15
Witt Rings and Brauer Groups

Quadratic forms can be classified by their Witt classes in Witt rings (of concerned fields). This is a slightly coarser classification than the one given by the Clifford algebras (of quadratic forms).

Associative algebras with unity can be studied by means of Brauer groups (of fields); for this one needs to know tensor products of algebras. These topics will be discussed in this chapter.

15.1 Quadratic forms

A *quadratic form* on a linear space V over a field $\mathbb{F}$ is a map $Q : V \to \mathbb{F}$ such that for any $\lambda \in \mathbb{F}$ and $\mathbf{x} \in V$

$$Q(\lambda \mathbf{x}) = \lambda^2 Q(\mathbf{x})$$

and such that the map

$$V \times V \to \mathbb{F}, \quad (\mathbf{x}, \mathbf{y}) \to Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})$$

is bilinear, that is, linear in both arguments. A linear space with a quadratic form on itself is called a *quadratic space*. A quadratic form obeys the parallelogram law

$$Q(\mathbf{x} + \mathbf{y}) + Q(\mathbf{x} - \mathbf{y}) = 2Q(\mathbf{x}) + 2Q(\mathbf{y}).$$

In characteristic $\neq 2$ the quadratic form may be recaptured from its symmetric bilinear form

$$B(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})]$$

since $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$. In characteristic $\neq 2$ the theory of quadratic forms is the same as the theory of symmetric bilinear forms, but in characteristic 2 there are quadratic forms which are not induced by any *symmetric* bilinear form.

Example. Consider the 2-dimensional space $\mathbb{F}_2^2$ over $\mathbb{F}_2 = \{0, 1\}$. For the quadratic form $Q : \mathbb{F}_2^2 \to \mathbb{F}_2$, $\mathbf{x} = (x_1, x_2) \to x_1 x_2$ there is no symmetric bilinear form B such that $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$. However, there is a bilinear form B, not symmetric, such that $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$, namely $B(\mathbf{x}, \mathbf{y}) = x_1 y_2$, but the matrix

$$B(\mathbf{e}_i, \mathbf{e}_j) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

cannot be symmetrized by adding an alternating matrix with entries in $\mathbb{F}_2 = \{0, 1\}$. ∎

Remark. We shall not be concerned with characteristic 2, or non-symmetric B, except at the end of this book when considering the relation between the exterior algebra and the Clifford algebras. The role of non-symmetric B will be described in Chevalley's construction $\mathcal{C}\ell(Q) \subset \mathrm{End}(\bigwedge V)$. ∎

A non-zero vector $\mathbf{x}$ is *null* or *isotropic* if $Q(\mathbf{x}) = 0$. A quadratic form is *anisotropic* if $Q(\mathbf{x}) = 0$ implies $\mathbf{x} = 0$, and *isotropic* if $Q(\mathbf{x}) = 0$ for some $\mathbf{x} \neq 0$. A bilinear form is *non-degenerate* if $B(\mathbf{x}, \mathbf{y}) = 0$ for all $\mathbf{y} \in V$ implies $\mathbf{x} = 0$. An anisotropic quadratic form is always non-degenerate.

A 2-dimensional isotropic but non-degenerate quadratic space is known as the *hyperbolic plane*. A hyperbolic plane has a quadratic form $y_1 y_2$ or equivalently $x_1^2 - x_2^2$ (choose $y_1 = x_1 + x_2$ and $y_2 = x_1 - x_2$ in characteristic $\neq 2$).

A subspace with a vanishing quadratic form is *totally isotropic*. In a non-degenerate quadratic space with a totally isotropic subspace S, there is another totally isotropic subspace S' such that $S \cap S' = \{0\}$ and $\dim S = \dim S'$. A non-degenerate quadratic space is *neutral* or *hyperbolic* if it is a direct sum of two totally isotropic subspaces (necessarily of the same dimension). A neutral quadratic space is even-dimensional.

