5

Relativity and spacetime

The geometric algebra of spacetime is called the *spacetime algebra*. Historically, the spacetime algebra was the first modern implementation of geometric algebra to gain widespread attention amongst the physics community. This is because it provides a *synthetic* framework for studying spacetime physics. There are two main approaches to the study of geometry, which can be loosely referred to as the algebraic and synthetic traditions. In the algebraic approach one works entirely with the components of a vector and manipulates these directly. Such an approach leads naturally to the subject of tensors, and places considerable emphasis on how coordinates transform under changes of frame. The synthetic approach, on the other hand, treats vectors as single, abstract entities x or a, and manipulates these directly. Geometric algebra follows in this tradition.

For much of modern physics the synthetic approach has come to dominate. The most obvious examples of this are classical mechanics and electromagnetism, both of which helped shape the development of abstract vector calculus. For these subjects, presentations typically perform all of the required calculations with the three-dimensional scalar and cross products. We have argued that geometric algebra provides extra efficiency and clarity, though it is not essential to a synthetic treatment of three-dimensional physics. But for spacetime calculations the cross product cannot be defined. Despite the obvious advantages of synthetic treatments, most relativity texts revert to a more basic, algebraic approach involving the components of 4-vectors and Lorentz-transform matrices. Such an approach has trouble encoding such basic notions as a plane in spacetime and, unsurprisingly, does a very poor job of handling the dynamics of extended bodies.

To develop a generally applicable algebra of vectors in spacetime one has little option but to use either geometric algebra, or the language of exterior forms (which is essentially a subset of geometric algebra which only employs the interior and exterior products). This is why relativistic physics still tends to dominate the literature of applications of geometric algebra. Many aspects of special relativity become clearer when viewed in the language of geometric algebra and, crucially, a wealth of new computational tools is provided which dramatically simplify relativistic problems.

5.1 An algebra for spacetime

It is not our intention in this chapter to give a fully self-contained introduction to relativity. Such an account can be found in the various books listed at the end of this chapter. In brief, a series of famous experiments conducted in the latter half of the nineteenth century showed that light did not appear to behave in quite the expected, Newtonian manner. This led Einstein to his 'second postulate', that the speed of light c is the same for all inertial (non-accelerating) observers. Combined with Einstein's 'first postulate', the principle of relativity, one is led inexorably to special relativity. The principle of relativity states simply that all inertial frames are equivalent for the purposes of physical experiment. An immediate consequence of these postulates is that the underlying geometry is no longer that of a (Euclidean) three-dimensional space, but instead the appropriate arena for physics is (Lorentzian) spacetime.

To understand why this is the case, suppose that a spherical flash of light is sent out from a source, and this event is described in two coordinate frames. We discuss the concept of a frame, as distinct from a single observer, later in this chapter. The frames are in relative motion, and their origins coincide with the location of the source at the moment the light is emitted. At this instant both frames also set their time measurements to zero. In the first frame the source is at rest and the light expands radially according to the equation

$$r = ct. \tag{5.1}$$

But the second frame must also record a radially expanding shell of light since the relative velocity of the source has no effect on the speed of light. The second frame therefore sees light expanding according to the equation

$$r' = ct'. \tag{5.2}$$

Since the two frames are in relative motion, points at a given fixed r cannot coincide with those at a fixed r'. So points reached at the same time in one frame are reached at *different* times in the second frame. But in both frames the light lies on a spherical expanding shell. So the one thing that is common to both frames is the value of

$$(ct)^2 - r^2 = (ct')^2 - (r')^2 = 0. \tag{5.3}$$

This defines the invariant interval of special relativity and is the fundamental algebraic concept we need to encode.

The preceding argument shows us that the algebra we need to construct is generated by four orthogonal vectors $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ satisfying the algebraic relations

$$\gamma_0^2 = 1, \quad \gamma_0 \cdot \gamma_i = 0, \quad \gamma_i \cdot \gamma_j = -\delta_{ij}, \tag{5.4}$$

where i and j run from 1 to 3. These are summarised in relativistic notation as

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+ - - -), \quad \mu, \nu = 0, \ldots, 3. \tag{5.5}$$

The notation $\{\gamma_\mu\}$ for a spacetime frame is a widely adopted convention in the spacetime algebra literature. The notation is borrowed from Dirac theory and we continue to employ it in this book. We have also chosen the 'particle physics' choice of signature, which has spacelike vectors with negative norm. General relativists often work with the opposite signature and swap all of the signs in $\eta_{\mu\nu}$. Both choices have their advocates and all (known) physical laws are independent of the choice of signature. Throughout we use Latin indices to denote the range 1–3 and Greek for the full spacetime range 0–3.

The $\{\gamma_\mu\}$ vectors are dimensionless, as is clear from their squares. Since we are in a space of mixed signature, we must adopt the conventions of section 4.3 and distinguish between a frame and its reciprocal. For the $\{\gamma_\mu\}$ frame the reciprocal frame vectors, $\{\gamma^\mu\}$, have $\gamma^0 = \gamma_0$ and $\gamma^i = -\gamma_i$. A general vector in the spacetime algebra can be constructed from the $\{\gamma_\mu\}$ vectors. A spacetime event, for example, is encoded in the vector x, which has coordinates x^μ in the $\{\gamma_\mu\}$ frame. Explicitly, the vector x is

$$x = x^\mu \gamma_\mu = ct\gamma_0 + x^i \gamma_i, \tag{5.6}$$

which has dimensions of distance. From this point on it will be convenient to work in units where the speed of light c is 1. Factors of c can then be inserted in any final result if the answer is required in different units. The mixed signature means that the square of a vector (a, say) is no longer necessarily positive, and instead we have

$$a^2 = aa = \epsilon |a^2|. \tag{5.7}$$

ϵ is the signature of the vector and can be ± 1 or 0. The mixed signature does not affect the validity of the axiomatic development and results of chapter 4, which made no reference to the signature.

5.1.1 The bivector algebra

There are $4 \times 3/2 = 6$ bivectors in our algebra. These fall into two classes: those that contain a timelike component (e.g. $\gamma_i \wedge \gamma_0$), and those that do not (e.g. $\gamma_i \wedge \gamma_j$). For any pair of orthogonal vectors a and b, $a \cdot b = 0$, we have

$$(a \wedge b)^2 = abab = -abba = -a^2 b^2. \tag{5.8}$$

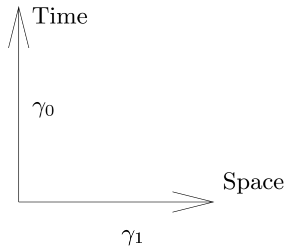

Figure 5.1 *A spacetime diagram.* Spacetime diagrams traditionally have the t axis vertical, so a suitable bivector for this plane is $\gamma_1\gamma_0$.

The two types of bivectors therefore have different signs of their squares. First, we have

$$(\gamma_i\wedge\gamma_j)^2 = -{\gamma_i}^2{\gamma_j}^2 = -1, \tag{5.9}$$

which is the familiar result for Euclidean bivectors. Each of these generates rotations in a plane. For bivectors containing a timelike component, however, we have

$$(\gamma_i\wedge\gamma_0)^2 = -{\gamma_i}^2{\gamma_0}^2 = +1. \tag{5.10}$$

Bivectors with positive square have a number of new properties. One immediate result we notice, for example, is that

$$\begin{aligned} \mathrm{e}^{\alpha\gamma_1\gamma_0} &= 1 + \alpha\gamma_1\gamma_0 + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!}\gamma_1\gamma_0 + \cdots \\ &= \cosh(\alpha) + \sinh(\alpha)\gamma_1\gamma_0. \end{aligned} \tag{5.11}$$

This shows us that we are dealing with *hyperbolic geometry*. This will prove crucial to our treatment of Lorentz transformations. Traditionally, spacetime diagrams are drawn with the time axis vertical (see figure 5.1). For these diagrams the 'right-handed' bivector is, for example, $\gamma_1\gamma_0$. These bivectors do not generate 90° rotations, however, as we now have

$$\gamma_0\cdot(\gamma_1\gamma_0) = -\gamma_1, \quad \gamma_1\cdot(\gamma_1\gamma_0) = -\gamma_0. \tag{5.12}$$

5.1.2 The pseudoscalar

We define the (grade-4) pseudoscalar I by

$$I = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{5.13}$$

In the literature the symbol i is often used for the pseudoscalar. We have departed from this practice to avoid confusion with the i of quantum theory. Using the latter symbol presents a potential problem because of the fact that the pseudoscalar anticommutes with vectors. The pseudoscalar defines an orientation for spacetime, and the reason for the above choice will emerge shortly. We still assume that $\{\gamma_1, \gamma_2, \gamma_3\}$ form a right-handed orthonormal set, as usual for a three-dimensional Cartesian frame. Since I is grade-4, it is equal to its own reverse:

$$\tilde{I} = \gamma_3\gamma_2\gamma_1\gamma_0 = I. \tag{5.14}$$

For relativistic applications we use the tilde ˜ to denote the reverse operation. The problem with the alternative symbol, the dagger †, is that it is usually reserved for a different role in relativistic quantum theory. The fact that $\tilde{I} = I$ makes it easy to compute the square of I :

$$I^2 = I\tilde{I} = (\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_3\gamma_2\gamma_1\gamma_0) = -1. \tag{5.15}$$

Multiplication of a bivector by I results in a multivector of grade $4 - 2 = 2$, so returns another bivector. This provides a map between bivectors with positive and negative squares, for example

$$I\gamma_1\gamma_0 = \gamma_1\gamma_0 I = \gamma_1\gamma_0\gamma_0\gamma_1\gamma_2\gamma_3 = -\gamma_2\gamma_3. \tag{5.16}$$

If we define $B_i = \gamma_i\gamma_0$ then the bivector algebra can be summarised by

$$\begin{aligned} B_i \times B_j &= \epsilon_{ijk}\, IB_k, \\ (IB_i)\times(IB_j) &= -\epsilon_{ijk}\, IB_k, \\ (IB_i)\times B_j &= -\epsilon_{ijk} B_k. \end{aligned} \tag{5.17}$$

These equations show that the pseudoscalar provides a natural complex structure for the set of bivectors. This in turn tells us that there is a complex structure hidden in the group of Lorentz transformations.

As well as the four vectors, we also have four trivectors in our algebra. The vectors and trivectors are interchanged by a duality transformation,

$$\gamma_1\gamma_2\gamma_3 = \gamma_0\gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_0 I = -I\gamma_0. \tag{5.18}$$

The pseudoscalar I *anticommutes* with vectors and trivectors, as we are in a space of even dimensions. As always, I commutes with all even-grade multivectors.

5.1.3 The spacetime algebra

Combining the preceding results, we arrive at an algebra with 16 terms. The $\{\gamma_\mu\}$ define an explicit basis for this algebra as follows:

1	$\{\gamma_\mu\}$	$\{\gamma_\mu\wedge\gamma_\nu\}$	$\{I\gamma_\mu\}$	I
1 scalar	4 vectors	6 bivectors	4 trivectors	1 pseudoscalar

This is the *spacetime algebra*, $\mathcal{G}(1,3)$. The structure of this algebra tells us practically all one needs to know about (flat) spacetime and the Lorentz transformation group. A general element of the spacetime algebra can be written as

$$M = \alpha + a + B + Ib + I\beta, \tag{5.19}$$

where α and β are scalars, a and b are vectors and B is a bivector. The reverse of this element is

$$\tilde{M} = \alpha + a - B - Ib + I\beta. \tag{5.20}$$

The vector generators of the spacetime algebra satisfy

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu}. \tag{5.21}$$

These are the defining relations of the Dirac matrix algebra, except for the absence of an identity matrix on the right-hand side. It follows that the Dirac matrices define a representation of the spacetime algebra. This also explains our notation of writing $\{\gamma_\mu\}$ for an orthonormal frame. But it must be remembered that the $\{\gamma_\mu\}$ are basis *vectors*, not a set of matrices in 'isospace'.

5.2 Observers, trajectories and frames

From a study of the literature on relativity one can easily form the impression that the subject is in the main concerned with transformations between frames. But it is the subject of relativistic dynamics that is of primary importance to us, and one aim of the spacetime algebra development is to minimise the use of coordinate frames. Instead, we aim to develop spacetime physics in a frame-free manner and, where necessary, then focus on the physics as seen from different observers. Developing relativistic physics in this manner has the added advantage of clarifying precisely which aspects of special relativity need modification to incorporate gravity.

