Chapter 2

General Properties and Fundamental Theorems

2.1 Introduction

On many occasions throughout this book, we will exploit a number of fundamental theorems satisfied by, and general properties of solutions of, the Stokes equations. These are collected and discussed at some length in this chapter. Theorems concerning energy dissipation are closely related to uniqueness proofs for solutions of the Stokes equations and are discussed in Section 2.2. Furthermore, they form the basis for *inclusion monotonicity theorems*, which place upper and lower bounds on energy dissipation rates, and thus the hydrodynamic drag and torque, for flows produced by the motion of rigid particles.

In Section 2.3, we discuss the *Lorentz reciprocal theorem*, a reciprocal relation between any two solutions of the Stokes equations. This theorem is essentially an analog of the more familiar Green's identity of potential theory, and the analogy carries as far as to the derivation of the integral representation for the velocity field. There are a host of other applications, but these will have to wait until subsequent chapters.

In Section 2.4, we present a derivation of the integral representation for Stokes flow. As in potential theory, these express the velocity fields as a weighted integral of the Green's function or fundamental solution. Instead of a formal derivation of the fundamental solution (which is possible with the use of Fourier transforms), we take the more elementary approach of guessing the form of the Green's function, and then verifying that it works. The integral representation then follows as a special case of the reciprocal theorem, *viz.*, the reciprocal relation between the Green's function and the velocity field of interest. The integral representation is used later on (especially in Part IV) as the basis for efficient solution strategies, especially in situations in which only particle motions are desired and the flow field in the background fluid is unimportant. Integral representations transform the governing three-dimensional partial differential equations into two-dimensional integral equations on the boundary of the fluid domain, and thus are particularly well suited for very large numerical computations. A special form of the integral representation for flows past particles in an unbounded domain will be derived for immediate use in the subsequent section on the multipole expansion.

The transition from the general theorems of this chapter to the specific theme of Chapter 3, flow past a single particle, occurs in the last section of Chapter 2, where we discuss the *multipole expansion* for the disturbance velocity field of a particle in an ambient flow field. Far away from the particle, the disturbance produced by the particle motion takes on a universal form (the multipole expansion), and as in electrostatics, follows from a Taylor series expansion of the Green's function in the integral representation, about a convenient reference point inside the particle. In place of moments of the surface charge distribution (net charge, dipole moment, quadrupole moment, *etc.*), we encounter moments of the surface traction (hydrodynamic drag on the particle, Stokes dipoles, Stokes quadrupoles, *etc.*).

2.2 Energy Dissipation Theorems

For a Newtonian fluid, the rate of energy dissipation due to viscosity, in a fluid region V is given by the expression [6]

$$\int_V \Phi_v \, dV = \int_V 2\mu \boldsymbol{e} : \nabla \boldsymbol{v} \, dV ,$$

where Φ_v is the rate of viscous energy dissipation per unit volume. For an incompressible Newtonian fluid, we may use the following identities:

$$\int_V 2\mu e_{ij} : \frac{\partial v_j}{\partial x_i} \, dV = \int_V 2\mu e_{ij} e_{ij} \, dV = \int_V \sigma_{ij} e_{ij} \, dV .$$

An analysis of energy dissipation yields two important theorems of Helmholtz [17] concerning the uniqueness of the Stokes solution and a minimum energy or variational principle.

2.2.1 Uniqueness

To prove uniqueness, assume that $\boldsymbol{v}$ and $\boldsymbol{v}'$ are two solutions of the Stokes equations that satisfy the same boundary conditions, so that $\boldsymbol{v} = \boldsymbol{v}'$ on S (the boundary of V). The rate of energy dissipated by the difference field, $\boldsymbol{v}' - \boldsymbol{v}$, may be manipulated as

$$\begin{aligned} 2\mu \int_V (e'_{ij} - e_{ij})(e'_{ij} - e_{ij}) \, dV &= 2\mu \int_V \left(\frac{\partial v'_i}{\partial x_j} - \frac{\partial v_i}{\partial x_j} \right) (e'_{ij} - e_{ij}) \, dV \\ &= 2\mu \oint_S (v'_i - v_i)(e'_{ij} - e_{ij}) n_j \, dS \\ &\quad - \mu \int_V (v'_i - v_i)(\nabla^2 v'_i - \nabla^2 v_i) \, dV \\ &= -\int_V (v'_i - v_i) \left(\frac{\partial p'}{\partial x_i} - \frac{\partial p}{\partial x_i} \right) dV \end{aligned}$$

$$
\begin{aligned}
&= -\oint_S (v_i' - v_i)(p' - p) n_i \, dS \\
&= 0 .
\end{aligned}
$$

The preceding steps used the symmetry of $\boldsymbol{e}$ and $\boldsymbol{e}'$, integration by parts, the boundary conditions on S, and the divergence theorem (with the incompressibility condition $\nabla \cdot \boldsymbol{v}' = \nabla \cdot \boldsymbol{v} = 0$). Since the volume integral on the left-hand side is non-negative, the preceding equation can be satisfied if and only if $\boldsymbol{e}' = \boldsymbol{e}$ throughout V. Therefore, $\boldsymbol{v}'$ and $\boldsymbol{v}$ differ by at most a rigid-body motion (see Exercise 2.1). However, the boundary condition $\boldsymbol{v}' = \boldsymbol{v}$ precludes all nonzero rigid-body motions, so that $\boldsymbol{v}' = \boldsymbol{v}$ throughout V.

2.2.2 Minimum Energy Dissipation

Assume that $\boldsymbol{v}$ is a solution of the Stokes equations, and consider another field $\boldsymbol{v}^*$ that is solenoidal, *i.e.*, $\nabla \cdot \boldsymbol{v}^* = 0$. As before, we require $\boldsymbol{v} = \boldsymbol{v}^*$ on S. According to the *minimum energy dissipation theorem*, the Stokes solution dissipates less energy than any other solenoidal field with the same boundary velocities. We may prove this by starting with the relation

$$\int_V (e_{ij}^* - e_{ij}) e_{ij} \, dV = 0 , \tag{2.1}$$

which may be established using the same steps as in the uniqueness derivation. The volume integral on the LHS can be interpreted as an inner product, in the space of second order tensor fields, so that we have just shown an orthogonality relation. Now an application of the Pythagorean theorem immediately gives the minimum principle. (The length of the hypotenuse is always greater than or equal to the length of the base, as shown in Figure 2.1). Indeed, the following derivation is just a proof of the Pythagorean theorem in the notation relevant to the problem at hand. We write the rate of energy dissipated by $\boldsymbol{v}^*$ as

$$
\begin{aligned}
2\mu \int_V e_{ij}^* e_{ij}^* \, dV &= 2\mu \int_V \{e_{ij}^* e_{ij}^* - (e_{ij}^* - e_{ij}) e_{ij}\} \, dV \\
&= 2\mu \int_V \{e_{ij} e_{ij} + (e_{ij}^* - e_{ij}) e_{ij}^*\} \, dV \\
&= 2\mu \int_V \{e_{ij} e_{ij} + (e_{ij}^* - e_{ij}) e_{ij}^* - (e_{ij}^* - e_{ij}) e_{ij}\} \, dV \\
&= 2\mu \int_V \{e_{ij} e_{ij} + (e_{ij}^* - e_{ij})(e_{ij}^* - e_{ij})\} \, dV ,
\end{aligned}
$$

which establishes that the dissipation rate is equal to that of $\boldsymbol{v}$ plus an integral of a non-negative integrand. This extra contribution is zero if and only if $\boldsymbol{e}^* = \boldsymbol{e}$, thus proving that the minimum dissipation occurs for $\boldsymbol{v}^* = \boldsymbol{v}$ by the same argument that ended the preceding section.

2.2.3 Lower Bounds on Energy Dissipation

By reversing the role of the velocity and stress fields, we may derive lower bounds for the rate of energy dissipation of Stokes velocity fields, as shown by

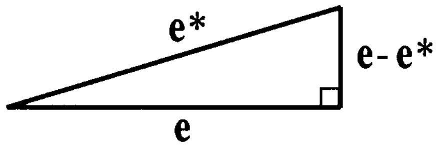

Figure 2.1: Minimum energy dissipation illustrated as a Pythagorean theorem.

Hill and Power [18]. Let $\boldsymbol{\sigma}'$ be an equilibrium stress field (*i.e.*, $\nabla \cdot \boldsymbol{\sigma}' = \mathbf{0}$). The field $\boldsymbol{e}' = (\boldsymbol{\sigma}' + p'\boldsymbol{\delta})/2\mu$ need not be associated with a velocity field.[1] We may obtain a lower bound for the energy dissipated by a solution of the Stokes equation in the same fluid domain, by considering a sequence of inequalities starting with $(\boldsymbol{e}' - \boldsymbol{e}) : (\boldsymbol{e}' - \boldsymbol{e}) \geq 0$:

$$\begin{aligned}
e'_{ij}e'_{ij} - e_{ij}e_{ij} &\geq 2e_{ij}(e'_{ij} - e_{ij}) \\
\sigma'_{ij}e'_{ij} - \sigma_{ij}e_{ij} &\geq 2e_{ij}(\sigma'_{ij} - \sigma_{ij}) \qquad (2.2) \\
\int_V (\sigma'_{ij}e'_{ij} - \sigma_{ij}e_{ij})\, dV &\geq 2\int_V \frac{\partial v_i}{\partial x_j}(\sigma'_{ij} - \sigma_{ij})\, dV \\
E' - E &\geq 2\oint_S v_i \sigma'_{ij} n_j \, dS - 2E \\
E &\geq 2\oint_S v_i \sigma'_{ij} n_j \, dS - E' \, .
\end{aligned}$$

In the first part of this sequence of inequalities, we used the condition on the traces of $\boldsymbol{e}$ and $\boldsymbol{e}'$, *e.g.*, ${e'}_{ii} = \delta_{ij}{e'}_{ij} = 0$. In the case of $\boldsymbol{v}'$ equaling the Stokes solution $\boldsymbol{v}$, the last integral is exactly $2E' = 2E$ and the equality is obtained.

2.2.4 Energy Dissipation in Particulate Flows

For flow past suspended particles, the previous results on energy dissipation further simplify into bounds on the hydrodynamic drag and torque, as discussed by Hill and Power [18]. The rate of energy dissipation is identical to the rate at which work is done on the particle surface, as can be seen by the sequence of equations,[2]

$$\int_V \sigma_{ij} e_{ij}\, dV = \int_V \sigma_{ij} \frac{\partial v_i}{\partial x_j}\, dV$$

[1]Here p' is determined by requiring $\boldsymbol{e}'$ to be traceless.

