10

Geometry

In the preceding chapters of this book we have dealt entirely with a single geometric interpretation of the elements of a geometric algebra. But the relationship between algebra and geometry is seldom unique. Geometric problems can be studied using a variety of algebraic techniques, and the same algebraic result can typically be pictured in a variety of different ways. In this chapter, we explore a range of alternative geometric systems, and discover how geometric algebra can be applied to each of them. We will find that there is no unique interpretation forced on the multivectors of a given grade. For example, to date we have viewed bivectors solely as directed plane segments. But in projective geometry a bivector represents a line, and in conformal geometry a bivector can represent a pair of points.

Ideas from geometry have always been a prime motivating factor in the development of mathematics. By the nineteenth century mathematicians were familiar with affine, Euclidean, spherical, hyperbolic, projective and inversive geometries. The unifying framework for studying these geometries was provided by the *Kleinian viewpoint*. Under this view a geometry consists of a space of points, together with a group of transformations mapping the points onto themselves. Any property of a particular geometry must be invariant under the action of the associated symmetry group. Klein was thus able to unite various geometries by describing how some symmetry groups are subgroups of larger groups. For example, Euclidean geometry is a subgeometry of affine geometry, because the group of Euclidean transformations is a subgroup of the group of affine transformations.

In this chapter we will see how the various classical geometries, and their associated groups, are handled in geometric algebra. But we will also go further by addressing the question of how to represent various geometric primitives in the most compact and efficient way. The Kleinian viewpoint achieves a united approach to classical geometry, but it does not help much when it comes to addressing problems of how to perform calculations efficiently. For example, circles are as much geometric primitives in Euclidean geometry as points, lines a planes. But how should circles be represented as algebraic entities? Storing a point and a radius is unsatisfactory, as this representation involves objects of different grades. In this chapter we answer this question by showing that both lines and circles are represented as *trivectors* in the conformal model of Euclidean geometry.

We begin with the study of projective geometry. The addition of an extra dimension allows us to create an algebra of incidence relations between points, lines and planes in space. We then return to Euclidean geometry, but rather than viewing this as a subgeometry of projective geometry (the Kleinian viewpoint), we will instead increase the dimension once more to establish a conformal representation of Euclidean geometry. The beauty of this construction is that the group of Euclidean transformations can now be formulated as a rotor group. Euclidean invariants are then constructed as inner products between multivectors. This framework allows us to extend the projective treatment of incidence relations to include circles and spheres.

A further attractive feature of the conformal model is that Euclidean, spherical and hyperbolic geometries are all handled in the same framework. This allows the Poincaré disc model of non-Euclidean geometry in the plane to be extended seamlessly to higher dimensions. Of particular importance is the clarification of the role of complex coordinates in planar non-Euclidean geometry. Much of their utility rests on features of the conformal group of the plane that do not extend naturally. Instead, we work within the framework of *real* geometric algebra to obtain results which are independent of dimension. Finally in this chapter we turn to spacetime geometry. The conformal model for spacetime is of considerable importance in formulations of supersymmetric theories of gravity, and also lies at the heart of the twistor program. We display some surprising links between these ideas and the multiparticle spacetime algebra described in chapter 9. Throughout this chapter we denote the vector space with signature p, q by $\mathcal{V}(p,q)$, and the geometric algebra of this space by $\mathcal{G}(p,q)$.

10.1 Projective geometry

There was a time when projective geometry formed a large part of undergraduate mathematics courses. For various reasons the subject fell out of fashion in the twentieth century, making way for the more relevant subject of differential geometry. But in recent years projective geometry has enjoyed a resurgence due to its importance in the computer graphics industry. For example, the routines at the core of the OpenGL graphics language are built on a projective representation of three-dimensional space.

The key idea in projective geometry is that points in space are represented as

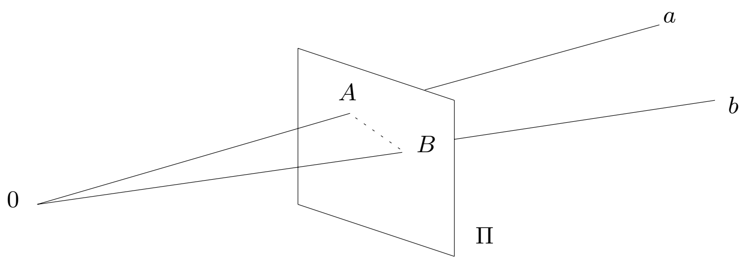

Figure 10.1 *Projective geometry.* Points in the projective plane are represented by vectors in a space one dimension higher. The plane Π does not intersect the origin 0.

vectors in a space of one dimension higher. For example, points in the projective plane are represented as vectors in three-dimensional space (see figure 10.1). The magnitude of the vector is unimportant, as both a and λa represent the same point. This representation of points is said to be *homogeneous*. The two key operations in projective geometry are the *join* and *meet*. The join of two points, for example, is the line between them. Forming the join raises the grade, and the join can usually be encoded algebraically via the exterior product (this was Grassmann's original motivation for introducing his exterior algebra). The meet is used for forming intersections, such as two lines in a plane meeting at a point. The meet is traditionally encoded via the notion of duality, and in geometric algebra the role of the meet is played by the inner product. Operations such as the meet and join do not depend on the metric, so in projective geometry we have a non-metric interpretation of the inner product. This is an important point. Some authors have argued that, because geometric algebra is built on a quadratic form, it is intimately tied to metric geometry. This view is incorrect, as we demonstrate below.

10.1.1 The projective line

The simplest place to start is with a one-dimensional line. The 'Euclidean' model of the line consists of labelling each point with a real number. But there are drawbacks with this representation of a line. Geometrically, all points on the line are equal. But algebraically there are two exceptional points on the line. The first is the origin, which is represented by the algebraically special number zero. The second is the point at infinity, which becomes important when we start to consider projective transformations. The resolution of both of these problems is to represent points in the line as vectors in two-dimensional space. In this way the point x is replaced by a pair of homogeneous coordinates (x_1, x_2), with

$$x = \frac{x_1}{x_2}. \tag{10.1}$$

One can immediately see that the origin is represented by the non-zero vector $(0, 1)$, and that the point at infinity is $(1, 0)$.

If the vectors $\{\mathsf{e}_1, \mathsf{e}_2\}$ denote an orthonormal frame for two-dimensional space, we can set

$$\boldsymbol{x} = x_1\mathsf{e}_1 + x_2\mathsf{e}_2. \tag{10.2}$$

The set of all non-zero vectors $\boldsymbol{x}$ constitute the projective line, RP^1. The fact that the origin is excluded implies that in projective spaces one loses linearity. This is obvious from the fact that $\boldsymbol{x}$ and $\lambda\boldsymbol{x}$ represent the same point, so linear combinations do not make geometric sense. Indeed, no geometric significance can be attached to the addition of two points in projective geometry. One cannot form midpoints, for example, as distances and angles are not projective invariants.

The projective group consists of the group of general linear transformations applied to vectors in projective space. For the case of the projective line this group is defined by transformations of the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}, \qquad ab - bc \neq 0. \tag{10.3}$$

In terms of points on the line, this transformation corresponds to

$$x \mapsto x' = \frac{ax + b}{cx + d}. \tag{10.4}$$

The group action includes dilations, inversions and translations. The last are obtained for the case $c = 0$, $a/d = 1$. The fact that translations become *linear* transformations in projective geometry is of considerable importance. In three-dimensional geometry, for example, both rotations and translations can be encoded as 4×4 matrices. While this may appear to be an overly-complicated representation, it makes stringing together a series of translations and rotations a straightforward exercise. This is important in computer graphics, and is the representation employed in all OpenGL routines.

In geometric algebra notation we write a general linear transformation as the map $\boldsymbol{x} \mapsto \mathsf{f}(\boldsymbol{x})$, where $\det(\mathsf{f}) \neq 0$. Valid geometric statements in projective geometry must be invariant under such transformations, which is a strong restriction. Inner products between projective vectors (points) are clearly not invariant under projective transformations. The outer product does transform sensibly, however, due to the properties of the outermorphism. For example, suppose that the points α and β are represented projectively by

$$\boldsymbol{a} = \alpha\mathsf{e}_1 + \mathsf{e}_2, \quad \boldsymbol{b} = \beta\mathsf{e}_1 + \mathsf{e}_2. \tag{10.5}$$

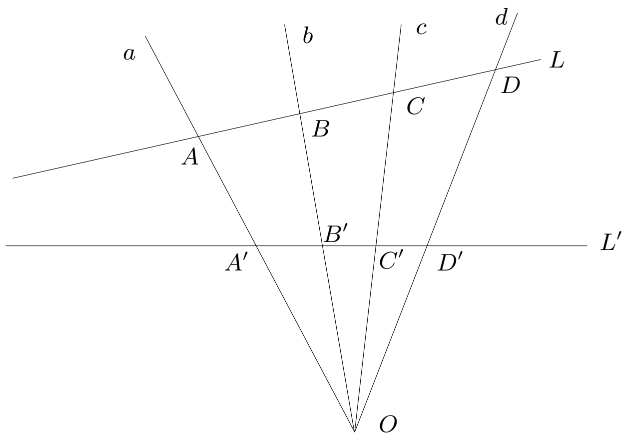

Figure 10.2 *The cross ratio.* Points on the lines L and L' represent two different projective views of the same vectors in space. The cross ratio of the four points is the same on both lines.

The outer product of these is

$$\boldsymbol{a}\wedge\boldsymbol{b} = (\alpha - \beta)\mathsf{e}_1\wedge\mathsf{e}_2, \tag{10.6}$$

which is controlled by the distance between the points on the line. Under a projective transformation in two dimensions

$$\mathsf{e}_1\wedge\mathsf{e}_2 \mapsto \mathsf{f}(\mathsf{e}_1\wedge\mathsf{e}_2) = \det\,(\mathsf{f})\,\mathsf{e}_1\wedge\mathsf{e}_2, \tag{10.7}$$

which is just an overall scaling.

The fact that distances between points are scaled under a projective transformation provides us with an important projective invariant for four points on a line. This is formed from ratios of lengths along a line. We must further ensure that the ratio is invariant under individual rescaling of individual vectors to be a true projective invariant. We therefore define the *cross ratio* of four points, A, B, C, D, by

$$(ABCD) = \frac{AC}{BC}\frac{BD}{AD} = \frac{\boldsymbol{a}\wedge\boldsymbol{c}}{\boldsymbol{b}\wedge\boldsymbol{c}}\frac{\boldsymbol{b}\wedge\boldsymbol{d}}{\boldsymbol{a}\wedge\boldsymbol{d}}, \tag{10.8}$$

where AB denotes the distance between A and B. Given any four points on a line, their cross ratio is a projective invariant (see figure 10.2). The figure illustrates one possible geometric interpretation of a projective transformation, which is that the line onto which points are projected is transformed to a new line. Invariants such as the cross ratio are important in computer vision where, for example, we seek to extract three-dimensional information from a series of two-dimensional scenes. Knowledge of invariants can help establish point matches between the scenes.

10.1.2 The projective plane

Rather more interesting than the case of a line is that of the projective plane. Points in the plane are now represented by vectors in the three-dimensional algebra $\mathcal{G}(3,0)$. Figure 10.1 shows that the line between the points a and b is the result of projecting the plane defined by a and b onto the projective plane. We therefore define the *join* of the points a and b by

$$\mathrm{join}(a,\, b) = a \wedge b. \tag{10.9}$$

Bivectors thus define lines in projective geometry. The line itself is recovered by solving the equation

$$a \wedge b \wedge x = 0. \tag{10.10}$$

This equation is solved by

$$x = \lambda a + \mu b, \tag{10.11}$$

which defines the set of projective points on the line joining A and B.

By taking exterior products of vectors we define (projectively) higher dimensional objects. For example, the join of a point a and a line $b \wedge c$ is the plane defined by the trivector $a \wedge b \wedge c$. Three points on a line cannot define a projected area, so for these we must have

$$a \wedge b \wedge c = 0 \quad \Rightarrow \quad a,\, b,\, c \text{ collinear}. \tag{10.12}$$

This was the condition used to recover the points x on the line $a \wedge b$. The join itself can be slightly more problematic. Given three points one cannot just write that their join is $a \wedge b \wedge c$, as the result may be zero. Instead the join is defined as the smallest subspace containing a, b and c. If they are collinear, then the join is the common line. This is well defined mathematically, but is hard to encode computationally. The problem is that the finite precision used on computers means that testing for zero is unreliable. Wherever possible it is safer to avoid defining the join and instead work with the exterior product.

Projective geometry deals with relationships that are invariant under projective transformations. The join is one such concept — as two points are transformed the line joining them transforms in the obvious way:

$$a \wedge b \mapsto \mathsf{f}(a) \wedge \mathsf{f}(b) = \mathsf{f}(a \wedge b). \tag{10.13}$$

So, for example, the statement that three points lie on a line ($a \wedge b \wedge c = 0$) is unchanged by a projective transformation. Similarly, the statement that three lines intersect at a point must also be a projective invariant. We therefore seek an algebraic encoding of the intersection of two lines. This is the called the *meet*, usually denoted with the $\vee$ symbol. Before we can encode this, however, we need to define the dual. In the projective plane, points and lines are represented as vectors and bivectors in $\mathcal{G}(3,0)$. We know that these can be interchanged via a duality transformation, which amounts to multiplying by the pseudoscalar I. In this way every point has a dual line, and vice versa. The geometric picture associated with duality depends on the embedding plane.

If we denote the dual of A by A^*, the meet $A \vee B$ is defined by the 'de Morgan' rule

$$(A \vee B)^* = A^* \wedge B^*. \tag{10.14}$$

For a pair of lines in a plane, this amounts to

$$A \vee B = -I(IA)\wedge(IB) = I\, A\times B = A\cdot(IB) = (IA)\cdot B. \tag{10.15}$$

These formulae show how the inner product can be used to encode the meet, without imposing a metric on projective space. The expression

$$A \vee B = I\, A\times B \tag{10.16}$$

shows how the construction works. In three dimensions, $A\times B$ is the plane perpendicular to A and B, and $I\, A\times B$ is the line perpendicular to this plane, through the origin. This is therefore the line common to both planes, so projectively gives the point of intersection of two lines.

The meet of two distinct lines in a plane always results in a non-zero point. If the lines are parallel then their meet returns the point at infinity. Parallelism is not a projective invariant, however, so under a projective transformation two parallel lines can transform to lines intersecting at a finite point. This illustrates the fact that the point at infinity does not necessarily stay at infinity under projective transformations. It is instructive to see how the meet itself transforms under a projective transformation. Using the results of section 4.4, we find that

$$\begin{aligned} A \vee B \mapsto \mathsf{f}(A) \vee \mathsf{f}(B) &= I\,\big(I\mathsf{f}(A)\big)\wedge\big(I\mathsf{f}(B)\big) \\ &= \det(\mathsf{f})^2\, I\,\bar{\mathsf{f}}^{-1}(IA)\wedge\bar{\mathsf{f}}^{-1}(IB) \\ &= \det(\mathsf{f})^2\, I\,\bar{\mathsf{f}}^{-1}\big((IA)\wedge(IB)\big) \\ &= \det(\mathsf{f})\,\mathsf{f}\big(I\,(IA)\wedge(IB)\big). \end{aligned} \tag{10.17}$$

We can summarise this result as

$$\mathsf{f}(A) \vee \mathsf{f}(B) = \det(\mathsf{f})\,\mathsf{f}(A \vee B). \tag{10.18}$$

But in projective geometry, a and λa represent the same point, so the factor of $\det(\mathsf{f})$ does not affect the resulting point. This confirms that under a projective transformation the meet transforms as required.

