2

Complex Numbers

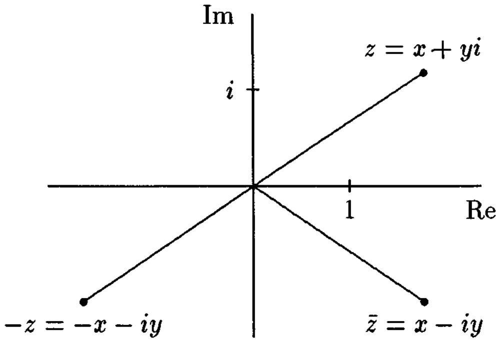
The feature distinguishing the complex numbers from the real numbers is that the complex numbers contain a square root of -1 called the *imaginary unit* $i = \sqrt{-1}$. [1] Complex numbers are of the form

$$z = x + iy$$

where $x, y \in \mathbb{R}$ and i satisfies $i^2 = -1$. The real numbers x, y are called the *real part* $x = \mathrm{Re}(z)$ and the *imaginary part* $y = \mathrm{Im}(z)$. To each ordered pair of real numbers x, y there corresponds a unique complex number $x + iy$.

A complex number $x + iy$ can be represented graphically as a point with rectangular coordinates (x, y). The xy-plane, where the complex numbers are represented, is called the *complex plane* $\mathbb{C}$. Its x-axis is the *real axis* and y-axis the *imaginary axis*.

A complex number $z = x + iy$ has an opposite $-z = -x - iy$ and a *complex conjugate* $\bar{z} = x - iy$, [2] obtained by changing the sign of the imaginary part.



1 Electrical engineers denote the square root of -1 by $j = \sqrt{-1}$.
2 In quantum mechanics the complex conjugate is denoted by $z^* = x - iy$.

The sum of two complex numbers is computed by adding separately the real parts and the imaginary parts:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2).$$

Addition of complex numbers can be illustrated by the parallelogram law of vector addition.

The product of two complex numbers is usually defined to be

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2),$$

although this result is also a consequence of distributivity, associativity and the replacement $i^2 = -1$.

Examples. 1. $i^3 = -i$, $i^4 = 1$, $i^5 = i$. 2. $(1 + i)^2 = 2i$. ∎

The product of a complex number $z = x + iy$ and its complex conjugate $\bar{z} = x - iy$ is a real number $z\bar{z} = x^2 + y^2$. Since this real number is non-zero for $z \neq 0$, we may introduce the inverse

$$z^{-1} = \frac{\bar{z}}{z\bar{z}}$$

or in coordinate form

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}.$$

Division is carried out as multiplication by the inverse: $z_1/z_2 = z_1z_2^{-1}$.

If we introduce polar coordinates r, φ in the complex plane by setting $x = r\cos\varphi$ and $y = r\sin\varphi$, then the complex number $z = x + iy$ can be written as

$$z = r(\cos\varphi + i\sin\varphi).$$

This is the *polar form* of z. [3] The distance $r = \sqrt{x^2 + y^2}$ from z to 0 is denoted by $|z|$ and called the *norm* of z. Thus [4]

$$|z| = \sqrt{z\bar{z}}.$$

The real number φ is called the *phase-angle* or *argument* of z [sometimes all the real numbers $\varphi + 2\pi k$, $k \in \mathbb{Z}$, are assigned to the same phase-angle].

The familiar addition rules for the sine and cosine result in the polar form of multiplication,

$$z_1z_2 = r_1r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)],$$

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3 Electrical engineers denote the polar form by $r\underline{/\varphi}$.

4 The scalar product $\mathrm{Re}(z_1\bar{z}_2)$ is compatible with the norm $|z|$. Incidentally, $\mathrm{Im}(z_1\bar{z}_2)$ measures the signed area of the parallelogram determined by z_1 and z_2.

of complex numbers

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1) \quad \text{and} \quad z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2).$$

Thus, the norm of a product is the product of the norms,

$$|z_1 z_2| = |z_1||z_2|,$$

and the phase-angle of a product is the sum of the phase-angles (mod 2π).

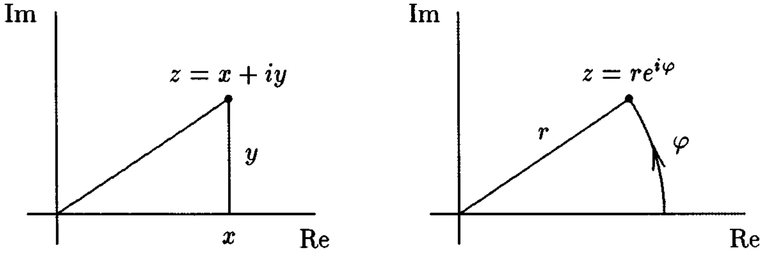
The exponential function can be defined everywhere in the complex plane by

$$\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots + \frac{z^k}{k!} + \ldots$$

We write $e^z = \exp(z)$. The series expansions of trigonometric functions result in *Euler's formula*

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

which allows us to abbreviate $z = r(\cos\varphi + i\sin\varphi)$ as $z = re^{i\varphi}$.