[2]Reading this equation from right to left motivates the definition given earlier for the rate of energy dissipation in the fluid element, for Stokes flows.

$$= \oint_S v_i \sigma_{ij} n_j \, dS \; .$$

Here the surface normal $\boldsymbol{n}$ points out of the fluid domain and into the particle, so that $\boldsymbol{F}$ and $\boldsymbol{T}$ below are the force and torque imparted by the particle to the fluid. On the surface of a rigid particle moving with the rigid-body motion, $\boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{x}$, where $\boldsymbol{U}$ is a constant translational velocity and $\boldsymbol{\omega}$ is a constant angular velocity, we get the equality

$$\begin{aligned} \oint_S v_i \sigma_{ij} n_j \, dS &= \oint_S \{U_i + (\boldsymbol{\omega} \times \boldsymbol{x})_i\} \, \sigma_{ij} n_j \, dS \\ &= U_i F_i + \omega_i T_i \; , \end{aligned}$$

where

$$\boldsymbol{F} = \oint \boldsymbol{\sigma} \cdot \boldsymbol{n} \, dS \qquad \text{and} \qquad \boldsymbol{T} = \oint \boldsymbol{x} \times (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \, dS \; .$$

From this result, the inequalities concerning energy dissipation rates may be expressed as

$$2(\boldsymbol{F}' \cdot \boldsymbol{U} + \boldsymbol{T}' \cdot \boldsymbol{\omega}) - E' \leq E = \boldsymbol{F} \cdot \boldsymbol{U} + \boldsymbol{T} \cdot \boldsymbol{\omega} \leq E^* \; . \tag{2.3}$$

From these inequalities, Hill and Power deduce the following statements, which we shall collectively call *inclusion monotonicity*:

1. If a body with surface S_2 is completely contained within a body with surface S_1, and if F_1 and F_2 are the drag on S_1 and S_2, respectively, then $F_1 \geq F_2$, the translations and external boundary being the same in the two problems.

2. The drag F_1 on a particle translating inside a rigid stationary container C_1 is greater than or equal to the drag on the same particle translating in a container C_2 if C_2 completely encloses C_1.

3. The drag on a body S is increased by the presence of other bodies that are either held fixed or free to move.

The proofs for the three are quite similar. For statement 1, we consider the actual Stokes solution $\boldsymbol{v}_2$ generated by translation of S_2. For $\boldsymbol{v}_2^*$, we use $\boldsymbol{v}_1$, the solution for translation of S_1, but augmented by the uniform translational velocity $\boldsymbol{U}$ in the region between the bodies, $S_1 - S_2$. Then the minimum dissipation theorem implies the inequality

$$E_2 = \boldsymbol{F}_2 \cdot \boldsymbol{U} \leq E_2^* \; .$$

However the rate-of-strain in $S_1 - S_2$ is zero, so $E_2^* = E_1$ and

$$E_2 = \boldsymbol{F}_2 \cdot \boldsymbol{U} \leq E_1 = \boldsymbol{F}_1 \cdot \boldsymbol{U} \; .$$

The work of Hill and Power has been extended in a number of directions. Minimum and maximum principles with more general boundary conditions, such as traction boundary conditions and normal flows through surfaces, may be found in the work of Keller, Rubenfeld, and Molyneux [22]. Nir, Weinberger, and Acrivos, [33] consider particles in a shear flow. We will consider this again in Chapter 3, after the discussion of *resistance tensors* for particles of arbitrary shape. Finlayson [14] considers the full Navier–Stokes equations, and shows that no variational principles exist unless either $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = \mathbf{0}$ or $\boldsymbol{v} \cdot \nabla \boldsymbol{v} = \mathbf{0}$. Variational theorems for non-Newtonian fluids are discussed in Johnson [20]. The inclusion monotonicity has also been established for sedimentation problems in Stokes flow by Weinberger [45]. We consider this case now in the framework of so-called mobility problems.

2.2.5 Energy Dissipation in Mobility Problems

In the preceding discussion we assumed boundary velocities and considered the resulting forces and torques in the usual mathematical format of boundary value problems, but actually, this is the inverse of most physical settings. Usually, the force and torque acting on the particle are given and the particle motion is to be determined; sedimentation of a particle of known mass illustrates this point. We shall call these physical problems *mobility* problems.

We start with inequality 2.2 encountered during the derivation of lower bounds to find

$$\begin{aligned}\int_V (\sigma'_{ij}e'_{ij} - \sigma_{ij}e_{ij})\, dV &\geq 2\int_V e_{ij}(\sigma'_{ij} - \sigma_{ij})\, dV \\ &= 2\oint_S v_i(\sigma'_{ij} - \sigma_{ij})n_j\, dS\end{aligned}$$

for the case $\nabla \cdot \boldsymbol{\sigma}' = \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}$. Let V be the volume of fluid between a particle and a container, the particle in rigid-body motion, and $\boldsymbol{v} = 0$ on the container surface. Then the preceding inequality becomes

$$E'(V) - E(V) \geq 2\left\{(\boldsymbol{F}' - \boldsymbol{F}) \cdot \boldsymbol{U} + (\boldsymbol{T}' - \boldsymbol{T}) \cdot \boldsymbol{\omega}\right\} .$$

Setting $\boldsymbol{F}' = \boldsymbol{F}$ and $\boldsymbol{T}' = \boldsymbol{T}$, we get

$$E(V) \leq E'(V) \leq E'(V') \qquad \text{for } V \subseteq V' .$$

In words, *on keeping the total force and torque affecting each particle constant, expanding the particles or adding no-slip boundaries (like a container) will reduce the dissipation by at least the amount that occurred in the excluded fluid volume.*

With this result, we can, for example, handle the question of whether a screw-like groove in a particle slows its sedimentation velocity due to its rotation. It does not. By filling the grooves with neutrally buoyant filler, we get a smooth particle *of the same weight or buoyancy* ($\boldsymbol{F} = \boldsymbol{F}'$) and a smaller fluid volume ($V \subseteq V'$). Then the energy dissipation argument becomes

$$\boldsymbol{F} \cdot \boldsymbol{U}' = E'(V') \geq E(V) = \boldsymbol{F} \cdot \boldsymbol{U} ;$$

so the settling rate (projection of the velocity in the direction of gravity) is greater for the grooved particle.

2.3 Lorentz Reciprocal Theorem

Consider a closed region of fluid V bounded by a surface S. Suppose that the velocity fields $\boldsymbol{v}$ and $\boldsymbol{v}'$ both satisfy the Stokes equations. We denote their respective stress fields by $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$. The *Lorentz reciprocal theorem* [29] states that

$$\oint_S \boldsymbol{v} \cdot (\boldsymbol{\sigma}' \cdot \boldsymbol{n})\, dS = \oint_S \boldsymbol{v}' \cdot (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, dS \; . \tag{2.4}$$

We choose to prove the relation, $\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}') = \nabla \cdot (\boldsymbol{v}' \cdot \boldsymbol{\sigma})$, since the theorem statement follows directly by application of the divergence theorem. Consider the reduction for $\boldsymbol{\sigma}' : \boldsymbol{e}$,

$$\begin{aligned} \sigma'_{ij} e_{ij} &= (-p'\delta_{ij} + 2\mu e'_{ij}) e_{ij} \\ &= 2\mu e'_{ij} e_{ij} \; . \end{aligned}$$

We have used the result $p'\delta_{ij}e_{ij} = p'e_{ii} = p'\,\nabla \cdot \boldsymbol{v} = 0$. Reversing the role of the primed and unprimed variables, we obtain also $\boldsymbol{\sigma} : \boldsymbol{e}' = 2\mu \boldsymbol{e} : \boldsymbol{e}'$, so that

$$\sigma'_{ij} e_{ij} = \sigma_{ij} e'_{ij} \; . \tag{2.5}$$

But consider the sequence of steps:

$$\begin{aligned} \sigma'_{ij} e_{ij} &= \frac{1}{2}\sigma'_{ij}\frac{\partial v_i}{\partial x_j} + \frac{1}{2}\sigma'_{ij}\frac{\partial v_j}{\partial x_i} \\ &= \sigma'_{ij}\frac{\partial v_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j}(\sigma'_{ij} v_i) - \left(\frac{\partial \sigma'_{ij}}{\partial x_j}\right) v_i \\ &= \frac{\partial}{\partial x_j}(\sigma'_{ij} v_i) \; . \end{aligned}$$

In the last step, we used the governing equation, $\nabla \cdot \boldsymbol{\sigma}' = \boldsymbol{0}$; for the penultimate step, we require $\sigma'_{ij} = \sigma'_{ji}$. If the same steps are applied to $\sigma_{ij} e'_{ij}$, it must reduce in an analogous fashion to $(\partial/\partial x_j)(\sigma_{ij} v'_i)$. So from Equation 2.5, we obtain the desired result, that for all points in V,

$$\nabla \cdot (\boldsymbol{v} \cdot \boldsymbol{\sigma}') = \nabla \cdot (\boldsymbol{v}' \cdot \boldsymbol{\sigma}) \; ,$$

thus completing the proof.

Note that if the assumptions $\nabla \cdot \boldsymbol{\sigma}_1 = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{\sigma}_2 = \boldsymbol{0}$ are relaxed, then the corresponding result is

$$\begin{aligned} &\oint_S \boldsymbol{v}_1 \cdot (\boldsymbol{\sigma}_2 \cdot \boldsymbol{n})\, dS - \int_V \boldsymbol{v}_1 \cdot (\nabla \cdot \boldsymbol{\sigma}_2)\, dV \\ &\quad = \oint_S \boldsymbol{v}_2 \cdot (\boldsymbol{\sigma}_1 \cdot \boldsymbol{n})\, dS - \int_V \boldsymbol{v}_2 \cdot (\nabla \cdot \boldsymbol{\sigma}_1)\, dV \; , \end{aligned} \tag{2.6}$$

which shall also prove to be quite useful.

In Exercise 2.2, we consider how the reciprocal theorem may be used to show that Stokes flow "transmits" unchanged the total force and torque from an inner closed surface to an outer enclosing surface.

2.4 Integral Representations

The integral representation for the velocity field can be viewed as a restatement of the governing equations from a three-dimensional partial differential equation to a two-dimensional integral equation for unknown densities over the boundary of the fluid domain. This reduction in dimensionality will be significant in the numerical solutions approach to the problems of microhydrodynamics. As a preliminary step, we derive the Green's function for Stokes flow.