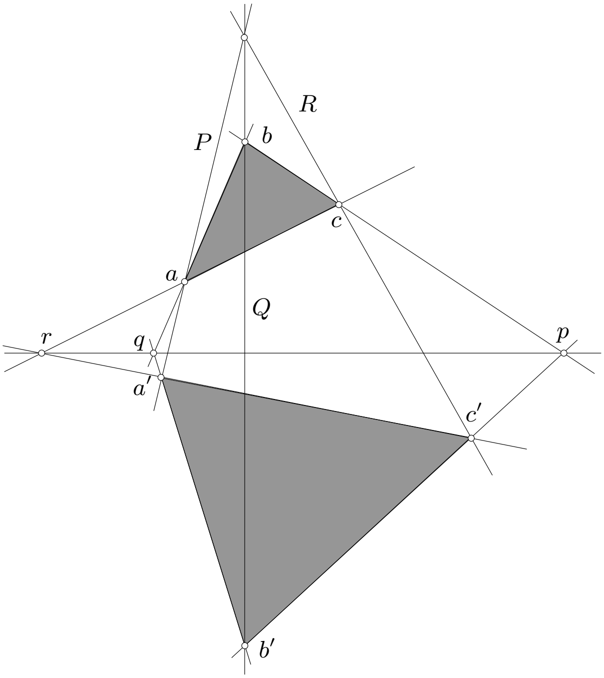

Figure 10.3 *Desargues' theorem.* The lines P, Q, R meet at a point if and only if the points p, q, r lie on a line. The two triangles are then projectively related.

The condition that three lines meet at a common point requires that the meet of two lines lies on a third line, which goes as

$$(A \vee B) \wedge C = (I\, A \times B) \wedge C = 0. \tag{10.19}$$

Dualising this result we obtain the condition

$$\langle (A \times B) C \rangle = \langle ABC \rangle = 0, \quad \Rightarrow \quad A,\, B,\, C \text{ coincident.} \tag{10.20}$$

This is an extremely simple algebraic encoding of the statement that three lines (represented by bivectors) all meet at a common point. Equations like this demonstrate how powerful geometric algebra can be when applied in a projective setting.

As an application consider Desargues' theorem, which is illustrated in figure 10.3. The points a, b, c and a', b', c' define two triangles. The associated lines are defined by

$$A = b \wedge c, \quad B = c \wedge a, \quad C = a \wedge b, \tag{10.21}$$

with the same definitions holding for A', B', C' in terms of a', b', c'. The two sets of vertices determine the lines

$$P = a\wedge a', \quad Q = b\wedge b', \quad R = c\wedge c', \tag{10.22}$$

and the two sets of lines determine the points

$$p = A\times A'\,I, \quad q = B\times B'\,I, \quad r = C\times C'\,I. \tag{10.23}$$

Desargues' theorem states that, if p, q, r lie on a common line, then P, Q and R all meet at a common point. The latter condition requires

$$\langle PQR\rangle = \langle a\wedge a'\, b\wedge b'\, c\wedge c'\rangle = 0. \tag{10.24}$$

Similarly, for p, q, r to fall on a line we form

$$\begin{aligned} p\wedge q\wedge r &= \langle A\times A'\,I\,B\times B'\,I\,C\times C'\,I\rangle_3 \\ &= -I\langle A\times A'\,B\times B'\,C\times C'\rangle. \end{aligned} \tag{10.25}$$

Desargues' theorem is then proved by the algebraic identity

$$\langle a\wedge b\wedge c\, a'\wedge b'\wedge c'\rangle\langle a\wedge a'\, b\wedge b'\, c\wedge c'\rangle = \langle A\times A'\, B\times B'\, C\times C'\rangle, \tag{10.26}$$

the proof of which is left as an exercise. The left-hand side vanishes if and only if the lines P, Q, R meet at a point. The right-hand side vanishes if and only if the points p, q, r lie on a line. This proves the theorem. The complex geometry illustrated in figure 10.3 has therefore been reduced to a straightforward algebraic identity.

We can find a simple generalisation of the cross ratio for the case of the projective plane. From the derivation of the cross ratio, it is clear that any analogous object for the plane must involve ratios of trivectors. These represent areas in the projective plane. For example, suppose we have six points in space with position vectors $a_1, \ldots, a_6$. These produce the six projected points $A_1, \ldots, A_6$. An invariant is formed by

$$\frac{a_5\wedge a_4\wedge a_3}{a_5\wedge a_1\wedge a_3}\frac{a_6\wedge a_2\wedge a_1}{a_6\wedge a_2\wedge a_4} = \frac{A_{543}}{A_{513}}\frac{A_{621}}{A_{624}}, \tag{10.27}$$

where A_{ijk} is the projected area of the triangle with vertices A_i, A_j, A_k. Again, elementary algebraic reasoning quickly yields a geometrically significant result.

10.1.3 Homogeneous coordinates and projective splits

In typical applications of projective geometry we are interested in the relationship between coordinates in an image plane (for example in terms of pixels relative to some origin) and the three-dimensional position vector. Suppose that the origin in the image plane is defined by the vector n, which is perpendicular to the plane. The line on the image plane from the origin to the image point is represented by the bivector $a\wedge n$ (see figure 10.4) . The vector OA belongs to a two-dimensional

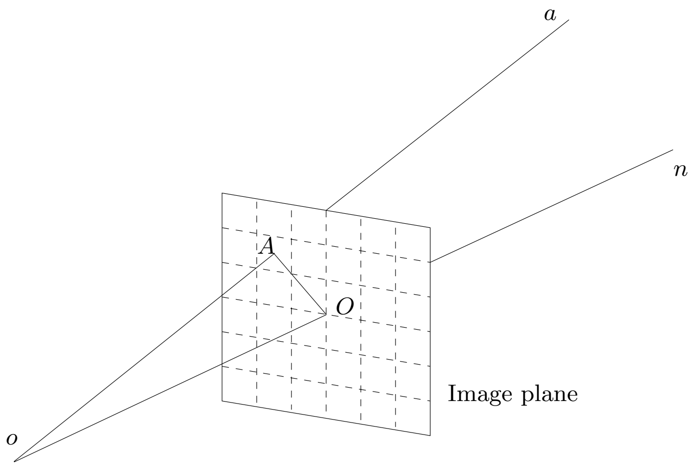

Figure 10.4 *The image plane.* Vectors in the image plane, OA, are described by bivectors in $\mathcal{G}(3,0)$. The point A can be expressed in terms of *homogeneous* coordinates in the image plane.

geometric algebra. We can relate this directly to the three-dimensional algebra by first writing

$$n + OA = \lambda a. \tag{10.28}$$

Contracting with n, we find that $\lambda = n^2(a\cdot n)^{-1}$. It follows that

$$OA = \frac{a\,n^2 - a\cdot n\,n}{a\cdot n} = \frac{a\wedge n}{a\cdot n}n. \tag{10.29}$$

If we now drop the final factor of n, we obtain a bivector that is homogeneous in both a and n. In this way we can directly represent the line OA in two dimensions with the bivector

$$A = \frac{a\wedge n}{a\cdot n}. \tag{10.30}$$

This is the *projective split*, first introduced in chapter 5 as a means of relating physics as seen by observers with different velocities.

The map of equation (10.30) relates bivectors in a higher dimensional space to vectors in a space of dimension one lower. If we introduce a coordinate frame $\{\mathsf{e}_i\}$, with e_3 in the n direction, we see that the coordinates of the image of $a = a_i\mathsf{e}_i$ are

$$A = \frac{a_1}{a_3}\mathsf{e}_1\mathsf{e}_3 + \frac{a_2}{a_3}\mathsf{e}_2\mathsf{e}_3 = A_1\boldsymbol{E}_1 + A_2\boldsymbol{E}_2. \tag{10.31}$$

This equation defines the *homogeneous coordinates* A_i:

$$A_i = \frac{a_i}{a_3}. \tag{10.32}$$

Homogeneous coordinates are independent of scale and it is these that are usually measured in a camera projection of a scene. The bivectors $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ act as generators for a two-dimensional geometric algebra. If the vectors in the projective space are all Euclidean, the $\boldsymbol{E}_i$ bivectors will have negative square. If necessary, this can be avoided by letting e_3 be an anti-Euclidean vector. The projective split is an elegant scheme for relating results in projective space to Euclidean space one dimension lower. Algebraically, the projective split rests on the isomorphism

$$\mathcal{G}^+(p+1, q) \simeq \mathcal{G}(q, p). \tag{10.33}$$

This states that the even subalgebra of the geometric algebra with signature $(p+1, q)$ is isomorphic to the algebra with signature (q, p). The projective split is not always the best way to map from projective space back to Euclidean space, however, as constructing a set of bivectors can be an unnecessary complication. Often it is simpler to choose an orthonormal frame, with n one of the frame vectors, and then scale all vectors x such that $n \cdot x = 1$.

10.1.4 Projective geometry in three dimensions

To handle complicated three-dimensional problems in a projective framework we require a four-dimensional geometric algebra. The basic elements of four-dimensional geometric algebra will be familiar from relativity and the spacetime algebra, though now the elements are given a projective interpretation. The algebra of a four-dimensional space contains six bivectors, which represent lines in three dimensions. As in the planar case, the important feature of the projective framework is that we are free from the restriction that all lines pass through the origin. The line through the points a and b is again represented by the bivector $a \wedge b$. This is a *blade*, as must be the case for any bivector representing a line. Any bivector blade $B = a \wedge b$ must satisfy the algebraic condition

$$B \wedge B = a \wedge b \wedge a \wedge b = 0, \tag{10.34}$$

which removes one degree of freedom from the six components needed to specify an arbitrary bivector. This is known at the Plücker condition. If the vector e_4 defines the projection into Euclidean space, the line $a \wedge b$ has coordinates

$$a \wedge b = (\boldsymbol{a} + \mathsf{e}_4) \wedge (\boldsymbol{b} + \mathsf{e}_4) = \boldsymbol{a} \wedge \boldsymbol{b} + (\boldsymbol{a} - \boldsymbol{b}) \wedge \mathsf{e}_4, \tag{10.35}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ denote vectors in the three-dimensional space. The bivector B therefore encodes a line as a combination of a tangent $(\boldsymbol{b} - \boldsymbol{a})$ and a moment $\boldsymbol{a} \wedge \boldsymbol{b}$. These are the Plücker coordinates for a line.

Given two lines as bivectors B and B', the test that they intersect in three dimensions is that their join does not span all of projective space, which implies that

$$B \wedge B' = 0. \tag{10.36}$$

This provides a projective interpretation for commuting bivectors in four dimensions. Commuting (orthogonal) bivectors have BB' equalling a multiple of the pseudoscalar. Projectively, these can be interpreted as two lines in three dimensions that do not share a common point. As mentioned earlier, the problem with a test such as equation (10.36) is that one can never guarantee to obtain zero when working to finite numerical precision. In practice, then, one tends to avoid trying to find the intersection of two lines in the three dimensions, unless there is good reason to believe that they intersect at a point.

The exterior product of three vectors in projective space results in the trivector encoding the plane containing the three points. One of the most frequently encountered problems is finding the point of intersection of a line L and a plane P. This is given by

$$x = P \cdot (IL), \tag{10.37}$$

where I is the four-dimensional pseudoscalar. This will always return a point, provided the line does not lie entirely in the plane. Similarly, the intersection of two planes in three dimensions must result in a line. Algebraically, this line is encoded by the bivector

$$L = (IP_1) \cdot P_2 = I\, P_1 \times P_2, \tag{10.38}$$

where P_1 and P_2 are the two planes. Such projective formulae are important in computer vision and graphics applications.

10.2 Conformal geometry

Projective geometry does provide an efficient framework for handling Euclidean geometry. Euclidean geometry is a subgeometry of projective geometry, so any valid result in the latter must hold in the former. But there are some limitations to the projective viewpoint. Euclidean concepts, like lengths and angles, are not straightforwardly encoded, and the related concepts of circles and spheres are equally awkward. Conformal geometry provides an elegant solution to this problem. The key is to introduce a further dimension of opposite signature, so that points in a space of signature (p, q) are modelled as null vectors in a space of signature $(p + 1, q + 1)$. That is, points in $\mathcal{V}(p, q)$ are represented by null vectors in $\mathcal{V}(p + 1, q + 1)$. Projective geometry is retained as a subset of conformal geometry, but the range of geometric primitives is extended to include circles and spheres.

We denote a point in $\mathcal{V}(p, q)$ by x, and its conformal representation by X. We continue to employ the spacetime notation of using the tilde symbol to denote

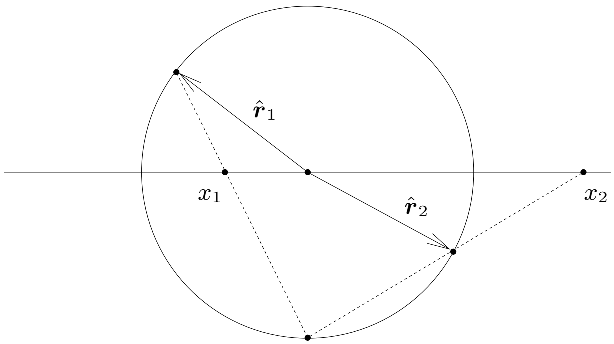

Figure 10.5 *A stereographic projection.* The line is mapped into the unit circle, so the points on the line x_1 and x_2 are mapped to the unit vectors $\hat{\boldsymbol{r}}_1$ and $\hat{\boldsymbol{r}}_2$. The origin and infinity are mapped to opposite points on the circle.

the reverse operation for a general multivector in any geometric algebra. A basis set of vectors for $\mathcal{G}(p,q)$ is denoted by $\{e_i\}$, and the two additional vectors $\{e,\bar{e}\}$ complete this to an orthonormal basis for $\mathcal{G}(p+1,q+1)$.

10.2.1 Stereographic projection of a line

We illustrate the general construction by starting with the simple case of a line. In projective geometry points on a line are modeled as two-dimensional vectors. The conformal model is established from a slightly different starting point, using the *stereographic projection*. Under a stereographic projection, points on a line are mapped to the unit circle in a plane (see figure 10.5). Points on the unit circle in two dimensions are represented by

$$\hat{\boldsymbol{r}} = \cos(\theta)\, e_1 + \sin(\theta)\, e_2. \tag{10.39}$$

The corresponding point on the line is given by

$$x = \frac{\cos(\theta)}{1+\sin(\theta)}. \tag{10.40}$$

This relation inverts simply to give

$$\cos(\theta) = \frac{2x}{1+x^2}, \quad \sin(\theta) = \frac{1-x^2}{1+x^2}. \tag{10.41}$$

So far we have achieved a representation of the line in terms of a circle in two dimensions. But the constraint that the vector has unit magnitude means that we have lost homogeneity. To get round this we introduce a third vector, $\bar{e}$, which has negative signature,

$$\bar{e}^2 = -1, \tag{10.42}$$

and we assume that $\bar{e}$ is orthogonal to e_1 and e_2. We can now replace the unit vector $\hat{\boldsymbol{r}}$ with the null vector X, where

$$X = \cos(\theta)\, e_1 + \sin(\theta)\, e_2 + \bar{e} = \frac{2x}{1+x^2} e_1 + \frac{1-x^2}{1+x^2} e_2 + \bar{e}. \tag{10.43}$$

The vector X satisfies $X^2 = 0$, so is null.