The exponential form of multiplication seems natural:

$$(r_1 e^{i\varphi_1})(r_2 e^{i\varphi_2}) = (r_1 r_2) e^{i(\varphi_1 + \varphi_2)}.$$

Powers and roots are computed as

$$(re^{i\varphi})^n = r^n e^{in\varphi} \quad \text{and} \quad \sqrt[n]{re^{i\varphi}} = \sqrt[n]{r} e^{i\varphi/n + i2\pi k/n}, \ k \in \mathbb{Z}_n.$$

Examples. $(1+i)^{-1} = \frac{1}{2}(1-i)$, $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$, $e^{i\pi/2} = i$. ∎

2.1 The field $\mathbb{C}$ versus the real algebra $\mathbb{C}$

Numbers are elements of a mathematical object called a field. In a field numbers can be both added and multiplied. The usual rules of addition

$a + b = b + a$	commutativity
$(a + b) + c = a + (b + c)$	associativity
$a + 0 = a$	zero 0
$a + (-a) = 0$	opposite $-a$ of a

are satisfied for all numbers a, b, c in a field $\mathbb{F}$. The multiplication satisfies

$$\begin{array}{ll} \left.\begin{array}{l}(a+b)c = ac+bc \\ a(b+c) = ab+ac\end{array}\right\} & \text{distributivity} \\ (ab)c = a(bc) & \text{associativity} \\ 1a = a & \text{unity } 1 \\ aa^{-1} = 1 & \text{inverse } a^{-1} \text{ of } a \neq 0 \\ ab = ba & \text{commutativity} \end{array}$$

for all numbers a, b, c in a field $\mathbb{F}$. The above rules of addition and multiplication make up the *axioms* of a **field** $\mathbb{F}$.

Examples of fields are the fields of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, rationals $\mathbb{Q}$, and the finite fields $\mathbb{F}_q$ where $q = p^m$ with a prime p. [5]

It is tempting to regard $\mathbb{R}$ as a unique subfield in $\mathbb{C}$. However, $\mathbb{C}$ contains several, infinitely many, subfields isomorphic to $\mathbb{R}$; choosing one means introducing a real linear structure on $\mathbb{C}$, obtained by restricting a in the product $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $(a, b) \to ab$ to be real, $a \in \mathbb{R}$. Such extra structure turns the field $\mathbb{C}$ into a real algebra $\mathbb{C}$.

Definition. An *algebra* over a field $\mathbb{F}$ is a linear space A over $\mathbb{F}$ together with a bilinear [6] function $A \times A \to A$, $(a, b) \to ab$. [7] ∎

To distinguish the field $\mathbb{C}$ from a real algebra $\mathbb{C}$ let us construct $\mathbb{C}$ as the set $\mathbb{R} \times \mathbb{R}$ of all ordered pairs of real numbers $z = (x, y)$ with addition and multiplication defined as

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \quad \text{and}$$
$$(x_1, y_1)(x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1).$$

The set $\mathbb{R} \times \mathbb{R}$ together with the above addition and multiplication rules makes up the *field* $\mathbb{C}$. The imaginary unit $(0, 1)$ satisfies $(0, 1)^2 = (-1, 0)$.

Since $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $(x_1, 0)(x_2, 0) = (x_1 x_2, 0)$, the real field $\mathbb{R}$ is contained in $\mathbb{C}$ as a subfield by $\mathbb{R} \to \mathbb{C}$, $x \to (x, 0)$. If we restrict multiplication so that one factor is in this distinguished copy of $\mathbb{R}$,

$$(\lambda, 0)(x, y) = (\lambda x, \lambda y),$$

then we actually introduce a *real linear structure* on the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. This

5 The finite fields $\mathbb{F}_q$, where $q = p^m$ with a prime p, are called Galois fields $GF(p^m)$.
6 Bilinear means linear with respect to both arguments. This implies distributivity. In other words, distributivity has no independent meaning for an algebra.
7 Note that associativity is not assumed.

real linear structure allows us to view the field of complex numbers intuitively as the complex plane $\mathbb{C}$. [8]

The above construction of $\mathbb{C}$ as the real linear space $\mathbb{R}^2$ brings in more structure than just the field structure: it makes $\mathbb{C}$ an *algebra* over $\mathbb{R}$. [9] We often identify $\mathbb{R}$ with the subfield $\{(x,0) \mid x \in \mathbb{R}\}$ of $\mathbb{C}$, and denote the standard basis of $\mathbb{R}^2$ by $1 = (1,0)$, $i = (0,1)$ in $\mathbb{C}$.

A function $\alpha : \mathbb{C} \to \mathbb{C}$ is an *automorphism of the field* $\mathbb{C}$ if it preserves addition and multiplication,

$$\alpha(z_1 + z_2) = \alpha(z_1) + \alpha(z_2),$$
$$\alpha(z_1 z_2) = \alpha(z_1)\alpha(z_2),$$

as well as the unity, $\alpha(1) = 1$. A function $\alpha : \mathbb{C} \to \mathbb{C}$ is an *automorphism of the real algebra* $\mathbb{C}$ if it preserves the real linear structure and multiplication (of complex numbers),

$$\alpha(z_1 + z_2) = \alpha(z_1) + \alpha(z_2), \quad \alpha(\lambda z) = \lambda\alpha(z), \quad \lambda \in \mathbb{R},$$
$$\alpha(z_1 z_2) = \alpha(z_1)\alpha(z_2),$$

as well as the unity, $\alpha(1) = 1$.

