17

Spin Groups and Spinor Spaces

We have already met in some lower-dimensional special cases the spinor spaces, minimal left ideals of Clifford algebras, and the spin groups, which operate on spinor spaces. In this chapter we shall study the general case of $\mathbb{R}^{p,q}$.

Spin Groups and the Two Exponentials

Review first the special case of the 3-dimensional Euclidean space $\mathbb{R}^3$.

17.1 Spin group Spin(3) and $SU(2)$

The traceless Hermitian matrices $x\sigma_1 + y\sigma_2 + z\sigma_3$, with x, y, $z \in \mathbb{R}$, represent vectors $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \in \mathbb{R}^3$. The group of unitary and unimodular matrices

$$SU(2) = \{U \in \mathrm{Mat}(2,\mathbb{C}) \mid U^\dagger U = I,\ \det U = I\}$$

represents the *spin group* $\mathbf{Spin}(3) = \{u \in \mathcal{C}\ell_3 \mid u\tilde{u} = 1,\ u\bar{u} = 1\}$ or

$$\mathbf{Spin}(3) = \{u \in \mathcal{C}\ell_3^+ \mid u\tilde{u} = 1\}.$$

Both these groups are isomorphic with the group of unit quaternions $S^3 = \{q \in \mathbb{H} \mid q\bar{q} = 1\}$. For an element $u \in \mathbf{Spin}(3)$ the mapping $\mathbf{r} \to u\mathbf{r}\tilde{u}$ is a rotation of $\mathbb{R}^3$. Every element of $SO(3)$ can be represented in this way by an element in $\mathbf{Spin}(3)$. In fact, there are two elements u and $-u$ in $\mathbf{Spin}(3)$ representing the same rotation of $\mathbb{R}^3$. This can be written as $\mathbf{Spin}(3)/\{\pm 1\} \simeq SO(3)$ and one can say that $\mathbf{Spin}(3)$ is a double covering of $SO(3)$.

17.2 The Lipschitz groups and the spin groups

The Lipschitz group $\mathbf{\Gamma}_{p,q}$, also called the Clifford group although invented by Lipschitz 1880/86, could be defined as the subgroup in $\mathcal{C}\ell_{p,q}$ generated by invertible vectors $\mathbf{x} \in \mathbb{R}^{p,q}$, or equivalently in either of the following ways:

$$\mathbf{\Gamma}_{p,q} = \{s \in \mathcal{C}\ell_{p,q} \mid \forall \mathbf{x} \in \mathbb{R}^{p,q},\ s\mathbf{x}\hat{s}^{-1} \in \mathbb{R}^{p,q}\}$$
$$\mathbf{\Gamma}_{p,q} = \{s \in \mathcal{C}\ell^{+}_{p,q} \cup \mathcal{C}\ell^{-}_{p,q} \mid \forall \mathbf{x} \in \mathbb{R}^{p,q},\ s\mathbf{x}s^{-1} \in \mathbb{R}^{p,q}\}.$$

Note the presence of the grade involution $s \to \hat{s}$, and/or the restriction to the even/odd parts $\mathcal{C}\ell^{\pm}_{p,q}$. For $s \in \mathbf{\Gamma}_{p,q}$, $s\tilde{s} \in \mathbb{R}$. The Lipschitz group has a normalized subgroup

$$\mathbf{Pin}(p,q) = \{s \in \mathbf{\Gamma}_{p,q} \mid s\tilde{s} = \pm 1\}.$$

The group $\mathbf{Pin}(p,q)$ has an even subgroup

$$\mathbf{Spin}(p,q) = \mathbf{Pin}(p,q) \cap \mathcal{C}\ell^{+}_{p,q}.$$

The spin group $\mathbf{Spin}(p,q)$ has a subgroup

$$\mathbf{Spin}_{+}(p,q) = \{s \in \mathbf{Spin}(p,q) \mid s\tilde{s} = 1\}.$$

Write $\mathbf{Spin}(n) = \mathbf{Spin}(n,0)$, and note that $\mathbf{Spin}_{+}(n) = \mathbf{Spin}(n)$. Because of the algebra isomorphisms $\mathcal{C}\ell^{+}_{p,q} \simeq \mathcal{C}\ell^{+}_{q,p}$ we have the group isomorphisms $\mathbf{Spin}(p,q) \simeq \mathbf{Spin}(q,p)$. However, in general $\mathbf{Pin}(p,q) \not\simeq \mathbf{Pin}(q,p)$. In particular, $\mathbf{Pin}(1) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbf{Pin}(0,1) \simeq \mathbb{Z}_4$.

The groups $\mathbf{Pin}(p,q)$, $\mathbf{Spin}(p,q)$, $\mathbf{Spin}_{+}(p,q)$ are two-fold covering groups of $O(p,q)$, $SO(p,q)$, $SO_{+}(p,q)$. Although $SO_{+}(p,q)$ is connected, its two-fold cover $\mathbf{Spin}_{+}(p,q)$ need not be connected. However, the groups $\mathbf{Spin}_{+}(p,q)$, $p+q \geq 2$, are connected with the exception of

$$\mathbf{Spin}_{+}(1,1) = \{x + y\mathbf{e}_{12} \mid x, y \in \mathbb{R};\ x^2 - y^2 = 1\},$$

which has two components, two branches of a hyperbola. The group

$$\mathbf{Spin}(1,1) = \{x + y\mathbf{e}_{12} \mid x, y \in \mathbb{R};\ x^2 - y^2 = \pm 1\}$$

has four components.