2.4.1 The Green's Function for Stokes Flow

We seek the solution to the following problem:

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \mu \nabla^2 \boldsymbol{v} = -\boldsymbol{F}\delta(\boldsymbol{x}) \ , \qquad \nabla \cdot \boldsymbol{v} = 0 \ . \tag{2.7}$$

For a more detailed exposition on the Dirac delta function, $\delta(\boldsymbol{x})$, and the theory of distributions, we refer the reader to Lighthill [28]. For our present purposes, the meaning of the first equation is

1. For $\boldsymbol{x} \neq \boldsymbol{0}$, $\nabla \cdot \boldsymbol{\sigma} = 0$.

2. For any volume V that encloses the point $\boldsymbol{x} = \boldsymbol{0}$, $\int_V \nabla \cdot \boldsymbol{\sigma}\, dV = -\boldsymbol{F}$.

It is possible to derive the solution formally by the Fourier transform (see Exercise 2.9), or by using the solution for a translating sphere (Exercise 2.8). Here, we take the easier path of picking the solution out of thin air and proving that it works. We claim that the fundamental solution consists of the velocity and pressure pair,

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{F} \cdot \frac{\mathcal{G}(\boldsymbol{x})}{8\pi\mu} \ , \qquad p(\boldsymbol{x}) = \boldsymbol{F} \cdot \frac{\mathcal{P}(\boldsymbol{x})}{8\pi\mu} \ , \tag{2.8}$$

where $\mathcal{G}(\boldsymbol{x})/8\pi\mu$ is a Green's dyadic. Many authors remove the factor of $8\pi\mu$ from the Green's dyadic (as we have done), so that the *Oseen tensor*, $\mathcal{G}(\boldsymbol{x})$, given by

$$\mathcal{G}_{ij}(\boldsymbol{x}) = \frac{1}{r}\delta_{ij} + \frac{1}{r^3}x_i x_j \ , \tag{2.9}$$

is purely a geometric quantity, *i.e.*, independent of fluid properties. In the literature we also encounter the terminology *Oseen-Burgers tensor* for the combination $\mathcal{G}(\boldsymbol{x})/8\pi\mu$ [5]. The pressure field of the Oseen tensor we denote by $\mathcal{P}(\boldsymbol{x})$, and it is given by

$$\mathcal{P}_j(\boldsymbol{x}) = 2\mu \frac{x_j}{r^3} + \mathcal{P}_j^\infty \ . \tag{2.10}$$

The proof consists of two steps. We first show that for $\boldsymbol{x} \neq \mathbf{0}$, Equations 2.9 and 2.10 satisfy the Stokes equations, and then show that if the fluid domain V encloses the origin, the divergence of the solution stress field behaves as a delta function.

By direct differentiation, noticing that $\partial r/\partial x_k = x_k/r$, we have for $\boldsymbol{x} \neq 0$,

$$\begin{aligned} \mathcal{G}_{ij,k} &= -\frac{1}{r^3}\delta_{ij}x_k + \frac{1}{r^3}(\delta_{ik}x_j + \delta_{jk}x_i) - \frac{3}{r^5}x_i x_j x_k \ , && (2.11) \\ \nabla^2 \mathcal{G}_{ij} &= \mathcal{G}_{ij,kk} = \frac{2}{r^3}\delta_{ij} - \frac{6}{r^5}x_i x_j \ . && (2.12) \end{aligned}$$

Notice that from Equation 2.11 we may also verify that the equation of continuity is satisfied. Simply replace the index k with i, so that

$$8\pi\mu\nabla\cdot\boldsymbol{v} = F_j\mathcal{G}_{ij,i} = F_j\left[-\frac{1}{r^3}\delta_{ij}x_i + \frac{1}{r^3}(\delta_{ii}x_j + \delta_{ji}x_i) - \frac{3}{r^5}x_i x_j x_i\right] = 0 \ ,$$

since $\delta_{ij}x_i = x_j$, $\delta_{ii} = 3$ and $x_i x_i = r^2$. This shows that $\boldsymbol{v}$ satisfies continuity for $\boldsymbol{x} \neq \mathbf{0}$. The integral of $\mathcal{G}(\boldsymbol{x})\cdot\boldsymbol{n}$ over a small sphere of radius ϵ about the origin is

$$\oint_S \left(\frac{1}{\epsilon}\delta_{ij} + \frac{1}{\epsilon}n_i n_j\right) n_j \, dS \ ,$$

which, taking into account the smallness of dS, behaves as $O(\epsilon)$ for small ϵ. Thus the origin does not contain a source or sink and continuity is satisfied everywhere.

For $\boldsymbol{x} \neq \mathbf{0}$, the Laplacian term may be written as

$$\nabla^2\mathcal{G}_{ij} = \frac{2}{r^3}\delta_{ij} - \frac{6}{r^5}x_i x_j \ , = -2\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) \ ,$$

so the Oseen tensor has the pressure field

$$\mathcal{P}_j = -2\mu\frac{\partial}{\partial x_j}\left(\frac{1}{r}\right) + \mathcal{P}_j^\infty = 2\mu\frac{x_j}{r^3} + \mathcal{P}_j^\infty$$

and

$$p = \frac{\boldsymbol{F}\cdot\boldsymbol{x}}{4\pi r^3} + p^\infty \ .$$

This completes the demonstration that for $\boldsymbol{x} \neq \mathbf{0}$, Equation 2.8 satisfies the Stokes equations.

Now consider the stress field, denoted by $\boldsymbol{F}\cdot\boldsymbol{\Sigma}$, of the Green's function. The stress field of the Oseen tensor is then the triadic $8\pi\mu\boldsymbol{\Sigma}$, and its expression follows directly from the above expression for $\mathcal{P}$ and $\nabla\mathcal{G}$ as

$$8\pi\mu\Sigma_{ijk} = -\mathcal{P}_j\delta_{ik} + \mu(\mathcal{G}_{ij,k} + \mathcal{G}_{kj,i}) = -6\mu\frac{x_i x_j x_k}{r^5} \ .$$

Consider the volume integral

$$\int_V \nabla\cdot\boldsymbol{\Sigma}\, dV$$

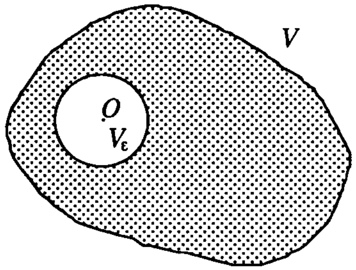

Figure 2.2: Breakdown of the fluid domain V into V_ϵ and $V - V_\epsilon$.

with V containing $\mathbf{0}$. Since $\nabla \cdot \boldsymbol{\Sigma} = \mathbf{0}$ everywhere except at the origin, the integral over V may be replaced with an integral over the volume of a sphere of radius ϵ, with ϵ arbitrarily small (see Figure 2.2). This volume integral can be converted to a surface integral using the divergence theorem and can be further reduced as follows:

$$F_j \oint_S \Sigma_{ijk} n_k \, dS(\boldsymbol{x}) = -\frac{3F_j}{4\pi} \oint_S \frac{n_i n_j n_k}{\epsilon^2} n_k \, dS = -F_i \ .$$

We have used $\boldsymbol{x} = \epsilon \boldsymbol{n}$ and the result (see Exercise 2.10) for the integral of $\boldsymbol{nn}$ over all orientations,

$$\int_0^{2\pi} \int_0^{\pi} n_i n_j \, \sin\theta \, d\theta \, d\phi = \frac{4\pi}{3} \delta_{ij} \ .$$

This completes the proof that $\boldsymbol{F} \cdot \mathcal{G}(\boldsymbol{x})/8\pi\mu$ is the Green's function for the Stokes equations.

Several comments concerning the Green's function are in order at this point. The term *Stokeslet* is often used for $\boldsymbol{F} \cdot \mathcal{G}(\boldsymbol{x})/8\pi\mu$. At other times, it is also called the *point-force solution*, the motivation for the terminology following naturally from the solution property, since the velocity field of the Green's function is force-free everywhere except at the origin, where it contains a force of strength $\boldsymbol{F}$. The analogy with the point charge solution in electrostatics is quite apparent. Finally, we may discard $\boldsymbol{F}$ from the problem statement, place the singular point at $\boldsymbol{\xi}$, and write the more general statements:

$$8\pi\mu \frac{\partial}{\partial x_k} \Sigma_{ijk}(\boldsymbol{x} - \boldsymbol{\xi}) = -\frac{\partial}{\partial x_i} \mathcal{P}_j(\boldsymbol{x} - \boldsymbol{\xi}) + \mu \nabla^2 \mathcal{G}_{ij} = -8\pi\mu\delta_{ij}\delta(\boldsymbol{x} - \boldsymbol{\xi}) \qquad (2.13)$$

$$\frac{\partial}{\partial x_i} \mathcal{G}_{ij}(\boldsymbol{x} - \boldsymbol{\xi}) = 0 \ . \qquad (2.14)$$

2.4.2 Integral Representation with Single and Double Layer Potentials

We now derive the integral representation for the velocity field. Consider the reciprocal theorem, Equation 2.6, with $\boldsymbol{v}_1$ replaced by $\mathcal{G}(\boldsymbol{\xi}-\boldsymbol{x})$ and $\boldsymbol{v}_2$ replaced by $\boldsymbol{v}$, the solution of the Stokes equations throughout a region V for which the integral representation is desired. Here, we fix $\boldsymbol{x}$ and use $\boldsymbol{\xi}$ as the integration variable in the reciprocal theorem. It is easy to show that the reciprocal theorem holds just as well for the dyadic component $\mathcal{G}_{i\ell}(\boldsymbol{\xi}-\boldsymbol{x})$, in place of the usual velocity component v_i. In fact, the index i undergoes the same set of steps, while the index ℓ is just carried along.