The field $\mathbb{C}$ has an infinity of automorphisms. In contrast, the only automorphisms of the real algebra $\mathbb{C}$ are the identity automorphism and complex conjugation.

Theorem. Complex conjugation is the only field automorphism of $\mathbb{C}$ which is different from the identity but preserves a fixed subfield $\mathbb{R}$.

Proof. First, note that $\alpha(i) = \pm i$ for any field automorphism α of $\mathbb{C}$, since $\alpha(i)^2 = \alpha(i^2) = \alpha(-1) = -1$. If $\alpha : \mathbb{C} \to \mathbb{C}$ is a field automorphism such that $\alpha(\mathbb{R}) \subset \mathbb{R}$, then $\alpha(x) = x$ for all $x \in \mathbb{R}$, because the only automorphism of the real field is the identity. It then follows that, for all $x + iy$ with $x, y \in \mathbb{R}$,

$$\alpha(x + iy) = \alpha(x) + \alpha(i)\alpha(y) = x + \alpha(i)y$$

where $\alpha(i) = \pm i$. The case $\alpha(i) = i$ gives the identity automorphism, and the case $\alpha(i) = -i$ gives complex conjugation. ∎

The other automorphisms of the field $\mathbb{C}$ send a real subfield $\mathbb{R}$ onto an isomorphic copy of $\mathbb{R}$, which is necessarily different from the original subfield $\mathbb{R}$. However, any field automorphism of $\mathbb{C}$ fixes point-wise the rational subfield $\mathbb{Q}$.

8 The geometric view of complex numbers is connected with the structure of $\mathbb{C}$ as a real algebra, and not so much as a field.

9 In the above construction we introduced a field structure into the real linear space $\mathbb{R}^2$ and arrived at an algebra $\mathbb{C}$ over $\mathbb{R}$, or equivalently at a field $\mathbb{C}$ with a distinguished subfield $\mathbb{R}$.

Example. It is known that there is a field automorphism of $\mathbb{C}$ sending $\sqrt{2}$ to $-\sqrt{2}$ and $\sqrt[4]{2}$ to $i\sqrt[4]{2}$, but no one has been able to construct such an automorphism explicitly since its existence proof calls for the axiom of choice. ∎

If a field automorphism of $\mathbb{C}$ is neither the identity nor a complex conjugation, then it sends some irrational numbers outside $\mathbb{R}$, and permutes an infinity of subfields all isomorphic with $\mathbb{R}$. Related to each real subfield there is a unique complex conjugation across that subfield, and all such automorphisms of finite order are complex conjugations for some real subfield. The image $\alpha(\mathbb{R})$ under such an automorphism α of a distinguished real subfield $\mathbb{R}$ is dense in $\mathbb{C}$ [in the topology of the metric $|z| = \sqrt{z\bar{z}}$ given by the complex conjugation across $\mathbb{R}$]. This can be seen as follows: An automorphism α must satisfy $\alpha(rx) = r\alpha(x)$ when $r \in \mathbb{Q}$. So if there is an irrational $x \in \mathbb{R}$ with $t = \alpha(x) \notin \mathbb{R}$, and necessarily $t \notin \mathbb{Q} + i\mathbb{Q}$, the image $\alpha(\mathbb{R})$ of $\mathbb{R}$ contains all numbers of the form $\alpha(r + sx) = r + st$ with $r, s \in \mathbb{Q}$. This is a dense set in $\mathbb{C}$.

The above discussion indicates that **there is no unique complex conjugation in the field of complex numbers**, and that the field structure of $\mathbb{C}$ does not fix by itself the subfield $\mathbb{R}$ of $\mathbb{C}$. The field injection $\mathbb{R} \to \mathbb{C}$ is an extra piece of structure added on top of the field $\mathbb{C}$. If a privileged real subfield $\mathbb{R}$ is singled out in $\mathbb{C}$, it brings along a real linear structure on $\mathbb{C}$, and a unique complex conjugation across $\mathbb{R}$, which then naturally imports a metric structure to $\mathbb{C}$.

Our main interest in complex numbers in this book is $\mathbb{C}$ as a real algebra, not so much as a field.

2.2 The double-ring ${}^2\mathbb{R}$ of $\mathbb{R}$

There is more than one interesting bilinear product (or algebra structure) on the linear space $\mathbb{R}^2$. For instance, component-wise multiplication

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$$

results in the double-ring ${}^2\mathbb{R}$ of $\mathbb{R}$. The only automorphisms of the real algebra ${}^2\mathbb{R}$ are the identity and the *swap*

$${}^2\mathbb{R} \to {}^2\mathbb{R},\ (\lambda, \mu) \to \mathrm{swap}(\lambda, \mu) = (\mu, \lambda).$$

The swap acts like the complex conjugation of $\mathbb{C}$, since

$$\mathrm{swap}[a(1,1) + b(1,-1)] = a(1,1) - b(1,-1).$$

The multiplicative unity $1 = (1,1)$ and the reflected element $j = (1,-1)$ are now related by $j^2 = 1$.

