9

Multiparticle states and quantum entanglement

The previous chapter dealt with the quantum theory of single particles in a background field. In this chapter we turn to the study of multiparticle quantum theory. In many ways, this subject is even more strange than the single-particle theory, as it forces us to face up to the phenomenon of quantum entanglement. The basic idea is simple enough to state. The joint state of a two-particle system is described by a *tensor product* state of the form $|\psi\rangle \otimes |\phi\rangle$. This is usually abbreviated to $|\psi\rangle|\phi\rangle$. Quantum theory allows for linear complex superpositions of multiparticle states, which allows us to consider states which have no classical counterpart. An example is the spin singlet state

$$|\varepsilon\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle|1\rangle - |1\rangle|0\rangle\big). \tag{9.1}$$

States such as these are referred to as being *entangled*. The name reflects the fact that observables for the two particles remain correlated, even if measurements are performed in such a way that communication between the particles is impossible. The rapidly evolving subject of *quantum information processing* is largely concerned with the properties of entangled states, and the prospects they offer for quantum computation.

Quantum entanglement is all around us, though rarely in a form we can exploit. Typically, a state may entangle with its environment to form a new *pure* state. (A pure state is one that can be described by a single wavefunction, which may or may not be entangled.) The problem is that our knowledge of the state of the environment is highly limited. All we can measure are the observables of our initial state. In this case the wavefunction formulation is of little practical value, and instead we have to consider equations for the evolution of the observables themselves. This is usually handled by employing a representation in terms of *density matrices*. These lead naturally to concepts of quantum statistical physics and quantum definitions of entropy.

In this chapter we explore how these concepts can be formulated in the language of geometric algebra. One of the essential mysteries of quantum theory is the origin of this tensor product construction. The tensor product is used in constructing both multiparticle states and many of the operators acting on these states. So the first challenge is to find a representation of the tensor product in terms of the geometric product. This is surprisingly simple to do, though only once we have introduced the idea of a relativistic configuration space. The geometric algebra of such a space is called the *multiparticle spacetime algebra* and it provides the ideal algebraic structure for studying multiparticle states and operators. This has applications in a wealth of subjects, from NMR spectroscopy to quantum information processing, some of which are discussed below. Most of these applications concern non-relativistic multiparticle quantum mechanics. Later in this chapter we turn to a discussion of the insights that this new approach can bring to relativistic multiparticle quantum theory. There we find a simple, geometric encoding of the Pauli principle, which opens up a route through to the full quantum field theory.

9.1 Many-body quantum theory

In order to set the context for this chapter, we start with a review of the basics of multiparticle quantum theory. We concentrate in particular on two-particle systems, which illustrate many of the necessary properties. The key concept is that the quantum theory of n-particles is *not* described by a set of n single wavefunctions. Instead, it is described by *one* wavefunction that encodes the entire state of the system of n particles. Unsurprisingly, the equations governing the evolution of such a wavefunction can be extraordinarily complex.

For a wide range of problems one can separate position degrees of freedom from internal (spin) degrees of freedom. This is typically the case in non-relativistic physics, particularly if the electromagnetic field can be treated as constant. In this case the position degrees of freedom are handled by the many-body Schrödinger equation. The spin degrees of freedom in many ways represent a cleaner system to study, as they describe the quantum theory of n two-state systems. This illustrates the two most important features of multiparticle quantum theory: the exponential increase in the size of state space, and the existence of entangled states.

9.1.1 The two-body Schrödinger equation

Two-particle states are described by a single wavefunction $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$. The joint vectors $(\boldsymbol{r}_1, \boldsymbol{r}_2)$ define an abstract six-dimensional configuration space over which ψ defines a complex-valued function. This sort of configuration space is a useful tool in classical mechanics, and in quantum theory it is indispensable. The

kinetic energy operator is given by the sum of the individual operators:

$$\hat{K} = -\frac{\hbar^2 \boldsymbol{\nabla}_1^2}{2m_1} - \frac{\hbar^2 \boldsymbol{\nabla}_2^2}{2m_2}. \tag{9.2}$$

The subscripts refer to the individual particles, and m_i is the mass of particle i. The two-particle Schrödinger equation is now

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 \boldsymbol{\nabla}_1^2}{2m_1}\psi - \frac{\hbar^2 \boldsymbol{\nabla}_2^2}{2m_2}\psi + V(\boldsymbol{r}_1, \boldsymbol{r}_2)\psi. \tag{9.3}$$

As a simple example, consider the bound state Coulomb problem

$$-\frac{\hbar^2 \boldsymbol{\nabla}_1^2}{2m_1}\psi - \frac{\hbar^2 \boldsymbol{\nabla}_2^2}{2m_2}\psi - \frac{q_1 q_2}{4\pi\epsilon_0 r}\psi = E\psi, \tag{9.4}$$

where r is the Euclidean distance between the points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. This problem is separated in a similar manner to the classical Kepler problem (see section 3.2). We introduce the vectors

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2, \quad \frac{\boldsymbol{R}}{\mu} = \frac{\boldsymbol{r}_1}{m_1} + \frac{\boldsymbol{r}_2}{m_2}, \tag{9.5}$$

where μ is the reduced mass. In terms of these new variables the Schrödinger equation becomes

$$-\frac{\hbar^2 \boldsymbol{\nabla}_r^2}{2\mu}\psi - \frac{\hbar^2 \boldsymbol{\nabla}_R^2}{2M}\psi - \frac{q_1 q_2}{4\pi\epsilon_0 r}\psi = E\psi. \tag{9.6}$$

We can now find separable solutions to this equation by setting

$$\psi(\boldsymbol{r}_1, \boldsymbol{r}_2) = \phi(\boldsymbol{r})\Psi(\boldsymbol{R}). \tag{9.7}$$

The wavefunction Ψ satisfies a free-particle equation, which corresponds classically to the motion of the centre of mass. The remaining term, $\phi(\boldsymbol{r})$, satisfies the equivalent single-particle equation, with the mass given by the reduced mass of the two particles.

This basic example illustrates how quantum mechanics accounts for multiparticle interactions. There is a single wavefunction, which simultaneously accounts for the properties of all of the particles. In many cases this wavefunction decomposes into the product of a number of simpler wavefunctions, but this is not always the case. One can construct states that cannot be decomposed into a single direct product state. An important example of this arises when the two particles in question are identical. In this case one can see immediately that if $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2)$ is an eigenstate of a two-particle Hamiltonian, then so to is $\psi(\boldsymbol{r}_2, \boldsymbol{r}_1)$. The operator that switches particle labels like this is called the particle interchange operator $\hat{P}$, and it commutes with all physically-acceptable Hamiltonians. Since it commutes with the Hamiltonian, and squares to the identity operation, there are two possible eigenstates of $\hat{P}$. These are

$$\psi_\pm = \psi(\boldsymbol{r}_1, \boldsymbol{r}_2) \pm \psi(\boldsymbol{r}_2, \boldsymbol{r}_1). \tag{9.8}$$

These two possibilities are the only ones that arise physically, and give rise to the distinction between fermions (minus sign) and bosons (plus sign). Here we see the first indications of some new physical possibilities entering in multiparticle interactions. Quantum theory remains linear, so one can form complex superpositions of the n-particle wavefunctions. These superpositions can have new properties not present in the single-particle theory.

9.1.2 Spin states

Ignoring the spatial dependence and concentrating instead on the internal spin degrees of freedom, a spin-1/2 state can be written as a complex superposition of 'up' and 'down' states, which we will denote as $|0\rangle$ and $|1\rangle$. Now suppose that a second particle is introduced, so that system 1 is in the state $|\psi\rangle$ and system 2 is in the state $|\phi\rangle$. The joint state of the system is described by the *tensor product* state

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle, \tag{9.9}$$

which is abbreviated to $|\psi\rangle|\phi\rangle$. The total set of possible states is described by the basis

$$\begin{aligned} |00\rangle &= |0\rangle|0\rangle, & |01\rangle &= |0\rangle|1\rangle, \\ |10\rangle &= |1\rangle|0\rangle, & |11\rangle &= |1\rangle|1\rangle. \end{aligned} \tag{9.10}$$

This illustrates an important phenomenon of multiparticle quantum theory. The number of available states grows as 2^n, so large systems have an enormously larger state space than their classical counterparts. Superpositions of these basis states will, in general, produce states which cannot be written as a single tensor product of the form $|\psi\rangle|\phi\rangle$. Such states are entangled. A standard example is the singlet state of equation (9.1). One feature of these entangled states is that they provide 'short-cuts' through Hilbert space between classical states. The speed-up this can offer is often at the core of algorithms designed to exploit the possibilities offered by quantum computation.

A challenge faced by theorists looking for ways to exploit these ideas is how best to classify multiparticle entanglement. The problem is to describe concisely the properties of a state that are unchanged under local unitary operations. Local operations consist of unitary transformations applied entirely to one particle. They correspond to operations applied to a single particle in the laboratory. Features of the state that are unchanged by these operations relate to joint properties of the particles, in particular how entangled they are.

To date, only two-particle (or 'bipartite') systems have been fully understood. A general state of two particles can be written

$$\Psi = \sum_{i,j} \alpha_{ij} |i\rangle \otimes |j\rangle, \tag{9.11}$$

where the $|i\rangle$ denote some orthonormal basis. The *Schmidt decomposition* (which is little more than a singular-value decomposition of α_{ij}) tells us that one can always construct a basis such that

$$\Psi = \sum_i \beta_i |i'\rangle \otimes |i'\rangle. \tag{9.12}$$

The β_i are all real parameters that tell us directly how much entanglement is present. These parameters are unchanged under local transformations of the state Ψ. An important example of the Schmidt decomposition, which we shall revisit frequently, is for systems of two entangled spinors. For these we find that a general state can be written explicitly as

$$\begin{aligned}|\psi\rangle =& \rho^{1/2} \mathrm{e}^{i\chi} \left(\cos(\alpha/2) \mathrm{e}^{i\tau/2} \begin{pmatrix} \cos(\theta_1/2)\mathrm{e}^{-i\phi_1/2} \\ \sin(\theta_1/2)\mathrm{e}^{i\phi_1/2} \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta_2/2)\mathrm{e}^{-i\phi_2/2} \\ \sin(\theta_2/2)\mathrm{e}^{i\phi_2/2} \end{pmatrix} \right. \\ &\left. + \sin(\alpha/2) \mathrm{e}^{-i\tau/2} \begin{pmatrix} \sin(\theta_1/2)\mathrm{e}^{-i\phi_1/2} \\ -\cos(\theta_1/2)\mathrm{e}^{i\phi_1/2} \end{pmatrix} \otimes \begin{pmatrix} \sin(\theta_2/2)\mathrm{e}^{-i\phi_2/2} \\ -\cos(\theta_2/2)\mathrm{e}^{i\phi_2/2} \end{pmatrix} \right). \end{aligned} \tag{9.13}$$

In this decomposition we arrange that $0 \leq \alpha \leq \pi/4$, so that the decomposition is unique (save for certain special cases).

