A History of Clifford Algebras

Clifford's geometric algebras were created by William K. Clifford in 1878/1882, when he introduced a new multiplication rule into Grassmann's exterior algebra $\bigwedge \mathbb{R}^n$, by means of an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ of $\mathbb{R}^n$. Clifford also classified his algebras into four classes according to the signs in $(\mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_n)^2 = \pm 1$ and $(\mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_n)\mathbf{e}_i = \pm\mathbf{e}_i(\mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_n)$. In the special case of $n = 3$, Clifford's construction embodied Hamilton's quaternions, as bivectors $i = \mathbf{e}_2\mathbf{e}_3$, $j = \mathbf{e}_3\mathbf{e}_1$, $k = \mathbf{e}_1\mathbf{e}_2$. Clifford algebras were independently rediscovered by Lipschitz 1880/1886, who also presented their first application to geometry, while exploring rotations of $\mathbb{R}^n$, in terms of $\mathbf{Spin}(n)$, a normalized subgroup of the Lipschitz group $\mathbf{\Gamma}_n^+$.

Spinor representations of the orthogonal Lie algebras, $B_\ell = so(2n+1)$ and $D_\ell = so(2n)$, were observed by E. Cartan in 1913, but without using the term "spinor". Two-valued spinor representations of the rotation groups $SO(n)$ were re-constructed recursively by Brauer & Weyl in 1935, but without using the term "Clifford algebra".

In the Schrödinger equation, Pauli 1927 replaced $\pi^2 = \vec{\pi}\cdot\vec{\pi}$, where $\vec{\pi} = -i\hbar\nabla - e\vec{A}$, by

$$\vec{\pi}^2 = \vec{\pi}\cdot\vec{\pi} + \vec{\pi}\wedge\vec{\pi},$$

where the exterior part does not vanish: $(\vec{\pi}\wedge\vec{\pi})\psi = -\hbar e\vec{B}\psi$. Pauli explained interaction of a spinning electron with a magnetic field $\vec{B}$ by means of a spinor-valued wave function $\psi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}^2$. Thus, besides tensors, nature required new kinds of objects: spinors, whose construction calls for Clifford algebras.

Clifford algebras are not only necessary but also offer advantages: their multivector structure enables controlling of subspaces without losing information about their orientations. Physicists are familiar with this advantage in the special case of 1-dimensional oriented subspaces, which they manipulate by vectors, not by projection operators, which forget orientations.

23.13 Algebras of Hamilton, Grassmann and Clifford

The first step towards a Clifford algebra was taken by Hamilton in 1843 (first published 1844), when he studied products of sums of squares and invented his quaternions while searching for multiplicative compositions of triplets in $\mathbb{R}^3$. The present formulation of vector algebra was extracted from the quaternion product of triplets/vectors $\mathbf{xy} = -\mathbf{x}\cdot\mathbf{y} + \mathbf{x}\times\mathbf{y}$ by Gibbs 1881-84 (first published 1901). Hamilton regarded quaternions as quotients of vectors and wrote a rotation in the form $\mathbf{y} = a\mathbf{x}$ using a unit quaternion $a \in \mathbb{H}$, $|a| = 1$. In such a rotation, the axis $\mathbf{a}$ had to be perpendicular to the vector $\mathbf{x}$ (which turned in the plane orthogonal to $\mathbf{a}$). Cayley 1845 published the formula for rotations

$$\mathbf{y} = a\mathbf{x}a^{-1}, \quad a = \cos(\frac{\alpha}{2}) + \frac{\mathbf{a}}{\alpha}\sin(\frac{\alpha}{2}),$$

about an arbitrary axis $\mathbf{a} \in \mathbb{R}^3$ by angle $\alpha = |\mathbf{a}|$ (Cayley ascribed the discovery to Hamilton). Cayley thus came into contact with half-angles and spin representation of rotations in $\mathbb{R}^3$. However, in 1840 Olinde Rodrigues had already recognized the relevancy of half-angles in his study on the composition of rotations in $\mathbb{R}^3$. Cayley 1855 also discovered the quaternionic representation $q \to aqb$ of rotations in $\mathbb{R}^4$, equivalent to the decomposition $\mathbf{Spin}(4) \simeq \mathbf{Spin}(3) \times \mathbf{Spin}(3)$.