5.2.1 Spacetime paths

Suppose that $x(\lambda)$ describes a curve in spacetime, where λ is some arbitrary, monotonically-increasing parameter along the curve. The tangent vector to the curve is

$$x' = \frac{dx(\lambda)}{d\lambda}. \tag{5.22}$$

Under a change of parameter from λ to τ the tangent vector becomes

$$\frac{dx}{d\tau} = \frac{d\lambda}{d\tau}\frac{dx}{d\lambda}. \tag{5.23}$$

It follows that

$$\left(\frac{dx}{d\tau}\right)^2 = \left(\frac{d\lambda}{d\tau}\right)^2 \left(\frac{dx}{d\lambda}\right)^2, \tag{5.24}$$

so the sign of $(x')^2$ is an invariant feature of the path. We assume for simplicity that this sign does not change along the path. As we are working in a space of mixed signature there are then three cases to consider.

The first possibility is that $(x')^2 > 0$, in which case the path is said to be *timelike*. Timelike trajectories are those followed by massive particles. For these paths we can define an invariant proper interval

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \left(\frac{dx}{d\lambda}\cdot\frac{dx}{d\lambda}\right)^{1/2} d\lambda. \tag{5.25}$$

It is straightforward to check that this interval is independent of how the path is parameterised. If we consider the simplest case of a particle (or observer) at rest in the γ_0 system, its spacetime trajectory can be written as $x = t\gamma_0$. In this case it is clear that the interval defines the elapsed time in the observer's rest frame. This must be true for all possible paths, so the interval (5.25) defines the time as measured along the path. This is called the *proper time*, and is usually given the symbol τ. The proper time defines a preferred parameter along the curve with the unique property that the velocity v,

$$v = \frac{dx}{d\tau} = \dot{x}, \tag{5.26}$$

satisfies

$$v^2 = 1. \tag{5.27}$$

Throughout we use dots to denote differentiation with respect to proper time τ. The unit timelike vector v then defines the instantaneous rest frame. The definition of 'proper time' makes it clear that in relativity observers moving in relative motion measure different times.

The second case to consider is that $(x')^2 = 0$. In this case the trajectory is said to be *lightlike* or *null*. Null trajectories are followed by massless (point) particles and (in the geometric optics limit) they define possible photon paths. There is no preferred parameter along these curves, and the proper distance (or time) measured along the curve is 0. Photons do still carry an intrinsic clock, defined by their frequency, but this can tick at an arbitrary rate.

The third possibility is that $(x')^2 < 0$, in which case the trajectory is said to be *spacelike*. As with timelike paths there is a preferred (affine) parameter along the path such that $(x')^2 = -1$. In this case the parameter defines the *proper distance*. Spacelike curves cannot arise for the trajectories of (known) particles, which are constrained to move at less than (or equal to) the speed of light. Events which are separated by spacelike intervals cannot be in causal

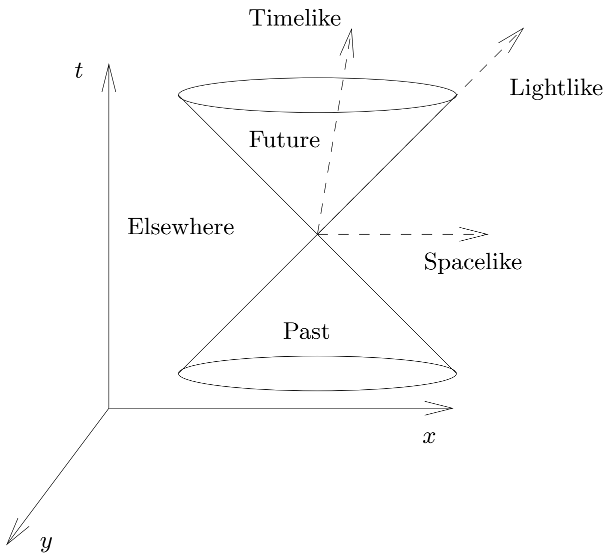

Figure 5.2 *Spacetime trajectories.* There are three different types of spacetime trajectory: timelike, lightlike and spacelike. The set of lightlike trajectories through a point separate spacetime into three regions: the past, the future and 'elsewhere'.

contact with each other and cannot exert any classical influence over each other. The three possibilities for spacetime trajectories are summarised in figure 5.2.

5.2.2 Spacetime frames

The subject of spacetime frames and coordinates dominates many discussions of the meaning of special relativity. The concept of a frame is distinct from that of an observer as it involves the notion of a coordinate lattice. We start with an inertial observer with constant velocity v. This velocity vector is then equated with the timelike vector e_0 from a spacetime frame $\{\mathsf{e}_\mu\}$. The remaining vectors e_i are chosen so that they form a right-handed set of orthonormal spacelike vectors perpendicular to $\mathsf{e}_0 = v$. The $\{\mathsf{e}_\mu\}$ then define a set of frame vectors satisfying

$$\mathsf{e}_\mu \cdot \mathsf{e}_\nu = \eta_{\mu\nu}. \tag{5.28}$$

So far these vectors are only defined at a single point on the observer's trajectory. We now assume that the vectors extend throughout all spacetime, so that any event can be given a set of spacetime coordinates

$$x^\mu = \mathsf{e}^\mu \cdot x. \tag{5.29}$$

Clearly these coordinates are a rather distinct concept from what an observer will actually measure, since the observer is constrained to remain in one place and only receives incoming photons. Frequently one sees discussions involving arrays of clocks all cleverly synchronised to read the time x^0 at each spatial location. But how such a frame is set up is not really the point. The assertion is that the coordinates as specified above are a reasonable model for the sort of distance and time measurements performed in a laboratory system using physical measuring devices. It is precisely this assertion that is challenged by general relativity, which insists that one talk entirely in terms of physically-defined coordinates, so that the x^μ defined above have no physical meaning. That said, for applications not involving gravity and for non-accelerating frames, we can safely identify the coordinates defined above with physical distances and times and will continue to do so in this chapter.

5.2.3 Relative vectors

Now suppose that we follow a timelike path with instantaneous velocity v, $v^2 = 1$. What sort of quantities do we measure? First we construct a frame of rest vectors $\{\mathsf{e}_i\}$ perpendicular to $v = \mathsf{e}_0$. We also take a point on the worldline as the spatial origin. Then a general event x can be decomposed in this frame as

$$x = t\mathsf{e}_0 + x^i\mathsf{e}_i, \tag{5.30}$$

where the time coordinate is

$$t = x\cdot\mathsf{e}_0 = x\cdot v \tag{5.31}$$

and spatial coordinates are

$$x^i = x\cdot\mathsf{e}^i. \tag{5.32}$$

Suppose now that the event is a point on the worldline of an object at rest in our frame. The three-dimensional vector to this object is

$$x^i\mathsf{e}_i = x\cdot\mathsf{e}^\mu\,\mathsf{e}_\mu - x\cdot\mathsf{e}^0\,\mathsf{e}_0 = x - x\cdot v\,v = x\wedge v\,v. \tag{5.33}$$

Wedging with v projects onto the components of the vector x in the rest frame of v. The key quantity is the spacetime bivector $x\wedge v$. We call this the *relative* vector and write

$$\boldsymbol{x} = x\wedge v. \tag{5.34}$$

With these definitions we have

$$xv = x\cdot v + x\wedge v = t + \boldsymbol{x}. \tag{5.35}$$

The invariant distance now decomposes as

$$\begin{aligned} x^2 = xvvx &= (x\cdot v + x\wedge v)(x\cdot v + v\wedge x) \\ &= (t + \boldsymbol{x})(t - \boldsymbol{x}) = t^2 - \boldsymbol{x}^2, \end{aligned} \tag{5.36}$$

recovering the invariant interval. A second observer with a different velocity performs a different split of x into time and space components. But the interval x^2 is the same for all observers as it manifestly does not depend on the choice of frame.

5.2.4 The even subalgebra

Each observer sees a set of relative vectors, which we model as spacetime bivectors. What algebraic properties do these have? To simplify matters, we take the timelike velocity vector to be γ_0 and introduce a standard frame of relative vectors

$$\boldsymbol{\sigma}_i = \gamma_i \gamma_0. \tag{5.37}$$

These define a set of spacetime bivectors representing timelike planes. (The notation is again borrowed from quantum mechanics and is commonplace in the spacetime algebra literature.) The $\{\boldsymbol{\sigma}_i\}$ satisfy

$$\begin{aligned} \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j &= \tfrac{1}{2}(\gamma_i\gamma_0\gamma_j\gamma_0 + \gamma_j\gamma_0\gamma_i\gamma_0) \\ &= \tfrac{1}{2}(-\gamma_i\gamma_j - \gamma_j\gamma_i) = \delta_{ij}. \end{aligned} \tag{5.38}$$

These act as vector generators for a three-dimensional algebra. This is the geometric algebra of the relative space in the rest frame defined by γ_0. Furthermore, the volume element of this algebra is

$$\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = (\gamma_1\gamma_0)(\gamma_2\gamma_0)(\gamma_3\gamma_0) = -\gamma_1\gamma_0\gamma_2\gamma_3 = I, \tag{5.39}$$

so the algebra of relative space shares the same pseudoscalar as spacetime. This was the reason for our earlier definition of I. Of course, we still have

$$\tfrac{1}{2}(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_i) = \epsilon_{ijk} I \boldsymbol{\sigma}_k, \tag{5.40}$$

so that both relative vectors and relative bivectors are spacetime bivectors.

The even-grade terms in the spacetime algebra define the *even subalgebra*. As we have just established, this algebra has precisely the properties of the algebra of three-dimensional (relative) space. The even subalgebra contains scalar and pseudoscalar terms, and six bivector terms. These are split into three timelike vectors and three spacelike vectors, which in turn become relative vectors and bivectors. This is called a *spacetime split*, and it is *observer-dependent*. Different velocity vectors generate different spacetime splits. Algebraically, this provides us with an extremely efficient tool for comparing physical effects in different frames.

Spacetime bivectors which are also used as relative vectors are written in bold. This conforms with our earlier usage of a bold face for vectors in three dimensions. There is a potential ambiguity here — how are we to interpret the expression $\boldsymbol{a}\wedge\boldsymbol{b}$? Our convention is that if all of the terms in an expression are bold, the dot and wedge symbols drop down to their three-dimensional meaning, otherwise they take their spacetime definition. This works pretty well in practice, though where necessary we will try to draw attention to the fact that this convention is in use.

5.2.5 Relative velocity

Suppose that an observer with constant velocity v measures the relative velocity of a particle with proper velocity $u(\tau) = \dot{x}(\tau)$, $u^2 = 1$. We have

$$uv = \frac{d}{d\tau}(x(\tau)v) = \frac{d}{d\tau}(t + \boldsymbol{x}), \tag{5.41}$$

where $t + \boldsymbol{x}$ is the description of the event x in the v frame. It follows that

$$\frac{dt}{d\tau} = u\cdot v, \quad \frac{d\boldsymbol{x}}{d\tau} = u\wedge v. \tag{5.42}$$

The relative velocity $\boldsymbol{u}$ as measured in the v frame is therefore

$$\boldsymbol{u} = \frac{d\boldsymbol{x}}{dt} = \frac{d\boldsymbol{x}}{d\tau}\frac{d\tau}{dt} = \frac{u\wedge v}{u\cdot v}. \tag{5.43}$$

This construction of the relative velocity is extremely elegant. It embodies the concept of relativity in its precise (anti)symmetry. If we interchange u and v the second observer measures precisely the same relative speed as the first, but in the opposite direction. Expressions like $u\wedge v/u\cdot v$ arise frequently in the subject of *projective geometry* (see section 10.1). The resulting bivector is homogeneous, which is to say we can rescale u and v and still recover the same result. So the choice of parameterisation of the two spacetime trajectories is irrelevant to their relative velocity. The relative velocity is determined solely by the spacetime trajectories themselves, and not by any evolution parameter.

The definition of the relative velocity ensures that the magnitude is

$$\frac{(u\wedge v)^2}{(u\cdot v)^2} = 1 - \frac{1}{(u\cdot v)^2} < 1, \tag{5.44}$$

so no two observers measure a relative velocity greater than the speed of light (which is 1 in our current choice of units). If we form the Lorentz factor γ using

$$\begin{aligned}\gamma^{-2} &= 1 - \boldsymbol{u}^2 \\ &= 1 + (u\cdot v)^{-2}[(uv - u\cdot v)(vu - v\cdot u)] = (u\cdot v)^{-2},\end{aligned} \tag{5.45}$$

we find that $\gamma = u\cdot v$. It follows that we can decompose the velocity as

$$u = uvv = (u\cdot v + u\wedge v)v = \gamma(1 + \boldsymbol{u})v, \tag{5.46}$$

which shows a neat split into a part $\gamma \boldsymbol{u} v$ in the rest space of v, and a part γv along v.

