# The optimal control curve

So now we know that if $\xi$ is a minimizer of (P) then it must be of the form

$$(1) \quad \xi(t) = \sum_{i \in \mathbb{N}_2} \left[ \cos(\sigma_i(\mu) t) a_i + \sin(\sigma_i(\mu) t) a_i' \right]$$

with $\sigma_1(\mu) \neq \sigma_2(\mu) \in \mathbb{N}$ and we have asserted that it is possible to choose $\mu \in \mathbb{R}^6$ such that the latter holds. If we plug $\xi$ into the energy functional $\mathcal{G}_p$, we have

$$\mathcal{G}(\xi) = \sigma_1(\mu)^2 \left[ G a_1 \cdot a_1 + G a_1' \cdot a_1' \right]$$

$$+ \sigma_2(\mu)^2 \left[ G a_2 \cdot a_2 + G a_2' \cdot a_2' \right]$$

$$(2) \quad = \frac{\sigma_1(\mu)}{2\pi} \left[ \left( \Delta_g^{-1/2} G \Delta_g^{-1/2} \right) \tilde{u}_{\sigma_1} \cdot \tilde{u}_{\sigma_1} + \left( \Delta_g^{-1/2} G \Delta_g^{-1/2} \right) \tilde{v}_{\sigma_1} \cdot \tilde{v}_{\sigma_1} \right]$$

$$+ \frac{\sigma_2(\mu)}{2\pi} \left[ \left( \Delta_g^{-1/2} G \Delta^{-1/2} \right) \tilde{u}_{\sigma_2} \cdot \tilde{u}_{\sigma_2} + \left( \Delta_g^{-1/2} G \Delta_g^{-1/2} \right) \tilde{v}_{\sigma_2} \cdot \tilde{v}_{\sigma_2} \right],$$

where $\tilde{u}_{\sigma_i} := \sqrt{2\pi \Delta_g \sigma_i}\, a_i$ and $\tilde{v}_{\sigma_i} := \sqrt{2\pi \Delta_g \sigma_i}\, a_i'$. In particular, we then have

$$(3) \quad \sqrt{\det \Delta_g} \left( \Delta_h \tilde{\Delta}_g \right)^{-1} \delta p = \tilde{v}_{\sigma_1} \wedge \tilde{u}_{\sigma_1} + \tilde{v}_{\sigma_2} \wedge \tilde{u}_{\sigma_2}.$$

The latter equation is independent of the values of $\sigma_i$, so we choose the one that minimize (2) which are $\sigma_1 = 2$ and $\sigma_1 = 1$. Recall that $\sigma_1(\mu) \geq \sigma_2(\mu)$.

Finally, we may permute the Fourier coefficients such that

$$G a_1 \cdot a_1 + G a_1' \cdot a_1' \leq G a_2 \cdot a_2 + G a_2' \cdot a_2'$$

to minimize the energy are last time. This proves the conjecture.

## RECONSTRUCTION

Suppose, we are given a non-simple bivector $\delta p$ as a net-displacement. Writing again $\omega := \sqrt{\det \Delta_g}\,(\Delta_h \tilde{\Delta}_g)^{-1} \tilde{\delta p}$, we now have

$$\omega = v_1 \wedge u_1 + v_2 \wedge u_2$$

such that $v_i, u_j$ are mutually orthogonal. Then, we see from (3) that we can set

$$a_1' := \frac{1}{\sqrt{4\pi}} \Delta_g^{-1/2} v_1$$

$$a_1 := \frac{1}{\sqrt{4\pi}} \Delta_g^{-1/2} u_1$$

$$a_2' := \frac{1}{\sqrt{\cancel{4}2\pi}} \Delta_g^{-1/2} v_2$$

$$a_2 := \frac{1}{\sqrt{2\pi}} \Delta_g^{-1/2} u_2$$

<u>Interesting fact:</u>

If the two orth. simple bivectors have equal magnitude, then the decomposition is not unique

What does that mean for us?

- where we assume that up to permutation

$$G a_1 \cdot a_1 + G a_1' \cdot a_1' \leq G a_2 \cdot a_2 + G a_2' \cdot a_2'.$$

Eventually, we set

$$\xi(t) := \cos(2t) a_1 + \sin(2t) a_1' + \cos(t) a_2 + \sin(2t) a_2'.$$

- Hence, the remaining but very crucial question is how to decompose any non-simple bivector into the sum of two orthogonal simple bivectors.