The reason for picking $\mathcal{G}(\boldsymbol{\xi}-\boldsymbol{x})$ as one of the two fields of the reciprocal relation becomes apparent when we recall that the divergence of its associated stress field is quite special: $\nabla_\xi \cdot \boldsymbol{\sigma}(\mathcal{G}) = -8\pi\mu\boldsymbol{\delta}\delta(\boldsymbol{\xi}-\boldsymbol{x})$. Equation 2.6 becomes

$$\begin{aligned} \oint_S \mathcal{G}_{i\ell}(\boldsymbol{\xi}-\boldsymbol{x})\,\sigma_{ik}(\boldsymbol{\xi})n_k\,dS(\boldsymbol{\xi}) &= 8\pi\mu\oint_S v_i(\boldsymbol{\xi})\,\boldsymbol{\Sigma}_{i\ell k}(\boldsymbol{\xi}-\boldsymbol{x})n_k\,dS(\boldsymbol{\xi}) \\ &+ 8\pi\mu\int_V v_i(\boldsymbol{\xi})\,\delta_{i\ell}\,\delta(\boldsymbol{\xi}-\boldsymbol{x})\,dV(\boldsymbol{\xi})\ , \end{aligned}$$

with the surface normal $\boldsymbol{n}$ *pointing out of the fluid region.* We may change the order of $\boldsymbol{\xi}$ and $\boldsymbol{x}$ in the arguments of $\mathcal{G}$ and $\boldsymbol{\Sigma}$ by recalling that $\mathcal{G}(\boldsymbol{\xi}-\boldsymbol{x}) = \mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})$ and $\boldsymbol{\Sigma}(\boldsymbol{\xi}-\boldsymbol{x}) = -\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi})$. The Kronecker delta leads to $v_i\delta_{i\ell} = v_\ell$ and finally, because of the properties of the Dirac delta function $\delta(\boldsymbol{\xi}-\boldsymbol{x})$, the volume integral simplifies to $\boldsymbol{v}(\boldsymbol{x})$ or zero, depending on whether $\boldsymbol{x}$ is inside or outside V. We simplify as just described and rearrange the preceding equation to obtain the following integral representation[3] for $\boldsymbol{v}$ (see the exercises for a corresponding representation of the associated pressure field):

$$\left.\begin{array}{ll} \text{if } \boldsymbol{x}\in V^o, & \boldsymbol{v}(\boldsymbol{x}) \\ \text{if } \boldsymbol{x}\notin \bar{V}, & \boldsymbol{0} \end{array}\right\} = -\oint_S [\boldsymbol{\sigma}(\boldsymbol{\xi})\cdot\hat{\boldsymbol{n}}(\boldsymbol{\xi})]\cdot\frac{\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})}{8\pi\mu}\,dS(\boldsymbol{\xi}) - \oint_S \boldsymbol{v}(\boldsymbol{\xi})\cdot\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi})\cdot\hat{\boldsymbol{n}}(\boldsymbol{\xi})\,dS(\boldsymbol{\xi})\ . \qquad (2.15)$$

Here, the surface normal $\hat{\boldsymbol{n}}$ *points out of the particle and into the fluid region.*[4] *The integral representation is a significant statement — a Stokes velocity field may be reconstructed throughout a region V using only values of the velocity and traction fields on the boundary of V.*

The first term on the RHS is a velocity field generated by a distribution of surface forces of strength $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\,dS$, since $\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})$ is the velocity field generated by a point force at $\boldsymbol{\xi}$. By analogy with potential theory, this integral is called the *single layer potential* [24, 49] (the single layer of charges distributed over the surface of a electrical conductor is replaced by a single layer of forces).

[3] This integral representation says nothing (yet) about the boundary points $\boldsymbol{x}\in\bar{V}\setminus V = \partial V$, as V^o is the *interior* of V, explicitly excluding the boundary points, while $\bar{V}$ is the *closure* of V, explicitly including the boundary points.

[4] Throughout this book, we shall distinguish such surface normals from the generic brand by using a caret.

The second term is called the *double layer potential*, also by analogy with potential theory. In electrostatics, this term would correspond to a double layer of positive and negative charges separated by an infinitesimal gap, or equivalently, a surface distribution of electric dipoles. The structure of the hydrodynamic double layer is somewhat richer, and its physical interpretation is more involved. More explicitly, if we write

$$8\pi\mu v_j \Sigma_{jik}\hat{n}_k = \mathcal{P}_i \boldsymbol{v}\cdot\hat{\boldsymbol{n}} + \mu(\mathcal{G}_{ji,k} + \mathcal{G}_{ki,j})v_j\hat{n}_k \ ,$$

then the double layer distribution can be interpreted as a distribution of sources (or sinks) of strength $\boldsymbol{v}\cdot\hat{\boldsymbol{n}}$ and a true "double layer" of Stokeslets. This can also be deduced from the fact that $\boldsymbol{\Sigma}$ as a "velocity" field does not satisfy continuity, since $\nabla\cdot\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi}) = -\boldsymbol{\delta}\delta(\boldsymbol{x}-\boldsymbol{\xi})$, but $\nabla\mathcal{G}$ does, and so the $\mathcal{P}$-term must correspond to the sources and sinks. In fact, we see that

$$\oint_S \mathcal{P}_i\hat{n}_j\, dS = 2\mu\oint_S \frac{x_i\hat{n}_j}{r^3}\, dS = \frac{8}{3}\pi\mu\delta_{ij} \ .$$

The term $\mathcal{G}_{ji,k}v_j\hat{n}_k$ corresponds to opposing Stokeslets of dipole strength $\boldsymbol{v}$ displaced in the direction of $\hat{\boldsymbol{n}}$, while the term $\mathcal{G}_{ki,j}v_j\hat{n}_k$ corresponds to opposing Stokeslets of dipole strength $|\boldsymbol{v}|\hat{\boldsymbol{n}}$ displaced in the direction of $\boldsymbol{v}$. This symmetric placement of point forces implies that no net force or torque is exerted on the fluid. (We shall examine such structures in more detail in the next section.) Finally, despite the presence of sources and sinks over S, the total mass within S is conserved because the total source and sink strength is

$$\oint_S \boldsymbol{v}\cdot\hat{\boldsymbol{n}}\, dS = 0 \ .$$

The integral representation 2.15 gives the velocity of interest for a point $\boldsymbol{x}$ in the fluid region and zero outside. If we take the view that S is a virtual surface that encloses a subdomain V of a much larger region in which $\boldsymbol{v}$ is a meaningful Stokes velocity field, then the two surface integrals combine to give the correct result in the subdomain, but not outside. In fact, it fails catastrophically outside this subdomain; instead of the correct value for $\boldsymbol{v}$, we get zero. This jump discontinuity across the boundary must come from the double layer term, for we can show that the single layer potential is continuous. The kernel of the single layer potential, $\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})$ behaves as $r_{\xi x}^{-1} = |\boldsymbol{x}-\boldsymbol{\xi}|^{-1}$ in the neighborhood of the point $\boldsymbol{\xi}$. In local polar coordinates about $\boldsymbol{\xi}$, $dS(\boldsymbol{\xi}) = r_{\xi x}dr_{\xi x}d\phi$, so the integrand of the single layer is nonsingular, and the velocity generated by the single layer distribution is continuous across the surface; letting $\boldsymbol{x}$ approach the surface from both sides, we simply get the limiting value that is exactly the value of the surface integral with $\boldsymbol{x}$ on S.

The double layer kernel, on the other hand, scales as $r_{\xi x}^{-2}$, so the integral exists as an improper integral for $\boldsymbol{x} = \boldsymbol{\eta}$ on S only if the dot product $\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})/r_{\xi\eta}$ vanishes as $\boldsymbol{\xi}$ approaches $\boldsymbol{\eta}$ along the surface. For "smooth" surfaces this is indeed the case; more explicitly, we say the surface is Lyapunov-smooth if $\boldsymbol{n}(\boldsymbol{\xi})\cdot(\boldsymbol{x}-\boldsymbol{\xi})/r_{\xi x} = O(r_{\xi x}^{\alpha})$ as $\boldsymbol{\xi}\to\boldsymbol{x}$, with $\alpha\in(0,1]$. Now as $\boldsymbol{x}$ approaches

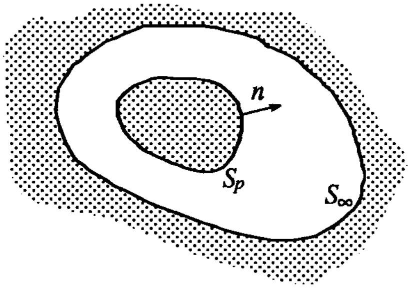

Figure 2.3: The region bounded by S_p and S_∞.

η from either side, the surface will locally look planar, and since the jump is due to the immediate neighborhood of η, it can be claimed that by symmetry that the jump is equal in magnitude whether x comes to sit on the surface from the inside or the outside. This behavior is completely analogous to the jump in the potential as one passes through an electric double layer. Physically, we expect a jump in the *normal* component of the velocity field as we pass through a surface distribution of sources and sinks, since the velocity fields will be directed in opposite directions on the two sides. A surface distribution of symmetric force dipoles also causes a jump, but only in the *tangential* component of the velocity field. Indeed, since $\nabla \mathcal{G}$ is solenoidal, the standard "pillbox" argument shows that the normal component of the velocity produced by a true bilayer of Stokeslets is continuous across the surface. In Exercises 2.13 and 2.14 we ask the reader to evaluate the jump explicitly for some simple double layer distributions, where the integrals can be evaluated directly, and in Exercise 2.15 we examine jumps in the normal and tangential velocity components of the double layer potential.

2.4.3 Representation of Flows Outside a Rigid Particle

The integral representation takes on a special form involving only the single layer potential, for disturbance flows created by a particle undergoing rigid-body motion $\boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{x}$ in an ambient flow field, $\boldsymbol{v}^\infty(\boldsymbol{x})$. The problem is meaningful only if $\boldsymbol{v}^\infty$ is a solution of the Stokes equation. For boundary conditions, we have

1. $\boldsymbol{v} = \boldsymbol{v}^{RBM} = \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{x}$ on the surface of the particle.

2. $\boldsymbol{v} = \boldsymbol{v}^\infty$ far away from the particle.

