6
The Fourth Dimension

In this chapter we study the geometry of the Euclidean space $\mathbb{R}^4$. The purpose is to help readers to get a solid view, or as solid a view as possible, of the first dimension beyond our ability to visualize. This is an important intermediate step in scrutinizing higher dimensions. We start by reviewing regular figures in lower dimensions.

6.1 Regular polygons in $\mathbb{R}^2$

The equilateral triangle, the square, the regular pentagon, ..., are regular polygons. We shall also call them a 3-cell, 4-cell, 5-cell, ..., denoted by $\{3\}$, $\{4\}$, $\{5\}$, ..., respectively. Therefore, we call a regular p-gon a p-cell, denoted by $\{p\}$. As p grows toward infinity, we get in the limit an ∞-cell, where the line is divided into line segments of equal length. As a degenerate case we get a 2-cell, which is bounded by 2 line segments in the same place. The interior angle of a regular p-gon at a vertex is $(1-2/p)\pi$.

6.2 Regular polyhedra in $\mathbb{R}^3$

A regular polyhedron is a convex polyhedron bounded by congruent regular polygons, for instance, by p-gons. The number of regular p-gons meeting at a vertex is the same, say q; it satisfies

$$q\left(1-\frac{2}{p}\right)\pi < 2\pi,$$

because the sum of angles of faces meeting at a vertex cannot exceed 2π. The above inequality can also be written in the form

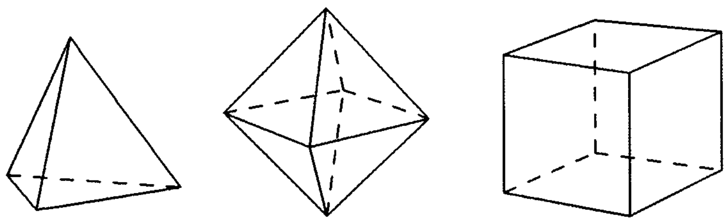
$$\frac{1}{p}+\frac{1}{q} > \frac{1}{2}.$$

The same result is obtained by inspection of the topological properties of a regular polyhedron: the numbers N_0, N_1, N_2 of vertices, edges and faces satisfy the Euler formula:

$$N_0 - N_1 + N_2 = 2.$$

On the other hand, each edge of a regular polyhedron is a boundary of two faces, each with p sides, so that $2N_1 = pN_2$; and a vertex is a meeting point of q edges, each with 2 end points, so that $qN_0 = 2N_1$. The above inequality is a consequence of the Euler formula and the equation

$$qN_0 = 2N_1 = pN_2.$$



3,3 3,4 4,3

A regular polyhedron ($p, q \geq 3$) must satisfy the foregoing inequality, and so only a few pairs p, q are possible. These regular polyhedra are called Platonic solids, or p, q-cells with *Schläfli* symbols $\{p, q\}$. There are five Platonic solids.

Name	$\{p,q\}$	N_0	N_1	N_2
Tetrahedron	$\{3,3\}$	4	6	4
Octahedron	$\{3,4\}$	6	12	8
Cube	$\{4,3\}$	8	12	6
Icosahedron	$\{3,5\}$	12	30	20
Dodecahedron	$\{5,3\}$	20	30	12

When $q = 2$ in the above inequality we get a dihedron with Schläfli symbol $\{p, 2\}$. A dihedron is bounded by two regular polygons positioned in the same place.

When a plane is covered by regular polygons so that at each vertex there meet q regular p-gons, we are solving the equation

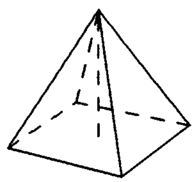
$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

There are three solutions to the above equation; they have Schläfli symbols

$\{4,4\}$, $\{3,6\}$ and $\{6,3\}$ corresponding to tilings of the plane by squares, equilateral triangles and regular hexagons. These regular tilings are called *tessellations*.

6.3 Regular polytopes in $\mathbb{R}^4$

A polyhedron is regular if its faces and vertices (= parts of the polyhedron near a vertex point) are regular. A regular polyhedron with Schläfli symbol $\{p,q\}$ has p-cells as faces and q-cells as vertices. A vertex is regular, if a plane cuts off a regular polygon whose central normal passes through the vertex.