The quaternion algebra $\mathbb{H}$ is isomorphic to the even Clifford algebras $\mathcal{C}\ell_3^+ \simeq \mathcal{C}\ell_{0,3}^+$ as well as to the proper ideals $\mathcal{C}\ell_{0,3}\frac{1}{2}(1 \pm \mathbf{e}_{123})$ of $\mathcal{C}\ell_{0,3}$. Hamilton also considered complex quaternions $\mathbb{C}\otimes\mathbb{H}$, isomorphic to the Clifford algebra $\mathcal{C}\ell_3$ on $\mathbb{R}^3$. Note the algebra isomorphisms $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$ and $\mathcal{C}\ell_{0,3} \simeq \mathbb{H}\oplus\mathbb{H}$.

Bivectors were introduced by H. Grassmann, when he created his exterior algebras in 1844. The exterior product of two vectors, the bivector $\mathbf{a}\wedge\mathbf{b}$, was interpreted geometrically as the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as edges, and two such exterior products were equal if their parallelograms lay in parallel planes and had the same area with the same sense of rotation (from $\mathbf{a}$ to $\mathbf{b}$). Thus the exterior product of two vectors was anticommutative, $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$. Using a basis $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$ for $\mathbb{R}^n$, the exterior algebra $\bigwedge\mathbb{R}^n$ had a basis

$$\begin{array}{l} 1 \\ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \\ \mathbf{e}_1\wedge\mathbf{e}_2, \mathbf{e}_1\wedge\mathbf{e}_3, \ldots, \mathbf{e}_1\wedge\mathbf{e}_n, \mathbf{e}_2\wedge\mathbf{e}_3, \ldots, \mathbf{e}_{n-1}\wedge\mathbf{e}_n \\ \quad\vdots \\ \mathbf{e}_1\wedge\mathbf{e}_2\wedge\ldots\wedge\mathbf{e}_n \end{array}$$

and was thereby of dimension 2^n.

W. K. Clifford studied compounds (tensor products) of two quaternion algebras, where quaternions of one algebra commuted with the quaternions of the other algebra, and applied exterior algebra to grade these tensor products of quaternion algebras. Clifford coined his **geometric algebra** in 1876 (first published in 1878). Clifford's geometric algebra was generated by n orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ which anticommuted $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ (like Grassmann) and satisfied all $\mathbf{e}_i^2 = -1$ (like Hamilton) [or then all $\mathbf{e}_i^2 = 1$ as in Clifford's paper 1882]. The number of independent products $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$, $i < j$, of degree 2 was $\frac{1}{2}n(n-1) = \binom{n}{2}$. Clifford summed up the numbers of independent products of various degrees $0, 1, 2, \ldots, n$ and thereby determined the dimension of his geometric algebra to be

$$1 + n + \binom{n}{2} + \ldots + 1 = 2^n.$$

Clifford distinguished four classes of these geometric algebras characterized by the signs of $(\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)\mathbf{e}_i = \pm\mathbf{e}_i(\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)$ and $(\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)^2 = \pm 1$. He also introduced two algebras of lower dimension 2^{n-1}, namely, the subalgebra of even elements and, for odd n, a reduced (non-universal) algebra obtained by putting $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n = \pm 1$ [instead of letting $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n$ be of degree n with $(\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)^2 = 1$].

23.14 Rudolf Lipschitz

Clifford's geometric algebra was reinvented in 1880, just two years after its first publication, by Rudolf Lipschitz, who later acknowledged Clifford's prior discovery in his book, see R. Lipschitz: *Untersuchungen über die Summen von Quadraten*, 1886. In his study on sums of squares, Lipschitz considered a representation of rotations by complex numbers and quaternions and generalized this to higher dimensional rotations in $\mathbb{R}^n$. Lipschitz thus gave the first geometric application of Clifford algebras in 1880. He expressed a rotation

$$\mathbf{y} = (I + A)(I - A)^{-1}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n,$$