Alternatively and equivalently we may consider pairs of real numbers $(a, b) \in \mathbb{R}^2$ as Study numbers

$$a + jb, \quad j^2 = 1, \; j \neq 1.$$

Study numbers have Study conjugate $(a + jb)^- = a - jb$, Lorentz squared norm $(a + jb)(a - jb) = a^2 - b^2$, and the hyperbolic polar form $a + jb = \rho(\cosh\chi + j\sinh\chi)$ for $a^2 - b^2 \geq 0$. [10] In products Lorentz squared norms are preserved and hyperbolic angles added. Study numbers have the matrix representation

$$a + jb \simeq \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Exercise 4

2.3 Representation by means of real 2×2-matrices

Complex numbers were constructed as ordered pairs of real numbers. Thus we can replace

$$z = x + iy \quad \text{in} \quad \mathbb{C} \quad \text{by} \quad \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{in} \quad \mathbb{R}^2,$$

making explicit the real linear structure on $\mathbb{C}$. The product of two complex numbers $c = a + ib$ and z,

$$cz = ax - by + i(bx + ay),$$

can be replaced by / factored as

$$\begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One is thus led to consider representing complex numbers by certain real 2×2-matrices in $\mathrm{Mat}(2, \mathbb{R})$: [11]

$$\mathbb{C} \to \mathrm{Mat}(2, \mathbb{R}), \quad a + ib \to \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

10 The linear space $\mathbb{R}^2$ endowed with an indefinite quadratic form $(a, b) \to a^2 - b^2$ is the hyperbolic quadratic space $\mathbb{R}^{1,1}$. The Clifford algebra of $\mathbb{R}^{1,1}$ is $\mathcal{C}\ell_{1,1}$ which has Study numbers as the even subalgebra $\mathcal{C}\ell_{1,1}^+$.

11 In this matrix representation, the complex conjugate of a complex number becomes the transpose of the matrix and the (squared) norm becomes the determinant. The norm is preserved under similarity transformations, but 'transposition = complex conjugation' is only preserved under similarities by orthogonal matrices.

The multiplicative unity 1 and the imaginary unit i in $\mathbb{C}$ are represented by the matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

However, this is not the only linear representation of $\mathbb{C}$ in $\mathrm{Mat}(2,\mathbb{R})$. A similarity transformation by an invertible matrix U, $\det U \neq 0$, sends the representative of the imaginary unit J to another 'imaginary unit' $J' = UJU^{-1}$ in $\mathrm{Mat}(2,\mathbb{R})$.

Example. Choosing $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, we find $J' = \left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right)$, and the matrix representation $x + iy \to \left(\begin{smallmatrix} x+y & -2y \\ y & x-y \end{smallmatrix}\right)$. ∎

Geometric Interpretation of $i = \sqrt{-1}$

In the rest of this chapter we shall study introduction of complex numbers by means of the Clifford algebra $\mathcal{C}\ell_2$ of the Euclidean plane $\mathbb{R}^2$. This approach gives the imaginary unit $i = \sqrt{-1}$ various geometrical meanings. We will see that i represents

(i) an oriented plane area in $\mathbb{R}^2$,
(ii) a quarter turn of $\mathbb{R}^2$.

The Euclidean plane $\mathbb{R}^2$ has a quadratic form

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \to |\mathbf{r}|^2 = x^2 + y^2.$$

We introduce an associative product of vectors such that

$$\mathbf{r}^2 = |\mathbf{r}|^2 \quad \text{or} \quad (x\mathbf{e}_1 + y\mathbf{e}_2)^2 = x^2 + y^2.$$

Using distributivity this results in the multiplication rules

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = 1, \quad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1.$$

The element $\mathbf{e}_1\mathbf{e}_2$ satisfies

$$\boxed{(\mathbf{e}_1\mathbf{e}_2)^2 = -1}$$

and therefore cannot be a scalar or a vector. It is an example of a bivector, the *unit bivector*. Denote it for short by $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$.

2.4 $\mathbb{C}$ as the even Clifford algebra $\mathcal{C}\ell_2^+$

The Clifford algebra $\mathcal{C}\ell_2$ is a 4-dimensional real algebra with a basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$. The basis elements obey the multiplication table

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

The basis elements span the subspaces consisting of [12]

1	$\mathbb{R}$	scalars
$\mathbf{e}_1, \mathbf{e}_2$	$\mathbb{R}^2$	vectors
$\mathbf{e}_{12}$	$\bigwedge^2 \mathbb{R}^2$	bivectors.

Thus, the Clifford algebra $\mathcal{C}\ell_2$ contains copies of $\mathbb{R}$ and $\mathbb{R}^2$, and it is a direct sum of its subspaces of elements of degrees 0,1,2:

$$\mathcal{C}\ell_2 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2.$$

The Clifford algebra is also a direct sum $\mathcal{C}\ell_2 = \mathcal{C}\ell_2^+ \oplus \mathcal{C}\ell_2^-$ of its

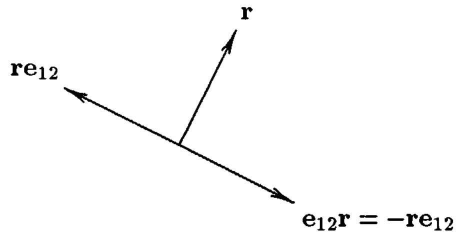
$$\begin{aligned} \text{even part} \quad \mathcal{C}\ell_2^+ &= \mathbb{R} \oplus \textstyle\bigwedge^2 \mathbb{R}^2, \\ \text{odd part} \quad \mathcal{C}\ell_2^- &= \mathbb{R}^2. \end{aligned}$$