A regular vertex

A polytope is a higher-dimensional analog of a polyhedron. A polytope is regular if its faces and vertices are regular. A 4-dimensional regular polytope with p,q-cells as faces must have q,r-cells as vertices. This drops the number of 4-dimensional regular polytopes from $5^2 = 25$ to 11. The sum of the solid angles of the faces meeting at a vertex cannot exceed 4π. As a consequence, there remain six possible combinations of p,q and q,r. A closer inspection shows that all these six combinations are in fact 4-dimensional regular polytopes; we shall call them p,q,r-cells with Schläfli symbols $\{p,q,r\}$.

$\{p,q,r\}$	N_0	N_1	N_2	N_3	Face	Vertex
$\{3,3,3\}$	5	10	10	5	Tetrahedron	Tetrahedron
$\{3,3,4\}$	8	24	32	16	Tetrahedron	Octahedron
$\{4,3,3\}$	16	32	24	8	Cube	Tetrahedron
$\{3,4,3\}$	24	96	96	24	Octahedron	Cube
$\{3,3,5\}$	120	720	1200	600	Tetrahedron	Icosahedron
$\{5,3,3\}$	600	1200	720	120	Dodecahedron	Tetrahedron

There are the regular simplex $\{3,3,3\}$ and the hypercube $\{4,3,3\}$, also called a tesseract. There is the octahedron analog $\{3,3,4\}$, a dipyramid with octahedron as a basis. There are the analogs of the icosahedron and the dodecahedron, $\{3,3,5\}$ and $\{5,3,3\}$; and there is an extra regular polytope $\{3,4,3\}$.

The 3-dimensional space can be filled with cubes, a configuration with

Schläfli symbol $\{4,3,4\}$. The 4-dimensional space can be filled with hypercubes, dipyramids and the extra regular polytope, configurations with Schläfli symbols $\{4,3,3,4\}$, $\{3,3,4,3\}$ and $\{3,4,3,3\}$.

In a higher-dimensional space, $n > 4$, there are only the regular simplex, dipyramid and hypercube, and it can only be filled with hypercubes.

6.4 The spheres

A circle with radius r in $\mathbb{R}^2$ has circumference $2\pi r$ and area πr^2. A sphere with radius r in $\mathbb{R}^3$ has surface $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$. A hypersphere with radius r in $\mathbb{R}^4$ has 3-dimensional surface $2\pi^2 r^3$ and 4-dimensional hypervolume $\frac{1}{2}\pi^2 r^4$. For lower-dimensional spheres we have the following table:

n	surface	volume
1	2	$2r$
2	$2\pi r$	πr^2
3	$4\pi r^2$	$\frac{4}{3}\pi r^3$
4	$2\pi^2 r^3$	$\frac{1}{2}\pi^2 r^4$
5	$\frac{8}{3}\pi^2 r^4$	$\frac{8}{15}\pi^2 r^5$

If the volume of the sphere in $\mathbb{R}^n$ is denoted by $\omega_n r^n$ then its surface is $n\omega_n r^{n-1}$. Observe a rule $m\omega_m r^{m-1} = 2\pi r \cdot \omega_n r^n$ between the surface in dimension $m = n+2$ and the volume in dimension n. This leads to the recursion

$$\omega_{n+2} = \frac{2\pi\omega_n}{n+2}$$

and the formula

$$\omega_n = \frac{\pi^{n/2}}{(n/2)!}$$

which can be computed for odd n by recalling that $(1/2)! = \sqrt{\pi}/2$.

6.5 Rotations in four dimensions

Let A be an antisymmetric 4×4-matrix, that is, $A \in \mathrm{Mat}(4,\mathbb{R})$, $A^\top = -A$. Then the matrix e^A represents a rotation of the 4-dimensional Euclidean space $\mathbb{R}^4$. In general, a rotation of $\mathbb{R}^4$ has two invariant planes which are completely orthogonal; in particular they have only one point in common. The antisymmetric matrix A has imaginary eigenvalues, say $\pm i\alpha$ and $\pm i\beta$, the eigenvalues of the rotation matrix e^A are unit complex numbers $e^{\pm i\alpha}$ and $e^{\pm i\beta}$, and the invariant planes turn by angles α and β under e^A. First, assume that $\alpha > \beta \geq 0$ (and $\alpha < \pi$). Each vector is turned through at least an angle β and at most an angle α. In the case $\beta = 0$ we have a simple rotation leaving one plane point-wise fixed. If β/α is rational, then $e^{tA} = I$ for some $t > 0$. If β/α is irrational, then $e^{tA} \neq I$ for any $t > 0$.