5.2.6 Momentum and wave vectors

The relativistic definitions of energy and momentum can be motivated in various ways. Perhaps the simplest is to consider photons with frequency ω and wavevector $\boldsymbol{k}$ measured in the γ_0 frame. From quantum theory, the energy and momentum are given by $\hbar\omega$ and $\hbar\boldsymbol{k}$ respectively. If we define the wavevector k by

$$k = \omega\gamma_0 + k^i\gamma_i, \tag{5.47}$$

then the energy-momentum vector for the photon is simply

$$p = \hbar k. \tag{5.48}$$

An observer with velocity v, as opposed to γ_0, measures energy and momentum given by

$$E = p\cdot v, \quad \boldsymbol{p} = p\wedge v. \tag{5.49}$$

We take this as the correct definition for massive particles as well. So a particle of rest mass m and velocity u has an energy-momentum vector $p = mu$. A spacetime split of this vector with the velocity vector v yields

$$pv = p\cdot v + p\wedge v = E + \boldsymbol{p}. \tag{5.50}$$

A significant feature of this definition is that the relative momentum is related to the velocity by

$$\boldsymbol{p} = mu\cdot v\, \boldsymbol{u} = \gamma m \boldsymbol{u}, \tag{5.51}$$

where again γ is the Lorentz factor. One sometimes sees this formula written in terms of a velocity-dependent mass $m' = \gamma m$, but we will not adopt this practice here.

From the definition of p we recover the invariant

$$m^2 = p^2 = pvvp = (E + \boldsymbol{p})(E - \boldsymbol{p}) = E^2 - \boldsymbol{p}^2. \tag{5.52}$$

Similarly, for a photon with wavevector k, $k^2 = 0$, we have

$$0 = kvvk = (\omega + \boldsymbol{k})(\omega - \boldsymbol{k}) = \omega^2 - \boldsymbol{k}^2. \tag{5.53}$$

This recovers the relation $|\boldsymbol{k}| = \omega$, which holds in all frames.

5.2.7 Proper acceleration

A final ingredient in the formulation of relativistic dynamics is the proper acceleration. A particle follows a trajectory $x(\tau)$, where τ is the proper time. The particle has velocity $v = \dot{x}$, $v^2 = 1$. The proper acceleration is simply

$$\dot{v} = \frac{dv}{d\tau}. \tag{5.54}$$

Since $v^2 = 1$, the velocity and acceleration are perpendicular

$$\frac{d}{d\tau}(v^2) = 0 = 2\dot{v}\cdot v. \tag{5.55}$$

In many physical phenomena it turns out that a more useful concept is provided by the *acceleration bivector*

$$B_v = \dot{v}\wedge v = \dot{v}v. \tag{5.56}$$

This bivector denotes the acceleration projected into the instantaneous rest frame of the particle. Typically this bivector multiplied by the rest mass is equated with a bivector encoding the forces acting on the particle. Any change in the parameter along the curve will rescale the velocity vector, so B_v can be written as

$$B_v = \frac{v'\wedge v}{(v\cdot v)^{3/2}}, \tag{5.57}$$

which is independent of the parameterisation of the trajectory.

Before applying the various preceding definitions to a range of dynamical problems, we turn to a discussion of the Lorentz transformations. This will pave the way for a powerful method for studying relativistic problems which is unique to geometric algebra.

5.3 Lorentz transformations

Lorentz transformations are usually expressed in the form of a coordinate transformation. We suppose that two inertial observers have set up 'coordinate lattices' in their own rest frames, as discussed in section 5.2.2. We denote these frames by S and S', and assume that they are set up such that their 1 and 2 axes coincide, but that S' moves at (scalar) velocity βc along the 3 axis as seen in the S frame. We denote the 0 and 3 components by t and z respectively. If the origins of the frames coincide at $t = t' = 0$, the coordinates of the same spacetime event as measured in the two frames are related by

$$t' = \gamma(t - \beta z), \quad x^{1'} = x^1, \quad x^{2'} = x^2, \quad z' = \gamma(z - \beta t), \tag{5.58}$$

where $\gamma = (1-\beta^2)^{-1/2}$ and β is the velocity in units of c ($\beta < 1$). The inverse relations are easily found to be

$$t = \gamma(t' + \beta z'), \quad x^1 = x^{1'}, \quad x^2 = x^{2'}, \quad z = \gamma(z' + \beta t'). \tag{5.59}$$

The arguments leading to these transformation laws are discussed in all introductory texts on relativity (see e.g. Rindler (1977) or French (1968)).

To get a clearer understanding of this transformation law we must first convert these relations into a transformation law for the frame vectors. The vector x has been decomposed in two frames, $\{\mathsf{e}_\mu\}$ and $\{\mathsf{e}'_\mu\}$, so that

$$x = x^\mu \mathsf{e}_\mu = x^{\mu\prime} \mathsf{e}'_\mu. \tag{5.60}$$

We then have, for example,

$$t = \mathsf{e}^0 \cdot x, \quad t' = \mathsf{e}^{0\prime} \cdot x. \tag{5.61}$$

Concentrating on the 0 and 3 components we have

$$t\mathsf{e}_0 + z\mathsf{e}_3 = t'\mathsf{e}'_0 + z'\mathsf{e}'_3, \tag{5.62}$$

and from this we derive the vector relations

$$\mathsf{e}'_0 = \gamma(\mathsf{e}_0 + \beta\mathsf{e}_3), \quad \mathsf{e}'_3 = \gamma(\mathsf{e}_3 + \beta\mathsf{e}_0). \tag{5.63}$$

These define the new frame in terms of the old. As a check the new frame vectors have the correct normalisation,

$$(\mathsf{e}'_0)^2 = \gamma^2(1-\beta^2) = 1, \quad (\mathsf{e}'_3)^2 = -1. \tag{5.64}$$

The geometry of this transformation is illustrated in figure 5.3.

We saw earlier that bivectors with positive square lead to hyperbolic geometry. This suggests that we introduce an 'angle' α with

$$\tanh(\alpha) = \beta \tag{5.65}$$

so that

$$\gamma = \big(1 - \tanh^2(\alpha)\big)^{-1/2} = \cosh(\alpha). \tag{5.66}$$

The vector e'_0 is now

$$\begin{aligned} \mathsf{e}'_0 &= \cosh(\alpha)\, \mathsf{e}_0 + \sinh(\alpha)\, \mathsf{e}_3 \\ &= \big(\cosh(\alpha) + \sinh(\alpha)\, \mathsf{e}_3\mathsf{e}_0\big)\mathsf{e}_0 \\ &= \exp(\alpha\, \mathsf{e}_3\mathsf{e}_0)\, \mathsf{e}_0, \end{aligned} \tag{5.67}$$

where we have expressed the scalar + bivector term as an exponential. Similarly, we have

$$\mathsf{e}'_3 = \cosh(\alpha)\, \mathsf{e}_3 + \sinh(\alpha)\, \mathsf{e}_0 = \exp(\alpha\, \mathsf{e}_3\mathsf{e}_0)\, \mathsf{e}_3. \tag{5.68}$$

Now recall that these are just two of four frame vectors, and the other pair

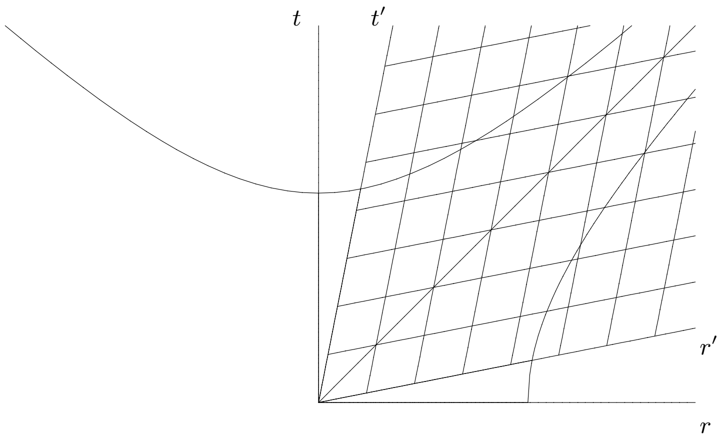

Figure 5.3 *A Lorentz transformation.* The transformation leaves the magnitude of a vector invariant. As the underlying geometry of a spacetime plane is Lorentzian, vectors of constant magnitude lie on hyperbolae, rather than circles. The transformed axes define a new coordinate grid.

are unchanged by the transformation. Since $\mathsf{e}_3\mathsf{e}_0$ anticommutes with e_0 and e_3, but commutes with e_1 and e_2, we can express the relationship between the two frames as

$$\mathsf{e}'_\mu = R\mathsf{e}_\mu \tilde{R}, \quad \mathsf{e}^{\mu\prime} = R\mathsf{e}^\mu \tilde{R}, \tag{5.69}$$

where

$$R = \mathrm{e}^{\alpha\,\mathsf{e}_3\mathsf{e}_0/2}. \tag{5.70}$$

The same rotor prescription introduced for rotations in Euclidean space also works for boosts in relativity! This is dramatically simpler than having to work with 4×4 Lorentz transform matrices.

5.3.1 Addition of velocities

As a simple example, suppose that we are in a frame with basis vectors $\{\gamma_\mu\}$. We observe two objects flying apart with 4-velocities

$$v_1 = \mathrm{e}^{\alpha_1\gamma_1\gamma_0/2}\gamma_0\mathrm{e}^{-\alpha_1\gamma_1\gamma_0/2} = \mathrm{e}^{\alpha_1\gamma_1\gamma_0}\gamma_0 \tag{5.71}$$

and

$$v_2 = \mathrm{e}^{-\alpha_2\gamma_1\gamma_0/2}\gamma_0\mathrm{e}^{\alpha_2\gamma_1\gamma_0/2} = \mathrm{e}^{-\alpha_2\gamma_1\gamma_0}\gamma_0. \tag{5.72}$$

What is the relative velocity they see for each other? We form

$$\frac{v_1 \wedge v_2}{v_1 \cdot v_2} = \frac{\langle e^{(\alpha_1+\alpha_2)\gamma_1\gamma_0}\rangle_2}{\langle e^{(\alpha_1+\alpha_2)\gamma_1\gamma_0}\rangle_0} = \frac{\sinh(\alpha_1+\alpha_2)\gamma_1\gamma_0}{\cosh(\alpha_1+\alpha_2)}. \tag{5.73}$$

Both observers therefore measure a relative velocity of

$$\tanh(\alpha_1+\alpha_2) = \frac{\tanh(\alpha_1)+\tanh(\alpha_2)}{1+\tanh(\alpha_1)\tanh(\alpha_2)}, \tag{5.74}$$

Addition of (collinear) velocities is achieved by adding hyperbolic angles, and not the velocities themselves. Replacing the tanh factors by the scalar velocities $u = c\tanh(\alpha)$ recovers the more familiar expression

$$u' = \frac{u_1+u_2}{1+u_1u_2/c^2}. \tag{5.75}$$

The surprising conclusion is that addition of velocities in spacetime is really a generalized rotation in a hyperbolic space! Quite dramatically different from the Newtonian prescription of simple vector addition of the velocities.

5.3.2 Photons, Doppler shifts and aberration

For many relativistic applications involving the properties of light it is sufficient to use a simplified model of a photon as a point particle following a null trajectory. The tangent vector to the path is the wavevector k. This provides for simple formulae for Doppler shifts and aberration. Suppose that two particles follow different worldlines and that particle 1 emits a photon which is received by particle 2 (see figure 5.4). The frequency seen by particle 1 is $\omega_1 = v_1 \cdot k$, and that by particle 2 is $\omega_2 = v_2 \cdot k$. The ratio of these describes the Doppler effect, often expressed as a redshift, z:

$$1+z = \frac{\omega_1}{\omega_2} = \frac{v_1 \cdot k}{v_2 \cdot k}. \tag{5.76}$$

This can be applied in many ways. For example, suppose that the emitter is receding in the γ_1 direction, and $v_2 = \gamma_0$. We have

$$k = \omega_2(\gamma_0+\gamma_1), \qquad v_1 = \cosh(\alpha)\,\gamma_0 - \sinh(\alpha)\,\gamma_1, \tag{5.77}$$

so that

$$1+z = \frac{\omega_2\big(\cosh(\alpha)+\sinh(\alpha)\big)}{\omega_2} = e^{\alpha}. \tag{5.78}$$

The velocity of the emitter in the γ_0 frame is $\tanh(\alpha)$, and it is easy to check that

$$e^{\alpha} = \left(\frac{1+\tanh(\alpha)}{1-\tanh(\alpha)}\right)^{1/2}. \tag{5.79}$$

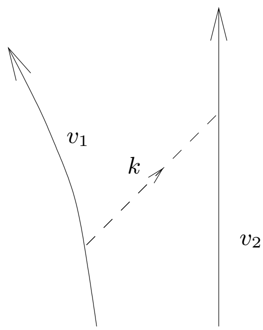

Figure 5.4 *Photon emission and absorption.* A photon is emitted by particle 1 and received by particle 2.

This formula recovers the standard expression for the relativistic Doppler effect:

$$\omega_2 = \left(\frac{1-\beta}{1+\beta}\right)^{1/2} \omega_1. \tag{5.80}$$

In its current form this formula is appropriate for a source and receiver moving away from each other at velocity βc. Had they been approaching each other the sign of β would be reversed, leading to an increased frequency at the receiver (a blueshift).