The even part is not only a subspace but also a subalgebra. It consists of elements of the form $x + y\mathbf{e}_{12}$ where $x, y \in \mathbb{R}$ and $\mathbf{e}_{12}^2 = -1$. Thus, the even subalgebra $\mathcal{C}\ell_2^+ = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ of $\mathcal{C}\ell_2$ is isomorphic to $\mathbb{C}$. The unit bivector $\mathbf{e}_{12}$ shares the basic property of the square root i of -1, that is $i^2 = -1$, and we could write $i = \mathbf{e}_{12}$. It should be noted, however, that our imaginary unit $\mathbf{e}_{12}$ anticommutes with $\mathbf{e}_1$ and $\mathbf{e}_2$ and thus $\mathbf{e}_{12}$ *anticommutes with every vector in the* $\mathbf{e}_1\mathbf{e}_2$*-plane*: [13]

$$\mathbf{r}\mathbf{e}_{12} = -\mathbf{e}_{12}\mathbf{r} \quad \text{for} \quad \mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \quad \text{and} \quad \mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2.$$

12 In higher dimensions the Clifford algebra $\mathcal{C}\ell_n$ of $\mathbb{R}^n$ is a sum of its subspaces of k-vectors: $\mathcal{C}\ell_n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \ldots \oplus \bigwedge^n \mathbb{R}^n$.

13 In a complex linear space, or complex algebra, where scalars are complex numbers, the imaginary unit commutes with all the vectors, $i\mathbf{r} = \mathbf{r}i$.



2.5 Imaginary unit = the unit bivector

Multiplying the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ by the unit bivector $\mathbf{e}_{12}$ gives another vector $\mathbf{r}\mathbf{e}_{12} = x\mathbf{e}_2 - y\mathbf{e}_1$ which is perpendicular to $\mathbf{r}$. The function $\mathbf{r} \to \mathbf{r}\mathbf{e}_{12}$ is a left turn, and the effect of two left turns $[\mathbf{e}_{12} \cdot \mathbf{e}_{12}]$ is to reverse direction $[-1]$; or, in a more picturesque way, is a *U*-turn. The statement '$\mathbf{e}_{12}^2 = -1$' is just an arithmetic version of the obvious geometric fact that the sum of two right angles, $90° + 90°$, is a straight angle, $180°$. In the vector plane $\mathbb{R}^2$ the sense of rotation depends on what side the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ is multiplied by $\mathbf{e}_{12}$ so that the rotation $\mathbf{r} \to \mathbf{e}_{12}\mathbf{r} = y\mathbf{e}_1 - x\mathbf{e}_2$ is clockwise and $\mathbf{r} \to \mathbf{r}\mathbf{e}_{12} = -y\mathbf{e}_1 + x\mathbf{e}_2$ is counter-clockwise.

In the complex plane $\mathbb{C} = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ both the rotations sending $z = x + y\mathbf{e}_{12}$ to $\mathbf{e}_{12}z$ and $z\mathbf{e}_{12}$ are counter-clockwise. Multiplying a complex number $z = x + y\mathbf{e}_{12}$ by the unit bivector $\mathbf{e}_{12}$ results in a left turn, $z\mathbf{e}_{12} = -y + x\mathbf{e}_{12}$, and the effect of two left turns $[\mathbf{e}_{12} \cdot \mathbf{e}_{12}]$ is direction reversal $[-1]$; that is a half-turn in the complex plane $\mathbb{C}$:

$$\begin{tikzcd} -z = z\mathbf{e}_{12}^2 & z\mathbf{e}_{12} \arrow[l] \\ {} \arrow[r] & z \arrow[u] \end{tikzcd}$$

The square root of -1 has two distinct geometric roles in $\mathbb{R}^2$: it is the generator of rotations, $i = \mathbf{e}_1\mathbf{e}_2 \in \mathcal{C}\ell_2^+$, and it represents a unit oriented plane area $\mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^2$. [14]

A complex number $z = x + y\mathbf{e}_{12} \in \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2$ is a sum of

- a real number $x = \mathrm{Re}(z)$ and
- a bivector $y\mathbf{e}_{12} = \mathbf{e}_{12}\,\mathrm{Im}(z)$.

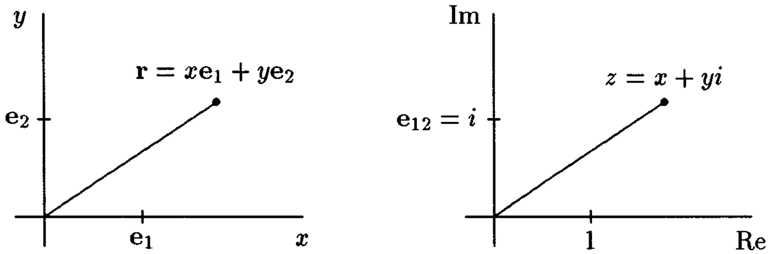
14 In an n-dimensional vector space $\mathbb{R}^n$ rotations can be represented by multiplications in Clifford algebras $\mathcal{C}\ell_n$, while certain simple elements of the exterior algebra $\bigwedge \mathbb{R}^n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \cdots \oplus \bigwedge^n \mathbb{R}^n$ represent oriented subspaces of dimensions $0, 1, 2, \ldots, n$.