(written here in modern notation with an antisymmetric matrix A) in the form $\mathbf{y} - A\mathbf{y} = \mathbf{x} + A\mathbf{x}$ or as $\mathbf{y} + \mathbf{y} \lrcorner \mathbf{B} = \mathbf{x} + \mathbf{B} \llcorner \mathbf{x}$ where $\mathbf{B} \in \bigwedge^2 \mathbb{R}^n$ is the bivector determined by $A\mathbf{x} = \mathbf{B} \llcorner \mathbf{x}$ $(= -\mathbf{x} \lrcorner \mathbf{B})$. Lipschitz rewrote $(I - A)\mathbf{y} = (I + A)\mathbf{x}$ using an even Clifford number $a \in \mathcal{C}\ell_{0,n}^+$ in the form $\mathbf{y}a = a\mathbf{x}$, thus representing the rotation as

$$\mathbf{y} = a\mathbf{x}a^{-1}, \quad a \in \mathbf{\Gamma}_{0,n}^+.$$

[Lipschitz wrote $ya_1 = ax$ where $x = \mathbf{x}\mathbf{e}_1^{-1}$, $y = \mathbf{y}\mathbf{e}_1^{-1}$, $a_1 = \mathbf{e}_1 a \mathbf{e}_1^{-1}$.] In modern terms (introduced by Chevalley), Lipschitz used the exterior exponential

$$a = e^{\wedge\mathbf{B}}, \quad e^{\wedge\mathbf{B}} = 1 + \mathbf{B} + \frac{1}{2}\mathbf{B} \wedge \mathbf{B} + \frac{1}{6}\mathbf{B}^{\wedge 3} + \cdots,$$

of the bivector **B** so that the normalized element $a/|a|$ was in the spin group $\mathbf{Spin}(n)$.

23.15 Theodor Vahlen

Vahlen 1897 found an explicit expression for the multiplication rule of two basis elements in $C\ell_{0,n}$

$$(\mathbf{e}_1^{\alpha_1}\mathbf{e}_2^{\alpha_2}\ldots\mathbf{e}_n^{\alpha_n})(\mathbf{e}_1^{\beta_1}\mathbf{e}_2^{\beta_2}\ldots\mathbf{e}_n^{\beta_n}) = (-1)^{\sum_{i\geq j}\alpha_i\beta_j}\mathbf{e}_1^{\alpha_1+\beta_1}\mathbf{e}_2^{\alpha_2+\beta_2}\ldots\mathbf{e}_n^{\alpha_n+\beta_n}$$

where the exponents are 0 or 1 (added here modulo 2, although for Vahlen 1+1 = 2, so that summation was over $i > j$). Vahlen's formula has frequently been reinvented afterwards: for positive metrics by Brauer & Weyl 1935, for arbitrary metrics by Deheuvels 1981 p. 294, disguised with index sets as in Chevalley 1946 p. 62, Artin 1957 p. 186 and Brackx & Delanghe & Sommen 1982 p. 2, or hidden among permutations as in Kähler 1960/62 and Delanghe & Sommen & Souček 1992, pp. 58-59.

Although Brauer & Weyl 1935 reinvented (in the case of the Clifford algebra $C\ell_n$) the above explicit multiplication formula of Vahlen, they did not observe the connection to the Walsh functions (discovered in the meantime by Walsh 1923). The connection to the Walsh functions was observed by Hagmark & Lounesto 1986.

In 1902, Vahlen initiated the study of Möbius transformations of vectors in $\mathbb{R}^n$ (or paravectors in $\mathbb{R}\oplus\mathbb{R}^n$) by 2×2-matrices with entries in $C\ell_{0,n}$. This study was re-initiated by Ahlfors in the 1980's.

23.16 Elie Cartan

Besides detecting spinors in 1913 (and pure spinors in 1938), Cartan made two other contributions to Clifford algebras: their periodicity of 8 and the triality of $\mathbf{Spin}(8)$.