Define the disturbance field $\boldsymbol{v}^D = \boldsymbol{v} - \boldsymbol{v}^\infty$. Consider a fluid region, V_f, bounded by the particle surface and a large surface, S_∞, of dimension R, which we intend to expand by taking the limit of large R (see Figure 2.3). The integral representation for $\boldsymbol{v}^D$ is

$$\left.\begin{array}{ll}\text{if } \boldsymbol{x} \in V_f, & \boldsymbol{v}^D(\boldsymbol{x}) \\ \text{if } \boldsymbol{x} \in V_p, & \mathbf{0}\end{array}\right\} = -\frac{1}{8\pi\mu}\oint_{S_p}(\boldsymbol{\sigma}^D \cdot \hat{\boldsymbol{n}}) \cdot \mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\, dS(\boldsymbol{\xi})$$
$$- \oint_{S_p}(\boldsymbol{v}^{RBM} - \boldsymbol{v}^{\infty}) \cdot \boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi}) \cdot \hat{\boldsymbol{n}}\, dS(\boldsymbol{\xi})$$
$$- \frac{1}{8\pi\mu}\oint_{S_\infty}(\boldsymbol{\sigma}^D \cdot \hat{\boldsymbol{n}}) \cdot \mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\, dS(\boldsymbol{\xi})$$
$$- \oint_{S_\infty} \boldsymbol{v}^D \cdot \boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi}) \cdot \hat{\boldsymbol{n}}\, dS(\boldsymbol{\xi})\ ,$$

where V_p denotes the region inside the particle, which of course is disjoint with V_f and $\hat{\boldsymbol{n}}$ points into V_f as before. We have also made use of the boundary condition on S_p. The integral over S_∞ will vanish in the limit of large R if $\boldsymbol{v}^D \to 0$ and $R\boldsymbol{\sigma}^D \to 0$, since $\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi}) \sim O(R^{-1})$, $\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi}) \sim O(R^{-2})$ and $dS \sim O(R^2)$. Now $\boldsymbol{v}^D \to 0$ is satisfied because of the problem statement. The other requirement also holds, since for large R we expect $\boldsymbol{\sigma}^D$ to scale as $\nabla\boldsymbol{v}^D$ and $O(\nabla\boldsymbol{v}^D) \sim O(\boldsymbol{v}^D)/R$.

The double layer potential in the integral over S_p may be eliminated as follows. Apply the velocity representation theorem to the field $\boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{x} - \boldsymbol{v}^{\infty}$ *inside the particle*:

$$\left.\begin{array}{ll}\text{if } \boldsymbol{x} \in V, & \mathbf{0} \\ \text{if } \boldsymbol{x} \in V_p, & \boldsymbol{v}^{RBM} - \boldsymbol{v}^{\infty}(\boldsymbol{x})\end{array}\right\} = -\frac{1}{8\pi\mu}\oint_{S_p}[(\boldsymbol{\sigma}^{RBM} - \boldsymbol{\sigma}^{\infty}) \cdot \boldsymbol{n}] \cdot \mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\, dS(\boldsymbol{\xi})$$
$$- \oint_{S_p}(\boldsymbol{v}^{RBM} - \boldsymbol{v}^{\infty}) \cdot \boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi}) \cdot \boldsymbol{n}\, dS(\boldsymbol{\xi})\ ,$$

with the surface normal $\boldsymbol{n}$ *now pointing into* V_p*, the region of interest*, and $\boldsymbol{\sigma}^{\infty}$ denoting the stress field of the ambient velocity field. The surface traction for rigid-body motion $\boldsymbol{\sigma}^{RBM} \cdot \boldsymbol{n}$ is just a constant pressure, $-p_0\boldsymbol{n}$, of no dynamic significance and contributes nothing to the integral. In fact, note that the integral of $\boldsymbol{n} \cdot \mathcal{G}$ over the surface of the particle is identically zero, since $\mathcal{G}$ is solenoidal.

We add the representations for $\boldsymbol{v}^D$ and $\boldsymbol{v}^{RBM} - \boldsymbol{v}^{\infty}$, keeping in mind $\hat{\boldsymbol{n}} = -\boldsymbol{n}$. The double layer terms cancel, and since $\boldsymbol{\sigma}^{\infty} + \boldsymbol{\sigma}^D = \boldsymbol{\sigma}$, and $\boldsymbol{v}^D = \boldsymbol{v} - \boldsymbol{v}^{\infty}$, we obtain the result

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}^{\infty}(\boldsymbol{x}) - \frac{1}{8\pi\mu}\oint_{S_p}(\boldsymbol{\sigma}(\boldsymbol{\xi}) \cdot \boldsymbol{n}) \cdot \mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\ , \tag{2.16}$$

which shows that flows past a rigid particle can be represented with just single layer potentials. This is consistent with our intuition that the disturbance caused by the relative motion of a rigid body through a viscous fluid can be represented by a collection of point forces imparted to the fluid at the particle surface, and what initially appeared to be a collection of local sources, sinks, and force dipoles add up over the entire particle surface to simply rigid-body motions and a reconstruction of the ambient field. This simple result does not hold for the disturbance velocity field produced by the relative motion of a viscous drop. In the next section, we show how this special representation leads to the multipole expansion, a universal form for the disturbance flow field far away from the particle.

2.5 The Multipole Expansion

There is a one-to-one correspondence between the multipole expansion of hydrodynamics and the more familiar form from electrostatics. At great distances from the particle, *i.e.*, $|\boldsymbol{x}| \gg |\boldsymbol{\xi}|$, we cannot distinguish between the points $\boldsymbol{\xi}$ on the surface of the particle and a reference origin $\mathbf{0}$ located at a convenient point inside the particle, so that $\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi}) \sim \mathcal{G}(\boldsymbol{x})$ may be moved outside the integral in the single layer potential. The integral of what remains, $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$, is simply the hydrodynamic drag on the particle, so that the single layer potential simplifies to

$$\boldsymbol{v}^D(\boldsymbol{x}) \sim -\boldsymbol{F}\cdot\frac{\mathcal{G}(\boldsymbol{x})}{8\pi\mu} ,$$

independent of the details of the particle shape. This is analogous to the single layer potential of electrostatics reducing to the field of a point charge far away from the conductor. The geometry and the analogy between hydrodynamics and electrostatics is illustrated in Figure 2.4. We may obtain higher order corrections for the disturbance field in which the coefficients are moments of the surface traction $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ and thus obtain the multipole expansion. Formally, we set $|\boldsymbol{x}| \gg |\xi|$ and take the Taylor series[5] of $\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})$ in $\boldsymbol{\xi}$ about $\boldsymbol{\xi}=\mathbf{0}$,

$$\begin{aligned}
\mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{\xi}) &= \sum_{n=0}^{\infty}\frac{1}{n!}(\boldsymbol{\xi}_c\cdot\nabla_\xi)^n\mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{\xi})|_{\xi=0} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\boldsymbol{\xi}\cdot\nabla)^n\mathcal{G}_{ij}(\boldsymbol{x}) \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\xi_{k_1}\cdots\xi_{k_n}\mathcal{G}_{ij,k_1\ldots k_n}(\boldsymbol{x}) ,
\end{aligned}$$

and insert the expansion into the velocity representation. The result is

$$\begin{aligned}
v_i(\boldsymbol{x}) - v_i^\infty(\boldsymbol{x}) &= -\frac{1}{8\pi\mu}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\oint_{S_p}[(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})_j\xi_{k_1}\cdots\xi_{k_n}]\,dS\;\mathcal{G}_{ij,k_1\ldots k_n}(\boldsymbol{x}) \\
&= -\frac{F_j}{8\pi\mu}\mathcal{G}_{ij}(\boldsymbol{x}) + \frac{D_{jk}}{8\pi\mu}\mathcal{G}_{ij,k}(\boldsymbol{x}) + \cdots ,
\end{aligned}$$

with

$$\begin{aligned}
F_j &= \oint_{S_p}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})_j\,dS , \\
D_{jk} &= \oint_{S_p}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})_j\xi_k\,dS .
\end{aligned}$$

[5]The subscript c is used to indicate that the variable in question is considered a constant on carrying out differentiations, to make the first sum unambiguous.

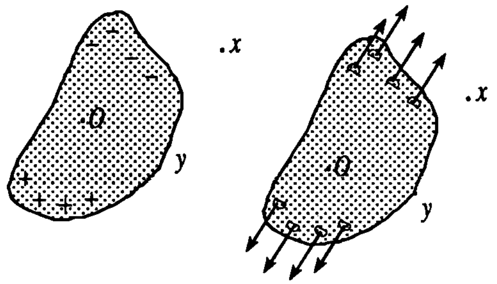

Figure 2.4: The multipole expansion for the electrostatic potential and the velocity field.

Thus in the far field, regardless of the details of the particle shape, all particle disturbance fields exhibit certain common features. The leading term, or monopole, will be a Stokeslet with coefficient $\boldsymbol{F}$ equal to the force exerted by the fluid on the particle. This field, decaying as $|\boldsymbol{x}|^{-1}$ away from the particle, will be present if and only if the particle and fluid exert a net force on each other. The next effect is a force dipole $\boldsymbol{D}$, a second-order tensor with a field that decays as $|\boldsymbol{x}|^{-2}$ away from the particle. Note that for Stokes flow the tensorial rank of the moment coefficients is one greater than the corresponding quantities in electrostatics, because the surface density $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ is a vector, whereas in electrostatics the surface charge density is a scalar.

The isotropic portion of $\boldsymbol{D}$, shown and subtracted in the equation below, is of no dynamic significance, since $\nabla\cdot\mathcal{G}=0$. We split the rest into a symmetric part called the *stresslet* and an antisymmetric part,

$$D_{jk}-\frac{1}{3}D_{ii}\delta_{jk}=S_{jk}+T_{jk}\ ,$$

with

$$\begin{aligned} S_{jk} &= \frac{1}{2}\oint_S [(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})_j\xi_k+(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})_k\xi_j]\,dS-\frac{1}{3}\oint_S(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\cdot\boldsymbol{\xi}\,dS\ \delta_{jk}\ . \\ T_{jk} &= \frac{1}{2}\oint_S [(\boldsymbol{\sigma}\cdot\boldsymbol{n})_j\xi_k-(\boldsymbol{\sigma}\cdot\boldsymbol{n})_k\xi_j]\,dS\ . \end{aligned}$$

The antisymmetric portion may be identified with the hydrodynamic torque on the particle. Explicitly, we define a relation[6] between the components of an antisymmetric tensor and a pseudovector, $\boldsymbol{T}$,

$$-\epsilon_{ijk}T_{jk}=-\epsilon_{ijk}\oint_S(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})_j\xi_k\,dS=T_i\ ,$$

[6]The alternating tensor $\boldsymbol{\epsilon}$ can be defined by the relationship $\boldsymbol{a}\times\boldsymbol{b}=\epsilon_{ijk}a_jb_k$ being valid for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, or by $\boldsymbol{\epsilon}=\boldsymbol{\delta}\times\boldsymbol{\delta}$.