By the Cayley-Hamilton theorem e^A is a linear combination of the matrices I, A, A^2 and A^3 so that

$$e^A = h_0 I + h_1 A + h_2 A^2 + h_3 A^3$$

and direct computation shows that

$$\begin{aligned} h_0 &= \frac{1}{\alpha^2 - \beta^2}(\alpha^2 \cos\beta - \beta^2 \cos\alpha), \\ h_1 &= \frac{1}{\alpha^2 - \beta^2}\left(\frac{\alpha^2}{\beta}\sin\beta - \frac{\beta^2}{\alpha}\sin\alpha\right), \\ h_2 &= \frac{1}{\alpha^2 - \beta^2}(\cos\beta - \cos\alpha), \\ h_3 &= \frac{1}{\alpha^2 - \beta^2}\left(\frac{1}{\beta}\sin\beta - \frac{1}{\alpha}\sin\alpha\right). \end{aligned}$$

Letting α now approach β and computing the coefficients in the limit give

$$\begin{aligned} \lim_{\alpha\to\beta} e^A = I&(\cos\alpha + \frac{\alpha}{2}\sin\alpha) \\ &+\frac{A}{\alpha}\left(\frac{3}{2}\sin\alpha - \frac{\alpha}{2}\cos\alpha\right) \\ &+\frac{A^2}{\alpha^2}\left(\frac{\alpha}{2}\sin\alpha\right) \\ &+\frac{A^3}{\alpha^3}\left(\frac{1}{2}\sin\alpha - \frac{\alpha}{2}\cos\alpha\right). \end{aligned}$$

Observe that in the limit $A^2 = -\alpha^2 I$, which cancels some terms and results in

$$\lim_{\alpha\to\beta} e^A = I\cos\alpha + \frac{A}{\alpha}\sin\alpha.$$

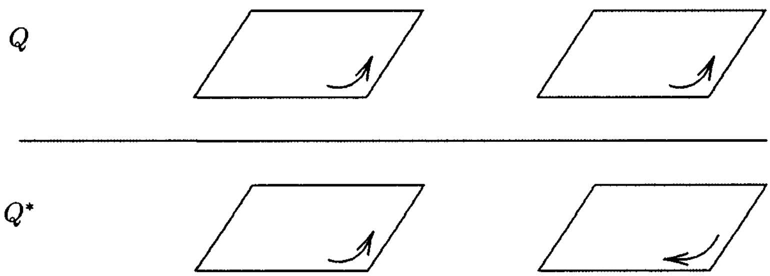
These rotations with only one rotation angle α have a whole bundle of invariant rotation planes. In fact, every point of $\mathbb{R}^4$ stays in some invariant plane, but not every plane of $\mathbb{R}^4$ is an invariant plane of e^A.

If a rotation U of $\mathbb{R}^4$ has rotation angles α and β we shall denote it by $U(\alpha, \beta)$. Consider the set $\mathcal{J} = \{U(\alpha,\beta) \in SO(4) \mid \alpha = \beta\}$ and the relation '$\sim$' in the set $\mathcal{J}' = \mathcal{J} \setminus \{I, -I\}$,

$$U \sim V \iff UV \in \mathcal{J},$$

which can be seen to be an equivalence relation. The equivalence class of a matrix $U \in \mathcal{J}'$ is the set $\{X \in \mathcal{J}' \mid X \sim U\}$. This equivalence class together with the center $\{I, -I\}$ of the rotation group $SO(4)$ forms a subgroup of $SO(4)$, denoted in the sequel by the letter Q. Also $(\mathcal{J} \setminus Q) \cup \{I, -I\}$ is a subgroup of $SO(4)$; denote it by Q^*. Observe that $UV = VU$ for $U \in Q$ and $V \in Q^*$. It can be shown that Q and Q^* are isomorphic to the group of unit quaternions $S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$.