Cartan 1908 p. 464, identified the Clifford algebras $C\ell_{p,q}$ as matrix algebras with entries in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{R}\oplus\mathbb{R}$, $\mathbb{H}\oplus\mathbb{H}$ and found a periodicity of 8. To decipher Cartan's notation:

h		
± 1	S_m	$\mathrm{Mat}(m,\mathbb{R})$
± 2	IS_m	$\mathrm{Mat}(m,\mathbb{C})$
± 3	QS_m	$\mathrm{Mat}(m,\mathbb{H})$
0	$2S_m$	${}^2\mathrm{Mat}(m,\mathbb{R})$
4	$2Q_m$	${}^2\mathrm{Mat}(m,\mathbb{H})$

where $h = 1 - p + q \pmod 8$. Clifford's original notion of 4 classes was thus refined to 8 classes (and generalized from $C\ell_n$ and $C\ell_{0,n}$ to $C\ell_{p,q}$). [20]

Cartan's periodicity of 8 for real Clifford algebras, with an involution, was extended by C.T.C Wall 1968 and Porteous 1969 (rediscovered by Harvey 1990). Wall considered real graded Clifford algebras, with an anti-involution, and found a 2-way periodicity of type $(8 \times 8)/2$, like the movements of a bishop on a chessboard. Porteous used the anti-involution to induce a scalar product for spinors, and classified the scalar products of spinors into 32 classes, according to the signature types (p, q) of real quadratic spaces $\mathbb{R}^{p,q}$.

In 1925, Cartan came into contact with the triality automorphism of $\mathbf{Spin}(8)$. Lounesto 1997 (in the first edition of this book) showed that triality is a restriction of a polynomial mapping $C\ell_8 \to C\ell_8$, of degree 2.

23.17 Ernst Witt

Witt 1937 started the modern algebraic theory of quadratic forms. He identified Clifford algebras of non-degenerate quadratic forms over arbitrary fields of characteristic $\neq 2$. The Witt ring $W(\mathbb{F})$, of a field $\mathbb{F}$, consists of similarity classes of non-singular quadratic forms over $\mathbb{F}$ (similar quadratic forms have isometric anisotropic parts). In characteristic $\neq 2$, the structure of Clifford algebras of certain quadratic forms was studied by Lee 1945/48 ($\mathbf{e}_i^2 = 1$), Chevalley 1946 ($\mathbf{e}_i^2 = -1$), and Kawada & Iwahori 1950 ($\mathbf{e}_i^2 = \pm 1$). These authors did not benefit the Witt ring (although they already had it at their disposal), and so they did not consider all the isometry classes of anisotropic quadratic forms.

Example. The Witt ring $W(\mathbb{F}_5)$ of the finite field $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ of characteristic 5 contains four isometry classes 0, $\langle 1 \rangle$, $\langle s \rangle$, $\langle 1, s \rangle$ where $s = 2$ or $s = 3$. Chevalley 1946, Lee and Kawada & Iwahori did not notice that none of the quadratic forms $x_1^2 + x_2^2$, $x_1^2 - x_2^2$, $-x_1^2 - x_2^2$ on the plane $\mathbb{F}_5^2$ is isometric with $x_1^2 + s x_2^2 \simeq \langle 1, s \rangle$ (but in fact they are all neutral, and thereby in the same isometry class as 0). A simpler example is the line $\mathbb{F}_5$ where the Clifford algebra of $2x^2 \simeq \langle 2 \rangle$ is the quadratic extension $\mathbb{F}_5(\sqrt{2})$ whereas the Clifford algebras of both $\pm x^2 \simeq \langle \pm 1 \rangle$ split $\mathbb{F}_5 \times \mathbb{F}_5$. ∎

20 Cartan's periodicity of 8 for Clifford algebras, from 1908, is often attributed to Bott, who was born in 1923 and proved his periodicity of homotopy groups of rotation groups in 1959.

23.18 Claude Chevalley

Chevalley 1954 constructed Clifford algebras as subalgebras of the endomorphism algebra of the exterior algebra, $\mathcal{C}\ell(Q) \subset \mathrm{End}(\bigwedge V)$, by means of a not necessarily symmetric bilinear form B on V such that $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$. By this construction, Chevalley managed to include the exceptional case of characteristic 2, and thus amended the work of Witt.

Chevalley 1954 went on further and gave the most general definition, as a factor algebra of the tensor algebra, $\mathcal{C}\ell(Q) = \otimes V/\mathcal{I}_Q$, valid also when ground fields are replaced by commutative rings. From the pedagogical point of view, this approach is forbidding, while it refers to the infinite-dimensional tensor algebra $\otimes V$.