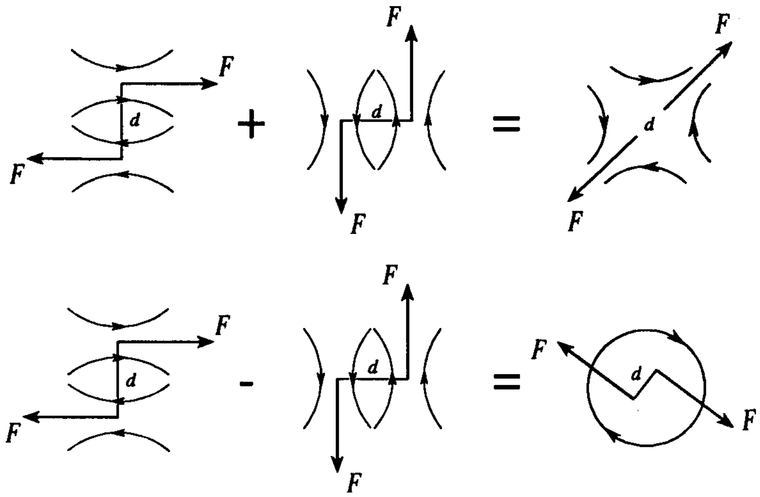

Figure 2.5: The stresslet and rotlet fields

which identifies $\boldsymbol{T}$ as the hydrodynamic torque exerted by the fluid on the particle. Equivalently, we may obtain the antisymmetric tensor components from the torque as

$$T_{jk} = -\frac{1}{2}\epsilon_{jk\ell}T_\ell \ .$$

The corresponding term in the multipole expansion may be written as

$$T_{jk}\mathcal{G}_{ij,k} = -\frac{1}{2}\epsilon_{jk\ell}T_\ell\mathcal{G}_{ij,k} = \frac{1}{2}(\boldsymbol{T}\times\nabla)_j\mathcal{G}_{ij} = \frac{1}{2}\boldsymbol{T}\cdot(\nabla\times\mathcal{G}) \ .$$

The antisymmetric dipole field can also be interpreted as the field of a point torque (also known as the *rotlet*) at the origin, since the torque on any surface enclosing the origin is precisely $\boldsymbol{T}$. This can also be seen by constructing a couple formed by two opposing Stokeslets of strength $\boldsymbol{F}$, separated by a displacement $\boldsymbol{d}$ with $\boldsymbol{d}\cdot\boldsymbol{F} = 0$ (see Figure 2.5). As we shrink the displacement, keeping $|\boldsymbol{d}||\boldsymbol{F}|$ constant, the antisymmetric portion approaches $\frac{1}{2}(F_jd_k - F_kd_j)\mathcal{G}_{ij,k}$.

The stresslet and symmetric dipole field are intimately connected with straining motions. Again referring to Figure 2.5, the field

$$\left[\frac{1}{2}(F_jd_k + F_kd_j) - \frac{1}{3}\boldsymbol{F}\cdot\boldsymbol{d}\,\delta_{jk}\right]\mathcal{G}_{ij,k}$$

produces a straining motion induced by Stokeslets oriented and displaced in the principal directions of the matrix $(F_jd_k + F_kd_j) - \frac{2}{3}\boldsymbol{F}\cdot\boldsymbol{d}\,\delta_{jk}$, as can be seen by a coordinate rotation, which transforms this matrix into diagonal form (in Figure 2.5, $\boldsymbol{F} \perp \boldsymbol{d}$ so the principal directions are at 45° between $\boldsymbol{F}$ and $\boldsymbol{d}$). We may draw the same conclusion starting with S_{jk}. (**Note:** Is the stresslet "transmitted" unchanged by Stokes flow? See Exercise 2.3).

A rigid particle in the ambient field $\boldsymbol{v}^{\infty}(\boldsymbol{x})$ may undergo a rigid body motion $\boldsymbol{U} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_0)$ that matches the ambient velocity and vorticity at $\boldsymbol{x}_0$, so that $\boldsymbol{F} = \boldsymbol{T} = \boldsymbol{0}$. Such particles produce a fairly weak disturbance of the ambient flow. However, rigid particles do not possess a mechanism for relieving the local straining motion in the fluid, hence in a general ambient field the stresslet cannot be nonzero. Thus in an arbitrary flow field, even a force-free and torque-free particle will produce a disturbance that decays as $|\boldsymbol{x}|^{-2}$ induced by the symmetric force dipole. This ultimately shows up as an increase in the rate of viscous dissipation of mechanical energy, and thus a suspension of rigid particles has an effective viscosity greater than that of the pure solvent. Indeed, the next example shows that the particle contribution to the effective stress in the suspension is connected intimately with the particle stresslets.

Example 2.1 Effective Stress in a Suspension of Rigid Particles [3, 9]

We start with the assumption that the macroscopic, observed stress tensor, *i.e.*, the effective stress in the suspension, is the ensemble average of the stress distribution in all realizations of the suspension. For homogeneous suspensions, this ensemble average is equivalent to a volume average over a volume that is large enough to contain a statistically significant number of particles, but is smaller than the scale of variations of interest in the macroscopic system. The formal expression for the effective stress, $\boldsymbol{\sigma}^{\text{eff}}$, can then be written as

$$\begin{aligned}
\sigma_{ij}^{\text{eff}} &= \frac{1}{V}\int_V \sigma_{ij}\, dV \\
&= \frac{1}{V}\int_{V-\Sigma V_n} (-p\delta_{ij} + 2\mu e_{ij})\, dV + \sum_n \int_{V_n} \sigma_{ij}\, dV \ ,
\end{aligned}$$

where the volume integral has been decomposed into the portion over the fluid domain in which we may use the constitutive equation for the fluid stress and the portions over each rigid particle. The stresses in the rigid particle are indeterminate, but we may use the following relation,

$$\begin{aligned}
\int_{V_n} \sigma_{ij}\, dV &= \int_{V_n} \frac{\partial}{\partial x_k}(\sigma_{ik}x_j)\, dV - \int_{V_n} \frac{\partial \sigma_{ik}}{\partial x_k} x_j\, dV \\
&= \oint_{S_n} \sigma_{ik}\hat{n}_k x_j\, dS - \int_{V_n} \frac{\partial \sigma_{ik}}{\partial x_k} x_j\, dV \ ,
\end{aligned}$$

to rewrite the stresses in terms of the surface tractions plus the first (volume) moment of $\nabla \cdot \boldsymbol{\sigma}$. If the particles are force-free, $\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{0}$ and the last term is dropped.

The integral of $\boldsymbol{e}$ over the fluid region may be rewritten as follows:

$$\int_{V-V_n} 2e_{ij}\, dV = \int_V 2e_{ij}\, dV - \sum_n \int_{V_n} 2e_{ij}\, dV$$

$$= \int_V 2e_{ij}\, dV - \sum_n \oint_{S_n} (v_i \hat{n}_j + v_j \hat{n}_i)\, dS \ ,$$

so that

$$\frac{1}{V}\int_{V-V_n} 2e_{ij}\, dV = 2 < e_{ij} > \; - \frac{1}{V}\sum_n \int_{S_n} (v_i \hat{n}_j + v_j \hat{n}_i)\, dS \ .$$

Combining these results together, we find that

$$\sigma_{ij}^{\text{eff}} = -p^{\text{eff}}\delta_{ij} + 2\mu < e_{ij} > + \sigma_{ij}^p \ ,$$

with the isotropic part of the stress lumped into an effective pressure and in which the particle contribution to the stress is given by

$$\sigma_{ij}^p = \frac{1}{V}\sum_n \oint_{S_n} [\sigma_{ik}\hat{n}_k x_j - \mu(v_i \hat{n}_j + v_j \hat{n}_i)]\, dS \ .$$

For rigid particles, the velocity at the particle surface is a rigid-body motion so the velocity terms in the integral vanish identically. The rigid particles' contributions to the stress are given by the force dipoles from each particle in the volume.

We may examine the antisymmetric and symmetric parts separately to shed more light on the nature of the particle contribution to the suspension stress. The antisymmetric part of σ^p is seen to be directly related to the torque on each particle; thus the suspension stress contains an antisymmetric portion if and only if couples are generated within each particle. Otherwise, the suspension stress is symmetric, with the extra "particle" contributions coming directly from the sum of the stresslets of each particle in the volume. After this discussion of the role of the stresslet in suspension rheology, it should come as no surprise that much of the theoretical investigations in suspension rheology centers around computations of $\boldsymbol{S}$ as a function of system configuration. ◊

Exercises

Exercise 2.1 Zero Rate-of-Strain Fields.
If the rate-of-strain tensor $\boldsymbol{e}$ is zero throughout a volume of fluid V, show that the velocity field is at most a rigid-body motion throughout V.
Hint: Starting with $v_{i,j} = -v_{j,i}$, show that $v_{i,jk} = -v_{j,ki}$, which then implies that $v_{i,jk} = 0$ throughout V, so that $\boldsymbol{v}$ is linear in $\boldsymbol{x}$. Since $\boldsymbol{e} = \boldsymbol{0}$, show that this linear field can only be a uniform translation plus a solid-body rotation.

Exercise 2.2 Transmission of Force and Torque.
Using the reciprocal theorem, show that Stokes flow "transmits" unchanged the total force and torque from an inner closed surface to an outer enclosing surface.
Hint: Consider the fluid region between the surfaces and choose for $\boldsymbol{v}_2$ a rigid-body motion.

Exercise 2.3 Transmission of the Stresslet.
Does Stokes flow "transmit" unchanged the stresslet on an inner closed surface to an outer enclosing surface?
Hint: Consider the fluid region between the surfaces and choose $\boldsymbol{v}_2$ equal to $\boldsymbol{e} \cdot \boldsymbol{x}$, with a constant rate-of-strain, $\boldsymbol{e}$.

Exercise 2.4 Solenoidal Fields with Uniform Translations.
In this and two subsequent exercises, we illustrate the use of the minimum and maximum energy dissipation principles to obtain bounds as in Hill and Power [18]. Our first task is to find solenoidal trial solutions that satisfy the boundary condition, $\boldsymbol{v} = \boldsymbol{U}$. Consider the following trial solution, consisting of a Stokeslet and a field that decays as r^{-n}:

$$\boldsymbol{v} = \boldsymbol{U}\left(C\frac{a}{r} + (1-C)\frac{a^n}{r^n}\right) + C\frac{\boldsymbol{U}\cdot\boldsymbol{x}\boldsymbol{x}}{a^2}\left(\frac{a^3}{r^3} - \frac{a^{n+2}}{r^{n+2}}\right) .$$

By inspection, we see that $\boldsymbol{v} = \boldsymbol{U}$ on the surface of a sphere of radius a. Since the Stokeslet is solenoidal, the trial solution will also be solenoidal if and only if the $O(r^{-n})$ field is solenoidal. Show that this requires $C = n/(2n-2)$. Only the case $n = 2$ is used in the example in Hill and Power.