Examples. 1. A Euclidean space $\mathbb{R}^n$ has an anisotropic (and positive definite) quadratic form on itself, i.e., $\mathbf{x} \neq 0$ implies $\mathbf{x} \cdot \mathbf{x} > 0$. This enables us to introduce the *norm* or *length* $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ of $\mathbf{x} \in \mathbb{R}^n$.
2. The real quadratic space $\mathbb{R}^{p,q}$ is non-degenerate and for non-zero p, q also isotropic (indefinite). The dimension of its maximal totally isotropic subspace is p or q according as $p \leq q$ or $p \geq q$, respectively. This number is called the *isotropy index* (or Witt index) of $\mathbb{R}^{p,q}$.
3. The quadratic form $x_1^2 + x_2^2$ on $\mathbb{F}_5^2$ is non-degenerate but isotropic, since $1^2 + 2^2 = 0 \bmod 5$. It is also neutral.
4. The quadratic forms $x_1^2 + 2x_2^2$ and $x_1^2 + 3x_2^2$ are anisotropic on $\mathbb{F}_5^2$. ∎

Two quadratic spaces (V, Q) and (V', Q') are *isometric* if there is a linear isomorphism $L : V \to V'$ such that $Q'(L\mathbf{x}) = Q(\mathbf{x})$, or equivalently, in characteristic $\neq 2$,

$$B'(L\mathbf{x}, L\mathbf{y}) = B(\mathbf{x}, \mathbf{y}) \quad \text{for all} \quad \mathbf{x}, \mathbf{y} \in V.$$

We will express the isometry as $(V, Q) \simeq (V', Q')$ or simply $Q \simeq Q'$. A self-isometry (or automorphism) of Q on V is a linear isomorphism $L : V \to V$ such that $Q(L(\mathbf{x})) = Q(\mathbf{x})$ for all $\mathbf{x} \in V$; these self-isometries of Q form the *orthogonal group* $O(V, Q)$.

Two vectors $\mathbf{x}, \mathbf{y}$ such that $B(\mathbf{x}, \mathbf{y}) = 0$ are said to be *orthogonal* (in the case of a symmetric B). If a quadratic space is a direct sum of two subspaces, $(V_1, Q_1) \oplus (V_2, Q_2)$, such that $B(\mathbf{x}_1, \mathbf{x}_2) = 0$ for all $\mathbf{x}_1 \in V_1$ and $\mathbf{x}_2 \in V_2$, it is an *orthogonal sum* denoted by $V_1 \perp V_2$ or $Q_1 \perp Q_2$.

There is also, for two quadratic spaces (V_1, Q_1) and (V_2, Q_2), the *tensor product* $V_1 \otimes V_2$ of dimension $(\dim V_1)(\dim V_2)$, with a quadratic form satisfying

$$Q(\mathbf{x}_1 \otimes \mathbf{x}_2) = Q_1(\mathbf{x}_1) Q_2(\mathbf{x}_2)$$

for decomposable elements $\mathbf{x}_1 \otimes \mathbf{x}_2$ with $\mathbf{x}_1 \in V_1$ and $\mathbf{x}_2 \in V_2$.

The symmetric matrix $B(\mathbf{e}_i, \mathbf{e}_j)$ can be diagonalized (in characteristic $\neq 2$); as a consequence any quadratic form is isometric to a diagonal form $d_1 x_1^2 + d_2 x_2^2 + \ldots + d_n x_n^2$ for some $d_1, d_2, \ldots, d_n \in \mathbb{F}$. We shall write

$$\langle d_1, d_2, \ldots, d_n \rangle \quad \text{to denote} \quad d_1 x_1^2 + d_2 x_2^2 + \ldots + d_n x_n^2.$$

The orthogonal sum and the tensor product appear in diagonal form as

$$\langle a_1, \ldots, a_m \rangle \perp \langle b_1, \ldots, b_n \rangle = \langle a_1, \ldots, a_m, b_1, \ldots, b_n \rangle,$$
$$\langle a_1, \ldots, a_m \rangle \otimes \langle b_1, \ldots, b_n \rangle = \langle a_1 b_1, \ldots, a_i b_j, \ldots, a_m b_n \rangle.$$