2.6 Even and odd parts

The Clifford algebra $\mathcal{C}\ell_2$ of $\mathbb{R}^2$ contains both the complex plane $\mathbb{C}$ and the vector plane $\mathbb{R}^2$ so that

$\mathbb{R}^2$ is spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$,
$\mathbb{C}$ is spanned by 1 and $\mathbf{e}_{12}$.

The only common point of the two planes is the zero 0. The two planes are both parts of the same algebra $\mathcal{C}\ell_2$. The vector plane $\mathbb{R}^2$ and the complex field $\mathbb{C}$ are incorporated as separate substructures in the Clifford algebra $\mathcal{C}\ell_2 = \mathcal{C}\ell_2^+ \oplus \mathcal{C}\ell_2^-$ so that the complex plane $\mathbb{C}$ is the *even part* $\mathcal{C}\ell_2^+$ and the vector plane $\mathbb{R}^2$ is the *odd part* $\mathcal{C}\ell_2^-$.



Vector plane $\mathbb{R}^2 = \mathcal{C}\ell_2^-$ Complex plane $\mathbb{C} = \mathcal{C}\ell_2^+$

The names even and odd mean that the elements are products of an even or odd number of vectors. Parity considerations show that

– complex number times complex number is a complex number,
– vector times complex number is a vector,
– complex number times vector is a vector, and
– vector times vector is a complex number.

The above observations can be expressed by the inclusions

$$\mathcal{C}\ell_2^+\mathcal{C}\ell_2^+ \subset \mathcal{C}\ell_2^+,$$
$$\mathcal{C}\ell_2^-\mathcal{C}\ell_2^+ \subset \mathcal{C}\ell_2^-,$$
$$\mathcal{C}\ell_2^+\mathcal{C}\ell_2^- \subset \mathcal{C}\ell_2^-,$$
$$\mathcal{C}\ell_2^-\mathcal{C}\ell_2^- \subset \mathcal{C}\ell_2^+.$$

By writing $(\mathcal{C}\ell_2)_0 = \mathcal{C}\ell_2^+$ and $(\mathcal{C}\ell_2)_1 = \mathcal{C}\ell_2^-$, this can be further condensed to $(\mathcal{C}\ell_2)_j(\mathcal{C}\ell_2)_k \subset (\mathcal{C}\ell_2)_{j+k}$, where j, k are added modulo 2. These observations are expressed by saying that the Clifford algebra $\mathcal{C}\ell_2$ has an *even-odd grading* or that it is graded over $\mathbb{Z}_2 = \{0, 1\}$. [15]

15 We have already met a $\mathbb{Z}_2$-graded algebra, namely the real algebra $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ with even part $\mathbb{R} = \mathrm{Re}(\mathbb{C})$ and odd part $i\mathbb{R} = i\,\mathrm{Im}(\mathbb{C})$.

2.7 Involutions and the norm

The Clifford algebra $\mathcal{C}\ell_2$ has three involutions similar to complex conjugation in $\mathbb{C}$. For an element $u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 \in \mathcal{C}\ell_2$, $\langle u\rangle_k \in \bigwedge^k \mathbb{R}^2$, we define

$$\begin{array}{ll} \text{grade involution} & \hat{u} = \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2, \\ \text{reversion} & \tilde{u} = \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2, \\ \text{Clifford-conjugation} & \bar{u} = \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2. \end{array}$$

The grade involution is an automorphism, $\widehat{uv} = \hat{u}\hat{v}$, while the reversion and the Clifford-conjugation are anti-automorphisms, $\widetilde{uv} = \tilde{v}\tilde{u}$, $\overline{uv} = \bar{v}\bar{u}$.

For a complex number $z = x + y\mathbf{e}_{12}$ the complex conjugation $z \to \bar{z} = x - y\mathbf{e}_{12}$ is a restriction of the Clifford-conjugation $u \to \bar{u}$ in $\mathcal{C}\ell_2$ and also of the reversion $u \to \tilde{u}$ in $\mathcal{C}\ell_2$. Likewise, the norm $|z| = \sqrt{x^2 + y^2}$ in $\mathbb{C}$, obtained as the square root of $z\bar{z} = x^2 + y^2$, is a restriction of the norm $|u| = \sqrt{\langle u\tilde{u}\rangle_0}$ in $\mathcal{C}\ell_2$.

A complex number is a product of its norm $r = |z|$ and its phase-factor $\cos\varphi + \mathbf{e}_{12}\sin\varphi$, where $x = r\cos\varphi$ and $y = r\sin\varphi$. The expression $z = r(\cos\varphi + \mathbf{e}_{12}\sin\varphi)$ can be abbreviated as $z = r\exp(\mathbf{e}_{12}\varphi)$, and read as '$r$ in phase φ.'

2.8 Vectors multiplied by complex numbers

The product of a vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ and a unit complex number $e^{\mathbf{i}\varphi} = \cos\varphi + \mathbf{i}\sin\varphi$, where for short $\mathbf{i} = \mathbf{e}_{12}$, is another vector in the $\mathbf{e}_1\mathbf{e}_2$-plane:

$$\mathbf{r}\cos\varphi + \mathbf{ri}\sin\varphi = \mathbf{r}e^{\mathbf{i}\varphi}.$$

The vector $\mathbf{ri} = x\mathbf{e}_2 - y\mathbf{e}_1$ is perpendicular to $\mathbf{r}$ so that a rotation to the left by $\pi/2$ carries $\mathbf{r}$ to $\mathbf{ri}$.