The equation $X^2 = 0$ is homogeneous. If it is satisfied for X, it is satisfied for λX. We can therefore move to a homogeneous representation and let both X and λX represent the same point. Multiplying by $(1+x^2)$ we establish the conformal representation

$$X = 2xe_1 + (1-x^2)e_2 + (1+x^2)\bar{e}. \tag{10.44}$$

This is the basic representation we use throughout. To establish a more general notation we first replace the vector e_2 by $-e$. We therefore have

$$e^2 = 1, \qquad \bar{e}^2 = -1, \qquad e \cdot \bar{e} = 0. \tag{10.45}$$

The vectors e and $\bar{e}$ are then the two extra vectors that extend the space $\mathcal{V}(p,q)$ to $\mathcal{V}(p+1,q+1)$. Frequently, it is more convenient to work with a null basis for the extra dimensions. We define

$$n = e + \bar{e}, \qquad \bar{n} = e - \bar{e}. \tag{10.46}$$

These vectors satisfy

$$n^2 = \bar{n}^2 = 0, \qquad n \cdot \bar{n} = 2. \tag{10.47}$$

The vector X is now

$$X = 2xe_1 + x^2 n - \bar{n}. \tag{10.48}$$

It is straightforward to confirm that this is a null vector. The set of all null vectors in this space form a cone, and the real number line is modelled by the intersection of this cone and a plane. The construction is illustrated in figure 10.6.

10.2.2 Conformal model of Euclidean space

The form of equation (10.48) generalises easily. If x is an element of $\mathcal{V}(p,q)$, we set

$$F(x) = X = x^2 n + 2x - \bar{n}, \tag{10.49}$$

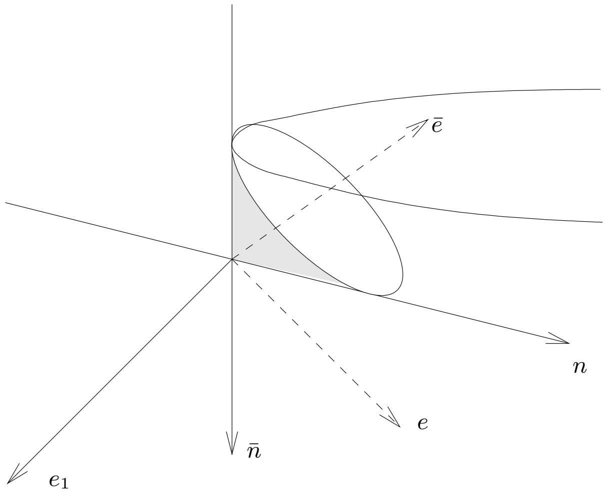

Figure 10.6 *The conformal model of a line.* Points on the line are represented by null vectors in three dimensions. These lie on a cone, and the intersection of the cone with a plane recovers the point.

which is a null vector in $\mathcal{V}(p+1, q+1)$. This vector can be obtained simply via the map,

$$F(x) = -(x-e)n(x-e), \tag{10.50}$$

which is a reflection of the null vector n in the plane perpendicular to $(x-e)$. The result must therefore be a new null vector. The presence of the vector e removes any ambiguity in handling the origin $x = 0$. The map $F(x)$ is non-linear so, as with projective geometry, we move to a non-linear representation of points in conformal geometry.

More generally, any null vector in $\mathcal{V}(p+1, q+1)$ can be written as

$$X = \lambda(x^2 n + 2x - \bar{n}), \tag{10.51}$$

with λ a scalar. This provides a projective map between $\mathcal{V}(p+1, q+1)$ and $\mathcal{V}(p, q)$. The family of null vectors, $\lambda(x^2 n + 2x - \bar{n})$, in $\mathcal{V}(p+1, q+1)$ correspond to the single point $x \in \mathcal{V}(p, q)$. Given an arbitrary null vector X, it is frequently useful to convert it to the standard form of equation (10.49). This is achieved by setting

$$X \mapsto -2\frac{X}{X \cdot n}. \tag{10.52}$$

This map is similar to that employed in constructing a standard embedding in projective geometry. The status of the vector n is clear here — it represents the point at infinity.

Given two null vectors X and Y, in standard form, their inner product is

$$\begin{aligned} X\cdot Y &= \big(x^2 n + 2x - \bar{n}\big)\cdot\big(y^2 n + 2y - \bar{n}\big) \\ &= -2x^2 - 2y^2 + 4x\cdot y \\ &= -2(x-y)^2. \end{aligned} \tag{10.53}$$

This result is of fundamental importance to the conformal model of Euclidean geometry. The inner product in conformal space encodes the *distance* between points in Euclidean space. It follows that any transformation of null vectors in $\mathcal{V}(p+1, q+1)$ which leaves inner products invariant can correspond to a transformation in $\mathcal{V}(p, q)$ which leaves angles and distances invariant. In the next section we discuss these transformations in detail.

10.3 Conformal transformations

The study of the main geometric primitives in conformal geometry is simplified by first understanding the nature of the conformal group. For points x, y in $\mathcal{V}(p, q)$ the definition of a conformal transformation is that it leaves angles invariant. So, if f is a map from $\mathcal{V}(p, q)$ to itself, then f is a conformal transformation if

$$\mathsf{f}(a)\cdot\mathsf{f}(b) = \lambda a\cdot b, \quad \forall a, b \in \mathcal{V}(p, q), \tag{10.54}$$

where

$$\mathsf{f}(a) = a\cdot\nabla f(x). \tag{10.55}$$

While $\mathsf{f}(a)$ is a linear map at each point x, the conformal transformation $f(x)$ is not restricted to being linear. Conformal transformations form a group, the conformal group, the main elements of which are translations, rotations, dilations and inversions. We now study each of these in turn.

10.3.1 Translations

To begin, consider the fundamental operation of translation in the space $\mathcal{V}(p, q)$. This is *not* a linear operation in $\mathcal{V}(p, q)$, but does become linear in the projective framework. In the conformal model we achieve a further refinement, as translations can now be handled by rotors. Consider the rotor

$$R = T_a = \mathrm{e}^{na/2}, \tag{10.56}$$

where $a \in \mathcal{V}(p, q)$, so that $a\cdot n = 0$. The generator for the rotor is a null bivector, so the Taylor series for T_a terminates after two terms:

$$T_a = 1 + \frac{na}{2}. \tag{10.57}$$

The rotor T_a transforms the null vectors n and $\bar{n}$ into

$$T_a n \tilde{T}_a = n + \tfrac{1}{2}nan + \tfrac{1}{2}nan + \tfrac{1}{4}nanan = n \tag{10.58}$$

and

$$T_a \bar{n} \tilde{T}_a = \bar{n} - 2a - a^2 n. \tag{10.59}$$

Acting on a vector $x \in \mathcal{V}(p,q)$ we similarly obtain

$$T_a x \tilde{T}_a = x + n(a \cdot x). \tag{10.60}$$

Combining these we find that

$$\begin{aligned} T_a F(x) \tilde{T}_a &= x^2 n + 2(x + a \cdot x\, n) - (\bar{n} - 2a - a^2 n) \\ &= (x+a)^2 n + 2(x+a) - \bar{n} \\ &= F(x+a), \end{aligned} \tag{10.61}$$

which performs the conformal version of the translation $x \mapsto x + a$. Translations are handled as rotations in conformal space, and the rotor group provides a double-cover representation of a translation. The identity

$$\tilde{T}_a = T_{-a} \tag{10.62}$$

ensures that the inverse transformation in conformal space corresponds to a translation in the opposite direction, as required.

10.3.2 Rotations

Next, suppose that we rotate the vector x about the origin in $\mathcal{V}(p,q)$. This is achieved with the rotor $R \in \mathcal{G}(p,q)$ via the familiar transformation $x \mapsto x' = Rx\tilde{R}$. The image of the transformed point is

$$\begin{aligned} F(x') &= x'^2 n + 2Rx\tilde{R} - \bar{n} \\ &= R(x^2 n + 2x - \bar{n})\tilde{R} = RF(x)\tilde{R}. \end{aligned} \tag{10.63}$$

This holds because R is an even element in $\mathcal{G}(p,q)$, so must commute with both n and $\bar{n}$. Rotations about the origin therefore take the same form in either space.

Suppose instead that we wish to rotate about the point $a \in \mathcal{V}(p,q)$. This can be achieved by translating a to the origin, rotating and then translating forward again. In terms of $X = F(x)$ the result is

$$X \mapsto T_a R T_{-a} X \tilde{T}_{-a} \tilde{R} \tilde{T}_a = R' X \tilde{R}. \tag{10.64}$$

The rotation is now controlled by the rotor

$$R' = T_a R \tilde{T}_a = \left(1 + \frac{na}{2}\right) R \left(1 + \frac{an}{2}\right). \tag{10.65}$$

So, as expected, the conformal model has freed us from treating the origin as a special point. Rotations about any point are handled in the same manner, and are still generated by a bivector blade. Similar observations hold for reflections, but we delay a full treatment of these until we have described how lines and surfaces are handled in the conformal model. The preceding formulae for translations and rotations form the basis of the subject of *screw theory*, which has its origins in the nineteenth century.

10.3.3 Inversions

Rotations and translations are elements of the Euclidean group, as they leave distances between points invariant. This is a subgroup of the larger conformal group, which only leaves angles invariant. The conformal group essentially contains two further transformations: inversions and dilations. An inversion in the origin consists of the map

$$x \mapsto \frac{x}{x^2}. \tag{10.66}$$

The conformal vector corresponding to the inverted point is

$$F(x^{-1}) = x^{-2}n + 2x^{-1} - \bar{n} = \frac{1}{x^2}(n + 2x - x^2\bar{n}). \tag{10.67}$$

But in conformal space points are represented homogeneously, so the pre-factor of x^{-2} can be ignored. In conformal space an inversion in the origin consists solely of the map

$$n \mapsto -\bar{n}, \qquad \bar{n} \mapsto -n. \tag{10.68}$$

This is generated by a reflection in e, since

$$-ene = -ee\bar{n} = -\bar{n}. \tag{10.69}$$

We can therefore write

$$-eF(x)e = x^2F(x^{-1}), \tag{10.70}$$

which shows that inversions in $\mathcal{V}(p,q)$ are represented as reflections in the conformal space $\mathcal{V}(p+1,q+1)$. As both X and $-X$ are homogeneous representations of the same point, it is irrelevant whether we take $-e(\ldots)e$ or $e(\ldots)e$ as the reflection. In the following we will use $e(...)e$ for convenience.

A reflection in e corresponds to an inversion in the origin in Euclidean space. To find the generator of an inversion in an arbitrary point a, we translate to the origin, invert and translate forward again. The resulting generator is then

$$T_a e T_{-a} = \left(1 + \frac{na}{2}\right) e \left(1 + \frac{an}{2}\right) = e - a - \frac{a^2}{2}n. \tag{10.71}$$

Now, recalling that $e = (n+\bar{n})/2$, the generating vector can also be written as

$$T_a e T_{-a} = \tfrac{1}{2}\big(n - F(a)\big) = \tfrac{1}{2}(n - A). \tag{10.72}$$

A reflection in $(n - F(a))$ therefore achieves an inversion about the point a in Euclidean space. As with translations, a nonlinear transformation in Euclidean space has been linearised by moving to a conformal representation of points. The generator of an inversion is a vector with positive square. In section 10.5.1 we see how these vectors are related to circles and spheres.

10.3.4 Dilations

A dilation in the origin is given by

$$x \mapsto x' = \mathrm{e}^{-\alpha} x, \tag{10.73}$$

where α is a scalar. Clearly, this transformation does not alter angles, so is a conformal transformation. The null vector corresponding to the transformed point is

$$F(x') = \mathrm{e}^{-\alpha}(x^2 \mathrm{e}^{-\alpha} n + 2x + \mathrm{e}^{\alpha} \bar{n}). \tag{10.74}$$

Clearly the map we need to achieve is

$$n \mapsto \mathrm{e}^{-\alpha} n, \qquad \bar{n} \mapsto \mathrm{e}^{\alpha} \bar{n}. \tag{10.75}$$

This transformation does not alter the inner product of n and $\bar{n}$, so can be represented with a rotor. As the vector x is unchanged, the rotor can only be generated by the timelike bivector $e\bar{e}$. If we set

$$N = e\bar{e} = \tfrac{1}{2}\bar{n}\wedge n \tag{10.76}$$

then N satisfies

$$Nn = -n = -nN, \qquad N\bar{n} = \bar{n} = -\bar{n}N, \qquad N^2 = 1. \tag{10.77}$$

We now introduce the rotor

$$D_\alpha = \mathrm{e}^{\alpha N/2} = \cosh(\alpha/2) + \sinh(\alpha/2)\, N. \tag{10.78}$$

This rotor satisfies

$$\begin{aligned} D_\alpha n \tilde{D}_\alpha &= \mathrm{e}^{-\alpha} n, \\ D_\alpha \bar{n} \tilde{D}_\alpha &= \mathrm{e}^{\alpha} \bar{n} \end{aligned} \tag{10.79}$$

and so carries out the required transformation. We can therefore write

$$F(\mathrm{e}^{-\alpha} x) = \mathrm{e}^{-\alpha} D_\alpha F(x) \tilde{D}_\alpha, \tag{10.80}$$

which confirms that a dilation in the origin is represented by a simple rotor in conformal space. To achieve a dilation about an arbitrary point a we form

$$D'_\alpha = T_a D_\alpha \tilde{T}_a = \mathrm{e}^{\alpha N'/2}, \tag{10.81}$$

where the generator is now

$$N' = T_a N \tilde{T}_a = \tfrac{1}{2} T_a \bar{n} \wedge n \tilde{T}_a = -\tfrac{1}{2} A \wedge n, \tag{10.82}$$

with $A = F(a)$. A dilation about a is therefore generated by

$$D'_\alpha = \exp(-\alpha A \wedge n/4) = \exp\left(\frac{\alpha}{2} \frac{A \wedge n}{A \cdot n}\right). \tag{10.83}$$

The generator is governed by two null vectors, one for the point about which the dilation is performed and one for the point at infinity.

10.3.5 Special conformal transformations

A special conformal transformation consists of an inversion in the origin, a translation and a further inversion in the origin. We can therefore handle these in terms of the representations we have already established. In Euclidean space the effect of a conformal transformation can be written as

$$x \mapsto \frac{x + ax^2}{1 + 2a \cdot x + a^2 x^2} = x \frac{1}{1 + ax} = \frac{1}{1 + xa} x. \tag{10.84}$$

The final expressions confirm that a special conformal transformation corresponds to a position-dependent rotation and dilation in Euclidean space, so does leave angles unchanged. To construct the equivalent rotor in $\mathcal{G}(p+1, q+1)$ we form

$$K_a = e T_a e = 1 - \frac{\bar{n} a}{2}, \tag{10.85}$$

which ensures that $K_a F(x) \tilde{K}_a$ is a special conformal transformation. Explicitly, we have

$$F\left(x \frac{1}{1 + ax}\right) = (1 + 2a \cdot x + a^2 x^2)^{-1} K_a F(x) \tilde{K}_a \tag{10.86}$$

and again we can ignore the pre-factor and use $K_a F(x) \tilde{K}_a$ as the homogeneous representation of the result of a special conformal transformation.

10.3.6 Euclidean transformations

The group of Euclidean transformations is a subgroup of the full conformal group. The additional restriction is that lengths as well as angles are invariant. Equation (10.53) showed that the inner product of two null vectors is related to the Euclidean distance between the corresponding points. To establish a homogeneous formula, we must write

$$|a - b|^2 = -2 \frac{A \cdot B}{A \cdot n \, B \cdot n}, \tag{10.87}$$

which is homogeneous on A and B. The Euclidean group can now be seen to be the subgroup of the conformal group which leaves n invariant. This is sensible, as the point at infinity should stay there under a Euclidean transformation. The Euclidean group is thus the *stability group* of a null vector in conformal space. The group of generators of reflections and rotations in conformal space which leave n invariant then provide a double cover of the Euclidean group. Equation (10.87) returns the Euclidean distance between points. If the vector n is replaced by e or $\bar{e}$ we can transform to distance measures in hyperbolic or spherical geometry. This makes it a simple exercise to attach different geometric pictures to algebraic results in conformal space.

