6

Geometric calculus

Geometric algebra provides us with an invertible product for vectors. In this chapter we investigate the new insights this provides for the subject of vector calculus. The familiar gradient, divergence and curl operations all result from the action of the vector operator, ∇. Since this operator is vector-valued, we can now form its geometric product with other multivectors. We call this the *vector derivative*. Unlike the separate divergence and curl operations, the vector derivative has the important property of being invertible. That is to say, Green's functions exist for ∇ which enable initial conditions to be propagated off a surface.

The synthesis of vector differentiation and geometric algebra described in this chapter is called '*geometric calculus*'. We will see that geometric calculus provides new insights into the subject of complex analysis and enables the concept of an analytic function to be extended to arbitrary dimensions. In three dimensions this generalisation gives rise to the angular eigenstates of the Pauli theory, and the spacetime generalisation of an analytic function defines the wavefunction for a massless spin-1/2 particle. Clearly there are many insights to be gained from a unified treatment of calculus based around the geometric product.

The early sections of this chapter discuss the vector derivative, and its associated Green's functions, in flat spaces. This way we can quickly assemble a number of results of central importance in later chapters. The generalisations to embedded surfaces and manifolds are discussed in the final section. This is a large and important subject, which has been widely discussed elsewhere. Our presentation here is kept brief, focusing on the key results which are required later in this book.

6.1 The vector derivative

The vector derivative is denoted with the symbol ∇ (or $\boldsymbol{\nabla}$ in two and three dimensions). Algebraically, this has all of the properties of a vector (grade-1) object in a geometric algebra. The operator properties of ∇ are contained in the definition that the inner product of ∇ with any vector a results in the *directional derivative* in the a direction. That is,

$$a\cdot\nabla F(x) = \lim_{\epsilon\mapsto 0}\frac{F(x+\epsilon a)-F(x)}{\epsilon}, \tag{6.1}$$

where we assume that this limit exists and is well defined. Suppose that we now define a constant coordinate frame $\{\mathsf{e}_k\}$ with reciprocal frame $\{\mathsf{e}^k\}$. Spatial coordinates are defined by $x^k = \mathsf{e}^k\cdot x$, and the summation convention is assumed except where stated otherwise. The vector derivative can be written

$$\nabla = \sum_k \mathsf{e}^k\frac{\partial}{\partial x^k} = \mathsf{e}^k\partial_k, \tag{6.2}$$

where we introduce the useful abbreviation

$$\partial_i = \frac{\partial}{\partial x^i}. \tag{6.3}$$

The frame decomposition $\nabla = \mathsf{e}^k\partial_k$ shows clearly how the the vector derivative combines the algebraic properties of a vector with the operator properties of the partial derivatives. It is a straightforward exercise to confirm that the definition of ∇ is independent of the choice of frame.

6.1.1 Scalar fields

As a first example, consider the case of a scalar field $\phi(x)$. Acting on ϕ, the vector derivative ∇ returns the *gradient*, $\nabla\phi$. This is the familiar grad operation. The result is a vector whose components in the $\{\mathsf{e}^k\}$ frame are the partial derivatives with respect to the x^k coordinates. The simplest example of a scalar field is the quantity $a\cdot x$, where a is a constant vector. We write $a\cdot x = x^j a_j$, so that the gradient becomes

$$\nabla(x\cdot a) = \mathsf{e}^i\frac{\partial x^j}{\partial x^i}a_j = \mathsf{e}^i a_j\delta_i^j. \tag{6.4}$$

But the right-hand side simply expresses the vector a in the $\{\mathsf{e}^k\}$ frame, so we are left with the frame-free result

$$\nabla(x\cdot a) = a. \tag{6.5}$$

This result is independent of both the dimensions and signature of the vector space. Many formulae for the vector derivative can be built up by combining this primitive result with the chain and product rules for differentiation. A particular application of this result is to the coordinates themselves,

$$\nabla x^k = \nabla(x\cdot \mathsf{e}^k) = \mathsf{e}^k, \tag{6.6}$$

a formula which generalises to curvilinear coordinate systems.

As a second example, consider the derivative of the scalar x^2. We first derive the result in coordinates before discussing a more elegant, frame-free derivation. We form

$$\begin{aligned}\nabla(x^2) &= \mathsf{e}^i \partial_i (x^j x^k) \mathsf{e}_j \cdot \mathsf{e}_k \\ &= \mathsf{e}^i \left(\frac{\partial x^j}{\partial x^i} x^k + \frac{\partial x^k}{\partial x^i} x^j \right) \mathsf{e}_j \cdot \mathsf{e}_k \\ &= x^k \mathsf{e}_k + x^j \mathsf{e}_j \\ &= 2x,\end{aligned} \tag{6.7}$$

which recovers the expected result. It is extremely useful to be able to perform such manipulations without reference to any coordinate frame. This requires a notation to keep track of which terms are being differentiated in a given expression. A suitable convention is to use overdots to define the scope of the vector derivative. With this notation we can write

$$\nabla(x^2) = \dot{\nabla}(\dot{x}\cdot x) + \dot{\nabla}(x\cdot\dot{x}) = 2\dot{\nabla}(\dot{x}\cdot x). \tag{6.8}$$

In the final term it is only the first factor of x which is differentiated, while the second is held constant. We can therefore apply the result of equation (6.5), which immediately gives $\nabla(x^2) = 2x$. More complex results can be built up in a similar manner.

In Euclidean spaces $\nabla\phi$ points in the direction of steepest increase of ϕ. This is illustrated in equation (6.5). To get the biggest increase in $a\cdot x$ for a given step size you must clearly move in the positive a direction, since moving in any orthogonal direction does not change the value. More generally, suppose $\nabla\phi = J$ and consider the contraction of this equation with the unit vector n,

$$n\cdot\nabla\phi = n\cdot J. \tag{6.9}$$

We seek the direction of n which maximises this value. Clearly in a Euclidean space this must be the J direction, so J points in the direction of greatest increase of ϕ. Also, setting n in the J direction shows that the magnitude of J is simply the derivative in the direction of steepest increase.

In mixed signature spaces, such as spacetime, this simple geometric picture can break down. As a simple example, consider a timelike plane defined by orthogonal basis vectors $\{\gamma_0, \gamma_1\}$, with $\gamma_0^2 = 1$ and $\gamma_1^2 = -1$. We introduce the scalar field

$$\phi = \langle x\gamma_0 x\gamma_0\rangle = (x^0)^2 + (x^1)^2. \tag{6.10}$$

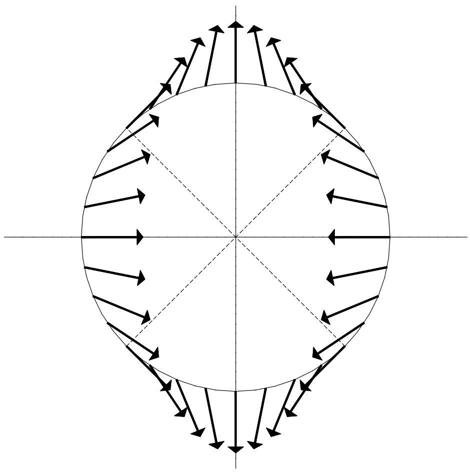

Figure 6.1 *Spacetime gradients.* The contours of the scalar field $\phi = \langle x\gamma_0 x\gamma_0\rangle$ define circles in spacetime. But the direction of the vector derivative is only in the outward normal direction along the 0 axis. Along the 1 axis the gradient points inwards, which reflects the opposite signature. Around the circle the gradient interpolates between these two extremes. At points where x is null the gradient vector is tangential to the circle.

Contours of constant ϕ are circles in the spacetime plane, so the direction of steepest increase points radially outwards. But if we form the gradient of ϕ we obtain

$$\nabla\phi = 2\dot{\nabla}\langle\dot{x}\gamma_0 x\gamma_0\rangle = 2\gamma_0 x\gamma_0. \tag{6.11}$$

Figure 6.1 shows the direction of this vector for various points on the unit circle. Clearly the vector does not point in the direction of steepest increase of ϕ. Instead, $\nabla\phi$ points in a direction 'normal' to tangent vectors in the circle. In mixed signature spaces, the 'normal' does not point in the direction our Euclidean intuition is used to. This example should be borne in mind when we consider directed integration in spaces of mixed signature. (This example may appear esoteric, but closed spacetime curves of this type are of considerable importance in some modern attempts to construct a quantum theory of gravity.)

6.1.2 Vector fields

Suppose now that we have a vector field $J(x)$. The full vector derivative ∇J contains two terms, a scalar and a bivector. The scalar term is the *divergence* of

$J(x)$. In terms of the constant frame vectors $\{\mathsf{e}_k\}$ we can write

$$\nabla\cdot J = \frac{\partial}{\partial x^k} e^k \cdot J = \frac{\partial J^k}{\partial x^k} = \partial_k J^k. \tag{6.12}$$

The divergence can also be defined in terms of the geometric product as

$$\nabla\cdot J = \tfrac{1}{2}(\nabla J + \dot{J}\dot{\nabla}). \tag{6.13}$$

The simplest example of the divergence is for the vector x itself, for which we find

$$\nabla\cdot x = \frac{\partial x^k}{\partial x^k} = n, \tag{6.14}$$

where n is the dimension of the space.

