widely studied in literature [2]–[7]. In [2]–[7], the antenna gain within 3dB beamwidth is approximated by a constant. The authors claimed that this approximation comes from the book [8]. Unfortunately, I cannot get this reference (fourth edition). But, I have the second edition of this book [9]. In [9], the maximum antenna (the gain at 0^o) is approximated by the constant that is mentioned in [2]–[7]. Moreover, a cosine-shaped antenna gain is proposed in [9]. Even if antenna gain is approximated as a constant in [8], cosine function seems be a better approximation than a constant. Therefore, I formulate UAV's antenna gain as a cosine function. On the other hand, the existing studies [1] made a common assumption that all UAVs have the identical height. In this report, we will study the UAVs with variant heights which is more reasonable in practice.

Notation We denote the positive integers by $\mathbb{N} = \{1, 2, 3, \dots\}$ and for $N \in \mathbb{N}$ the first N positive integers by $[N] = \{1, 2, \dots, N\}$. We will denote by $\mathbf{x} \in \mathbb{R}^d$ a $d-$dimensional real-valued row vector $\mathbf{x} = (x_1, \dots, x_d)$ and by bold capital letters a matrix $\mathbf{X} = \mathbb{R}^{d\times N}$. We denote by $(\cdot)^T$ the transpose of a vector or matrix. For convenience, let $B(\mathbf{c}, r) = \{\boldsymbol{\omega} \mid \|\boldsymbol{\omega} - \mathbf{c}\|^2 \le r\}$ be the open ball in $\mathbb{R}^d$ centered at $\mathbf{c} \in \mathbb{R}^d$ with radius $r \ge 0$. Note that $B(\mathbf{c}, 0)$ is an empty set. We denote by ∂V the boundary of the set $V \subset \mathbb{R}^d$ and by V^c its complement. In particular, $\partial B(\mathbf{c}, r)$ is the sphere with center at $\mathbf{c}$ and radius r. We will denote the positive numbers by $\mathbb{R}_+ := \{a \in \mathbb{R} \mid a > 0\}$. Let equation $E\mathbf{q} + F = 0$, where $E \in \mathbb{R}^{1\times 2}$ is a 1×2 vector and $F \in \mathbb{R}$ is a scalar, define the perpendicular bisector hyperplane between the two points. Then, the equations $E\mathbf{q} + F \ge 0$ and $E\mathbf{q} + F \le 0$ define two half spaces. Moreover, for two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, we denote the generated half space between them, which contains $\mathbf{a} \in \mathbb{R}^d$ by $HS(\mathbf{a}, \mathbf{b})$.

2 System model

We consider a network of several GTs at zero elevation on the target region $\Omega \subset \mathbb{R}^2$ and N low-altitude UAVs in the sky. Let $\mathbf{P} = (\mathbf{p}_1^T, \ldots, \mathbf{p}_N^T)$ be the UAV (ground) horizontal deployment, where $\mathbf{p}_n = (x_n, y_n) \in \mathbb{R}^2$ is the projected location on the ground Ω, and $\mathbf{h} = (h_1, \ldots, h_N)$ be the UAV vertical (height) deployment, where $h_n \in \mathbb{R}_+$ is UAV n's height. In particular, UAV n's 3-dimensional location is defined as $\mathbf{q}_n = (\mathbf{p}_n, h_n) = (x_n, y_n, h_n)$. The elevation angle, $\theta \in [0, \frac{\pi}{2}]$, from a GT at $\boldsymbol{\omega}$ to UAV n is defined as the angle between the ground distance $\overline{\boldsymbol{\omega}\mathbf{p}_n}$ to the flight height h_n.

As Koyuncu et.al. [1], we consider a fixed-rate variable-power transmission scenario where a GT at location ω wishes to communicate with bit-rate R_b in bits/s, and transmits with power P_{TX} in J/s.

To capture the power falloff versus distance along with the random attenuation about the path loss from shadowing, we adopt the propagation model [10, (2.51)] as:

$$PL_{dB} = 10 \log_{10} K - \underbrace{10\alpha \log_{10} \frac{d}{d_0}}_{\text{terrestrial path-loss}} - \psi_{dB}, \tag{1}$$

where K is a unitless constant depending on the antenna characteristics, d_0 is a reference distance, $\alpha \geq 1$ is the terrestrial path loss exponent, and ψ_{dB} is a Gaussian random variable following $\mathcal{N}\left(0, \sigma^2_{\psi_{dB}}\right)$. This air-to-ground or terrestrial path loss model is widely used for UAV basestation path-loss models [11]. Practical values are between 2 and 6 and depends on the distance d and the shadowing due to low altitude. For common practical measurements see for example [12]. Typically maximal altitudes for UAV are < 1000m, due to flight zone restrictions of aircrafts. Therefore, the received power at UAV n can be represented as

$$P_{RX} = P_{TX} G_{TX} G_{RX} PL = \frac{P_{TX} G_{TX} G_{RX} K d_0^{\alpha}}{d^{\alpha} 10^{\frac{\psi_{dB}}{10}}}, \tag{2}$$

where

$$d((\mathbf{p}, h), (\boldsymbol{\omega}, 0)) = \sqrt{\|\mathbf{p} - \boldsymbol{\omega}\|^2 + h^2} = \sqrt{(p_x - \omega_x)^2 + (p_y - \omega_y)^2 + h^2} \tag{3}$$

is the Euclidean distance between the UAV at $\mathbf{q} = (\mathbf{p}, h)$ and the ground terminal at $\boldsymbol{\omega} = (\omega_x, \omega_y)$ and G_{TX} and G_{RX} are the antenna gains of the transmitter and the receiver, respectively. In this report, we focus on omnidirectional transmitter (GT) antennas and directional receiver (UAV) antennas. The antenna gains are formulated as the positive real numbers

$$G_{\mathrm{GT}} > 0 \quad , \quad G_{\mathrm{UAV}} = \cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{h}{d((p, h), (\omega, 0))}. \tag{4}$$

see [13, pp.52]. We model here the GT antennas as perfect omni-directional, with an isotropic gain. The combined antenna intensity is then proportional to

$$G = G_{\mathrm{UAV}} G_{\mathrm{GT}} K = \frac{h G_{\mathrm{GT}} K}{d((p, h), (\omega, 0))} \tag{5}$$

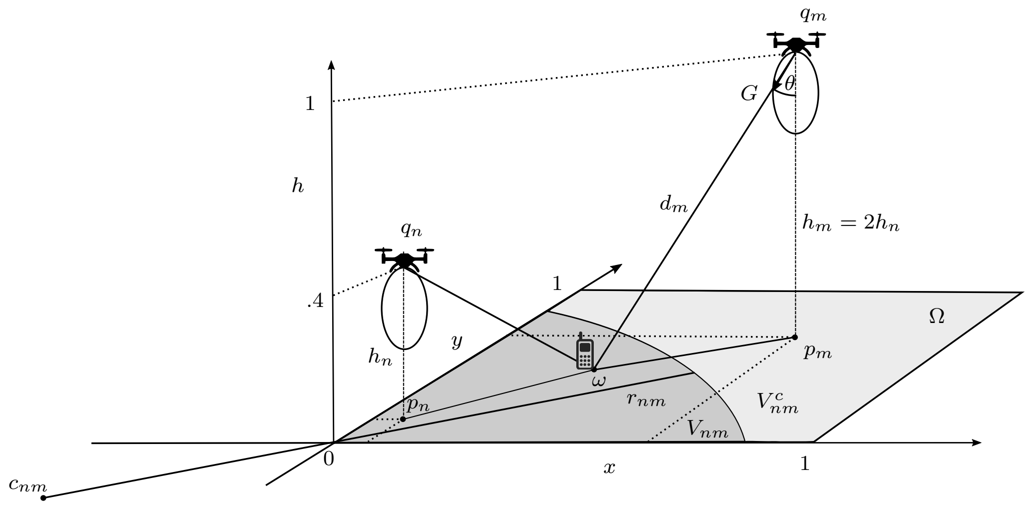

Figure 1: UAV deployment with directed antenna beam for $\alpha = 2$ and $N = 2$ for a uniform GT distribution.

see Figure 1. Accordingly, the received power (2) can be rewritten as

$$P_{RX} = \frac{P_{TX} h G_{\mathrm{GT}} K d_0^\alpha}{d((p,h),(\omega,0))^{\alpha+1} 10^{\frac{\psi_{dB}}{10}}}. \tag{6}$$

To achieve the reliable communication between GT and UAV with capacity (bit-data-rate) at least R_b, we have

$$B \log_2 \left(1 + \frac{P_{RX}}{N_0}\right) \geq R_b, \tag{7}$$

where B is the channel bandwidth and N_0 is the noise power. The minimum transmission power is then given by

$$P_{TX} = \frac{\left(2^{\frac{R_b}{B}} - 1\right) N_0 d((p,h),(\omega,0))^{\alpha+1} 10^{\frac{\psi_{dB}}{10}}}{h G_{\mathrm{GT}} K d_0^\alpha}. \tag{8}$$

The expectation of the minimum transmitter power of the nth UAV or user in the nth cell is then

$$\begin{aligned}
\mathbb{E}[P_{TX,n}] &= \frac{\left(2^{\frac{R_b}{B}} - 1\right) d((p_n,h_n),(\omega,0))^{\alpha+1}}{h_n G_{\mathrm{GT}} K d_0^\alpha} \frac{N_0}{\sqrt{2\pi}\sigma_{\psi_{dB}}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\psi_{dB}^2}{2\sigma_{\psi_{dB}}^2} + \ln(10)\frac{\psi_{dB}}{10}\right) d\psi_{dB} \\
&= \frac{\left(2^{\frac{R_b}{B}} - 1\right) d((p_n,h_n),(\omega,0))^{\alpha+1} N_0 \exp\left(-\frac{\sigma_{\psi_{dB}}^2 (\ln 10)^2}{200}\right)}{h_n G_{\mathrm{GT}} K d_0^\alpha} \\
&= \beta \cdot \frac{d((p_n,h_n),(\omega,0))^{2\gamma}}{h_n} =: P(\omega, p_n, h_n)
\end{aligned} \tag{9}$$

where the fixed parameters

$$\beta = \frac{(2^{\frac{R_b}{B}} - 1)N_0 \exp(-\frac{\sigma^2_{\psi_{dB}}(\ln 10)^2}{200})}{G_{\mathrm{GT}} K d_0^\alpha} \quad , \quad \gamma = \frac{\alpha+1}{2} \tag{10}$$

are independent of the UAVs and will be chosen by design.

3 Optimal Static UAV deployment

To define an optimization problem over the UAVs we will need a *single-objective function*, depending on the N UAVs ground positions $\mathbf{P} = (\mathbf{p}_1^T, \mathbf{p}_2^T, \dots, \mathbf{p}_N^T)$ and flight heights $\mathbf{h} = (h_1, h_2, \dots, h_N)$. We will define the average power over all ground terminals (GTs) distributed by λ in Ω as

$$\bar{P}(\mathbf{P}, \mathbf{h}) = \int_\Omega \min_n \{P(\boldsymbol{\omega}, \mathbf{p}_n, h_n)\} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}, \tag{11}$$

where we assume that each GT at ω will chose the UAV which requires the lowest transmit power. Here we assume a *continuous distribution* of GTs, given by $\lambda : \Omega \to [0, 1]$ with $\int_\Omega \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} = 1$, as used in [14]–[17]. The power function P can be interpret as a "distance" or *distortion* function, which measures the required power (distortion) between user at $\boldsymbol{\omega}$ to the nth UAV, which we seek to minimize. We also assume that the communication between all users and UAVs is orthogonal, i.e., separated in frequency or time (slotted protocols). An *optimal static deployment* is then the minimization of the *average power* $\bar{P}$ over all UAVs allowed locations and associated parameters

$$\min_{\mathbf{P} \in \Omega^N, \mathbf{h} \in \mathbb{R}_+^N} \bar{P}(\mathbf{P}, \mathbf{h}), \tag{12}$$

which defines a *locational-parameter optimization problem*. Here we associate to the locations $\mathbf{p}_n$ of the facilities (UAV ground positions) an additional parameter, given by the flight height h_n. Although h_n is a location parameter in space, the power is not a monotone function of the height, due to the directional antenna effect.

To find local extrema of (12) analytically, we will need that the single-objective function $\bar{P}$ is continuously differentiable at any point in $\mathcal{Q}^N = \Omega^N \times \mathbb{R}_+^N$, i.e., the gradient exists and is a continuous function, as was shown for piecewise continuous non-decreasing distortion functions in the Euclidean metric over Ω^N [18, Thm.2.2] and [19]. We could extend this result in [20] to continuous distortion functions on $\Omega \times \mathcal{Q}$ for arbitrary $D-$dimensional parameter sets $\mathcal{Q} \subset \mathbb{R}^D$, which are multivariate polynomials of finite degree if restricted to Ω. Then the necessary condition for a local extrema is the vanishing of the gradient at a critical point[1]. To derive the partial derivatives we will need to rewrite the integral kernel, which is the minimum of N continuous functions, as a sum of N integrals by using generalized Voronoi tessellations of Ω.