Examples. 1. The real quadratic space $\mathbb{R}^{p,p}$ is neutral and an orthogonal sum of p copies of hyperbolic planes $\mathbb{R}^{1,1}$ each with a quadratic form $\langle 1, -1 \rangle$.
2. The hyperbolic plane over $\mathbb{F}$, $\operatorname{char} \mathbb{F} \neq 2$, is isometric with $x_1 x_2$ and $x_1^2 - x_2^2 \simeq \langle 1, -1 \rangle$.
3. In $\mathbb{F}_5$, $\langle 1 \rangle \not\simeq \langle 2 \rangle$ since $2 \notin \mathbb{F}_5^{\square}$, the set of non-zero squares, but $\langle 2 \rangle \simeq \langle 3 \rangle$ since $2x_1^2 = 3(2x_1)^2$.
4. The quadratic forms $\langle 1, 2 \rangle$ and $\langle 1, 3 \rangle$ are isometric on $\mathbb{F}_5^2$ as can be seen by the identity $x_1^2 + 2x_2^2 = x_1^2 + 3(2x_2)^2 \bmod 5$ and the linear isomorphism $(x_1, x_2) \to (x_1, 2x_2)$.
5. The quadratic forms $\langle 1, 1 \rangle$ and $\langle 1, -1 \rangle$ are isometric on $\mathbb{F}_5^2$ since $x_1^2 - x_2^2 = x_1 + (2x_2)^2$. ∎

15.2 Witt rings

Two quadratic spaces V, Q and V', Q' are said to be in the same *Witt class* if $Q \perp (-Q')$ is neutral. A non-degenerate quadratic space is an orthogonal sum $V = V_a \perp V_h$ of an anisotropic subspace V_a and a neutral subspace V_h. The anisotropic part V_a is unique up to isometry. It follows that V and V' are in the same Witt class iff $Q_a \perp (-Q'_a)$ is neutral, or equivalently iff $Q_a \simeq Q'_a$. This results in a correspondence

$$\boxed{\text{Witt class of } V = V_a \oplus V_h \longleftrightarrow \text{isometry class of } V_a}$$

exactly one anisotropic isometry class being included in each Witt class.

Examples. Let the ground field be $\mathbb{F}_5$.
1. From the orthogonal sum $\langle 1\rangle \perp \langle 1,2\rangle \simeq \langle 1,1,2\rangle$ we can cancel the hyperbolic plane $\langle 1,1\rangle$ to extract the anisotropic part $\langle 2\rangle$. So the sum $\langle 1\rangle \perp \langle 1,2\rangle$ is in the Witt class of $\langle 2\rangle$.
2. From the tensor product $\langle 1,2\rangle \otimes \langle 3\rangle \simeq \langle 3, 2\cdot 3 = 1\rangle$ there is nothing to cancel since $\langle 3,1\rangle \simeq \langle 1,2\rangle$ is anisotropic. So $\langle 1,2\rangle \otimes \langle 3\rangle$ is in the Witt class of $\langle 1,2\rangle$. ∎

The orthogonal sum $\perp$ and the tensor product $\otimes$ provide the set of all the Witt classes over $\mathbb{F}$ with a ring structure yielding the *Witt ring* $W(\mathbb{F})$.

The opposite of Q is represented by $-Q$ in $W(\mathbb{F})$. The neutral quadratic forms, in particular $\langle 0\rangle$, represent the zero of $W(\mathbb{F})$. The 1-dimensional form $\langle 1\rangle$ corresponds to the multiplicative unity of $W(\mathbb{F})$. The zero $\langle 0\rangle$ and the unity $\langle 1\rangle$ are the only idempotents in $W(\mathbb{F})$. The even-dimensional quadratic forms induce an ideal of $W(\mathbb{F})$.