The groups $\mathbf{Spin}(n)$, $n \geq 3$, and $\mathbf{Spin}_{+}(n-1,1) \simeq \mathbf{Spin}_{+}(1,n-1)$, $n \geq 4$, are simply connected and therefore universal covering groups of $SO(n)$ and $SO_{+}(n-1,1) \simeq SO_{+}(1,n-1)$. However, the maximal compact subgroup of $SO_{+}(3,3)$ is $SO(3) \times SO(3)$ which has a four-fold universal cover $\mathbf{Spin}(3) \times \mathbf{Spin}(3)$. Consequently, $\mathbf{Spin}_{+}(3,3)$ is not simply connected, but rather doubly connected, and therefore not a universal cover of $SO_{+}(3,3)$.

17.3 The two exponentials of bivectors

The Lie algebra of $\mathbf{Spin}_+(p,q)$ is the space of bivectors $\bigwedge^2 \mathbb{R}^{p,q}$. For two bivectors $\mathbf{A}, \mathbf{B} \in \bigwedge^2 \mathbb{R}^{p,q}$ the commutator is again a bivector,

$$\mathbf{AB} - \mathbf{BA} \in \bigwedge^2 \mathbb{R}^{p,q}.$$

This can be seen by considering the reverse of

$$\mathbf{AB} = \langle \mathbf{AB} \rangle_0 + \langle \mathbf{AB} \rangle_2 + \langle \mathbf{AB} \rangle_4 \in \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^{p,q} \oplus \bigwedge^4 \mathbb{R}^{p,q}$$

for which

$$(\mathbf{AB})\tilde{} = \langle \mathbf{AB} \rangle_0 - \langle \mathbf{AB} \rangle_2 + \langle \mathbf{AB} \rangle_4$$

and on the other hand

$$(\mathbf{AB})\tilde{} = \tilde{\mathbf{B}}\tilde{\mathbf{A}} = (-\mathbf{B})(-\mathbf{A}) = \mathbf{BA}.$$

The exponentials of bivectors generate the group $\mathbf{Spin}_+(p,q)$.

In this section we consider two different exponentials of bivectors, the ordinary or Clifford exponential

$$e^{\mathbf{B}} = 1 + \mathbf{B} + \frac{1}{2}\mathbf{B}^2 + \frac{1}{6}\mathbf{B}^3 + \dots,$$

where $\mathbf{B}^2 = \mathbf{BB}$, and the exterior exponential

$$e^{\wedge \mathbf{B}} = 1 + \mathbf{B} + \frac{1}{2}\mathbf{B}^{\wedge 2} + \frac{1}{6}\mathbf{B}^{\wedge 3} + \dots,$$

where $\mathbf{B}^{\wedge 2} = \mathbf{B} \wedge \mathbf{B}$. The series of the exterior exponential is finite. The ordinary exponential is always in the spin group, that is,

$$e^{\mathbf{B}} \in \mathbf{Spin}_+(p,q) \quad \text{for} \quad \mathbf{B} \in \bigwedge^2 \mathbb{R}^{p,q}.$$

The exterior exponential is in the Lipschitz group, if it is invertible in the Clifford algebra,

$$e^{\wedge \mathbf{B}} \in \mathbf{\Gamma}_{p,q} \quad \text{for} \quad \mathbf{B} \in \bigwedge^2 \mathbb{R}^{p,q} \quad \text{such that} \quad e^{\wedge \mathbf{B}} e^{\wedge(-\mathbf{B})} \neq 0.$$

Note that $e^{\wedge \mathbf{B}} \wedge e^{\wedge(-\mathbf{B})} = 1$, and so the exterior inverse of $e^{\wedge \mathbf{B}}$ is $e^{\wedge(-\mathbf{B})}$. The reverse of $s = e^{\wedge \mathbf{B}}$ is $\tilde{s} = e^{\wedge(-\mathbf{B})}$, and so the exterior inverse $s^{\wedge(-1)}$ of s equals $\tilde{s}$. The ordinary inverse of s, in the Clifford algebra $\mathcal{C\ell}_3$, is given by

$$s^{-1} = \frac{\tilde{s}}{s\tilde{s}}$$

where $s\tilde{s} \in \mathbb{R}$.