10.4 Geometric primitives in conformal space

Now that we have seen how points are encoded in conformal space, we can begin to build up more complex geometric objects. As in projective geometry, we expect that a multivector blade L will encode a geometric object via the equation

$$L \wedge X = 0, \qquad X^2 = 0. \tag{10.88}$$

The question, then, is what type of object does each grade of multivector return. One important result we can exploit is that $X^2 = 0$ is unchanged if $X \mapsto RX\tilde{R}$. So, if a geometric object is specified by L via equation (10.88), it follows that

$$R(L \wedge X)\tilde{R} = (RL\tilde{R}) \wedge (RX\tilde{R}) = 0. \tag{10.89}$$

We can therefore transform the object L with a general element of the conformal group to obtain a new object. Similar considerations hold for incidence relations. Since conformal transformations only preserve angles, and do not necessarily map straight lines to straight lines, the range of objects we can describe by simple blades is clearly going to be larger than in projective geometry.

10.4.1 Bivectors and points

A pair of points in Euclidean space are represented by two null vectors in a space of two dimensions higher. We know that the inner product in this space returns information about distances. The next question to ask is what is the significance of the outer product of two vectors. If A and B are null vectors, we form the bivector

$$G = A \wedge B. \tag{10.90}$$

The bivector G has magnitude

$$G^2 = (AB - A \cdot B)(-BA + A \cdot B) = (A \cdot B)^2, \tag{10.91}$$

which shows that G is *timelike*, borrowing the terminology of special relativity. It follows that G contains a pair of null vectors. If we look for solutions to the equation

$$G\wedge X = 0, \qquad X^2 = 0, \tag{10.92}$$

the only solutions are the two null vectors contained in G. These are precisely A and B, so the bivector encodes the two points directly. In the conformal model, no information is lost in forming the exterior product of two null vectors. Spacelike bivectors, with $B^2 < 0$, do not contain any null vectors, so in this case there are no solutions to $B\wedge X = 0$ with $X^2 = 0$. The critical case of $B^2 = 0$ implies that B contains a single null vector.

Given a timelike bivector, $B^2 > 0$, we require an efficient means of finding the two null vectors in the plane. This can be achieved without solving any quadratic equations as follows. Pick an arbitrary vector a, with a partial projection in the plane, $a\cdot B \neq 0$. If the underlying space is Euclidean, one can use the vector $\bar{e}$, since all timelike bivectors contain a factor of this. Now remove the component of a outside the plane by defining

$$a' = a - a\wedge\hat{B}\,\hat{B}, \tag{10.93}$$

where $\hat{B} = B/|B|$ is normalised so that $\hat{B}^2 = 1$. If a' is already null then it defines one of the required vectors. If not, then one can form two null vectors in the B plane by writing

$$A_{\pm} = a' \pm a'\hat{B}. \tag{10.94}$$

One can easily confirm that $A_{\pm}$ are both null vectors, and so return the desired points.

10.4.2 Trivectors, lines and circles

If a bivector now only represents a pair of points, the obvious question is how do we describe a line? Suppose we construct the line through the points a and b in $\mathcal{V}(p,q)$. A point on the line is given by

$$x = \lambda a + (1-\lambda)b. \tag{10.95}$$

The conformal version of this line is

$$\begin{aligned} F(x) &= \big(\lambda^2 a^2 + 2\lambda(1-\lambda)a\cdot b + (1-\lambda)^2 b\big)n + 2\lambda a + 2(1-\lambda)b - \bar{n} \\ &= \lambda A + (1-\lambda)B + \tfrac{1}{2}\lambda(1-\lambda)A\cdot B\, n, \end{aligned} \tag{10.96}$$

and any multiple of this encodes the same point on the line. It is clear, then, that a conformal point X is a linear combination of A, B and n, subject to the constraint that $X^2 = 0$. This is summarised by

$$(A\wedge B\wedge n)\wedge X = 0, \qquad X^2 = 0. \tag{10.97}$$

So it is *trivectors* that represent lines in conformal geometry. This illustrates a general feature of the conformal model — geometric objects are represented by multivectors of one grade higher than their projective counterpart. The extra degree of freedom is absorbed by the constraint that $X^2 = 0$.

As stated above, if we apply a conformal transformation to a trivector representing a line, we must obtain a new line. But there is no reason to expect this to be straight. To see what else can result, consider a simple inversion in the origin. Suppose that (x_1, x_2) denote a pair of Cartesian coordinates for the Euclidean plane, and consider the line $x_1 = 1$. Points on the line have components $(1, x_2)$, with $-\infty \leq x_2 \leq +\infty$. The image of this line under an inversion in the origin has coordinates (x_1', x_2'), where

$$x_1' = \frac{1}{1+x_2^2}, \qquad x_2' = \frac{x_2}{1+x_2^2}. \tag{10.98}$$

It is now straightforward to show that

$$(x_1' - \tfrac{1}{2})^2 + (x_2')^2 = \left(\tfrac{1}{2}\right)^2. \tag{10.99}$$

Hence inversion of a line produces a *circle*, centred on $(1/2, 0)$ and with radius $1/2$.

It follows that a general trivector in conformal space can encode a circle, with a line representing the special case of infinite radius. This is entirely sensible, as three distinct points are required to specify a circle. The points define a plane, and any three non-collinear points in a plane specify a unique circle. So, given three points A_1, A_2, A_3, the circle through all three is defined by

$$A_1 \wedge A_2 \wedge A_3 \wedge X = 0, \tag{10.100}$$

together with the restriction (often unstated) that $X^2 = 0$. The trivector

$$L = A_1 \wedge A_2 \wedge A_3 \tag{10.101}$$

therefore encodes a unique circle in conformal geometry. The test that the points lie on a straight line is that the circle passes through the point at infinity,

$$L \wedge n = 0 \quad \Rightarrow \quad \text{straight line}. \tag{10.102}$$

This explains why our earlier derivation of the line through A_1 and A_2 led to the trivector $A_1 \wedge A_2 \wedge n$, which explicitly includes the point at infinity. Unlike tests for linear dependence, testing for zero in equation (10.102) is numerically acceptable. The reason is that the magnitude of $L \wedge n$ controls the deviation from straightness. If precision is limited, one can then define how close $L \wedge n$ should be to zero in order for the line to be treated as straight. This is quite different to linear independence, where the concept of 'nearly independent' makes no sense.

Given that a trivector L encodes a circle, we should expect to be able to extract the key geometric properties of the circle directly from L. In particular, we seek

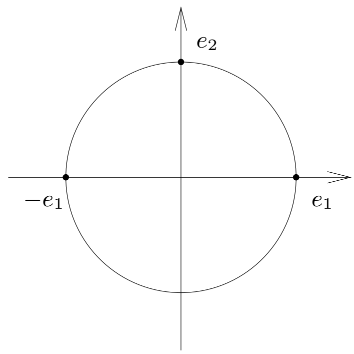

Figure 10.7 *The unit circle.* Three reference points are marked on the circle.

expressions for the centre and radius of the circle. (The plane containing the circle is specified by the 4-vector $L \wedge n$, as we explain in the following section.) Any circle in a plane can be mapped onto any other by a translation and a dilation. Under that latter we find that

$$L \wedge n \mapsto (D_\alpha L \tilde{D}_\alpha) \wedge n = \mathrm{e}^{\alpha} D_\alpha (L \wedge n) \tilde{D}_\alpha. \tag{10.103}$$

It follows that $(L \wedge n)^2$ scales as the inverse square of the radius. Next, consider the unit circle in the circle in the xy plane, and take as three points on the circle those shown in figure 10.7. The trivector for this circle is

$$L_0 = F(e_1) \wedge F(e_2) \wedge F(-e_1) = 16 e_1 e_2 \bar{e}. \tag{10.104}$$

It follows that

$$\frac{L_0^2}{(L_0 \wedge n)^2} = -1, \tag{10.105}$$

which is (minus) the square of the radius of the unit circle. We can translate and dilate this into any circle we choose, so the radius ρ of the circle encoded by the trivector L is given by

$$\rho^2 = -\frac{L^2}{(L \wedge n)^2}. \tag{10.106}$$

This is a further illustration of how metric information is carried around in the homogeneous framework of the conformal model. If L represents a straight line we know that $L \wedge n = 0$, so the radius we obtain is infinite.

Similar reasoning produces a formula for the centre of a circle. Essentially the only objects we have to work with are L and n. If we form LnL for the case of the unit circle we obtain

$$L_0 n L_0 \propto e_1 e_2 \bar{e} n \bar{e} e_1 e_2 = -\bar{n}. \tag{10.107}$$

But $\bar{n}$ is the null vector for the origin, so this expression has returned the desired point. Again, we can translate and dilate this result to obtain an arbitrary circle, and we find in general that the centre C of the circle L is obtained by

$$C = LnL. \tag{10.108}$$

We will see in section 10.5.5 that the operation $L \ldots L$ generates a reflection in a circle. Equation (10.108) then says that the centre of a circle is the image of the point at infinity under a reflection in the circle.

10.4.3 4-vectors, spheres and planes

We can apply the same reasoning for lines and circles to the case of planes and spheres and, for mixed signature spaces, hyperboloids. Suppose initially that the points a, b, c define a plane in $\mathcal{V}(p,q)$, so that an arbitrary point in the plane is given by

$$x = \alpha a + \beta b + \gamma c, \qquad \alpha + \beta + \gamma = 1. \tag{10.109}$$

The conformal representation of x is

$$X = \alpha A + \beta B + \gamma C + \delta n, \tag{10.110}$$

where $A = f(a)$ etc., and

$$\delta = \tfrac{1}{2}(\alpha\beta A\cdot B + \alpha\gamma A\cdot C + \beta\gamma B\cdot C). \tag{10.111}$$

Varying α and β, together with the freedom to scale $F(x)$, now produces general null combinations of the vectors A, B, C and n. The equation for the plane can then be written

$$A\wedge B\wedge C\wedge n\wedge X = 0. \tag{10.112}$$

The plane passes through the points defined by A, B, C and the point at infinity n. We can therefore see that a general plane in conformal space is defined by four points.

If the four points in question do not lie on a (flat) plane, then the 4-vector formed from their outer product defines a *sphere*. To see this we again consider inversion in the origin, this time applied to the $x_1 = 1$ plane. A point on the plane has coordinates $(1, x_2, x_3)$, and under an inversion this maps to the point with coordinates

$$x_1' = \frac{1}{1+x_2^2+x_3^2}, \quad x_2' = \frac{y}{1+x_2^2+x_3^2}, \quad x_3' = \frac{z}{1+x_2^2+x_3^2}. \tag{10.113}$$

The new coordinates satisfy

$$\left(x_1' - \tfrac{1}{2}\right)^2 + (x_2')^2 + (x_3')^2 = \left(\tfrac{1}{2}\right)^2, \tag{10.114}$$

which is the equation of a sphere. Inversion thus interchanges planes and spheres. In particular, the point at infinity n is transformed to the origin $\bar{n}$ under inversion, which is now one of the points on the sphere.

Given any four distinct points $A_1, \dots, A_4$, not all on a line or circle, the equation of the unique sphere through all four points is

$$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge X = P \wedge X = 0, \tag{10.115}$$

so the sphere is defined by the 4-vector $P = A_1 \wedge A_2 \wedge A_3 \wedge A_4$. The sphere is flat (a plane) if it passes through the point at infinity, the test for which is

$$A_1 \wedge A_2 \wedge A_3 \wedge A_4 \wedge n = P \wedge n = 0. \tag{10.116}$$

The 4-vector P contains all of the relevant geometric information for a sphere. The radius of the sphere ρ is given by

$$\rho^2 = \frac{P^2}{(P \wedge n)^2}, \tag{10.117}$$

as is easily confirmed for the case of the unit sphere, $P = e_1 e_2 e_3 \bar{e}$. Similarly, the centre of the sphere $C = F(c)$ is given by

$$C = PnP. \tag{10.118}$$

These formulae are the obvious generalisations of the results derived for circles.

10.5 Intersection and reflection in conformal space

One of the most significant advantages of the conformal approach to Euclidean geometry is the ease with which it solves complicated intersection problems. So, for example, finding the circle of intersection of two spheres is now no more complicated than finding the line of intersection of two planes. In addition, the concept of reflection is generalised in conformal space to include reflection in a sphere. This provides a very compact means of encoding the key concepts of inversive geometry.

10.5.1 Duality in conformal space

The concept of duality is key to intersecting objects in projective space, and the same is true in conformal space. Suppose that we start with the Euclidean plane, modelled in $\mathcal{G}(3,1)$. Duality in this algebra interchanges spacelike and timelike bivectors. It also maps trivectors to vectors, and vice versa. A trivector encodes a line, or circle, so the dual of the circle C is a vector c, where

$$c = C^* = IC \tag{10.119}$$

and I is the pseudoscalar for $\mathcal{G}(3,1)$. The equation for the circle, $X\wedge C = 0$, can now be written in dual form and reduces to

$$X\cdot c = -I(X\wedge C) = 0. \tag{10.120}$$

The radius of the circle is now given by

$$\rho^2 = \frac{c^2}{(c\cdot n)^2}, \tag{10.121}$$

as the vector dual to a circle has positive signature. This picture provides us with an alternative view of the concept of a point as being a circle of zero radius.

Similar considerations hold for spheres in three-dimensional space. These are represented as 4-vectors in $\mathcal{G}(4,1)$, so their dual is a vector. We write

$$s = S^* = IS, \tag{10.122}$$

where I is the pseudoscalar, so that the equation of a sphere becomes

$$X\cdot s = I(X\wedge S) = 0. \tag{10.123}$$

The radius of the sphere is again given by

$$\rho^2 = \frac{s^2}{(s\cdot n)^2}, \tag{10.124}$$

so that points are spheres of zero radius. One can see that this is sensible by considering an alternative equation for a sphere. Suppose we are interested in the sphere with centre C and radius ρ^2. The equation for this can be written

$$-2\frac{X\cdot C}{X\cdot n\, C\cdot n} = \rho^2. \tag{10.125}$$

Rearranging, this equation becomes

$$X\cdot(2C + \rho^2 C\cdot n\, n) = 0, \tag{10.126}$$

and if C is in standard form, $C = F(c)$, we obtain

$$X\cdot(F(c) - \rho^2 n) = 0. \tag{10.127}$$

We can therefore identify $s = S^*$ with the vector $F(c) - \rho^2 n$, which neatly encodes the centre and radius of the sphere in a single vector. Whether the 4-vector S or its dual vector s is most useful depends on whether the sphere is specified by four points lying on it, or by its centre and radius. For a given sphere s we can now write

$$s = \lambda(2C + \rho^2 C\cdot n\, n). \tag{10.128}$$

It is then straightforward to confirm that the radius is given by equation (10.124). The centre of the circle can be recovered from

$$\frac{C}{C\cdot n} = \frac{s}{s\cdot n} - \frac{\rho^2}{2}n = \frac{sns}{2(s\cdot n)^2}. \tag{10.129}$$

The sns form for the centre of a sphere is dual to the SnS expression found in equation (10.118).

10.5.2 Intersection of two lines in a plane

As a simple example of intersection in the conformal model, consider the intersection of two lines in a Euclidean plane. The lines are described by trivectors L_1 and L_2 in $\mathcal{G}(3,1)$. The intersection is described by the bivector

$$B = (L_1^* \wedge L_2^*)^* = I(L_1 \times L_2), \tag{10.130}$$

where I is the conformal pseudoscalar. The bivector B can contain zero, one or two points, depending on the sign of its square, as described in section 10.4.1. This is to be expected, as distinct circles can intersect at a maximum of two points. If the lines are both straight, then one of the points of intersection will be at infinity, and $B \wedge n = 0$.