[1] Note, if $\nabla \bar{P}$ is not continuous in $\mathcal{Q}^N$ than any jump-point is a potential critical point and has to be checked individually.

First we will derive the tessellation of Ω by generalized Voronoi diagrams in d dimensions for any height parameters $h_n \in \mathbb{R}_+$ associate to the generating (ground) points p_n, which are special cases of *Möbius diagrams*, introduced in [21] and [22].

Lemma 1. *Let $\mathbf{P} = (\mathbf{p}_1^T, \mathbf{p}_2^T, \dots, \mathbf{p}_N^T) \subset \Omega^N \subset (\mathbb{R}^d)^N$ for $d \in \{1,2\}$ be the ground positions and $\mathbf{h} = (h_1, \dots, h_N) \in \mathbb{R}_+^N$ the associated flight heights of the N UAVs. Then the total average transmit power to serve any ground terminal in Ω distributed by λ for $\beta = 1$ is given by*

$$\bar{P}(\mathbf{P}, \mathbf{h}) = \int_\Omega \min_n \left\{ \frac{(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^\gamma}{h_n} \right\} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} = \sum_{n=1}^N \int_{\mathcal{V}_n} \frac{(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^\gamma}{h_n} \lambda(\omega) d\omega \tag{13}$$

where $\gamma = (1+\alpha)/2$ is given by the path-loss exponent $\alpha \geq 1$. The generalized Voronoi regions for the ground terminals are given by

$$\mathcal{V}_n = \mathcal{V}_n(\mathbf{P}, \mathbf{h}) = \bigcap_{m \neq n} \mathcal{V}_{nm}(\mathbf{P}, \mathbf{h}) \tag{14}$$

with the dominance regions of $\mathbf{p}_n$ over $\mathbf{p}_m$ are given by

$$\mathcal{V}_{nm} = \Omega \cap \begin{cases} HS(\mathbf{p}_n, \mathbf{p}_m) & , h_m = h_n \\ B(\mathbf{c}_{nm}, r_{nm}) & , h_n < h_m \\ B^c(\mathbf{c}_{nm}, r_{nm}) & , h_n > h_m \end{cases} \tag{15}$$

where the center and radii of the balls are given by

$$\mathbf{c}_{nm} = \frac{\mathbf{p}_n - h_{nm}\mathbf{p}_m}{1 - h_{nm}} \quad , \quad r_{nm} = \left(\frac{h_{nm}}{(1 - h_{nm})^2} \|\mathbf{p}_n - \mathbf{p}_m\|^2 + h_n^2 \frac{h_{nm}^{-\alpha} - 1}{1 - h_{nm}} \right)^{1/2} \tag{16}$$

with the relative flight heights between UAV n and UAV m given by

$$h_{nm} = \left(\frac{h_n}{h_m} \right)^{\frac{2}{1+\alpha}}. \tag{17}$$

Remark. If two UAVs have the same ground position, i.e., $\mathbf{p}_n = \mathbf{p}_m$, then both of them are active, if the relative height difference is not to large. In all other cases, each UAV will be active, as long as it is not to high or to low. Let us note, that the individual transmit powers of each UAV are only bounded by $P(\mathbf{P}, \mathbf{h})$. To bound the individual average transmit powers of the nth UAV, one needs to bound the maximal and minimal flight heights, since it holds

$$\bar{P}_n(\mathbf{P}, \mathbf{h}) = \int_{V_n} (h_n^{-1/\gamma} \|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^{2 - \frac{1}{\gamma}})^\gamma \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{18}$$

where h_n^α will dominate the power for large flight heights. It can be also seen that the radius of the nth GT cells will increase if all relative flight heights increase.

Proof. The minimization of the performance functions over Ω defines an assignment rule for a generalized Voronoi diagram $\mathcal{V}(\mathbf{P},\mathbf{h}) = \{\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_N\}$ where

$$\mathcal{V}_n = \mathcal{V}_n(\mathbf{P},\mathbf{h}) := \left\{ \boldsymbol{\omega} \in \Omega \;\middle|\; \frac{(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^\gamma}{h_n} \le \frac{(\|\mathbf{p}_m - \boldsymbol{\omega}\|^2 + h_m^2)^\gamma}{h_m}, m \neq n \right\} \tag{19}$$

$$= \left\{ \boldsymbol{\omega} \in \Omega \;\middle|\; a_n \|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + b_n \le a_m \|\mathbf{p}_m - \boldsymbol{\omega}\|^2 + b_m, m \neq n \right\} \tag{20}$$

is the nth Voronoi region. Here we denoted the weights by the positive numbers

$$a_n = h_n^{-\frac{1}{\gamma}}, \quad b_n = h_n^{2-\frac{1}{\gamma}} \tag{21}$$

which define a *compoundly weighted power distance*, see [23, (3.1.12)]. The resulting diagram is also called a *Möbius diagram* and its bisectors are circles or lines in $\mathbb{R}^2$ as we will show below, see also [21], [22]. The nth Voronoi region is defined by $N-1$ inequalities, which can be written as the intersection of the $N-1$ *dominance regions* of $\mathbf{p}_n$ over $\mathbf{p}_m$, given by

$$\mathcal{V}_{nm} = \left\{ \boldsymbol{\omega} \in \Omega \;\middle|\; a_n \|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + b_n \le a_m \|\mathbf{p}_m - \boldsymbol{\omega}\|^2 + b_m \right\}. \tag{22}$$

If $h_n = h_m$ then $a_n = a_m$ and $b_n = b_m$, such that $\mathcal{V}_{nm} = HS(\mathbf{p}_n, \mathbf{p}_m)$, the left half-space between $\mathbf{p}_n$ and $\mathbf{p}_m$. For $a_n \neq a_m$ we can rewrite the inequality as

$$\begin{aligned}
a_n \|\mathbf{p}_n\|^2 + a_n \|\boldsymbol{\omega}\|^2 - 2a_n \langle \mathbf{p}_n, \omega \rangle - a_m \|\mathbf{p}_m\|^2 - a_m \|\boldsymbol{\omega}\|^2 + 2a_m \langle \mathbf{p}_m, \boldsymbol{\omega} \rangle + (b_n - b_m) &\le 0 \\
(a_n - a_m) \|\boldsymbol{\omega}\|^2 - 2 \langle a_n p_n - a_m p_m, \boldsymbol{\omega} \rangle + a_n \|\mathbf{p}_n\|^2 - a_m \|\mathbf{p}_m\|^2 + (b_n - b_m) &\le 0 \\
(a_n - a_m) \left[\|\boldsymbol{\omega}\|^2 - 2 \langle \mathbf{c}_{nm}, \boldsymbol{\omega} \rangle + \frac{a_n^2 \|\mathbf{p}_n\|^2 + a_m^2 \|\mathbf{p}_m\|^2 - a_n a_m (\|\mathbf{p}_n\|^2 + \|\mathbf{p}_m\|^2)}{(a_n - a_m)^2} + \frac{b_n - b_m}{a_n - a_m} \right] &\le 0
\end{aligned}$$

where the center point is given by

$$\mathbf{c}_{nm} = \frac{a_n \mathbf{p}_n - a_m \mathbf{p}_m}{a_n - a_m} = a_n \frac{\mathbf{p}_n - h_{nm} \mathbf{p}_m}{a_n - a_m} = \frac{\mathbf{p}_n - h_{nm} \mathbf{p}_m}{1 - h_{nm}} \tag{23}$$

where we introduced the *relative flight height* between the nth and mth UAV

$$h_{nm} := \frac{a_m}{a_n} = \left(\frac{h_n}{h_m} \right)^{\frac{2}{1+\alpha}} > 0. \tag{24}$$

If $0 < a_n - a_m$, which is equivalent to $h_n < h_m$, then this defines a ball (disc) and for $h_n > h_m$ its complement. Hence we get

$$\mathcal{V}_{nm} = \begin{cases} B(\mathbf{c}_{nm}, r_{nm}) = \{ \boldsymbol{\omega} \in \Omega \mid \|\boldsymbol{\omega} - \mathbf{c}_{nm}\| < r_{nm} \}, & h_n < h_m \\ HS(\mathbf{p}_n, \mathbf{p}_m) = \{ \boldsymbol{\omega} \in \Omega \mid \|\boldsymbol{\omega} - \mathbf{p}_n\| \le \|\boldsymbol{\omega} - \mathbf{p}_m\| \}, & h_n = h_m \\ B^c(\mathbf{c}_{nm}, r_{nm}) = \{ \boldsymbol{\omega} \in \Omega \mid \|\boldsymbol{\omega} - \mathbf{c}_{nm}\| > r_{nm} \}, & h_n > h_m \end{cases} \tag{25}$$

where the radius square is given by

$$r_{nm}^2 = \frac{-2a_n a_m \langle \mathbf{p}_n, \mathbf{p}_m \rangle + a_n a_m (\|\mathbf{p}_n\|^2 + \|\mathbf{p}_m\|^2)}{(a_n - a_m)^2} - \frac{b_n - b_m}{a_n - a_m} \tag{26}$$

$$= a_n a_m \frac{\|\mathbf{p}_n - \mathbf{p}_m\|^2}{(a_n - a_m)^2} + \frac{b_m - b_n}{a_n - a_m} = \frac{a_n}{a_m} \frac{\|\mathbf{p}_n - \mathbf{p}_m\|^2}{(1 - \frac{a_n}{a_m})^2} + \frac{b_m - b_n}{a_n - a_m} \tag{27}$$

The second summand can be written as

$$\frac{b_m - b_n}{a_n - a_m} = \frac{h_m^{2-\frac{2}{1+\alpha}} - h_n^{2-\frac{2}{1+\alpha}}}{h_n^{-\frac{2}{1+\alpha}} - h_m^{-\frac{2}{1+\alpha}}} = \frac{h_m^{2-\frac{2}{1+\alpha}} - h_n^{2-\frac{2}{1+\alpha}}}{h_n^{-\frac{2}{1+\alpha}}\left(1 - \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}}\right)} \tag{28}$$

$$= \frac{h_m^2 \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}} - h_n^2}{1 - \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}}} = \frac{h_n^2 \left(\left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}-2} - 1\right)}{1 - \left(\frac{h_n}{h_m}\right)^{\frac{2}{1+\alpha}}} = h_n^2 \frac{h_{nm}^{-\alpha} - 1}{1 - h_{nm}}. \tag{29}$$

For any $\alpha \geq 1$, we have $h_{nm} = (h_n/h_m)^{2/(1+\alpha)} < 1$ if $h_n < h_m$ and $h_{nm} > 1$ if $h_m > h_n$. In both cases (29) is positive, which implies a radius $r_{nm} > 0$ whenever $\mathbf{p}_n \neq \mathbf{p}_m$. Hence, in this case the radius-square is

$$r_{nm}^2 = \frac{h_{nm}}{(1 - h_{nm})^2} \|\mathbf{p}_n - \mathbf{p}_m\|^2 + h_n^2 \frac{h_{nm}^{-\alpha} - 1}{1 - h_{nm}} \tag{30}$$

■

Example 1. We plotted in Figure 1 for $N = 2$ and $\Omega = [0,1]^2$ the GT cells for a uniform distribution with UAVs placed on

$$\mathbf{p}_1 = (0.1, 0.2), h_1 = 0.5, \quad \text{and} \quad \mathbf{p}_2 = (0.6, 0.6), h_2 = 1 \tag{31}$$

If the second UAV reaches an altitude of $h_2 \geq 2.3$ its Voroni cell $\mathcal{V}_2 = \mathcal{V}_{2,1}$ will be empty and hence become inactive.

3.1 Static UAV Deployment: Local Optimality

To find the global optimal deployment for N static UAVs we have to minimize the average power consumption for all GTs in Ω over its locational-parameter set $\mathcal{Q}^N$, i.e., we have to solve the following non-convex $N-$*facility locational-parameter optimization problem* [20]

$$\min_{\mathbf{Q} \in \mathcal{Q}^N} \bar{P}(\mathbf{Q}) = \min_{\mathbf{P} \in \Omega^N, \mathbf{h} \in \mathbb{R}_+^N} \sum_{n=1}^{N} \int_{\mathcal{V}_n(\mathbf{P},\mathbf{h})} h_n^{-1} (\|p_n - \boldsymbol{\omega}\|^2 + h_n^2)^\gamma \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{32}$$

where $\mathcal{V}_n(\mathbf{p}, h)$ are the Möbius regions given in (15) for each fixed $\mathbf{q} = (\mathbf{p}, h)$.