The Euclidean spaces $\mathbb{R}^n$. The bivector $\mathbf{B}$ can be written as a sum

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \ldots + \mathbf{B}_\ell$$

of at most $\ell = \lfloor n/2 \rfloor$ simple bivectors $\mathbf{B}_i$, $\mathbf{B}_i \wedge \mathbf{B}_i = 0$, which are mutually completely orthogonal so that their planes have only one point in common, $\mathbf{B}_i \wedge \mathbf{B}_j = \mathbf{B}_i\mathbf{B}_j$, $i \neq j$. This decomposition is unique unless $\mathbf{B}_i^2 = \mathbf{B}_j^2$. Notwithstanding, the product

$$(1+\mathbf{B}_1) \wedge (1+\mathbf{B}_2) \wedge \ldots \wedge (1+\mathbf{B}_\ell) = (1+\mathbf{B}_1)(1+\mathbf{B}_2)\ldots(1+\mathbf{B}_\ell)$$

depends only on $\mathbf{B}$ and equals the exterior exponential $e^{\wedge\mathbf{B}}$. The square norm of $e^{\wedge\mathbf{B}}$ is seen to be $|e^{\wedge\mathbf{B}}|^2 = (1-\mathbf{B}_1^2)(1-\mathbf{B}_2^2)\ldots(1-\mathbf{B}_\ell^2)$.

The Cayley transform. An antisymmetric $n \times n$-matrix A is sent by the Cayley transform to the rotation matrix

$$U = (I+A)(I-A)^{-1} \in SO(n).$$

There corresponds to A a bivector $\mathbf{B} \in \bigwedge^2 \mathbb{R}^n$ such that $A(\mathbf{x}) = \mathbf{B} \llcorner \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^{p,q}$. If $\mathbf{y} = U\mathbf{x}$, then $\mathbf{y} - A\mathbf{x} = \mathbf{x} + A\mathbf{x}$, or equivalently

$$\mathbf{y} + \mathbf{y} \lrcorner \mathbf{B} = \mathbf{x} + \mathbf{B} \llcorner \mathbf{x}. \tag{1}$$

Next, compute $s \wedge (\mathbf{x} + \mathbf{B} \llcorner \mathbf{x}) = s \wedge \mathbf{x} + s \wedge (\mathbf{B} \llcorner \mathbf{x})$ for $s = e^{\wedge\mathbf{B}}$. Sum up

$$\frac{1}{k!}(\underbrace{\mathbf{B} \wedge \mathbf{B} \wedge \ldots \wedge \mathbf{B}}_{k}) \wedge (\mathbf{B} \llcorner \mathbf{x}) = \frac{1}{(k+1)!}(\underbrace{\mathbf{B} \wedge \mathbf{B} \wedge \ldots \wedge \mathbf{B}}_{k+1}) \llcorner \mathbf{x}$$

for $k = 0, 1, 2, \ldots, \ell$ to obtain $s \wedge (\mathbf{B} \llcorner \mathbf{x}) = s \llcorner \mathbf{x}$. Since $s \wedge \mathbf{x} + s \llcorner \mathbf{x} = s\mathbf{x}$, it follows that $s \wedge (\mathbf{x} + \mathbf{B} \llcorner \mathbf{x}) = s\mathbf{x}$. Similarly, $s \wedge (\mathbf{y} + \mathbf{y} \lrcorner \mathbf{B}) = s \wedge \mathbf{y} - s \llcorner \mathbf{y} = \mathbf{y} \wedge s + \mathbf{y} \lrcorner s = \mathbf{y}s$. Therefore, the equation (1) is equivalent to $s\mathbf{x} = \mathbf{y}s$ or

$$U(\mathbf{x}) = s\mathbf{x}s^{-1}.$$

This representation of rotations was first discovered by R. Lipschitz 1880/86.

Thus we have the following result: An antisymmetric $n \times n$-matrix A and the rotation matrix $U = (I+A)(I-A)^{-1} \in SO(n)$ correspond, respectively, to the bivector $\mathbf{B} \in \bigwedge^2 \mathbb{R}^n$, $A(\mathbf{x}) = \mathbf{B} \llcorner \mathbf{x}$, and its exterior exponential $s = e^{\wedge\mathbf{B}} \in \mathbf{\Gamma}_n^+$, which is the unique element of $\mathbf{\Gamma}_n^+$, with scalar part 1, inducing the rotation U, $U(\mathbf{x}) = s\mathbf{x}s^{-1}$. For every rotation $U \in SO(n)$, which does not rotate any plane by a half-turn (all eigenvalues are different from -1), there is a unique element $s \in \mathbf{\Gamma}_n^+$, $\langle s \rangle_0 = 1$, such that $U(\mathbf{x}) = s\mathbf{x}s^{-1}$.

For an element $s \in \mathbf{\Gamma}_n^+$, $s\tilde{s} \in \mathbb{R}$, $s\tilde{s} > 0$. Therefore $|s| = \sqrt{s\tilde{s}}$, and

$$\frac{s}{|s|} \in \mathbf{Spin}(n) \quad \text{for} \quad s = e^{\wedge\mathbf{B}}.$$

Every element $u \in \mathbf{Spin}(n)$, $\langle u \rangle_0 \neq 0$, can be written in the form

$$u = \pm \frac{e^{\wedge \mathbf{B}}}{|e^{\wedge \mathbf{B}}|},$$

which corresponds to the rotation $U = (I+A)(I-A)^{-1} \in SO(n)$. This should be contrasted with the fact that every element in $\mathbf{Spin}(n)$ can be written in the form $e^{\mathbf{B}/2}$, which corresponds to the rotation e^A in $SO(n)$.

