A GENERALIZED NONCONVEX DUALITY WITH ZERO GAP AND APPLICATIONS

Phan Thien Thach[1]
Institute of Mathematics, Hanoi, Vietnam

1. INTRODUCTION

In this report we present a duality with zero gap for nonconvex optimization problems. The first class of nonconvex problems where local optima may not be global, is a quasiconvex minimization over a convex set. For this class we obtain a generalized Kuhn-Tucker condition, and the duality is similar as Fenchel-Moreau-Rockafellar duality scheme. By the duality one could reduce the problem to solving a system of inequations. This result provides a simple proof to prove that the complexity of a quasiconvex minimization over a convex set is equivalent to the oracles introduced recently by Grotschel, Lovasz and Schriver [3] . For more general nonconvex problems such as a quasiconvex maximization over a compact set or a general minimization over the complement of a convex set we could obtain a duality with zero gap as well. If we consider a minimization of the difference of two convex functions as a particular case of reverse convex programs then the duality introduced by Toland [12] and Hiriart-Urruty [2] can be obtained from our scheme. A zero gap in primal-dual pairs allows us to develop primal-dual algorithms which are very suitable when the dual is simpler than the primal.

This report consists of 6 sections. In Section 2 we introduce the so-called quasiconjugate and the so-called quasisubdifferential, which are basic materials in the nonconvex duality. In Section 3 we present a duality for a quasiconvex minimization over a convex set. We shall see that this problem is of a convex type, although a local optimum may not be global. In Section 4 we present a duality with zero gap for nonconvex type problems. In Section 5 we give several applications. We draw some conclusions in Section 6.

---

[1] A fellow of AvH-Foundation at Trier University

2. QUASICONJUGATES AND QUASISUBDIFFERENTIALS

The basic materials in Duality Theory are concepts of subdifferentials, Fenchel conjugate, polar sets ... and these concepts are related to level sets, epigraphs, decreasing directions, convex hulls ... . For a generalized duality we introduce the so-called quasiconjugate and quasisubdifferential for quasiconvex functions.

Let $f: R^n \to \bar{R}$ be an arbitrary function.

<u>Definition 1</u> (see Thach [5]). The quasiconjugate of f is a function $f^H: R^n \to \bar{R}$ defined as follows

$$f^H(y) = \begin{cases} -\inf\{f(x): y^T x \geq 1\} & \text{if } y \neq 0 \\ -\sup\{f(x): x \in R^n\} & \text{if } y = 0 \end{cases}$$

<u>Definition 2</u> (see Thach [9]). A vector $y \in R^n$ is called a quasisubdifferential of f at $x \in R^n$ if $y^T x = 1$ and $f(x) = -f^H(y)$.

The set of quasisubdifferentials of f at x is denoted by $\partial^H f(x)$. Function f is quasisubdifferntiable at x if $\partial^H f(x) \neq \emptyset$.

We restrict our attention into the following classes of functions which are <u>large enough</u> for optimization problems:

$G^U := \{ f: R^n \to \bar{R}$ such that f is upper semi-continuous (usc), and $f(0) = \inf\{f(x): x \in R^n\} \}$

$G^L := \{ f: R^n \to \bar{R}$ such that f is lower semi-continuous (lsc), $f(0) = \inf\{f(x): x \in R^n \setminus \{0\}\}$ and $f(x) \to \sup\{f(x): x \in R^n\}$ as $\|x\| \to \infty$

$G = G^U \cap G^L$.

In the sequel we introduce some of basic properties of quasiconjugates, and quasisubdifferentials which show the relations between these concepts and the level sets, the decreasing directions and the quasi-convex hulls of functions.

<u>Theorem 1</u> (see Thach [5,9,10]).

i) f^H is a quasiconvex function and $\partial^H f(x)$ is a convex set;

ii) If $f \in G^U$ then $f^H \in G^L$ and $\{x: f(x) < t\}^o = \{y: f^H(y) \leq -t\}$ $\forall t$;

iii) If $f \in G^L$ then $f^H \in G^U$ and $\text{int}\{x: f(x) \leq t\}^o = \{y: f^H(y) < -t\}$ $\forall t$;

iv) If $f \in G^L \cup G^U$ then f^{HH} is the greatest quasiconvex function majorized by f (or the quasiconvex hull of f).