The remaining, antisymmetric, term defines the exterior derivative of the vector field. In terms of coordinates this can be written

$$\nabla\wedge J = \mathsf{e}^i\wedge(\partial_i J) = \mathsf{e}^i\wedge\mathsf{e}^j\,\partial_i J_j. \tag{6.15}$$

The components are the antisymmetrised terms in $\partial_i J_j$. In three dimensions these are the components of the curl, though $\boldsymbol{\nabla}\wedge\boldsymbol{J}$ is a bivector, rather than an (axial) vector. (In this chapter we write vectors in two and three dimensions in bold face.) The three-dimensional curl requires a duality operation to return a vector,

$$\mathrm{curl}(\boldsymbol{J}) = -I\,\boldsymbol{\nabla}\wedge\boldsymbol{J}. \tag{6.16}$$

The exterior derivative generalises the curl to arbitrary dimensions.

As an example, consider the exterior derivative of the position vector x. We find that

$$\nabla\wedge x = \mathsf{e}^i\wedge\mathsf{e}_i = \mathsf{e}^i\wedge\mathsf{e}^j\,(\mathsf{e}_i\cdot\mathsf{e}_j) = 0, \tag{6.17}$$

which follows because $\mathsf{e}^i\wedge\mathsf{e}^j$ is antisymmetric on i and j, whereas $\mathsf{e}_i\cdot\mathsf{e}_j$ is symmetric. Again, we can give an algebraic definition of the exterior derivative in terms of the geometric product as

$$\nabla\wedge J = \tfrac{1}{2}(\nabla J - \dot{J}\dot{\nabla}). \tag{6.18}$$

Equations (6.13) and (6.18) combine to give the familiar decomposition of a geometric product:

$$\nabla J = \nabla\cdot J + \nabla\wedge J. \tag{6.19}$$

So, for example, we have $\nabla x = n$.

6.1.3 Multivector fields

The preceding definitions extend simply to the case of the vector derivative acting on a multivector field. We have

$$\nabla A = \mathsf{e}^k \partial_k A, \tag{6.20}$$

and for an r-grade multivector field A_r we write

$$\nabla \cdot A_r = \langle \nabla A_r \rangle_{r-1}, \tag{6.21}$$

$$\nabla \wedge A_r = \langle \nabla A_r \rangle_{r+1}. \tag{6.22}$$

These define the interior and exterior derivatives respectively. The interior derivative is often referred to as the divergence, and the exterior derivative is sometimes called the curl. This latter name conflicts with the more familiar meaning of 'curl' in three dimensions, however, and we will avoid this name where possible.

An important result for the vector derivative is that the exterior derivative of an exterior derivative always vanishes,

$$\begin{aligned} \nabla \wedge (\nabla \wedge A) &= \mathsf{e}^i \wedge \partial_i (\mathsf{e}^j \wedge \partial_j A) \\ &= \mathsf{e}^i \wedge \mathsf{e}^j \wedge (\partial_i \partial_j A) = 0. \end{aligned} \tag{6.23}$$

This follows because $\mathsf{e}^i \wedge \mathsf{e}^j$ is antisymmetric on i, j, whereas $\partial_i \partial_j A$ is symmetric, due to the fact that partial derivatives commute. Similarly, the divergence of a divergence vanishes,

$$\nabla \cdot (\nabla \cdot A) = 0, \tag{6.24}$$

which is proved in the same way, or by using duality. (By convention, the inner product of a vector and a scalar is zero.)

Because ∇ is a vector, it does not necessarily commute with other multivectors. We therefore need to be careful in describing the scope of the operator. We use the following series of conventions to clarify the scope:

(i) In the absence of brackets, ∇ acts on the object to its immediate right.
(ii) When the ∇ is followed by brackets, the derivative acts on all of the terms in the brackets.
(iii) When the ∇ acts on a multivector to which it is not adjacent, we use overdots to describe the scope.

The 'overdot' notation was introduced in the previous section, and is invaluable when differentiating products of multivectors. For example, with this notation we can write

$$\nabla (AB) = \nabla AB + \dot{\nabla} A \dot{B}, \tag{6.25}$$

which encodes a version of the product rule. If necessary, the overdots can be replaced with partial derivatives by writing

$$\dot{\nabla} A \dot{B} = \mathsf{e}^k A \, \partial_k B. \tag{6.26}$$

Later in this chapter we also employ the overdot notation for linear functions. Suppose that $\mathsf{f}(a)$ is a position-dependent linear function. We write

$$\dot{\nabla} \dot{\mathsf{f}}(a) = \nabla \mathsf{f}(a) - \mathsf{e}^k \mathsf{f}(\partial_k a), \tag{6.27}$$

so that $\dot{\nabla}\dot{\mathsf{f}}(a)$ only differentiates the position dependence in the linear function, and not in its argument.

We can continue to build up a series of useful basic results by differentiating various multivectors that depend linearly on x. For example, consider

$$\nabla \, x \cdot A_r = \mathsf{e}^k \, \mathsf{e}_k \cdot A_r, \tag{6.28}$$

where A_r is a grade-r multivector. Using the results of section 4.3.2 we find that

$$\begin{aligned} \nabla \, x \cdot A_r &= r A_r, \\ \nabla \, x \wedge A_r &= (n-r) A_r, \\ \dot{\nabla} A_r \dot{x} &= (-1)^r (n-2r) A_r, \end{aligned} \tag{6.29}$$

where n is the dimension of the space.

6.2 Curvilinear coordinates

So far we have only expressed the vector derivative in terms of a fixed coordinate frame (which is usually chosen to be orthonormal). In many applications, however, it is more convenient to work in a *curvilinear* coordinate system, where the frame vectors vary from point to point. A general set of coordinates consist of a set of scalar functions $\{x^i(x)\}$, $i = 1, \ldots, n$, defined over some region. In this region we can equally write $x(x^i)$, expressing the position vector x parametrically in terms of the coordinates. If one of the coordinates is varied and all of the others are held fixed we specify an associated coordinate curve. The derivatives along these curves specify a set of frame vectors by

$$\mathsf{e}_i(x) = \frac{\partial x}{\partial x^i} = \lim_{\epsilon \mapsto 0} \frac{x(x^1, \ldots, x^i + \epsilon, \ldots, x^n) - x}{\epsilon}, \tag{6.30}$$

where the ith coordinate is varied and all others are held fixed. The derivative in the e_i direction, $\mathsf{e}_i \cdot \nabla$, is found by moving a small amount along e_i. But this is precisely the same as varying the x^i coordinate with all others held fixed. We therefore have

$$\mathsf{e}_i \cdot \nabla = \frac{\partial}{\partial x^i} = \partial_i. \tag{6.31}$$

In order that the coordinate system be valid over a given region we require that throughout this region

$$\mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \cdots \wedge \mathsf{e}_n \neq 0. \tag{6.32}$$

As this quantity can never pass through zero it follows that the frame has the same orientation throughout the valid region.

We can construct a second frame directly from the coordinate functions by defining

$$\mathsf{e}^i = \nabla x^i. \tag{6.33}$$

From their construction we see that the $\{\mathsf{e}^i\}$ vectors have vanishing exterior derivative:

$$\nabla \wedge \mathsf{e}^i = \nabla \wedge (\nabla x^i) = 0. \tag{6.34}$$

As the notation suggests, the two frames defined above are reciprocal to one another. This is straightforward to check:

$$\mathsf{e}_i \cdot \mathsf{e}^j = \mathsf{e}_i \cdot \nabla x^j = \frac{\partial x^j}{\partial x^i} = \delta_i^j. \tag{6.35}$$

This result is very useful because, when working with curvilinear coordinates, one usually has simple expressions for either $x^i(x)$ or $x(x^i)$, but rarely both. Fortunately, only one is needed to construct a set of frame vectors, and the reciprocal frame can then be constructed algebraically (see section 4.3). This construction provides a simple geometric picture for the gradient in a general space. Suppose we view the coordinate $x^1(x)$ as a scalar field. The contours of constant x^1 are a set of $(n-1)$-dimensional surfaces. The remaining coordinates $x^2, \ldots, x^n$ define a set of directions in this surface. At each point on the surface of constant x^1 the vector ∇x^1 is orthogonal to all of the directions in the surface. In Euclidean spaces this vector is necessarily orthogonal (normal) to the surface. In other spaces this construct defines what we mean by normal.

Now suppose we have a function $F(x)$ that is expressed in terms of the coordinates as $F(x^i)$. A simple application of the chain rule gives

$$\nabla F = \nabla x^i \, \partial_i F = \mathsf{e}^i \partial_i F. \tag{6.36}$$

This is consistent with the decomposition

$$\nabla = \mathsf{e}^i \frac{\partial}{\partial x^i} = \mathsf{e}^i \partial_i = \mathsf{e}^i \mathsf{e}_i \cdot \nabla, \tag{6.37}$$

which holds as the $\{\mathsf{e}_i\}$ and $\{\mathsf{e}^i\}$ are reciprocal frames.

6.2.1 Tensor analysis

A consequence of curvilinear frame vectors is that one has to be careful when working entirely in terms of coordinates, as is the case in tensor analysis. The problem is that for a vector, for example, we have $J = J^i \mathsf{e}_i$. If we just keep the coordinates J^i we lose the information about the position dependence in the coordinate frame. When formulating the derivative of J in tensor analysis we must introduce *connection coefficients* to keep track of the derivatives of the frame vectors. This can often complicate derivations.