Aberration formulae can be obtained in a similar manner. Suppose that observer 1 has velocity γ_0, and that this observer receives photons at an angle θ to the 1 axis in the 12 plane. The photons are therefore on a null trajectory with tangent vector

$$n = \gamma_0 - \cos(\theta)\,\gamma_1 - \sin(\theta)\,\gamma_2, \tag{5.81}$$

and the γ_0 observer recovers the angle θ via

$$\tan(\theta) = \frac{n\cdot\gamma_2}{n\cdot\gamma_1}. \tag{5.82}$$

Suppose now that a second observer moves with velocity β relative to the first along the 1 axis. This observer's velocity is

$$v = \mathsf{e}_0 = \cosh(\alpha)\,\gamma_0 + \sinh(\alpha)\,\gamma_1 \tag{5.83}$$

and the frame vectors for this observer are

$$\mathsf{e}_1 = \cosh(\alpha)\,\gamma_1 + \sinh(\alpha)\,\gamma_0, \quad \mathsf{e}_2 = \gamma_2, \quad \mathsf{e}_3 = \gamma_3. \tag{5.84}$$

According to this observer the photons arrive at an angle

$$\tan(\theta') = \frac{n\cdot\mathsf{e}_2}{n\cdot\mathsf{e}_1} = \frac{\sin(\theta)}{\cosh(\alpha)\,\cos(\theta) + \sinh(\alpha)}. \tag{5.85}$$

A straightforward rearrangement gives

$$\cos(\theta') = \frac{\cosh(\alpha)\, \cos(\theta) + \sinh(\alpha)}{\cosh(\alpha) + \sinh(\alpha)\, \cos(\theta)} = \frac{\cos(\theta) + \beta}{1 + \beta \cos(\theta)}, \tag{5.86}$$

so observers in relative motion measure different angles to a fixed light source. This effect can be seen in observations of stars from the Earth. The Earth's orbital velocity around the sun has a β of roughly 10^{-4} so to a good approximation we have

$$\cos(\theta') \approx \cos(\theta) + \beta \sin^2(\theta). \tag{5.87}$$

The aberration angle $\phi = \theta - \theta'$ satisfies the approximate formula

$$\phi \approx \beta \sin(\theta), \tag{5.88}$$

which implies that the aberration varies over a year as θ varies through a complete cycle. This variation was first observed by James Bradley in 1727 and was explained in terms of a particle model of light. Bradley was able to use his data to give an improved estimate of the speed of light, though the full relativistic relation of (5.86) cannot be checked in this manner.

5.4 The Lorentz group

The full Lorentz group consists of the transformation group for vectors that preserves lengths and angles. These include reflections and rotations. A reflection in the hyperplane perpendicular to n is achieved by

$$a \mapsto -nan^{-1}. \tag{5.89}$$

The n^{-1} is necessary to accommodate both timelike $n^2 > 0$ and spacelike $n^2 < 0$ cases. We cannot have null n, as the inverse does not exist. A timelike n generates time-reversal transformations, whereas spacelike reflections preserve time-ordering. Pairs of either of these result in a transformation which preserves time-ordering. However, a combination of one spacelike and one timelike reflection does not preserve the time-ordering. The full Lorentz group therefore contains four sectors (table 5.1).

The structure of the Lorentz group is easily understood in the spacetime algebra. We concentrate on even numbers of reflections, which have determinant +1 and correspond to type I and type IV transformations. The remaining types are obtained from these by a single extra reflection. If we combine even numbers of reflections we arrive at a transformation of the form

$$a \mapsto \psi a \psi^{-1}, \tag{5.90}$$

where ψ is an even multivector. This expression is currently too general, as we

	Parity preserving	Space reflection
Time order preserving	I Proper orthochronous	II I with space reflection
Time reversal	III I with time reversal	IV I with $a \mapsto -a$

Table 5.1 *The full Lorentz group.* The group of Lorentz transformations falls into four disjoint sectors. Sectors I and IV have determinant $+1$, whereas II and III have determinant -1. Both I and II preserve time-ordering, and the proper orthochronous transformations (type I) are simply-connected to the identity.

have not ensured that the right-hand side is a vector. To see how to do this we decompose ψ into invariant terms. We first note that

$$\psi\tilde{\psi} = (\psi\tilde{\psi})^{\sim} \tag{5.91}$$

so $\psi\tilde{\psi}$ is even-grade and equal to its own reverse. It can therefore only contain a scalar and a pseudoscalar,

$$\psi\tilde{\psi} = \alpha_1 + I\alpha_2 = \rho \mathrm{e}^{I\beta}, \tag{5.92}$$

where $\rho \neq 0$ in order for ψ^{-1} to exist. We can now define a rotor R by

$$R = \psi(\rho \mathrm{e}^{I\beta})^{-1/2}, \tag{5.93}$$

so that

$$R\tilde{R} = \psi\tilde{\psi}(\rho \mathrm{e}^{I\beta})^{-1} = 1, \tag{5.94}$$

as required. We now have

$$\psi = \rho^{1/2}\mathrm{e}^{I\beta/2}R, \qquad \psi^{-1} = \rho^{-1/2}\mathrm{e}^{-I\beta/2}\tilde{R} \tag{5.95}$$

and our general transformation becomes

$$a \mapsto \mathrm{e}^{I\beta/2}Ra\mathrm{e}^{-I\beta/2}\tilde{R} = \mathrm{e}^{I\beta}Ra\tilde{R}. \tag{5.96}$$

The term $Ra\tilde{R}$ is necessarily a vector as it is equal to its own reverse, so we must restrict β to either 0 or π, leaving the transformation

$$a \mapsto \pm Ra\tilde{R}. \tag{5.97}$$

The transformation $a \mapsto Ra\tilde{R}$ preserves causal ordering as well as parity. Transformations of this type are called ‘proper orthochronous’ transformations.

We can prove that transformations parameterised by rotors are proper orthochronous by starting with the velocity γ_0 and transforming it to $v = R\gamma_0\tilde{R}$. We require that the γ_0 component of v is positive, that is,

$$\gamma_0 \cdot v = \langle \gamma_0 R \gamma_0 \tilde{R} \rangle > 0. \tag{5.98}$$

Decomposing in the γ_0 frame we can write

$$R = \alpha + \boldsymbol{a} + I\boldsymbol{b} + I\beta \tag{5.99}$$

and we find that

$$\langle \gamma_0 R \gamma_0 \tilde{R} \rangle = \alpha^2 + \boldsymbol{a}^2 + \boldsymbol{b}^2 + \beta^2 > 0 \tag{5.100}$$

as required. Our rotor transformation law describes the group of proper orthochronous transformations, often called the *restricted Lorentz group*. These are the transformations of most physical relevance. The negative sign in equation (5.97) corresponds to $\beta = \pi$ and gives class-IV transformations.

5.4.1 Invariant decomposition and fixed points

Every rotor in spacetime can be written in terms of a bivector as

$$R = \pm e^{B/2}. \tag{5.101}$$

(The minus sign is rarely required, and does not affect the vector transformation law.) We can understand many of the features of spacetime transformations and rotors through the properties of the bivector B. The bivector B can be decomposed in a Lorentz-invariant manner by first writing

$$B^2 = \langle B^2 \rangle_0 + \langle B^2 \rangle_4 = \rho e^{I\phi}, \tag{5.102}$$

and we will assume that $\rho \neq 0$. (The case of a null bivector is treated slightly differently.) We now define

$$\hat{B} = \rho^{-1/2} e^{-I\phi/2} B, \tag{5.103}$$

so that

$$\hat{B}^2 = \rho^{-1} e^{-I\phi} B^2 = 1. \tag{5.104}$$

With this we can now write

$$B = \rho^{1/2} e^{I\phi/2} \hat{B} = \alpha \hat{B} + \beta I \hat{B}, \tag{5.105}$$

which decomposes B into a pair of bivector blades, $\alpha\hat{B}$ and $\beta I \hat{B}$. Since

$$\hat{B}(I\hat{B}) = (I\hat{B})\hat{B} = I, \tag{5.106}$$

the separate bivector blades commute. The rotor R now decomposes into

$$R = e^{\alpha\hat{B}/2} e^{\beta I \hat{B}/2} = e^{\beta I \hat{B}/2} e^{\alpha \hat{B}/2}, \tag{5.107}$$

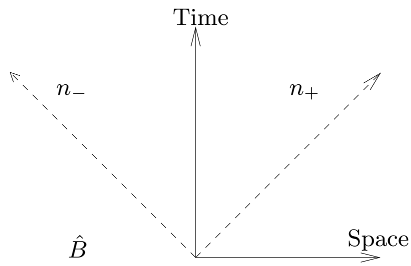

Figure 5.5 *A timelike plane.* Any timelike plane $\hat{B}$, $\hat{B}^2 = 1$, contains two null vectors n_+ and n_-. These can be normalised so that $n_+ \wedge n_- = 2\hat{B}$.

exhibiting an *invariant* split into a boost and a rotation. The boost is generated by $\hat{B}$ and the rotation by $I\hat{B}$.

For every timelike bivector $\hat{B}$, $\hat{B}^2 = 1$, we can construct a pair of null vectors $n_\pm$ satisfying

$$\hat{B} \cdot n_\pm = \pm n_\pm. \tag{5.108}$$

These are necessarily null, since

$$n_+ \cdot n_+ = (B \cdot n_+) \cdot n_+ = B \cdot (n_+ \wedge n_+) = 0, \tag{5.109}$$

with the same holding for n_-. The two null vectors can also be chosen so that

$$n_+ \wedge n_- = 2\hat{B}, \tag{5.110}$$

so that they form a null basis for the timelike plane defined by $\hat{B}$ (see figure 5.5).

The null vectors $n_\pm$ anticommute with $\hat{B}$ and therefore commute with $I\hat{B}$. The effect of the Lorentz transformation on $n_\pm$ is therefore

$$\begin{aligned} Rn_\pm \tilde{R} &= \mathrm{e}^{\alpha \hat{B}/2} n_\pm \mathrm{e}^{-\alpha \hat{B}/2} \\ &= \cosh(\alpha)\, n_\pm + \sinh(\alpha)\, \hat{B} \cdot n_\pm \\ &= \mathrm{e}^{\pm\alpha} n_\pm. \end{aligned} \tag{5.111}$$

The two null directions are therefore just scaled — their direction is unchanged. It follows that every Lorentz transformation has two invariant null directions. The case where the bivector generator itself is null, $B^2 = 0$, corresponds to the special situation where these two null directions coincide.

5.4.2 The celestial sphere

One way to visualise the effect of Lorentz transformations is through their effect on the past light-cone (see figure 5.6). Each null vector on the past light-cone maps to a point on the sphere S^- — the *celestial sphere* for the observer. Suppose

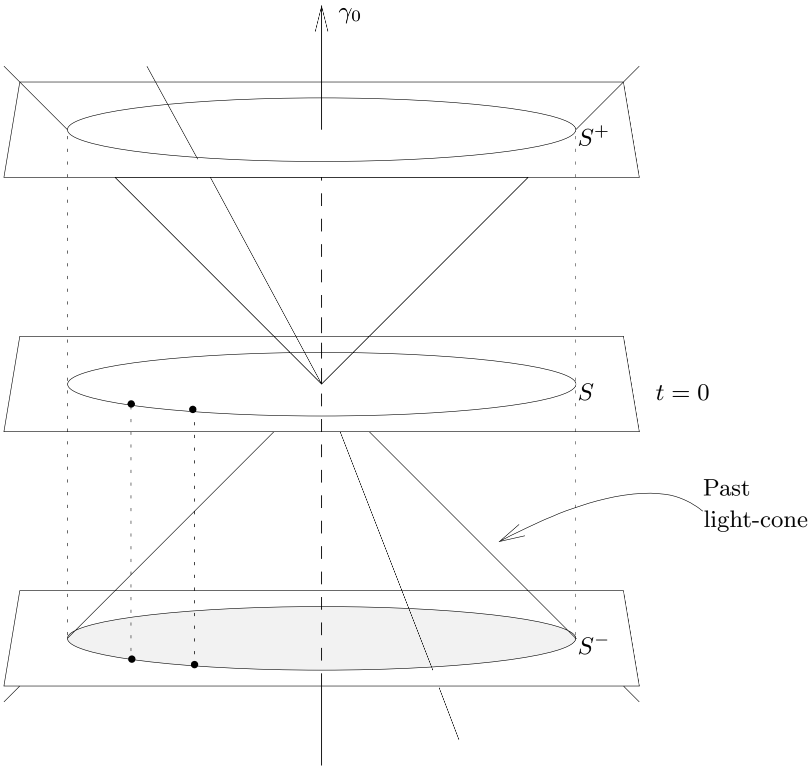

Figure 5.6 *The celestial sphere.* Each observer sees events in their past light-cone, which can be viewed as defining a sphere (shown here as a circle in a plane).

then that light is received along the null vector n, with the observer's velocity chosen to be γ_0. The relative vector in the γ_0 frame is $n\wedge\gamma_0$. This has magnitude

$$(n\wedge\gamma_0)^2 = (n\cdot\gamma_0)^2 - n^2\gamma_0^2 = (n\cdot\gamma_0)^2. \tag{5.112}$$

We therefore define the unit relative vector $\boldsymbol{n}$ by the projective formula

$$\boldsymbol{n} = \frac{n\wedge\gamma_0}{n\cdot\gamma_0}. \tag{5.113}$$

Observers passing through the same spacetime point at different velocities see different celestial spheres. If a second observer has velocity $v = R\gamma_0\tilde{R}$, the unit relative vectors in this observer's frame are formed from $n\wedge v/n\cdot v$. These can be brought to the γ_0 frame for comparison by forming

$$\boldsymbol{n}' = \tilde{R}\frac{n\wedge v}{n\cdot v}R = \frac{n'\wedge\gamma_0}{n'\cdot\gamma_0}, \tag{5.114}$$

where $n' = \tilde{R}nR$. The effects of Lorentz transformations can be visualised simply by moving around points on the celestial sphere with the map $n \mapsto \tilde{R}nR$. We know immediately, then, that two directions remain invariant and so describe the same points on the celestial spheres of two observers.