A vector $z \in R^n$ is called a decreasing direction of a function f at

$x \in R^n$ if there is $t > 0$ such that $f(x+t.z) < f(x)$. The set of all decreasing directions of f at x is denoted by $H(f,x)$.

Theorem 2 (see Thach [9]). Let $f \in G^U$ and x be a vector such that $f(x) > f(0)$. Then,

i) $\partial^H f(x) \neq \emptyset \iff f^{HH}(x) = f(x) \iff 0 \notin \text{conv } H(f,x)$;

ii) If $\partial^H f(x) \neq 0$ then $\partial^H f(x)$ is a compact convex set and

$$\text{conv } H(f,x) = \{z: \sup\{y^T z: y \in \partial^H f(x)\} < 0\} .$$

The following theorem gives a relation between the Fenchel conjugate and the quasiconjugate, and the subdifferential and the quasisubdifferential in the class of convex functions.

Theorem 3 (see Thach [9]) Suppose that f is a lsc convex function.

i) For every $y \in R^n \setminus \{0\}$ either $f^H(y) = f^*(y) = \infty$ or there is $\bar{t} \geq 0$ such that $f^H(y) = f^*(\bar{t}.y) - \bar{t} = \inf\{f^*(t.y) - t : t \geq 0\}$ where f^* denotes the Fenchel conjugate of f.

ii) If $0 \in \text{int dom} f$ then for every $x \in \text{dom} f$ such that $f(x) > f(0)$ one has $\text{cone } \partial^H f(x) = \text{cone } \partial f(x)$.

3. A DUALITY SCHEME FOR A QUASICONVEX MINIMIZATION OVER A CONVEX SET

Consider a problem

$$\min\{f(x) : x \in D\} \tag{1}$$

where $f \in G$ is a quasiconvex function, D a closed convex set such that $0 \in D$. Since $f \in G$ and D is closed, this problem is solvable (see e.g. Thach [5]). Problem (1) may have many local optima which are not global optima. Denote

$$D^H = \{y: y^T x \geq 1 \quad \forall x \in D\}$$
$$N(x,D) = \{y: y^T(z-x) \leq 0 \ \forall z \in D\} \quad .$$

Theorem 4 (see Thach [9]). A generalized Kuhn-Tucker condition:

$$0 \in \partial^H f(x) + N(x,D)$$

is sufficient for the global optimality of $x \in D$ and it is satisfied at at least a vector x in D.

The dual of problem (1), by definition, is

$$\min\{f^H(y) : y \in D^H\} . \tag{2}$$

Since f belongs to G, f^H belongs to G. This problem is also a quasiconvex minimization over a convex set.

Theorem 5 (see Thach [9]). min(1) = -min(2).

By virtue of Theorem 5, a vector $(x,y) \in D \times D^H$ satisfies $f(x) \leqslant -f^H(y)$ if and only if x is optimal to problem (1) and y is optimal to problem (2). Problem (1) is reduced to finding a vector in the set

$$A := \{(x,y) : x \in D,\ y \in D^H,\ -f(x) \geqslant f^H(y)\}\ .$$

Since the set of optimal solutions in problem (1) and the set of optimal solutions in problem (2) are convex sets, A is convex. Assume that f(x) can be polynomially computed and we have a polynomial subroutine to test if a given vector x belongs to D. Then, the complexity of problem (1) is equivalent to the complexity of the oracles (see Grotschel, Lovasz and Schriver [3], Lovasz [4]). Indeed, in order to check if $(x,y) \in A$ we have to check if $x \in D$, $y \in D^H$ and $-f(x) \geqslant f^H(y)$. Problem of checking if $x \in D$, by assumption, can be realized by a polynomial subroutine. By definition of D^H, problem of checking if $y \in D^H$ is the validity oracle. Since