Examples. 1. The Witt ring $W(\mathbb{F}_5)$ contains four anisotropic isometry classes $\langle 0\rangle$, $\langle 1\rangle$, $\langle 2\rangle$, $\langle 1,2\rangle$. The addition and multiplication tables of $W(\mathbb{F}_5)$ are

$\perp$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 1,2\rangle$
$\langle 1\rangle$	$\langle 0\rangle$	$\langle 1,2\rangle$	$\langle 2\rangle$
$\langle 2\rangle$	$\langle 1,2\rangle$	$\langle 0\rangle$	$\langle 1\rangle$
$\langle 1,2\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle 0\rangle$

$\otimes$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 1,2\rangle$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 2\rangle$	$\langle 1,2\rangle$
$\langle 2\rangle$	$\langle 2\rangle$	$\langle 1\rangle$	$\langle 1,2\rangle$
$\langle 1,2\rangle$	$\langle 1,2\rangle$	$\langle 1,2\rangle$	$\langle 0\rangle$

The Witt ring $W(\mathbb{F}_5)$ is isomorphic to the group algebra $\mathbb{Z}_2[\mathbb{F}_5^\times/\mathbb{F}_5^\square]$; and the additive group of $W(\mathbb{F}_5)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, but as a ring $W(\mathbb{F}_5) \not\simeq {}^2\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$.
2. The 1-dimensional line $\mathbb{F}_5$ with quadratic form $\langle 2\rangle$ has the quadratic field extension $\mathbb{F}_5(\sqrt{2}) \simeq \mathbb{F}_{25}$ as its Clifford algebra. The Clifford algebras of both $\langle 1\rangle$ and $\langle -1\rangle$ split as the double-ring ${}^2\mathbb{F}_5 = \mathbb{F}_5 \times \mathbb{F}_5$. Therefore, as algebras $\mathcal{C}\ell(\langle 2\rangle, \mathbb{F}_5) \not\simeq \mathcal{C}\ell(\langle 1\rangle, \mathbb{F}_5) \simeq \mathcal{C}\ell(\langle -1\rangle, \mathbb{F}_5)$. [1]

1 Denoting the Clifford algebra of the n-dimensional quadratic space $p\langle 1\rangle \perp q\langle -1\rangle$ by

3. The Witt ring $W(\mathbb{F}_7)$ contains four anisotropic isometry classes $\langle 0\rangle$, $\langle 1\rangle$, $\langle 1,1\rangle$, $\langle 1,1,1\rangle$, and $W(\mathbb{F}_7) \simeq \mathbb{Z}_4$. ∎

The finite fields $\mathbb{F}_q$. The Witt ring of $\mathbb{F}_q$, $q = p^m$, $\operatorname{char} p \neq 2$, contains four anisotropic isometry classes

$$\begin{array}{llll} \langle 0\rangle,\ \langle 1\rangle,\ \langle s\rangle,\ \langle 1,s\rangle & \text{where} \quad s \notin \mathbb{F}_q^{\square} & \text{for} & p = 1 \bmod 4, \\ \langle 0\rangle,\ \langle 1\rangle,\ \langle 1,1\rangle,\ \langle 1,1,1\rangle & & \text{for} & p = 3 \bmod 4. \end{array}$$

The corresponding Witt rings are $W(\mathbb{F}_q) \simeq \mathbb{Z}_2[\mathbb{F}_q^\times/\mathbb{F}_q^\square]$, $q = 1 \bmod 4$, and $W(\mathbb{F}_q) \simeq \mathbb{Z}_4$, $q = 3 \bmod 4$. All quadratic forms over the finite fields are isotropic in dimensions $n > 3$.

The real field $\mathbb{R}$. A field $\mathbb{F}$ is *ordered* if there is a subset $P \subset \mathbb{F}$ (of positive numbers) such that for all $a, b \in P$ also $a + b, ab \in P$, and, for all $a \in \mathbb{F}$ exactly one of $a \in P$, $a = 0$, and $-a \in P$ holds. The statement $a - b \in P$ is also written $a > b$. An ordered field $\mathbb{F}$ has an *absolute value* $\mathbb{F} \to P$, $x \to |x|$ defined by setting $|0| = 0$, $|x| = x$ for $x > 0$, and $|x| = -x$ for $-x > 0$.