There are two cases of the vector derivative in curvilinear coordinates that do not require connection coefficients. The first is the exterior derivative, for which we can write

$$\nabla\wedge J = \nabla\wedge(J_i \mathsf{e}^i) = (\nabla J_i)\wedge \mathsf{e}^i. \tag{6.38}$$

It follows that the exterior derivative has coordinates $\partial_i J_j - \partial_j J_i$ regardless of chosen coordinate system. The second exception is provided by the divergence of a vector. We have

$$\nabla\cdot J = \nabla\cdot(J^i \mathsf{e}_i). \tag{6.39}$$

If we define the volume factor V by

$$\mathsf{e}_1\wedge\mathsf{e}_2\wedge\cdots\wedge\mathsf{e}_n = IV, \tag{6.40}$$

where I is the unit pseudoscalar, we can write (following section 4.3)

$$\mathsf{e}_i = (-1)^{i-1}\mathsf{e}^n\wedge\mathsf{e}^{n-1}\wedge\cdots\wedge\check{\mathsf{e}}^i\wedge\cdots\wedge\mathsf{e}^1\, IV. \tag{6.41}$$

Recalling that each of the e^i vectors has vanishing exterior derivative, one can quickly establish that

$$\nabla\cdot J = \frac{1}{V}\frac{\partial}{\partial x^i}\big(VJ^i\big). \tag{6.42}$$

Similarly, the Laplacian ∇^2 can be written as

$$\nabla^2\phi = \frac{1}{V}\frac{\partial}{\partial x^i}\left(Vg^{ij}\frac{\partial\phi}{\partial x^j}\right), \tag{6.43}$$

where $g^{ij} = \mathsf{e}^i\cdot\mathsf{e}^j$.

6.2.2 Orthogonal coordinates in three dimensions

A number of the most useful coordinate systems are orthogonal systems of coordinates in three dimensions. For these systems a number of special results hold. We define a set of orthonormal vectors by first introducing the magnitudes

$$h_i = |\mathsf{e}_i| = (\mathsf{e}_i\cdot\mathsf{e}_i)^{1/2}. \tag{6.44}$$

In terms of these we can write (no sums implied)

$$\mathsf{e}_i = h_i\hat{\mathsf{e}}_i, \quad \mathsf{e}^i = \frac{1}{h_i}\hat{\mathsf{e}}_i. \tag{6.45}$$

We now use the $\{\hat{\mathsf{e}}_i\}$ as our coordinate frame and, since this frame is orthonormal, we can work entirely with lowered indices. For a vector $\boldsymbol{J}$ we have

$$\boldsymbol{J} = J_i \hat{\mathsf{e}}_i = \sum_{i=1}^{3} \frac{J_i}{h_i} \mathsf{e}_i. \tag{6.46}$$

It follows that we can write

$$\boldsymbol{\nabla} \cdot \boldsymbol{J} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial x_1}(h_2 h_3 J_1) + \frac{\partial}{\partial x_2}(h_3 h_1 J_2) + \frac{\partial}{\partial x_3}(h_1 h_2 J_3) \right). \tag{6.47}$$

A compact formula for the Laplacian is obtained by replacing each J_i term with $1/h_i \, \partial_i \phi$,

$$\begin{aligned}\boldsymbol{\nabla}^2 \phi = &\frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial x_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial x_1} \right) + \frac{\partial}{\partial x_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial x_2} \right) \right. \\ &\left. + \frac{\partial}{\partial x_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial x_3} \right) \right). \end{aligned} \tag{6.48}$$

The components of the curl can be found in a similar manner. A number of useful curvilinear coordinate systems are summarised below.

Cartesian coordinates

These are the basic starting point for all other coordinate systems. We introduce a constant, right-handed orthonormal frame $\{\boldsymbol{\sigma}_i\}$, $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = I$. This notation for a Cartesian frame is borrowed from quantum theory and is very useful in practice. The coordinates in the $\{\boldsymbol{\sigma}_i\}$ frame are written, following standard notation, as (x, y, z). To avoid confusion between the scalar coordinate x and the three-dimensional position vector we write the latter as $\boldsymbol{r}$. That is,

$$\boldsymbol{r} = x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2 + z\boldsymbol{\sigma}_3. \tag{6.49}$$

Since the frame vectors are orthonormal we have $h_1 = h_2 = h_3 = 1$, so the divergence and Laplacian take on their simplest forms.

Cylindrical polar coordinates

These are denoted (ρ, ϕ, z) with ρ and ϕ the standard two-dimensional polar coordinates

$$\rho = \left(x^2 + y^2\right)^{1/2}, \qquad \tan\phi = \frac{y}{x}. \tag{6.50}$$

The coordinates lie in the ranges $0 \le r < \infty$ and $0 \le \phi < 2\pi$. The coordinate vectors are

$$\begin{aligned} \hat{\mathsf{e}}_\rho &= \cos(\phi)\, \boldsymbol{\sigma}_1 + \sin(\phi)\, \boldsymbol{\sigma}_2, \\ \hat{\mathsf{e}}_\phi &= -\sin(\phi)\, \boldsymbol{\sigma}_1 + \cos(\phi)\, \boldsymbol{\sigma}_2, \\ \hat{\mathsf{e}}_z &= \boldsymbol{\sigma}_3. \end{aligned} \tag{6.51}$$

We have adopted the common convention of labelling the frame vectors with the associated coordinate. The magnitudes are $h_\rho = 1$, $h_\phi = \rho$ and $h_z = 1$, and the frame vectors satisfy

$$\hat{\mathsf{e}}_\rho \hat{\mathsf{e}}_\phi \hat{\mathsf{e}}_z = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = I \tag{6.52}$$

and so form a right-handed set in the order (ρ, ϕ, z).

Spherical polar coordinates

Spherical polar coordinates arise in many problems in physics, particularly quantum mechanics and field theory. They are typically labelled (r, θ, ϕ) and are defined by

$$r = |\boldsymbol{r}| = (\boldsymbol{r}\cdot\boldsymbol{r})^{1/2}, \quad r\cos(\theta) = z, \quad \tan(\phi) = \frac{y}{x}. \tag{6.53}$$

The coordinate ranges are $0 \le r < \infty$, $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. The ϕ coordinate is ill defined along the z axis — a reflection of the fact that it is impossible to construct a global coordinate system over the surface of a sphere. The inverse relation giving $\boldsymbol{r}(r, \theta, \phi)$ is often useful,

$$\boldsymbol{r} = r\sin(\theta)(\cos(\phi)\,\boldsymbol{\sigma}_1 + \sin(\phi)\,\boldsymbol{\sigma}_2) + r\cos(\theta)\,\boldsymbol{\sigma}_3. \tag{6.54}$$

This expression makes it a straightforward exercise to compute the orthonormal frame vectors, which are

$$\begin{aligned}
\hat{\mathsf{e}}_r &= \sin(\theta)(\cos(\phi)\,\boldsymbol{\sigma}_1 + \sin(\phi)\,\boldsymbol{\sigma}_2) + \cos(\theta)\,\boldsymbol{\sigma}_3 = r^{-1}\boldsymbol{r}, \\
\hat{\mathsf{e}}_\theta &= \cos(\theta)(\cos(\phi)\,\boldsymbol{\sigma}_1 + \sin(\phi)\,\boldsymbol{\sigma}_2) - \sin(\theta)\,\boldsymbol{\sigma}_3, \\
\hat{\mathsf{e}}_\phi &= -\sin(\phi)\,\boldsymbol{\sigma}_1 + \cos(\phi)\,\boldsymbol{\sigma}_2.
\end{aligned} \tag{6.55}$$

The associated normalisation factors are

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin(\theta). \tag{6.56}$$

The orthonormal vectors satisfy $\hat{\mathsf{e}}_r\hat{\mathsf{e}}_\theta\hat{\mathsf{e}}_\phi = I$ so that $\{\hat{\mathsf{e}}_r, \hat{\mathsf{e}}_\theta, \hat{\mathsf{e}}_\phi\}$ form a right-handed orthonormal frame. This frame can be obtained from the $\{\mathsf{e}_i\}$ frame through the application of a position-dependent rotor, so that $\hat{\mathsf{e}}_r = R\boldsymbol{\sigma}_3\tilde{R}$, $\hat{\mathsf{e}}_\theta = R\boldsymbol{\sigma}_1\tilde{R}$ and $\hat{\mathsf{e}}_\phi = R\boldsymbol{\sigma}_2\tilde{R}$. The rotor is then given by

$$R = \exp(-I\boldsymbol{\sigma}_3\phi/2)\exp(-I\boldsymbol{\sigma}_2\theta/2). \tag{6.57}$$

Spheroidal coordinates

These coordinates turn out to be useful in a number of problems in gravitation and electromagnetism involving rotating sources. We introduce a vector $\boldsymbol{a}$, so that $\pm\boldsymbol{a}$ denote the foci of a family of ellipses. The distances from the foci are given by

$$r_1 = |\boldsymbol{r} + \boldsymbol{a}|, \quad r_2 = |\boldsymbol{r} - \boldsymbol{a}|. \tag{6.58}$$

From these we define the orthogonal coordinates

$$u = \tfrac{1}{2}(r_1 + r_2), \quad v = \tfrac{1}{2}(r_1 - r_2). \tag{6.59}$$

The coordinate system is completed by rotating the ellipses around the $\boldsymbol{a}$ axis. This defines an oblate spheroidal coordinate system. Prolate spheroidal coordinates are formed by starting in a plane, defining (u_1, u_2) as above, and rotating this system around the minor axis.