9.1.3 Pure and mixed states

So far the discussion has focused entirely on *pure* states, which can be described in terms of a single wavefunction. For many applications, however, such a description is inappropriate. Suppose, for example, that we are studying spin states in an NMR experiment. The spin states are only partially coherent, and one works in terms of ensemble averages. For example, the average spin vector (or polarisation) is given by

$$\boldsymbol{p} = \frac{1}{n} \sum_{i=1}^{n} \hat{\boldsymbol{s}}_i. \tag{9.14}$$

Unless all of the spin vectors are precisely aligned (a coherent state), the polarisation vector will not have unit length and so cannot be generated by a single wavefunction. Instead, we turn to a formulation in terms of density matrices. The density matrix for a normalised pure state is

$$\hat{\rho} = |\psi\rangle\langle\psi|, \tag{9.15}$$

which is necessarily a Hermitian matrix. All of the observables associated with the state $|\psi\rangle$ can be obtained from the density matrix by writing

$$\langle\psi|\hat{Q}|\psi\rangle = \mathrm{tr}(\hat{\rho}\hat{Q}). \tag{9.16}$$

For an incoherent mixture (a mixed state) the density matrix is the weighted sum of the matrices for the pure states:

$$\hat{\rho} = \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i|. \tag{9.17}$$

The real coefficients satisfy

$$\sum_{i=1}^{n} p_i = 1, \tag{9.18}$$

which ensures that the density matrix has unit trace. The definition of $\hat{\rho}$ ensures that all observables are constructed from the appropriate averages of the pure states. In principle, the state of any system is described by a Hermitian density matrix, which is constrained to be positive-semidefinite and to have unit trace. All observables are then formed according to equation (9.16).

The need for a density matrix can be seen in a second way, as a consequence of entanglement. Suppose that we are interested in the state of particle 1, but that this particle has been allowed to entangle with a second particle 2, forming the pure state $|\psi\rangle$. The density matrix for the two-particle system is again described by equation (9.15). But we can only perform measurements of particle 1. The effective density matrix for particle 1 is obtained by performing a *partial trace* of $\hat{\rho}$ to trace out the degrees of freedom associated with particle 2. We therefore define

$$\hat{\rho}_1 = \mathrm{tr}_2 \hat{\rho}, \tag{9.19}$$

where the sum runs over the space of particle 2. One can easily check that, in the case where the particles are entangled, $\hat{\rho}_1$ is no longer the density matrix for a pure state. The most extreme example of this is the singlet state (9.1) mentioned in the introduction. In the obvious basis, the singlet state can be written as

$$|\varepsilon\rangle = \frac{1}{\sqrt{2}}(0,\, 1,\, -1,\, 0)^{\dagger}. \tag{9.20}$$

The density matrix for this state is

$$\hat{\rho} = |\varepsilon\rangle\langle\varepsilon| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{9.21}$$

This is appropriate for a pure state, as the matrix satisfies $\hat{\rho}^2 = \hat{\rho}$. But if we now form the partial trace over the second particle we are left with

$$\hat{\rho}_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9.22}$$

This is the density matrix for a totally unpolarised state, which is to be expected, since there can be no directional information in the singlet state. Clearly, $\hat{\rho}_1$ cannot be generated by a single-particle pure state.

9.2 Multiparticle spacetime algebra

The key to constructing a suitable geometric framework for multiparticle quantum theory involves the full, relativistic spacetime algebra. This is because it is only the relativistic treatment which exposes the nature of the $\boldsymbol{\sigma}_i$ as spacetime bivectors. This is crucial for determining their algebraic properties as further particles are added. The n-particle spacetime algebra is the geometric algebra of $4n$-dimensional relativistic configuration space. We call this the multiparticle spacetime algebra. A basis is for this is constructed by taking n sets of basis vectors $\{\gamma_\mu^a\}$, where the superscript labels the particle space. These satisfy the orthogonality conditions

$$\gamma_\mu^a \gamma_\nu^b + \gamma_\nu^b \gamma_\mu^a = \begin{cases} 0 & a \neq b \\ 2\eta_{\mu\nu} & a = b \end{cases}, \tag{9.23}$$

which are summarised in the single formula

$$\gamma_\mu^a \cdot \gamma_\nu^b = \delta^{ab} \eta_{\mu\nu}. \tag{9.24}$$

There is nothing uniquely quantum-mechanical in this construction. A system of three classical particles could be described by a set of three trajectories in a single space, or by one path in a nine-dimensional space. The extra dimensions label the properties of each individual particle, and are not to be thought of as existing in anything other than a mathematical sense. One unusual feature concerning relativistic configuration space is that it requires a separate copy of the time dimension for each particle, as well as the three spatial dimensions. This is required in order that the algebra is fully Lorentz-covariant. The presence of multiple time coordinates can complicate the evolution equations in the relativistic theory. Fortunately, the non-relativistic reduction does not suffer from this problem as all of the individual time coordinates are identified with a single absolute time.

As in the single-particle case, the even subalgebra of each copy of the spacetime algebra defines an algebra for relative space. We perform all spacetime splits with the vector γ_0, using a separate copy of this vector in each particle's space. A basis set of relative vectors is then defined by

$$\boldsymbol{\sigma}_i^a = \gamma_i^a \gamma_0^a. \tag{9.25}$$

Again, superscripts label the particle space in which the object appears, and subscripts are retained for the coordinate frame. We do not enforce the summation convention for superscripted indices in this chapter. If we now consider

bivectors from spaces 1 and 2, we find that the basis elements satisfy

$$\boldsymbol{\sigma}_i^1 \boldsymbol{\sigma}_j^2 = \gamma_i^1 \gamma_0^1 \gamma_j^2 \gamma_0^2 = \gamma_i^1 \gamma_j^2 \gamma_0^2 \gamma_0^1 = \gamma_j^2 \gamma_0^2 \gamma_i^1 \gamma_0^1 = \boldsymbol{\sigma}_j^2 \boldsymbol{\sigma}_i^1. \tag{9.26}$$

The basis elements *commute*, rather than anticommute. This solves the problem of how to represent the tensor product in geometric algebra. The geometric product $\boldsymbol{\sigma}_i^a \boldsymbol{\sigma}_j^b$ *is* the tensor product. Since single particle states are constructed out of geometric algebra elements, this gives a natural origin for tensor product states in the multiparticle case. This property only holds because the relative vectors $\boldsymbol{\sigma}_i^a$ are constructed as spacetime bivectors.

The pseudoscalar for each particle space is defined in the obvious way, so that

$$I^a = \gamma_0^a \gamma_1^a \gamma_2^a \gamma_3^a. \tag{9.27}$$

Relative bivectors in each space take the form $I^a \boldsymbol{\sigma}_k^a$. Wherever possible we abbreviate these by dropping the first particle label, so that

$$I\boldsymbol{\sigma}_k^a = I^a \boldsymbol{\sigma}_k^a. \tag{9.28}$$

The reverse operation in the multiparticle spacetime algebra is denoted with a tilde, and reverses the order of products of all relativistic vectors. Wherever possible we use this operation when forming observables. The Hermitian adjoint in each space can be constructed by inserting appropriate factors of γ_0^a.

9.2.1 Non-relativistic states and the correlator

In the single-particle theory, non-relativistic states are constructed from the even subalgebra of the Pauli algebra. A basis for these is provided by the set $\{1, I\boldsymbol{\sigma}_k\}$. When forming multiparticle states we take tensor products of the individual particle states. Since the tensor product and geometric product are equivalent in the multiparticle spacetime algebra, a complete basis is provided by the set

$$\{1,\ I\boldsymbol{\sigma}_k^1,\ I\boldsymbol{\sigma}_k^2,\ I\boldsymbol{\sigma}_j^1\, I\boldsymbol{\sigma}_k^2\}. \tag{9.29}$$

But these basis elements span a 16-dimensional real space, whereas the state space for two spin-1/2 particles is a four-dimensional complex space — only eight real degrees of freedom. What has gone wrong? The answer lies in our treatment of the complex structure. Quantum theory works with a single unit imaginary i, but in our two-particle algebra we now have two bivectors playing the role of i: $I\boldsymbol{\sigma}_3^1$ and $I\boldsymbol{\sigma}_3^2$. Right-multiplication of a state by either of these has to result in the same state in order for the geometric algebra treatment to faithfully mirror standard quantum mechanics. That is, we must have

$$\psi I\boldsymbol{\sigma}_3^1 = \psi I\boldsymbol{\sigma}_3^2. \tag{9.30}$$

Rearranging this, we find that

$$\psi = -\psi I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^2 = \psi \tfrac{1}{2}(1 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^2). \tag{9.31}$$

This tells us what we must do. If we define

$$E = \tfrac{1}{2}(1 - I\boldsymbol{\sigma}_3^1 \, I\boldsymbol{\sigma}_3^2), \tag{9.32}$$

we find that

$$E^2 = E. \tag{9.33}$$

So right-multiplication by E is a *projection operation*. If we include this factor on the right of all states we halve the number of (real) degrees of freedom from 16 to the expected 8.

The spacetime algebra representation of a direct-product two-particle Pauli spinor is now given by $\psi^1\phi^2 E$, where ψ^1 and ϕ^2 are spinors (even multivectors) in their own spaces. A complete basis for two-particle spin states is provided by

$$\begin{aligned} |0\rangle|0\rangle &\leftrightarrow E, \\ |0\rangle|1\rangle &\leftrightarrow -I\boldsymbol{\sigma}_2^2 \, E, \\ |1\rangle|0\rangle &\leftrightarrow -I\boldsymbol{\sigma}_2^1 \, E, \\ |1\rangle|1\rangle &\leftrightarrow I\boldsymbol{\sigma}_2^1 \, I\boldsymbol{\sigma}_2^2 \, E. \end{aligned} \tag{9.34}$$

We further define

$$J = EI\boldsymbol{\sigma}_3^1 = EI\boldsymbol{\sigma}_3^2 = \tfrac{1}{2}(I\boldsymbol{\sigma}_3^1 + I\boldsymbol{\sigma}_3^2), \tag{9.35}$$

so that

$$J^2 = -E. \tag{9.36}$$

Right-sided multiplication by J takes on the role of multiplication by the quantum imaginary i for multiparticle states.