$$\begin{aligned} & f^H(y) \leqslant -f(x) \\ \Longleftrightarrow\ & f(x) \leqslant \inf\{f(z) : y^T z \geqslant 1\} \\ \Longleftrightarrow\ & \sup\{ y^T z : f(z) < f(x)\} \leqslant 1, \end{aligned}$$

problem of checking if $f^H(y) \leqslant -f(x)$ is the optimization oracle. If an oracle can be solved by a polynomial algorithm then all the oracles can (see Grotschel, Lovasz and Schriver [3], Lovasz [4]) and we have a polynomial subroutine to check if $(x,y) \in A$. Since A is convex, problem of finding $(x,y) \in A$ is an oracle.

For further results in the duality scheme in quasiconvex minimization over a convex set we can see Thach [9].

4. A DUALITY SCHEME FOR NONCONVEX TYPE OPTIMIZATION PROBLEMS

Consider a problem

$$\max\{ f(x) : x \in D\} \tag{3}$$

where $f \in G^U$ is quasiconvex, and D is a compact set such that $0 \in \operatorname{conv} D$. Since f is usc and D is compact, this problem is solvable. Suppose that f is nonconstant on D.

Theorem 6 (see Thach [9]). A vector $z \in D$ is a global optimal solution to problem (3) if and only if f is quasisubdifferentiable at z and there is $y \in \partial^H f(z)$ such that

$$f^H(y) = \min\{f^H(v) : v \notin \operatorname{int} D^o\}\ .$$

The dual of problem (3), by inition, is

$$\min \{ f^H(v) : v \notin \operatorname{int} D^o \}. \tag{4}$$

This is a quasiconvex minimization over the complement of a convex set.

<u>Theorem 7</u> (see Thach [9]). max(3)=-min(4).

If $f(x)=\max\{0, c^Tx\}$ (hence $f \in G^U$) and D is convex then problem (3) is a convex program which maximizes c^Tx over a convex set D. The dual (4) then becomes

$$\begin{aligned} &\min \{ t>0 : t.c \notin \operatorname{int} D^o \} \\ \Leftrightarrow\ &\max \{ t : t.c \in D^o \} \\ \Leftrightarrow\ &1/\min \{ t : c^Tx \leqslant t \quad \forall x \in D \} . \end{aligned} \tag{5}$$

The dual (5) for a general convex program has been introduced by Tind and Wolsey [11] .

Now we consider a reverse convex constraint problem

$$\min \{ f(x) : x \notin \operatorname{int} D \} \tag{6}$$

where $f \in G^L$ and D is a closed convex set containing 0 in its interior. This problem is solvable (see e.g. Thach [5]).

<u>Theorem 8</u> (see Thach [9]). A vector z is optimal to problem (6) if and only if there is a vector $y \in \partial^H f(z) \cap D^o$ such that

$$f^H(y) = \max \{ f^H(v) : v \in D^o \}.$$

The dual of (6), by definition, is

$$\max \{ f^H(v) : v \in D^o \} . \tag{7}$$

This is a quasiconvex maximization over a convex set.

<u>Theorem 9</u> (see Thach [9]). min(6)=-max(7).

Problem (6) is more general than the following problem

$$\min \{ h_1(x) - h_2(x) : x \in R^n \}$$

where h_1 is an arbitrary function and h_2 is a finite convex function. Indeed, problem (8) can be reduced to

$$\min \{ f(x,t) : (x,t) \notin \operatorname{int} D \} \tag{9}$$

where $f(x,t)=h_1(x)-t$ and $D=\{(x,t) : h_2(x)-t-h_2(0)-1 \leqslant 0\}$. In Thach [10], the dual of (9) is

$$\max\{h_1^*(-y/r)+1/r : r < 0,\ h_2^*(-y/r)+r \leqslant -1/r \}$$

$$\Longleftrightarrow \max\{h_1^*(y)-h_2^*(y): y \in \operatorname{dom} h_2^*\}$$

$$\Longleftrightarrow \min\{h_1^*(y)-h_2^*(y): y \in \operatorname{dom} h_2^*\}\ . \tag{10}$$

The dual (10) for problem (8) was introduced by Toland[12] and Hiriart-Urruty[2].