In an ordered field, $\mathbb{F}^\square \subset P$. If all the positive numbers have square roots, then there is a unique ordering with $P = \mathbb{F}^\square$. The following holds for any ordered field $\mathbb{F}$ such that $P = \mathbb{F}^\square$, but we shall only consider the real field $\mathbb{R}$.

There exist exactly two anisotropic forms on $\mathbb{R}^n$, namely the positive definite $\langle 1,1,\ldots,1\rangle$ and the negative definite $\langle -1,-1,\ldots,-1\rangle$. [2] A non-degenerate quadratic form on $\mathbb{R}^n$ is isometric to

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2, \quad p + q = n,$$

which we abbreviate as $p\langle 1\rangle \perp q\langle -1\rangle$. The real quadratic space with this quadratic form is denoted by $\mathbb{R}^{p,q}$. The integer $p - q$ is called the *signature* of $\mathbb{R}^{p,q}$.

The signature map sending $\mathbb{R}^{p,q}$ to $p-q$ gives a ring isomorphism $W(\mathbb{R}) \simeq \mathbb{Z}$. As a consequence, the Clifford algebras of non-degenerate real quadratic spaces can be listed by the symbols $\mathcal{C}\ell_{p,q}$, denoted more fully by $\mathcal{C}\ell_{p,q}(\mathbb{R}^n) = \mathcal{C}\ell(\mathbb{R}^{p,q})$ or $\mathcal{C}\ell(p\langle 1\rangle \perp q\langle -1\rangle, \mathbb{R}^n)$.

The complex field $\mathbb{C}$. The Witt ring of $\mathbb{C}$ contains only two anisotropic isometry classes, namely $\langle 0\rangle$ and $\langle 1\rangle$, and $W(\mathbb{C}) \simeq \mathbb{Z}_2$. We only have to distinguish between even- and odd-dimensional spaces over $\mathbb{C}$.

Exercises 1,2

$\mathcal{C}\ell_{p,q}(\mathbb{F}^n)$, this example shows that the notion $\mathcal{C}\ell_{p,q}$ does not reach all the Clifford algebras over arbitrary fields $\mathbb{F}$.

2 The real linear space $\mathbb{R}^n$ with the positive definite quadratic form $n\langle 1\rangle = \langle 1,1,\ldots,1\rangle$ is the *Euclidean space* $\mathbb{R}^n$.

15.3 Algebras

An algebra A over a field $\mathbb{F}$ is a linear (that is a vector) space A over $\mathbb{F}$ together with a bilinear map $A \times A \to A$, $(a,b) \to ab$, the algebra product. Bilinearity means distributivity $(a+b)c = ac + bc$, $a(b+c) = ab + ac$ and $(\lambda a)b = a(\lambda b) = \lambda(ab)$ for all $a, b, c \in A$ and $\lambda \in \mathbb{F}$.

Examples. 1. The 2-dimensional real linear space $\mathbb{R}^2$ together with the product $(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$ results in the real algebra of complex numbers $\mathbb{C}$.
2. The double-ring ${}^2\mathbb{F}$ of a field $\mathbb{F}$ has a product $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$ making it a 2-dimensional algebra over the subfield denoted by $\mathbb{F}(1,1) = \{(\lambda, \lambda) \mid \lambda \in \mathbb{F}\}$.
3. The matrix algebra of real 2×2-matrices $\mathrm{Mat}(2, \mathbb{R})$ is a 4-dimensional real associative algebra with unity I.
4. The real linear space $\mathbb{R}^3$ together with the cross product $\mathbf{a} \times \mathbf{b}$ is a (non-associative) Lie algebra. ∎

An algebra is without *zero-divisors* if $ab = 0$ implies $a = 0$ or $b = 0$. In a *division algebra* $\mathbb{D}$ the equations $ax = b$ and $ya = b$ have unique solutions x, y for all non-zero $a, b \in \mathbb{D}$. A division algebra is without zero-divisors, and conversely, every finite-dimensional algebra without zero-divisors is a division algebra. If a division algebra is associative, then it has a multiplicative unity and each non-zero element admits a unique inverse (on both sides).