Each rotation $L \in SO(4)$ of $\mathbb{R}^4$ can be written in the form $L = UV$, where $U \in Q$, $V \in Q^*$. The rotation angles of L are $\alpha \pm \beta$ when the rotation angles of U and V are α and β. A pair of completely orthogonal planes, both with a fixed sense of rotation, induces a pair of senses of rotations for all pairs of completely orthogonal planes. There are two classes of such pairs of oriented planes: those of the type Q and those of type Q^*.



Furthermore, we have an isomorphism of algebras,

$$\mathbb{H} \simeq \{\lambda q \mid \lambda > 0,\, q \in Q\} \cup \{0\},$$

which we shall regard as an identification. Introduce the algebra

$$\mathbb{H}^* = \{\lambda q \mid \lambda > 0,\, q \in Q^*\} \cup \{0\}.$$

and observe an isomorphism of algebras, $\mathbb{H} \simeq \mathbb{H}^*$.

6.6 Rotating ball in $\mathbb{R}^4$

A rotating ball in $\mathbb{R}^3$ has an axis of rotation, like the axis going through the North and South Poles, and a plane of rotation, like the plane of the equator. A rotating ball in $\mathbb{R}^4$ has two planes of rotation, which are completely orthogonal to each other in the sense that they have only one point in common. Let the angular velocities in these planes be bivectors ω_1 and ω_2. The total angular velocity is a bivector $\omega = \omega_1 + \omega_2$. The velocity $\mathbf{v}$ of a point $\mathbf{x}$ on the surface of the ball is

$$\mathbf{v} = \mathbf{x} \lrcorner\, \omega_1 + \mathbf{x} \lrcorner\, \omega_2.$$

Assume that φ is the angle between the direction $\mathbf{x}$ and the plane of ω_1. Then

$$|\mathbf{v}| = |\mathbf{x}|\sqrt{|\omega_1|^2(\cos\varphi)^2 + |\omega_2|^2(\sin\varphi)^2}.$$

Therefore, the local angular velocity $|\mathbf{v}|/|\mathbf{x}|$ is always between $|\omega_1|$ and $|\omega_2|$.

If $|\omega_1| = |\omega_2|$, then every point on the sphere is rotating at the same velocity and furthermore every point is travelling along some great circle, that is, everybody is living on an equator!

6.7 The Clifford algebra $\mathcal{C}\ell_4$

The Clifford algebra $\mathcal{C}\ell_4$ of $\mathbb{R}^4$ with an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is generated by the relations

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = \mathbf{e}_4^2 = 1 \quad \text{and} \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j.$$

It is a 16-dimensional algebra with basis consisting of

scalar	1
vectors	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
bivectors	$\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}$
3-vectors	$\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{134}, \mathbf{e}_{234}$
volume element	$\mathbf{e}_{1234}$

where $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j$ for $i \neq j$ and $\mathbf{e}_{1234} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$.

An arbitrary element $u \in \mathcal{C}\ell_4$ is a sum of its k-vector parts:

$$u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 + \langle u\rangle_3 + \langle u\rangle_4 \quad \text{where} \quad \langle u\rangle_k \in \bigwedge^k \mathbb{R}^4.$$

There are three important involutions of $\mathcal{C}\ell_4$:

$$\begin{aligned}
\hat{u} &= \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2 - \langle u\rangle_3 + \langle u\rangle_4 && \text{grade involution}\\
\tilde{u} &= \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2 - \langle u\rangle_3 + \langle u\rangle_4 && \text{reversion}\\
\bar{u} &= \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2 + \langle u\rangle_3 + \langle u\rangle_4 && \text{Clifford-conjugation.}
\end{aligned}$$

The Clifford algebra $\mathcal{C}\ell_4$ is isomorphic to the real algebra of 2×2-matrices $\mathrm{Mat}(2, \mathbb{H})$ with quaternions as entries,

$$\mathbf{e}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \ \mathbf{e}_3 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \ \mathbf{e}_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

6.8 Bivectors in $\bigwedge^2 \mathbb{R}^4 \subset \mathcal{C}\ell_4$

The essential difference between 3-dimensional and 4-dimensional spaces is that bivectors are no longer products of two vectors. Instead, bivectors are sums of products of two vectors in $\mathbb{R}^4$. In the 3-dimensional space $\mathbb{R}^3$ there are only *simple* bivectors, that is, all the bivectors represent a plane. In the 4-dimensional space $\mathbb{R}^4$ this is not the case any more.