Since the unit bivector $\mathbf{i}$ anticommutes with every vector $\mathbf{r}$ in the $\mathbf{e}_1\mathbf{e}_2$-plane, the rotated vector could also be expressed as

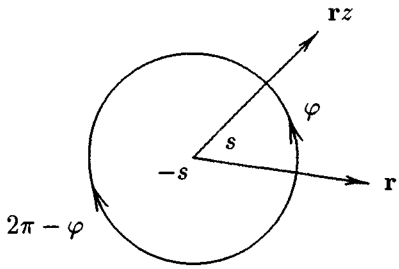
$$\mathbf{r}\cos\varphi + \mathbf{ri}\sin\varphi = \mathbf{r}\cos\varphi - \mathbf{ir}\sin\varphi = e^{-\mathbf{i}\varphi}\mathbf{r}.$$

Furthermore, we have $\cos\varphi + \mathbf{i}\sin\varphi = (\cos\frac{\varphi}{2} + \mathbf{i}\sin\frac{\varphi}{2})^2$ and thus the rotated vector also has the form $s^{-1}\mathbf{r}s$ where $s = e^{\mathbf{i}\varphi/2}$ and $s^{-1} = e^{-\mathbf{i}\varphi/2}$. The rotation of $\mathbf{r}$ to the left by the angle φ will then result in $\mathbf{r}z = z^{-1}\mathbf{r} = s^{-1}\mathbf{r}s$ where $z = e^{\mathbf{i}\varphi}$, $z^{-1} = e^{-\mathbf{i}\varphi}$ and $s^2 = z$. There are two complex numbers s and $-s$ which result in the same rotation $s^{-1}\mathbf{r}s = (-s)^{-1}\mathbf{r}(-s)$. In other words, there are two complex numbers which produce the same final result but via different actions.

$s = e^{\mathbf{i}\varphi/2}$

$-s = e^{-\mathbf{i}(2\pi-\varphi)/2} = e^{\mathbf{i}\varphi/2}e^{-\mathbf{i}\pi}$

$\boxed{e^{\mathbf{i}\pi} = -1}$



2.9 The group Spin(2)

The unit complex numbers $z \in \mathbb{C}$, $|z| = 1$, form the *unit circle* $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, which with multiplication of complex numbers as the product becomes the *unitary group* $U(1) = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$. A counter-clockwise rotation of the complex plane $\mathbb{C}$ by an angle φ can be represented by complex number multiplication:

$$x + iy \to (\cos\varphi + i\sin\varphi)(x + iy), \quad \cos\varphi + i\sin\varphi \in U(1).$$

A counter-clockwise rotation of the vector plane $\mathbb{R}^2$ by an angle φ can be represented by a matrix multiplication:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \in SO(2)$$

where $SO(2) = \{R \in \mathrm{Mat}(2,\mathbb{R}) \mid R^{\mathsf{T}}R = I,\ \det R = 1\}$, the rotation group. The rotation group $SO(2)$ is isomorphic to the unitary group $U(1)$.

Rotations of $\mathbb{R}^2$ can also be represented by Clifford multiplication: [16]

$$x\mathbf{e}_1 + y\mathbf{e}_2 \to (\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})^{-1}(x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})$$

where $\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2} \in \mathbf{Spin}(2) = \{s \in \mathcal{C}\ell_2^+ \mid s\bar{s} = 1\}$, the spin group. The fact that two opposite elements of the spin group $\mathbf{Spin}(2)$ represent the same rotation in $SO(2)$ is expressed by saying that $\mathbf{Spin}(2)$ is a *two-fold* [17] cover of $SO(2)$, and written as $\mathbf{Spin}(2)/\{\pm 1\} \simeq SO(2)$. Although $SO(2)$ and $\mathbf{Spin}(2)$ act differently on $\mathbb{R}^2$, they are isomorphic as abstract groups, that is,

16 We use this particular form to represent the rotation because the expression $x\mathbf{e}_1 + y\mathbf{e}_2 \to (\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})^{-1}(x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\frac{\varphi}{2} + \mathbf{e}_{12}\sin\frac{\varphi}{2})$ can be generalized to higher dimensions. The expression $x\mathbf{e}_1 + y\mathbf{e}_2 \to (x\mathbf{e}_1 + y\mathbf{e}_2)(\cos\varphi + \mathbf{e}_{12}\sin\varphi)$ is not generalizable to higher-dimensional rotations.

17 You are already familiar with two-fold covers: 1. A position of the hands of your watch corresponds to two positions of the Sun. 2. A rotating mirror turns half the angle of the image. 3. Circulating a coin one full turn around another makes the coin turn twice around its center.

$\mathbf{Spin}(2) \simeq SO(2)$. [18]

Exercise 6

History

Imaginary numbers first appeared around 1540, when Tartaglia and Cardano expressed real roots of a cubic equation in terms of conjugate complex numbers. The first one to represent complex numbers by points on a plane was a Norwegian surveyor, Caspar Wessel, in 1798. He posited an imaginary axis perpendicular to the axis of real numbers. This configuration came to be known as the Argand diagram, although Argand's contribution was an interpretation of $i = \sqrt{-1}$ as a rotation by a right angle in the plane. Complex numbers got their name from Gauss, and their formal definition as pairs of real numbers is due to Hamilton in 1833 (first published 1837).