An algebra with a multiplicative unity is said to *admit inverses* if each non-zero element admits an inverse.

Examples. 1. The quaternions $\mathbb{H}$ form a real associative but non-commutative division algebra with unity 1.
2. Define the following product for pairs of quaternions:

$$(x_1, y_1) \circ (x_2, y_2) = (x_1x_2 - \bar{y}_2y_2, y_2x_1 + y_1\bar{x}_2).$$

This makes the real linear space $\mathbb{H} \times \mathbb{H}$ a real algebra, the Cayley algebra of octonions $\mathbb{O}$. The Cayley algebra is non-associative, $a \circ (b \circ c) \neq (a \circ b) \circ c$, but alternative, $(a \circ a) \circ b = a \circ (a \circ b)$, $a \circ (b \circ b) = (a \circ b) \circ b$. It is a division algebra with unity 1.
3. Consider a 3-dimensional real algebra with basis $\{1, i, j\}$ such that 1 is the unity and $i^2 = j^2 = -1$ but $ij = ji = 0$. The algebra is commutative, non-associative and non-alternative. It admits inverses, but the inverses of the elements of the form $xi + yj$ are not unique, $(xi + yj)^{-1} = \lambda(yi - xj) - \frac{ix + jy}{x^2 + y^2}$. It has by definition zero-divisors, and cannot be a division algebra. ∎

An isomorphism or anti-isomorphism of algebras A and B is a linear isomorphism $f : A \to B$ such that

$$f(xy) = f(x)f(y) \quad \text{or} \quad f(xy) = f(y)f(x),$$

respectively. An *automorphism* or *anti-automorphism* of an algebra A is an isomorphism or anti-isomorphism $A \to A$, respectively. An automorphism or anti-automorphism f of A such that $f(f(x)) = x$ for all $x \in A$ is an *involution* or an *anti-involution*, respectively.

The only algebra automorphisms of $\mathbb{C}$, regarded as a real algebra, are the identity and the complex conjugation $z \to \bar{z}$.

The only automorphisms of the real algebra ${}^2\mathbb{R}$ are the identity and the *swap*

$${}^2\mathbb{R} \to {}^2\mathbb{R}, \ (\lambda, \mu) \to \text{swap}(\lambda, \mu) = (\mu, \lambda).$$

The swap acts like the complex conjugation of $\mathbb{C}$, since

$$\text{swap}[\lambda(1,1) + \mu(1,-1)] = \lambda(1,1) - \mu(1,-1).$$

Two automorphisms or anti-automorphisms α, β of an algebra A are said to be *similar* if there is an automorphism γ of A such that $\alpha\gamma = \gamma\beta$. If no such γ exists then α and β are said to be *dissimilar*.

The identity automorphism is similar only to itself. Consequently, the two involutions of the real algebra $\mathbb{C}$ are dissimilar, and the two involutions of the real algebra ${}^2\mathbb{R}$ are dissimilar.

Exercises 3,4

15.4 Tensor products of algebras, Brauer groups

The tensor product of two algebras A and B over a field $\mathbb{F}$ is the linear space $A \otimes B$ made into an algebra with the product satisfying

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb')$$

for $a, a' \in A$ and $b, b' \in B$. This algebra is also denoted by $A \otimes B$ or, to emphasize the ground field, by $A \otimes_{\mathbb{F}} B$.

In the special case of finite-dimensional associative algebras with multiplicative unity, the statement $C = A \otimes B$ can be tested by the following conditions posed on the subalgebras A and B of C:

(i) $ab = ba$ for any $a \in A$ and $b \in B$,
(ii) C is generated as an algebra by A and B,
(iii) $\dim C = (\dim A)(\dim B)$.

Examples. 1. $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \simeq \mathrm{Mat}(2, \mathbb{C})$ the real matrix algebra of 2×2-matrices with complex numbers as entries.