Lorentz signatures. In the Lorentz signatures the decomposition

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \ldots + \mathbf{B}_\ell$$

still exists and can be used to test invertibility of $\mathbf{e}^{\wedge \mathbf{B}}$. The exterior exponential

$$e^{\wedge \mathbf{B}} = (1+\mathbf{B}_1)(1+\mathbf{B}_2)\ldots(1+\mathbf{B}_\ell)$$

is invertible in the Clifford algebra if $\mathbf{B}_i^2 \neq 1$ for all i. In other words, $e^{\wedge \mathbf{B}} \in \mathbf{\Gamma}_{n-1,1}^{+}$ if all $\mathbf{B}_i^2 \neq 1$.

Indefinite metrics. Every isometry U of $\mathbb{R}^{p,q}$, connected to the identity of $SO_+(p,q)$, is an exponential of an antisymmetric transformation A of $\mathbb{R}^{p,q}$, $U = e^A$, if and only if

$$\mathbb{R}^{p,q} \quad \text{is} \quad \mathbb{R}^{n,0},\ \mathbb{R}^{0,n},\ \mathbb{R}^{n-1,1} \quad \text{or} \quad \mathbb{R}^{1,n-1},$$

see M. Riesz 1958/93 pp. 150-152. In these Euclidean and Lorentz signatures there is always a bivector $\mathbf{B}$, $\mathbf{B} \llcorner \mathbf{x} = A(\mathbf{x})$ such that $U(\mathbf{x}) = e^{\mathbf{B}}\mathbf{x}e^{-\mathbf{B}}$, see M. Riesz 1958/93 p. 160.

Given a bivector $\mathbf{B}$ one can, in general, find other bivectors $\mathbf{F}$ such that $e^{\mathbf{B}} = -e^{\mathbf{F}}$ and hence $e^{\mathbf{B}}\mathbf{x}e^{-\mathbf{B}} = e^{\mathbf{F}}\mathbf{x}e^{-\mathbf{F}}$. The only exceptions concern the following cases:

$$\begin{array}{llllll}
\mathbb{R}^{1,1} & \text{for all} & \mathbf{B} & & & \\
\mathbb{R}^{2,1} & \text{and} & \mathbb{R}^{1,2} & \text{for all} & \mathbf{B} \neq 0 & \text{such that} \quad \mathbf{B}^2 \geq 0 \\
\mathbb{R}^{3,1} & \text{and} & \mathbb{R}^{1,3} & \text{for all} & \mathbf{B} \neq 0 & \text{such that} \quad \mathbf{B}^2 = 0,
\end{array}$$

see M. Riesz 1958/93 p. 172.

To summarize with special cases: All the elements of the compact spin groups $\mathbf{Spin}(n)$ are exponentials of bivectors [when $n \geq 2$]. Among the other spin groups the same holds only for $\mathbf{Spin}_+(n-1,1) \simeq \mathbf{Spin}_+(1,n-1)$, $n \geq 5$. In particular, the two-fold cover $\mathbf{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$ of the Lorentz group $SO_+(1,3)$ contains elements which are not exponentials of bivectors: take $(\gamma_0+\gamma_1)\gamma_2 \in \bigwedge^2 \mathbb{R}^{1,3}$, $[(\gamma_0+\gamma_1)\gamma_2]^2 = 0$, then $-e^{(\gamma_0+\gamma_1)\gamma_2} = -1-(\gamma_0+\gamma_1)\gamma_2 \neq e^{\mathbf{B}}$ for any $\mathbf{B} \in \bigwedge^2 \mathbb{R}^{1,3}$. [1] However, all the elements of $\mathbf{Spin}_+(1,3)$ are of the

1 In contrast, $-e^{(\mathbf{e}_1+\mathbf{e}_5)\mathbf{e}_2} = -1-(\mathbf{e}_1+\mathbf{e}_5)\mathbf{e}_2 = e^{(\mathbf{e}_1+\mathbf{e}_5)\mathbf{e}_2+\pi\mathbf{e}_{34}}$ in $\mathbf{Spin}_+(4,1) \simeq Sp(2,2)$.

form $\pm e^{\mathbf{B}}$, $\mathbf{B} \in \bigwedge^2 \mathbb{R}^{1,3}$. Therefore, the exponentials of bivectors do not form a group.

Every element L of the Lorentz group $SO_+(1,3)$ is an exponential of an antisymmetric matrix, $L = e^A$, $gA^\top g^{-1} = -A$; a similar property is not shared by $SO_+(2,2)$. There are elements in $\mathbf{Spin}_+(2,2)$ which cannot be written in the form $\pm e^{\mathbf{B}}$, $\mathbf{B} \in \bigwedge^2 \mathbb{R}^{2,2}$; for instance $\pm \mathbf{e}_{1234} e^{\beta \mathbf{B}}$, $\mathbf{B} = \mathbf{e}_{12} + 2\mathbf{e}_{14} + \mathbf{e}_{34}$, $\beta > 0$, see M. Riesz 1958/93 p. 168-171. [2]

Lower-dimensional spin groups. The dimension of the Lie group $\mathbf{Spin}(n)$ is $\frac{1}{2}n(n-1)$. The groups $\mathbf{Spin}(n)$, $n \leq 6$, and $\mathbf{Spin}_+(p,q)$, $p+q \leq 6$, are identified in Table 1.