To verify this result, consider the case of two straight lines, both passing through the origin, and with the first line in the a direction and the second in the b direction. With suitable normalisation we can write

$$L_1 = aN, \qquad L_2 = bN, \tag{10.131}$$

where $N = e\bar{e}$. The intersection of L_1 and L_2 is controlled by

$$B = I\, a \wedge b \propto N \tag{10.132}$$

and the bivector N contains the null vectors n and $\bar{n}$. This confirms that the lines intersect at the origin and infinity. Applying conformal transformations to this result ensures that it holds for all lines in a plane, whether the lines are straight or circular. The formulae for L_1 and L_2 also show that their inner product is related to the angle between the lines,

$$\langle L_1 L_2 \rangle = a\cdot b. \tag{10.133}$$

We can therefore write

$$\cos(\theta) = \frac{\langle L_1 L_2 \rangle}{|L_1|\,|L_2|}, \tag{10.134}$$

where $|L| = \sqrt{(L^2)}$. This equation returns the angle between two lines. The quantity is invariant under the full conformal group, and not just the Euclidean group, because angles are conformal invariants. It follows that the same formula must hold even if L_1 and L_2 describe circles. The angle between two circles is the angle made by their tangent vectors at the point of intersection. Two circles intersect at a right angle, therefore, if

$$\langle L_1 L_2 \rangle = 0. \tag{10.135}$$

This result can equally be expressed in terms of the dual vectors l_1 and l_2.

10.5.3 Intersection of a line and a surface

Now suppose that the 4-vector P defines a plane or sphere in three-dimensional Euclidean space, and we wish to find the point of intersection with a line described by the trivector L. The algebra proceeds entirely as expected and we arrive at the bivector

$$B = (P^* \wedge L^*)^* = (IP) \cdot L = I \langle PL \rangle_3. \tag{10.136}$$

This bivector can again describe zero, one or two points, depending on the sign of its square. This setup describes all possible intersections between lines or circles, and planes or spheres — an extremely wide range of applications. Precisely the same algebra enables us to answer whether a ring in space intersects a given plane, or whether a straight line passes through a sphere.

10.5.4 Surface intersections

Next, suppose we wish to intersect two surfaces in three dimensions. Suppose that these are spheres defined by the 4-vectors S_1 and S_2. Their intersection is described by the trivector

$$L = I(S_1 \times S_2). \tag{10.137}$$

This trivector directly encodes the circle formed from the intersection of two spheres. As with the bivector case, the sign of L^2 defines whether or not two surfaces intersect. If $L^2 > 0$ then the surfaces do intersect. If $L^2 = 0$ then the surfaces intersect at a point. Tests such as this are extremely helpful in graphics applications.

We can similarly express the intersection in terms of the dual vectors s_1 and s_2 as

$$L = I\, s_1 \wedge s_2. \tag{10.138}$$

As a check, the point X lies on both spheres if

$$X \cdot s_1 = X \cdot s_2 = 0. \tag{10.139}$$

It follows that

$$X \cdot (s_1 \wedge s_2) = X \cdot s_1\, s_2 - X \cdot s_2\, s_1 = 0. \tag{10.140}$$

The dual result is that $X\wedge(I\, s_1\wedge s_2) = 0$, which confirms that X lies in the space defined by the trivector L.

10.5.5 Reflections in conformal space

At various points in previous sections we have obtained formulae which generate reflections. We now discuss these more systematically. In section 2.6 we established that the vector obtained by reflecting a in the hyperplane perpendicular to l, $l^2 = 1$, is $-lal$. But this formula assumes that the line and plane intersect at the origin. We seek a more general expression, valid for an arbitrary line and plane. Let P denote the plane and L the line we wish to reflect in the plane, then the obvious candidate for the reflected line L' is

$$L' = PLP. \tag{10.141}$$

(The sign of this is irrelevant in conformal space.) To verify that this is correct, suppose that L passes through the origin in the a direction,

$$L = aN_3 \tag{10.142}$$

and the plane P is defined by the origin and the directions b and c,

$$P = b\wedge cN. \tag{10.143}$$

In this case

$$L' = b\wedge c\, a\, b\wedge c\, N = \big(-(I_3\, b\wedge c)a(I_3\, b\wedge c)\big)N, \tag{10.144}$$

where I_3 is the three-dimensional pseudoscalar. This result achieves the required result. The vector a is reflected in the $b\wedge c$ plane to obtain the desired direction. The outer product with N then defines the line through the origin with the required direction. Equation (10.141) is correct at the origin, so therefore holds for all lines and planes, by conformal invariance.

There are a number of significant consequences of equation (10.141). The first is that it recovers the correct line in three dimensions without having to to find the point of reflection. The second is that it is straightforward to chain together multiple reflections by forming successive products with planes. In this way complicated reflections can be easily composed, all the time keeping track of the direction and position of the resultant line. A further consequence is that the same reflection formula must hold for higher dimensional objects. Suppose, for example, we wish to reflect the sphere S in the plane P. The result is

$$S' = PSP. \tag{10.145}$$

This type of equation is extremely useful in dealing with wave propagation, where a wavefront is modelled as a series of expanding spheres.

Conformal invariance of the reflection formula (10.141) ensures that the same formula holds for reflection in a circle, or in a sphere. For example, suppose we wish to carry out a reflection in the unit circle in two-dimensional Euclidean space. The circle is defined by $L_0 = e_1 e_2 \bar{e}$, and the dual vector is

$$IL_0 = e. \tag{10.146}$$

Reflection in the unit circle is therefore performed by the operation

$$M \mapsto eMe. \tag{10.147}$$

This is an inversion, as discussed in section 10.3.3. In this manner, the main results of inversive geometry are easily formulated in terms of reflections in conformal space.

10.6 Non-Euclidean geometry

The sudden growth in the subject of geometry in the nineteenth century was stimulated in part by the discovery of geometries with very different properties to Euclidean space. These were obtained by a simple modification of Euclid's *parallel postulate*. For Euclidean geometry this states that, given any line l and a point P not on the line, there exists a unique line through P in the plane of l and P which does not meet l. This is then a line parallel to l. For many centuries this postulate was viewed as problematic, as it cannot be easily experimentally verified. As a result, mathematicians attempted to remove the parallel postulate by proving it from the remaining, uncontroversial, postulates of Euclidean geometry. This enterprise proved fruitless, and the reason why was discovered by Lobachevskii and Bolyai in the 1820s. One can replace the parallel postulate with a different postulate, and obtain a new, mathematically acceptable geometry.

There are in fact two alternative geometries one can obtain, by replacing the statement that there is a *single* line through P which does not intersect l with either an infinite number or zero. The case of an infinite number produces *hyperbolic* geometry, which is the non-Euclidean geometry constructed by Lobachevskii and Bolyai. (In this section 'non-Euclidean' usually refers to the hyperbolic case.) The case of zero lines produces spherical geometry. Intuitively, the spherical case corresponds to space curling up, so that all (straight) lines meet somewhere, and the hyperbolic case corresponds to space curving outwards, so that lines do not meet. From the more modern perspective of Riemannian geometry, we are talking about homogeneous, isotropic spaces, which have no preferred points or directions. These can have positive, zero or negative curvature, corresponding to spherical, Euclidean and hyperbolic geometries. Today, the question of which of these correctly describes the universe on the largest scales remains an outstanding problem in cosmology.

An extremely attractive feature of the conformal model of Euclidean geometry

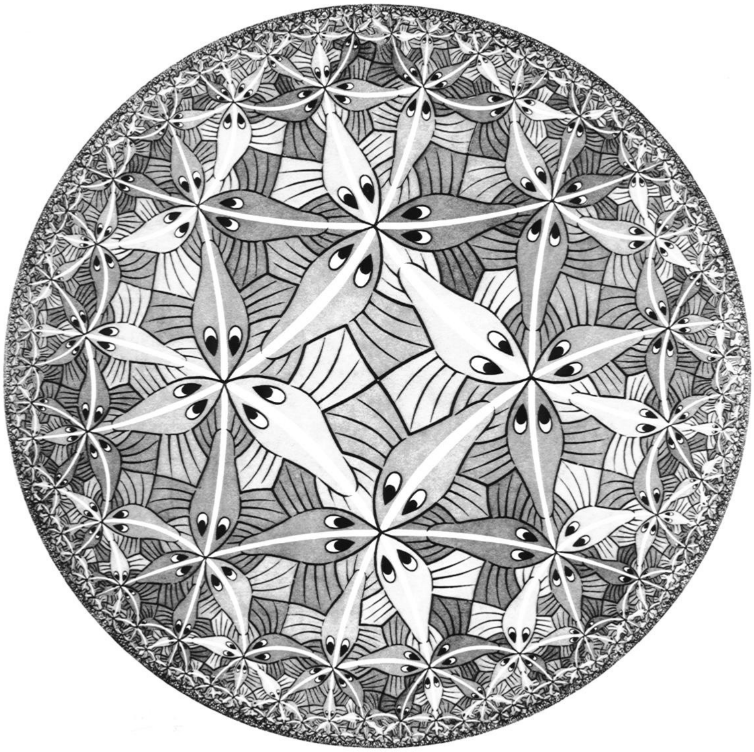

Figure 10.8 *Circle limit III* by Maurits Escher. ©2002 Cordon Art B.V., Baarn, Holland.

is that, with little modification, it can be applied to both hyperbolic and spherical geometries as well. In essence, the geometry reduces to a choice of the point at infinity, which in turn fixes the distance measure. This idea replaces the concept of the *absolute conic*, adopted in classical projective geometry as a means of imposing a distance measure. In this section we illustrate these ideas with a discussion of the conformal approach to planar hyperbolic geometry. As a concrete model of this we concentrate on the Poincaré disc. This version of hyperbolic geometry is mathematically very appealing, and also gives rise to some beautiful graphic designs, as popularised in the prints of Maurits Escher (see figure 10.8).

10.6.1 The Poincaré disc

The Poincaré disc $\mathcal{D}$ consists of the set of points in the plane a distance $r < 1$ from the origin. At first sight this may not appear to be homogeneous, but in

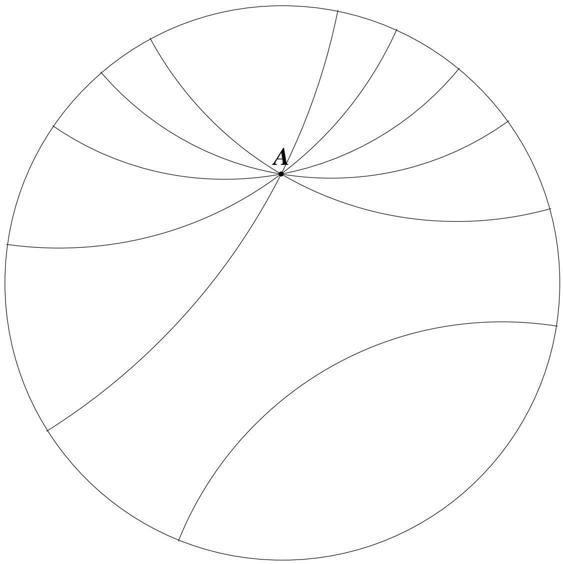

Figure 10.9 *The Poincaré disc.* Points inside the disc represent points in a hyperbolic space. A set of d-lines are also shown. These are (Euclidean) circles that intersect the unit circle at right angles. The d-lines through A illustrate the parallel postulate for hyperbolic geometry.

fact the nature of the geometry will ensure that there is nothing special about the origin. Note that points on the unit circle $r = 1$ are *not* included in this model of hyperbolic geometry. The key to this geometry is the concept of a non-Euclidean straight line. These are called d-lines, and represent geodesics in hyperbolic geometry. A d-line consists of a section of a Euclidean circle which intersects the unit circle at a right angle. Examples of d-lines are illustrated in figure 10.9. Given any two points in the Poincaré disc there is a unique d-line through them, which represents the 'straight' line between the points. It is now clear that for any point not on a given d-line l, there are an infinite number of d-lines through the point which do not intersect l.

We can now begin to encode these concepts in the conformal setting. We continue to denote points in the plane with homogeneous null vectors in precisely the same manner as the Euclidean case. Suppose, then, that X and Y are the conformal vectors representing two points in the disc. The set of all circles through these two points consists of trivectors of the form $X \wedge Y \wedge A$, where A is an additional point. But we require that the d-line intersects the unit circle at right angles. The unit circle is described by the trivector Ie, where I is the pseudoscalar in $\mathcal{G}(3,1)$. If a line L is perpendicular to the unit circle it satisfies

$$(Ie)\cdot L = I(e\wedge L) = 0. \tag{10.148}$$

It follows that all d-lines contain a factor of e. The d-line through X and Y must therefore be described by the trivector

$$L = X\wedge Y\wedge e. \tag{10.149}$$

One can see now that a general scheme is beginning to emerge. Everywhere in the Euclidean treatment that the vector n appears it is replaced in hyperbolic geometry by the vector e. This vector represents the circle at infinity.

Given a pair of d-lines, they can either miss each other, or intersect at a point in the disc $\mathcal{D}$. If they intersect, the angle between the lines is given by the Euclidean formula

$$\cos(\theta) = \frac{L_1\cdot L_2}{|L_1|\,|L_2|}. \tag{10.150}$$

It follows that angles are preserved by a general conformal transformation in hyperbolic geometry. A non-Euclidean transformation takes d-lines to d-lines. The transformation must therefore map (Euclidean) circles to circles, while preserving orthogonality with e. The group of non-Euclidean transformations must therefore be the subgroup of the conformal group which leaves e invariant. This is confirmed in the following section, where we find the appropriate distance measure for non-Euclidean geometry.

The fact that the point at infinity is represented by e, as opposed to n in the Euclidean counterpart, provides an additional operation in non-Euclidean geometry. This is inversion in e:

$$X \mapsto eXe. \tag{10.151}$$

As all non-Euclidean transformations leave e invariant, all geometric relations remain unchanged under this inversion. Geometrically, the interpretation of the inversion is quite clear. It maps everything inside the Poincaré disc to a 'dual' version outside the disc. In this dual space incidence relations and distances are unchanged from their counterparts inside the disc.

10.6.2 Non-Euclidean translations and distance

The key to finding the correct distance measure in non-Euclidean geometry is to first generalise the concept of a translation. Given points X and Y we know that the d-line connecting them is defined by $X\wedge Y\wedge e$. This is the non-Euclidean concept of a straight line. A non-Euclidean translation must therefore move points along this line. Such a transformation must take X to Y, but must also leave e invariant. The generator for such a transformation is the bivector

$$B = (X\wedge Y\wedge e)e = Le, \tag{10.152}$$

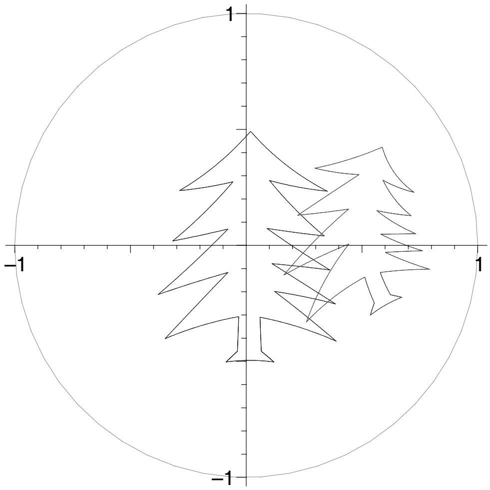

Figure 10.10 *A non-Euclidean translation.* The figure near the origin is translated via a boost to give the distorted figure on the right. This distortion in the Poincaré disc is one way of visualising the effect of a Lorentz boost in spacetime.

where $L = X \wedge Y \wedge e$. We find immediately that

$$B^2 = L^2 > 0, \tag{10.153}$$

so non-Euclidean translations are *hyperbolic* transformations, as one might expect. An example of such a translation is shown in figure 10.10.