If we define

$$\hat{\boldsymbol{r}}_1 = \frac{\boldsymbol{r}+\boldsymbol{a}}{r_1}, \quad \hat{\boldsymbol{r}}_2 = \frac{\boldsymbol{r}-\boldsymbol{a}}{r_2}, \tag{6.60}$$

we see that

$$\mathsf{e}^u = \tfrac{1}{2}(\hat{\boldsymbol{r}}_1 + \hat{\boldsymbol{r}}_2), \quad \mathsf{e}^v = \tfrac{1}{2}(\hat{\boldsymbol{r}}_1 - \hat{\boldsymbol{r}}_2), \tag{6.61}$$

which are clearly orthogonal. The normalisation factors are found from

$$h_u^2 = \frac{u^2 - v^2}{u^2 - a^2}, \quad h_v^2 = \frac{u^2 - v^2}{a^2 - v^2}. \tag{6.62}$$

If we align $\boldsymbol{a}$ with the 3 axis and let ϕ take its spherical-polar meaning, the coordinate frame is completed with the vector $\hat{\mathsf{e}}_\phi$, and

$$h_\phi^2 = (u^2 - a^2)(a^2 - v^2). \tag{6.63}$$

The frame vectors satisfy $\hat{\mathsf{e}}_u \hat{\mathsf{e}}_\phi \hat{\mathsf{e}}_v = I$. The hyperbolic nature of the coordinate system is often best expressed by redefining the u and v coordinates as $a\cosh(w)$ and $a\cos(\vartheta)$ respectively.

6.3 Analytic functions

The vector derivative combines the algebraic properties of geometric algebra with vector calculus in a simple and natural way. In this section we show how the vector derivative can be used to extend the definition of an analytic function to arbitrary dimensions. We start by considering the vector derivative in two dimensions to establish the link with complex analysis.

6.3.1 Analytic functions in two dimensions

Suppose that $\{\mathsf{e}_1, \mathsf{e}_2\}$ define an orthonormal frame in two dimensions. This is identified with the Argand plane by singling out e_1 as the real axis. We denote coordinates by (x, y) and write the position vector as $\boldsymbol{r}$:

$$\boldsymbol{r} = x\mathsf{e}_1 + y\mathsf{e}_2. \tag{6.64}$$

With this notation the vector derivative is

$$\boldsymbol{\nabla} = \mathsf{e}_1 \frac{\partial}{\partial x} + \mathsf{e}_2 \frac{\partial}{\partial y}. \tag{6.65}$$

In section 2.3.3 we showed that complex numbers sit naturally within the geometric algebra of the plane. The pseudoscalar is the bivector $I = \mathsf{e}_1\mathsf{e}_2$, which satisfies $I^2 = -1$. Complex numbers therefore map directly onto even-grade elements in the algebra by identifying the unit imaginary i with I. The position vector $\boldsymbol{r}$ is mapped onto a complex number by pre-multiplying by the vector representing the real axis:

$$z = x + Iy = \mathsf{e}_1\boldsymbol{r}. \tag{6.66}$$

Now suppose we introduce the complex field $\psi = u + Iv$. The vector derivative applied to ψ yields

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathsf{e}_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\mathsf{e}_2. \tag{6.67}$$

The terms in brackets are precisely the ones that vanish in the Cauchy–Riemann equations. The statement that ψ is an *analytic function* (a function that satisfies the Cauchy–Riemann equations) reduces to the equation

$$\boldsymbol{\nabla}\psi = 0. \tag{6.68}$$

This is the fundamental equation which can be generalised immediately to higher dimensions. These generalisations invariably turn out to be of mathematical and physical importance, and it is is no exaggeration to say that equations of the type of equation (6.68) are amongst the most studied in physics.

To complete the link with complex analysis we recall that the complex partial derivative ∂_z is defined by the properties

$$\frac{\partial z}{\partial z} = 1, \qquad \frac{\partial z^\dagger}{\partial z} = 0 \tag{6.69}$$

with the complex conjugate satisfying

$$\frac{\partial z}{\partial z^\dagger} = 0, \qquad \frac{\partial z^\dagger}{\partial z^\dagger} = 1. \tag{6.70}$$

From these we see that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial z^\dagger} = \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right). \tag{6.71}$$

An analytic function is one that depends on z alone. That is, we can write $\psi(x + Iy) = \psi(z)$. The function is therefore independent of $z^\dagger$, and we have

$$\frac{\partial \psi(z)}{\partial z^\dagger} = 0. \tag{6.72}$$

This summarises the content of the Cauchy–Riemann equations, though this fact is often obscured by the complex limiting argument favoured in many textbooks. Comparing the preceding forms, we see that this equation is equivalent to

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right)\psi = \tfrac{1}{2}\mathsf{e}_1\boldsymbol{\nabla}\psi = 0, \tag{6.73}$$

recovering our earlier equation.

It is instructive to see why solutions to $\boldsymbol{\nabla}\psi = 0$ can be constructed as power series in z. We first see that

$$\boldsymbol{\nabla} z = \boldsymbol{\nabla}(\mathsf{e}_1 \boldsymbol{r}) = 2\mathsf{e}_1 \cdot \boldsymbol{\nabla}\boldsymbol{r} - \mathsf{e}_1 \boldsymbol{\nabla}\boldsymbol{r} = 2\mathsf{e}_1 - 2\mathsf{e}_1 = 0. \tag{6.74}$$

This little manipulation drives most of analytic function theory! It follows immediately, for example, that

$$\boldsymbol{\nabla}(z - z_0)^n = n\boldsymbol{\nabla}(\mathsf{e}_1 \boldsymbol{r} - z_0)(z - z_0)^{n-1} = 0, \tag{6.75}$$

so a Taylor series expansion in z about z_0 automatically returns an analytic function. We will delay looking at poles until we have introduced the subject of directed integration.

6.3.2 Generalized analytic functions

There are two problems with the standard presentation of complex analytic function theory that prevent a natural generalisation to higher dimensions:

(i) Both the vector operator $\boldsymbol{\nabla}$ and the functions it operates on are mapped into the same algebra by picking out a preferred direction for the real axis. This only works in two dimensions.
(ii) The 'complex limit' argument does not generalise to higher dimensions. Indeed, one can argue that it is not wholly satisfactory in two dimensions, as it confuses the concept of a directional derivative with the concept of being independent of $z^\dagger$.

These problems are solved by keeping the derivative operator $\boldsymbol{\nabla}$ as a vector, while letting it act on general multivectors. The analytic requirement is then replaced with the equation $\boldsymbol{\nabla}\psi = 0$. Functions satisfying this equation are said to be *monogenic*. If ψ contains all grades it is clear that both the even-grade and odd-grade components must satisfy this equation independently. Without loss of generality, we can therefore assume that ψ has even grade.

We can construct monogenic functions by following the route which led to the conclusion that z is analytic in two dimensions. We recall that $\boldsymbol{\nabla}\boldsymbol{r} = 3$ and

$$\boldsymbol{\nabla}(\boldsymbol{a}\boldsymbol{r}) = -\boldsymbol{a}. \tag{6.76}$$

It follows that

$$\psi = \boldsymbol{r}\boldsymbol{a} + 3\boldsymbol{a}\boldsymbol{r} \tag{6.77}$$

is a monogenic for any constant vector $\boldsymbol{a}$. The main difference with complex analysis is that we cannot derive new monogenics simply from power series in this solution, due to the lack of commutativity. One can construct monogenic functions from series of geometric products, but a more instructive route is to classify monogenics via their angular properties.

First we assume that Ψ is a monogenic containing terms which scale uniformly with $\boldsymbol{r}$. If we introduce polar coordinates we can then write

$$\Psi(\boldsymbol{r}) = r^l \psi(\theta, \phi). \tag{6.78}$$

The function $\psi(\theta, \phi)$ then satisfies

$$l r^{l-1} \mathsf{e}_r \psi + r^l \boldsymbol{\nabla} \psi(\theta, \phi) = 0. \tag{6.79}$$

It follows that ψ satisfies the angular eigenvalue equation

$$-\boldsymbol{r} \wedge \boldsymbol{\nabla} \psi = l\psi. \tag{6.80}$$

These angular eigenstates play a key role in the Pauli and Dirac theories of the electron. Since Ψ satisfies $\boldsymbol{\nabla}\Psi = 0$, it follows that

$$\boldsymbol{\nabla}^2 \Psi = 0. \tag{6.81}$$

So each component of Ψ (in a constant basis) satisfies Laplace's equation. It follows that each component of ψ is a spherical harmonic, and hence that l is an integer. We can construct a monogenic by starting with the function $(x + yI\boldsymbol{\sigma}_3)^l$, which is the three-dimensional extension of the complex analytic function z^l. In terms of polar coordinates

$$(x + yI\boldsymbol{\sigma}_3)^l = r^l \sin^l(\theta) \, \mathrm{e}^{l\phi I \boldsymbol{\sigma}_3}, \tag{6.82}$$

which gives us our first angular monogenic function

$$\psi_l^l = \sin^l(\theta) \, \mathrm{e}^{l\phi I \boldsymbol{\sigma}_3}. \tag{6.83}$$

The remaining monogenic functions are constructed from this by acting with an operator which, in quantum terms, lowers the eigenvalue of the angular momentum around the z axis. These are discussed in more detail in section 8.4.1.