This procedure extends simply to higher multiplicities. All that is required is to find the 'quantum correlator' E_n satisfying

$$E_n I\boldsymbol{\sigma}_3^a = E_n I\boldsymbol{\sigma}_3^b = J_n \quad \text{for all } a, b. \tag{9.37}$$

E_n can be constructed by picking out the $a = 1$ space, say, and correlating all the other spaces to this, so that

$$E_n = \prod_{b=2}^{n} \tfrac{1}{2}(1 - I\boldsymbol{\sigma}_3^1 \, I\boldsymbol{\sigma}_3^b). \tag{9.38}$$

The value of E_n is independent of which of the n spaces is singled out and correlated to. The complex structure is defined by

$$J_n = E_n I\boldsymbol{\sigma}_3^a, \tag{9.39}$$

where $I\boldsymbol{\sigma}_3^a$ can be chosen from any of the n spaces. To illustrate this consider

the case of $n = 3$, where

$$\begin{aligned} E_3 &= \tfrac{1}{4}(1 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^2)(1 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^3) \\ &= \tfrac{1}{4}(1 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^2 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^3 - I\boldsymbol{\sigma}_3^2\, I\boldsymbol{\sigma}_3^3) \end{aligned} \tag{9.40}$$

and

$$J_3 = \tfrac{1}{4}(I\boldsymbol{\sigma}_3^1 + I\boldsymbol{\sigma}_3^2 + I\boldsymbol{\sigma}_3^3 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^2\, I\boldsymbol{\sigma}_3^3). \tag{9.41}$$

Both E_3 and J_3 are symmetric under permutations of their indices.

9.2.2 Operators and observables

All of the operators defined for the single-particle spacetime algebra extend naturally to the multiparticle algebra. In the two-particle case, for example, we have

$$i\hat{\sigma}_k \otimes \hat{\mathsf{I}}|\psi\rangle \;\leftrightarrow\; I\boldsymbol{\sigma}_k^1\psi, \tag{9.42}$$

$$\hat{\mathsf{I}} \otimes i\hat{\sigma}_k|\psi\rangle \;\leftrightarrow\; I\boldsymbol{\sigma}_k^2\psi, \tag{9.43}$$

where $\hat{\mathsf{I}}$ is the 2×2 identity matrix and a factor of E is implicit in the spinor ψ. For the Hermitian operators we form, for example,

$$\hat{\sigma}_k \otimes \hat{\mathsf{I}}|\psi\rangle \;\leftrightarrow\; -I\boldsymbol{\sigma}_k^1\psi J = \boldsymbol{\sigma}_k^1\psi\boldsymbol{\sigma}_3^1. \tag{9.44}$$

This generalises in the obvious way, so that

$$\hat{\mathsf{I}} \otimes \cdots \otimes \hat{\sigma}_k^a \otimes \cdots \otimes \hat{\mathsf{I}}|\psi\rangle \;\leftrightarrow\; \boldsymbol{\sigma}_k^a\psi\boldsymbol{\sigma}_3^a. \tag{9.45}$$

We continue to adopt the j symbol as a convenient shorthand notation for the complex structure, so

$$i|\psi\rangle \;\leftrightarrow\; j\psi = \psi J = \psi I\boldsymbol{\sigma}_3^a. \tag{9.46}$$

The quantum inner product is now

$$\langle\psi|\phi\rangle \leftrightarrow 2^{n-1}\big(\langle\phi E\tilde{\psi}\rangle - \langle\phi J\tilde{\psi}\rangle j\big). \tag{9.47}$$

The factor of E in the real part is not strictly necessary as it is always present in the spinors, but including it does provide a neat symmetry between the real and imaginary parts. The factor of 2^{n-1} guarantees complete consistency with the standard quantum inner product, as it ensures that the state E has unit norm.

Suppose that we now form the observables in the two-particle case. We find that

$$\langle\psi|\,\hat{\sigma}_k \otimes \hat{\mathsf{I}}\,|\psi\rangle \leftrightarrow -2I\boldsymbol{\sigma}_k^1\cdot(\psi J\tilde{\psi}) \tag{9.48}$$

and

$$\langle\psi|\,\hat{\sigma}_j \otimes \hat{\sigma}_k\,|\psi\rangle \leftrightarrow -2(I\boldsymbol{\sigma}_j^1\, I\boldsymbol{\sigma}_k^2)\cdot(\psi E\tilde{\psi}). \tag{9.49}$$

All of the observables one can construct are therefore contained in the multivectors $\psi E\tilde{\psi}$ and $\psi J\tilde{\psi}$. This generalises to arbitrary particle numbers. To see why, we use the fact that any density matrix can be expanded in terms of products of Hermitian operators, as in the two-particle expansion

$$\hat{\rho} = |\psi\rangle\langle\psi| = \frac{1}{4}\big(\hat{\mathsf{I}}\otimes\hat{\mathsf{I}} + a_k\,\hat{\sigma}_k\otimes\hat{\mathsf{I}} + b_k\,\hat{\mathsf{I}}\otimes\hat{\sigma}_k + c_{jk}\,\hat{\sigma}_j\otimes\hat{\sigma}_k\big). \tag{9.50}$$

The various coefficients are found by taking inner products with the appropriate combinations of operators. Each of these corresponds to picking out a term in $\psi E\tilde{\psi}$ or $\psi J\tilde{\psi}$. If an even number of Pauli matrices is involved we pick out a term in $\psi E\tilde{\psi}$, and an odd number picks out a term in $\psi J\tilde{\psi}$. In general, $\psi E\tilde{\psi}$ contains terms of grades 0, 4, ..., and $\psi J\tilde{\psi}$ contains terms of grade 2, 6, ... These account for all the coefficients in the density matrix, and hence for all the observables that can be formed from ψ.

An advantage of working directly with the observables $\psi E\tilde{\psi}$ and $\psi J\tilde{\psi}$ is that the partial trace operation has a simple interpretation. If we want to form the partial trace over the ath particle, we simply remove all terms from the observables with a contribution in the ath particle space. No actual trace operation is required. Furthermore, this operation of discarding information is precisely the correct physical picture for the partial trace operation — we are discarding the (often unknown) information associated with a particle in one or more spaces. A minor complication in this approach is that $\psi J\tilde{\psi}$ gives rise to anti-Hermitian terms, whereas the density matrix is Hermitian. One way round this is to correlate all of the pseudoscalars together and then dualise all bivectors back to vectors. This is the approach favoured by Havel and coworkers in their work on NMR spectroscopy. Alternatively, one can simply ignore this feature and work directly with the observables $\psi E\tilde{\psi}$ and $\psi J\tilde{\psi}$. When presented with a general density matrix one often needs to pull it apart into sums of terms like this anyway (the product operator expansion), so it makes sense to work directly with the multivector observables when they are available.

9.3 Systems of two particles

Many of the preceding ideas are most simply illustrated for the case of a system of two particles. For these, the Schmidt decomposition of equation (9.13) provides a useful formulation for a general state. The geometric algebra version of this is rather more compact, however, as we now establish. First, we define the spinor

$$\psi(\theta,\phi) = \mathrm{e}^{-\phi I\boldsymbol{\sigma}_3/2}\,\mathrm{e}^{-\theta I\boldsymbol{\sigma}_2/2}. \tag{9.51}$$

We also need a representation of the state orthogonal to this, which is

$$\begin{pmatrix}\sin(\theta/2)\mathrm{e}^{-i\phi/2}\\ -\cos(\theta/2)\mathrm{e}^{i\phi/2}\end{pmatrix} \leftrightarrow \psi(\theta,\phi)I\boldsymbol{\sigma}_2. \tag{9.52}$$

Now we are in a position to construct the multiparticle spacetime algebra version of the Schmidt decomposition. We replace equation (9.13) with

$$\begin{aligned}\psi =&\rho^{1/2}\Big(\cos(\alpha/2)\psi^1(\theta_1,\phi_1)\psi^2(\theta_2,\phi_2)\mathrm{e}^{J\tau/2}\\ &+\sin(\alpha/2)\psi^1(\theta_1,\phi_1)\psi^2(\theta_2,\phi_2)I\boldsymbol{\sigma}_2^1\, I\boldsymbol{\sigma}_2^2\mathrm{e}^{-J\tau/2}\Big)\mathrm{e}^{J\chi}E\\ =&\rho^{1/2}\psi^1(\theta_1,\phi_1)\psi^2(\theta_2,\phi_2)\mathrm{e}^{J\tau/2}\left(\cos(\alpha/2)+\sin(\alpha/2)I\boldsymbol{\sigma}_2^1\, I\boldsymbol{\sigma}_2^2\right)\mathrm{e}^{J\chi}E. \quad (9.53)\end{aligned}$$

We now define the individual rotors

$$R=\psi(\theta_1,\phi_1)\mathrm{e}^{I\boldsymbol{\sigma}_3\tau/4},\quad S=\psi(\theta_2,\phi_2)\mathrm{e}^{I\boldsymbol{\sigma}_3\tau/4}, \quad (9.54)$$

so that the wavefunction ψ simplifies to

$$\psi=\rho^{1/2}R^1S^2\big(\cos(\alpha/2)+\sin(\alpha/2)I\boldsymbol{\sigma}_2^1I\boldsymbol{\sigma}_2^2\big)\mathrm{e}^{J\chi}E. \quad (9.55)$$

This gives a compact, general form for an arbitrary two-particle state. The degrees of freedom are held in an overall magnitude and phase, two separate rotors in the individual particle spaces, and a single entanglement angle α. In total this gives nine degrees of freedom, so one must be redundant. This redundancy lies in the single-particle rotors. If we take

$$R\mapsto R\mathrm{e}^{I\boldsymbol{\sigma}_3\beta},\quad S\mapsto S\mathrm{e}^{-I\boldsymbol{\sigma}_3\beta} \quad (9.56)$$

then the overall wavefunction ψ is unchanged. In practice this redundancy is not a problem, and the form of equation (9.55) turns out to be extremely useful.

9.3.1 Observables for two-particle states

The individual rotors R^1 and S^2 generate rotations in their own spaces. These are equivalent to local unitary transformations. The novel features associated with the observables for a two-particle system arise from the entanglement angle α. To study this we first form the bivector observable $\psi J\tilde{\psi}$:

$$\begin{aligned}\psi J\tilde{\psi}=&R^1S^2\big(\cos(\alpha/2)+\sin(\alpha/2)I\boldsymbol{\sigma}_2^1\, I\boldsymbol{\sigma}_2^2\big)J\big(\cos(\alpha/2)+\sin(\alpha/2)I\boldsymbol{\sigma}_2^1\, I\boldsymbol{\sigma}_2^2\big)\tilde{R}^1\tilde{S}^2\\ =&\tfrac{1}{2}R^1S^2\big(\cos^2(\alpha/2)-\sin^2(\alpha/2)\big)(I\boldsymbol{\sigma}_3^1+I\boldsymbol{\sigma}_3^2)\tilde{R}^1\tilde{S}^2\\ =&\tfrac{1}{2}\cos(\alpha)\big((RI\boldsymbol{\sigma}_3\tilde{R})^1+(SI\boldsymbol{\sigma}_3\tilde{S})^2\big), \quad (9.57)\end{aligned}$$

where we have assumed that $\rho=1$. This result extends the definition of the spin bivector to multiparticle systems. One can immediately see that the lengths of the bivectors are no longer fixed, but instead depend on the entanglement. Only in the case of zero entanglement are the spin bivectors unit length.