Theorem 1. *Let $\alpha \geq 1$ be fixed. Then a point $\mathbf{q}^* = (\mathbf{p}^*, h^*)$ with Voronoi tessellation $\mathcal{V}^* = \mathcal{V}(\mathbf{p}^*, h^*) = (\mathcal{V}_1^*, \ldots, \mathcal{V}_N^*)$ given by Lemma 1 is a critical point of (32) if and only if for each $n \in [N]$ it holds*

$$0 = \int_{\mathcal{V}_n^*} (\mathbf{p}_n^* - \boldsymbol{\omega})(\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\frac{\alpha-1}{2}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{33}$$

$$0 = \int_{\mathcal{V}_n^*} (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 + h_n^{*2})^{\frac{\alpha-1}{2}} \cdot (\|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 - \alpha h_n^{*2}) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \tag{34}$$

Proof. Since the power function

$$P(\boldsymbol{\omega}, \mathbf{p}, h) = (h_n^{-\frac{2}{1+\alpha}} \|\mathbf{p} - \boldsymbol{\omega}\|^2 + h_n^{\frac{2\alpha}{1+\alpha}})^{\frac{\alpha+1}{2}} \tag{35}$$

is a polynomial in $\boldsymbol{\omega}$ of degree less than $1+\alpha$ for each fixed $\mathbf{q} = (\mathbf{p}, h)$, the average distortion function is continuous differentiable, and we obtain by [20, Thm.1] for the partial derivatives

$$\frac{\partial \bar{P}(\mathbf{Q})}{\partial q_{n,i}} = \int_{\mathcal{V}_n(\mathbf{Q})} \frac{\partial P(\boldsymbol{\omega}, \mathbf{q})}{\partial q_{n,i}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \quad , \quad i \in \{1,2,3\}, n \in \{1, 2, \ldots, N\}. \tag{36}$$

Hence, a critical point is only attained if all its partial derivatives are vanishing, i.e., for $n = 1, 2, \ldots, N$

$$\begin{aligned}
0 \overset{!}{=} \nabla_n \bar{P}(\mathbf{p}, h) &= \begin{pmatrix} \int_{\mathcal{V}_n} h_n^{-1} \gamma 2 (p_n - \boldsymbol{\omega})(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma - 1} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \int_{\mathcal{V}_n} \left[-h_n^{-2} (\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma} + 2\gamma (\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma-1} \right] \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{pmatrix} \in \mathbb{R}^3 \\
\Leftrightarrow 0 &= \begin{pmatrix} \int_{\mathcal{V}_n} (\mathbf{p}_n - \boldsymbol{\omega})(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma-1} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \int_{\mathcal{V}_n} (\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\gamma-1} \cdot \left(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2 - 2\gamma h_n^2 \right) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{pmatrix} \\
\Leftrightarrow 0 &= \begin{pmatrix} \int_{\mathcal{V}_n} (\mathbf{p}_n - \boldsymbol{\omega})(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\frac{\alpha-1}{2}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \\ \int_{\mathcal{V}_n} (\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h_n^2)^{\frac{\alpha-1}{2}} \cdot (\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 - \alpha h_n^2) \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega} \end{pmatrix} \qquad (37)
\end{aligned}$$

■

When UAV vertical deployment $\mathbf{h}$ and cell partition are fixed, the distortion is a convex function. Similarly, when UAV horizontal deployment $\mathbf{p}$ and cell partition are fixed, the distortion is also convex. In what follows, we proposed a Lloyd-like algorithm to optimize the UAV deployments. The proposed algorithm iterates between three steps: (1) Partition optimization: Partitioning is done by LVDs; (2) The horizontal deployment optimization: each sensor moves horizontally via gradient descent. (3) The vertical deployment optimization: each sensor moves vertically via gradient descent.

If $\alpha > 1$ the integral kernel will be a polynomial in h_n and p_n of order > 1, which results in multiple solutions for (37). A closed form solution for $\mathbf{p}_n^*$ and h_n^* is hence not possible for arbitrary α. Let us investigate some tractable special cases.

Corollary 1. *If $\alpha = 1$, then all critical points of (32) must satisfy*

$$\mathbf{p}_n^* = \frac{\int_{\mathcal{V}_n^*} \boldsymbol{\omega}\lambda(\boldsymbol{\omega})d\boldsymbol{\omega}}{\int_{\mathcal{V}_n^*} \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}} \quad , \quad h_n^* = \sqrt{\frac{\int_{\mathcal{V}_n^*} \|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}}{\int_{\mathcal{V}_n^*} \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}}} \tag{38}$$

with $\mathcal{V}_n^* = \mathcal{V}_n(\mathbf{p}^*, h^*)$ given by Lemma 1 for all $n \in [N]$.

Proof. If $\alpha = 1$ then $\gamma = 1$ and we get from (37)

$$0 = \mathbf{p}_n^* \int_{\mathcal{V}_n^*} \lambda(\boldsymbol{\omega})d\boldsymbol{\omega} - \int_{\mathcal{V}_n^*} \omega\lambda(\boldsymbol{\omega})d\boldsymbol{\omega} \quad \Leftrightarrow \quad \mathbf{p}_n^* = \frac{\int_{\mathcal{V}_n^*} \omega\lambda(\boldsymbol{\omega})d\boldsymbol{\omega}}{\int_{\mathcal{V}_n^*} \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}} \tag{39}$$

$$0 = \int_{\mathcal{V}_n^*} \|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 \lambda(\boldsymbol{\omega})d\boldsymbol{\omega} - h_n^2 \int_{\mathcal{V}_n^*} \lambda(\boldsymbol{\omega})d\boldsymbol{\omega} \quad \Leftrightarrow \quad h_n^* = \sqrt{\frac{\int_{\mathcal{V}_n^*} \|\mathbf{p}_n^* - \boldsymbol{\omega}\|^2 \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}}{\int_{\mathcal{V}_n^*} \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}}} \tag{40}$$

■

Remark. For $\alpha = 1$, the optimal ground user tessellation of the UAVs is by (38) centroidal. However, the shape of the user cells depend on the heights, which are different and therefore the cells are spherical and not polyhedral.

3.1.1 Same flight heights

If $h_n = h > 0$ for all n, then the locational-parameter problem becomes

$$\min_{\mathbf{p} \in \Omega^N, h \in \mathbb{R}_+} \bar{P}(\mathbf{p}, h) \tag{41}$$

Since the minimization of the distortion functions in (13) does not depend on a **common** flight height, the Voronoi diagram will be independent of h, in which case the Möbius diagram becomes the ordinary Euclidean Voronoi diagram by Lemma 1. Hence, for all h the Voronoi diagram is $\mathcal{V}(\mathbf{p}) = \mathcal{V}(\mathbf{p}, h)$, i.e., independent of h. This allows to separate the optimization in (41) in an optimization over the ground positions and an optimization over the common flight height $h > 0$. An optimal height is then given if the gradient vanishes

$$\frac{\partial \bar{P}(\mathbf{p}, h)}{\partial h} = \sum_n \int_{\mathcal{V}_n} \left[-h^{-2}(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h^2)^\gamma + 2\gamma(\|\mathbf{p}_n - \boldsymbol{\omega}\|^2 + h^2)^{\gamma-1} \right] \lambda(\boldsymbol{\omega})d\boldsymbol{\omega}$$

$$\Leftrightarrow \quad 0 = \sum_n \int_{\mathcal{V}_n(\mathbf{P})} (\|\boldsymbol{\omega} - \mathbf{p}_n\|^2 + h^2)^{\gamma-1} \cdot ((2\gamma - 1)h^2 - \|\boldsymbol{\omega} - \mathbf{p}_n\|^2)\lambda(\boldsymbol{\omega})d\boldsymbol{\omega} \tag{42}$$

Hence we have shown

Corollary 2. *The optimization problem*

$$\min_{h \in \mathbb{R}_+} \bar{P}(\mathbf{P}, h) \tag{43}$$

has for each fixed ground positions $\mathbf{P} = (\mathbf{p}_1^T, \ldots, \mathbf{p}_N^T) \in \Omega^N$ *and* $\alpha \geq 1$ *a local optimal common flight height* h^**, if it holds*

$$h^* = \sqrt{\frac{\sum_n \int_{\mathcal{V}_n} \|\boldsymbol{\omega} - \mathbf{p}_n\|^2 \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}}{\alpha \sum_n \int_{\mathcal{V}_n} (\|\boldsymbol{\omega} - \mathbf{p}_n\|^2 + h^{*2})^{\frac{\alpha-1}{2}} \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}}}. \tag{44}$$

If $\alpha = 1$ *then this reduces to the global optimal flight height for* $\mathbf{p}$

$$h^{**} = \sqrt{\sum_n \int_{\mathcal{V}_n} \|\boldsymbol{\omega} - \mathbf{p}_n\|^2 \lambda(\boldsymbol{\omega}) d\boldsymbol{\omega}}. \tag{45}$$

Remark. To find the optimal ground positions for a common flight height $h_n = h$ and $\alpha > 1$ was investigated in [24, Sec.III]. The authors could show asymptotic results for $N \to \infty$ and the one-dimensional case, which reduces to the uniform quantizer $\mathbf{P}^* = (1/2N, 3/2N, \ldots, 2N - 1/2N)$. However, they did not considered the optimal flight height for a given ground position.

4 The Optimal UAV deployment over one-dimensional ground

In this section, we discuss the optimal UAV deployment when users are placed on a one-dimensional ground and restrict them to a convex target area $\Omega = [s, t]$. Under such circumstances, UAVs' ground projections are degenerated to scalars, i.e., $p_n \in [s, t]$, $\forall\, n \in \{1, \ldots, N\}$. If we shift the target area Ω by an arbitrary number $a \in \mathbb{R}$, then the power consumption, i.e., the objective function, will not change if we shift all UAV ground positions by the same number a. Hence, if we set $a = -s$ we can shift any deployment $[s, t]$ to $[0, A]$ where $A = t - s$ without changing the average power consumption. Hence we will only consider here target areas of the form $[0, A]$ with $A > 0$.

4.1 Optimal UAV deployment for common height and uniform user density

If $d = 1$ the ground users will be located on a line and we have $\mathbf{p}_n = x_n \in \mathbb{R}$ for each n. Let us assume a uniform distribution on Ω, i.e. $\lambda(\omega) = 1/\mu(\Omega)$ where μ is the Lebesgue measure and $0 < A = \mu(\Omega) < \infty$. Let us first recall the uniform scalar quantizer for the interval $[0, A]$.

Lemma 2 (Uniform Scalar Quantization)**.** *Let* N *be a positive integer and* $A > 0$*. Then the optimal* $N-$*point quantizer in* $[0, A]$ *with respect to the Euclidean norm and* *for a uniform density,* $\lambda(\omega) = 1/A$ *for each* $\omega \in [0, A]$*, is the unique global minimizer of*

$$H_V(\mathbf{x}^*) = \frac{A^2}{12N^2} = \min_{\mathbf{x}\in[0,A]^N} H_V(\mathbf{x}) \quad \textit{with} \quad H_V(\mathbf{x}) = \frac{1}{A}\sum_{n=1}^{N}\int_{\mathcal{V}_n(\mathbf{x})}(x_n-\omega)^2 d\omega \tag{46}$$

and is given point wise by

$$x_n^* = \frac{(n-1/2)A}{N} \quad , \quad 1 \le n \le N. \tag{47}$$

Proof. For a proof see Appendix A. See also the scalar quantization theorem in [25] or [26]. The proof of [26, Thm.1] uses the optimal Euclidean embedding and a sharp reverse Hölder inequality, which allows one to prove the result without derivations (take the homogeneous special case). ■

Theorem 2. *Let* $N \in \mathbb{N}$ *and* $\Omega = [0, A]$ *for some* $A > 0$*. The* global optimal deployment *for* N *UAVs located at* $\mathbf{q}_n^* = (x_n^*, h^*) \in \Omega \times \mathbb{R}_+$ *with path loss* $\alpha \ge 1$*, common flight height* $h^* = h_n^*$*, and a uniform ground user density is given by uniform ground positions*

$$\forall\, n\colon x_n^* = \frac{(n-1/2)A}{N}, \tag{48}$$

and optimal common height

$$h^* = \frac{A}{2N}g(\alpha) \quad \textit{with} \quad g(\alpha) := \arg\min_{h>0}\int_0^1 \frac{(\omega^2+h^2)^{\frac{1+\alpha}{2}}}{h}d\omega. \tag{49}$$

with minimal average power consumption

$$\bar{P}(\mathbf{x}^*, h^*) = \left(\frac{A}{2N}\right)^{\alpha}\int_0^1 \frac{(\omega^2+g^2(\alpha))^{\frac{1+\alpha}{2}}}{g(\alpha)}d\omega. \tag{50}$$