5.4.3 Relativistic visualisation

We have endeavoured to separate the concept of a single observer from that of a coordinate lattice. A clear illustration of this distinction arises when one studies how bodies appear when seen by different observers. Concentrating purely on coordinates leads directly to the conclusion that there is a measurable Lorentz contraction in the direction of motion of a body moving relative to some coordinate system. But when we consider what two different observers actually *see*, the picture is rather different.

Suppose that two observers in relative motion observe a sphere. The sphere and one of the observers are both at rest in the γ_0 system. This observer sees the edge of the sphere as a circle defined by the unit vectors

$$\boldsymbol{n} = \sin(\theta)(\cos(\phi)\,\boldsymbol{\sigma}_1 + \sin(\phi)\,\boldsymbol{\sigma}_2) + \cos(\theta)\,\boldsymbol{\sigma}_3, \quad 0 \leq \phi < 2\pi. \tag{5.115}$$

The angle θ is fixed so the sphere subtends an angle 2θ on the sky and is centred on the 3 axis (see figure 5.7). The incoming photon paths from the sphere are defined by the family of null vectors

$$n = (1 - \boldsymbol{n})\gamma_0. \tag{5.116}$$

Now suppose that a second observer has velocity $\beta = \tanh(\alpha)$ along the 1 axis, so

$$v = \cosh(\alpha)\,\gamma_0 + \sinh(\alpha)\,\gamma_1 = R\gamma_0\tilde{R}, \tag{5.117}$$

where $R = \exp(\alpha\,\gamma_1\gamma_0/2)$. To compare what these two observers see we form

$$\begin{aligned} n' = \tilde{R}nR = {}&\cosh(\alpha)\big(1 + \beta\sin(\theta)\,\cos(\phi)\big)\gamma_0 - \cosh(\alpha)\big(\sin(\theta)\,\cos(\phi) + \beta\big)\gamma_1 \\ &- \sin(\theta)\,\sin(\phi)\,\gamma_2 - \cos(\theta)\,\gamma_3. \end{aligned} \tag{5.118}$$

And from this the new unit relative outward vector is

$$\boldsymbol{n}' = \frac{\cosh(\alpha)\big(\sin(\theta)\,\cos(\phi) + \beta\big)\boldsymbol{\sigma}_1 + \sin(\theta)\,\sin(\phi)\,\boldsymbol{\sigma}_2 + \cos(\theta)\,\boldsymbol{\sigma}_3}{\cosh(\alpha)\big(1 + \beta\sin(\theta)\,\cos(\phi)\big)}. \tag{5.119}$$

Now consider the vector

$$\boldsymbol{c} = \boldsymbol{\sigma}_3 + \sinh(\alpha)\,\cos(\theta)\,\boldsymbol{\sigma}_1. \tag{5.120}$$

This vector satisfies

$$\boldsymbol{c}\cdot\boldsymbol{n}' = \cosh(\alpha)\,\cos(\theta), \tag{5.121}$$

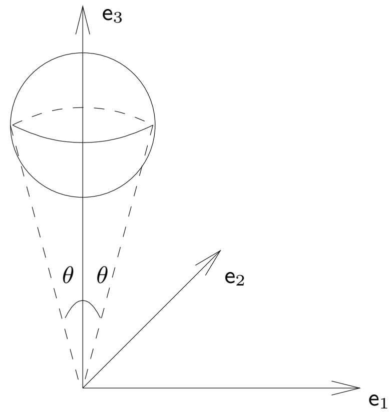

Figure 5.7 *Relativistic visualization of a sphere.* The sphere is at rest in the γ_0 frame with its centre a unit distance along the 3 axis. The sphere is simultaneously observed by two observers placed at the spatial origin. One observer is at rest in the γ_0 system, and the other is moving along the 1 axis.

which is independent of ϕ. It follows that, from the point of view of the second observer, all points on the edge of the sphere subtend the same angle to $\boldsymbol{c}$. So the vector $\boldsymbol{c}$ must lie at the centre of a circle, and the second observer still sees the edge of the sphere as circular. That is, both observers see the sphere as a sphere, and there is no observable contraction along the direction of motion. The only difference is that the moving observer sees the angular diameter of the sphere reduced from 2θ to $2\theta'$, where

$$\begin{aligned} \cos(\theta') &= \frac{\cos(\theta)\,\cosh(\alpha)}{\left(1+\sinh^2(\alpha)\,\cos^2(\theta)\right)^{1/2}}, \\ \tan(\theta') &= \frac{\tan(\theta)}{\gamma}. \end{aligned} \tag{5.122}$$

More generally, moving observers see solid objects as rotated, as opposed to contracted along their direction of motion. Visualising Lorentz transformations of solid objects has now been discussed by various authors (see Rau, Weiskopf & Ruder (1998)). But the original observation that spheres remain spheres for observers in relative motion had to wait until 1959 — more than 50 years after the development of special relativity! The first authors to point out this invisibility of the Lorentz contraction were Terrell (1959) and Penrose (1959). Both authors based their studies on the fact that the Lorentz group is isomorphic to the conformal group acting on the surface of a sphere. This type of geometry is discussed in chapter 10.

5.4.4 Pure boosts and observer splits

Suppose we are travelling with velocity u and want to boost to velocity v. We seek the rotor for this which contains no additional rotational factors. We have

$$v = Lu\tilde{L} \tag{5.123}$$

with $La_\perp\tilde{L} = a_\perp$ for any vector outside the $u\wedge v$ plane. It is clear that the appropriate bivector for the rotor is $u\wedge v$, and as this anticommutes with u and v we have

$$v = Lu\tilde{L} = L^2 u \quad \Rightarrow L^2 = vu. \tag{5.124}$$

The solution to this is

$$L = \frac{1+vu}{[2(1+u\cdot v)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge u}{|v\wedge u|}\right), \tag{5.125}$$

where the angle α is defined by $\cosh(\alpha) = u\cdot v$.

Now suppose that we start in the γ_0 frame and some arbitrary rotor R takes this to $v = R\gamma_0\tilde{R}$. We know that the pure boost for this transformation is

$$L = \frac{1+v\gamma_0}{[2(1+v\cdot\gamma_0)]^{1/2}} = \exp\left(\frac{\alpha}{2}\frac{v\wedge\gamma_0}{|v\wedge\gamma_0|}\right), \tag{5.126}$$

where $v\cdot\gamma_0 = \cosh(\alpha)$. Now define the further rotor U by

$$U = \tilde{L}R, \qquad U\tilde{U} = \tilde{L}R\tilde{R}L = 1. \tag{5.127}$$

This satisfies

$$U\gamma_0\tilde{U} = \tilde{L}vL = \gamma_0, \tag{5.128}$$

so $U\gamma_0 = \gamma_0 U$. We must therefore have $U = \exp(I\boldsymbol{b}/2)$, where $I\boldsymbol{b}$ is a relative bivector, and U generates a pure rotation in the γ_0 frame. We now have

$$R = LU, \tag{5.129}$$

which decomposes R into a relative rotation and boost. Unlike the invariant decomposition into a boost and rotation of equation (5.107), the boost L and rotation U will not usually commute. The fact that the LU decomposition initially singled out the γ_0 vector shows that the decomposition is frame-dependent. Both the invariant split of equation (5.107) and the frame-dependent split of equation (5.129) are useful in practice.

5.5 Spacetime dynamics

Dynamics in spacetime is traditionally viewed as a hard subject. This need not be the case, however. We have now established that Lorentz transformations which preserve parity and causal structure can be described with rotors. By parameterising the motion in terms of rotors many equations are considerably simplified, and can be solved in new ways. This provides a simple understanding of the Thomas precession, as well as a new formulation of the Lorentz force law for a particle in an electromagnetic field.

5.5.1 Rotor equations and Fermi transport

A spacetime trajectory $x(\tau)$ has a future-pointing velocity vector $\dot{x} = v$. This is normalised to $v^2 = 1$ by parameterising the curve in terms of the proper time. This suggests an analogy with rigid-body dynamics. We write

$$v = R\gamma_0\tilde{R}, \tag{5.130}$$

which keeps v future-pointing and normalised. This moves all of the dynamics into the rotor $R = R(\tau)$, and this is the key idea which simplifies much of relativistic dynamics. The next quantity we need to find is the acceleration

$$\dot{v} = \frac{d}{d\tau}(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}. \tag{5.131}$$

But just as in three dimensions, $\dot{R}\tilde{R}$ is of even grade and is equal to minus its reverse, so can only contain bivector terms. We therefore have

$$\begin{aligned}\dot{v} &= \dot{R}\tilde{R}v - v\dot{R}\tilde{R} \\ &= 2(\dot{R}\tilde{R})\cdot v.\end{aligned} \tag{5.132}$$

This equation is consistent with the fact that $v\cdot\dot{v} = 0$, which follows from $v^2 = 1$.

If we now form the acceleration bivector we obtain

$$\dot{v}v = 2(\dot{R}\tilde{R})\cdot v\, v. \tag{5.133}$$

This determines the projection of the bivector into the instantaneous rest frame defined by v. In this frame the projected bivector is purely timelike and corresponds to a pure boost. The remaining freedom in $\dot{R}\tilde{R}$ corresponds to an additional rotation in R which does not change v.

For the purposes of determining the velocity and trajectory of a particle the component of $\dot{R}\tilde{R}$ perpendicular to v is of no relevance. In some applications, however, it is useful to attach physical significance to the comoving frame vectors $\{\mathsf{e}_\mu\}$,

$$\mathsf{e}_\mu = R\gamma_\mu\tilde{R}, \tag{5.134}$$

which have $\mathsf{e}_0 = v$. The spatial set of vectors $\{\mathsf{e}_i\}$ satisfy $\mathsf{e}_i\cdot v = 0$ and span the instantaneous rest space of v. In this case, the dynamics of the e_i can be used to determine the component of $\dot{R}\tilde{R}$ which is not fixed by v alone.

The vectors $\{\mathsf{e}_i\}$ are carried along the trajectory by the rotor R. They are said to be *Fermi-transported* if their transformation from one instant to the next is

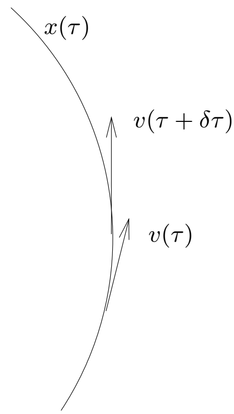

Figure 5.8 *The proper boost.* The change in velocity from τ to $\tau + \delta\tau$ should be described by a rotor solely in the $\dot{v}\wedge v$ plane.

a pure boost in the v frame. In this case the $\{\mathsf{e}_i\}$ vectors remain 'as constant as possible', subject to the constraint $\mathsf{e}_i \cdot v = 0$. For example, the direction defined by the angular momentum of an inertial guidance gyroscope (supported at its centre of mass so there are no torques) is Fermi-transported along the path of the gyroscope through spacetime.