The remaining observables are contained in

$$\psi E\tilde{\psi}=\tfrac{1}{2}R^1S^2\Big(1-I\boldsymbol{\sigma}_3^1I\boldsymbol{\sigma}_3^2+\sin(\alpha)(I\boldsymbol{\sigma}_2^1I\boldsymbol{\sigma}_2^2-I\boldsymbol{\sigma}_1^1I\boldsymbol{\sigma}_1^2)\Big)\tilde{R}^1\tilde{S}^2. \quad (9.58)$$

To make this result clearer we introduce the notation

$$A_k = RI\boldsymbol{\sigma}_k\tilde{R}, \quad B_k = SI\boldsymbol{\sigma}_k\tilde{S}, \tag{9.59}$$

so that

$$2\psi E\tilde{\psi} = 1 - A_3^1 B_3^2 + \sin(\alpha)(A_2^1 B_2^2 - A_1^1 B_1^2). \tag{9.60}$$

The scalar part confirms that the state is normalised correctly. The 4-vector part contains an interesting new term, which goes as $A_2^1 B_2^2 - A_1^1 B_1^2$. None of the individual A_1, A_2, B_1, or B_2 bivectors is accessible to measurement in the single-particle case as they are not phase-invariant. But in the two-particle case these terms do start to influence the observables. This is one of essential differences between classical and quantum models of spin.

9.3.2 Density matrices and probabilities

Now that we have all of the observables, we have also found all of the terms in the density matrix. Of particular interest are the results of partial traces, where we discard the information associated with one of the particles. If we throw out all of the information about the second particle, for example, what remains is the single-particle density matrix

$$\hat{\rho} = \tfrac{1}{2}(1 + \boldsymbol{p}), \tag{9.61}$$

where the polarisation vector is given by

$$\boldsymbol{p} = \cos(\alpha) R\boldsymbol{\sigma}_3\tilde{R}. \tag{9.62}$$

This vector no longer has unit length, so the density matrix is that of a mixed state. Entanglement with a second particle has led to a loss of coherence of the first particle. This process, by which entanglement produces decoherence, is central to attempts to explain the emergence of classical physics from quantum theory.

For two particles we see that there is a symmetry between the degree of entanglement. If we perform a partial trace over particle 1, the polarisation vector for the second particle also has its length reduced by a factor of $\cos(\alpha)$. More generally the picture is less simple, and much work remains in understanding entanglement beyond the bipartite case.

A further application of the preceding is to calculate the overlap probability for the inner product of two states. Given two normalised states we have

$$P(\psi, \phi) = |\langle\psi|\phi\rangle|^2 = \mathrm{tr}(\hat{\rho}_\psi \hat{\rho}_\phi). \tag{9.63}$$

The degrees of freedom in the density matrices are contained in $\psi E\tilde{\psi}$ and $\psi J\tilde{\psi}$, with equivalent expressions for ϕ. When forming the inner product between two

density matrices, the only terms that can arise are inner products between these observables. A little work confirms that we can write, in the n-particle case,

$$P(\psi,\phi) = 2^{n-2}\langle(\psi E\tilde{\psi})(\phi E\tilde{\phi})\rangle - 2^{n-2}\langle(\psi J\tilde{\psi})(\phi J\tilde{\phi})\rangle. \tag{9.64}$$

Expressions like this are unique to the geometric algebra approach. The expression confirms that once one has found the two multivector observables for a state, one has all of the available information to hand.

As an example, suppose that we are presented with two *separable* states, ψ and ϕ. For separable states we know that the observables take the forms

$$2\psi J\tilde{\psi} = A^1 + B^2, \qquad 2\psi E\tilde{\psi} = 1 - A^1B^2 \tag{9.65}$$

and

$$2\phi J\tilde{\phi} = C^1 + D^2, \qquad 2\phi E\tilde{\phi} = 1 - C^1D^2, \tag{9.66}$$

where each of the A^1, B^2, C^1 and D^2 are unit bivectors. We can now write

$$\begin{aligned} P(\psi,\phi) &= \tfrac{1}{4}\langle(1 - A^1B^2)(1 - C^1D^2) - (A^1 + B^2)(C^1 + D^2)\rangle \\ &= \tfrac{1}{4}(1 + A\cdot C\, B\cdot D - A\cdot C - B\cdot D) \\ &= \tfrac{1}{2}(1 - A\cdot C)\,\tfrac{1}{2}(1 - B\cdot D). \end{aligned} \tag{9.67}$$

This confirms the probability is the product of the separate single-particle probabilities. If one of the states is entangled this result no longer holds, as we see in the following section.

9.3.3 The singlet state

As a further example of entanglement we now study some of the properties of the non-relativistic spin singlet state. This is

$$|\varepsilon\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle|1\rangle - |1\rangle|0\rangle\big). \tag{9.68}$$

This is represented in the two-particle spacetime algebra by the multivector

$$\varepsilon = \frac{1}{\sqrt{2}}\big(I\boldsymbol{\sigma}^1_2 - I\boldsymbol{\sigma}^2_2\big)E. \tag{9.69}$$

The properties of ε are more easily seen by writing

$$\varepsilon = \tfrac{1}{2}(1 + I\boldsymbol{\sigma}^1_2\, I\boldsymbol{\sigma}^2_2)\tfrac{1}{2}(1 + I\boldsymbol{\sigma}^1_3\, I\boldsymbol{\sigma}^2_3)\sqrt{2}\, I\boldsymbol{\sigma}^1_2, \tag{9.70}$$

which shows how ε contains the commuting idempotents $(1 + I\boldsymbol{\sigma}^1_2\, I\boldsymbol{\sigma}^2_2)/2$ and $(1 + I\boldsymbol{\sigma}^1_3\, I\boldsymbol{\sigma}^2_3)/2$. Identifying these idempotents tells us immediately that

$$I\boldsymbol{\sigma}^1_2\varepsilon = \tfrac{1}{2}(I\boldsymbol{\sigma}^1_2 - I\boldsymbol{\sigma}^2_2)\tfrac{1}{2}(1 + I\boldsymbol{\sigma}^1_3\, I\boldsymbol{\sigma}^2_3)\sqrt{2}I\boldsymbol{\sigma}^1_2 = -I\boldsymbol{\sigma}^2_2\varepsilon \tag{9.71}$$

and

$$I\boldsymbol{\sigma}^1_3\varepsilon = -I\boldsymbol{\sigma}^2_3\varepsilon. \tag{9.72}$$

If follows that

$$I\boldsymbol{\sigma}_1^1\varepsilon = I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_2^1\varepsilon = -I\boldsymbol{\sigma}_2^2\, I\boldsymbol{\sigma}_3^1\varepsilon = I\boldsymbol{\sigma}_2^2\, I\boldsymbol{\sigma}_3^2\varepsilon = -I\boldsymbol{\sigma}_1^2\varepsilon. \tag{9.73}$$

Combining these results, if M^1 is an arbitrary even element in the Pauli algebra ($M^1 = M_0 + M_k I\boldsymbol{\sigma}_k^1$), ε satisfies

$$M^1\varepsilon = \tilde{M}^2\varepsilon. \tag{9.74}$$

Here M^1 and M^2 denote the same multivector, but expressed in space 1 or space 2.

Equation (9.74) provides a novel demonstration of the rotational invariance of ε. Under a joint rotation in two-particle space, a spinor ψ transforms to $R^1R^2\psi$, where R^1 and R^2 are copies of the same rotor but acting in the two different spaces. From equation (9.74) it follows that, under such a rotation, ε transforms as

$$\varepsilon \mapsto R^1R^2\varepsilon = R^1\tilde{R}^1\varepsilon = \varepsilon, \tag{9.75}$$

so that ε is a genuine two-particle rotational scalar.

If we now form the observables from ε we find that

$$2\varepsilon E\tilde{\varepsilon} = 1 + \sum_{k=1}^{3} I\boldsymbol{\sigma}_k^1\, I\boldsymbol{\sigma}_k^2 \tag{9.76}$$

and

$$\varepsilon J\tilde{\varepsilon} = 0. \tag{9.77}$$

The latter has to hold, as there are no rotationally-invariant bivector observables. Equation (9.76) identifies a new two-particle invariant, which we can write as

$$\sum_{k=1}^{3} I\boldsymbol{\sigma}_k^1\, I\boldsymbol{\sigma}_k^2 = 2\varepsilon\tilde{\varepsilon} - 1. \tag{9.78}$$

This is invariant under joint rotations in the two particles spaces. This multivector equation contains the essence of the matrix result

$$\sum_{k=1}^{3} \hat{\sigma}_{k\,a'}^{a}\, \hat{\sigma}_{k\,b'}^{b} = 2\delta_{b'}^{a}\, \delta_{a'}^{b} - \delta_{a'}^{a}\, \delta_{b'}^{b}, \tag{9.79}$$

where a, b, a', b' label the matrix components. In standard quantum mechanics this invariant would be thought of as arising from the 'inner product' of the spin vectors $\hat{\sigma}_i^1$ and $\hat{\sigma}_i^2$. Here, we have seen that the invariant arises in a completely different way, as a component of the multivector $\varepsilon\tilde{\varepsilon}$.

The fact that $\varepsilon J\tilde{\varepsilon} = 0$ confirms that the reduced density matrix for either particle space is simply one-half of the identity matrix, as established in equation (9.22). It follows that all directions are equally likely. If we align our measuring apparatus along some given axis and measure the state of particle 1,

then both up and down have equal probabilities of 1/2. Suppose now that we construct a joint measurement on the singlet state. We can model this as the overlap probability between ψ and the separable state

$$\phi = R^1 S^2 E. \tag{9.80}$$

Denoting the spin directions by

$$RI\boldsymbol{\sigma}_3\tilde{R} = P, \quad SI\boldsymbol{\sigma}_3\tilde{S} = Q, \tag{9.81}$$

we find that, from equation (9.64),

$$\begin{aligned} P(\psi,\phi) &= \langle \tfrac{1}{2}(1 - P^1 Q^2)\tfrac{1}{2}(1 + I\boldsymbol{\sigma}_k^1\, I\boldsymbol{\sigma}_k^2)\rangle \\ &= \tfrac{1}{4}(1 - P\cdot(I\boldsymbol{\sigma}_k)\, Q\cdot(I\boldsymbol{\sigma}_k)) \\ &= \tfrac{1}{4}\big(1 - \cos(\theta)\big) \end{aligned} \tag{9.82}$$

where θ is the angle between the spin bivectors P and Q. So, for example, the probability that both measurements result in the particles having the same spin ($\theta = 0$) is zero, as expected. Similarly, if the measuring devices are aligned, the probability that particle 1 is up and particle 2 is down is 1/2, whereas if there was no entanglement present the probability would be the product of the separate single-particle measurements (resulting in 1/4).