For $\alpha \in \{1, 3, 5\}$ *we can derive the integral minimization in closed form*

$$g(1) = \frac{1}{\sqrt{3}}, \quad g(3) = \frac{\sqrt{\sqrt{32/5}-1}}{3}, \quad g(5) = \sqrt{\frac{(\frac{32}{7})^{1/3}-1}{5}} \tag{51}$$

which is decreasing with increasing α*.*

Proof. The average-communication power is given for a common height $h = h_n$ by

$$\bar{P}(\mathbf{q}) = \frac{1}{hA}\int_{\mathcal{V}_n(\mathbf{x})}((x_n-\omega)^2+h^2)^{\gamma}d\omega \tag{52}$$

Then the optimal ground positions $x_1, \dots, x_n^*$ need to satisfy by Theorem 1

$$0 = \int_{\mathcal{V}_n(\mathbf{x}^*)}(x_n^*-\omega)((x_n^*-\omega)^2+h^2)^{\gamma-1}d\omega \quad , \quad 1 \le n \le N \tag{53}$$

Since the Voronoi regions are intervals given by Lemma 1 and (145) as

$$\mathcal{V}_n(\mathbf{x}^*) = [b_{n-1}, b_n] \quad \text{with} \quad b_n = \begin{cases} 0, & n = 0 \\ A, & n = N \\ \frac{x_{n+1}+x_n}{2}, & \text{else} \end{cases} \tag{54}$$

we get by substituting $\tilde{\omega} = x_n^* - \omega$ for each n

$$0 = -\int_{x_n - b_{n-1}}^{x_n - b_n} (\tilde{\omega}^2 + h^2)^{\gamma-1} \tilde{\omega} d\tilde{\omega} = \int_{x_n - b_n}^{x_n - b_{n-1}} (\tilde{\omega}^2 + h^2)^{\gamma-1} \tilde{\omega} d\tilde{\omega} \tag{55}$$

Since the integral kernel $f(\omega) = (\omega^2 + h^2)^{\gamma-1}\omega$ is anti-symmetric in ω it can only vanish if the integral boundaries have different signs, i.e., since $b_{n-1} \leq b_n$ we have

$$0 = \int_{x_n^* - b_n}^{0} f(\omega) d\omega + \int_0^{x_n^* - b_{n-1}} f(\omega) d\omega \tag{56}$$

Moreover, we get by the anti-symmetry of f and a substituting by $\tilde{\omega} = -\omega$

$$\int_{x_n^* - b_n}^{0} f(\omega) d\omega = -\int_{x_n^* - b_n}^{0} f(-\omega) d\omega = \int_0^{x_n^* - b_n} f(-\omega) d\omega = -\int_0^{b_n - x_n^*} f(\tilde{\omega}) d\tilde{\omega} \tag{57}$$

Hence, we get by inserting in (56)

$$\int_0^{b_n - x_n^*} f(\omega) d\omega = \int_0^{x_n^* - b_{n-1}} f(\omega) d\omega \tag{58}$$

Hence it must hold for $2 \leq n \leq N-1$

$$b_n - x_n^* = \frac{x_{n+1}^* - x_n^*}{2} = \frac{x_n^* - x_{n-1}^*}{2} = x_n^* - b_{n-1} \Leftrightarrow x_n^* = \frac{x_{n+1}^* + x_{n-1}^*}{2} \tag{59}$$

and for $n = 1$ and $n = N$

$$b_1 - x_1 * = \frac{x_2^* - x_1^*}{2} = x_1^* \quad \Leftrightarrow \quad x_1^* = \frac{x_2^*}{3} \tag{60}$$

$$b_N - x_N^* = A - x_N^* = x_N^* - b_{N-1} = \frac{x_N^* - x_{N-1}^*}{2} \quad \Leftrightarrow \quad x_N^* = \frac{2A + x_{N-1}^*}{3} \tag{61}$$

Hence, by the same argument as in the proof Appendix A of Lemma 2 we get the uniform scalar quantizer

$$x_n^* = \frac{(n - 1/2)A}{N} \quad \text{and} \quad \Delta = \frac{A}{N} \tag{62}$$

To obtain the optimal height we need to minimize (52) over $h > 0$ with the uniform quantizer. In fact, for each ground cell we get for the average power by substituting wit $\tilde{\omega} = \omega - x_n^*$

$$\bar{P}_n(x_n^*, h) = \frac{1}{hA} \int_{b_{n-1}}^{b_n} (\omega - x_n^*)^2 + h^2)^{\gamma} d\omega = \frac{1}{hA} \int_{b_{n-1} - x_n^*}^{b_n - x_n^*} (\tilde{\omega}^2 + h^2)^{\gamma} d\omega \tag{63}$$

where $b_n = (x^*_{n+1}+x^*_n)/2$ and $b_{n-1}-x^*_n = (x^*_{n-1}-x^*_n)/2 = -(x_{n+1}-x^*_n)/2 = -(b_n-x^*_n)$ which is $A/2N$ by (62) for ever $n \in [N]$ s.t.

$$= \frac{1}{hA} \int_{-A/2N}^{A/2N} (\omega^2 + h^2)^\gamma d\omega = \frac{2}{A} \int_0^{A/2N} \frac{(\omega^2 + h^2)^\gamma}{h} d\omega \tag{64}$$

Hence we get

$$h^*(\alpha, N, A) = \arg\min_{h>0} \int_0^{A/2N} \frac{(\omega^2 + h^2)^{\frac{1+\alpha}{2}}}{h} d\omega \tag{65}$$

We will separate the dependence of α from N and A. Let us substitute $\tilde{\omega} = N2\omega/A$ and $\tilde{h} = N2h/A$ which gives

$$\int_0^{A/2N} \frac{(\omega^2 + h^2)^{\frac{1+\alpha}{2}}}{h} d\omega = \int_0^1 \frac{(A^2\tilde{\omega}^2/4N^2 + A^2\tilde{h}^2/4N^2)^{\frac{1+\alpha}{2}}}{A\tilde{h}/2N} \frac{A}{2N} d\tilde{\omega} \tag{66}$$

$$= \left(\frac{A}{2N}\right)^{1+\alpha} \int_0^1 \frac{(\omega^2 + \tilde{h}^2)^{\frac{1+\alpha}{2}}}{\tilde{h}^2} d\omega. \tag{67}$$

Hence, we get

$$h^* = \frac{A}{2N} g(\alpha) \quad \text{with} \quad g(\alpha) := \arg\min_{h>0} \int_0^1 \frac{(\omega^2 + h^2)^{\frac{1+\alpha}{2}}}{h} d\omega \tag{68}$$

where the integral $f(h)$ can only be solved in closed form for integer valued α.

We will show that f is actually a strict convex function in $(0, \infty)$, i.e., its second derivative is strictly positive. Then, there exists only one critical point in $\mathbb{R}_+$ and hence only one global optimal height h^*. It is obvious that f is continuous in $\mathbb{R}_+$ since the integral kernel function and the integral is continuous. Let us calculate the first and second derivative

$$f'(h) = \int_0^1 2\gamma(\omega^2 + h^2)^{\gamma-1} - h^{-2}(\omega^2 + h^2)^\gamma d\omega d\omega \tag{69}$$

$$= \int_0^1 (\omega^2 + h^2)^{\gamma-1}((2\gamma-1) - h^{-2}\omega^2) d\omega = \frac{1}{h^{\frac{\alpha+1}{2}}} \int_0^{\frac{1}{h}} (\omega^2 + 1)^{\frac{\alpha-1}{2}} (\alpha - \omega^2)$$

Hence, $f'(h)$ can only vanish if $h < 1/\sqrt{\alpha}$, which is a upper bound on $g(\alpha)$.

$$f''(h) = \int_0^1 4h\gamma(\gamma - 1)(\omega^2 + h^2)^{\gamma-2} + 2^{-1}h^{-3}(\omega^2 + h^2)^\gamma - 2h^{-1}\gamma(\omega^2 + h^2)^{\gamma-1} \tag{70}$$

$$= \int (\omega^2 + h^2)^{\gamma-2} \left[4h\gamma(\gamma - 1) + h^{-3}\omega^4 + 2\omega^2 h^{-1} + h - 2\gamma h^{-1}\omega^2 - 2\gamma h\right] d\omega$$

$$\gamma = 1 + \epsilon \rightarrow = \int (\omega^2 + h^2)^{\epsilon-1} \left[h^{-2}\omega^4 - 2\omega^2 h^{-1}\epsilon + h(-1 - 2\epsilon + 4(1 + \epsilon)\epsilon)\right] \tag{71}$$

subsitute again ω by ω/h gets

$$= h^{2\epsilon-6} \int_0^{1/h} (\omega^2+1)^{\epsilon-1}[h^4\omega^4 - 2\omega^2\epsilon h^4 + h^4(4\epsilon^2+2\epsilon-1)]d\omega \tag{72}$$

$$= h^{2\epsilon-2} \int_0^{1/h} (\omega^2+1)^{\epsilon-1}[(\omega^2-2\epsilon)^2 + 2\omega^2\epsilon + 2\epsilon - 1]d\omega \tag{73}$$

which shows strict positivity for each $h > 0$ if $\epsilon > 1/3$. Since we know that there are only critical points in $(0, 1/\sqrt{1+2\epsilon})$ we need to check positivity for $\epsilon \in [0, 1/3]$ only in this range.

Let us set $0 < x = h^2$ in (69), then we get for $\gamma \in \mathbb{N}$ a polynomial in ω of degree 2γ and in x of degree γ. Furthermore, resolving for x only allows positive real roots, which might not always exist. The integral does not have a closed form for arbitrary γ. Since we only are interested in $\alpha \in [1,6]$ we will calculate by hand for $\alpha \in \{1,2,3\}$, yielding to the integral kernel polynomials

$$(\omega^2 - 1x)(\omega^2+x)^0 = \omega^2 - x \tag{74}$$

$$(\omega^2 - 3x)(\omega^2+x)^1 = \omega^4 - 2\omega^2 x - 3x^2 \tag{75}$$

$$(\omega^2 - 5x)(\omega^2+x)^2 = \omega^6 - 3\omega^4 x - 9\omega^2x^2 - 5x^3 \tag{76}$$

which yield with the definite integrals to

$$0 = \omega(\frac{\omega^2}{3} - x)\Big|_{\omega=1} \tag{77}$$

$$0 = \omega(\frac{\omega^4}{5} - \frac{2\omega^2 x}{3} - 3x^2)\Big|_{\omega=1} \tag{78}$$

$$0 = \omega(\frac{\omega^6}{7} - \frac{3\omega^4 x}{5} - 3\omega^2x^2 - 5x^3)\Big|_{\omega=1} \tag{79}$$

Solving the first for x yields to the only feasible solution

$$x = \frac{1}{3} \quad \Rightarrow \quad g^*(1) = \frac{1}{\sqrt{3}} \approx 0.577. \tag{80}$$

The second yields for x to

$$x_{1,2} = -\frac{1}{9} \pm \sqrt{\frac{1}{81} + \frac{1}{15}} = \frac{\pm\sqrt{32/5} - 1}{9} \tag{81}$$

Since only positive roots are allowed we get as only feasible solution

$$g^*(3) = \frac{\sqrt{\sqrt{32/5} - 1}}{3} \approx 0.412. \tag{82}$$

Finally, for $\alpha = 5$ we get the cubic equation (79)

$$5x^3 + 3x^2 + \frac{3}{5}x - \frac{1}{7} = 0 \tag{83}$$

Then the discriminant is given by, see for example [27, p. 2.3.2],

$$\Delta = q^2 + 4p^3 \quad \text{with} \quad q = \frac{2b^3 - 9abc + 27a^2 d}{27a^3}, p = \frac{3ac - b^2}{9a^2} \tag{84}$$

Let us identify $a = 5, b = 3, c = 3/5$ and $d = -1/7$, then we get

$$q = \frac{6 \cdot 9 - 9 \cdot 9 - 27 \cdot 5^2 \cdot 1/7}{27 \cdot 5^3} = -\frac{3}{3 \cdot 5 \cdot 25} - \frac{1}{5 \cdot 7} = -\frac{32}{25 \cdot 35} \tag{85}$$

$$\Delta = q^2 + 4\left(\frac{3 \cdot 3 - 9}{9 \cdot 5^2}\right)^3 = q^2 > 0 \tag{86}$$

which indicates only one real-valued root, given by

$$x = \alpha_+^{1/3} + \alpha_-^{1/3} - \frac{b}{3a} \quad \text{with} \quad \alpha_\pm = \frac{-q \pm \sqrt{\Delta}}{2} = \{0, \frac{32}{25 \cdot 35}\} \tag{87}$$

which computes to

$$x = \left(\frac{32}{5^3 \cdot 7}\right)^{1/3} - \frac{1}{5} = \frac{(\frac{32}{7})^{1/3} - 1}{5} \Rightarrow g^*(5) = \sqrt{\frac{(\frac{32}{7})^{1/3} - 1}{5}} \approx 0.363. \tag{88}$$

■

Remark. If Ω is a union of M disjoint intervals, then the optimal deployment can not necessarily uniform quantize each interval, if the intervals have different size. The distortion measure will be larger in such cases.