Table 1. Spin Groups $\mathbf{Spin}_+(p,q)$, $p+q \leq 6$.

$q \backslash p$	0	1	2	3	4	5	6
0	$\{\pm 1\}$	$O(1)$	$U(1)$	$Sp(2)$	${}^2Sp(2)$	$Sp(4)$	$SU(4)$
1	$O(1)$	$GL(1,\mathbb{R})$	$Sp(2,\mathbb{R})$	$Sp(2,\mathbb{C})$	$Sp(2,2)$	$SU^*(4)$	
2	$U(1)$	$Sp(2,\mathbb{R})$	${}^2Sp(2,\mathbb{R})$	$Sp(4,\mathbb{R})$	$SU(2,2)$		
3	$Sp(2)$	$Sp(2,\mathbb{C})$	$Sp(4,\mathbb{R})$	$SL(4,\mathbb{R})$			
4	${}^2Sp(2)$	$Sp(2,2)$	$SU(2,2)$				
5	$Sp(4)$	$SU^*(4)$					
6	$SU(4)$						

Note that $\mathbf{Spin}_+(p,q) = \{s \in \mathcal{C}\ell^+_{p,q} \mid s\tilde{s} = 1\}$ for $p+q \leq 5$. In dimension 6 the group $\{s \in \mathcal{C}\ell^+_6 \mid s\tilde{s} = 1\} \simeq U(4)$ has a proper subgroup $\mathbf{Spin}(6) \simeq SU(4)$. The groups $\mathbf{Spin}(7)$ and $\mathbf{Spin}(8)$ are not directly related to classical matrix groups; their study will be postponed till the discussion on triality.

In the case of the complex quadratic spaces $\mathbb{C}^n$ we define the complex pin group slightly differently: [3]

$$\mathbf{Pin}(n,\mathbb{C}) = \{s \in \mathcal{C}\ell(\mathbb{C}^n) \mid s\tilde{s} = 1;\ \forall \mathbf{x} \in \mathbb{C}^n,\ s\mathbf{x}\hat{s}^{-1} \in \mathbb{C}^n\}.$$

2 Riesz also showed, by the same construction on pp. 170-171, that there are bivectors which cannot be written as sums of simple and completely orthogonal bivectors; for instance $\mathbf{B} = \mathbf{e}_{12} + 2\mathbf{e}_{14} + \mathbf{e}_{34} \in \bigwedge^2 \mathbb{R}^{2,2}$.

3 The structures of square classes are different for $\mathbb{R}$ and $\mathbb{C}$. In $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, $\mathbb{R}^\times = \pm \mathbb{R}^\square$, $\mathbb{R}^\square = \{\lambda^2 \mid \lambda \in \mathbb{R}^\times\}$; so to pick up one representative out of each square class we set $s\tilde{s} = \pm 1$. In contrast, in $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $\mathbb{C}^\times = \mathbb{C}^\square$; so to pick up one representative out of each square class we set $s\tilde{s} = 1$.

The complex spin groups $\mathbf{Spin}(n, \mathbb{C})$, $n \leq 6$, are seen to be as follows:

$$\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \{\pm 1\} & O(1,\mathbb{C}) & GL(1,\mathbb{C}) & Sp(2,\mathbb{C}) & {}^2Sp(2,\mathbb{C}) & Sp(4,\mathbb{C}) & SL(4,\mathbb{C}) \end{array}$$

We also define the Lipschitz group $\$\Gamma_{q+1,p}$ for paravectors in $\mathbb{R} \oplus \mathbb{R}^{p,q}$ as the group containing the products of invertible paravectors, or equivalently,

$$\$\Gamma_{q+1,p} = \{s \in C\ell_{p,q} \mid \forall x \in \mathbb{R} \oplus \mathbb{R}^{p,q},\ sx\hat{s}^{-1} \in \mathbb{R} \oplus \mathbb{R}^{p,q}\}.$$

For any non-null paravector $a \in \mathbb{R} \oplus \mathbb{R}^{p,q}$, the mapping $x \to ax\hat{a}^{-1}$ is a special orthogonal transformation of $\mathbb{R} \oplus \mathbb{R}^{p,q}$ with metric $x \to x\bar{x}$. Therefore $\$\Gamma_{q+1,p} \simeq \mathbf{\Gamma}^+_{q+1,p}$. Note that $\mathbf{\Gamma}_{p,q} \subset \$\Gamma_{q+1,p}$ and $\$\Gamma^{\pm}_{q+1,p} = \mathbf{\Gamma}^{\pm}_{p,q}$. The normalized subgroup $\$pin(q+1,p) = \{s \in \$\Gamma_{q+1,p} \mid s\bar{s} = \pm 1\}$ is isomorphic to $\mathbf{Spin}(q+1,p)$.