6.3.3 The spacetime vector derivative

To construct the vector derivative in spacetime suppose that we introduce the orthonormal frame $\{\gamma_\mu\}$ with associated coordinates x^μ. We can then write

$$\nabla = \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma_0 \frac{\partial}{\partial t} + \gamma^i \frac{\partial}{\partial x^i}. \tag{6.84}$$

This derivative is the key operator in all relativistic field theories, including electromagnetism and Dirac theory. If we post-multiply by γ_0 we see that

$$\nabla \gamma_0 = \partial_t + \gamma^i \gamma_0 \partial_i = \partial_t - \boldsymbol{\nabla}, \tag{6.85}$$

where $\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \partial_i$ is the vector derivative in the relative space defined by the γ_0 vector. Similarly,

$$\gamma_0 \nabla = \partial_t + \boldsymbol{\nabla}. \tag{6.86}$$

These equations are consistent with

$$\nabla x = \nabla(\gamma_0 \gamma_0 x) = (\partial_t - \boldsymbol{\nabla})(t - \boldsymbol{r}) = 4, \tag{6.87}$$

where x is the spacetime position vector. The spacetime vector derivative satisfies

$$\nabla^2 = \frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2, \tag{6.88}$$

which is the fundamental operator describing waves travelling at the speed of light. The spacetime monogenic equation $\nabla \psi = 0$ is discussed in detail in chapters 7 and 8. We only note here that, if ψ is an even-grade element of the spacetime algebra, the monogenic equation is precisely the wave equation for a massless spin-1/2 particle.

6.3.4 Characteristic surfaces and propagation

The fact that ∇^2 can give rise to either elliptic or hyperbolic operators, depending on signature, suggests that the propagator theory for ∇ will depend strongly on the signature. This is confirmed by a simple argument which can be modified to apply to most first-order differential equations. Suppose we have a generic equation of the type

$$\nabla \psi = f(\psi, x), \tag{6.89}$$

where ψ is some multivector field, $f(\psi, x)$ is a known function and x is the position vector in an n-dimensional space. We are presented with data on some $(n-1)$-dimensional surface, and wish to propagate these initial conditions away from the surface. If surfaces exist for which this is not possible they are known as *characteristic surfaces*. Suppose that we construct a set of independent tangent vectors in the surface, $\{e_1, \ldots, e_{n-1}\}$. Knowledge of ψ on the surface enables us to calculate each of the directional derivatives $e_i \cdot \nabla \psi$, $i = 1, \ldots, n-1$. We now form the normal vector

$$n = I\, e_1 \wedge e_2 \wedge \cdots \wedge e_{n-1}, \tag{6.90}$$

where I is the pseudoscalar for the space. Pre-multiplying equation (6.89) with n we obtain

$$n \cdot \nabla \psi = -n \wedge \nabla \psi + n f(\psi, x). \tag{6.91}$$

But we have

$$n\wedge\nabla\psi = I(e_1\wedge e_2\wedge\cdots\wedge e_{n-1})\cdot\nabla\psi$$
$$= I\sum_{i=1}^{n-1}(-1)^{i+1-n}(e_1\wedge\cdots\wedge\check{e}_i\wedge\cdots\wedge e_{n-1})\,e_i\cdot\nabla\psi, \tag{6.92}$$

which is constructed entirely from known derivatives of ψ. Equation (6.91) then tells us how to propagate ψ in the n direction. The only situation in which we can fail to propagate ψ is when n still lies in the surface. This happens if n is linearly dependent on the surface tangent vectors. If this is the case we have

$$n\wedge(e_1\wedge e_2\wedge\cdots\wedge e_{n-1}) = 0. \tag{6.93}$$

But this implies that

$$(I^{-1}n)\wedge n = I^{-1}n\cdot n = 0. \tag{6.94}$$

We therefore only fail to propagate when $n^2 = 0$, so characteristic surfaces are always null surfaces. This possibility can only arise in mixed signature spaces, and unsurprisingly the propagators in these spaces can have quite different properties to their Euclidean counterparts.

6.4 Directed integration theory

The true power of geometric calculus begins to emerge when we study directed integration theory. This provides a very general and powerful integral theorem which enables us to construct Green's functions for the vector derivative in various spaces. These in turn can be used to generalise the many powerful results from complex function theory to arbitrary spaces.

6.4.1 Line integrals

The simplest integrals to start with are line integrals. The line integral of a multivector field $F(x)$ along a line $x(\lambda)$ is defined by

$$\int F(x)\frac{dx}{d\lambda}\,d\lambda = \int F\,dx = \lim_{n\mapsto\infty}\sum_{i=1}^{n}\bar{F}^i\Delta x^i. \tag{6.95}$$

In the final expression a set of successive points along the curve $\{x_i\}$ are introduced, with x_0 and x_n the endpoints, and

$$\Delta x^i = x_i - x_{i-1}, \quad \bar{F}^i = \tfrac{1}{2}\big(F(x_{i-1}) + F(x_i)\big). \tag{6.96}$$

If the curve is closed then $x_0 = x_n$. The result of the integral is independent of the way we choose to parameterise the curve, provided the parameterisation respects the required ordering of points along the curve. Curves that double back on themselves are handled by referring to the parameterised form $x(\lambda)$, which tells us how the curve is traversed.

The definition of the integral (6.95) looks so standard that it is easy to overlook the key new feature, which is that dx is a *vector-valued measure*, and the product $F\,dx$ is a geometric product between multivectors. This small extension to scalar integration is sufficient to bring a wealth of new features. We refer to dx, and its multivector-valued extensions, as a *directed measure*. The fact that dx is no longer a scalar means that equation (6.95) is not the most general line integral we can form. We can also consider integrals of the form

$$\int F(x)\frac{dx}{d\lambda}G(x)\,d\lambda = \int F(x)\,dx\,G(x), \tag{6.97}$$

and more generally we can consider sums of terms like these. The most general form of line integral can be written

$$\int \mathsf{L}(\partial_\lambda x; x)\,d\lambda = \int \mathsf{L}(dx), \tag{6.98}$$

where $\mathsf{L}(a) = \mathsf{L}(a;x)$ is a multivector-valued linear function of a. The position dependence in L can often be suppressed to streamline the notation.

Suppose now that the field F is replaced by the vector-valued function $v(x)$. We have

$$\int v\,dx = \int v\cdot dx + \int v\wedge dx, \tag{6.99}$$

which separates the directed integral into scalar and bivector-valued terms. If v is the unit tangent vector along the curve then the scalar integral returns the arc length. In many applications the scalar and bivector integrals are considered separately. But to take advantage of the most powerful integral theorems in geometric calculus we need to use the combined form, containing a geometric product with the directed measure.

6.4.2 Surface integrals

The natural extension of a line integral is to a directed surface integral. Suppose now that the the multivector-valued field F is defined over a two-dimensional surface embedded in some larger space. If the surface is parameterised by two coordinates $x(x^1, x^2)$ we define the directed measure by the bivector

$$dX = \frac{\partial x}{\partial x^1}\wedge\frac{\partial x}{\partial x^2}\,dx^1\,dx^2 = e_1\wedge e_2\,dx^1\,dx^2, \tag{6.100}$$

where $e_i = \partial_i x$. This measure is independent of how the surface is parameterised, provided we orient the coordinate vectors in the desired order. Sometimes more than one coordinate patch will be needed to parameterise the entire surface, but

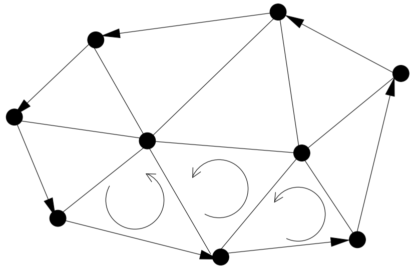

Figure 6.2 *A triangulated surface.* The surface is represented by a series of points, and each set of three adjacent points defines a triangle, or simplex. As more points are added the simplices become a closer fit to the true surface. Each simplex is given the same orientation by ensuring that for adjacent simplices, the common edge in traversed in opposite directions.

the directed measure dX is still defined everywhere. A directed surface integral then takes the form

$$\int F\,dX = \int Fe_1 \wedge e_2\,dx^1\,dx^2, \tag{6.101}$$

or a sum of such terms if more than one coordinate patch is required. Again, we form the geometric product between the integrand and the measure. As in the case of a line integral, this is not the most general surface integral that can be considered, as the integrand can multiply the measure from the left or the right, giving rise to different integrals.

As an example of a surface integral, consider a closed surface in three dimensions, with unit outward normal $\boldsymbol{n}$. We let F be given by the bivector-valued function $\phi \boldsymbol{n} I^{-1}$, where ϕ is a scalar field. The surface integral is then

$$\oint \phi \boldsymbol{n} I^{-1}\,dX = \oint \phi |dS|. \tag{6.102}$$

Here $|dS| = I^{-1}\boldsymbol{n}\,dX$ is the scalar-valued measure over the surface. The directed measure is usually chosen so that $\boldsymbol{n}\,dX$ has the same orientation as I. As a second example, suppose that $F = 1$. In this case we can show that

$$\oint dX = 0, \tag{6.103}$$

which holds for any closed surface (see later). If the surface is open, the result of the directed surface integral depends entirely on the boundary, since all the internal simplices cancel out. This result is sometimes called the vector area, though in geometric algebra the result is a bivector.