Let us set $\beta = 1 = A$. Then the optimal UAV deployment is pictured in Figure 2 for $N = 2$ and $N = 4$. The maximal evaluation angle $\theta_{\max}$ is hereby constant for each UAV and does not change if the number of UAVs. Moreover, it is also independent of A and β, since

$$\cos(\theta_{\max}) = \frac{\hat{h}}{\hat{d}_n/2} = \frac{AN}{N\sqrt{3}A} = \frac{1}{\sqrt{3}}. \tag{89}$$

4.2 Optimal UAV deployment for different heights and uniform user density

To extend the previous results to different heights we need first to verify that the optimal deployment of $N-$UAVs involves only active UAVs.

Lemma 3. *Let $\Omega = [0, A]$ for some $A > 0$. The optimal deployment $\mathbf{Q}^* \in (\Omega \times \mathbb{R}_+)^N$ of N UAVs for a uniform ground user density has optimal ground cells $\mathcal{V}_n(\mathbf{Q}^*) = [b_{n-1}, b_n]$ with $0 \leq b_{n-1} < b_n \leq A$ for $n \in [N]$, i.e., each optimal ground cell has positive measure and is served by exactly one UAV.*

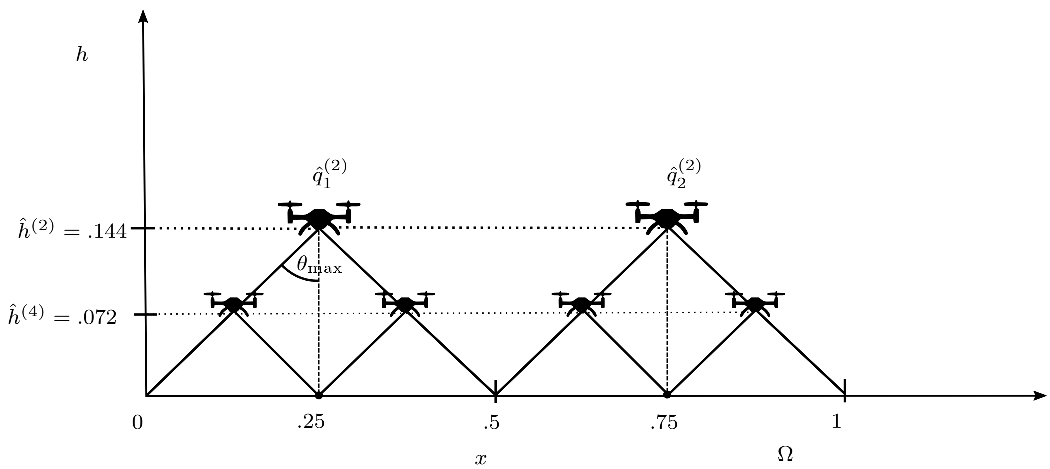

Figure 2: UAV deployment in one dimension for $\Omega = [0,1]$ for $\alpha = 1$ and $N = 2,4$ for a uniform GT distribution by (103).

Remark. For any optimal deployment all UAVs are active, which is intuitively, since we want to use all UAV antennas for optimal coverage.

Proof. Although, this statement seems to be trivial, it is not straight forward to show, since the power function P in (9) is not strictly monotone increasing in $\mathbf{q}_n = (p_n, h_n)$ for any user $\omega \in \Omega$ and UAV $n \in [N]$. To bypass this non-monotonicity property of $P(\mathbf{Q}, \omega)$ we will use the quantization relaxation for the average power consumption $\bar{P}$ in (11) to solve the optimal deployment problem (12). We define, as in quantization theory, see for example [25], an $N-$*point quantizer* for Ω, by a (disjoint) partition $\mathcal{R} = \{\mathcal{R}_n\}_{n=1}^N \subset \Omega$ of Ω and assign to each partition cell $\mathcal{R}_n$ a reproduction point $\mathbf{q}_n \in \mathcal{Q} = \Omega \times \mathbb{R}_+$. The assignment rule or *quantization rule* can be anything and does not have to depend on the performance function P or whatsoever. Minimizing over all quantizer, that is, over all partitions and possible reproduction points will yield to the optimal quantizer, which is by definition the optimal deployment (reproduction points) which generate the generalized Voronoi diagram as the optimal partition (tesselation[2]). This holds for any density function $\lambda(\omega)$ and target area Ω. To see this[3] let us start with any quantizer $(\mathbf{Q}, \mathcal{R})$ for Ω yielding to the average power

$$\begin{aligned}\bar{P}(\mathbf{Q}, \mathcal{R}) = \sum_{n=1}^{N} \int_{\mathcal{R}_n} P(\mathbf{q}_n, \omega)\lambda(\omega) d\omega &\geq \sum_{n=1}^{N} \int_{\mathcal{R}_n} \left(\min_{m\in[N]} P(\mathbf{q}_m, \omega) \right) \lambda(\omega) d\omega \qquad (90)\\ &= \int_{\Omega} \min_{m\in[N]} P(\mathbf{q}_m, \omega)\lambda(\omega) d\omega = \sum_{n} \int_{\mathcal{V}_n(\mathbf{Q})} P(\mathbf{q}_n, \omega)\lambda(\omega) d\omega\end{aligned}$$

where the first inequality is only achieved if for any $\omega \in \mathcal{R}_n$ we have chosen $\mathbf{q}_n$ to be the optimal quantization point with respect to P, or vice versa, if every $\mathbf{q}_n$ is

[2]Since we take here the continuous case, the integral will not distinguish between open or closed sets.

[3]We use the same argumentation as in the prove of [26, Prop.1].

optimal for every $\omega \in \mathcal{R}_n$, which is the definition of the generalized Voronoi cell $\mathcal{V}(\mathbf{Q})$. Therefore, minimizing over all partitions gives equality, i.e.

$$\min_{\mathcal{R}} \bar{P}(\mathbf{Q}, \mathcal{R}) = \bar{P}(\mathbf{Q}, \mathcal{V}(\mathbf{Q})) \tag{91}$$

for any reproduction points $\mathbf{Q} \in \mathcal{Q}^N$. Hence, we have shown that the optimal quantizer problem is equivalent to the optimal deployment problem

$$\min_{\mathbf{Q}\in\mathcal{Q}^N} \min_{\mathcal{R}\in\Omega^N} \bar{P}(\mathbf{Q}, \mathcal{R}) = \min_{\mathbf{Q}\in\mathcal{Q}^N} \bar{P}(\mathbf{Q}, \mathcal{V}(\mathbf{Q})) = \bar{P}(\mathbf{Q}^*, \mathcal{V}(\mathbf{Q}^*)). \tag{92}$$

We need to show, that for the optimal $N-$UAV deployment $\mathbf{Q}^*$ we have $\mu(\mathcal{V}_n(\mathbf{Q}^*)) > 0$ for all $n \in [N]$. Let us first show that each cell is indeed a closed interval, i.e., $\mathcal{V}_n(\mathbf{Q}^*) = [b_{n-1}, b_n]$ with $0 \le b_{n-1} \le b_n \le A$.

By the definition of the Moebius diagrams in Lemma 1, each dominance region is either one interval (if it is a ball not contained in the target region or a halfspace) or two intervals (if its a ball contained in the target region), we can not have more than $K_n \le 2N - 2$ disjoint closed intervals for each Moebius (Voronoi) region. Therefore, the nth optimal Voronoi regions is given as $\mathcal{V}_n^* = \mathcal{V}_n(p^*, h^*) = \bigcup_{k=1}^{K_n} v_{n,k}$, where $v_{n,k} = [a_{n,k-1}, a_{n,k}]$ is a continuous interval and $2N - 2 \ge K_n \ge 1$.

Let us assume at least one UAV has a disconnected region, i.e. $K_n > 1$ for some $n \in [N]$. Then we will re-arrange the partition $\mathcal{V}(\mathbf{p}^*, \mathbf{h}^*)$ by concatenating the K_n intervals $v_{n,k}$ to $\mathcal{R}_n = [b_{n-1}, b_n]$ such that for all $n \in [N]$ it holds $\mu(\mathcal{R}_n) = b_n - b_{n-1}$ where we ordered them such that $b_{n-1} \le b_n$ for all $n \in [N]$ and set $b_0 = 0$ and $b_N = A$. Then we move each p_n^* to the center of the new arranged cells, i.e., $\tilde{p}_n = \frac{b_n + b_{n-1}}{2}$ without changing the associated heights. This therefore defines a new partition $\mathcal{R} = \{\mathcal{R}_n(\tilde{\mathbf{p}}, \mathbf{h}^*)\}$ and ground positions $\tilde{\mathbf{p}} = (\tilde{p}_1, \dots, \tilde{p}_N)$. The average power consumption is then

$$\bar{P}(\tilde{\mathbf{p}}^*, \mathbf{h}^*, \mathcal{R}) = \sum_{n=1}^{N} \int_{b_{n-1}}^{b_n} \frac{((\tilde{p}_n - \omega)^2 + h_n^{*2})^\gamma}{h_n^*} d\omega \tag{93}$$

Substituting ω with $\tilde{p}_n - \omega$ we get

$$\bar{P}(\tilde{\mathbf{p}}^*, \mathbf{h}^*, \mathcal{R}) = 2 \sum_{n=1}^{N} \int_{0}^{\frac{b_n - b_{n-1}}{2}} \frac{(\omega^2 + h_n^{*2})^\gamma}{h_n^*} d\omega \tag{94}$$

Since the height is not changing and the function $(\omega^2 + h_n^{*2})^\gamma$ is strictly monotone increasing in ω for each $\gamma > 0$, it holds for the average power of the nth cell (with $K_n > 1$)

$$2 \int_{0}^{\frac{b_n - b_{n-1}}{2}} \frac{(\omega^2 + h_n^{*2})^\gamma}{h_n^*} d\omega < \sum_{k=1}^{K_n} \int_{a_{n,k}-p_n^*}^{a_{n,k-1}-p_n^*} \frac{(\omega^2 + h_n^{*2})^\gamma}{h_n^*} d\omega \tag{95}$$

since the non-zero gaps in $\mathcal{V}_n^* - p_n^*$ will lead to larger ω and therefore to a strictly larger average power consumption, whereas the other cells with $K_n = 1$ will not change the average power. Therefore the deployment $(\tilde{\mathbf{p}}, \mathbf{h}^*)$ with convex partition sets $\mathcal{R}(\tilde{\mathbf{p}}, \mathbf{h}^*)$ has a smaller power consumption which contradicts to the assumption that $(\mathbf{p}^*, \mathbf{h}^*)$ is the optimal deployment (92). Hence $K_n = 1$ for each $n \in [N]$ and every $\gamma \geq 1$.

Now, we have the optimal deployment $\mathbf{Q}^*$ with $\mathcal{V}_n^* = \{[b_{n-1}, b_n]\}_{n=1}^n$. Let us show that for each $n \in [N]$ it also holds $b_{n-1} < b_n$. By (92) we get

$$\min_{\mathcal{R}} \bar{P}(\mathbf{Q}^*, \mathcal{R}) = \min_{\mathcal{R}} \sum_n \int_{\mathcal{R}_n} P(\mathbf{q}_n^*, \boldsymbol{\omega}) d\boldsymbol{\omega} = \bar{P}(\mathbf{Q}^*, \mathcal{V}(\mathbf{Q}^*)). \tag{96}$$

If $N = 1$ there is nothing to optimize and we have $\mathcal{R} = \mathcal{R}_1 = \Omega = [a, b]$. Furthermore, we know by Theorem 2, that the optimal $\mathbf{q}^{(1)*} = (p_1^{(1)*}, h_1^{(1)*})$ for $N = 1$ is the uniform quantizer given by $p_1^{(1)*} = a + A/2$ with $A = b - a$ and $h^* = h_1^{(1)*} = \operatorname{argmin}_{h>0} \int_0^{A/2} \frac{(\omega^2+h^2)^\gamma}{h} d\omega$. Let us note, that the average power $\bar{P}(p_1^{(1)*}, \cdot)$ is convex (second-derivative is always positive) in h and hence there is only one critical point h^* in $\mathbb{R}_+$, which is the global minimum. Now assume, we add one more quantizer $\mathbf{q}_2^{(2)} = (p_2^{(2)}, h^*)$ with same height and partitioning Ω by $\mathcal{R}_\epsilon$ given by $R_1 = [a, a+\epsilon]$ and $R_2 = [a+\epsilon, b]$ for some $\epsilon \in (0, A)$. If we minimize only over the ground positions, we get

$$\bar{P}(\mathbf{p}^{(2)*}, h^*, \mathcal{R}_\epsilon) := \min_{p_1} \int_a^{a+\epsilon} P(p_1, h^*, \omega) d\omega + \min_{p_2} \int_{a+\epsilon}^b P(p_2, h^*, \omega) d\omega \tag{97}$$