Chevalley 1954 introduced exterior exponentials of bivectors and used them to scrutinize the Lipschitz group, unfairly naming it a 'Clifford group'. Thus, there were two exponentials such that

$$e^{\mathbf{B}} \in \mathbf{Spin}(n) \quad \text{and} \quad \frac{e^{\wedge\mathbf{B}}}{|e^{\wedge\mathbf{B}}|} \in \mathbf{Spin}(n) \quad \text{for} \quad \mathbf{B} \in \bigwedge^2 \mathbb{R}^n.$$

23.19 Marcel Riesz

Marcel Riesz 1958, pp. 61-67, reconstructed Grassmann's exterior algebra from the Clifford algebra, in any characteristic $\neq 2$, by

$$\mathbf{x} \wedge u = \frac{1}{2}(\mathbf{x}u + (-1)^k u\mathbf{x}),$$

where $\mathbf{x} \in V$ and $u \in \bigwedge^k V$. Riesz' contribution enhanced Chevalley's result of 1946, which related exterior products of vectors to antisymmetrized Clifford products of vectors (Chevalley's result was valid only in characteristic 0).

Clifford algebras admit a **parity grading** or **even-odd grading** with even and odd parts, $\mathcal{C}\ell(Q) = \mathcal{C}\ell^+(Q) \oplus \mathcal{C}\ell^-(Q)$. Riesz's construction of 1958 showed that there also exists a **dimension grading**

$$\mathcal{C}\ell(Q) = \mathbb{K} \oplus V \oplus \bigwedge^2 V \oplus \ldots \oplus \bigwedge^n V$$

in all characteristics $\neq 2$ (because there is a privileged linear isomorphism between a Clifford algebra and the exterior algebra). Thus, bivectors exist in all characteristics $\neq 2$.

M. Riesz 1947 expressed the Maxwell stress tensor as $T_{\mu\nu} = -\frac{1}{2}\langle \mathbf{e}_\mu \mathbf{F} \mathbf{e}_\nu \mathbf{F}\rangle_0$.

The first one to consider spinors as elements in a **minimal left ideal** of a Clifford algebra was M. Riesz 1947 (although the special case of pure spinors had been considered earlier by Cartan 1938).

23.20 Atiyah & Bott & Shapiro

In 1964, Atiyah & Bott & Shapiro reconsidered spinor spaces as modules over a Clifford algebra, instead of regarding them as minimal left ideals in the Clifford algebra. This permitted differentiation of spinor valued functions on manifolds, not just on flat spaces.

They re-identified the definite real Clifford algebras $\mathcal{C}\ell_n$ and $\mathcal{C}\ell_{0,n}$ as matrix algebras with entries in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{R}\oplus\mathbb{R}$, $\mathbb{H}\oplus\mathbb{H}$ (identified by Cartan in 1908). They rediscovered the periodicity of 8 (found by Cartan in 1908) with respect to the graded tensor product (used earlier by Chevalley in 1955). They emphasized the importance of the $\mathbb{Z}_2$-graded structure (the even-odd parity in Clifford's works) and used it to simplify Chevalley's approach to Lipschitz groups (the role of parity grading and/or grade involution in reflections was not observed by Chevalley).

Atiyah & Bott & Shapiro's paper of 1964 is known for its dirty joke: they attributed the introduction of $\mathbf{Pin}(n)$ to Serre from France, where pronunciation of 'pin group' in English sounds the same as 'pine groupe' in French: pine [pin] is a slang expression for male genitals.

E. Kähler 1960/62 introduced a second product for Cartan's exterior differential forms making Grassmann's exterior algebra isomorphic with a Clifford algebra. This re-interpretation of Chevalley's definition of the Clifford algebra (extending $\mathbf{x}u = \mathbf{x} \lrcorner u + \mathbf{x} \wedge u$) was renamed as *Kähler–Atiyah algebra* by W. Graf 1978 (Graf's Kähler–Atiyah algebra was again reinvented and applied to the Kähler–Dirac equation by Salingaros & Wene 1985).