Exercise 2.5 Upper Bounds.
Evaluate energy dissipation rates of the solenoidal fields, $\boldsymbol{v}^{(n)}$, of Exercise 2.4,

$$\boldsymbol{v}^{(n)} = \boldsymbol{U}\left(\frac{n}{2(n-1)}\frac{a}{r} + \frac{n-2}{2(n-1)}\frac{a^n}{r^n}\right) + \frac{\boldsymbol{U}\cdot\boldsymbol{x}\boldsymbol{x}}{a^2}\frac{n}{2(n-1)}\left(\frac{a^3}{r^3} - \frac{a^{n+2}}{r^{n+2}}\right) .$$

With the notation, $E^{(n)} = E(\boldsymbol{v}^{(n)})$, your answers for $n = 2$ and $n = 3$ should be $E^{(2)} = \frac{56}{9}\pi\mu a U^2$ and $E^{(3)} = 6\pi\mu a U^2$. Show that $E^{(3)}$ is the least upper bound from this family of trial solutions. We will see in Chapter 3 that the case $n = 3$ is in fact the exact solution for the translating sphere.

Exercise 2.6 Lower Bounds.
Show that the traction field

$$\frac{a}{\mu}\sigma'_{ij} = \alpha\frac{U_k x_k x_i x_j}{a^3}\frac{a^5}{r^5} + \beta\left(2\frac{U_k x_k x_i x_j}{a^3}\frac{a^6}{r^6} - \frac{U_i x_j + U_j x_i}{2a}\frac{a^4}{r^4}\right)$$

satisfies $\nabla \cdot \boldsymbol{\sigma}' = \boldsymbol{0}$, and thus the condition of the dissipation theorem. Here, α and β are constants that we will use later on to tighten the bound from below. Show that

$$\begin{aligned}
\boldsymbol{F}' &= \oint_S \boldsymbol{\sigma}\cdot\boldsymbol{n}\, dS = \frac{4}{3}\pi\mu a U\alpha \\
E' &= \frac{4}{9}\pi\mu a U^2\left(\alpha^2 + \alpha\beta + \frac{5}{6}\beta^2\right) \\
2\boldsymbol{F}'\cdot\boldsymbol{U}' - E' &= \frac{4}{9}\pi\mu a U^2\left(6\alpha - (\alpha^2 + \alpha\beta + \frac{5}{6}\beta^2)\right) \\
&= \frac{4}{9}\pi\mu a U^2\left(\frac{90}{7} - (\alpha + \frac{\beta}{2} - 3)^2 - \frac{7}{12}(\beta + \frac{18}{7})^2\right) .
\end{aligned}$$

The greatest lower bound $2\boldsymbol{F}' \cdot \boldsymbol{U}' - E' = (40/7)\pi\mu a U^2$ occurs for $\alpha = 30/7$, $\beta = -18/7$. In the original work of Hill and Power [18], the upper bound (Exercise 2.5) was set with $n = 2$ so that the bounds for the force on a translating sphere are given as

$$\frac{40}{7}\pi\mu a U \le |F| \le \frac{56}{9}\pi\mu a U \ .$$

In Chapter 3, we will derive the exact result, $F = 6\pi\mu a U$.

Exercise 2.7 Energy Dissipation Rates for the Translating Sphere.
In Chapter 3, we will show that the velocity field generated by a translating sphere is

$$\boldsymbol{u} = \boldsymbol{U}\left(\frac{3a}{4r} + \frac{a^3}{4r^3}\right) + \frac{\boldsymbol{U}\cdot\boldsymbol{x}\boldsymbol{x}}{r^2}\left(\frac{3a}{4r} - \frac{3a^3}{4r^3}\right) ,$$

where a is the sphere radius. Consider $\boldsymbol{v}^* = (1 - C)\boldsymbol{u} + C\boldsymbol{v}^{(2)}$, with $\boldsymbol{v}^{(2)}$ as defined in Exercise 2.5. Evaluate $E(\boldsymbol{v}^*)$ and show that it is given by

$$E^* = E(\boldsymbol{u}) + \{E(\boldsymbol{v}) - E(\boldsymbol{u})\}\, C^2 = 6\pi\mu a U^2 + \frac{2}{9}\pi\mu a U^2 C^2 \ .$$

Note that $E(\boldsymbol{v}) > E(\boldsymbol{u})$, in accordance with the minimum energy dissipation principle for Stokes flow.

Exercise 2.8 Translating Sphere Shrunk to a Point.
Calculate the tractions on the translating sphere (see previous exercise), and integrate to get the total force. Keeping the force constant, let the sphere shrink to a point. What happens to the translation velocity and rate of energy dissipation? Do you recognize the resulting flow field?

Exercise 2.9 Derivation of the Oseen Tensor by Fourier Transform.
Define the Fourier transform pair,

$$\begin{aligned} \hat{\boldsymbol{v}}(\boldsymbol{k}) &= \left(\frac{1}{2\pi}\right)^{3/2} \int e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, \boldsymbol{v}(\boldsymbol{x})\, dV(\boldsymbol{x}) \\ \boldsymbol{v}(\boldsymbol{x}) &= \left(\frac{1}{2\pi}\right)^{3/2} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \hat{\boldsymbol{v}}(\boldsymbol{k})\, dV(\boldsymbol{k}) \ , \end{aligned}$$

and apply the Fourier tranform to Equation 2.7. The result is (with F_j eliminated)

$$ik_i \frac{\hat{\mathcal{P}}_j}{8\pi\mu} - \frac{k^2}{8\pi}\hat{\mathcal{G}}_{ij} = -\delta_{ij}\left(\frac{1}{2\pi}\right)^{3/2} , \qquad k_i\hat{\mathcal{G}}_{ij} = 0 \ .$$

Take the dot product with k_i in the transformed equation of motion to eliminate $\hat{\mathcal{G}}$ and show that this gives

$$\frac{\hat{\mathcal{P}}_j}{8\pi\mu} = ik_j\left(\frac{1}{2\pi}\right)^{3/2} k^{-2} \ .$$

Show that this leads to the correct result for the pressure,

$$\mathcal{P}_j(\boldsymbol{x}) = -2\mu \frac{\partial r^{-1}}{\partial x_j} \ .$$

Hint: First show by direct integration in $\boldsymbol{k}$-space, with $\boldsymbol{x} = (0, 0, x_3)$, that

$$\mathcal{F}\left\{\frac{1}{4\pi r}\right\} = \left(\frac{1}{2\pi}\right)^{3/2} k^{-2} \ ,$$

and then apply the rule for transforming derivatives. Note that the preceding inverse transform constitutes the formal derivation of the Green's function for the Laplace equation.

Eliminate $\hat{\mathcal{P}}$ from the transformed equation of motion and show that

$$\frac{\hat{\mathcal{G}}_{ij}}{8\pi} = \delta_{ij}\left(\frac{1}{2\pi}\right)^{3/2} k^{-2} - \left(\frac{1}{2\pi}\right)^{3/2} \frac{k_i k_j}{k^4} \ .$$

Hint: Once again, the first term on the RHS is the transform of the Green's function for the Laplace equation, and after inversion, contributes $(4\pi r)^{-1}\delta_{ij}$. What remains is to show that

$$\mathcal{F}^{-1}\left\{\left(\frac{1}{2\pi}\right)^{3/2} \frac{k_i k_j}{k^4}\right\} = \frac{\delta_{ij}}{8\pi r} - \frac{x_i x_j}{8\pi r^3} \ .$$

Using symmetry arguments, deduce that the inverse tranform must be of the form

$$\mathcal{F}^{-1}\left\{\left(\frac{1}{2\pi}\right)^{3/2} \frac{k_i k_j}{k^4}\right\} = C_1 \frac{\delta_{ij}}{8\pi r} + C_2 \frac{x_i x_j}{8\pi r^3} \ ,$$

where C_1 and C_2 are numerical constants. Take the trace of both sides to get one condition, $3C_1 + C_2 = 2$. Show by direct integration in $\boldsymbol{k}$-space that for $i = j = 3$, and with $\boldsymbol{x} = (0, 0, x_3)$, the equation yields another necessary condition, $C_1 + C_2 = 0$. The two equations for C_1 and C_2 imply that $C_1 = -C_2 = 1$ and this completes the formal derivation of the Oseen tensor.

The following definite integrals will be useful in this exercise:

$$\begin{aligned} \int_0^\infty \frac{\sin k}{k} dk &= \frac{\pi}{2} \\ \int_0^\infty \frac{1 - \cos k}{k^2} dk &= \frac{\pi}{2} \\ \int_0^\infty \frac{k - \sin k}{k^3} dk &= \frac{\pi}{4} \ . \end{aligned}$$

The second and third integrals actually follow from the first (integration by parts).

Exercise 2.10 Angular Integration of the Dyadic $\boldsymbol{nn}$.
Show by direct integration in spherical polar coordinates that

$$\int_0^{2\pi} \int_0^\pi n_i n_j \sin\theta \, d\theta \, d\phi = \frac{4\pi}{3}\delta_{ij} \ .$$

Alternate derivations: Since the integral is isotropic, it must equal $C\delta_{ij}$. The constant C can be evaluated by considering the trace of the tensor. You could also use the divergence theorem on a unit sphere ($\boldsymbol{r} = \boldsymbol{n}$) in the form $\oint dS\, \boldsymbol{n}\boldsymbol{r} = \int dV\, \nabla \boldsymbol{r} = \ldots$

Exercise 2.11 Constant Single Layer Density on a Circular Disk.
Consider a *constant* single layer density $\boldsymbol{\psi}$ distributed over a circular disk of radius b. Show by direct integration in cylindrical coordinates that the velocity field generated by this distribution is continuous passing through the disk. Are the tractions also continuous?

Exercise 2.12 A Special Single Layer Distribution on a Disk.
Consider the single layer density

$$\boldsymbol{\psi}(\boldsymbol{\xi}) = \frac{\boldsymbol{F}}{2\pi b\sqrt{b^2 - (\xi_1^2 + \xi_2^2)}}$$

distributed over a circular disk of radius b. Show by direct integration in cylindrical coordinates that the velocity field generated by this distribution is continuous passing through the disk, and that in fact,

$$\begin{aligned}
&\lim_{z\to 0^+}\left\{-\int\!\!\int \boldsymbol{\psi}(\boldsymbol{\xi})\cdot\frac{\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})}{8\pi\mu}\,d\xi_1\,d\xi_2\right\}\\
&= \lim_{z\to 0^-}\left\{-\int\!\!\int \boldsymbol{\psi}(\boldsymbol{\xi})\cdot\frac{\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})}{8\pi\mu}\,d\xi_1\,d\xi_2\right\}\\
&= \begin{cases} \frac{\boldsymbol{F}}{16\mu b} & \text{if } \boldsymbol{F} \parallel \boldsymbol{e}_z, \\ \frac{3\boldsymbol{F}}{32\mu b} & \text{if } \boldsymbol{F} \perp \boldsymbol{e}_z. \end{cases}
\end{aligned}$$

This shows that our special choice for $\boldsymbol{\psi}(\boldsymbol{\xi})$ leads to the velocity field generated by a circular disk in uniform translation, and that F is the drag on the disk. Note that the tractions are singular at the rim, but the singularity is an integrable one and the hydrodynamic drag is finite.