Example. The bivector $\mathbf{B} = \mathbf{e}_{12} + \mathbf{e}_{34} \in \bigwedge^2 \mathbb{R}^4$ is not simple. For all simple elements the square is real, but $\mathbf{B}^2 = -2 + 2\mathbf{e}_{1234} \notin \mathbb{R}$. ∎

If the square of a bivector is real, then it is simple. [1]

Usually a bivector in $\bigwedge^2 \mathbb{R}^4$ can be uniquely written as a sum of two simple bivectors, which represent completely orthogonal planes. There is an exception to this uniqueness, crucial to the study of four dimensions: If the simple components of a bivector have equal squares, that is equal norms, then the decomposition to a sum of simple components is not unique.

Example. The bivector $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4$ can also be decomposed into a sum of two completely orthogonal bivectors as follows:

$$\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4 = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_3)(\mathbf{e}_2 + \mathbf{e}_4) + \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_3)(\mathbf{e}_2 - \mathbf{e}_4).$$ ∎

6.9 The group Spin(4) and its Lie algebra

The group $\mathbf{Spin}(4) = \{s \in \mathcal{C}\ell_4^+ \mid s\tilde{s} = 1\}$ is a two-fold covering group of the rotation group $SO(4)$ so that the map

$$\mathbb{R}^4 \to \mathbb{R}^4,\ \mathbf{x} \to s\mathbf{x}s^{-1}, \quad \text{where} \quad s \in \mathbf{Spin}(4),$$

is a rotation, and each rotation can be so represented, the same rotation being obtained by s and $-s$. The Lie algebra of $\mathbf{Spin}(4)$ is the subspace of bivectors $\bigwedge^2 \mathbb{R}^4$ with commutator product as the product. The two sets of basis bivectors

$$\begin{matrix} \frac{1}{4}(\mathbf{e}_{23} + \mathbf{e}_{14}) & & \frac{1}{4}(\mathbf{e}_{23} - \mathbf{e}_{14}) \\ \frac{1}{4}(\mathbf{e}_{31} + \mathbf{e}_{24}) & \text{and} & \frac{1}{4}(\mathbf{e}_{31} - \mathbf{e}_{24}) \\ \frac{1}{4}(\mathbf{e}_{12} + \mathbf{e}_{34}) & & \frac{1}{4}(\mathbf{e}_{12} - \mathbf{e}_{34}) \end{matrix}$$

in $\bigwedge^2 \mathbb{R}^4 \subset \mathcal{C}\ell_4$ both span a Lie algebra isomorphic to the subspace $\bigwedge^2 \mathbb{R}^3 \subset \mathcal{C}\ell_3$ with basis $\{\frac{1}{2}\mathbf{e}_{23}, \frac{1}{2}\mathbf{e}_{31}, \frac{1}{2}\mathbf{e}_{12}\}$, that is, they satisfy the same commutation relations. In other words, the Lie algebras

$$\frac{1}{2}(1 - \mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4 \quad \text{and} \quad \frac{1}{2}(1 + \mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4$$

1 Although the square of a 3-vector is real, it need not be simple. For instance, $\mathbf{V} = \mathbf{e}_{123} + \mathbf{e}_{456} \in \bigwedge^3 \mathbb{R}^6$ is not simple [this can be seen by computing $\mathbf{V}\mathbf{e}_i\mathbf{V}^{-1}$, $i = 1, 2, \ldots, 6$, and observing that they are not all vectors].

are both isomorphic to $\bigwedge^2 \mathbb{R}^3$. The two subspaces $\frac{1}{2}(1 \pm \mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4$ of $C\ell_4$ annihilate each other, and consequently,

$$\bigwedge^2 \mathbb{R}^4 \simeq \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3.$$

At the group level this means the isomorphism

$$\mathbf{Spin}(4) \simeq \mathbf{Spin}(3) \times \mathbf{Spin}(3)$$

where $\mathbf{Spin}(3) \simeq S^3 \simeq SU(2)$.