In order to construct proofs of some of the more important results it is necessary to express the surface integral (6.101) in terms of a limit of a sum. This involves the idea of a triangulated surface (figure 6.2). A set of points are chosen

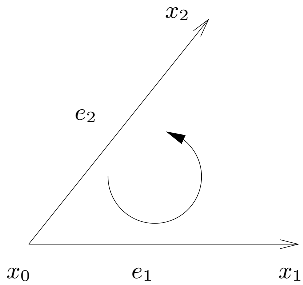

Figure 6.3 *A planar simplex.* The points x_0, x_1, x_2 define a triangle. The order specifies how the boundary is traversed, which defines an orientation for the simplex.

on the surface, and adjacent sets of three points define a series of planar triangles, or *simplices*. As more points are added these triangles become smaller and are an ever better model for the surface. (In computer graphics programs this is precisely how 'smooth' surfaces are represented internally.) Each simplex has an orientation attached such that, for a pair of adjacent simplices, the common edge is traversed in opposite directions. In this way an initial simplex builds up to define an orientation for the entire surface. For some surfaces, such as the Mobius strip, it is not possible to define a consistent orientation over the entire surface. For these it is not possible to define a directed integral, so our presentation is restricted to *orientable* surfaces.

Suppose now that the three points x_0, x_1, x_2 define the corners of a simplex, with orientation specified by traversing the edges in the order $x_0 \mapsto x_1 \mapsto x_2$ (see figure 6.3). We define the vectors

$$e_1 = x_1 - x_0, \qquad e_2 = x_2 - x_0. \tag{6.104}$$

The surface measure is then defined by

$$\Delta X = \tfrac{1}{2} e_1 \wedge e_2 = \tfrac{1}{2}(x_1 \wedge x_2 + x_2 \wedge x_0 + x_0 \wedge x_1). \tag{6.105}$$

ΔX has the orientation defined by the boundary, and an area equal to that of the simplex. The final expression makes it clear that ΔX is invariant under even permutations of the vertices. With this definition of ΔX we can express the surface integral (6.101) as the limit:

$$\int F\, dX = \lim_{n \mapsto \infty} \sum_{k=1}^{n} \bar{F}^k \Delta X^k. \tag{6.106}$$

The sum here runs over all simplices making up the surface, and for each simplex $\bar{F}$ is the average value of F over the simplex. For well-behaved integrals the value in the limit is independent of the precise nature of the limiting process.

6.4.3 n-dimensional surfaces

The simplex structure introduced in the previous section provides a means of defining a directed integral for any dimension of surface. We discretise the surface by considering a series of points, and adjacent sets of points are combined to define a simplex. Suppose that we have an n-dimensional surface, and that one simplex for the discretised surface has vertices $x_0, \ldots, x_n$, with the order specifying the desired orientation. For this simplex we define vectors

$$e_i = x_i - x_0, \quad i = 1, \ldots, n, \tag{6.107}$$

and the directed volume element is

$$\Delta X = \frac{1}{n!} e_1 \wedge \cdots \wedge e_n. \tag{6.108}$$

A point in the simplex can be described in terms of coordinates $\lambda^1, \ldots, \lambda^n$ by writing

$$x = x_0 + \sum_{i=1}^{n} \lambda^i e_i. \tag{6.109}$$

Each coordinate lies in the range $0 \leq \lambda^i \leq 1$, and the coordinates also satisfy

$$\sum_{i=1}^{n} \lambda^i \leq 1. \tag{6.110}$$

Now suppose we have a multivector field $F(x)$ defined over the surface. We denote the value at each vertex by $F_i = F(x_i)$. A new function $f(x)$ is then introduced which linearly interpolates the F_i over the simplex. This can be written

$$f(x) = F_0 + \sum_{i=1}^{n} \lambda^i (F_i - F_0). \tag{6.111}$$

As the number of points increases and the simplices grow smaller, $f(x)$ becomes an ever better approximation to $F(x)$, and the triangulated surface approaches the true surface.

The directed integral of F over the surface is now approximated by the integral of f over each simplex in the surface. To evaluate the integral over each simplex we use the λ^i as coordinates, so that

$$dX = e_1 \wedge \cdots \wedge e_n \, d\lambda^1 \cdots d\lambda^n. \tag{6.112}$$

It is then a straightforward exercise in integration to establish that

$$\int dX = \Delta X \tag{6.113}$$

and

$$\int \lambda^i \, dX = \frac{1}{n+1} \Delta X, \quad \forall \lambda^i. \tag{6.114}$$

Combining these two results we find that the integral of $f(x)$ over a single simplex evaluates to

$$\int f \, dX = \frac{1}{n+1} \left(\sum_{i=0}^{n} F_i \right) \Delta X. \tag{6.115}$$

The function is therefore replaced by its average value over the simplex. We write this as $\bar{F}$. Summing over all the simplices making up the surface we can now define

$$\int F \, dX = \lim_{n \mapsto \infty} \sum_{k=1}^{n} \bar{F}^k \, \Delta X^k, \tag{6.116}$$

where k runs over all of the simplices in the surface. More generally, suppose that $\mathsf{L}(A_n)$ is a position-dependent linear function of a grade-n multivector A_n. We can then write

$$\int \mathsf{L}(dX) = \lim_{n \mapsto \infty} \sum_{k=1}^{n} \bar{\mathsf{L}}^k(\Delta X^k), \tag{6.117}$$

with $\bar{\mathsf{L}}^k(\Delta X^k)$ the average value of $\mathsf{L}(\Delta X^k)$ over the vertices of each simplex.

6.4.4 The fundamental theorem of geometric calculus

Most physicists are familiar with a number of integral theorems, including the divergence and Stokes' theorems, and the Cauchy integral formula of complex analysis. We will now show that these are all special cases of a more general theorem in geometric calculus. In this section we will sketch of proof of this important theorem. Readers who are not interested in the details of the proof may want to jump straight to the following section, where some applications are discussed. The proof given here uses simplices and triangulated surfaces, which means that it is relevant to methods of discretising integrals for numerical computation.

We start by introducing a notation for simplices which helps clarify the nature of the boundary operator. We let $(x_0, x_1, \ldots, x_k)$ denote the k-simplex defined by the $k+1$ points $x_0, \ldots, x_k$. This is abbreviated to

$$(x)_{(k)} = (x_0, x_1, \ldots, x_k). \tag{6.118}$$

The order of points is important, as it specifies the orientation of the simplex. If any two adjacent points are swapped then the simplex changes sign. The boundary operator for a simplex is denoted by ∂ and is defined by

$$\partial(x)_{(k)} = \sum_{i=0}^{k}(-1)^i(x_0, \ldots, \check{x}_i, \ldots, x_k)_{(k-1)}, \tag{6.119}$$

where the check denotes that the term is missing from the product. So, for example,

$$\partial(x_0, x_1) = (x_1) - (x_0), \tag{6.120}$$

which returns the two points at the end of a line segment. The boundary of a boundary vanishes,

$$\partial\partial(x)_{(k)} = 0. \tag{6.121}$$

Proofs of this can be found in most differential geometry textbooks.

So far we have dealt only with ordered lists of points, not geometric sums or products. To add some geometry we introduce the operator Δ which returns the directed content of a simplex,

$$\Delta(x)_{(k)} = \frac{1}{k!}(x_1 - x_0)\wedge(x_2 - x_0)\wedge\cdots\wedge(x_k - x_0). \tag{6.122}$$

This is the result of integrating the directed measure over a simplex

$$\int_{(x)_{(k)}} dX = \Delta(x)_{(k)} = \Delta X. \tag{6.123}$$

The directed content of a boundary vanishes,

$$\Delta(\partial(x)_{(k)}) = 0. \tag{6.124}$$

As an example, consider a planar simplex consisting of three points. We have

$$\partial(x_0, x_1, x_2) = (x_1, x_2) - (x_0, x_2) + (x_0, x_1). \tag{6.125}$$

So the directed content of the boundary is

$$\Delta(\partial(x_0, x_1, x_2)) = (x_2 - x_1) - (x_2 - x_0) + (x_1 - x_0) = 0. \tag{6.126}$$

The general result of equation (6.124) can be established by induction from the case of a triangle. These results are sufficient to establish that the directed integral over the surface of a simplex is zero:

$$\oint_{\partial(x)_{(k)}} dS = \sum_{i=0}^{k}(-1)^i \int_{(\check{x}_i)_{(k-1)}} dX = \Delta(\partial(x)_{(k)}) = 0. \tag{6.127}$$

A general volume is built up from a chain of simplices. Simplices in the chain are defined such that, at any common boundary, the directed areas of the bounding faces of two simplices are equal and opposite. It follows that the surface integrals over two simplices cancel out over their common face. The surface integral over the boundary of the volume can therefore be replaced by the sum of the surface integrals over each simplex in the chain. If the boundary is closed we establish that

$$\oint dS = \lim_{n\mapsto\infty} \sum_{a=1}^{n} \oint dS^a = 0. \tag{6.128}$$

The sum runs over each simplex in the surface, with a labeling the simplex. It is implicit in this proof that the surface bounds a volume which can be filled by a connected set of simplices. So, as well as being oriented, the surface must be closed and simply connected.