To ensure Fermi-transport of $R\gamma_i\tilde{R}$ we need to ensure that the rotor describes pure boosts from one instant to the next (see figure 5.8). To first order in $\delta\tau$ we have

$$v(\tau + \delta\tau) = v(\tau) + \delta\tau\, \dot{v}. \tag{5.135}$$

The pure boost between $v(\tau)$ and $v(\tau + \delta\tau)$ is determined by the rotor

$$L = \frac{1 + v(\tau + \delta\tau)v(\tau)}{[2(1 + v(\tau + \delta\tau)\cdot v(\tau))]^{1/2}} = 1 + \tfrac{1}{2}\delta\tau\, \dot{v}v, \tag{5.136}$$

to first order in $\delta\tau$. But since

$$R(\tau + \delta\tau) = R(\tau) + \delta\tau\dot{R}(\tau) = (1 + \delta\tau\dot{R}\tilde{R})R(\tau), \tag{5.137}$$

the additional rotation that takes the $\{\mathsf{e}_i\}$ frame from τ to $\tau + \delta\tau$ is described by the rotor $1 + \delta\tau\dot{R}\tilde{R}$. Equating this to the pure boost L of equation (5.136), we find that the correct expression to ensure Fermi-transport of the $\{\mathsf{e}_i\}$ is

$$\dot{R}\tilde{R} = \tfrac{1}{2}\dot{v}v. \tag{5.138}$$

This is as one would expect. The bivector describing the change in the rotor is simply the acceleration bivector, which is the acceleration seen in the instantaneous rest frame.

Under Fermi-transport the $\{\mathsf{e}_i\}$ frame vectors satisfy

$$\dot{\mathsf{e}}_i = 2(\dot{R}\tilde{R})\cdot \mathsf{e}_i = -\mathsf{e}_i\cdot(\dot{v}v). \tag{5.139}$$

This leads directly to the definition of the *Fermi derivative*

$$\frac{Da}{D\tau} = \dot{a} + a\cdot(\dot{v}v). \tag{5.140}$$

The Fermi derivative of a vector vanishes if the vector is Fermi-transported along the worldline. The derivative preserves both the magnitude a^2 and $a \cdot v$. The former holds because

$$\frac{d}{d\tau}(a^2) = -2a\cdot\big(a\cdot(\dot{v}\wedge v)\big) = 0. \tag{5.141}$$

Conservation of $a \cdot v$ is also straightforward to check:

$$\begin{aligned}\frac{d}{d\tau}(a\cdot v) &= -\big(a\cdot(\dot{v}v)\big)\cdot v + a\cdot\dot{v} \\ &= -a\cdot\dot{v} + a\cdot v\, \dot{v}\cdot v + a\cdot\dot{v} = 0.\end{aligned} \tag{5.142}$$

It follows that if a starts perpendicular to v it remains so. In the case where $a\cdot v = 0$ the Fermi derivative takes on the simple form

$$\frac{Da}{D\tau} = \dot{a} + a\cdot\dot{v}\, v = \dot{a} - \dot{a}\cdot v\, v = \dot{a}\wedge v\, v. \tag{5.143}$$

This is the projection of $\dot{a}$ perpendicular to v, as expected. The Fermi derivative extends simply to multivectors as follows:

$$\frac{DM}{D\tau} = \frac{dM}{d\tau} + M\times(\dot{v}v). \tag{5.144}$$

Derivatives of this type are important in gauge theories and gravity.

5.5.2 Thomas precession

As an application, consider a particle in a circular orbit (figure 5.9). The worldline is

$$x(\tau) = t(\tau)\gamma_0 + a(\cos(\omega t)\gamma_1 + \sin(\omega t)\gamma_2), \tag{5.145}$$

and the velocity is

$$v = \dot{x} = \dot{t}\big(\gamma_0 + a\omega(-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2)\big). \tag{5.146}$$

The relative velocity as seen in the γ_0 frame, $\boldsymbol{v} = v\wedge\gamma_0/v\cdot\gamma_0$, has magnitude $|\boldsymbol{v}| = a\omega$. We therefore introduce the hyperbolic angle α, with

$$\tanh(\alpha) = a\omega, \quad \dot{t} = \cosh(\alpha). \tag{5.147}$$

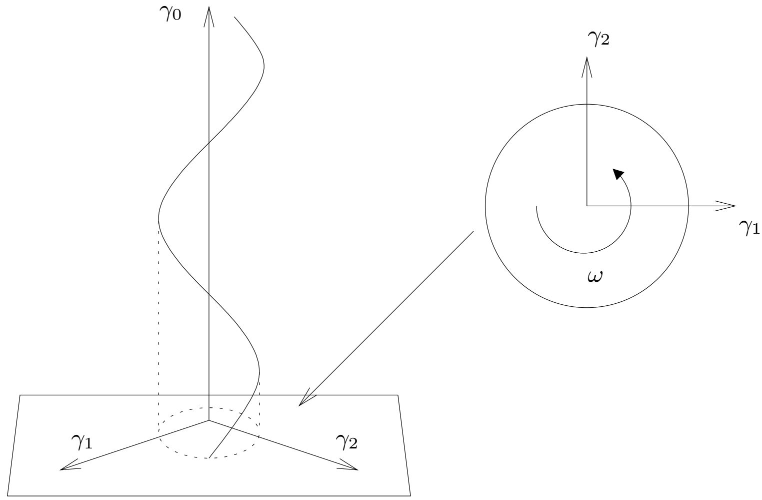

Figure 5.9 *Thomas precession.* The particle follows a helical worldline, rotating at a constant rate in the γ_0 frame.

The velocity is now

$$\begin{aligned} v &= \cosh(\alpha)\,\gamma_0 + \sinh(\alpha)\big(-\sin(\omega t)\gamma_1 + \cos(\omega t)\gamma_2\big) \\ &= \mathrm{e}^{\alpha \boldsymbol{n}/2}\gamma_0 \mathrm{e}^{-\alpha \boldsymbol{n}/2}, \end{aligned} \tag{5.148}$$

where

$$\boldsymbol{n} = -\sin(\omega t)\boldsymbol{\sigma}_1 + \cos(\omega t)\boldsymbol{\sigma}_2. \tag{5.149}$$

This form of time dependence in the rotor is inconvenient to work with. To simplify, we write

$$\boldsymbol{n} = \mathrm{e}^{-\omega t I \boldsymbol{\sigma}_3}\boldsymbol{\sigma}_2 = R_\omega \boldsymbol{\sigma}_2 \tilde{R}_\omega, \tag{5.150}$$

where $R_\omega = \exp(-\omega t I\boldsymbol{\sigma}_3/2)$. We now have

$$\mathrm{e}^{\alpha \boldsymbol{n}/2} = \exp(\alpha R_\omega \boldsymbol{\sigma}_2 \tilde{R}_\omega/2) = R_\omega R_\alpha \tilde{R}_\omega, \tag{5.151}$$

where

$$R_\alpha = \exp(\alpha \boldsymbol{\sigma}_2/2). \tag{5.152}$$

The velocity is now given by

$$v = R_\omega R_\alpha \tilde{R}_\omega \gamma_0 R_\omega \tilde{R}_\alpha \tilde{R}_\omega = R_\omega R_\alpha \gamma_0 \tilde{R}_\alpha \tilde{R}_\omega. \tag{5.153}$$

The final expression follows because R_ω commutes with γ_0.

We can now see that the rotor for the motion must have the form

$$R = R_\omega R_\alpha \Phi, \tag{5.154}$$

where Φ is a rotor that commutes with γ_0. We want R to describe Fermi transport of the $\{\mathsf{e}_i\}$, so we must have $\dot{v}v = 2\dot{R}\tilde{R}$. We begin by forming the acceleration bivector $\dot{v}v$. We can simplify this derivation by writing $v = R_\omega v_\alpha \tilde{R}_\omega$, where $v_\alpha = R_\alpha \gamma_0 \tilde{R}_\alpha$. We then find that

$$\begin{aligned}\dot{v}v &= R_\omega \big(2(\tilde{R}_\omega \dot{R}_\omega)\cdot v_\alpha\, v_\alpha\big)\tilde{R}_\omega \\ &= -\omega\, \cosh(\alpha)\, R_\omega \big((I\boldsymbol{\sigma}_3)\cdot v_\alpha\, v_\alpha\big)\tilde{R}_\omega \\ &= \omega\, \sinh(\alpha)\, \cosh(\alpha)\, R_\omega \big(-\cosh(\alpha)\,\boldsymbol{\sigma}_1 + \sinh(\alpha)\, I\boldsymbol{\sigma}_3\big)\tilde{R}_\omega.\end{aligned} \tag{5.155}$$

We also form the rotor equivalent, $2\dot{R}\tilde{R}$, which is

$$\begin{aligned}2\dot{R}\tilde{R} &= 2\dot{R}_\omega \tilde{R}_\omega + 2R_\omega R_\alpha \dot{\Phi}\tilde{\Phi}\tilde{R}_\alpha \tilde{R}_\omega \\ &= -\omega \cosh(\alpha)\, I\boldsymbol{\sigma}_3 + 2R_\omega R_\alpha \dot{\Phi}\tilde{\Phi}\tilde{R}_\alpha \tilde{R}_\omega.\end{aligned} \tag{5.156}$$

Equating the two preceding results we find that

$$\begin{aligned}2\dot{\Phi}\tilde{\Phi} &= \omega \cosh^2(\alpha)\, \tilde{R}_\alpha \big(-\sinh(\alpha)\,\boldsymbol{\sigma}_1 + \cosh(\alpha)\, I\boldsymbol{\sigma}_3\big) R_\alpha \\ &= \omega \cosh^2(\alpha)\, I\boldsymbol{\sigma}_3.\end{aligned} \tag{5.157}$$

The solution with $\Phi = 1$ at $t = 0$ is $\Phi = \exp(\omega \cosh(\alpha) t I\boldsymbol{\sigma}_3/2)$, so the full rotor is

$$R = \mathrm{e}^{-\omega t I\boldsymbol{\sigma}_3/2}\mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\mathrm{e}^{\cosh(\alpha)\omega t I\boldsymbol{\sigma}_3/2}. \tag{5.158}$$

This form of the rotor ensures that the $\mathsf{e}_i = R\gamma_i\tilde{R}$ are Fermi transported. The fact that the 'internal' rotation rate $\omega \cosh(\alpha)$ differs from ω is due to the fact that the acceleration is formed in the instantaneous rest frame v and not the fixed γ_0 frame. This difference introduces a precession — the *Thomas precession*. We can see this effect by imagining the vector γ_1 being transported around the circle. The rotated vector is

$$\mathsf{e}_1 = R\gamma_1\tilde{R}. \tag{5.159}$$

In the low velocity limit $\cosh(\alpha) \mapsto 1$ the vector γ_1 continues to point in the γ_1 direction and the frame does not rotate, as we would expect. At larger velocities, however, the frame starts to precess. After time $t = 2\pi/\omega$, for example, the γ_1 vector is transformed to

$$\mathsf{e}_1(2\pi/\omega) = \mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}\mathrm{e}^{2\pi\cosh(\alpha) I\boldsymbol{\sigma}_3}\gamma_1 \mathrm{e}^{-\alpha\boldsymbol{\sigma}_2/2}. \tag{5.160}$$

Dotting this with the initial vector $\mathsf{e}_1(0) = \gamma_1$ we see that the vector has precessed through an angle

$$\theta = 2\pi(\cosh(\alpha) - 1). \tag{5.161}$$

This shows that the effect is of order $|\boldsymbol{v}|^2/c^2$. The form of the Thomas precession justifies one of the relativistic corrections to the spin-orbit coupling in the Pauli theory of the electron.

5.5.3 The Lorentz force law

The non-relativistic form of the Lorentz force law for a particle of charge q is

$$\frac{d\boldsymbol{p}}{dt} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{5.162}$$

where the $\times$ here denotes the vector cross product, and all relative vectors are expressed in some global Newtonian frame, which we will take to be the γ_0 frame. We seek a covariant relativistic version of this law. The quantity $\boldsymbol{p}$ on the left-hand side is the relative vector $p\wedge\gamma_0$. Since $dt = \gamma d\tau$, we must multiply through by $\gamma = v\cdot\gamma_0$ to convert the derivative into one with respect to proper time. The first term on the right-hand side then includes

$$\begin{aligned} v\cdot\gamma_0\, \boldsymbol{E} &= \tfrac{1}{4}\big(\boldsymbol{E}(v\gamma_0 + \gamma_0 v) + (v\gamma_0 + \gamma_0 v)\boldsymbol{E}\big) \\ &= \tfrac{1}{4}\big((\boldsymbol{E}v - v\boldsymbol{E})\gamma_0 - \gamma_0(\boldsymbol{E}v - v\boldsymbol{E})\big) \\ &= (\boldsymbol{E}\cdot v)\wedge\gamma_0. \end{aligned} \tag{5.163}$$

Recall at this point that $\boldsymbol{E}$ is a spacetime bivector built from the $\boldsymbol{\sigma}_k = \gamma_k\gamma_0$, so $\boldsymbol{E}$ *anticommutes* with γ_0.