6.10 The mapping $\mathbf{F} \to (1+\mathbf{F})(1-\mathbf{F})^{-1}$ for $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$

The exponential $e^{\mathbf{F}/2} \in \mathbf{Spin}(4)$ of a bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ corresponds to the rotation $e^A \in SO(4)$, where $A(\mathbf{x}) = \mathbf{F} \llcorner \mathbf{x}$, for $\mathbf{x} \in \mathbb{R}^4$. Every rotation of $\mathbb{R}^4$ can be so represented, and the two elements $\pm e^{\mathbf{F}/2}$ represent the same rotation.

The exterior exponential $e^{\mathbf{F}} = 1 + \mathbf{F} + \frac{1}{2}\mathbf{F} \wedge \mathbf{F}$ of a bivector $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ is a multiple of an element in $\mathbf{Spin}(4)$, that is,

$$\frac{e^{\mathbf{F}}}{|e^{\mathbf{F}}|} \in \mathbf{Spin}(4).$$

Up to a sign, every element in $\mathbf{Spin}(4)$ can be so represented, except $\pm\mathbf{e}_{1234}$. The exterior exponential $e^{\mathbf{F}}$ of the bivector $\mathbf{F}$ corresponds to the rotation $(I+A)(I-A)^{-1}$; every rotation of $\mathbb{R}^4$ can be so represented, except $-I$.

The above observations raise the question: What is the rotation corresponding to $(1+\mathbf{F})(1-\mathbf{F})^{-1} \in \mathbf{Spin}(4)$? This is an interesting and non-trivial question in dimension 4. [2] Here follows the answer.

Let $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$. The antisymmetric function induced by $\mathbf{F}$ is denoted by A, that is, $A(\mathbf{x}) = \mathbf{F} \llcorner \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^4$. Write $s = (1+\mathbf{F})(1-\mathbf{F})^{-1}$. The rotation induced by $s \in \mathbf{Spin}(4)$ is denoted by $U \in SO(4)$, that is, $U = (I+A)(I-A)^{-1}$. In other words, $U(\mathbf{x}) = s\mathbf{x}s^{-1}$ for all $\mathbf{x} \in \mathbb{R}^4$. The following cases can be distinguished:

(i) If $\mathbf{F} \in \bigwedge^2 \mathbb{R}^3$ then $U = \left(\dfrac{I+A}{I-A}\right)^2$.

(ii) If $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ is simple, then $U = \left(\dfrac{I+A}{I-A}\right)^2$.

(iii) If $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ is isoclinic, then $U = \dfrac{I+2A}{I-2A}$.

2 It is also a non-trivial question in dimension 5. In dimension 6, $(1+\mathbf{F})(1-\mathbf{F})^{-1} \notin \mathbf{Spin}(6)$.

(iv) In the case of an arbitrary $\mathbf{F} \in \bigwedge^2 \mathbb{R}^4$ we cannot express U as a rational function of A [although U still has the same eigenplanes as A]. Instead,

$$U = \frac{A^4 + B^4 - 2A^2B^2 + 6A^2 - 2B^2 + I + 4A(A^2 - B^2 + I)}{A^4 + B^4 - 2A^2B^2 - 2A^2 - 2B^2 + I},$$

where $B(\mathbf{x}) = (\mathbf{F}\mathbf{e}_{1234}) \llcorner \mathbf{x}$, the dual of A. The denominator of U is a multiple of the identity I. [3]

Summary

There are three different kinds of rotations in four dimensions depending on the values of the rotation angles α, β satisfying $\pi > \alpha \geq \beta \geq 0$. Let $R : \mathbb{R}^4 \to \mathbb{R}^4$ be a rotation and $\mathbf{a}$ a non-zero vector with iterated images $\mathbf{b} = R(\mathbf{a})$, $\mathbf{c} = R(\mathbf{b})$, $\mathbf{d} = R(\mathbf{c})$. In general, $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are linearly independent, that is, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} \neq 0$. In the case of a simple rotation with $\beta = 0$, only the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly independent, that is, $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \neq 0$ but $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} = 0$. In the case of an isoclinic [4] rotation with $\alpha = \beta$, only the vectors $\mathbf{a}, \mathbf{b}$ are linearly independent, that is, $\mathbf{a} \wedge \mathbf{b} \neq 0$ but $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = 0$ and $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{d} = 0$.