We next define

$$\hat{B} = \frac{B}{|B|}, \qquad \hat{B}^2 = 1, \tag{10.154}$$

so that we can write

$$Y = \mathrm{e}^{\alpha \hat{B}/2} X \mathrm{e}^{-\alpha \hat{B}/2}. \tag{10.155}$$

By varying α we obtain the set of points along the d-line through X and Y. To obtain a distance measure, we first require a formula for α. If we decompose X into

$$X = X\hat{B}^2 = X \cdot \hat{B}\, \hat{B} + X \wedge \hat{B}\, \hat{B} \tag{10.156}$$

we obtain

$$Y = X \wedge \hat{B}\, \hat{B} + \cosh(\alpha)\, X \cdot \hat{B}\, \hat{B} - \sinh(\alpha)\, X \cdot \hat{B}. \tag{10.157}$$

The right-hand side must give zero when contracted with Y, so

$$\langle X \wedge \hat{B}\, \hat{B} \wedge Y \rangle + \cosh(\alpha) \langle X \cdot \hat{B}\, \hat{B} \cdot Y \rangle + \sinh(\alpha)\, (X \wedge Y) \cdot \hat{B} = 0. \tag{10.158}$$

To simplify this equation we first find

$$X\wedge\hat{B} = \frac{X\wedge(X\wedge Y\wedge e\,e)}{|B|} = \frac{e\cdot X\,L}{|L|} \tag{10.159}$$

and

$$(X\wedge Y)\cdot\hat{B} = \frac{L^2}{|B|} = |L|. \tag{10.160}$$

It follows that

$$e\cdot X\,e\cdot Y + \cosh(\alpha)(X\cdot Y - e\cdot X\,e\cdot Y) + \sinh(\alpha)\,|L| = 0, \tag{10.161}$$

the solution to which is

$$\cosh(\alpha) = 1 - \frac{X\cdot Y}{X\cdot e\,Y\cdot e}. \tag{10.162}$$

The half-angle formula is more relevant for the distance measure, and we find that

$$\sinh^2(\alpha/2) = -\frac{X\cdot Y}{2X\cdot e\,Y\cdot e}. \tag{10.163}$$

This closely mirrors the Euclidean expression, with n replaced by e.

There are a number of obvious properties that a distance measure must satisfy. Among these is the additive property that

$$d(X_1,X_2) + d(X_2,X_3) = d(X_1,X_3) \tag{10.164}$$

for any three points X_1, X_2, X_3 in this order along a d-line. Returning to the translation formula of equation (10.155), suppose that Z is a third point along the line, beyond Y. We can write

$$Z = \mathrm{e}^{\beta\hat{B}/2}Y\mathrm{e}^{-\beta\hat{B}/2} = \mathrm{e}^{(\alpha+\beta)\hat{B}}X\mathrm{e}^{-(\alpha+\beta)\hat{B}/2}. \tag{10.165}$$

Clearly it is hyperbolic angles that must form the appropriate distance measure. No other function satisfies the additive property. We therefore define the non-Euclidean distance by

$$d(x,y) = 2\sinh^{-1}\left(-\frac{X\cdot Y}{2X\cdot e\,Y\cdot e}\right)^{1/2}. \tag{10.166}$$

In terms of the position vectors x and y in the Poincaré disc we can write

$$d(x,y) = 2\sinh^{-1}\left(\frac{|x-y|^2}{(1-x^2)(1-y^2)}\right)^{1/2}, \tag{10.167}$$

where the modulus refers to the Euclidean distance. The presence of the arcsinh function in the definition of distance reflects the fact that, in hyperbolic geometry, generators of translations have positive square and the appropriate distance measure is the hyperbolic angle. Similarly, in spherical geometry translations correspond to rotations, and it is the trigonometric angle which plays the role of distance. Euclidean geometry is therefore unique in that the generators of translations are *null* bivectors. For these, combining translations reduces to the addition of bivectors, and hence we recover the standard definition of Euclidean distance.

10.6.3 Metrics and physical units

The derivation of the non-Euclidean distance formula of equation (10.166) forces us to face an issue that has been ignored to date. Physical distances are dimensional quantities, whereas our formulae for distances in both Euclidean and non-Euclidean geometries are manifestly dimensionless, as they are homogeneous in X. To resolve this we cannot just demand that the vector x has dimensions, as this would imply that the conformal vector X contained terms of mixed dimensions. Neither can this problem be circumvented by assigning dimensions of distance to $\bar{n}$ and $(\text{distance})^{-1}$ to n, as then e has mixed dimensions, and the non-Euclidean formula of (10.166) is non-sensical.

The resolution is to introduce a fundamental length scale, λ, which is a positive scalar with the dimensions of length. If the vector x has dimensions of length, the conformal representation is then given by

$$X = \frac{1}{2\lambda^2}\left(x^2 n + 2\lambda x - \lambda^2 \bar{n}\right). \tag{10.168}$$

This representation ensures that X remains dimensionless, and is nothing more than the conformal representation of x/λ. Physical distances can then be converted into a dimensionally meaningful form by including appropriate factors of λ. Curiously, the introduction of λ into the spacetime conformal model has many similarities to the introduction of a cosmological constant $\Lambda = \lambda^2$.

We can make contact with the metric encoding of distance by finding the infinitesimal distance between the points x and $x + dx$. This defines the line element

$$ds^2 = 4\lambda^4 \frac{dx^2}{(\lambda^2 - x^2)^2}, \tag{10.169}$$

where the factors of λ have been included and x is assumed to have dimensions of distance. This line element is more often seen in polar coordinates, where it takes the form

$$ds^2 = \frac{4\lambda^4}{(\lambda^2 - r^2)^2}(dr^2 + r^2 d\theta^2). \tag{10.170}$$

This is the line element for a space of constant negative curvature, expressed in terms of conformal coordinates. The coordinates are conformal because the line element is that of a flat space multiplied by a scaling function. The geodesics in this geometry are precisely the d-lines in the Poincaré disc. The Riemann curvature for this metric shows that the space has uniform negative curvature, so the space is indeed homogeneous and isotropic — there are no preferred points or directions. The centre of the disc is not a special point, and indeed it can be translated to any other point by 'boosting' along a d-line.

10.6.4 Midpoints and circles in non-Euclidean geometry

Now that we have a conformal encoding of a straight line and of distance in non-Euclidean geometry, we can proceed to discuss concepts such as the midpoint of two points, and of the set of points a constant distance from a given point (a non-Euclidean circle). Suppose that A and B are the conformal vectors of two points in the Poincaré disc. Their midpoint C lies on the line $L = A \wedge B \wedge e$ and is equidistant from both A and B. The latter condition implies that

$$\frac{C \cdot A}{C \cdot e \, A \cdot e} = \frac{C \cdot B}{C \cdot e \, B \cdot e}. \tag{10.171}$$

Both of the conditions for C are easily satisfied by setting

$$C = \frac{A}{2A \cdot e} + \frac{B}{2B \cdot e} + \alpha e, \tag{10.172}$$

where α must be chosen such that $C^2 = 0$. Normalising to $C \cdot e = -1$ we find that the midpoint is

$$C = -\frac{1}{\sqrt{1+\delta}} \left(\frac{A}{2A \cdot e} + \frac{B}{2B \cdot e} + \big(\sqrt{1+\delta} - 1\big) e \right), \tag{10.173}$$

where

$$\delta = -\frac{A \cdot B}{2A \cdot e \, B \cdot e}. \tag{10.174}$$

An equation such as this is rather harder to achieve without access to the conformal model.

Next suppose we wish to find the set of points a constant (non-Euclidean) distance from the point C. This defines a non-Euclidean circle with centre C. From equation (10.166), any point X on the circle must satisfy

$$-\frac{X \cdot C}{2X \cdot e \, C \cdot e} = \text{constant} = \alpha^2, \tag{10.175}$$

so that the radius is $\sinh^{-1}(\alpha)$. It follows that

$$X \cdot (C + 2\alpha^2 C \cdot e \, e) = 0. \tag{10.176}$$

If we define s by

$$s = C + 2\alpha^2 C \cdot e \, e \tag{10.177}$$

we see that $s^2 > 0$, and the circle is defined by $X \cdot s = 0$. But this is precisely the formula for a circle in Euclidean geometry, so non-Euclidean circles still appear as ordinary circles when plotted in the Poincaré disc. The only difference is the

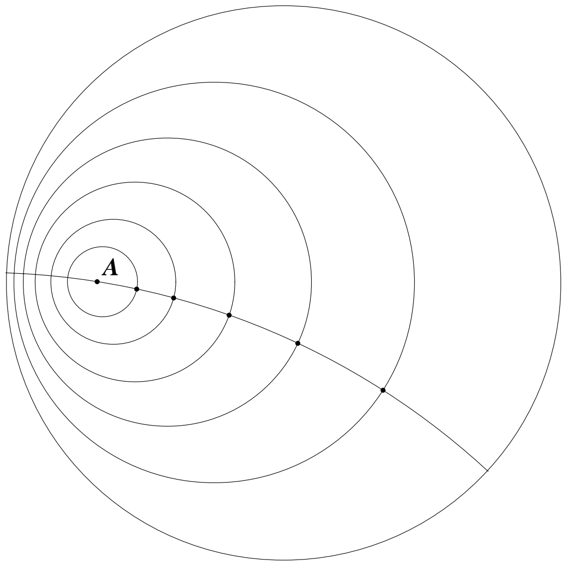

Figure 10.11 *Non-Euclidean circles.* A series of non-Euclidean circles with differing radii are shown, all about the common centre A. A d-line through A is also shown. This intersects each circle at a right angle.

interpretation of their centre. The Euclidean centre of the circle s, defined by sns, does not coincide with the non-Euclidean centre C. This is illustrated in figure 10.11.

Suppose that A, B and C are three points in the Poincaré disc. We can still define the line L through these points by

$$L = A \wedge B \wedge C, \tag{10.178}$$

and this defines the circle through the three points regardless of the geometry we are working in. All that is different in the two geometries is the position of the midpoint and the size of the radius. The test that the three points lie on a d-line is simply that $L \wedge e = 0$. Again, the Euclidean formula holds, but with n replaced by e. Similar comments apply to other operations in conformal space, such as reflection. Given a line L, points are reflected in this line by the map $X \mapsto LXL$. This formula is appropriate in both Euclidean and non-Euclidean geometry. In the non-Euclidean case it is not hard to verify that LXL corresponds to first finding the d-line through X intersecting L at right angles, and then finding the point on this line an equal non-Euclidean distance on the other side. This is as one would expect for the definition of reflection in a line.

10.6.5 A unified framework for geometry

We have so far seen how Euclidean and hyperbolic geometries can both be handled in terms of null vectors in conformal space. The key concept is the vector representing the point at infinity, which remains invariant under the appropriate symmetry group. The full conformal group of a space with signature (p, q) is the orthogonal group $O(p+1, q+1)$. The group of Euclidean transformations is the subgroup of $O(p+1, q+1)$ that leaves the vector n invariant. The hyperbolic group is the subgroup of $O(p+1, q+1)$ which leaves e invariant. For the case of planar geometry, with signature $(2, 0)$, the hyperbolic group is $O(2, 1)$. The Killing form for this group is non-degenerate (see chapter 11), which makes hyperbolic geometry a useful way of compactifying a flat space.

The remaining planar geometry to consider is spherical geometry. By now, it should come as little surprise that spherical geometry is handled in the conformal framework in terms of transformations which leave the vector $\bar{e}$ invariant. For the case of the plane, the conformal algebra has signature $(3, 1)$, with $\bar{e}$ the basis vector with negative signature. The subgroup of the conformal group which leaves $\bar{e}$ invariant is therefore the orthogonal group $O(3, 0)$, which is the group one expects for a 2-sphere. The distance measure for spherical geometry is

$$d(x, y) = 2\lambda \sin^{-1}\left(-\frac{X\cdot Y}{2X\cdot\bar{e}\, Y\cdot\bar{e}}\right)^{1/2}, \qquad (10.179)$$

with $\bar{e}$ replacing n in the obvious manner. To see that this expression is correct, suppose that we write

$$\frac{X}{X\cdot\bar{e}} = \hat{x} - \bar{e}, \qquad (10.180)$$

where $\hat{x}$ is a unit vector built in the three-dimensional space spanned by the vectors e_1, e_2 and e. With $Y/Y\cdot\bar{e}$ written in the same way we find that

$$-\frac{X\cdot Y}{2X\cdot\bar{e}\, Y\cdot\bar{e}} = \frac{1 - \hat{x}\cdot\hat{y}}{2} = \sin^2(\theta/2), \qquad (10.181)$$

where θ is the angle between the unit vectors on the 2-sphere. The distance measure is then precisely the angle θ multiplied by the dimensional quantity λ, which represents the radius of the sphere.

Conformal geometry provides a unified framework for the three types of planar geometry because in all cases the conformal groups are the same. That is, the group of transformations of sphere that leave angles in the sphere unchanged is the same as for the plane and the hyperboloid. In all cases the group is $O(3, 1)$. The geometries are then recovered by a choice of distance measure. In classical projective geometry the distance measure is defined by the introduction of the *absolute conic*. All lines intersect this conic in a pair of points. The distance between two points A and B is then found from the four-point ratio between A, B, and the two points of intersection of the line through A and B and the absolute conic. In this way all geometries are united in the framework of projective geometry. But there is a price to pay for this scheme — all coordinates have to be complex, to ensure that all lines intersect the conic in two points. Recovering a real geometry is then rather clumsy. In addition, the conformal group is not a subgroup of the projective group, so much of the elegant unity exhibited by the three geometries is lost. Conformal geometry is a more powerful framework for a unified treatment of these geometries. Furthermore, the conformal approach can be applied to spaces of any dimension with little modification. Trivectors represent lines and circles, 4-vectors represent planes and spheres, and so on.

So far we have restricted ourselves to a single view of the various geometries, but the discussion of the sphere illustrates that there are many different ways of representing the underlying geometry. To begin with, we have plotted points on the Euclidean plane according the the formula

$$x = -\frac{X \wedge N}{X \cdot n} N, \tag{10.182}$$

where $N = e\bar{e}$. This is the natural scheme for plotting on a Euclidean piece of paper, as it ensures that the angle between lines on the paper is the correct angle in each of the three geometries. Euclidean geometry plotted in this way recovers the obvious standard picture of Euclidean geometry. Hyperbolic geometry led to the Poincaré disc model, in which hyperbolic lines appear as circles. For spherical geometry the 'straight lines' are great circles on a sphere. On the plane these also plot as circles. This time the condition is that all circles intersect the unit circle at antipodal points. This then defines the spherical line between two points (see figure 10.12). This view of spherical geometry is precisely that obtained from a stereographic projection of the sphere onto the plane. This is not a surprise, as the conformal model was initially constructed in terms of a stereographic projection, with the $\bar{e}$ vector then enabling us to move to a homogeneous framework. In this representation of spherical geometry the map

$$X \mapsto \bar{e} X \bar{e} \tag{10.183}$$

is a symmetry operation. This maps points to their antipodal opposites on the sphere. In the planar view this transformation is an inversion in the unit circle, followed by a reflection in the origin.