By Theorem 2 we get for the optimal uniform ground positions for each **independent** integral

$$p_1^{(2)*} = \frac{2a+\epsilon}{2} \quad \text{and} \quad p_2^{(2)*} = \frac{a+b+\epsilon}{2} \tag{98}$$

inserting and substituting with $\tilde{\omega} = \omega - p_i^{(2)*}$ we get with $h^* = \sqrt{c}$

$$\begin{aligned}
\bar{P}(\mathbf{p}^{(2)*}, h^*, \mathcal{R}_\epsilon) &= \frac{1}{\sqrt{c}} \left(\int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} (\tilde{\omega}^2 + c)^\gamma d\tilde{\omega} + \int_{-\frac{b-(a+\epsilon)}{2}}^{\frac{b-(a+\epsilon)}{2}} (\tilde{\omega}^2 + c)^\gamma d\omega \right) && (99)\\
&= \frac{2}{\sqrt{c}} \left(\int_0^{\epsilon/2} (\omega^2 + c)^\gamma d\omega + \int_0^{\frac{b-a}{2} - \frac{\epsilon}{2}} (\omega^2 + c)^\gamma d\omega \right) && (100)
\end{aligned}$$

Since $(\omega^2 + c)^\gamma < ((\omega+\epsilon)^2 + c)^\gamma$ for each $\epsilon > 0$ and $\gamma > 0$ we have

$$< \frac{2}{\sqrt{c}} \left(\int_{\frac{b-a}{2} - \frac{\epsilon}{2}}^{\frac{b-a}{2}} (\omega^2 + c)^\gamma d\omega + \int_0^{\frac{b-a}{2} - \frac{\epsilon}{2}} (\omega^2 + c)^\gamma d\omega \right) = \bar{P}(\mathbf{q}^{(1)*}, [a, b]) \tag{101}$$

for any $b - a > \epsilon > 0$. Hence, if we also minimize over the heights, this can only yield to a smaller value

$$\min_{\mathbf{p} \in \Omega^2, \mathbf{h} \in \mathbb{R}_+^2} \min_{\mathcal{R}_\epsilon = \{\mathcal{R}_{\epsilon,n}\}_{n=1}^2} \sum_{n=1}^2 \int_{R_n} P(p_n, h_n, \omega) d\omega \leq \bar{P}(\mathbf{p}^{(2)*}, h^*, \mathcal{R}_\epsilon) < \min_{p \in \Omega, h \in \mathbb{R}+} \bar{P}(p, h, \omega) d\omega$$

Hence, if we add a UAV, we can split an arbitrary ground cell $[a, b]$ with $b > a$ in two cells with positive measure and obtain a strictly smaller power consumption. Therefore, the global minimum can only be attained if all UAVs are active, i.e., all optimal ground cells have positive measure $\mu(\mathcal{R}_n^*) > 0$. Since the optimal partition is by (92) a Möbius diagram, each optimal ground cell is given as $\mathcal{R}_n^* = \mathcal{V}_n(\mathbf{q}^*) = [b_{n-1}, b_n]$ with $0 \leq b_{n-1} < b_n \leq A$. ∎

Now we are now ready to extend Theorem 2 to arbitrary heights.

Theorem 3. *Over the uniform distributed one-dimensional ground, $\Omega = [0, A]$, the minimum total power of N UAVs is given by a N-level uniform quantizer*

$$\bar{P}(\mathbf{p}^*, h^*) = \frac{(\mu(\Omega))^{2\gamma-1}}{N^{2\gamma-1}} \int_{-1/2}^{1/2} \frac{(\omega^2 + g^2(\gamma))^\gamma}{g(\gamma)} d\omega \tag{102}$$

$$\text{with} \quad \forall\, n\colon p_n^* = s + \frac{(n-1/2)A}{N}, h_n^* = \frac{g(\alpha)\mu(\Omega)}{N} \tag{103}$$

where $\mathbf{q}_n^* = (p_n^*, h_n^*)$ is UAV n's optimal deployment, and $g(\alpha) \triangleq \arg\min_h \int_0^1 \frac{(\omega^2+h^2)^{\frac{1+\alpha}{2}}}{h} d\omega$ is a function of α.

Proof. We can generalize the prove of Theorem 2 to optimal UAV ground positions at different heights by only using the all active property of the optimal UAV deployment from Lemma 3. We also know, that the optimal ground cells are closed non-vanishing intervals, given by $\mathcal{V}_n(\mathbf{q}^*) = [b_{n-1}^*, b_n^*]$ with $b_{n-1}^* < b_n^*$. Hence, for the optimal ground positions x_n^* at the optimal height h_n^* it must hold

$$0 = \int_{b_{n-1}^*}^{b_n^*} (x_n^* - \omega)((x_n^* - \omega)^2 + {h_n^*}^2)^{\gamma-1} d\omega \quad , \quad 1 \leq n \leq N \tag{104}$$

where we get by substituting $\tilde{\omega} = x_n^* - \omega$ for each n

$$0 = -\int_{x_n^*-b_{n-1}^*}^{x_n^*-b_n^*} (\tilde{\omega}^2 + {h_n^*}^2)^{\gamma-1} \tilde{\omega} d\tilde{\omega} = \int_{x_n^*-b_n^*}^{x_n^*-b_{n-1}^*} (\omega^2 + {h_n^*}^2)^{\gamma-1} \omega d\omega \tag{105}$$

Since the integral kernel $f(\omega) = (\omega^2 + {h_n^*}^2)^{\gamma-1}\omega$ is anti-symmetric in ω, (105) can only vanish if the integral boundaries have different signs, i.e.,

$$0 = \int_{x_n^*-b_n^*}^{0} f(\omega) d\omega + \int_0^{x_n^*-b_{n-1}^*} f(\omega) d\omega \quad \Leftrightarrow \quad \int_0^{b_n^*-x_n^*} f(\omega) d\omega = \int_0^{x_n^*-b_{n-1}^*} f(\omega) d\omega \tag{106}$$

Hence it must hold for $1 \leq n \leq N$

$$b_n^* - x_n^* = x_n^* - b_{n-1}^* \Leftrightarrow x_n^* = \frac{b_{n-1}^* + b_n^*}{2} \tag{107}$$

where for $n = 1$ and $n = N$ we get $b_0 = 0$ and $b_N = A$, since $\{[b_{n-1}^*, b_n^*]\}$ is a partition of $[0, A]$. Hence the optimal ground positions are the centroids of the cells. Let us set

$\mu_n = b_n^* - b_{n-1}^*$ for $n \in [N]$, then we get by substituting $\frac{2(x_n^*-\omega)}{\mu_n} = \tilde{\omega}$ and $h_n^* = \tilde{h}_n \mu_n/2$ in the optimal average power consumption

$$\bar{P}(\mathbf{p}^*, \mathbf{h}^*) = \sum_{n=1}^{N} \int_{b_{n-1}^*}^{b_n^*} \frac{((x_n^* - \omega)^2 + {h_n^*}^2)^\gamma}{h_n^*} \frac{d\omega}{A} = \sum_n \int_1^{-1} -\frac{(\mu_n^2 \tilde{\omega}^2/4 + \tilde{h}_n^2 \mu_n^2/4)^\gamma}{g(\alpha)\mu_n/2} \frac{\mu_n}{2A} \tag{108}$$

$$= \frac{2}{A} \sum_n \int_0^1 \frac{(\omega^2 + \tilde{h}_n^2)^\gamma}{\tilde{h}} d\omega \cdot \mu_n^{1+\alpha} 2^{-1-\alpha} \tag{109}$$

where we inserted (107) to get $2(x_n^* - b_{n-1}^*)/\mu_n = 1 = -2(x_n^* - b_n^*)/\mu_n$. Since we do not know what $\tilde{h}_n$ is, we will lower bound over all possible optimal $\tilde{h}_n$, which is certainly the case if we minimize each nth integral over $\tilde{h}_n$

$$\geq \frac{1}{2^\alpha A} \sum_n \left(\min_{\tilde{h}_n > 0} \int_0^1 \frac{(\omega^2 + \tilde{h}_n^2)^\gamma}{\tilde{h}_n} d\omega \right) \mu_n^{1+\alpha}. \tag{110}$$

By Theorem 2 we know that the minimum of the user integrals is achieved for $\tilde{h}_n = g(\alpha)$, which leaves only an uncertainty for μ_n. Hence we get

$$= \frac{1}{2^\alpha A} \int_0^1 \frac{(\omega^2 + g^2(\alpha))^\gamma}{g(\alpha)} d\omega \cdot \sum_n \mu_n^{1+\alpha} \cdot \left(\frac{\sum_n 1^q}{N} \right)^{p/q} \tag{111}$$

which can be lower bounded by Hölder inequality with $p = 1 + \alpha$ and $q = (1+\alpha)/\alpha$ to

$$\geq \frac{1}{2^\alpha A} \int_0^1 \frac{(\omega^2 + g^2(\alpha))^\gamma}{g(\alpha)} d\omega \cdot \underbrace{(\sum_n \mu_n)^{1+\alpha}}_{=A^{1+\alpha}} N^{-\alpha} = \left(\frac{A}{2N} \right)^\alpha \int_0^1 \frac{(\omega^2 + g^2(\alpha))^\gamma}{g(\alpha)} d\omega \tag{112}$$

which can be achieved if and only if $\mu_n = A/N$. But then this is the average power for the common height case (50) , and hence, the only (global) optimal deployment is the common height deployment with uniform ground positions given in Theorem 2. ∎

4.3 Non-uniform distribution over the one-dimensional ground

Note that a great number of literature assumes the common flight height for convenience when they derive the optimal UAV deployment or trajectory. However, in this section, we show via an example that the common flight height is not a necessary condition for the optimal deployment over the space including but not limited to the non-uniform distributed one-dimensional ground.

In what follows, we provide an example to compare the best deployment with common flight height and an alternative deployment with different flight heights.

Consider two UAVs over a one-dimensional ground $[0, 1]$ with path loss parameter $\gamma = 1$ and a non-uniform density function

$$\lambda(\omega) = \begin{cases} 4 & \text{if } 0 \leq \omega \leq 0.2 \\ \frac{1}{4} & \text{if } 0.2 < \omega \leq 1 \\ 0 & otherwise \end{cases} \tag{113}$$

Let $q_1 = (p_1, h_1)$ and $q_2 = (p_2, h_2)$ be the deployment of UAV 1 and 2. First, we derive the best deployment with common flight height assumption, i.e., $h_1 = h_2 = h$. Since the common flight height, the generalized Voronoi partition is degenerated to Voronoi partition, $V_n(p.h) = \{\omega | \|\omega - p_n\| \leq \|\omega - p_m\|, \ \forall \ m \neq n\}$. Without loss of generality, we assume $0 < p_1 < p_2 < 1$, and thus the Voronoi partition can be represented as $V_1(p, h) = [0, b]$ and $V_2(p, h) = [b, 1]$, where the boundary is

$$b = \frac{p_1 + p_2}{2}. \tag{114}$$

(i) If $b \leq 0.2$, the distortions can be rewritten as $D(p, h) = 4\int_0^b \frac{(p_1-\omega)^2+h^2}{h} + 4\int_b^{0.2} \frac{(p_2-\omega)^2+h^2}{h} + \frac{1}{4}\int_{0.2}^1 \frac{(p_2-\omega)^2+h^2}{h}$. The best deployment satisfies zero-gradient, i.e.,

$$\frac{\partial D}{\partial p_1} = 4\int_0^b 2\frac{(p_1 - \omega)}{h} d\omega = 0 \tag{115}$$

$$\frac{\partial D}{\partial p_2} = 4\int_b^{0.2} 2\frac{(p_2 - \omega)}{h} d\omega + \frac{1}{4}\int_{0.2}^1 2\frac{(p_2 - \omega)}{h} d\omega = 0 \tag{116}$$

$$\frac{\partial D}{\partial h} = 4\int_0^b \frac{h^2 - (p_1 - \omega)^2}{h^2} d\omega + 4\int_b^{0.2} \frac{h^2 - (p_2 - \omega)^2}{h^2} d\omega + \frac{1}{4}\int_{0.2}^1 \frac{h^2 - (p_2 - \omega)^2}{h^2} d\omega = 0 \tag{117}$$

Solving (114), (115), and (116), we get a quadratic function $8b^2 - 3b + 0.4 = 0$ which has no real root. Therefore, $b \leq 0.2$ is not feasible.
(ii) If $b > 0.2$, the distortions can be rewritten as $D(p, h) = 4\int_0^{0.2} \frac{(p_1-\omega)^2+h^2}{h} + \frac{1}{4}\int_{0.2}^b \frac{(p_1-\omega)^2+h^2}{h} + \frac{1}{4}\int_b^1 \frac{(p_2-\omega)^2+h^2}{h}$. The best deployment satisfies zero-gradient, i.e.,

$$\frac{\partial D}{\partial p_1} = 4\int_0^{0.2} 2\frac{(p_1 - \omega)}{h} d\omega \frac{1}{4}\int_{0.2}^b 2\frac{(p_1 - \omega)}{h} d\omega = 0 \tag{118}$$

$$\frac{\partial D}{\partial p_2} = 4\int_b^1 2\frac{(p_2 - \omega)}{h} d\omega = 0 \tag{119}$$

$$\frac{\partial D}{\partial h} = 4\int_0^{0.2} \frac{h^2 - (p_1 - \omega)^2}{h^2} d\omega + \frac{1}{4}\int_{0.2}^b \frac{h^2 - (p_1 - \omega)^2}{h^2} d\omega + \frac{1}{4}\int_b^1 \frac{h^2 - (p_2 - \omega)^2}{h^2} d\omega = 0 \tag{120}$$

Solving (114), (118), (119), and (120), we get $p_1^* = \frac{3\sqrt{5.8}-7}{2}$, $p_2^* = \frac{\sqrt{5.8}-1}{2}$, and $h^* = 0.096$. Hence, the minimum distortion with the common flight height is $D(p^*, h^*) = 0.247$. Next, let $\widetilde{p} = p^* = (p_1^*, p_2^*)$ and $\widetilde{h} = (0.05, 0.2)$ be an alternative deployment.