In general, a rotation of $\mathbb{R}^4$ has six parameters, computed as

$$(3 + 2 - 1) + 2 = 6.$$

The number 3 comes from picking up a unit vector $\mathbf{a}$; the number 2 comes from picking up a unit vector $\mathbf{b}$ in the orthogonal complement of $\mathbf{a}$; the unit bivector $\mathbf{ab} = \mathbf{a} \wedge \mathbf{b}$ fixes a plane but the same plane is obtained by rotating $\mathbf{a}$ and $\mathbf{b}$ in the plane of $\mathbf{a} \wedge \mathbf{b}$, thus subtract 1; then finally add 2 for the two rotation parameters/angles α and β. On the other hand, an isoclinic rotation has three parameters, computed as

$$(3 - 1) + 1 = 3.$$

The number 3 comes from picking up a unit vector $\mathbf{a}$ in S^3; but in an isoclinic rotation $\mathbf{a}$ stays in a plane or a great circle S^1, so subtract 1; and finally add 1 for the rotation/angle $\alpha = \beta$.

A simple bivector, an exterior product of two vectors, corresponds to simple

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3 In dimension 5 the rotation U is given by the same expression, when

$$B(\mathbf{x}) = \left(\mathbf{F}\frac{\mathbf{F} \wedge \mathbf{F}}{|\mathbf{F} \wedge \mathbf{F}|}\right) \llcorner \mathbf{x}.$$

The denominator is no longer a multiple of I, although it still commutes with the numerator by virtue of $AB = BA$.

4 An isoclinic rotation with equal rotation angles corresponds to a multiplication by a quaternion.

rotation turning only one plane. A simple bivector multiplied by one of the idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{1234})$ corresponds to an isoclinic rotation. An isoclinic rotation has an infinity of rotation planes, and in fact, each vector is in some invariant rotation plane of an isoclinic rotation.

The two-fold cover $\mathbf{Spin}(4)$ of $SO(4)$ has three different subgroups isomorphic to $\mathbf{Spin}(3)$, each with a Lie algebra

$$\bigwedge^2 \mathbb{R}^3, \quad \frac{1}{2}(1+\mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4, \quad \frac{1}{2}(1-\mathbf{e}_{1234}) \bigwedge^2 \mathbb{R}^4.$$

There is an automorphism of $\mathbf{Spin}(4)$ which swaps the last two copies of $\mathbf{Spin}(3)$, but there is no automorphism of $\mathbf{Spin}(4)$ swapping the first copy of $\mathbf{Spin}(3)$ with either of the other two copies.

Exercises

1. Compute the squares of $\frac{1}{2}(1+\mathbf{e}_{12}+\mathbf{e}_{34} \pm \mathbf{e}_{1234})$.
2. Take a vector $\mathbf{a} \in \mathbb{R}^4$ and a bivector $\mathbf{B} = \alpha\mathbf{e}_{12} + \beta\mathbf{e}_{34} \in \bigwedge^2 \mathbb{R}^4$. Show that $\mathbf{BaB} \in \mathbb{R}^4$.
3. Compute $\exp(\alpha\mathbf{e}_{12} + \beta\mathbf{e}_{34})$.
4. Let $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$. Compute $\mathbf{A} = \mathbf{ae}_{123}$ and $\mathbf{B} = \mathbf{be}_{123}$. Determine $\frac{1}{2}(1+\mathbf{e}_{1234})\mathbf{A}$ and $\frac{1}{2}(1-\mathbf{e}_{1234})\mathbf{B}$, and show that these bivectors commute.
5. Compute $\mathbf{C} = \frac{1}{2}(1+\mathbf{e}_{1234})\mathbf{A} + \frac{1}{2}(1-\mathbf{e}_{1234})\mathbf{B}$, and express $\exp(\mathbf{C})$ using $|\mathbf{a}|$ and $|\mathbf{b}|$. What are the two rotation angles of the rotation $\mathbb{R}^4 \to \mathbb{R}^4$, $\mathbf{x} \to c\mathbf{x}c^{-1}$ where $c = \exp(\mathbf{C})$?
6. Consider the Lie algebra $\bigwedge^2 \mathbb{R}^4$ with the commutator product $[a,b] = ab - ba$, and its three subalgebras spanned by