Exercises

1. $(3+4i)^{-1}$, $\sqrt{3+4i}$, $\sqrt[4]{-4}$, $\sqrt[3]{-i}$, $\log(-1+i)$.
2. Let $z_k = e^{i\,2\pi k/n}$, $k = 1, 2, \ldots, n-1$. Compute $(1-z_1)(1-z_2)\cdots(1-z_{n-1})$.
3. An *ordering* of a field $\mathbb{F}$ is an assignment of a subset $P \subset \mathbb{F}$ such that

   (i) $0 \notin P$,
   (ii) for all non-zero $a \in \mathbb{F}$ either $a \in P$ or $-a \in P$, but not both,
   (iii) $a + b \in P$ and $ab \in P$ for all $a, b \in P$.

   It is customary to call P the set of *positive* numbers, and the set $-P = \{-a \mid a \in P\}$ the set of *negative* numbers. The statement $a - b \in P$ is also written $a > b$ (and $a - b \in P \cup \{0\}$ is written $a \geq b$). Show that the field $\mathbb{C}$ cannot be ordered.
4. Two automorphisms α, β of an algebra are similar if there exists an intertwining automorphism γ such that $\alpha\gamma = \gamma\beta$. The identity automorphism is similar only to itself.

a) Show that the two involutions of the real algebra $\mathbb{C}$ are dissimilar, and that the two involutions of the real algebra ${}^2\mathbb{R}$ are dissimilar.

b) Show that the two involutions $\alpha(\lambda, \mu) = (\mu, \lambda)$ and $\beta(\lambda, \mu) = (\bar{\mu}, \bar{\lambda})$ are similar involutions of the real or complex algebra ${}^2\mathbb{C}$ [that is, find an intertwining automorphism γ of ${}^2\mathbb{C}$ such that $\alpha\gamma = \gamma\beta$].

5. A rotation is called rational if it sends a vector with rational coordinates to

18 Both $SO(2)$ and $\mathbf{Spin}(2)$ are homeomorphic to S^1.

another vector with rational coordinates. Determine all the rational rotations of $\mathbb{R}^2$. Hint: $R \in SO(2) \setminus \{-I\}$ can be written in the form $R = (I + A)(I - A)^{-1}$ where $A^\top = -A$.

6. Write $\tilde{u} = \langle u \rangle_0 + \langle u \rangle_1 - \langle u \rangle_2$ for $u = \langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 \in \mathcal{C}\ell_2$, $\langle u \rangle_k \in \bigwedge^k \mathbb{R}^2$. Let $\mathbf{Pin}(2) = \{u \in \mathcal{C}\ell_2 \mid \tilde{u}u = 1\}$, $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \to R(\mathbf{x}) = u\mathbf{x}u^{-1}$, and $O(2) = \{R \in \mathrm{Mat}(2, \mathbb{R}) \mid R^\top R = I\}$. Show that $\mathbf{Pin}(2)/\{\pm 1\} \simeq O(2)$ and $\mathbf{Pin}(2) \simeq O(2)$.
7. Show that a 2-dimensional real algebra with unity 1 is both commutative and associative. Hint: First show that there is a basis $\{1, a\}$ such that $a^2 = \alpha 1,\ \alpha \in \mathbb{R}$.
8. Show that a 2-dimensional real algebra with unity 1 and no zero-divisors [$ab = 0$ implies $a = 0$ or $b = 0$] is isomorphic to $\mathbb{C}$.

Solutions

1. $\frac{1}{5}(3 - 4i)$, $\pm(2 + i)$, $\pm 1 \pm i$, $\sqrt[3]{-i} = \{i, \pm\frac{\sqrt{3}}{2} - i\frac{1}{2}\}$, $\log(-1 + i) = \frac{1}{2}\log 2 + i\frac{3\pi}{4} + i\,2\pi k$.
2. Note that the roots of $x^n - 1 = 0$ are $z_k = e^{i\,2\pi k/n}$, $k = 0, 1, \ldots, n-1$. Therefore $(x - z_0)(x - z_1)(x - z_2)\cdots(x - z_{n-1}) = x^n - 1$. Define $f(x) = (x - z_1)(x - z_2)\cdots(x - z_{n-1})$ which equals
$$f(x) = \frac{x^n - 1}{x - 1} \quad \text{for} \quad x \neq 1$$
and $f(x) = x^{n-1} + \ldots + x + 1$ in general. Compute $f(1) = n$.
3. In an ordered field non-zero numbers have positive squares, and the sum of such squares is positive, and therefore non-zero. The equality $i^2 + 1 = 0$ in $\mathbb{C}$ can also be written as $i^2 + 1^2 = 0$, which excludes the inequality $i^2 + 1^2 > 0$. Consequently, it is impossible to order the field $\mathbb{C}$.
4. b) Choose $\gamma(\lambda, \mu) = (\bar{\lambda}, \mu)$ or $\gamma(\lambda, \mu) = (\lambda, \bar{\mu})$ to find $\alpha\gamma = \gamma\beta$.

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