By straightforward calculation, we get the corresponding distortion $D(\widetilde{p}, \widetilde{h}) = 0.192$ which is smaller than that of $D(p^*, h^*)$. In sum, the common flight height is not a necessary condition for the optimal deployment. According to our simulations, there is a big performance gap between the "optimal" deployment with common flight height and the real optimal deployment.

5 Algorithms for deriving Möbius diagrams

To obtain the ground terminal cells we need to derive for a given UAV deployment $\mathbf{Q} = (\mathbf{P}, \mathbf{H})$ its Möbius diagram. Since our performance function is a special case of a Möbius diagram, we might be simplify the algorithms for this non-affine construction [22].

Let us observe that for each $n \neq m$ we have by Lemma 1

$$c_{nm} = c_{mn} \quad , \quad r_{nm} = r_{mn} \tag{121}$$

and by (15)

$$V_{nm} = V_{mn}^c. \tag{122}$$

Hence, we will reorder the N coordinates $q_1, \dots, q_N$ in ascending order of their heights, i.e.,

$$\tilde{Q} = (\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_N) \quad , \quad \tilde{h}_1 \geq \tilde{h}_2 \geq \cdots \geq \tilde{h}_N. \tag{123}$$

This allows to derive two symmetric centroid and radii matrices

$$\tilde{C} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \dots & \tilde{c}_{1N} \\ \tilde{c}_{21} & \tilde{c}_{22} & \dots & \tilde{c}_{2N} \\ \vdots & & \ddots & \vdots \\ \tilde{c}_{N1} & \tilde{c}_{N2} & \dots & \tilde{c}_{NN} \end{pmatrix} \quad , \quad \tilde{R} = \begin{pmatrix} \tilde{r}_{11} & \tilde{r}_{12} & \dots & \tilde{r}_{1N} \\ \tilde{r}_{21} & \tilde{r}_{22} & \dots & \tilde{r}_{2N} \\ \vdots & & \ddots & \vdots \\ \tilde{r}_{N1} & \tilde{r}_{N2} & \dots & \tilde{r}_{NN} \end{pmatrix} \tag{124}$$

Then nth row will generate the nth Möbius region by

$$\tilde{V}_n = \Big(\bigcap_{1 \leq i < n} B(\tilde{c}_{ni}, \tilde{r}_{ni}) \Big) \cap \Big(\bigcap_{n < i \leq N} B(\tilde{c}_{ni}, \tilde{r}_{ni})^c \Big) \tag{125}$$

Obviously, if the centroids of two balls have distance larger then the sum of their radii, then the intersection will be empty. We will therefore need to calculate for $n < i < j \leq N$

$$d_{i,j}^{(n)} = |\tilde{c}_{ni} - \tilde{c}_{nj}| \quad , \quad \tilde{r}_{i,j}^{(n)} = \tilde{r}_{ni} + \tilde{r}_{nj} \tag{126}$$

Furthermore, if $\tilde{r}_{ni} = \min\{\tilde{r}_{ni}, \tilde{r}_{nj}\} \leq d_{i,j}^{(n)}$, then C_{ni} is contained in C_{nj}

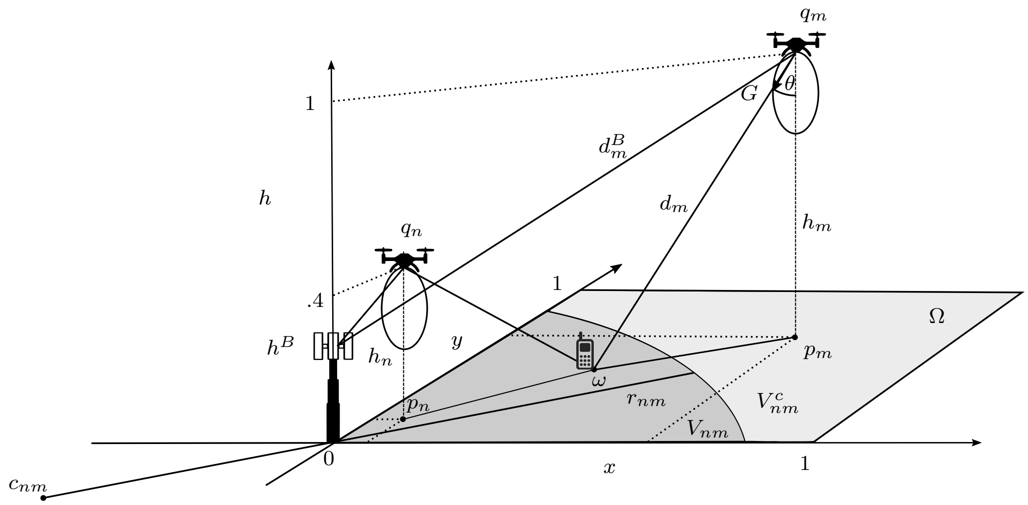

Figure 3: UAV base-station deployment with directed antenna beam for $\alpha = 2$ and $N = 2$ for a uniform GT distribution and perfect antenna alignment to the ground base-station at $q^B = (0, 0, h^B)$.

6 UAV as flying Base-Stations

We will extend our UAV setup to a flying base-station scenario, where each UAV act as a relay between the base-station and a ground terminal user, see for example [12]. We assume that the GTs are placed on Ω according to a time-invariant density function $f : \mathbb{R}^2 \to \mathbb{R}$, where $\int_\Omega f(\omega)d\omega = 1$ [14]–[17]. Further, the N UAVs are positioned at $(\mathbf{P}, \mathbf{H})$ and will relay the communication from a base-station at $q^B = (p_x^B, p_y^B, h^B)$ as in Figure 3. Thus, the total UAV transmit power or average GT transmit power can be rewritten as

$$D(\mathbf{p}, \mathbf{h}) = \int_\Omega \min_n \left\{ \beta \frac{(\|p_n - \omega\|^2 + h_n^2)^{\frac{1+\alpha}{2}}}{h_n} + \underbrace{\left\| q_n - q^B \right\|^\alpha}_{=d_n^B} \right\} f(\omega)\, d\omega, \tag{127}$$

where we assume that the UAV has a directed perfectly aligned antenna to the base station, such that for $\theta = 0$ by (4) it holds for the antenna relative radiation intensity $G_{BS} = 1$. Hence, only the path loss exponent remains. By setting the base-station at $q^B = (0, 0, h^B)$ we get

$$D(\mathbf{p}, \mathbf{h}) = \int_\Omega \min_n \left\{ \beta \frac{(\|p_n - \omega\|^2 + h_n^2)^{\frac{1+\alpha}{2}}}{h_n} + (\|p_n\|^2 + (h_n - h^B)^2)^{\frac{\alpha}{2}} \right\} f(\omega)\, d\omega. \tag{128}$$

7 Dynamic UAV deployment

We assume that the GTs are placed on Ω according to a periodic time-variant density function $f : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$, where $\int_\Omega f(\omega, t)d\omega = 1, \ \forall\ t \in [0, T]$ and

T is the period [1]. To keep track of the UAV trajectory over $[0,T]$, we combine $\mathbf{p}$ and $\mathbf{h}$ by $q(t)=(q_1(t),\ldots,q_N(t))$, where $q_n(t)=(q_{nx}(t),q_{ny}(t),q_{nh}(t))=(p_{nx}(t),p_{ny}(t),h_n(t))$ is UAV n's location at time instance t. In particular, $q(0)=(q_1(0),\ldots,q_N(0))$ represents the initial UAV deployment. For convenience, we define $d\left(q,\omega\right)=\sqrt{\left(q_x-\omega_x\right)^2+\left(q_y-\omega_y\right)^2+q_h^2}$ as the distance between 3D point $q=(q_x,q_y,q_h)$ and ground point $\omega=(\omega_x,\omega_y)$. Then, the total communication energy consumption over one period can be rewritten as

$$E_c\left(q\right)=\beta\int_{\tau=0}^{T}\int_{\Omega}\min_{n}\left\{\frac{\left[d\left(q_n(\tau),\omega\right)\right]^{1+\alpha}}{q_{nh}\left(\tau\right)}\right\}f\left(\omega,\tau\right)d\omega d\tau. \tag{129}$$

Let $\mathbf{U}(t)=(u_1(t),\ldots,u_N(t))$ be UAVs' dynamic control, where $u_n(t)=[u_{nx}(t),u_{ny}(t),u_{nh}(t)]=\left[\frac{dq_{nx}(t)}{dt},\frac{dq_{ny}(t)}{dt},\frac{dq_{nh}(t)}{dt}\right]$ is UAV n's velocity. As a result, UAV n's locations can be represented as

$$q_n(t)=q_n(0)+\int_{\tau=0}^{t}u_n(\tau)d\tau, \tag{130}$$

Like [16], [17], we formulate the motion energy over one period as a function of velocity, i.e.,

$$E_m\left(\mathbf{U}\right)=\xi\int_{\tau=0}^{T}\mathbf{U}_n^T(\tau)\mathbf{U}(\tau)d\tau=\xi\int_{\tau=0}^{T}\sum_{n=1}^{N}u_n^2(\tau)d\tau, \tag{131}$$

where ξ is a constant which characterizes moving efficiency. Thus, the total energy consumption over one period (or performance cost) can be rewritten as

$$\begin{aligned}\bar{J}\left(q,\mathbf{U}\right)=&E_c\left(q\right)+E_m\left(\mathbf{U}\right)\\=&\beta\int_{\tau=0}^{T}\int_{\Omega}\min_{n}\left\{\frac{\left[d\left(q_n(\tau),\omega\right)\right]^{1+\alpha}}{q_{nh}\left(\tau\right)}\right\}f\left(\omega,\tau\right)d\omega d\tau+\xi\int_{\tau=0}\sum_{n=1}^{N}u_n^2(\tau)d\tau\\=&\beta\int_{\tau=0}^{T}\sum_{n=1}^{N}\left[\int_{V_n(q(\tau))}\frac{\left[d\left(q_n(\tau),\omega\right)\right]^{1+\alpha}}{q_{nh}\left(\tau\right)}f\left(\omega,\tau\right)d\omega+\frac{\xi}{\beta}u_n^2(\tau)\right]d\tau,\end{aligned} \tag{132}$$

where $V_n\left(q(\tau)\right)$ is UAV n's LVD generated by $q\left(\tau\right)$. To omit the units of all quantities for brevity, we define

$$J\left(q,\mathbf{U}\right)=\int_{\tau=0}^{T}\sum_{n=1}^{N}\left[\int_{V_n(q(\tau))}\frac{\left[d\left(q_n(\tau),\omega\right)\right]^{1+\alpha}}{q_{nh}\left(\tau\right)}f\left(\omega,\tau\right)d\omega+\lambda u_n^2(\tau)\right]d\tau, \tag{133}$$

where $\lambda=\frac{\xi}{\beta}$ is a constant. In what follows, we concentrate on the optimization of $J\left(q,\mathbf{U}\right)$ over $\mathbf{U}$. By using the imbedding principle, we include (133) into a larger class of functions [28]

$$J(q(t),t,\{\mathbf{U}(\tau)\}_{t\leq\tau\leq T})=\int_{\tau=t}^{T}\sum_{n=1}^{N}\left[\int_{V_n(q(\tau))}\frac{\left[d\left(q_n(\tau),\omega\right)\right]^{1+\alpha}}{q_{nh}\left(\tau\right)}f\left(\omega,\tau\right)d\omega+\lambda u_n^2(\tau)\right]d\tau. \tag{134}$$