Idempotents, Left Ideals and Spinors

Review first the Clifford algebra $C\ell_3$ of the Euclidean space $\mathbb{R}^3$.

17.4 Pauli spinors

In the non-relativistic theory of the electron, spinors are regarded as columns

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{where} \quad \psi_1, \psi_2 \in \mathbb{C}.$$

We shall instead introduce spinors as square matrices

$$\psi \simeq \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}.$$

If we multiply ψ on the left by an arbitrary element u in $C\ell_3$ we obtain another element $u\psi = \varphi$ in $C\ell_3$ whose matrix is also of spinor type:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix}.$$

The spinors make up a *left ideal* S of $C\ell_3$, that is,

$$\text{for all} \quad u \in C\ell_3 \quad \text{and} \quad \psi \in S \quad \text{we also have} \quad u\psi \in S.$$

The left ideal S contains no left ideals of $C\ell_3$ other than the zero ideal $\{0\}$ and S itself. Such a left ideal is called *minimal* in $C\ell_3$.

The element $f = \frac{1}{2}(1 + \mathbf{e}_3)$ is an *idempotent*, that is, $f^2 = f$, which is *primitive* in $C\ell_3$, that is, it is not a sum of two annihilating idempotents, $f \neq f_1 + f_2$, $f_1 f_2 = f_2 f_1 = 0$. The left ideal $S = C\ell_3 f$ can be provided with a right linear structure over the division ring $\mathbb{D} = f\mathcal{C}\ell_3 f$ as follows: $S \times \mathbb{D} \to S$, $(\psi, \lambda) \to \psi\lambda$. With this right linear structure over $\mathbb{D} \simeq \mathbb{C}$ the left ideal S becomes a *spinor space*.

17.5 Primitive idempotents and minimal left ideals

An orthonormal basis of $\mathbb{R}^{p,q}$ induces a basis of $\mathcal{C}\ell_{p,q}$, called the standard basis. Take a non-scalar element $\mathbf{e}_T$, $\mathbf{e}_T^2 = 1$, from the standard basis of $\mathcal{C}\ell_{p,q}$. Set $e = \frac{1}{2}(1 + \mathbf{e}_T)$ and $f = \frac{1}{2}(1 - \mathbf{e}_T)$, then $e + f = 1$ and $ef = fe = 0$. So $\mathcal{C}\ell_{p,q}$ decomposes into a sum of two left ideals $\mathcal{C}\ell_{p,q} = \mathcal{C}\ell_{p,q}e \oplus \mathcal{C}\ell_{p,q}f$, where $\dim \mathcal{C}\ell_{p,q}e = \dim \mathcal{C}\ell_{p,q}f = \frac{1}{2}\mathcal{C}\ell_{p,q} = 2^{n-1}$. Furthermore, if $\{\mathbf{e}_{T_1}, \mathbf{e}_{T_2}, \ldots, \mathbf{e}_{T_k}\}$ is a set of non-scalar basis elements such that

$$\mathbf{e}_{T_i}^2 = 1 \quad \text{and} \quad \mathbf{e}_{T_i}\mathbf{e}_{T_j} = \mathbf{e}_{T_j}\mathbf{e}_{T_i},$$

then letting the signs vary independently in the product $\frac{1}{2}(1 \pm \mathbf{e}_{T_1})\frac{1}{2}(1 \pm \mathbf{e}_{T_2}) \ldots \frac{1}{2}(1 \pm \mathbf{e}_{T_k})$, one obtains 2^k idempotents which are mutually annihilating and sum up to 1. The Clifford algebra $\mathcal{C}\ell_{p,q}$ is thus decomposed into a direct sum of 2^k left ideals, and by construction, each left ideal has dimension 2^{n-k}. In this way one obtains a minimal left ideal by forming a maximal product of non-annihilating and commuting idempotents.

The Radon-Hurwitz number r_i for $i \in \mathbb{Z}$ is given by

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

and the recursion formula

$$r_{i+8} = r_i + 4.$$

For the negative values of i one may observe that $r_{-1} = -1$ and $r_{-i} = 1 - i + r_{i-2}$ for $i > 1$.

Theorem. In the standard basis of $\mathcal{C}\ell_{p,q}$ there are always $k = q - r_{q-p}$ non-scalar elements $\mathbf{e}_{T_i}$, $\mathbf{e}_{T_i}^2 = 1$, which commute, $\mathbf{e}_{T_i}\mathbf{e}_{T_j} = \mathbf{e}_{T_j}\mathbf{e}_{T_i}$, and generate a group of order 2^k. The product of the corresponding mutually non-annihilating idempotents,

$$f = \frac{1}{2}(1 + \mathbf{e}_{T_1})\frac{1}{2}(1 + \mathbf{e}_{T_2}) \ldots \frac{1}{2}(1 + \mathbf{e}_{T_k}),$$

is primitive in $\mathcal{C}\ell_{p,q}$. Thus, the left ideal $S = \mathcal{C}\ell_{p,q}f$ is minimal in $\mathcal{C}\ell_{p,q}$. ∎

Examples. 1. In the case of $\mathbb{R}^{0,7}$ we have $k = 7 - r_7 = 4$. Therefore the idempotent $f = \frac{1}{2}(1 + \mathbf{e}_{124})\frac{1}{2}(1 + \mathbf{e}_{235})\frac{1}{2}(1 + \mathbf{e}_{346})\frac{1}{2}(1 + \mathbf{e}_{457})$ is primitive in $\mathcal{C}\ell_{0,7} \simeq {}^2\mathrm{Mat}(8, \mathbb{R})$.