5. APPLICATIONS

Application 1 (see Thach[9]). We are given m+1 closed convex sets $X_0, X_1, \ldots, X_m$ in R^n such that $0 \in \operatorname{int} X_0$ and $0 \notin X_i$ ($i=1,\ldots,m$). By setting

$$Y_0 = X_0^o\ ,\ Y_i = X_i^H \quad i=1,\ldots,m,$$

we have the following primal-dual pair.

Primal: To find the biggest open ball centered at 0 which is contained in X_0 and does not intersect with X_i ($i=1,\ldots,m$).

Dual: To find the smallest closed ball centered at 0 which contains Y_0 and intersects with Y_i ($i=1,\ldots,m$).

For the special cases where m=0 and X_0 is either an ellipsoid or a polytope we can see in Thach[5].

Application 2 (see Thach and Tuy[7]).

Primal : $\displaystyle\max_{g(x)\leqslant 1}\ \max_{h(z)\leqslant 1}\ z^T Ax$

where A is a nonsingular $n \times n$-matrix, g(.) and h(.) are convex functions defined as follows

$$g(x) = \sup\{u^T Mx:\ u \in U\}$$

$$h(z) = \sup\{z^T Nv:\ v \in V\}$$

with U, V - compact convex sets containing 0 in their interiors in R^p, R^q and M, N - matrices of the sizes $p \times n$, $n \times q$, respectively. This problem is a bilinear program in $R^n \times R^n$.

Dual: $\displaystyle\min_{u \notin \operatorname{int} U}\ \max_{v \in V}\ u^T(M(A^{-1})N)v$

This is a minimax problem with a reverse convex constraint in $R^p \times R^q$.

Application 3 (see Burkard, Oettli and Thach[1]). In R^3 we are given n vectors $T_i=(a_i,b_i,c_i)$ ($i=1,\ldots,n$). Each vector T_i is associated with an weight w_i.

Primal: To find vectors $T_{i_1},\dots,T_{i_k}$ such that the sum of their weights is less than or equal to 1 and the length of vector $T=T_{i_1}+\dots+T_{i_k}$ is maximized.

This problem is NP-hard and it can be formulated as follows

$$\max\{(a^Tx)^2+(b^Tx)^2+(c^Tx)^2:\ w^Tx\leqslant 1,\ x\in\{0,1\}^n\}$$

where $a=(a_1,\dots,a_n)$, $b=(b_1,\dots,b_n)$, $c=(c_1,\dots,c_n)$ and $w=(w_1,\dots,w_n)$.

Dual: $\min\{r^2+s^2+t^2:(r,s,t)\notin \operatorname{int} L\}$
where $L=\{(r,s,t):\max\{(r.a+s.b+t.c)^Tx:w^Tx\leqslant 1,x\in\{0,1\}^n\}\leqslant 1\}$.

The dual is in R^3. Because of the small dimension the dual can be practically solved.

Application 4 (see Tuy, Migdalas and Värbrand [13]).

Primal: $\min\ c_1^Tx+d_1^Ty$ s.t. $A_1x+B_1y\leqslant g_1,\ x\in R^p_+,\ y\in R^q_+$
where y solves
$\min\ c_2^Tx+d_2^Ty$
s.t. $A_2x+B_2y\leqslant g_2$

This is a two-level optimization problem. By setting

$$f(x)=\min\{d_2^Ty:\ A_2x+B_2y\leqslant g_2,\ y\geqslant 0\}$$

we can rewrite this problem as a reverse convex program in R^{p+q} :

$$\begin{aligned}\min\quad c_1^Tx+d_1^Ty\quad \text{s.t.}\quad & A_1x+B_1y\leqslant g_1\\ & A_2x+B_2y\leqslant g_2\\ & x\geqslant 0,\ y\geqslant 0\\ & \text{and } d_2^Ty\leqslant f(x).\end{aligned}$$