2. $\mathrm{Mat}(p,\mathbb{R}) \otimes \mathrm{Mat}(q,\mathbb{R}) \simeq \mathrm{Mat}(pq,\mathbb{R})$.
3. $\mathrm{Mat}(p,\mathbb{R}) \otimes \mathrm{Mat}(q,\mathbb{C}) \simeq \mathrm{Mat}(pq,\mathbb{C})$. ∎

Exercises 5,6

A two-sided ideal T of an algebra A is a subalgebra such that $ta \in T$ and $at \in T$ for all $t \in T$ and $a \in A$. An algebra A is called *simple* if it has no two-sided ideals other than 0 and A. The *center* $\mathrm{Cen}(A)$ of an algebra A consists of the elements commuting with all the elements of A:

$$\mathrm{Cen}(A) = \{c \in A \mid ac = ca \text{ for all } a \in A\}.$$

An algebra with multiplicative unity 1 is called *central* if $\mathrm{Cen}(A) = \mathbb{F} \cdot 1 \simeq \mathbb{F}$. A finite-dimensional central simple associative $\mathbb{F}$-algebra A with multiplicative unity is isomorphic to $\mathrm{Mat}(d,\mathbb{D})$ for some suitable division ring $\mathbb{D}$ (and division algebra over $\mathbb{F}$).

The *opposite* A^{opp} of an algebra A is the linear space A with a new product $\mathrm{opp}[ab]$ of $a, b \in A$ given by $\mathrm{opp}[ab] = ba$. Two central simple associative $\mathbb{F}$-algebras A and B are in the same *Brauer class* if $A \otimes B^{\mathrm{opp}} \simeq \mathrm{Mat}(d,\mathbb{F})$ for some integer d. The tensor product of algebras induces a product for Brauer classes, making the set of Brauer classes a group, called the *Brauer group* $Br(\mathbb{F})$ of the field $\mathbb{F}$.

Examples. $Br(\mathbb{R}) \simeq \{\mathbb{R}, \mathbb{H}\}$, $Br(\mathbb{C}) \simeq \{\mathbb{C}\}$, $Br(\mathbb{F}_5) \simeq \{\mathbb{F}_5\}$. ∎

An algebra A is graded over $\mathbb{Z}_2 = \{0,1\}$ if it is a direct sum of two subalgebras $A = A_0 \oplus A_1$ so that $A_i A_j \subset A_{i+j}$ [the indices are added modulo 2]. For two graded algebras $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ the *graded tensor product* $A \hat{\otimes} B$ is the linear space $A \otimes B$ provided with the product determined by the formula

$$(a \otimes b)(a' \otimes b') = (-1)^{ij}(aa') \otimes (bb')$$

for homogeneous $a' \in A_i$ and $b \in B_j$. The graded opposite A^{opp} of a graded algebra $A = A_0 \oplus A_1$ is the linear space A with a new product $\mathrm{opp}[ab]$ of $a, b \in A$ given by $\mathrm{opp}[ab] = (-1)^{ij} ba$ for homogeneous elements $a \in A_i$ and $b \in B_j$.

Exercises

1. Determine the addition and the multiplication tables of the anisotropic isometry classes $\langle 0\rangle$, $\langle 1\rangle$, $\langle 1,1\rangle$, $\langle 1,1,1\rangle$ of $W(\mathbb{F}_7)$.
2. Identify as matrix algebras all the Clifford algebras of non-degenerate quadratic forms over $\mathbb{F}_5$.

3. Show that the two involutions $\alpha(\lambda,\mu)=(\mu,\lambda)$ and $\beta(\lambda,\mu)=(\bar{\mu},\bar{\lambda})$ are similar involutions of the real or complex algebra ${}^2\mathbb{C}$.
4. Consider the four anti-involutions of $\mathrm{Mat}(2,\mathbb{R})$ sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \quad \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Determine which ones of these four anti-involutions are similar or dissimilar to each other. Hint: keep track of what happens to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with squares I, I, and $-I$.
5. Show that $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathbb{C}\oplus\mathbb{C}$.
6. Show that $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H}\simeq\mathrm{Mat}(4,\mathbb{R})$.