Some consequences of equation (9.82) run counter to our intuitions about locality and causality. In particular, it is impossible to reproduce the statistics of equation (9.82) if we assume that the individual particles both know which spin state they are in prior to measurement. These contradictions are embodied in the famous *Bell inequalities*. The behaviour of entangled states has now been tested experimentally, and the results confirm all of the predictions of quantum mechanics. The results are unchanged even if the measurements are performed in such a way that the particles cannot be in causal contact. This does not provide any conflict with special relativity, as entangled states cannot be used to exchange classical information at faster than the speed of light. The reason is that the presence of entanglement can only be inferred when the separate measurements on the two subsystems are compared. Without knowing which measurements observer 1 is performing, observer 2 cannot extract any useful classical information from an entangled state.

For many years the properties of entangled states were explored largely as a theoretical investigation into the nature of quantum theory. Now, however, physicists are starting to view quantum entanglement as a resource that can be controlled in the laboratory. To date our control of entangled states is limited, but it is improving rapidly, and many predict that before long we will see the first viable quantum computers able to exploit this new resource.

9.4 Relativistic states and operators

The ideas developed for the multiparticle Pauli algebra extend immediately to the relativistic domain. A single-particle relativistic state is described by an arbitrary even element of the full spacetime algebra. Accordingly, a two-particle state is constructed from the tensor product of two such states. This results is a space of of $8 \times 8 = 64$ real dimensions. Post-multiplying the direct-product space by the quantum correlator E reduces to 32 real dimensions, which are equivalent to the 16 complex dimensions employed in standard two-particle relativistic quantum theory. All the single-particle operators and observables discussed in section 8.2 extend in fairly obvious ways.

To begin, the individual matrix operators have the equivalent action

$$\begin{aligned} \hat{\gamma}_\mu \otimes \hat{\mathsf{I}}|\psi\rangle &\leftrightarrow \gamma^1_\mu \psi \gamma^1_0, \\ \hat{\mathsf{I}} \otimes \hat{\gamma}_\mu|\psi\rangle &\leftrightarrow \gamma^2_\mu \psi \gamma^2_0, \end{aligned} \tag{9.83}$$

where $\hat{\mathsf{I}}$ denotes the 4×4 identity matrix. The multiparticle spacetime algebra operators *commute*, as they must in order to represent the tensor product. The result of the action of $\gamma^1_\mu \psi \gamma^1_0$, for example, does not take us outside the two-particle state space, since the factor of γ^1_0 on the right-hand side commutes with the correlator E. The remaining matrix operators are easily constructed now, for example

$$\hat{\gamma}_\mu \hat{\gamma}_\nu \otimes \hat{\mathsf{I}}|\psi\rangle \leftrightarrow \gamma^1_\mu \gamma^1_\nu \psi. \tag{9.84}$$

The role of multiplication by the unit imaginary i is still played by right-multiplication by J, and the individual helicity projection operators become

$$\hat{\gamma}_5 \otimes \hat{\mathsf{I}}|\psi\rangle \leftrightarrow -I^1 \psi J = \psi \boldsymbol{\sigma}^1_3. \tag{9.85}$$

Relativistic observables are also constructed in a similar manner to the single-particle case. We form geometric products $\psi \Sigma \tilde{\psi}$, where Σ is any combination of γ_0 and γ_3 from either space. The result is then guaranteed to be Lorentz-covariant and phase-invariant. The first observable to consider is the multivector

$$\psi\tilde{\psi} = \psi E \tilde{\psi} = \langle \psi E \tilde{\psi} \rangle_{0,8} + \langle \psi E \tilde{\psi} \rangle_4. \tag{9.86}$$

The grade-0 and grade-8 terms are the two-particle generalisation of the scalar + pseudoscalar combination $\psi\tilde{\psi} = \rho \exp(i\beta)$ found at the single-particle level. The 4-vector part generalises the entanglement terms found in the non-relativistic case. This allows for a relativistic definition of entanglement, which is important for a detailed study of the relationship between locality and entanglement.

Next, we form two-particle current and spin vectors:

$$\mathcal{J} = \langle \psi(\gamma^1_0 + \gamma^2_0)\tilde{\psi} \rangle_1, \tag{9.87}$$

$$s = \langle \psi(\gamma^1_3 + \gamma^2_3)\tilde{\psi} \rangle_1. \tag{9.88}$$

(The calligraphic symbol $\mathcal{J}$ is used to avoid confusion with the correlated bivector J.) The full observables will contain grade-1 and grade-5 terms. For direct-product states the latter are seen to arise from the presence of a β factor in either of the single-particle states. Finally, we can also define the spin bivector S by

$$S = \langle \psi J \tilde{\psi} \rangle_2. \tag{9.89}$$

These expressions show how easy it is to generalise the single-particle formulae to the multiparticle case.

9.4.1 The relativistic singlet state

In the non-relativistic theory the spin singlet state has a special significance, both in being maximally entangled, and in its invariance under joint rotations in the two-particle space. An interesting question is whether we can construct a relativistic analogue that plays the role of a Lorentz singlet. Recalling the definition of ε (9.69), the property that ensured ε was a singlet state was that

$$I\boldsymbol{\sigma}_k^1 \varepsilon = -I\boldsymbol{\sigma}_k^2 \varepsilon, \qquad k = 1, \ldots, 3. \tag{9.90}$$

In addition to (9.90) a relativistic singlet state, which we will denote as η, must satisfy

$$\boldsymbol{\sigma}_k^1 \eta = -\boldsymbol{\sigma}_k^2 \eta, \qquad k = 1, \ldots, 3. \tag{9.91}$$

It follows that η satisfies

$$I^1 \eta = \boldsymbol{\sigma}_1^1 \boldsymbol{\sigma}_2^1 \boldsymbol{\sigma}_3^1 \eta = -\boldsymbol{\sigma}_3^2 \boldsymbol{\sigma}_2^2 \boldsymbol{\sigma}_1^2 \eta = I^2 \eta. \tag{9.92}$$

For this to hold, η must contain a factor of $(1 - I^1 I^2)$. We can therefore construct a Lorentz single state by multiplying ε by $(1 - I^1 I^2)$, and we define

$$\eta = (I\boldsymbol{\sigma}_2^1 - I\boldsymbol{\sigma}_2^2)\tfrac{1}{2}(1 - I\boldsymbol{\sigma}_3^1\, I\boldsymbol{\sigma}_3^2)\tfrac{1}{2}(1 - I^1 I^2). \tag{9.93}$$

This is normalised so that $2\langle \eta E \tilde{\eta} \rangle = 1$. The properties of η can be summarised as

$$M^1 \eta = \tilde{M}^2 \eta, \tag{9.94}$$

where M is an even multivector in either the particle-1 or particle-2 spacetime algebra. The proof that η is a relativistic invariant now reduces to the simple identity

$$R^1 R^2 \eta = R^1 \tilde{R}^1 \eta = \eta, \tag{9.95}$$

where R is a single-particle relativistic rotor.

Equation (9.94) can be seen as originating from a more primitive relation

between vectors in the separate spaces. Using the result that $\gamma_0^1\gamma_0^2$ commutes with η, we can derive

$$\begin{aligned}\gamma_\mu^1\eta\gamma_0^1 &= \gamma_\mu^1\gamma_0^1\gamma_0^2\eta\gamma_0^2\gamma_0^1\gamma_0^1\\ &= \gamma_0^2(\gamma_\mu\gamma_0)^1\eta\gamma_0^2\\ &= \gamma_\mu^2\eta\gamma_0^2.\end{aligned} \tag{9.96}$$

For an arbitrary vector a we can now write

$$a^1\eta\gamma_0^1 = a^2\eta\gamma_0^2. \tag{9.97}$$

Equation (9.94) follows immediately from equation (9.97) by writing

$$\begin{aligned}a^1b^1\eta &= a^1b^2\eta\gamma_0^2\gamma_0^1\\ &= b^2a^2\eta\gamma_0^2\gamma_0^2\\ &= b^2a^2\eta.\end{aligned} \tag{9.98}$$

Equation (9.97) can therefore be viewed as the fundamental property of the relativistic invariant η.

The invariant η can be used to construct a series of observables that are also invariant under coupled rotations in the two spaces. The first is

$$2\eta E\tilde{\eta} = (1 - I^1I^2) - (\boldsymbol{\sigma}_k^1\,\boldsymbol{\sigma}_k^2 - I\boldsymbol{\sigma}_k^1\,I\boldsymbol{\sigma}_k^2). \tag{9.99}$$

The scalar and pseudoscalar (grade-8) terms are clearly invariants, and the 4-vector term, $(\boldsymbol{\sigma}_k^1\,\boldsymbol{\sigma}_k^2 - I\boldsymbol{\sigma}_k^1\,I\boldsymbol{\sigma}_k^2)$, is a Lorentz invariant because it is a contraction over a complete bivector basis in the two spaces. Next we consider the multivector

$$\begin{aligned}2\eta\gamma_0^1\gamma_0^2\tilde{\eta} &= \gamma_0^1\gamma_0^2 - I^1I^2\gamma_k^1\gamma_k^2 - I^1I^2\gamma_0^1\gamma_0^2 - \gamma_k^1\gamma_k^2)\\ &= (\gamma_0^1\gamma_0^2 - \gamma_k^1\gamma_k^2)(1 - I^1I^2).\end{aligned} \tag{9.100}$$

The essential invariant here is the bivector

$$K = \gamma_\mu^1\wedge\gamma^{\mu 2}, \tag{9.101}$$

and the invariants from (9.100) are simply K and KI^1I^2. The bivector K takes the form of a 'doubling' bivector, which will be encountered again in section 11.4.

From the definition of K in equation (9.101), we find that

$$\begin{aligned}K\wedge K &= -2\gamma_0^1\gamma_0^2\gamma_k^1\gamma_k^2 + (\gamma_k^1\gamma_k^2)\wedge(\gamma_j^1\gamma_j^2)\\ &= 2(\boldsymbol{\sigma}_k^1\,\boldsymbol{\sigma}_k^2 - I\boldsymbol{\sigma}_k^1\,I\boldsymbol{\sigma}_k^2),\end{aligned} \tag{9.102}$$

which recovers the grade-4 invariant found in equation (9.99). The full set of two-particle invariants constructed from K are summarised in table 9.1. These invariants are regularly employed in constructing interaction terms in multiparticle wave equations.