23.21 The Maxwell Equations

In the special case of a homogeneous and isotropic medium, Maxwell equations can be condensed into a single equation. This has been done by means of complex vectors (Silberstein 1907), complex quaternions (Silberstein 1912/1914), spinors (Laporte & Uhlenbeck 1931) and in terms of Clifford algebras. Juvet & Schidlof 1932, Mercier 1935 and Riesz 1958 condensed the Maxwell equations into a single equation by bivectors in the Clifford algebra $\mathcal{C}\ell_{1,3}$ of $\mathbb{R}^{1,3}$. In the Clifford algebra $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$, the single Maxwell equation

$$\partial \mathbf{F} = \mathbf{J},$$

where $\partial = \nabla - \mathbf{e}_0\partial_0$ and $\mathbf{F} = \mathbf{E}\mathbf{e}_0 - \mathbf{B}\mathbf{e}_{123} \in \bigwedge^2 \mathbb{R}^{3,1}$, could be decomposed into two parts

$$\partial \lrcorner \mathbf{F} = \mathbf{J}, \quad \partial \wedge \mathbf{F} = 0.$$

Similarly, $\partial \mathbf{A} = -\mathbf{F}$ could be decomposed into two parts, $\partial \wedge \mathbf{A} = -\mathbf{F}$ and the Lorenz gauge/condition $\partial \cdot \mathbf{A} = 0$ discovered by the Danish physicist Ludwig Lorenz and not by the Dutch physicist H. A. Lorentz. [21] Marcel Riesz 1947 expressed the Maxwell stress tensor as

$$T_{\mu\nu} = -\frac{1}{2}\langle \mathbf{e}_\mu \mathbf{F} \mathbf{e}_\nu \mathbf{F} \rangle_0$$

and Hestenes 1966, p. 31, introduced the vectors $\mathbf{T}_\mu = -\frac{1}{2}\mathbf{F}\mathbf{e}_\mu\mathbf{F}$ for which $T_{\mu\nu} = \mathbf{T}_\mu \cdot \mathbf{e}_\nu = \mathbf{e}_\mu \cdot \mathbf{T}_\nu$ and the mapping $T\mathbf{x} = -\frac{1}{2}\mathbf{F}\mathbf{x}\mathbf{F}$ where $(T\mathbf{x})^\mu = T^\mu{}_\nu x^\nu$. [Juvet & Schidlof 1932, p. 141, gave

$$T_{\mu\nu} = F_\mu{}^\lambda F_{\lambda\nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

but did not observe that $T\mathbf{x} = -\frac{1}{2}\mathbf{F}\mathbf{x}\mathbf{F}$.] Note that $\mathbf{T}_0 = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)\mathbf{e}_0 + \mathbf{E} \times \mathbf{B}$.

23.22 Spinors in ideals, as operators and recovered

Juvet 1930 and Sauter 1930 replaced column spinors by square matrices in which only the first column was non-zero – thus spinor spaces became minimal left ideals in a **matrix algebra**. Marcel Riesz 1947 was the first one to consider spinors as elements in a minimal left ideal of a **Clifford algebra** (although the special case of pure spinors had been considered earlier by Cartan 1938).

Gürsey 1956/58 rewrote the Dirac equation with 2×2 quaternion matrices in $\mathrm{Mat}(2, \mathbb{H})$ (see also Gsponer & Hurni 1993). Kustaanheimo 1964 presented the spinor regularization of the Kepler motion, the KS-transformation, which emphasized the operator aspect of spinors. This led David Hestenes 1966-74 to a reformulation of the Dirac theory, where the role of spinors [in columns $\mathbb{C}^4$ or in minimal left ideals of the complex Clifford algebra $\mathbb{C} \otimes C\ell_{1,3} \simeq \mathrm{Mat}(4, \mathbb{C})$] was taken over by operators in the even subalgebra $C\ell^+_{1,3}$ of the real Clifford algebra $C\ell_{1,3} \simeq \mathrm{Mat}(2, \mathbb{H})$.

Spinors were reconstructed from their bilinear covariants by Y. Takahashi 1983 and J. Crawford 1985, in the case of the electron. Lounesto 1993 generalized the reconstruction of spinors to the null case of the neutron, and predicted existence of a new particle residing in between electrons and neutrons.

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