We now have three separate geometries, all with conformal representations in the plane such that the true angle between lines is the same as that measured on the plane. The price for such a representation is that straight lines in spherical and hyperbolic geometries do not appear straight in the plane. But we could equally choose to replace the map of equation (10.182) with an alternative rule of how to plot the null vector X on a planar piece of paper. The natural alternatives to consider are replacing the vector n with e and $\bar{e}$. In total we then have three different planar realisations of each of the two-dimensional geometries. First,

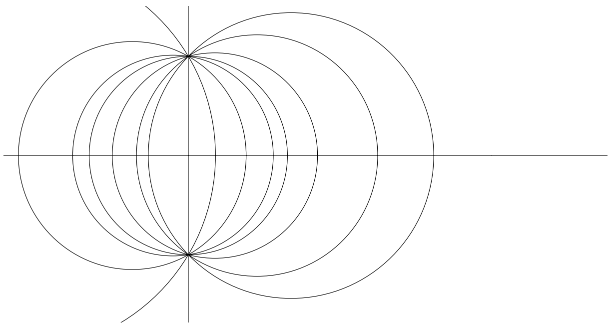

Figure 10.12 *Stereographic view of spherical geometry.* All great circles on the 2-sphere project onto circles in the plane which intersect the unit circle (shown in bold) at antipodal points. A series of such lines are shown.

suppose we define

$$y = \frac{X \wedge N}{X \cdot e} N. \tag{10.184}$$

In terms of the vector x we have

$$y = \frac{2x}{1 - x^2}, \tag{10.185}$$

which represents a radial rescaling. Euclidean straight lines now appear as hyperbolae or ellipses, depending on whether or not the original line intersected the disc. If the line intersected the disc then the map of equation (10.185) has two branches and defines a hyperbola. If the line misses the disc then an ellipse is obtained. In all cases the image lines pass through the origin, as this is the image of the point at infinity.

The fact that the map of equation (10.185) is two-to-one means it has little use as a version of Euclidean geometry. It is better suited to hyperbolic geometry, as one might expect, as the Poincaré disc is now mapped onto the entire plane. Hyperbolic straight lines now appear as (single-branch) hyperbolae on the Euclidean page, all with their asymptotes crossing at the origin. If the dual space outside the disc is included in the map, then this generates the second branch of each hyperbola. Points then occur in pairs, with each point paired with its image under reflection in the origin. Finally, we can consider spherical geometry as viewed on a plane through the map of equation (10.185). This defines a standard projective map between a sphere and the plane. Antipodal points on the sphere define the same point on the plane and spherical straight lines appear as straight lines.

Similarly, we can consider plotting vectors in the plane according to

$$y = -\frac{X \wedge N}{X \cdot \bar{e}} N = -\frac{F(x) \wedge N}{F(x) \cdot \bar{e}} N \tag{10.186}$$

or in terms of the vector x

$$y = \frac{2x}{1 + x^2}. \tag{10.187}$$

This defines a one-to-one map of the unit disc onto itself, and a two-to-one map of the entire plane onto the disc. Euclidean straight lines now appear plotted as ellipses inside the unit disc. This construction involves forming a stereographic projection of the plane onto the 2-sphere, so that lines map to circles on the sphere. The sphere is then mapped onto the plane by viewing from above, so that circles on the sphere map to ellipses. All ellipses pass through the origin, as this is the image of the point at infinity.

Similar comments apply to spherical geometry. Spherical lines are great circles on the sphere, and viewed in the plane according to equation (10.187) great circles appear as ellipses centred on the origin and touching the unit circle at their endpoints. The two-to-one form of the projection means that circle intersections are not faithfully represented in the disc as some of the apparent intersections are actually caused by points on opposite sides of the plane. Finally, we consider plotting hyperbolic geometry in the view of equation (10.187). The disc maps onto itself, so we do have a faithful representation of hyperbolic geometry. This is a representation in which hyperbolic lines appear straight on the page, though angles are not rendered correctly, and non-Euclidean circles appear as ellipses.

As well as viewing each geometry on the Euclidean plane, we can also picture the geometries on a sphere or a hyperboloid. The spherical picture is obtained in equation (10.180), and the hyperboloid view is similarly obtained by setting

$$\frac{X}{X \cdot e} = \hat{x} + e, \tag{10.188}$$

where $\hat{x}^2 = -1$. The set of $\hat{x}$ defines a pair of hyperbolic sheets in the space defined by the vectors $\{e_1, e_2, \bar{e}\}$. The fact that two sheets are obtained explains why some views of hyperbolic geometry end up with points represented twice. So, as well as three geometries (defined by a transformation group) and a variety of plotting schemes, we also have a choice of space to draw on, providing a large number of alternative schemes for studying the three geometries. At the back of all of this is a single algebraic scheme, based on the geometric algebra of conformal space. Any algebraic result involving products of null vectors immediately produces a geometric theorem in each geometry, which can be viewed in a variety of different ways.

10.7 Spacetime conformal geometry

As a final application of the conformal approach to geometry we turn to spacetime. The conformal geometric algebra for a spacetime with signature $(1, 3)$ is the six-dimensional algebra with signature $(2, 4)$. The algebra $\mathcal{G}(2, 4)$ contains 64 terms, which decompose into graded subspaces of dimensions 1, 6, 15, 20, 15, 6 and 1. As a basis for this space we use the standard spacetime algebra basis $\{\gamma_\mu\}$, together with the additional vectors $\{e, \bar{e}\}$. The pseudoscalar I is defined by

$$I = \gamma_0\gamma_1\gamma_2\gamma_3 e\bar{e}. \tag{10.189}$$

This has negative norm, $I^2 = -1$. The conformal algebra allows us to simply encode ideas such as closed circles in spacetime, or light-spheres centred on an arbitrary point.

The conformal algebra of spacetime also arises classically in a slightly different setting. In conformal geometry, circles and spheres are represented homogeneously as trivectors and 4-vectors. These are unoriented because L and $-L$ are used to encode the same object. A method of dealing with oriented spheres was developed by Sophus Lie and is called Lie sphere geometry. A sphere in three dimensions can be represented by a vector s in the conformal algebra $\mathcal{G}(4, 1)$, with $s^2 > 0$. Lie sphere geometry is obtained by introducing a further basis vector of negative signature, f, and replacing s by the null vector

$$\bar{s} = s + |s|f, \qquad \bar{s}^2 = 0. \tag{10.190}$$

Now the spheres encoded by s and $-s$ have different representations as null vectors in a space of signature $(4, 2)$. This algebra is ideally suited to handling the contact geometry of spheres. The signature shows that this space is isomorphic to the conformal algebra of spacetime, so in a sense the introduction of the vector f can be thought of as introducing a time direction. A sphere can then be viewed as a light-sphere allowed to grow for a certain time. Orientation for spheres is then handled by distinguishing between incoming and outgoing light-spheres.

The conformal geometry of spacetime is a rich and important subject. The Poincaré group of spacetime translations and rotations is a subgroup of the full conformal group, but in a number of subjects in theoretical physics, including supersymmetry and supergravity, it is the full conformal group that is relevant. One reason is that conformal symmetry is present in most massless theories. This symmetry then has consequences that can carry over to the massive regime. We will not develop the classical approach to spacetime conformal geometry further here. Instead, we concentrate on an alternative route through to conformal geometry, which unites the multiparticle spacetime algebra of chapter 9 with the concept of a *twistor*.

10.7.1 The spacetime conformal group

For most of this chapter we have avoided detailed descriptions of the relationships between the groups involved in the geometric algebra formulation of conformal geometry. For the following, however, it is helpful to have a clearer picture of precisely how the various groups fit together. The subject of Lie groups in general is discussed in chapter 11. The spacetime conformal group $C(1,3)$ consists of spacetime maps $x \mapsto f(x)$ that preserve angles. This is the definition first encountered in section 10.3. The group of orthogonal transformations $O(2,4)$ is a double-cover representation of the conformal group, because in conformal space both X and $-X$ represent the same spacetime point. As with Lorentz transformations, we are typically interested in the restricted conformal group. This consists of transformations that preserve orientation and time sense, and contains translations, proper orthochronous rotations, dilations and special conformal transformations. The restricted orthogonal group, $SO^+(2,4)$, is a double-cover representation of the restricted conformal group.

We can form a double-cover representation of $SO^+(2,4)$ by writing all restricted orthogonal transformations as rotor transformations $a \mapsto Ra\tilde{R}$. The group of conformal rotors, denoted $\text{spin}^+(2,4)$, is therefore a four-fold covering of the restricted conformal group. The rotor group in $\mathcal{G}(2,4)$ is isomorphic to the Lie group $\text{SU}(2,2)$. It follows that the action of the restricted conformal group can be represented in terms of complex linear transformations of four-dimensional vectors, in a complex space of signature $(2,2)$. This is the basis of the *twistor* program, initiated by Roger Penrose. Twistors were introduced as objects describing the geometry of spacetime at a 'pre-metric' level, one of the aims being to provide a route to a quantum theory of gravity. Instead of points and a metric, twistors represent incidence relations between null rays. Spacetime points and their metric relations then emerge as a secondary concept, corresponding to the points of intersection of null lines.

As a first step in understanding the twistor program, we establish a concrete representation of the conformal group within the spacetime algebra. The key to this is the observation that the spinor inner product

$$\langle\tilde{\psi}\phi\rangle_q = \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle I\boldsymbol{\sigma}_3 \tag{10.191}$$

defines a complex space with precisely the required metric. The complex structure is represented by right-multiplication by combinations of 1 and $I\boldsymbol{\sigma}_3$, as discussed in chapter 8. We continue to refer to ψ and ϕ as spinors, as they are acted on by a spin representation of the restricted conformal group. To establish a representation in terms of operators on ψ, we first form a representation of the bivectors in $\mathcal{G}(2,4)$ as

$$\begin{aligned} e\gamma_\mu &\leftrightarrow \gamma_\mu\psi\gamma_0 I\boldsymbol{\sigma}_3 = \gamma_\mu\psi I\gamma_3, \\ \bar{e}\gamma_\mu &\leftrightarrow I\gamma_\mu\psi\gamma_0. \end{aligned} \tag{10.192}$$

A representation of the even subalgebra of $\mathcal{G}(2,4)$, and hence an arbitrary rotor, can be constructed from these bivectors. The representation of each of the operations in the restricted conformal group can now be constructed from the rotors found in section 10.3. We use the same symbol for the spinor representation of the transformations as the vector case. A translation by the vector a has the spin representation

$$T_a(\psi) = \psi + a\psi I\gamma_3 \tfrac{1}{2}(1+\boldsymbol{\sigma}_3). \tag{10.193}$$

The spinor inner product of equation (10.191) is invariant under this transformation. To confirm this, suppose that we set

$$\psi' = T_a(\psi) \quad \text{and} \quad \phi' = T_a(\phi). \tag{10.194}$$

The quantum inner product contains the terms

$$\begin{aligned}\langle\tilde{\psi}'\phi'\rangle &= \langle\big(\phi + a\phi I\gamma_3\tfrac{1}{2}(1+\boldsymbol{\sigma}_3)\big)\big(\tilde{\psi} - \tfrac{1}{2}(1-\boldsymbol{\sigma}_3)I\gamma_3\tilde{\psi}a\big)\rangle \\ &= \langle\tilde{\psi}\phi\rangle\end{aligned} \tag{10.195}$$

and

$$\begin{aligned}\langle\tilde{\psi}'\phi' I\boldsymbol{\sigma}_3\rangle &= \langle\big(\phi + a\phi I\gamma_3\tfrac{1}{2}(1+\boldsymbol{\sigma}_3)\big)I\boldsymbol{\sigma}_3\big(\tilde{\psi} - \tfrac{1}{2}(1-\boldsymbol{\sigma}_3)I\gamma_3\tilde{\psi}a\big)\rangle \\ &= \langle\tilde{\psi}\phi I\boldsymbol{\sigma}_3\rangle.\end{aligned} \tag{10.196}$$

It follows that

$$\langle\tilde{\psi}'\phi'\rangle_q = \langle\tilde{\psi}\phi\rangle_q, \tag{10.197}$$

as expected.

The spinor representation of a rotation about the origin is precisely the spacetime algebra rotor, so we can write

$$R_0(\psi) = R\psi, \tag{10.198}$$

where R_0 denotes a rotation in the origin, and R is a spacetime rotor. Rotations about arbitrary points are constructed from combinations of translations and rotations. The dilation $x \mapsto \exp(\alpha)x$ has the spinor representation

$$D_\alpha(\psi) = \psi e^{\alpha\boldsymbol{\sigma}_3/2}. \tag{10.199}$$

This represents a dilation in the origin. Dilations about a general point are also obtained from a combination of translations and a dilation in the origin. The representation of the restricted conformal group is completed by the special conformal transformations, which are represented by

$$K_a(\psi) = \psi - a\psi I\gamma_3\tfrac{1}{2}(1-\boldsymbol{\sigma}_3). \tag{10.200}$$

It is a routine exercise to confirm that the preceding operations do form a spin representation of the restricted conformal group.

The full conformal group includes inversions. These can be represented as *antiunitary* operators. An inversion in the origin is represented by

$$\psi \mapsto \psi' = \psi I \boldsymbol{\sigma}_2. \tag{10.201}$$

The effect of this on the inner product of equation (10.191) is that we form

$$\langle \tilde{\psi}' \phi' \rangle_q = \langle \tilde{\phi} \psi \rangle_q = \big(\langle \tilde{\psi} \phi \rangle_q\big)^{\sim}. \tag{10.202}$$

This representation of an inversion in the origin satisfies

$$D_\alpha(\psi I \boldsymbol{\sigma}_2) = D_{-\alpha}(\psi) I \boldsymbol{\sigma}_2, \tag{10.203}$$

as required.

10.7.2 Multiparticle representation of conformal vectors

We have defined a carrier space for a spin-1/2 representation of the spacetime conformal group. A vector representation of the conformal groups can therefore be constructed from quadratic combinations of spinors. Spinors can be thought of as belonging to a complex four-dimensional space. The tensor product space therefor contains 16 complex degrees of freedom. This decomposes into a ten-dimensional symmetric space and six-dimensional antisymmetric space. The six complex degrees of freedom in the antisymmetric representation are precisely the dimensions required to construct a conformal vector. The ten-dimensional symmetric space has 20 real degrees of freedom, and forms a representation of trivectors in conformal spacetime.

In principle, then, we will form complex vectors in conformal spacetime. But for a special class of spinor the conformal vector is real. If we translate a constant spinor by the position vector $r = x^\mu \gamma_\mu$ we form the object

$$T_r(\psi) = \psi + r\psi I \gamma_3 \tfrac{1}{2}\left(1 + \boldsymbol{\sigma}_3\right), \tag{10.204}$$

which is the spacetime algebra version of a *twistor*. A twistor is essentially a spacetime algebra spinor with a particular position dependence. The key to constructing a real conformal vector from an antisymmetric pair of twistors is to impose the conditions that they are both null, and orthogonal. Suppose that we set

$$\mathsf{X} = T_r(\psi), \quad \mathsf{Z} = T_r(\phi). \tag{10.205}$$

The conditions that these generate a real conformal vector are then

$$\langle \tilde{\mathsf{X}} \mathsf{X} \rangle_q = \langle \tilde{\mathsf{Z}} \mathsf{Z} \rangle_q = \langle \tilde{\mathsf{X}} \mathsf{Z} \rangle_q = 0. \tag{10.206}$$

The position dependence in X and Z does not affect the inner product, so the same conditions must also be satisfied by ψ and ϕ. Choosing appropriate spinors satisfying these relationships essentially amounts to a choice of origin. The most straightforward way to satisfy the requirements is to set

$$\mathsf{X} = \omega \tfrac{1}{2}\left(1-\boldsymbol{\sigma}_3\right) + r\omega I\gamma_3 \tfrac{1}{2}\left(1+\boldsymbol{\sigma}_3\right) \tag{10.207}$$

and

$$\mathsf{Z} = \kappa \tfrac{1}{2}\left(1-\boldsymbol{\sigma}_3\right) + r\kappa I\gamma_3 \tfrac{1}{2}\left(1+\boldsymbol{\sigma}_3\right), \tag{10.208}$$

where ω and κ are Pauli spinors (spinors in the spacetime algebra that commute with γ_0).