For the magnetic term in equation (5.162) we first replace the cross product by the equivalent three-dimensional expression $(I\boldsymbol{B})\cdot\boldsymbol{v}$. Expanding out, and expressing in the full spacetime algebra, we obtain

$$\begin{aligned} \tfrac{1}{2}v\cdot\gamma_0(I\boldsymbol{B}\boldsymbol{v} - \boldsymbol{v}I\boldsymbol{B}) &= \tfrac{1}{4}\big(I\boldsymbol{B}(v\gamma_0 - \gamma_0 v) - (v\gamma_0 - \gamma_0 v)I\boldsymbol{B}\big) \\ &= \tfrac{1}{4}\big((I\boldsymbol{B}v - vI\boldsymbol{B})\gamma_0 - \gamma_0(I\boldsymbol{B}v - vI\boldsymbol{B})\big) \\ &= \big((I\boldsymbol{B})\cdot v\big)\wedge\gamma_0, \end{aligned} \tag{5.164}$$

where we use the fact that γ_0 *commutes* with $I\boldsymbol{B}$. Combining equations (5.163) and (5.164) we can now write the Lorentz force law (5.162) in the form

$$\frac{d\boldsymbol{p}}{d\tau} = \dot{p}\wedge\gamma_0 = q\big((\boldsymbol{E} + I\boldsymbol{B})\cdot v\big)\wedge\gamma_0. \tag{5.165}$$

We next define the *Faraday bivector* F by

$$F = \boldsymbol{E} + I\boldsymbol{B}. \tag{5.166}$$

This is the covariant form of the electromagnetic field strength. It unites the electric and magnetic fields into a single spacetime structure. We study this in greater detail in chapter 7. The Lorentz force law can now be written

$$\dot{p}\wedge\gamma_0 = q(F\cdot v)\wedge\gamma_0. \tag{5.167}$$

The rate of working on the particle is $q\boldsymbol{E}\cdot\boldsymbol{v}$, so

$$\frac{dp_0}{dt} = q\boldsymbol{E}\cdot\boldsymbol{v}. \tag{5.168}$$

Here, $p_0 = p\cdot\gamma_0$ is the particle's energy in the γ_0 frame. Multiplying through by $v\cdot\gamma_0$, we find

$$\dot{p}\cdot\gamma_0 = q\boldsymbol{E}\cdot(v\wedge\gamma_0) = q(F\cdot v)\cdot\gamma_0. \tag{5.169}$$

In the final step we have used $(I\boldsymbol{B})\cdot(v\wedge\gamma_0) = 0$. Adding this equation to equation (5.167), and multiplying on the right by γ_0, we find

$$\dot{p} = qF\cdot v. \tag{5.170}$$

Recalling that $p = mv$, we arrive at the relativistic form of the *Lorentz force law*,

$$m\dot{v} = qF\cdot v. \tag{5.171}$$

This is *manifestly* Lorentz covariant, because no particular frame is picked out. The acceleration bivector is

$$\dot{v}v = \frac{q}{m}F\cdot v\, v = \frac{q}{m}(F\cdot v)\wedge v = \frac{q}{m}\boldsymbol{E}_v, \tag{5.172}$$

where $\boldsymbol{E}_v$ is the relative electric field in the v frame. A charged point particle only responds to the instantaneous electric field in its frame. Algebraically, this bivector is

$$\boldsymbol{E}_v = \tfrac{1}{2}(F - vFv). \tag{5.173}$$

So $\boldsymbol{E}_v$ is the component of the bivector F which anticommutes with v.

Now suppose that we parameterise the velocity with a rotor, so that $v = R\gamma_0\tilde{R}$. We have

$$\dot{v} = 2\dot{R}\tilde{R}v = 2(\dot{R}\tilde{R})\cdot v = \frac{q}{m}F\cdot v. \tag{5.174}$$

The simplest form of the rotor equation comes from equating the projected terms:

$$\dot{R} = \frac{q}{2m}FR. \tag{5.175}$$

This is not the most general possibility as we could include an extra multiple of $F\wedge v\, v$. The rotor determined by equation (5.175) will not, in general, describe Fermi-transport of the $R\gamma_i\tilde{R}$ vectors. However, equation (5.175) is sufficient to determine the velocity of the particle, and is certainly the simplest form of rotor equation to work with. As we now demonstrate, the rotor equation (5.175) is remarkably efficient when it comes to solving the dynamical equations.

5.5.4 Constant field

Motion in a constant field is easy to solve for now. We can immediately integrate the rotor equation to give

$$R = \exp\left(\frac{q}{2m}F\tau\right)R_0. \tag{5.176}$$

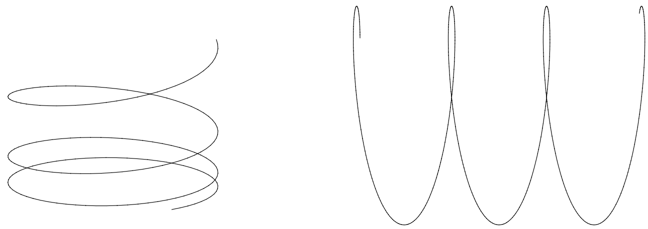

Figure 5.10 *Particle in a constant field.* The general motion is a combination of linear acceleration and circular motion. The plot on the left has $\boldsymbol{E}$ and $\boldsymbol{B}$ colinear. The plot on the right has $\boldsymbol{E}$ entirely in the $I\boldsymbol{B}$ plane, giving rise to cycloids.

To proceed and recover the trajectory we form the invariant decomposition of F. We first write

$$F^2 = \langle F^2 \rangle_0 + \langle F^2 \rangle_4 = \rho \mathrm{e}^{I\theta}, \tag{5.177}$$

so that we can set

$$F = \rho^{1/2} \mathrm{e}^{I\theta/2} \hat{F} = \alpha \hat{F} + I\beta \hat{F}, \tag{5.178}$$

where $\hat{F}^2 = 1$. (If F is null a slightly different procedure is followed.) We now have

$$R = \exp\left(\frac{q}{2m}\alpha\hat{F}\tau\right) \exp\left(\frac{q}{2m}I\beta\hat{F}\tau\right) R_0. \tag{5.179}$$

Next we decompose the initial velocity $v_0 = R_0 \gamma_0 \tilde{R}_0$ into components in and out of the $\hat{F}$ plane:

$$v_0 = \hat{F}^2 v_0 = \hat{F}\,\hat{F}\cdot v_0 + \hat{F}\,\hat{F}\wedge v_0 = v_{0\parallel} + v_{0\perp}. \tag{5.180}$$

Now $v_{0\parallel} = \hat{F}\,\hat{F}\cdot v_0$ anticommutes with $\hat{F}$, and $v_{0\perp}$ commutes with $\hat{F}$, so

$$\dot{x} = \exp\left(\frac{q}{m}\alpha\hat{F}\tau\right) v_{0\parallel} + \exp\left(\frac{q}{m}I\beta\hat{F}\tau\right) v_{0\perp}. \tag{5.181}$$

This integrates immediately to give the particle history

$$x - x_0 = \frac{\mathrm{e}^{q\alpha\hat{F}\tau/m} - 1}{q\alpha/m} \hat{F}\cdot v_0 - \frac{\mathrm{e}^{q\beta I\hat{F}\tau/m} - 1}{q\beta/m} (I\hat{F})\cdot v_0. \tag{5.182}$$

The first term gives linear acceleration and the second is periodic and drives rotational motion (see figure 5.10). One has to be slightly careful integrating the velocity equation in the case where either α or β is zero, which corresponds to perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

5.5.5 Particle in a Coulomb field

As a further application we consider the case of a charged point particle moving in a central Coulomb field. If relativistic effects are ignored the problem reduces to the inverse-square force law described in section 3.2.1. We therefore expect that the relativistic description will add additional perturbative effects to the elliptic and hyperbolic orbits found in the inverse-square case. We assume for simplicity that the central charge has constant velocity γ_0 and is placed at the origin. The electromagnetic field is

$$F = \frac{Q\boldsymbol{x}}{4\pi\epsilon_0 r^3}, \tag{5.183}$$

where $\boldsymbol{x} = x\wedge\gamma_0$ and $r^2 = \boldsymbol{x}^2$. In this section all bold symbols denote relative vectors in the γ_0 frame. The question of how to generalise the non-relativistic definitions of centre of mass and relative separation turns out to be surprisingly complex and is not tackled here. Instead we will simply assume that the source of the Coulomb field is far heavier than the test charge so that the source's motion can be ignored.

There are two constants of motion for this force law. The first is the energy

$$E = mv\cdot\gamma_0 + \frac{qQ}{4\pi\epsilon_0 r}. \tag{5.184}$$

If the charges are opposite, qQ is negative and the potential is attractive. The force law can now be written in the γ_0 frame as

$$m\frac{d^2\boldsymbol{x}}{d\tau^2} = \frac{qQ\boldsymbol{x}}{4\pi\epsilon_0 r^3}\left(\frac{E}{m} - \frac{qQ}{4\pi\epsilon_0 mr}\right). \tag{5.185}$$

The second conserved quantity is the angular momentum, which is conserved for any central force, as is the case in equation (5.185). If we define the spacetime bivector $L = x\wedge p$ we find that

$$\dot{L} = qx\wedge(F\cdot v). \tag{5.186}$$

It follows that the trivector $L\wedge\gamma_0$ is conserved. Equivalently, we can define the relative bivector

$$I\boldsymbol{l} = L\wedge\gamma_0\,\gamma_0, \tag{5.187}$$

so that the relative vector $\boldsymbol{l}$ is conserved. This is the relative angular momentum vector and satisfies $\boldsymbol{x}\cdot\boldsymbol{l} = 0$. It follows that the test particle's motion takes place in a constant plane as seen from the source charge.

In order to integrate the rotor equation we need to find a way to express the field as a function of the particle's proper time. This is achieved by introducing an angular measure in the plane of motion. Suppose that we align the 3 axis with $\boldsymbol{l}$, so that we can write

$$\hat{\boldsymbol{x}}(\tau) = \boldsymbol{\sigma}_1 \exp\bigl(I\boldsymbol{\sigma}_3\theta(\tau)\bigr), \tag{5.188}$$

where $\hat{\boldsymbol{x}}$ is the unit relative vector $\boldsymbol{x}/r$. It follows that

$$\boldsymbol{l}^2 = m^2 r^4 \dot{\hat{\boldsymbol{x}}}^2 = m^2 r^4 \dot{\theta}^2. \tag{5.189}$$

If we set $l = |\boldsymbol{l}|$ we have $l = mr^2\dot{\theta}$, which enables us to express the Coulomb field as

$$F = \frac{Qm\dot{\theta}\boldsymbol{\sigma}_1 \exp\bigl(I\boldsymbol{\sigma}_3\theta(\tau)\bigr)}{4\pi\epsilon_0 l}. \tag{5.190}$$

If we now let

$$\kappa = \frac{qQ}{4\pi\epsilon_0 l} \tag{5.191}$$

the rotor equation takes on the simple form

$$\frac{dR}{d\theta} = \frac{\kappa}{2}\boldsymbol{\sigma}_1 \exp(I\boldsymbol{\sigma}_3\theta)R. \tag{5.192}$$

Re-expressing the differential equation in terms of θ is a standard technique for solving inverse-square problems in non-relativistic physics. But this technique fails to give a simple solution to the relativistic equation (5.185). Instead, we see that the technique gives a simple solution to the relativistic problem only if applied directly to the rotor equation.

To solve equation (5.192) we first set

$$R = \exp(-I\boldsymbol{\sigma}_3\theta/2)U. \tag{5.193}$$

It follows that

$$\frac{dU}{d\theta}\tilde{U} = \tfrac{1}{2}(\kappa\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_3), \tag{5.194}$$

which integrates straightforwardly. The full rotor is then

$$R = \mathrm{e}^{-I\boldsymbol{\sigma}_3\theta/2}\mathrm{e}^{A\theta/2}R_0, \tag{5.195}$$

where

$$A = \kappa\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_3. \tag{5.196}$$

The initial conditions can be chosen such that $\theta(0) = 0$, which tells us how to align the 1 axis. The rotor R_0 then specifies the initial velocity v_0. If we are not interested in transporting a frame, R_0 can be set equal to a pure boost from γ_0 to v_0.

With the rotor equation now solved, the velocity can be integrated to recover the trajectory. Clearly, different types of path are obtained for the different signs of $A^2 = \kappa^2 - 1$. The equation relating r and θ is found from the relation

$$-\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{m}{l}\hat{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}}. \tag{5.197}$$

To evaluate the right-hand side we need

$$\begin{aligned}\hat{\boldsymbol{x}}\cdot\dot{\boldsymbol{x}} &= \langle \mathrm{e}^{-I\boldsymbol{\sigma}_3\theta/2}\boldsymbol{\sigma}_1\mathrm{e}^{I\boldsymbol{\sigma}_3\theta/2}R\gamma_0\tilde{R}\gamma_0\rangle \\ &= -\langle\gamma_1\mathrm{e}^{A\theta/2}v_0\mathrm{e}^{-A\theta/2}\rangle \\ &= \langle \mathrm{e}^{-A\theta}\gamma^1 v_0\rangle. \end{aligned} \tag{5.198}$$

It follows that

$$-\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{m}{l}\langle \mathrm{e}^{-A\theta}\gamma^1 v_0\rangle. \tag{5.199}$$

For a given l and v_0 this integrates to give the trajectory in the $\boldsymbol{Il}$ plane.