Next, we return to equation (6.114) and introduce a constant vector b. If we define $b_i = b\cdot e_i$ we see that

$$\sum_{i=1}^{k} b_i \lambda^i = b\cdot(x - x_0), \tag{6.129}$$

which is valid for all vectors x in the simplex of interest. Multiplying equation (6.114) by b_i and summing over i we obtain

$$\int\limits_{(x)_{(k)}} b\cdot(x - x_0)\, dX = \frac{1}{k+1} \sum_{i=1}^{k} b\cdot e_i\, \Delta X, \tag{6.130}$$

where the integral runs over a simplex defined by $k+1$ vertices. A simple reordering yields

$$\begin{aligned} \int b\cdot x\, dX &= \frac{1}{k+1}\left(\sum_{i=1}^{k} b\cdot(x_i - x_0) + (k+1) b\cdot x_0\right)\Delta X \\ &= b\cdot\bar{x}\,\Delta X, \end{aligned} \tag{6.131}$$

where $\bar{x}$ is the vector representing the (geometric) centre of the simplex,

$$\bar{x} = \frac{1}{k+1}\sum_{i=0}^{k} x_i. \tag{6.132}$$

Now suppose we have a k-simplex specified by the $k+1$ points $(x_0, \ldots, x_k)$ and we form the directed surface integral of $b\cdot x$. We obtain

$$\oint\limits_{\partial(x)_{(k)}} b\cdot x\, dS = \frac{1}{k+1}\sum_{i=0}^{k}(-1)^i b\cdot(x_0 + \cdots \check{x}_i \cdots + x_n)\Delta(\check{x}_i)_{(k-1)}. \tag{6.133}$$

To evaluate the final sum we need the result that

$$\sum_{i=0}^{k}(-1)^i b\cdot(x_0 + \cdots \check{x}_i \cdots + x_n)\Delta(\check{x}_i)_{(k-1)} = \frac{1}{k!} b\cdot(e_1 \wedge \cdots \wedge e_n). \tag{6.134}$$

The proof of this result is purely algebraic and is left as an exercise. We have now established the simple result that

$$\oint_{\partial(x)_{(k)}} b\cdot x\, dS = b\cdot(\Delta X), \tag{6.135}$$

where $\Delta X = \Delta((x)_{(k)})$. The order and orientations in this result are important. The simplex $(x)_{(k)}$ is oriented, and the order of points specifies how the boundary is traversed. With dS the oriented element over each boundary, and ΔX the volume element for the simplex, we find that the correct expression for the surface integral is $b\cdot(\Delta X)$.

We are now in a position to apply these results to the interpolated function $f(x)$ of equation (6.111). Suppose that we are working in a (flat) n-dimensional space and consider a simplex with points $(x_0, \ldots, x_n)$. The simplex is chosen such that its volume is non-zero, so the n vectors $e_i = x_i - x_0$ define a (non-orthonormal) frame. We therefore write

$$\mathsf{e}_i = x_i - x_0, \tag{6.136}$$

and introduce the reciprocal frame $\{\mathsf{e}^i\}$. These vectors satisfy

$$\mathsf{e}^i\cdot(x - x_0) = \lambda^i. \tag{6.137}$$

It follows that the surface integral of $f(x)$ over the simplex is given by

$$\begin{aligned}\oint_{\partial(x)_{(k)}} f(x)dS &= \sum_{i=1}^{n}(F_i - F_0)\oint \mathsf{e}^i\cdot(x - x_0)dS \\ &= \sum_{i=1}^{n}(F_i - F_0)\mathsf{e}^i\cdot(\Delta X).\end{aligned} \tag{6.138}$$

But if we consider the directional derivatives of $f(x)$ we find that

$$\frac{\partial f(x)}{\partial \lambda^i} = F_i - F_0. \tag{6.139}$$

The result of the surface integral can therefore be written

$$\begin{aligned}\oint_{\partial(x)_{(k)}} f(x)dS &= \sum_{i=1}^{n}(F_i - F_0)\mathsf{e}^i\cdot(\Delta X) \\ &= \sum_{i=1}^{n}\frac{\partial f}{\partial \lambda^i}\mathsf{e}^i\cdot(\Delta X) = \dot{f}\dot{\nabla}\cdot(\Delta X).\end{aligned} \tag{6.140}$$

Here we have used the result that $\nabla = \mathsf{e}^i\partial_i$, which follows from using the λ^i as a set of coordinates.

We now consider a chain of simplices, and add the result of equation (6.140) over each simplex in the chain. The interpolated function $f(x)$ takes on the same value over the common boundary of two adjacent simplices, since $f(x)$ is only defined by the values at the common vertices. In forming a sum over a chain, all of the internal faces cancel and only the surface integral over the boundary remains. We therefore arrive at

$$\oint f(x)\, dS = \sum_a \dot{f}\dot{\nabla}\cdot(\Delta X^a), \tag{6.141}$$

with the sum running over all of the simplices in the chain. Taking the limit as more points are added and each simplex is shrunk in size we arrive at our first statement of the fundamental theorem,

$$\oint_{\partial V} F\, dS = \int_V \dot{F}\dot{\nabla}\, dX. \tag{6.142}$$

We have replaced the interpolated function f with F, which is obtained in the limit as more points are added. We have also used the fact that ∇ lies entirely within the space defined by the pseudoscalar measure dX to remove the contraction on the right-hand side and write a geometric product.

The above proof is easily adapted for the case where the function sits to the right of the measure, giving

$$\oint_{\partial V} dS\, G = \int_V \dot{\nabla}\, dX\, \dot{G}. \tag{6.143}$$

Since ∇ is a vector, the commutation properties with dX will depend on the dimension of the space. A yet more general statement of the fundamental theorem can be constructed by introducing a linear function $\mathsf{L}(A_{n-1}) = \mathsf{L}(A_{n-1}; x)$. This function takes a multivector A_{n-1} of grade $n-1$ as its linear argument, and returns a general multivector. L is also position-dependent, and its linear interpolation over a simplex is defined by

$$L(A) = \mathsf{L}(A; x_0) + \sum_{i=1}^{n} \lambda^i \big(\mathsf{L}(A; x_i) - \mathsf{L}(A, x_0)\big). \tag{6.144}$$

The linearity of $L(A)$ means that sums and integrals can be moved inside the argument, and we establish that

$$\begin{aligned}\oint L(dS) &= \mathsf{L}\left(\oint dS; x_0\right) + \sum_{i=1}^{n} \mathsf{L}\left(\oint \lambda^i dS; x_i\right) - \sum_{i=1}^{n} \mathsf{L}\left(\oint \lambda^i dS; x_0\right) \\ &= \sum_{i=1}^{n} \mathsf{L}(\mathsf{e}^i \Delta X; x_i) - \mathsf{L}(\mathsf{e}^i \Delta X; x_0) \\ &= \dot{L}(\dot{\nabla}\Delta X).\end{aligned} \tag{6.145}$$

There is no position dependence in the final term as the derivative is constant over the simplex. Building up a chain of simplices and taking the limit we prove the general result

$$\oint_{\partial V} \mathsf{L}(dS) = \int_V \dot{\mathsf{L}}(\dot{\nabla} dX). \tag{6.146}$$

This holds for any linear function $\mathsf{L}(A_{n-1})$ integrated over a closed region of an n-dimensional flat space. This is still not the most general statement of the fundamental theorem, as we will later prove a version valid for surfaces embedded in a curved space, but equation (6.146) is sufficient to make contact with the main integral theorems of vector calculus.

6.4.5 The divergence and Green's theorems

To see the fundamental theorem of geometric calculus in practice, first consider the scalar-valued function

$$\mathsf{L}(A) = \langle JAI^{-1} \rangle. \tag{6.147}$$

Here J is a vector, and I is the (constant) unit pseudoscalar for the n-dimensional space. The argument A is a multivector of grade $n-1$. Equation (6.146) gives

$$\int_V \langle \dot{J}\dot{\nabla} dX I^{-1} \rangle = \int_V \nabla\cdot J \, |dX| = \oint_{\partial V} \langle J dS I^{-1} \rangle, \tag{6.148}$$

where $|dX| = I^{-1}dX$ is the scalar measure over the volume of interest. The normal to the surface, n is defined by

$$n|dS| = dS\, I^{-1}, \tag{6.149}$$

where $|dS|$ is the scalar-valued measure over the surface. This definition ensures that, in Euclidean spaces, $n\, dS$ has the orientation defined by I, and in turn that n points outwards. With this definition we arrive at

$$\int_V \nabla\cdot J \, |dX| = \oint_{\partial V} n\cdot J \, |dS|, \tag{6.150}$$

which is the familiar divergence theorem. This way of writing the theorem hides the fact that $n|dS|$ should be viewed as a single entity, which can be important in spaces of mixed signature.

Now return to the fundamental theorem in the form of equation (6.143), and let G equal the vector $\boldsymbol{J}$ in two-dimensional Euclidean space. We find that

$$\oint_{\partial V} dS\, \boldsymbol{J} = \int_V \dot{\boldsymbol{\nabla}}\, dX\, \dot{\boldsymbol{J}} = -\int_V \boldsymbol{\nabla} \boldsymbol{J}\, dX, \tag{6.151}$$

where we have used the fact that dX is a pseudoscalar, so it anticommutes with vectors in two dimensions. Introducing Cartesian coordinates we have $dX = I dx\, dy$, so

$$\oint_{\partial V} dS\, \boldsymbol{J} = -\int_V \boldsymbol{\nabla} \boldsymbol{J} I\, dx\, dy. \tag{6.152}$$

If we let $\boldsymbol{J} = P\mathsf{e}_1 + Q\mathsf{e}_2$ and take the scalar part of both sides, we prove Green's theorem in the plane

$$\oint P dx + Q dy = \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy. \tag{6.153}$$

The line integral is taken around the perimeter of the area in a positive sense, as specified by $I = \mathsf{e}_1 \mathsf{e}_2$.