To construct a vector from the two twistors X and Z we form their antisymmetrised tensor product in the multiparticle spacetime algebra. We therefore construct the multivector

$$\psi_r = (\mathsf{X}^1\mathsf{Z}^2 - \mathsf{Z}^1\mathsf{X}^2)E, \tag{10.209}$$

where the notation follows section 9.2. If we now make use of the results in table 9.2 we find that

$$\psi_r = (r\cdot r\,\epsilon - r^1\eta\gamma_0^1 J - \bar{\epsilon})\langle I\boldsymbol{\sigma}_2\tilde{\kappa}\omega\rangle_q, \tag{10.210}$$

where η is the Lorentz singlet state defined in equation (9.93), and ϵ and $\bar{\epsilon}$ are defined by

$$\epsilon = \eta\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1), \quad \bar{\epsilon} = \eta\tfrac{1}{2}(1-\boldsymbol{\sigma}_3^1). \tag{10.211}$$

The two-particle state ψ closely resembles our standard encoding of a point as a null vector in conformal space. The singlet state ϵ represents the point at infinity, and is the spacetime algebra version of the infinity twistor. The opposite ideal, $\bar{\epsilon}$, represents the origin ($r = 0$).

More generally, given arbitrary single-particle spinors, we arrive at a complex six-dimensional vector. Restricting to the real subspace, a general point in this space can be written as the state

$$\psi_P = (V-W)\epsilon + P^1\eta\gamma_0^1 + (V+W)\bar{\epsilon}, \tag{10.212}$$

where

$$P = T\gamma_0 + X\gamma_1 + Y\gamma_2 + Z\gamma_3. \tag{10.213}$$

To form the inner product of such states we require the results that

$$\langle\tilde{\epsilon}\epsilon\rangle_q = \langle\tilde{\bar{\epsilon}}\bar{\epsilon}\rangle_q = 0, \qquad 4\langle\tilde{\epsilon}\bar{\epsilon}\rangle_q = 1. \tag{10.214}$$

Now forming the quantum norm for the state ψ_P we find that

$$2\langle\tilde{\psi}_P\psi_P\rangle_q = T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2. \tag{10.215}$$

So (V, W, T, X, Y, Z) are the coordinates of a six-dimensional vector in a space with signature $(2, 4)$. This establishes the map between a two-particle antisymmetrised spinor and a conformal vector.

Our 'real' state ψ_r can be cast into standard form by removing the complex factor on the right-hand side and setting

$$\psi_r \mapsto \frac{\psi_r}{4\langle\tilde{\psi}_r\epsilon\rangle_q}. \tag{10.216}$$

Once this is done, all reference to the original ω and κ spinors is removed. The inner product between two two-particle states ψ_r and ϕ_s, where ϕ_s represents the point s, returns

$$-\frac{\langle\tilde{\psi}_r\phi_s\rangle_q}{4\langle\tilde{\psi}_r\epsilon\rangle_q\langle\tilde{\phi}_s\epsilon\rangle_q} = (r-s)\cdot(r-s). \tag{10.217}$$

The multiparticle inner product therefore recovers the square of the spacetime distance between points. This result is one reason why points are encoded through pairs of *null* twistors.

We have now established a complete representation of conformal vectors for spacetime in terms of antisymmetrised products of a class of spinors, each evaluated in a single copy of the spacetime algebra. We should now check that our representation of the conformal group through its action on spinors induces the correct vector representation in the two-particle algebra. We start with our standard multiparticle representation of a conformal vector as

$$\psi_r = r\cdot r\,\epsilon - r^1\eta\gamma_0^1 J - \bar{\epsilon}. \tag{10.218}$$

The first operation to consider is a translation. The spinor representation of a translation by a induces the map

$$\psi_r \mapsto \psi_r' = T_{a^1}T_{a^2}\psi_r. \tag{10.219}$$

After some algebra we establish that

$$\psi_r' = (r+a)\cdot(r+a)\,\epsilon - (r+a)^1\eta\gamma_0^1 J - \bar{\epsilon}, \tag{10.220}$$

as required.

Next consider a Lorentz rotation centred on the origin. These are easily accomplished as they correspond to multiplying the single-particle spinor by the appropriate rotor. This induces the map

$$\begin{aligned}\psi_r \mapsto R^1R^2\psi_r &= r\cdot r\,R^1R^2\epsilon - R^1r^1R^2\eta\gamma_0^1 J - R^1R^2\bar{\epsilon}\\ &= r\cdot r\,\epsilon - (Rr\tilde{R})^1\eta\gamma_0^1 J - \bar{\epsilon},\end{aligned} \tag{10.221}$$

which achieves the desired rotation. Reflections in planes through the origin are equally easily achieved through the single-particle antiunitary operation

$$\psi \mapsto Ia\psi\gamma_2, \tag{10.222}$$

where a is the normal vector to the plane of reflection. Applied to the two-particle state we obtain

$$\psi_r \mapsto a\cdot a\big(r\cdot r\,\epsilon + (ara^{-1})^1\eta\gamma_0^1 J - \bar{\epsilon}\big), \tag{10.223}$$

which is the conformal representation of the reflected vector $-ara^{-1}$. As we also have a representation of translations, we can rotate and reflect about an arbitrary point.

Inversions in the origin are handled in conformal space by an operation that swaps the vectors representing the origin and infinity. In the multiparticle setting we must therefore interchange ϵ and $\bar{\epsilon}$, which is achieved by right-multiplication by $\boldsymbol{I\sigma}_2^1\boldsymbol{I\sigma}_2^2$,

$$\begin{aligned}\psi_r \mapsto \psi_r \boldsymbol{I\sigma}_2^1\,\boldsymbol{I\sigma}_2^2 &= -r\cdot r\,\bar{\epsilon} + r^1\eta\gamma_0^1 J + \epsilon\\ &= -r\cdot r(r'\cdot r'\,\epsilon - (r')^1\eta\gamma_0^1 J - \bar{\epsilon}),\end{aligned} \tag{10.224}$$

where $r' = r/(r{\cdot}r)$. Dilations in the origin are performed in a similar manner, this time by scaling ϵ and $\bar{\epsilon}$ through opposite amounts. This is successfully achieved by the two-particle map induced by equation (10.199),

$$\psi_r \mapsto \psi_r' = \psi_r \mathrm{e}^{\alpha/2(\boldsymbol{\sigma}_3^1 + \boldsymbol{\sigma}_3^2)}. \tag{10.225}$$

Special conformal transformations are also handled in the obvious way as the two-particle extension of the K_a operator of equation (10.200). This completes the description of the conformal group in the two-particle spacetime algebra setting.

Conformal spacetime geometry can be formulated in an entirely 'quantum' language in terms of multiparticle states built from spinor representations of the conformal group. This link between multiparticle quantum theory and conformal geometry is quite remarkable, and is the basis for the twistor programme. But one obvious question remains — is this abstract quantum-mechanical formulation necessary, if all one is interested is the conformal geometric algebra of spacetime? If the twistor programme is simply a highly convoluted way of discussing conformal geometric algebra, then the answer is no. The question is whether there is anything more fundamental about the quantum framework of the twistor approach.

Advocates of the twistor program would argue that the route we have followed here, which embeds a twistor within the spacetime algebra, reverses the logic which initially motivates twistors. The idea is that they exist at a pre-metric level, so that the spacetime interval between points emerges from a particular two-particle quantum inner product. This hints at a route to a quantum theory of gravity, where distance becomes a quantum observable. But much of the initial promise of this work remains unfulfilled, and twistors are no longer the most popular candidate for a quantum theory of gravity. For classical applications to real spacetime geometry it does appear that all twistor methods have direct counterparts in the geometric algebra $\mathcal{G}(2,4)$, and the latter approach avoids much of the additional formal baggage required when employing twistors.

10.8 Notes

The authors would like to thank Joan Lasenby for her help in writing this chapter. The subjects discussed in this chapter range from the foundations of algebraic geometry, dating back to the nineteenth century and before, through to some very modern applications. An excellent introduction to geometry is the book *Geometry* by Brannan, Esplen & Gray (1999). Projective geometry is described in the classic text by Semple & Kneebone (1998), and Lie sphere geometry is described by Cecil (1992). A valuable tool for studying two-dimensional geometry is the software package *Cinderella*, written by Richter-Gebert and Kortenkamp. This package was used to produce a number of the illustrations in this chapter.

The geometric algebra formulation of projective geometry is described in the pair of important papers 'The design of linear algebra and geometry' by Hestenes and 'Projective geometry with Clifford algebra' by Hestenes & Ziegler (both 1991). These papers also include preliminary discussions of conformal geometry, though the approach is different to that taken here. Projective geometry is particularly relevant to the field of computer graphics, and some applications of geometric algebra in this area are discussed in the papers by Stevenson & Lasenby (1998) and Perwass & Lasenby (1998).

The systematic study of conformal geometry with geometric algebra was only initiated in the 1990s and is one of the fastest developing areas of current research. Some of the earliest developments are contained in *Clifford Algebra to Geometric Calculus* by Hestenes & Sobczyk (1984), and in the paper 'Distance geometry and geometric algebra' by Dress & Havel (1993), which emphasises the role of the conformal metric. Uncovering the roles of the various geometric primitives in conformal space was initiated by Hestenes (2001) in the paper 'Old wine in new bottles: a new algebraic framework for computational geometry' and is described in detail in the papers by Hestenes, Li & Rockwood (1999a,b). Applications to the study of surfaces are described in the paper 'Surface evolution and representation using geometric algebra' by Lasenby & Lasenby (2000b), and a range of further applications are discussed in the proceedings of the 2001 conference *Applications of Geometric Algebra in Computer Science and Engineering* (Dorst, Doran & Lasenby, 2002). The rapid development of the subject has meant that a consistent notation is yet to be established by all authors.

The unification of Euclidean and non-Euclidean geometry in the conformal framework is also described in the series of papers by Hestenes, Li & Rockwood (1999a,b) and in a separate paper by Li (2001). The development in this chapter goes further than these papers in giving a concrete realisation of traditional methods within the geometric algebra framework. Twistor techniques are described in volume II of *Spinors and Space-time* by Penrose & Rindler (1986). A preliminary discussion of how twistors are incorporated into spacetime algebra is contained in the paper '2-spinors, twistors and supersymmetry in the spacetime algebra' by Lasenby, Doran & Gull (1993b). The multiparticle description of conformal vectors is discussed in the paper 'Applications of geometric algebra in physics and links with engineering' by Lasenby & Lasenby (2000a). Due to a printing error all dot products in this paper appear as deltas, though once one knows this the paper is readable!

10.9 Exercises

10.1 Let A, B, C, D denote four points on a line, and write their cross ratio as $(ABCD)$. Given that $(ABCD) = k$, prove that

$$(BACD) = (ABDC) = 1/k$$

and

$$(ACBD) = (DBCA) = 1 - k.$$

10.2 Prove that the cross ratio of four collinear points is a projective invariant, regardless of the size of the space containing the line.

10.3 Given four points in a plane, no three of which are collinear, prove that there exists a projective transformation that maps these to any second set of four points, where again no three are collinear.

10.4 The vectors a, b, c, a', b', c' all belong to $\mathcal{G}(3,0)$. From these we define the bivectors

$$A = b\wedge c, \quad B = c\wedge a, \quad C = a\wedge b,$$

with the same definitions holding for A', B', C'. Prove that

$$\langle A\times A'\, B\times B'\, C\times C'\rangle = \langle a\wedge b\wedge c\, a'\wedge b'\wedge c'\rangle\langle a\wedge a'\, b\wedge b'\, c\wedge c'\rangle.$$

This proves Desargues' theorem for two triangles in a common plane. Does the theorem still hold in three dimensions when the triangles lie on different planes?

10.5 Given six vectors $a_1, \dots, a_6$ representing points in the projective plane, prove that

$$\frac{a_5\wedge a_4\wedge a_3}{a_5\wedge a_1\wedge a_3}\frac{a_6\wedge a_2\wedge a_1}{a_6\wedge a_2\wedge a_4} = \frac{A_{543}}{A_{513}}\frac{A_{621}}{A_{624}},$$

where A_{ijk} is the area of the triangle whose vertices are described projectively by the vectors a_i, a_j, a_k. How does this ratio of areas transform under a projective transformation?

10.6 A Möbius transformation in the complex plane is defined by

$$z \mapsto z' = \frac{az+b}{cz+d},$$

where a, b, c, d are complex numbers. Prove that, viewed as a map of the complex plane onto itself, a Möbius transformation is a conformal transformation. Can all conformal transformations in the plane be represented as Möbius transformations? If not, which operation is missing?

10.7 Find the general form of the rotor, in conformal space, for a rotation through θ in the $a \wedge b$ plane, about the point with position vector a.

10.8 A special conformal transformation in Euclidean space corresponds to a combination of an inversion in the origin, a translation by b and a further inversion in the origin. Prove that the result of this can be written

$$x \mapsto= x \frac{1}{1+bx}.$$

Hence show that the linear function $\mathsf{f}(a) = a \cdot \nabla x$ is given by

$$\mathsf{f}(a) = \frac{(1+bx)a(1+xb)}{(1+2b\cdot x + b^2x^2)^2}.$$

Why does this transformation leave angles unchanged?

10.9 Given a conformal bivector B, with $B^2 > 0$, why does this encode a pair of Euclidean points? Prove that the midpoint of these two points is described by

$$C = BnB.$$

10.10 Two circles in a Euclidean plane are described by conformal trivectors L_1 and L_2. By expressing the dual vectors l_1 and l_2 in terms of the centre and radius of the circles, confirm directly that the circles intersect at right angles if

$$l_1 \cdot l_2 = 0.$$

10.11 The conformal vector X denotes a point lying on the circle L, $L \wedge X = 0$, where L is a trivector. Prove that the tangent vector T to the circle at X can be written

$$T = (X \cdot L) \wedge n.$$

10.12 A non-Euclidean translation along the line through X and Y is generated by the bivector $B = Le$, where

$$L = X \wedge Y \wedge e.$$

Prove that the hyperbolic angle α which takes us from X to Y is given by

$$\cosh(\alpha) = 1 - \frac{X \cdot Y}{X \cdot e \, Y \cdot e}.$$

10.13 The line element over the Poincaré disc is defined by

$$ds^2 = \frac{1}{1-r^2}(dr^2 + r^2 d\theta^2),$$

where r and θ are polar coordinates and $r < 1$. Prove that geodesics in this geometry all intersect the circle $r = 1$ at right angles.

10.14 Suppose that ψ is an even element of the spacetime algebra. This is acted on by the following linear transformations:

$$\begin{aligned}
R_0(\psi) &= R\psi, \\
T_a(\psi) &= \psi + a\psi I\gamma_3 \tfrac{1}{2}\left(1 + \boldsymbol{\sigma}_3\right), \\
D_\alpha(\psi) &= \psi \mathrm{e}^{\alpha\boldsymbol{\sigma}_3/2}, \\
K_a(\psi) &= \psi - a\psi I\gamma_3 \tfrac{1}{2}\left(1 - \boldsymbol{\sigma}_3\right),
\end{aligned}$$

where R is a spacetime rotor. Prove that this set of linear transformations generate a representation of the restricted conformal group of spacetime.