Exercise 2.13 Jump in the Double Layer Potential with Constant Density on a Plane.
Consider a *constant* double layer density $\boldsymbol{\varphi}$ distributed over a coordinate plane (see Figure 2.6). Show by direct integration in cylindrical coordinates that at $\boldsymbol{x} = (x, y, z)$ the velocity field generated by the double layer distribution is

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e_3\cdot\boldsymbol{\Sigma}(\boldsymbol{x},\boldsymbol{\xi}))\cdot\boldsymbol{\varphi}\; d\xi_1 d\xi_2 = \begin{cases} \boldsymbol{\varphi}/2 & \text{if } z > 0 \\ \boldsymbol{0} & \text{if } z = 0 \\ -\boldsymbol{\varphi}/2 & \text{if } z < 0. \end{cases}$$

Note that the jump across the surface is equal to $|\boldsymbol{\varphi}|$ and that the jumps from the surface to either side are equal.
Hint: Without loss of generality, put $\boldsymbol{x}$ on the z-axis.

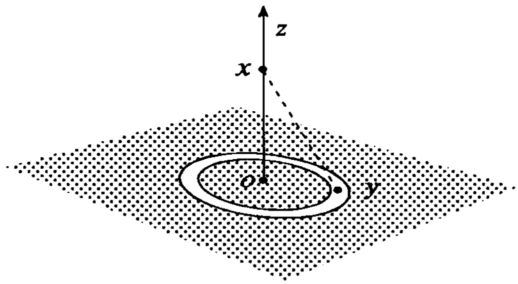

Figure 2.6: The geometry of the double layer distribution over an infinite plane.

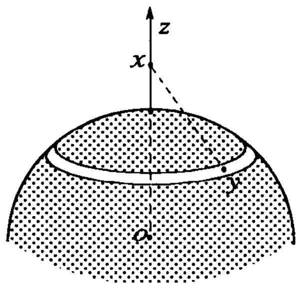

Figure 2.7: The geometry of the double layer distribution over a sphere surface.

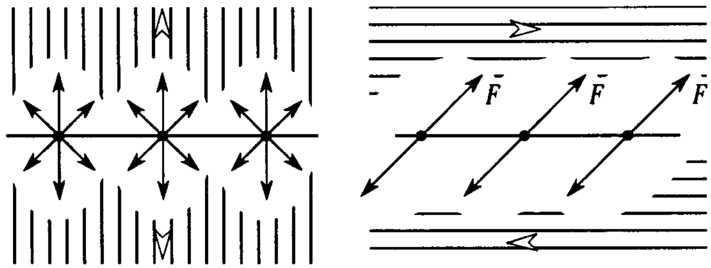

Figure 2.8: Distribution of sources/sinks and a true bilayer of Stokeslets.

Exercise 2.14 Jump in the Double Layer Potential with Constant Density on a Sphere.
Consider a *constant* double layer density φ distributed over the surface of a unit sphere. Show by direct integration in spherical polar coordinates that as the reference point $\boldsymbol{x}$ slides down the z-axis and passes through the "north pole" the velocity field generated by the double layer distribution jumps from zero to $-\varphi/2$ to $-\varphi$ for $\boldsymbol{x}$ just outside the sphere, on the sphere surface, and just inside the sphere surface, respectively.

Exercise 2.15 Normal and Tangential Components of the Double Layer Potential.
Consider the decomposition of the double layer kernel into the Oseen pressure field and the rate-of-strain field of the Oseen tensor,

$$8\pi\mu(\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi})\cdot\hat{\boldsymbol{n}})\cdot\boldsymbol{\varphi} = (\boldsymbol{\varphi}\cdot\hat{\boldsymbol{n}})\mathcal{P}(\boldsymbol{x}-\boldsymbol{\xi}) + \mu(\boldsymbol{\varphi}\hat{\boldsymbol{n}}+\hat{\boldsymbol{n}}\boldsymbol{\varphi}) : \nabla\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi}) \ .$$

The first term corresponds to sources and sinks, depending on the sign of the density, $\boldsymbol{\varphi}$, while the second group corresponds to a true bilayer of Stokeslets. From the figure, we can guess right away that the source/sink term gives a jump in the normal, but continuous tangential velocities, while for the bilayer the opposite occurs. Verify this by considering a local analysis using a constant density on a plane. In that situation, show also that only the tangential component of double layer density, $\boldsymbol{\varphi}^{\perp} = \boldsymbol{\varphi} - \boldsymbol{\varphi}\cdot\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}$, contributes to the velocity field of the bilayer distribution.

Exercise 2.16 Self-Adjointness of the Oseen Tensor.
Use the Lorentz reciprocal theorem, Equation 2.6, and the fundamental property,

$$\frac{\partial}{\partial x_k}\Sigma_{ijk}(\boldsymbol{x}-\boldsymbol{\xi}) = -\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{\xi}) \ ,$$

to deduce that the Green's function for Stokes flow is *self-adjoint*:

$$\mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{y}) = \mathcal{G}_{ji}(\boldsymbol{y}-\boldsymbol{x}) \ .$$

In words: the i-th component of the velocity field at $\boldsymbol{x}$ produced by a point force disturbance at $\boldsymbol{y}$, with the force directed along the j-th coordinate direction, is equal to the j-th component of the velocity field at $\boldsymbol{y}$ produced by a point force disturbance at $\boldsymbol{x}$, with the force directed along the i-th coordinate direction. This reciprocal property is closely tied to the concept of a self-adjoint operator, with $\mathcal{G}(\boldsymbol{x}-\boldsymbol{y})$ as the kernel of the integral operator.

Exercise 2.17 An Application of the Lorentz Reciprocal Theorem.
Consider a rigid particle of arbitrary shape in an ambient flow field, $\boldsymbol{v}^\infty(\boldsymbol{x})$. We derive in this exercise Brenner's [10] result for the drag on the particle. Suppose that $\boldsymbol{v}'$ is the solution for a sphere translating with steady velocity $\boldsymbol{U}$ and $\boldsymbol{\sigma}'$ is the associated stress field. Denote the disturbance velocity and stress fields for the solution with the arbitrary ambient flow by $\boldsymbol{v}^D$ and $\boldsymbol{\sigma}^D$. An application of the Lorentz reciprocal theorem yields the following relation:

$$\oint_{S_p} \boldsymbol{v}' \cdot (\boldsymbol{\sigma}^D \cdot \hat{\boldsymbol{n}})\, dS = \oint_{S_p} \boldsymbol{v}^D \cdot (\boldsymbol{\sigma}' \cdot \hat{\boldsymbol{n}})\, dS \ .$$

Upon insertion of the boundary conditions, $\boldsymbol{v}' = \boldsymbol{U}$ and $\boldsymbol{v}^D = -\boldsymbol{v}^\infty$ on S_p, this simplifies to

$$\boldsymbol{U} \cdot \oint_{S_p} \boldsymbol{\sigma}^D \cdot \hat{\boldsymbol{n}}\, dS = -\oint_{S_p} \boldsymbol{v}^\infty \cdot (\boldsymbol{\sigma}' \cdot \hat{\boldsymbol{n}})\, dS \ .$$

Show that this implies that the drag on the particle in an arbitrary ambient field satisfying $\nabla \cdot \boldsymbol{\sigma}^\infty = \boldsymbol{0}$ may be related directly to that ambient field by the expression,

$$F_i(\boldsymbol{v}^\infty) = -\oint_{S_p} w_{ij} v_j^\infty \, dS \ ,$$

where $U_i w_{ij} = (\boldsymbol{\sigma}' \cdot \hat{\boldsymbol{n}})_j$ is the j-th component of the surface traction from the problem of the translating particle. The important conclusion here is: Given the surface tractions for the translation problem, one can compute directly the hydrodynamic drag for the same particle in an arbitrary field.

Using the development as a guide, derive a similar relation between the torque on the particle in an arbitrary ambient field and the solution for the rotating particle.

A final note: Using concepts from linear operator theory, we can also use this idea in reverse. In Part IV, we show how the previous equation may be interpreted as a linear functional mapping vectors from a Hilbert space to a scalar. According to the Riesz representation theorm, w_{ij} uniquely plays the role given above, so any algorithm for calculating F_i, when cast in the form above, identifies the surface traction for particles in rigid-body motion.

Exercise 2.18 Double Layers and Multipole Expansion.
The multipole expansion was derived starting from the single layer representation, assuming that there are no sources or sinks within the particle considered.

The general integral representation, on the other hand, also includes a double layer term. Inspect the effects of this on the multipole expansion, specifically for the first few terms. (An explicit formula for the contribution of a double layer to the stresslet will be given later, in Chapter 16.)

Exercise 2.19 Stokes Solutions from Stokes Solutions
Assume that the (velocity and pressure field) pair $(\boldsymbol{u}, p)$ satisfies the Stokes equations. Show that the pair $(\boldsymbol{\sigma}(;\boldsymbol{u}), \mu\nabla p)$ is also a Stokes solution, one order higher in tensorial rank. (The order can be reduced for example by taking dot products with some fixed vector.)
Hint: Take the Laplacian of $\boldsymbol{\sigma}(;\boldsymbol{u})$, writing out the stress field in terms of $\boldsymbol{u}$ and p and observing that the latter of these is harmonic (*i.e.*, its Laplacian vanishes identically). Use the fact that $\nabla^2\boldsymbol{u}$ can be expressed in terms of p according to Stokes equations.

Exercise 2.20 Integral Representation for the Pressure Field. The pressure field for the Oseen tensor has been given in the text. It is conventional to choose the indeterminate constant (base pressure) so that the pressure field will be decaying as infinity is approached, so $\mathcal{P}_j^\infty = 0$. Interpret the single layer as a superposition of Stokeslets and construct the corresponding pressure field also by superposition. Recall that $8\pi\mu\boldsymbol{\Sigma}$ is the stress field of the Oseen tensor, and use the previous exercise to express the related pressure field as a gradient of the pressure field used in the first superposition above. Show that this tensorial pressure field is a symmetric tensor.

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