Invariant	Type of interaction	Grade
1	Scalar	0
K	Vector	2
$K \wedge K$	Bivector	4
$I^1 I^2 K$	Pseudovector	6
$I^1 I^2$	Pseudoscalar	8

Table 9.1 *Relativistic invariants in the two-particle algebra.*

9.4.2 Multiparticle wave equations

The question of how to construct a valid, relativistic, multiparticle wave equation has troubled physicists almost from the moment Dirac proposed his equation. The question is far from settled, and the current preferred option is to ignore the question where possible and instead work within the framework of perturbative quantum field theory. This approach runs into difficulties when analysing bound states, however, and for these problems the need for a suitable wave equation is particularly acute. The main candidate for a relativistic two-particle system is the Bethe–Salpeter equation. Written in the multiparticle spacetime algebra, this equation is

$$(j\hat{\nabla}_r^1 - m_1)(j\hat{\nabla}_s^2 - m_2)\psi(r,s) = \mathcal{I}(r,s)\psi(r,s) \tag{9.103}$$

where $\mathcal{I}(r,s)$ is an integral operator representing the interparticle interaction, and ∇_r^1 and ∇_s^2 denote vector derivatives with respect to r^1 and s^2 respectively. The combined vector

$$x = r^1 + s^2 = r^\mu \gamma_\mu^1 + s^\mu \gamma_\mu^2 \tag{9.104}$$

is the full position vector in eight-dimensional configuration space.

One slightly unsatisfactory feature of equation (9.103) is that it is not first-order. This has led researchers to propose a number of alternative equations, typically with the aim of providing a more detailed analysis of two-body bound state systems such as the hydrogen atom, or positronium. One such equation is

$$(\nabla_r^1 \psi \gamma_0^1 + \nabla_s^2 \psi \gamma_0^2) J = (m_1 + m_2)\psi. \tag{9.105}$$

As well as being first order, this equation also has the required property that it is satisfied by direct products of single-particle solutions. But a problem is that any distinction between the particle masses has been lost, since only the total mass enters. A second candidate equation, which does keep the masses distinct, is

$$\left(\frac{\nabla_r^1}{m_1} + \frac{\nabla_s^2}{m_2}\right)\psi(x)J = \psi(x)(\gamma_0^1 + \gamma_0^2). \tag{9.106}$$

This equation has a number of attractive features, not least of which is that the mass enters in a manner that is highly suggestive of gravitational interactions. A potential weakness of this equation is that the state space can no longer be restricted to sums of direct products of individual states. Instead we have to widen the state space to include the entire (correlated) even subalgebra of the two-particle spacetime algebra. This doubles the number of degrees of freedom, and it is not clear that this doubling can be physical.

Practically all candidate two-particle wave equations have difficulties in performing a separation into centre-of-mass and relative coordinates. This is symptomatic of the fact that the centre of mass cannot be defined sensibly even in classical relativistic dynamics. Usually some approximation scheme has to be employed to avoid this problem, even when looking for bound state solutions. While the question of finding a suitable wave equation remains an interesting challenge, one should be wary of the fact that the mass term in the Dirac equation is essentially a remainder from a more complicated interaction with the Higgs boson. The electroweak theory immediately forces us to consider particle doublets, and it could be that one has to consider multiparticle extensions of these in order to arrive at a satisfactory theory.

9.4.3 The Pauli principle

In quantum theory, indistinguishable particles must obey either Fermi–Dirac or Bose–Einstein statistics. For fermions this requirement results in the Pauli exclusion principle that no two particles can occupy a state in which their properties are identical. The Pauli principle is usually enforced in one of two ways in relativistic quantum theory. At the level of multiparticle wave mechanics, antisymmetrisation is enforced by using a Slater determinant representation of a state. At the level of quantum field theory, however, antisymmetrisation is a consequence of the anticommutation of the creation and annihilation operators for fermions. Here we are interested in the former approach, and look to achieve the antisymmetrisation in a simple geometrical manner.

We start by introducing the grade-4 multivector

$$I_P = \Gamma_0\Gamma_1\Gamma_2\Gamma_3, \tag{9.107}$$

where

$$\Gamma_\mu = \frac{1}{\sqrt{2}}\left(\gamma_\mu^1 + \gamma_\mu^2\right). \tag{9.108}$$

It is a simple matter to verify that I_P has the properties

$$I_P^2 = -1 \tag{9.109}$$

and

$$I_P\gamma_\mu^1 I_P = \gamma_\mu^2, \quad I_P\gamma_\mu^2 I_P = \gamma_\mu^1. \tag{9.110}$$

It follows that I_P functions as a geometrical version of the particle exchange operator. In particular, acting on the eight-dimensional position vector $x = r^1 + s^2$ we find that

$$I_P x I_P = r^2 + s^1 \tag{9.111}$$

where

$$r^2 = \gamma_\mu^2 r^\mu, \quad s^1 = \gamma_\mu^1 s^\mu. \tag{9.112}$$

So I_P can be used to interchange the coordinates of particles 1 and 2. Next we must confirm that I_P is independent of the choice of initial frame. Suppose that instead we had started with the rotated frame $\{R\gamma_\mu\tilde{R}\}$, with

$$\Gamma'_\mu = \frac{1}{\sqrt{2}}\left(R^1\gamma_\mu^1\tilde{R}^1 + R^2\gamma_\mu^2\tilde{R}^2\right) = R^1R^2\Gamma_\mu\tilde{R}^2\tilde{R}^1. \tag{9.113}$$

The new Γ'_μ vectors give rise to the rotated 4-vector

$$I'_P = R^1R^2I_P\tilde{R}^2\tilde{R}^1. \tag{9.114}$$

But, acting on a bivector in particle space 1, we find that

$$I_P a^1\wedge b^1 I_P = -(I_P a^1 I_P)\wedge(I_P b^1 I_P) = -a^2\wedge b^2, \tag{9.115}$$

and the same is true of an arbitrary even element in either space. More generally, the operation $M \mapsto I_P M I_P$ applied to an even element in one of the particle spaces flips it to the other particle space and changes sign, while applied to an odd element it just flips the particle space. It follows that

$$I_P\tilde{R}^2\tilde{R}^1 = \tilde{R}^1 I_P\tilde{R}^1 = \tilde{R}^1\tilde{R}^2 I_P, \tag{9.116}$$

and substituting this into (9.114) we find that $I'_P = I_P$. It follows that I_P is independent of the chosen orthonormal frame, as required.

We can now use the 4-vector I_P to encode the Pauli exchange principle geometrically. Let $\psi(x)$ be a wavefunction for two electrons. The state

$$\psi(x)' = -I_P\psi(I_P x I_P)I_P, \tag{9.117}$$

then swaps the position dependence, and interchanges the space of the multivector components of ψ. The antisymmetrised state is therefore

$$\psi_-(x) = \psi(x) + I_P\psi(I_P x I_P)I_P. \tag{9.118}$$

For n-particle systems the extension is straightforward, as we require that the wavefunction is invariant under the interchange enforced by the I_Ps constructed from each pair of particles.

For a single Dirac particle the probability current $J = \psi\gamma_0\tilde{\psi}$ has zero divergence, and can therefore be used to define streamlines. These are valuable for understanding a range of phenomena, such as wavepacket tunnelling and spin

measurement. We now illustrate how these ideas extend to the multiparticle domain. The two-particle current is

$$\mathcal{J} = \langle \psi(\gamma_0^1 + \gamma_0^2)\tilde{\psi} \rangle_1, \tag{9.119}$$

as defined in equation (9.87). The vector $\mathcal{J}$ has components in both particle-1 and particle-2 spaces, which we write as

$$\mathcal{J} = \mathcal{J}_1^1 + \mathcal{J}_2^2. \tag{9.120}$$

For sums of separable solutions to the single-particle equations, the individual currents are both conserved:

$$\nabla^1 \cdot \mathcal{J}_1^1 = \nabla^2 \cdot \mathcal{J}_2^2 = 0. \tag{9.121}$$

It follows that the full current $\mathcal{J}$ is conserved in 8-dimensional space, so its streamlines never cross there. The streamlines of the individual particles, however, are obtained by integrating $\mathcal{J}_1$ and $\mathcal{J}_2$ in a single spacetime, and these can cross if plotted in the same space. For example, suppose that the wavefunction is just

$$\psi = \phi^1(r^1)\chi^2(s^2)E, \tag{9.122}$$

where ϕ and χ are Gaussian wavepackets moving in opposite directions. Since the distinguishable case is assumed, no Pauli antisymmetrisation is used. One can easily confirm that for this case the streamlines and the wavepackets simply pass straight through each other.

But suppose now that we assume indistinguishability, and apply the Pauli symmetrisation procedure to the wavefunction of equation (9.122). We arrive at the state

$$\psi = \left(\phi^1(r^1)\chi^2(s^2) - \chi^1(r^2)\phi^2(s^1)\right) E, \tag{9.123}$$

from which we form $\mathcal{J}_1$ and $\mathcal{J}_2$, as before. Figure 9.1 shows the streamlines that result from these currents. In the left-hand plot both particles are in the same spin state. The corrugated appearance of the lines near the origin is the result of the streamlines having to pass through a region of highly oscillatory destructive interference, since the probability of both particles occupying the same position (the origin) with the same spin state is zero. The right-hand plot is for two particles in different spin states. Again, the streamlines are seen to repel. The reason for this can be found in the symmetry properties of the two-particle current. Given that the wavefunction ψ has been antisymmetrised according to equation (9.118), the current must satisfy

$$I_P \mathcal{J}(I_P x I_P) I_P = \mathcal{J}(x). \tag{9.124}$$

It follows that at the same spacetime position, encoded by $I_P x I_P = x$ in the two-particle algebra, the two currents $\mathcal{J}_1$ and $\mathcal{J}_2$ are equal. Hence, if two streamlines

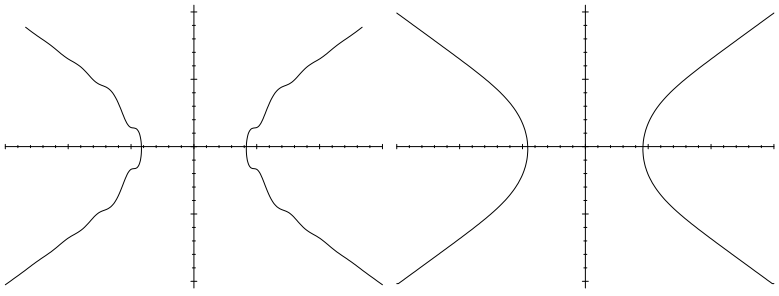

Figure 9.1 *Streamlines for an antisymmetrised two-particle wavefunction.* The wavefunction is $\psi = \big(\phi^1(r^1)\chi^2(s^2) - \chi^1(r^2)\phi^2(s^1)\big)E$. The individual wavepackets pass through each other, but the streamlines from separate particles do not cross. The left-hand figure has both particles with spins aligned in the $+z$ direction. The right-hand figure shows particles with opposite spins, with ϕ in the $+z$ direction, and χ in the $-z$ direction.

ever met, they could never separate again. For the simulations presented here, the symmetry of the set-up implies that the spatial currents at the origin are both zero. As the particles approach the origin, they are forced to slow up. The delay means that they are then swept back in the direction they have just come from by the wavepacket travelling through from the other side. This repulsion has its origin in indistinguishability, and the spin of the states exerts only a marginal effect.