2. In the case of $\mathbb{R}^{2,1}$ we have $k = 1 - r_{-1} = 1 - (r_7 - 4) = 2$. Therefore the idempotent $f = \frac{1}{2}(1 + \mathbf{e}_1)\frac{1}{2}(1 + \mathbf{e}_{23}) = \frac{1}{4}(1 + \mathbf{e}_1 + \mathbf{e}_{23} + \mathbf{e}_{123})$ is primitive in $\mathcal{C}\ell_{2,1} \simeq {}^2\mathrm{Mat}(2, \mathbb{R})$. ∎

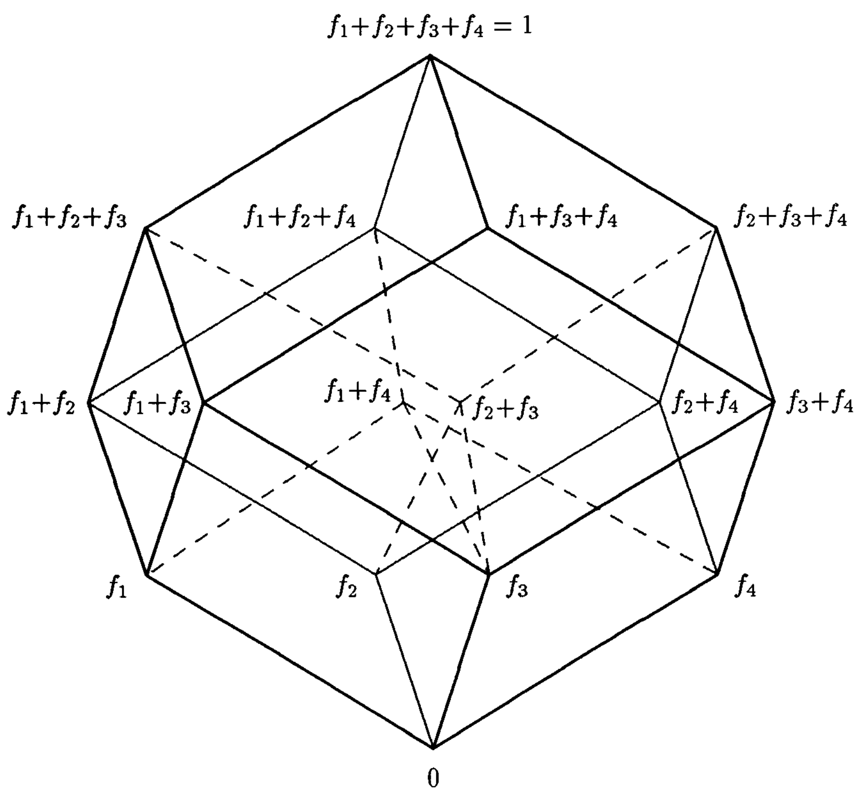
If e and f are commuting idempotents of a ring R, then ef and $e + f - ef$ are also idempotents of R. The idempotents ef and $e + f - ef$ are a greatest lower bound and a least upper bound relative to the partial ordering given by

$$e \leq f \quad \text{if and only if} \quad ef = fe = e.$$

A set of commuting idempotents induces a lattice of idempotents.

Example. In the Clifford algebra $\mathcal{C}\ell_{3,1}$, $k = 1 - r_{-2} = 1 - (r_6 - 4) = 2$. Since $2^k = 4$, $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4, \mathbb{R})$ and there are $2^{2^k} = 16$ commuting idempotents in the lattice generated by the following four mutually annihilating primitive idempotents:

$$f_1 = \tfrac{1}{2}(1 + \mathbf{e}_1)\tfrac{1}{2}(1 + \mathbf{e}_{24}), \quad f_2 = \tfrac{1}{2}(1 - \mathbf{e}_1)\tfrac{1}{2}(1 + \mathbf{e}_{24}),$$
$$f_3 = \tfrac{1}{2}(1 + \mathbf{e}_1)\tfrac{1}{2}(1 - \mathbf{e}_{24}), \quad f_4 = \tfrac{1}{2}(1 - \mathbf{e}_1)\tfrac{1}{2}(1 - \mathbf{e}_{24}).$$



The lattice induced by the primitive idempotents f_1, f_2, f_3, f_4 looks like a rhombidodecahedron, see diagram. ▮