Suppose, for example, that we are interested in bound states. For these we must have $A^2 < 0$, which implies that $\kappa^2 < 1$. We write

$$|A| = (1-\kappa^2)^{1/2} \tag{5.200}$$

for the magnitude of A. To simplify the equations we will assume that $\tau = 0$ corresponds to a point on the trajectory where $\boldsymbol{v}$ is perpendicular to $\boldsymbol{x}$. In this case we have

$$v_0 = \cosh(\alpha_0)\,\gamma_0 + \sinh(\alpha_0)\,\gamma_2 \tag{5.201}$$

so that the trajectory is determined by

$$-\frac{d}{d\theta}\left(\frac{1}{r}\right) = \frac{m}{l|A|}\big(\kappa\cosh(\alpha_0)+\sinh(\alpha_0)\big)\sin(|A|\theta). \tag{5.202}$$

The magnitude of the angular momentum is given by $l = mr_0\sinh(\alpha_0)$, which can be used to write

$$m\big(\kappa\cosh(\alpha_0)+\sinh(\alpha_0)\big) = (E^2 - m^2|A|^2)^{1/2}. \tag{5.203}$$

The trajectory is then given by

$$\frac{l|A|^2}{r} = -\kappa E + (E^2-m^2|A|^2)^{1/2}\cos(|A|\theta), \tag{5.204}$$

and since this represents a bound state, κ must be negative. The fact that the angular term goes as $\cos(|A|\theta)$ shows that this equation specifies a precessing ellipse (figure 5.11). The precession rate of the ellipse can be found simply using the technique of section 3.3.

5.5.6 The gyromagnetic moment

Particles with non-zero spin have a magnetic moment which is proportional to the spin. In non-relativistic physics we write this as $\boldsymbol{m} = \gamma\boldsymbol{s}$, where γ is the gyromagnetic ratio and $\boldsymbol{s}$ is the spin (which has units of angular momentum). The gyromagnetic ratio is usually written in the form

$$\gamma = g\frac{q}{2m}, \tag{5.205}$$

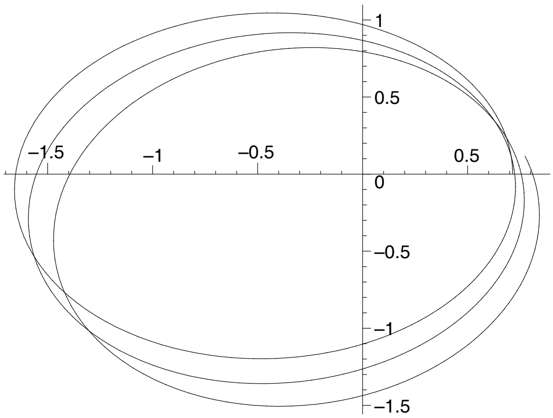

Figure 5.11 *Motion in a Coulomb field.* For bound orbits ($E < m$) the particle's motion is described by a precessing ellipse. The plot is for $|A| = 0.95$. The units are arbitrary.

where m is the particle mass, q is the charge and g is the (reduced) gyromagnetic ratio. The last is determined experimentally via the precession of the spin vector which, in classical physics, obeys

$$\dot{\boldsymbol{s}} = g\frac{q}{2m}(I\boldsymbol{B})\cdot\boldsymbol{s}. \tag{5.206}$$

We seek a relativistic extension of this equation. We start by introducing the relativistic spin vector s, which is perpendicular to the velocity v, so $s \cdot v = 0$. For a particle at rest in the γ_0 frame we have $s = \boldsymbol{s}\gamma_0$. The particle's spin will interact with the magnetic field only in the instantaneous rest frame, so we should regard equation (5.206) as referring to this frame.

Given that $\boldsymbol{s} = s\gamma_0$ we find that

$$\begin{aligned}(I\boldsymbol{B})\cdot\boldsymbol{s} &= \langle (F\wedge\gamma_0)\gamma_0 s\gamma_0\rangle_2 \\ &= (F\cdot s)\wedge\gamma_0.\end{aligned} \tag{5.207}$$

So, for a particle at rest in the γ_0 frame, equation (5.206) can be written

$$\frac{ds}{dt} = g\frac{q}{2m}(F\cdot s)\wedge\gamma_0\,\gamma_0. \tag{5.208}$$

To write down an equation which is valid for arbitrary velocity we must replace the two factors of γ_0 on the right-hand side with the velocity v. On the left-hand side we need the derivative of s which preserves $s \cdot v = 0$. This is the Fermi derivative of section 5.5.1, which tells us that the relativistic form of the spin precession equation is

$$\dot{s} + s\cdot(\dot{v}v) = g\frac{q}{2m}(F\cdot s)\wedge v\, v. \tag{5.209}$$

This equation tells us how much the spin vector rotates, relative to a Fermi-transported frame, which is physically sensible. We can eliminate the acceleration bivector $\dot{v}v$ by using the relativistic Lorentz force law to find

$$\begin{aligned}\dot{s} &= g\frac{q}{2m}(F\cdot s)\wedge v\, v - \frac{q}{m}s\cdot(F\cdot v\, v)\\ &= \frac{q}{2m}\big(g(F\cdot s)\wedge v + 2(F\cdot s)\cdot v\big)v\\ &= \frac{q}{m}F\cdot s + (g-2)\frac{q}{2m}(F\cdot s)\wedge v\, v.\end{aligned} \tag{5.210}$$

This is called the Bargmann–Michel–Telegdi equation.

For the value $g = 2$, the Bargmann–Michel–Telegdi equation reduces to

$$\dot{s} = \frac{q}{m}F\cdot s, \tag{5.211}$$

which has the same form as the Lorentz force law. In this sense, $g = 2$ is the most natural value of the gyromagnetic ratio of a point particle in relativistic physics. Ignoring quantum corrections, this is indeed found to be the value for an electron. Quantum corrections tell us that for an electron $g = 2(1+\alpha/2\pi+\cdots)$. The corrections are due to the fact that the electron is never truly isolated and constantly interacts with virtual particles from the quantum vacuum.

Given a velocity v and a spin vector s, with $v\cdot s = 0$ and s normalised to $s^2 = -1$, we can always find a rotor R such that

$$v = R\gamma_0\tilde{R}, \quad s = R\gamma_3\tilde{R}. \tag{5.212}$$

For these we have

$$\dot{v} = 2(\dot{R}\tilde{R})\cdot v, \quad \dot{s} = 2(\dot{R}\tilde{R})\cdot s. \tag{5.213}$$

For a particle with $g = 2$, this pair of equations reduces to the single rotor equation (5.175). The simple form of this equation further justifies the claim that $g = 2$ is the natural, relativistic value of the gyromagnetic ratio. This also means that once we have solved the rotor equation, we can simultaneously compute both the trajectory and the spin precession of a classical relativistic particle with $g = 2$.

5.6 Notes

There are many good introductions to special relativity. Standard references include the books by French (1968), Rindler (1977) and d'Inverno (1992). Practically all introductory books make heavy use of coordinate geometry. Geometric algebra was first systematically applied to the study of relativistic physics in the book *Space-Time Algebra* by Hestenes (1966). Since this book was published in 1966 many authors have applied spacetime algebra techniques to relativistic physics. The two most significant papers are again by Hestenes, 'Proper particle mechanics' and 'Proper dynamics of a rigid point particle' (1974a,b). These papers detail the use of rotor equations for solving problems in electrodynamics, and much of section 5.5 follows their presentation.

5.7 Exercises

5.1 Suppose that the spacetime bivector $\hat{B}$ satisfies $\hat{B}^2 = 1$. By writing $\hat{B} = \boldsymbol{a} + I\boldsymbol{b}$ in the γ_0 frame, show that we can write

$$\hat{B} = \cosh(u)\hat{\boldsymbol{a}} + \sinh(u)I\hat{\boldsymbol{b}} = \mathrm{e}^{uI\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}}\hat{\boldsymbol{a}},$$

where $\hat{\boldsymbol{a}}^2 = \hat{\boldsymbol{b}}^2 = 1$. Hence explain why we can write $\hat{B} = R\boldsymbol{\sigma}_3\tilde{R}$. By considering the null vectors $\gamma_0 \pm \gamma_3$, prove that we can always find two null vectors satisfying

$$\hat{B}\cdot n_\pm = \pm n_\pm.$$

5.2 The boost L from velocity u to velocity v satisfies

$$v = Lu\tilde{L} = L^2 u,$$

with $L\tilde{L} = 1$. Prove that a solution to this equation is

$$L = \frac{1 + vu}{[2(1 + v\cdot u)]^{1/2}}.$$

Is this solution unique? Show further that this solution can be written in the form

$$L = \exp\left(\frac{\alpha}{2}\frac{v\wedge u}{|v\wedge u|}\right),$$

where $\alpha > 0$ satisfies $\cosh(\alpha) = u\cdot v$.

5.3 *Compton scattering* occurs when a photon scatters off an electron. If we ignore quantum effects this can be modelled as a relativistic collison process. The incident photon has wavelength λ_0 in the frame in which the electron is initially stationary. Show that the wavelength after scattering, λ, satisfies

$$\lambda - \lambda_0 = \frac{2\pi\hbar}{mc}\big(1 - \cos(\theta)\big),$$

where θ is the angle through which the photon scatters.

5.4 A relativistic particle has velocity $v = R\gamma_0\tilde{R}$. Show that v satisfies the Lorentz force equation $m\dot{v} = qF\cdot v$ if R satisfies

$$\dot{R} = \frac{q}{2m}FR.$$

Show that the solution to this for a constant field is

$$R = \exp(qF\tau/2m)R_0.$$

Given that F is *null*, $F^2 = 0$, show that v is given by the polynomial

$$v = v_0 + \tau\frac{q}{m}F\cdot v_0 - \tau^2\frac{q^2}{4m^2}Fv_0F.$$

Suppose now that $F = \boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2$ and the particle is initially at rest in the γ_0 frame. Sketch the resultant motion in the $\gamma_1\gamma_3$ plane.

5.5 One way to construct the Fermi derivative of a vector a is to argue that we should 'de-boost' the vector at proper time $\tau + \delta\tau$ before comparing it with $a(\tau)$. Explain why this leads us to evaluate

$$\lim_{\delta\tau\to 0}\frac{1}{\delta\tau}\big(\tilde{L}a(\tau+\delta\tau)L - a(\tau)\big),$$

and confirm that this evaluates to $\dot{a} + a\cdot(\dot{v}v)$.

5.6 A frame is Fermi-transported along the worldline of a particle with velocity $v = R\gamma_0\tilde{R}$. The rotor R is decomposed into a rotation and boost in the γ_0 frame as $R = LU$. Show that the rotation U satisfies

$$2\dot{U}\tilde{U} = -(\tilde{L}\dot{L} + \gamma_0\tilde{L}\dot{L}\gamma_0).$$

What is the interpretation of the right-hand side in terms of the γ_0 frame?

5.7 The bivector $B = a\wedge b$ is Fermi-transported along a worldline by Fermi-transporting the two vectors a and b. Show that B remains a blade, and that the bivector satisfies

$$\frac{dB}{d\tau} + B\times(\dot{v}v) = 0.$$

5.8 A point particle with a gyromagnetic ratio $g = 2$ is in a circular orbit around a central Coulomb field. Show that in one complete orbit the spin vector rotates in the plane $A = \kappa\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_3$ by an amount $2\pi|A|$, where

$$\kappa = \frac{qQ}{4\pi\epsilon_0 l},$$

and l is the angular momentum.

5.9 Show that the Bargmann–Michel–Telegdi equation of (5.210) for a relativistic point particle with spin vector s can be written

$$\dot{s} = \frac{q}{m}\big(F + \tfrac{1}{2}(g-2)F\wedge v\, v\big)\cdot s.$$

Given that $v = R\gamma_0\tilde{R}$ and $s = R\gamma_3\tilde{R}$, show that the rotor R satisfies the equation

$$\dot{R} = \frac{q}{2m}FR + \frac{q}{4m}(g-2)RI\boldsymbol{B}_0,$$

where

$$I\boldsymbol{B}_0 = (\tilde{R}FR)\wedge\gamma_0\,\gamma_0.$$

Assuming that the electromagnetc field F is constant, prove that $\boldsymbol{B}_0$ is also constant. Hence study the precession of s for a particle with a gyromagnetic ratio $g \neq 2$.