9.5 Two-spinor calculus

The ideas introduced in this chapter can be employed to construct a geometric algebra version of the two-spinor calculus developed by Penrose & Rindler (1984). The building blocks of their approach are two-component complex spinors, denoted κ^A and $\bar{\omega}^{A'}$. Indices are raised and lowered with the antisymmetric tensor ϵ_{AB}. In the spacetime algebra version both κ^A and κ_A have the same multivector equivalent, which we write as

$$\kappa^A \leftrightarrow \kappa\tfrac{1}{2}(1+\boldsymbol{\sigma}_3). \tag{9.125}$$

The presence of the idempotent $(1+\boldsymbol{\sigma}_3)/2$ allows us to restrict κ to the Pauli-even algebra, as any Pauli-odd terms can be multiplied on the right by $\boldsymbol{\sigma}_3$ to convert them back to the even subspace. This ensures that κ has four real degrees of freedom, as required. Under a Lorentz transformation the full spinor transforms

to

$$R\kappa\tfrac{1}{2}(1+\boldsymbol{\sigma}_3)=\kappa'\tfrac{1}{2}(1+\boldsymbol{\sigma}_3), \tag{9.126}$$

where R is a Lorentz rotor. If we decompose the rotor R into Pauli-even and Pauli-odd terms, $R=R_+ + R_-$, then κ' is given by

$$\kappa' = R_+\kappa + R_-\kappa\boldsymbol{\sigma}_3. \tag{9.127}$$

The decomposition into Pauli-even and Pauli-odd terms is frame-dependent, as it depends on the choice of the γ_0 direction. But by augmenting κ with the $(1+\boldsymbol{\sigma}_3)/2$ idempotent we ensure that the full object is a proper Lorentz-covariant spinor.

The opposite idempotent, $(1-\boldsymbol{\sigma}_3)/2$, also generates a valid two-spinor which belongs to a second linear space (or module). This is the $\bar{\omega}^{A'}$ spinor in the notation of Penrose & Rindler, which we translate to

$$\bar{\omega}^{A'} \leftrightarrow = -\omega I\boldsymbol{\sigma}_2\tfrac{1}{2}(1-\boldsymbol{\sigma}_3). \tag{9.128}$$

The factor of $-I\boldsymbol{\sigma}_2$ is a matter of convention, and is inserted to simplify some of the later expressions. Under a Lorentz transformation we see that the Pauli-even element ω transforms as

$$\omega \mapsto \omega' = R_+\omega - R_-\omega\boldsymbol{\sigma}_3. \tag{9.129}$$

So κ and ω have different transformation laws: they belong to distinct carrier spaces of representations of the Lorentz group.

The power of the two-spinor calculus is the ease with which vector and tensor objects are generated from the basic two-spinors. As emphasised by Penrose & Rindler, this makes the calculus equally useful for both classical and quantum applications. It is instructive to see how this looks from the geometric algebra point of view. Unsurprisingly, what we discover is that the two-spinor calculus is a highly abstract and sophisticated means of introducing the geometric product to tensor manipulations. Once this is understood, much of the apparatus of the two-spinor calculus can be stripped away, and one is left with the now familiar spacetime algebra approach to relativistic physics.

9.5.1 Two-spinor observables

In two-spinor calculus one forms tensor objects from pairs of two-spinors, for example $\kappa^A\bar{\kappa}^{A'}$. To formulate this in the multiparticle spacetime algebra we simply multiply together the appropriate spinors, putting each spinor in its own copy of the spacetime algebra. In this way we replicate the tensor product implicit in writing $\kappa^A\bar{\kappa}^{A'}$. The result is that we form the object

$$\kappa^A\bar{\kappa}^{A'} \leftrightarrow -\kappa^1\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\kappa^2 I\boldsymbol{\sigma}_2^2\tfrac{1}{2}(1-\boldsymbol{\sigma}_3^2)\tfrac{1}{2}(1-I\boldsymbol{\sigma}_3^1\,I\boldsymbol{\sigma}_3^2). \tag{9.130}$$

$\frac{1}{2}(1+\boldsymbol{\sigma}_3^1)\frac{1}{2}(1-\boldsymbol{\sigma}_3^2)E = -\frac{1}{2}(\gamma_0^1+\gamma_3^1)I\boldsymbol{\sigma}_2^1\bar{\epsilon}\gamma_0^1$
$\frac{1}{2}(1-\boldsymbol{\sigma}_3^1)\frac{1}{2}(1+\boldsymbol{\sigma}_3^2)E = -\frac{1}{2}(\gamma_0^1-\gamma_3^1)I\boldsymbol{\sigma}_2^1\epsilon\gamma_0^1$
$\frac{1}{2}(1+\boldsymbol{\sigma}_3^1)\frac{1}{2}(1+\boldsymbol{\sigma}_3^2)E = -\frac{1}{2}(\boldsymbol{\sigma}_1^1+I\boldsymbol{\sigma}_2^1)\epsilon$
$\frac{1}{2}(1-\boldsymbol{\sigma}_3^1)\frac{1}{2}(1-\boldsymbol{\sigma}_3^2)E = -\frac{1}{2}(-\boldsymbol{\sigma}_1^1+I\boldsymbol{\sigma}_2^1)\bar{\epsilon}$

Table 9.2 *Two-spinor identities.* The identities listed here can be used to convert any expression involving a pair of two-spinors into an equivalent multivector.

As it stands this looks rather clumsy, but the various idempotents hide what is really going on. The key is to expose the Lorentz singlet structure hidden in the combination of idempotents. To achieve this we define two new Lorentz singlet states

$$\epsilon = \eta\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1), \quad \bar{\epsilon} = \eta\tfrac{1}{2}(1-\boldsymbol{\sigma}_3^2), \tag{9.131}$$

where η is the Lorentz singlet defined in equation (9.93). These new states both satisfy the essential equation

$$M^1\epsilon = \tilde{M}^2\epsilon, \quad M^1\bar{\epsilon} = \tilde{M}^2\bar{\epsilon}, \tag{9.132}$$

where M is an even-grade multivector. The reason is that any idempotents applied on the right of η cannot affect the result of equation (9.94). Expanding out in full, and rearranging the idempotents, we find that

$$\begin{aligned}\epsilon &= (I\boldsymbol{\sigma}_2^1 - I\boldsymbol{\sigma}_2^2)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^2)E,\\ \bar{\epsilon} &= (I\boldsymbol{\sigma}_2^1 - I\boldsymbol{\sigma}_2^2)\tfrac{1}{2}(1-\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1-\boldsymbol{\sigma}_3^2)E.\end{aligned} \tag{9.133}$$

These relations can manipulated to give, for example,

$$\begin{aligned}I\boldsymbol{\sigma}_2^1\epsilon &= -(1+I\boldsymbol{\sigma}_2^1\,I\boldsymbol{\sigma}_2^2)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^2)E,\\ \boldsymbol{\sigma}_1^1\epsilon &= -(1-I\boldsymbol{\sigma}_2^1\,I\boldsymbol{\sigma}_2^2)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^2)E.\end{aligned} \tag{9.134}$$

It follows that

$$\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^2)E = -\tfrac{1}{2}(\boldsymbol{\sigma}_1^1+I\boldsymbol{\sigma}_2^1)\epsilon. \tag{9.135}$$

There are four such identities in total, which are listed in table 9.2.

The results given in table 9.2 enable us to immediately convert any two-spinor expression into an equivalent multivector in the spacetime algebra. For example, returning to equation (9.130), we form

$$\begin{aligned}-\kappa^1\kappa^2 I\boldsymbol{\sigma}_2^2\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1-\boldsymbol{\sigma}_3^2)E &= \kappa^1\kappa^2\tfrac{1}{2}(\gamma_0^1+\gamma_3^1)\bar{\epsilon}\gamma_0^1\\ &= \tfrac{1}{2}\big(\kappa(\gamma_0+\gamma_3)\tilde{\kappa}\big)^1\bar{\epsilon}\gamma_0^1.\end{aligned} \tag{9.136}$$

The key term in this expression is the null vector $\kappa(\gamma_0 + \gamma_3)\tilde{\kappa}$, which is constructed in the familiar manner for relativistic observables. A feature of the two-spinor calculus is that it lends itself to formulating most quantities in terms of null vectors. The origin of these can be traced back to the original $(1 \pm \boldsymbol{\sigma}_3)/2$ idempotents, which contain the null vector $\gamma_0 \pm \gamma_3$. These are rotated and dilated onto spacetime null vectors through the application of a spinor.

9.5.2 The two-spinor inner product

A Lorentz-invariant inner product for a pair of two-spinors is constructed from the antisymmetric combination

$$\kappa^A \omega_A = -\kappa_0 \omega_1 + \kappa_1 \omega_0, \tag{9.137}$$

where the subscripts here denote complex components of a two-spinor. The result of the inner product is a Lorentz-invariant complex scalar. The antisymmetry of the inner product tells us that we should form the equivalent expression

$$\begin{aligned}(\kappa^1\omega^2 - \kappa^2\omega^1)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^1)\tfrac{1}{2}(1+\boldsymbol{\sigma}_3^2)E &= -\tfrac{1}{2}\big(\kappa(\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2)\tilde{\omega} - \omega(\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2)\tilde{\kappa}\big)^1\epsilon \\ &= -\langle\kappa(\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2)\tilde{\omega}\rangle_{0,4}^1\epsilon. \end{aligned} \tag{9.138}$$

The antisymmetric product picks out the scalar and pseudoscalar parts of the quantity $\kappa(\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2)\tilde{\omega}$. This is sensible, as these are the two terms that are invariant under Lorentz transformations.

The fact that we form a scalar + pseudoscalar combination reveals a second important feature of the two-spinor calculus, which is that the unit imaginary is a representation of the spacetime pseudoscalar. The complex structure therefore has a concrete, geometric significance, which is one reason why two-spinor techniques have proved popular in general relativity, for example. Further insight into the form of the two-spinor inner product is gained by assembling the full even multivector

$$\psi = \kappa\tfrac{1}{2}(1+\boldsymbol{\sigma}_3) + \omega I\boldsymbol{\sigma}_2\tfrac{1}{2}(1-\boldsymbol{\sigma}_3). \tag{9.139}$$

The essential term in the two-spinor inner product is now reproduced by

$$\begin{aligned}\psi\tilde{\psi} &= -\kappa\tfrac{1}{2}(1+\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_2\tilde{\omega} + \omega I\boldsymbol{\sigma}_2\tfrac{1}{2}(1-\boldsymbol{\sigma}_3)\tilde{\kappa} \\ &= -\langle\kappa(\boldsymbol{\sigma}_1 + I\boldsymbol{\sigma}_2)\tilde{\omega}\rangle_{0,4}, \end{aligned} \tag{9.140}$$

so the inner products pick up both the scalar and pseudoscalar parts of a full Dirac spinor product $\psi\tilde{\psi}$. This form makes the Lorentz invariance of the product quite transparent. Interchanging κ and ω in ψ of equation (9.139) is achieved by right-multiplication by $\boldsymbol{\sigma}_1$, which immediately reverses the sign of $\psi\tilde{\psi}$.