17.6 Spinor spaces

For a primitive idempotent $f \in \mathcal{C}\ell_{p,q}$ the division ring $\mathbb{D} = f\mathcal{C}\ell_{p,q}f$ is isomorphic to

$$\begin{array}{lll} \mathbb{R} & \text{for} & p-q = 0,1,2 \bmod 8 \\ \mathbb{C} & \text{for} & p-q = 3 \bmod 4 \\ \mathbb{H} & \text{for} & p-q = 4,5,6 \bmod 8 \end{array}$$

and the map

$$S \times \mathbb{D} \to S, \ (\psi, \lambda) \to \psi\lambda$$

defines a *right* $\mathbb{D}$-linear structure on the minimal left ideal $S = \mathcal{C}\ell_{p,q}f$. Provided with this right $\mathbb{D}$-linear structure the minimal left ideal S becomes a *spinor space.* [4]

The spinor space provides an *irreducible representation*

$$\mathcal{C}\ell_{p,q} \to \mathrm{End}_{\mathbb{D}}(S), \ u \to \gamma(u), \quad \gamma(u)\psi = u\psi,$$

of $\mathcal{C}\ell_{p,q}$. This representation is also faithful for all simple Clifford algebras $\mathcal{C}\ell_{p,q}$, $p-q \neq 1 \bmod 4$.

Next, we construct a faithful representation for semi-simple Clifford algebras $\mathcal{C}\ell_{p,q}$, $p-q = 1 \bmod 4$, which are direct sums of two simple ideals $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots n})\mathcal{C}\ell_{p,q}$. Take a primitive idempotent f and an idempotent $e = f + \hat{f}$ in $\mathcal{C}\ell_{p,q}$. The ring $\mathbb{E} = e\mathcal{C}\ell_{p,q}e$ is the direct sum $\mathbb{E} = \mathbb{D} \oplus \hat{\mathbb{D}}$, $\hat{\mathbb{D}} = \{\hat{\lambda} \mid \lambda \in \mathbb{D}\}$, isomorphic to the *double ring* ${}^2\mathbb{D}$ of the division ring $\mathbb{D}$, more precisely,

$$\begin{array}{lll} \mathbb{R} \oplus \mathbb{R} & \text{for} & p-q = 1 \bmod 8 \\ \mathbb{H} \oplus \mathbb{H} & \text{for} & p-q = 5 \bmod 8. \end{array}$$

To find a faithful representation for a semi-simple Clifford algebra $\mathcal{C}\ell_{p,q}$ with $p-q = 1 \bmod 4$ take a left ideal $S \oplus \hat{S}$ where $\hat{S} = \{\hat{\psi} \mid \psi \in S\}$. The map

$$(S \oplus \hat{S}) \times \mathbb{E} \to S \oplus \hat{S}, \ (\psi, \lambda) \to \psi\lambda$$

defines a right $\mathbb{E}$-linear structure on $S \oplus \hat{S}$. Provided with this right $\mathbb{E}$-linear

4 Similarly, beginning with a minimal left ideal of the even subalgebra $\mathcal{C}\ell^+_{p,q}$ we obtain an *even spinor space*. The dimension of the even spinor space is lower than the dimension of the spinor space, when $p-q = 0 \bmod 4$. In this case, even spinors are called semi-spinors.

structure the left ideal $S \oplus \hat{S}$ of $\mathcal{C}\ell_{p,q}$ becomes a *double spinor space.* [5] The double spinor space provides a faithful but reducible representation

$$\mathcal{C}\ell_{p,q} \to \mathrm{End}_{\mathbb{E}}(S \oplus \hat{S}),\ u \to \gamma(u), \quad \gamma(u)\psi = u\psi,$$

for a semi-simple $\mathcal{C}\ell_{p,q}$, $p - q = 1 \bmod 4$.

In order to be able to consider faithful representations of simple and semi-simple Clifford algebras at the same time, we adopt the following notation:

$$\begin{array}{cccc} \check{\mathbb{D}} & \text{is} & \mathbb{D} & \text{or} \quad \mathbb{D} \oplus \hat{\mathbb{D}} \\ \check{S} & \text{is} & S & \text{or} \quad S \oplus \hat{S} \end{array}$$

according as $\mathcal{C}\ell_{p,q}$ is simple or semi-simple, respectively. Thus, the ring $\check{\mathbb{D}}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{R}$ or ${}^2\mathbb{H}$. In this way we have a faithful representation

$$\mathcal{C}\ell_{p,q} \to \mathrm{End}_{\check{\mathbb{D}}}\,\check{S},\ u \to \gamma(u), \quad \gamma(u)\psi = u\psi,$$

for all $\mathcal{C}\ell_{p,q}$. However, this representation is reducible in the cases $p - q = 1 \bmod 4$.

Questions

1. Do the exponentials of bivectors form a group?
2. Do the exterior exponentials of bivectors form a group?
3. Are $\mathbf{Spin}_+(p,q)$, $p + q \geq 3$, $p, q \neq 2$, universal covers of $SO_+(p,q)$?
4. Are double spinor spaces needed to construct a faithful representation for $\mathcal{C}\ell_{2,5}$?

Answers

1. No. 2. No. 3. No. 4. Yes.

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[5] Similarly, by doubling of a minimal left ideal of the even subalgebra $\mathcal{C}\ell_{p,q}^+$ we obtain a *double even spinor space*.

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