Note that (134) represents the performance cost over the time interval $[t, T]$ with starting state (deployment), $q(t)$, and control history (velocities) $\{\mathbf{U}(\tau)\}_{t\leq\tau\leq T}$. Our final goal is to optimize $J(q(0), 0, \{\mathbf{U}(\tau)\}_{0\leq\tau\leq T})$ where $q(0)$ is the initial UAV deployment. The minimum performance cost is then

$$J^*\left(q(t), t\right) = \min_{\{\mathbf{U}(\tau)\}_{t\leq\tau\leq T}} J(q(t), t, \{\mathbf{U}(\tau)\}_{t\leq\tau\leq T})$$
$$= \min_{\{\mathbf{U}(\tau)\}_{t\leq\tau\leq T}} \left\{ \int_{\tau=t}^{T} \sum_{n=1}^{N} \left[\int_{V_n(q(\tau))} \frac{\left[d\left(q_n(\tau), \omega\right)\right]^{1+\alpha}}{q_{nh}\left(\tau\right)} f\left(\omega, \tau\right) d\omega + \lambda u_n^2(\tau) \right] d\tau \right\}. \tag{135}$$

For convenience, we define $J_q^*\left(q(t), t\right) \triangleq \frac{\partial J^*}{\partial q}\left(q(t), t\right) = \left[\frac{\partial J^*}{\partial q_1}\left(q(t), t\right), \ldots, \frac{\partial J^*}{\partial q_N}\left(q(t), t\right)\right]$ and $J_t^*\left(q(t), t\right) \triangleq \frac{\partial J^*}{\partial t}\left(q(t), t\right)$. The Hamilton-Jacobi-Bellman equation [28], [29] for (134) is

$$J_t^*\left(q(t), t\right) + \inf_{\mathbf{U}(t)} \left\{\mathcal{H}\left(q(t), \mathbf{U}(t), J_q^*, t\right)\right\} = 0 \tag{136}$$

where

$$\mathcal{H}\left(q(t), \mathbf{U}(t), J_q^*, t\right) = \mathcal{F}\left(q(t), \mathbf{U}(t)\right) + J_q^*\left(q(t), t\right) \cdot \mathbf{U}\left(t\right) \tag{137}$$

and

$$\mathcal{F}\left(q(t), \mathbf{U}(t)\right) = \sum_{n=1}^{N} \left[\int_{V_n(q(t))} \left(\frac{\left[d\left(q_n(t), \omega\right)\right]^{1+\alpha}}{q_{nh}\left(t\right)} \right) f\left(\omega, t\right) d\omega + \lambda u_n^2(t) \right]. \tag{138}$$

The boundary value for this partial differential equation is

$$J^*\left(q(T), T\right) = 0. \tag{139}$$

Differentiating $\mathcal{H}$ with respect to $u_n(t)$, we get a necessary condition for infimum

$$\frac{\partial \mathcal{H}}{\partial u_n(t)}\left(q(t), \mathbf{U}(t), J_q^*, t\right) = 2\lambda u_n(t) + \frac{\partial J^*}{\partial q_n}\left(q(t), t\right) = 0 \Rightarrow \frac{\partial J^*}{\partial q_n}\left(q(t), t\right) = -2\lambda u_n(t) \tag{140}$$

Since it's a necessary condition for the optimal solution, we have

$$\frac{\partial J^*}{\partial q_n}\left(q^*(t), t\right) = -2\lambda u_n^*(t) \tag{141}$$

The partial derivative $J_t^*\left(q^*(t), t\right)$ can be calculated as

$$J_t^*\left(q^*(t), t\right) = -\sum_{n=1}^{N} \left[\int_{V_n(q^*(t))} \frac{\left[d\left(q_n^*(t), \omega\right)\right]^{1+\alpha}}{q_{nh}^*\left(t\right)} f\left(\omega, t\right) d\omega + \lambda \left(u_n^*(t)\right)^2 \right] \tag{142}$$

Replacing (140) in (136) and (137), we get ...
One possible solution for (**??**) is ?
To be continued ...

8 Simulation Results

9 Future Work

In the future, we will take (1) maximum transmitter power, (2) motion energy, and (3) time-variant density function into consideration.

A Proof of Uniform Scalar Quantizer

Proof of Lemma 2. By the *conservation law of mass* [18, Thm.2.2], the partial derivatives of $H_V(\mathbf{x})$ after x_n are given by

$$\frac{\partial H_V(\mathbf{x})}{\partial x_n} = \int_{\mathcal{V}_n(\mathbf{x})} \frac{\partial (x_n - \omega)^2}{\partial x_n} d\omega \quad \Rightarrow \quad 0 \stackrel{!}{=} \int_{\mathcal{V}_n(\mathbf{x}^*)} (x_n^* - \omega) d\omega \Leftrightarrow x_n^* = \frac{\int_{\mathcal{V}_n(\mathbf{x}^*)} \omega d\omega}{\int_{\mathcal{V}_n(\mathbf{x}^*)} d\omega} \tag{143}$$

which is the centroid of the optimal Voronoi region $\mathcal{V}_n(\mathbf{x}^*)$. Since the Voronoi region depend on all x_n we need the explicit parameterization of the regions. W.l.o.g. we can label the N quantizer points such that their ground positions are in increasing order, i.e., $0 \leq x_1 < x_2 < \cdots < x_N \leq A$, where we assume that all N points are different and contained in the target region $\Omega = [0, A]$, otherwise not all points would be active and the number of effective points would be less than N. We will show later, that inactive points will increase the objective function and therefore not yield the global minimum. Each Euclidean Voronoi region is by definition the intersection of $N-1$ dominance regions $\mathcal{V}_{nm}(\mathbf{x}) = [0, (x_m + x_n)/2]$ for $n < m$ and $[(x_n + x_m)/2, A]$ for $n > m$ given by

$$\mathcal{V}_n(\mathbf{x}) = \bigcap_m \mathcal{V}_{nm}(\mathbf{x}) = \bigcap_{n<m} [0, \frac{x_m + x_n}{2}] \bigcap_{n>m} [\frac{x_n + x_m}{2}, A] \tag{144}$$

$$= [b_{n-1}, b_n] = \begin{cases} [0, \frac{x_2 + x_1}{2}] & , n = 1 \\ [\frac{x_N + x_{N-1}}{2}, A] & , n = N \\ [\frac{x_n + x_{n-1}}{2}, \frac{x_{n+1} + x_n}{2}] & , \text{else} \end{cases} \tag{145}$$

The integral of the centroid (143) can then be calculated as

$$x_n^* = \frac{\frac{1}{2}\omega^2 \Big|_{a_n^*}^{b_n^*}}{b_n^* - a_n^*} = \frac{1}{2} \frac{(b_n^*)^2 - (a_n^*)^2}{b_n^* - a_n^*} = \frac{b_n^* + a_n^*}{2}. \tag{146}$$

For $1 < n < N$ we get

$$x_n^* = \frac{x_{n+1}^* + 2x_n^* + x_{n-1}^*}{4} = \frac{x_{n+1}^* + x_{n-1}^*}{4} + \frac{x_n^*}{2} \quad \Rightarrow \quad x_n^* = \frac{x_{n+1}^* + x_{n-1}^*}{2}. \tag{147}$$

For $n = 1$ and $n = N$ we get

$$x_1^* = \frac{1}{3} x_2^* \quad , \quad x_N^* = \frac{1}{3}(2A + x_{N-1}^*). \tag{148}$$

The two equations contain 4 unknowns and are therefore under-determined. To resolve the positions, we need to extract from (147) two more equations in the same 4 unknowns x_1, x_2, x_N, x_{N-1} (Let us omit the star-index for now). We prove by induction that it holds for each $N-1 \geq m \geq 2$

$$P(m) :\Leftrightarrow x_1 = m x_m - (m-1) x_{m+1}. \tag{149}$$

Let us resolve (147) to

$$x_{n-1} = 2x_n - x_{n+1} \quad , \quad N-1 \geq n \geq 2 \tag{150}$$

then for $m = n = 2$ this proofs the induction start $P(2)$. Let us assume $P(m)$ is true, than we get with $m = n - 1 \geq 2$ in (150)

$$x_1 = m x_m - (m-1)x_{m+1} = m(2x_{m+1} - x_{m+2}) - (m-1)x_{m+1} = (m+1)x_{m+1} - m x_{m+2}$$

which proofs $P(m+1)$ and therefore the assumption $P(m)$ for all $N-1 \geq m \geq 2$. Similar we can show for $N-1 \geq m \geq 2$ the assertion

$$\tilde{P}(m) :\Leftrightarrow x_N = m x_{N-(m-1)} - (m-1)x_{N-m}. \tag{151}$$

Resolving (150) to $x_{n+1} = 2x_n - x_{n-1}$ yields for $m = 2$ and $n = N - m + 1$ the induction start $\tilde{P}(2)$ and with $\tilde{P}(m)$ and $n = N - m$ this shows $\tilde{P}(m+1)$. Then from (148), $P(N-1)$, and $\tilde{P}(N-1)$ we get the two equations

$$x_1 = (N-1)(3x_N - 2A) - (N-2)x_N = (2N-1)x_N - 2(N-1)A \tag{152}$$
$$x_N = (N-1)x_2 - (N-2)x_1 = (N-1)3x_1 - (N-2)x_1 = (2N-1)x_1 \tag{153}$$

which yields to

$$x_1 = (2N-1)(2N-1)x_1 - 2(N-1)A \Leftrightarrow A = \frac{4N^2 - 4N + 1 - 1}{2(N-1)} x_1 \Leftrightarrow x_1 = \frac{A}{2N}. \tag{154}$$

Inserting in (153) gives

$$x_N = \frac{(2N-1)A}{2N}. \tag{155}$$

Together with (150) this shows $x_n = (2n-1)A/(2N)$ for $n = 1, 2, \ldots, N$ and $N = 1, 2, 3$. For $N \geq 4$ we need to derive the pairwise ground distances, given for $N-1 \geq m \geq 3$ by

$$\Delta_m = x_m - x_{m-1} \overset{P(m-1)}{=} x_m - \frac{x_1 + (m-2)x_m}{m-1} = \frac{x_m - x_1}{m-1} \tag{156}$$
$$\Delta_m = x_m - x_{m-1} \overset{\tilde{P}(N-m+1)}{=} x_m + \frac{x_N - (N-m+1)x_m}{N-m} = \frac{x_N - x_m}{N-m} \tag{157}$$

Eliminating x_m yields to

$$(m-1)\Delta_m + x_1 = x_N - \Delta_m(N-m) \Leftrightarrow \Delta_m = \frac{x_N - x_1}{N-1} \tag{158}$$

Finally, inserting (155) and (154) yields to

$$\Delta_m = \Delta = \frac{(2N-1)A - A}{2N(N-1)} = \frac{A}{N} \tag{159}$$

and since $x_2 - x_1 = A/N = x_N - x_{N-1}$ by (148) and (153) resp. (155) (159) holds for $2 \leq m \leq N$. Hence, we get for $n = 1, 2, \ldots, N$ and any $N \geq 1$

$$x_n^* = x_1^* + (n-1)\Delta = \frac{(2n-1)A}{2N}. \tag{160}$$

To derive the optimal height we need to calculate $H_V(\mathbf{x}^*)$. Calculating explicitly the integrals we get for the Euclidean distortion

$$AH_V(\mathbf{x}^*) := (\frac{1}{3}\Big((\omega - x_1^*)^3\Big|_0^{b_1} + (\omega - x_N^*)^3\Big|_{b_{N-1}}^{A}\Big) + \sum_{n=2}^{N-1}\frac{1}{3}(\omega - x_n^*)^3\Big|_{b_{n-1}}^{b_n} \tag{161}$$

with the bisectors given by (145) and (160)

$$b_{n-1} = \frac{x_n^* + x_{n-1}^*}{2} = \frac{(2n-1+(2(n-1)-1)))A}{2\cdot 2N} = \frac{(n-1)A}{N} \quad \text{and} \quad b_0 = 0, b_N = A. \tag{162}$$

The difference is then given by

$$\frac{1}{3}\sum_n\Big(\Big(\frac{(2n-(2n-1))A}{2N}\Big)^3 - \Big(\frac{(2n-2-(2n-1))A}{2N}\Big)^3\Big) = \sum_n \frac{2A^3}{3\cdot 8N^3} = (N-2)\frac{A^3}{12N^3}$$

which yields with the first and last cell

$$\frac{1}{3}\Big(\Big(\frac{2A-A}{2N}\Big)^3 - \frac{-A^3}{(2N)^3} + \Big(\frac{2NA-(2N-1)A}{2N}\Big)^3 - \Big(\frac{2(N-1)A-(2N-1)A}{2N}\Big)^3\Big) = \frac{4}{3}\Big(\frac{A}{2N}\Big)^3$$

to

$$H_V(\mathbf{x}^*) = \frac{1}{A}\left((N-2)\frac{A^3}{12N^3} + 2\frac{A^3}{12N^3}\right) = \frac{A^2}{12N^2}. \tag{163}$$

Now we see, that if there are inactive quantization points, N would decrease and the power consumption increase. Therefore, inactive point configurations can be seen as local minima and the all active case as the global minima. ∎

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