9.5.3 Spin-frames and the null tetrad

An important concept in the two-spinor calculus is that of a *spin-frame*. This consists of a pair of two-spinors, κ^A and ω^A say, normalised such that $\kappa^A\omega_A = 1$. In terms of the spinor ψ of equation (9.139), this normalisation condition amounts to saying that ψ satisfies $\psi\tilde{\psi} = 1$. A normalised spin-frame is therefore the two-spinor encoding of a spacetime rotor. This realisation also sheds light on the associated concept of a *null tetrad*. In terms of the spin frame $\{\kappa^A, \omega^A\}$, the associated null tetrad is defined as follows:

$$\begin{aligned} l^a = \kappa^A\bar{\kappa}^{A'} &\leftrightarrow \big(\kappa(\gamma_0+\gamma_3)\tilde{\kappa}\big)^1\bar{\epsilon}\gamma_0^1, \\ n^a = \omega^A\bar{\omega}^{A'} &\leftrightarrow \big(\omega(\gamma_0+\gamma_3)\tilde{\omega}\big)^1\bar{\epsilon}\gamma_0^1, \\ m^a = \kappa^A\bar{\omega}^{A'} &\leftrightarrow \big(\kappa(\gamma_0+\gamma_3)\tilde{\omega}\big)^1\bar{\epsilon}\gamma_0^1, \\ \bar{m}^a = \omega^A\bar{\kappa}^{A'} &\leftrightarrow \big(\omega(\gamma_0+\gamma_3)\tilde{\kappa}\big)^1\bar{\epsilon}\gamma_0^1. \end{aligned} \tag{9.141}$$

In each case we have projected into a single copy of the spacetime algebra to form a geometric multivector. To simplify these expressions we introduce the rotor R defined by

$$R = \kappa\tfrac{1}{2}(1+\boldsymbol{\sigma}_3) + \omega I\boldsymbol{\sigma}_2\tfrac{1}{2}(1-\boldsymbol{\sigma}_3). \tag{9.142}$$

It follows that

$$\begin{aligned} R(\gamma_1+I\gamma_2)\tilde{R} &= -\kappa\gamma_1(1+\boldsymbol{\sigma}_3)I\boldsymbol{\sigma}_2\tilde{\omega} \\ &= \kappa(\gamma_0+\gamma_3)\tilde{\omega}. \end{aligned} \tag{9.143}$$

The null tetrad induced by a normalised spin-frame can now be written in the spacetime algebra as

$$\begin{aligned} l &= R(\gamma_0+\gamma_3)\tilde{R}, & m &= R(\gamma_1+I\gamma_2)\tilde{R}, \\ n &= R(\gamma_0-\gamma_3)\tilde{R}, & \bar{m} &= R(\gamma_1-I\gamma_2)\tilde{R}. \end{aligned} \tag{9.144}$$

(One can chose alternative normalisations, if required). The complex vectors m^a and $\bar{m}^a$ of the two-spinor calculus have now been replaced by vector + trivector combinations. This agrees with the earlier observation that the imaginary scalar in the two-spinor calculus plays the role of the spacetime pseudoscalar. The multivectors in a null tetrad satisfy the anticommutation relations

$$\{l, n\} = 4, \quad \{m, \bar{m}\} = 4, \quad \text{all others} = 0. \tag{9.145}$$

These relations provide a framework for the formulation of supersymmetric quantum theory within the multiparticle spacetime algebra.

9.6 Notes

The multiparticle spacetime algebra was introduced in the paper 'States and operators in the spacetime algebra' by Doran, Lasenby & Gull (1993a). Since its introduction the multiparticle spacetime algebra has been developed by a range of researchers. For introductions see the papers by Parker & Doran (2002) and Havel & Doran (2000a,2002b). Of particular interest are the papers by Somaroo et al. (1998,1999) and Havel et al. (2001), which show how the multiparticle spacetime algebra can be applied to great effect in the theory of quantum information processing. These researchers were primarily motivated by the desire to create quantum gates in an NMR environment, though their observations can be applied to quantum computation in general. For a good introduction into the subject of quantum information, we recommend the course notes made available by Preskill (1998).

The subject of relativistic multiparticle quantum theory has been tackled by many authors. The most authoritative discussions are contained in the papers by Salpeter & Bethe (1951), Salpeter (1952), Breit (1929) and Feynman (1961). A more modern perspective is contained in the discussions in Itzykson & Zuber (1980) and Grandy (1991). For more recent attempts at constructing a two-particle version of the Dirac equation, see the papers by Galeao & Ferreira (1992), Cook (1988) and Koide (1982). A summary of the multiparticle spacetime algebra approach to this problem is contained in Doran et al.(1996b).

The two-spinor calculus is described in the pair of books '*Spinors and Spacetime*' volumes I and II by Penrose & Rindler (1984,1986). The spacetime algebra version of two-spinor calculus is described in more detail in 'Geometric algebra and its application to mathematical physics' by Doran (1994), with additional material contained in the paper '2-spinors, twistors and supersymmetry in the spacetime algebra' by Lasenby et al. (1993b). The conventions adopted in this book differ slightly from those adopted in many of the earlier papers.

9.7 Exercises

9.1 Explain how the two-particle Schrödinger equation for the Coulomb problem is reduced to the effective single-particle equation

$$-\frac{\hbar^2\boldsymbol{\nabla}^2}{2\mu}\psi - \frac{q_1 q_2}{4\pi\epsilon_0 r}\psi = E\psi,$$

where μ is the reduced mass.

9.2 Given that $\psi(\theta,\phi) = \exp(-\phi I\boldsymbol{\sigma}_3/2)\exp(-\theta I\boldsymbol{\sigma}_2/2)$, prove that

$$\begin{pmatrix} \sin(\theta/2)\mathrm{e}^{-i\phi/2} \\ -\cos(\theta/2)\mathrm{e}^{i\phi/2} \end{pmatrix} \leftrightarrow \psi(\theta,\phi)I\boldsymbol{\sigma}_2.$$

Confirm that this state is orthogonal to $\psi(\theta,\phi)$.

9.3 The interaction energy of two dipoles is given classically by

$$E = \frac{\mu_0}{4\pi}\left(\frac{\boldsymbol{\mu}_1\cdot\boldsymbol{\mu}_2}{r^3} - 3\frac{\boldsymbol{\mu}_1\cdot\boldsymbol{r}\,\boldsymbol{\mu}_2\cdot\boldsymbol{r}}{r^5}\right),$$

where $\boldsymbol{\mu}_i$ denotes the magnetic moment of particle i. For a quantum system of spin 1/2 particles we replace the magnetic moment vectors with the operators $\hat{\mu}_k = (\gamma\hbar/2)\hat{\sigma}_k$. Given that $\boldsymbol{n} = \boldsymbol{r}/r$, show that the Hamiltonian operator takes the form of the 4-vector

$$H = -\frac{d}{4}\left(\sum_{k-1}^{3} I\boldsymbol{\sigma}_k^1\, I\boldsymbol{\sigma}_k^2 - 3\, I\boldsymbol{n}^1\, I\boldsymbol{n}^2\right)$$

and find an expression for d. Can you solve the two-particle Schrödinger equation with this Hamiltonian?

9.4 ψ and ϕ are a pair of non-relativistic multiparticle states. Prove that the overlap probability between the two states can be written

$$P(\psi,\phi) = \frac{\langle(\psi E\tilde{\psi})(\phi E\tilde{\phi})\rangle - \langle(\psi J\tilde{\psi})(\phi J\tilde{\phi})\rangle}{2\langle\psi E\tilde{\psi}\rangle\langle\phi E\tilde{\phi}\rangle}.$$

9.5 Investigate the properties of the $l = 1$, $m = 0$ state

$$|\psi\rangle = |0\rangle|1\rangle + |1\rangle|0\rangle.$$

Is this state maximally entangled?

9.6 The β_μ operators that act on states in the two-particle relativistic algebra are defined by:

$$\beta_\mu(\psi) = \tfrac{1}{2}\left(\gamma_\mu^1\psi\gamma_0^1 + \gamma_\mu^2\psi\gamma_0^2\right).$$

Verify that these operators generate the *Duffin–Kemmer* ring

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \eta_{\nu\rho}\beta_\mu + \eta_{\nu\mu}\beta_\rho.$$

9.7 The multiparticle wavefunction ψ is constructed from superpositions of states of the form $\phi^1(r^1)\chi^2(s^2)$, where ϕ and χ satisfy the single-particle Dirac equation. Prove that the individual currents $\mathcal{J}_1^1$ and $\mathcal{J}_2^2$ are conserved, where

$$\mathcal{J}_1^1 + \mathcal{J}_2^2 = \langle\psi(\gamma_0^1 + \gamma_0^2)\tilde{\psi}\rangle_1.$$

9.8 In the two-spinor calculus the two-component complex vector κ^A is acted on by a 2×2 complex matrix R. Prove that R is a representation of the Lorentz rotor group if $\det\mathsf{R} = 1$. (This defines the Lie group Sl(2,C).) Hence establish that the antisymmetric combination $\kappa^0\omega^1 - \kappa^1\omega^0$ is a Lorentz scalar.

9.9 The two-spinor calculus version of the Dirac equation is

$$\nabla^{A'A}\kappa_A = \mu\bar{\omega}^{A'},$$
$$\nabla^{AA'}\bar{\omega}_{A'} = \mu\kappa^A,$$

where $\mu = m/\sqrt{2}$. Prove that these equations are equivalent to the single equation $\nabla\psi I\boldsymbol{\sigma}_3 = m\psi\gamma_0$ and give an expression for ψ in terms of κ^A and $\bar{\omega}_{A'}$.

9.10 A null tetrad is defined by the set

$$l = R(\gamma_0 + \gamma_3)\tilde{R}, \qquad m = R(\gamma_1 + I\gamma_2)\tilde{R},$$
$$n = R(\gamma_0 - \gamma_3)\tilde{R}, \qquad \bar{m} = R(\gamma_1 - I\gamma_2)\tilde{R}.$$

Prove that these satisfy the anticommutation relations

$$\{l, n\} = 4, \quad \{m, \bar{m}\} = 4, \quad \text{all others} = 0.$$