$$\begin{array}{ll} \mathcal{V}: & \frac{1}{2}\mathbf{e}_{23},\ \frac{1}{2}\mathbf{e}_{31},\ \frac{1}{2}\mathbf{e}_{12} \\ \mathcal{I}_1: & \frac{1}{4}(\mathbf{e}_{23}-\mathbf{e}_{14}),\ \frac{1}{4}(\mathbf{e}_{31}-\mathbf{e}_{24}),\ \frac{1}{4}(\mathbf{e}_{12}-\mathbf{e}_{34}) \\ \mathcal{I}_2: & \frac{1}{4}(\mathbf{e}_{23}+\mathbf{e}_{14}),\ \frac{1}{4}(\mathbf{e}_{31}+\mathbf{e}_{24}),\ \frac{1}{4}(\mathbf{e}_{12}+\mathbf{e}_{34}), \end{array}$$

   each isomorphic to $\bigwedge^2 \mathbb{R}^3$. Show that there is no automorphism of the Lie algebra $\bigwedge^2 \mathbb{R}^4$ which permuts $\mathcal{V}, \mathcal{I}_1, \mathcal{I}_2$ cyclically or swaps $\mathcal{V}$ for $\mathcal{I}_1$ or $\mathcal{I}_2$.
7. In two dimensions we can place 4 circles of radius r inside a square of side $4r$, and put a circle of radius $(\sqrt{2}-1)r$ in the middle of the 4 circles. In three dimensions we can place 8 spheres of radius r inside a cube of side $4r$, and put a sphere of radius $(\sqrt{3}-1)r$ in the middle of the 8 circles. In n dimensions we can place 2^n spheres of radius r inside a hypercube of side $4r$, and put a sphere of radius $(\sqrt{n}-1)r$ in the middle of the 2^n spheres.

Dimensions 2 and 3 differ topologically: in dimension 3 one can see the middle sphere from outside the cube. Let the dimension be progressively increased. In some dimension the middle sphere actually emerges out of the hypercube. In some dimension the middle sphere becomes even bigger than the hypercube, in the sense that its volume is larger than the volume of the hypercube. Determine those dimensions.

Solutions

1. $\mathbf{e}_{1234}$, $\mathbf{e}_{12} + \mathbf{e}_{34}$.
3. $\cos\alpha\cos\beta + \mathbf{e}_{12}\sin\alpha\cos\beta + \mathbf{e}_{34}\cos\alpha\sin\beta + \mathbf{e}_{1234}\sin\alpha\sin\beta$.
5. The rotation angles are $\alpha = (|\mathbf{a}| + |\mathbf{b}|)/2$ and $\beta = (|\mathbf{a}| - |\mathbf{b}|)/2$, and

$$\frac{1}{2}(1+\mathbf{e}_{1234})\Big(\cos|\mathbf{a}| + \frac{\mathbf{A}}{|\mathbf{a}|}\sin|\mathbf{a}|\Big) + \frac{1}{2}(1-\mathbf{e}_{1234})\Big(\cos|\mathbf{b}| + \frac{\mathbf{B}}{|\mathbf{b}|}\sin|\mathbf{b}|\Big)$$
$$= \cos\alpha\cos\beta - \mathbf{e}_{1234}\sin\alpha\sin\beta$$
$$+\mathbf{C}\frac{\alpha - \beta\mathbf{e}_{1234}}{\alpha^2 - \beta^2}(\sin\alpha\cos\beta + \mathbf{e}_{1234}\cos\alpha\sin\beta).$$

7. In dimension 9 the middle sphere touches the surface of the hypercube, and in dimension 10 it emerges out of the hypercube. In dimension 1206 the volume of the middle sphere is larger than the volume of the hypercube.

Bibliography

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