Solutions

1.

$\perp$	$\langle 1\rangle$	$\langle 1,1\rangle$	$\langle 1,1,1\rangle$
$\langle 1\rangle$	$\langle 1,1\rangle$	$\langle 1,1,1\rangle$	$\langle 0\rangle$
$\langle 1,1\rangle$	$\langle 1,1,1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$
$\langle 1,1,1\rangle$	$\langle 0\rangle$	$\langle 1\rangle$	$\langle 1,1\rangle$

$\otimes$	$\langle 1\rangle$	$\langle 1,1\rangle$	$\langle 1,1,1\rangle$
$\langle 1\rangle$	$\langle 1\rangle$	$\langle 1,1\rangle$	$\langle 1,1,1\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 0\rangle$	$\langle 1,1\rangle$
$\langle 1,1,1\rangle$	$\langle 1,1,1\rangle$	$\langle 1,1\rangle$	$\langle 1\rangle$

2. A non-degenerate quadratic form $\langle a_1,a_2,\ldots,a_n\rangle$ over $\mathbb{F}_5$ is isometric to $p\langle 1\rangle\perp q\langle 2\rangle$ where the numbers p and q mean, respectively, occurrences of $1,4$ and $2,3$ in $a_1,a_2,\ldots,a_n$. The Clifford algebra $\mathcal{C\ell}(p\langle 1\rangle\perp q\langle 2\rangle,\mathbb{F}_5^n)$ is isomorphic, as an associative algebra, to the matrix algebra

$$\begin{array}{ll} \mathrm{Mat}(2^{n/2},\mathbb{F}_5) & p \text{ and } q \text{ even} \\ \mathrm{Mat}(2^{n/2},\mathbb{F}_5) & p \text{ and } q \text{ odd} \\ {}^2\mathrm{Mat}(2^{(n-1)/2},\mathbb{F}_5) & p \text{ odd, } q \text{ even} \\ \mathrm{Mat}(2^{(n-1)/2},\mathbb{F}_5(\sqrt{2})) & p \text{ even, } q \text{ odd.} \end{array}$$

For instance, $\mathcal{C\ell}(\langle 1,2\rangle,\mathbb{F}_5^2)\simeq\mathrm{Mat}(2,\mathbb{F}_5)$ by the correspondences

$$\mathrm{e}_1\simeq\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathrm{e}_2\simeq\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$$

of an orthogonal basis $\{e_1, e_2\}$ of $\langle 1, 2\rangle$ on $\mathbb{F}_5^2$.

3. Choose $\gamma(\lambda, \mu) = (\bar{\lambda}, \mu)$ or $\gamma(\lambda, \mu) = (\lambda, \bar{\mu})$ to find $\alpha\gamma = \gamma\beta$.
4. Only two of the anti-involutions are similar,
$$\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \quad \beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix},$$
as can be seen by choosing the intertwining automorphism
$$\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$
for which $\alpha\gamma = \gamma\beta$.
5. We must have $1 \otimes 1 \simeq (1, 1)$ and may choose $i \otimes i \simeq (1, -1)$ or $i \otimes i \simeq (-1, 1)$. If we choose the latter, we may still choose $1 \otimes i \simeq (i, i)$, $i \otimes 1 \simeq (i, -i)$ or $1 \otimes i \simeq (i, -i)$, $i \otimes 1 \simeq (i, i)$ or opposites of both.
6. Choose for $a = a_0 + ia_1 + ja_2 + ka_3$, $b = b_0 + ib_1 + jb_2 + kb_3$ in $\mathbb{H}$ the matrix representations
$$a \simeq \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{pmatrix}, \quad b \simeq \begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ -b_1 & b_0 & -b_3 & b_2 \\ -b_2 & b_3 & b_0 & -b_1 \\ -b_3 & -b_2 & b_1 & b_0 \end{pmatrix},$$
and check that the matrices commute and form two isomorphic images of the ring $\mathbb{H}$.

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