With a suitable transformation we can assume that

$$0\in \operatorname{int}\{(x,y):\ f(x)-d_2^Ty\leqslant 0\}\ .$$

Dual: $\max\{F(v,r):\ r<0,\ f^*(-v)\leqslant -1/r\}$
where $F(v,r)=\inf\{c_1^Tx+d_1^Ty:\ A_1x+B_1y\leqslant g_1,\ A_2x+B_2y\leqslant g_2,\ x^Tv+r.d_2^Ty\geqslant 1,\ x\geqslant 0,\ y\geqslant 0\}$. The dual is a quasiconvex maximization over a convex set in R^{p+1}.

For further applications we can see in Refs. [5-10,13].

6. CONCLUSIONS

In a quasiconvex minimization over a convex set although a local minimum may not global, we obtain a convex type duality scheme. In convex type problems, the optimality criterion at a feasible solution z is of the form

$$0 \in A(z) \qquad (11)$$

where A(z) is a convex set depending on z in the dual space. In nonconvex type problems the global optimality criterion at a feasible solution z is of the form

$$C(z) \subset A(z) \qquad (12)$$

where C(z), A(z) are convex sets depending on z in the dual space (see Thach 10). Of course criterion (12) becomes criterion (11) when C(z) is reduced to $\{0\}$. Thus, in some senses, criterion (12) is a generalization of criterion (11). But the duality obtained from (12) is quite different from the duality obtained from (11). The duality obtained from (12) is of the form max=-min (max=max or min=min), whereas the duality obtained from (11) is of the form min=max (max=-max or min=-min). By the duality we can reduce a convex type optimization problem to solving a system of inequations but we could not do the similar thing for a nonconvex type optimization problems.

REFERENCES

1. R.E.Burkard, W. Oettli and P.T.Thach, Dual solution methods for two discrete optimization problems in the space, Preprint, 1991.

2. J.B. Hiriart-Urruty, Generalized differentiability, duality and optimization for problems dealing with the difference of convex functions, Lecture Notes in Economic and Mathematical System, Ed. J. Ponstains 256(1984)37-70.

3. M.Grötschel, L.Lovasz and A.Schrijver, Geometric algorithms and and combinatorial optimization, Springer Verlag, Berlin 1988.

4. L.Lovasz, Geometric algorithms and algorithmic geometry, Plenary Lecture at the International Congress of Mathematicians, Kyoto, Japan, 1990.

5. P.T.Thach, Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a reverse convex constraint and applications, Journal of Mathematical Analysis and Applications 159(1991)199-322.

6. P.T.Thach, R.E.Burkard and W.Oettli, Mathematical programs with a two-dimensional reverse convex constraint, Journal of Global Optimization 1(1991)145-154.

7. P.T.Thach and H.Tuy, Dual outer approximation methods for concave programs and reverse convex programs, Report 90-30, Institute of Human and Social Sciences, Tokyo Institute of Technology, 1990.

8. P.T.Thach, A generalized duality and applications, Report 90-31, Institute of Human and Social Sciences, Tokyo Institute of Technology, 1990.

9. P.T.Thach, A nonconvex duality with zero gap and applications, Preprint, Department of Mathematics, Trier University, 1991.

10. P.T.Thach, Global optimality criterions and a duality with a zero gap in nonconvex optimization problems, Preprint, Department of Mathematics, Trier University, 1991.

11. J.Tind and L.A.Wolsey, An elementary survey of general duality in mathematical programming, Mathematical Programming 21(1981) 241-261.

12. J.F.Toland, Duality in nonconvex optimization, Journal of Mathematical Analysis and Applications 66(1978)399-415.

13. H.Tuy, A.Migdalas and P.Wärbrand, A global optimization approach for the linear two-level program, Report 90-17, Department of Mathematics, Linköping University, 1990.