6.4.6 Cauchy's integral formula

The fundamental theorem of geometric calculus enables us to view the Cauchy integral theorem of complex variable theory in a new light. We let ψ denote an even-grade multivector, which therefore commutes with dX, so we can write

$$\int \boldsymbol{\nabla} \psi\, dX = \oint d\boldsymbol{s}\, \psi = \oint \frac{\partial \boldsymbol{r}}{\partial \lambda} \psi\, d\lambda. \tag{6.154}$$

In the final expression λ is a parameter along the (closed) curve. Now recall from section 6.3.1 that we form the complex number z by $z = \mathsf{e}_1 \boldsymbol{r}$. We therefore have

$$\oint \psi dz = \int \mathsf{e}_1 \boldsymbol{\nabla} \psi\, dX, \tag{6.155}$$

where the term on the left is now a complex line integral. The condition that ψ is analytic can be written $\boldsymbol{\nabla}\psi = 0$ so we have immediately proved that the line integral of an analytic function around a closed curve always vanishes.

Cauchy's integral formula states that, for an analytic function,

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz, \tag{6.156}$$

where the contour C encloses the point a and is traversed in a positive sense. The precise form of the contour is irrelevant, because the difference between two contour integrals enclosing a is a contour integral around a region not enclosing a (see figure 6.4). In such a region $f(z)/(z-a)$ is analytic so the difference has zero contribution.

To understand Cauchy's theorem in terms of geometric calculus we need to focus on the properties of the Cauchy kernel $1/(z-a)$. We first write

$$\frac{1}{z-a} = \frac{(z-a)^\dagger}{|(z-a)|^2} = \frac{\boldsymbol{r} - \boldsymbol{a}}{(\boldsymbol{r} - \boldsymbol{a})^2} \mathsf{e}_1, \tag{6.157}$$

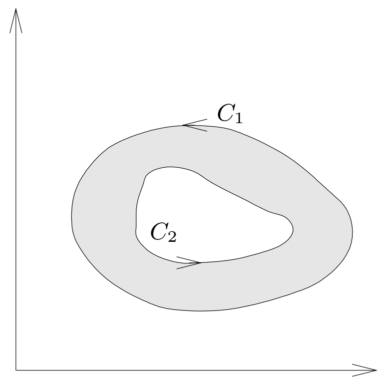

Figure 6.4 *Contour integrals in the complex plane.* The two contours C_1 and C_2 can be deformed into one another, provided the function to be integrated has no singularities in the intervening region. In this case the difference vanishes, by Cauchy's theorem.

where $\boldsymbol{a} = \mathsf{e}_1 a$ is the vector corresponding to the complex number a. The essential quantity here is the vector $(\boldsymbol{r}-\boldsymbol{a})/(\boldsymbol{r}-\boldsymbol{a})^2$, which we can write as

$$\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2} = \boldsymbol{\nabla} \ln|\boldsymbol{r}-\boldsymbol{a}|. \tag{6.158}$$

But $\ln|\boldsymbol{r}-\boldsymbol{a}|$ is the Green's function for the Laplacian operator in two dimensions,

$$\boldsymbol{\nabla}^2 \ln|\boldsymbol{r}-\boldsymbol{a}| = 2\pi\delta(\boldsymbol{r}-\boldsymbol{a}). \tag{6.159}$$

It follows that the vector part of the Cauchy kernel satisfies

$$\boldsymbol{\nabla}\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2} = 2\pi\delta(\boldsymbol{r}-\boldsymbol{a}). \tag{6.160}$$

The Cauchy kernel is the Green's function for the two-dimensional vector derivative! The existence of this Green's function proves that the vector derivative is invertible, which is not true of its separate divergence and curl components.

The Cauchy integral formula now follows from the fundamental theorem of geometric calculus in the form of equation (6.155),

$$\begin{aligned}\oint \frac{f(z)}{z-a}dz &= \mathsf{e}_1 \int \boldsymbol{\nabla}\left(\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\mathsf{e}_1 f(x)\right) dX \\ &= \mathsf{e}_1 \int \left(2\pi\delta(x-\boldsymbol{a})\mathsf{e}_1 f(z) + \boldsymbol{\nabla} f(z)\frac{\boldsymbol{r}-\boldsymbol{a}}{(\boldsymbol{r}-\boldsymbol{a})^2}\mathsf{e}_1\right) I|dX| \\ &= 2\pi I f(a), \end{aligned} \tag{6.161}$$

where we have assumed that f is analytic, $\boldsymbol{\nabla} f(z) = 0$. We can now understand precisely the roles of each term in the theorem:

(i) The dz encodes the tangent vector and forms a *geometric* product in the integrand.
(ii) The $(z-a)^{-1}$ is the Green's function for the vector derivative $\boldsymbol{\nabla}$ and ensures that the area integral only picks up the value at a.
(iii) The I (which replaces i) comes from the directed volume element $dX = I\,dx\,dy$.

Much of this is hidden in conventional accounts, but all of these insights are crucial to generalising the theorem. Indeed, we have already proved a more general theorem in two dimensions applying to non-analytic functions. For these we can now write, following section 6.3.1,

$$2\pi I f(a) = \oint \frac{f}{z-a} dz - 2\int \frac{\partial f}{\partial z^\dagger} \frac{1}{z-a} I|dX|. \tag{6.162}$$

A second key ingredient in complex analysis is the series expansion of a function. In particular, if $f(z)$ is analytic apart from a pole of order n at $z=a$, the function has a Laurent series of the form

$$f(z) = \frac{a_{-n}}{(z-a)^n} \cdots \frac{a_{-1}}{z-a} + \sum_{i=0}^{\infty} a_i (z-a)^i. \tag{6.163}$$

The powerful residue theorem states that for such a function

$$\oint_C f(z)\,dz = 2\pi i a_{-1}. \tag{6.164}$$

We now have a new interpretation for the residue term in a Laurent expansion — it is a weighted Green's function. The residue theorem just recovers the weight! Geometric calculus unifies the theory of poles and residues, supposedly unique to complex analysis, with that of Green's functions and δ-functions.

We now have an alternative picture of complex variable theory in terms of Green's functions and surface data. Suppose, for example, that we start with a function $f(x)$ on the real axis. We seek to propagate this function into the upper half-plane, subject to the boundary conditions that f falls to zero as $|z| \mapsto \infty$. The Cauchy formula tells us that we should propagate according to the formula

$$f(a) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x-a}\,dx. \tag{6.165}$$

But suppose now that we form the Fourier transform of the initial function $f(x)$,

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \bar{f}(k) \mathrm{e}^{ikx}. \tag{6.166}$$

We now have

$$f(a) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \bar{f}(k) \int_{-\infty}^{\infty} \frac{\mathrm{e}^{ikx}}{x-a}\,dx. \tag{6.167}$$

Now we only close the x integral in the upper half-plane for positive k. For negative k there is no residue term, since a lies in the the upper half-plane. The Cauchy integral formula now returns

$$f(a) = \int_0^\infty \frac{dk}{2\pi} \bar{f}(k) \mathrm{e}^{ika}. \tag{6.168}$$

This shows that only the part of the function consistent with the desired boundary conditions is propagated in the positive y direction. The remaining part of the function propagates in the $-y$ direction, if similar boundary conditions are imposed in the lower half plane. In this way the boundary conditions and the Green's function between them specify precisely which parts of a function are propagated in the desired direction. No restrictions are placed on the boundary values $f(x)$, which need not be part of an analytic function.

A second example, which generalises nicely, is the unit circle. Suppose we have initial data $f(\theta)$ defined over the unit circle. We write $f(\theta)$ as

$$f(\theta) = \sum_{-\infty}^{\infty} f_n \mathrm{e}^{in\theta}. \tag{6.169}$$

The terms in $\exp(in\theta)$ are replaced by z^n over the unit circle, and we then choose whether to evaluate in interior or exterior closure of the Cauchy integral. The result is that only the negative powers are propagated outwards from the circle, resulting in the function

$$f(z) = \sum_{n=1}^{\infty} f_{-n} z^{-n}, \qquad |z| > 1. \tag{6.170}$$

(The constant component f_0 is technically propagated as well, but this can be removed trivially.) These observations are simple from the point of view of complex variable theory, but are considerably less obvious in propagator theory.

6.4.7 Green's functions in Euclidean spaces

The extension of complex variable theory to arbitrary Euclidean spaces is now straightforward. The analogue of an analytic function is a multivector ψ satisfying $\nabla\psi = 0$. We choose to work with even-grade multivectors to simplify matters. The fundamental theorem states that

$$\oint_{\partial V} dS\, \psi = \int \nabla\psi\, dX = 0. \tag{6.171}$$

where we have used the fact that ψ commutes with the pseudoscalar measure dX. For any *monogenic* function ψ, the directed integral of ψ over a closed surface must vanish.

The Green's function for the vector derivative in n dimensions is simply

$$G(x;y) = \frac{1}{S_n} \frac{x-y}{|x-y|^n}, \tag{6.172}$$

where x and y are vectors and S_n is the surface area of the unit ball in n-dimensional space. The Green's function satisfies

$$\nabla G(x;y) = \nabla \cdot G(x;y) = \delta(x-y). \tag{6.173}$$

In order to allow for the lack of commutativity between G and ψ we use the fundamental theorem in the form

$$\begin{aligned}\oint_{\partial V} G\, dS\, \psi &= \int_V (\dot{G}\dot{\nabla}\, \psi + G\, \nabla \psi)\, dX \\ &= \int_V \dot{G}\dot{\nabla}\psi\, dX,\end{aligned} \tag{6.174}$$

where we have used the fact that ψ is a monogenic function. Setting G equal to the Green's function of equation (6.172) we find that Cauchy's theorem in n dimensions can be written in the form

$$\psi(y) = \frac{1}{IS_n} \oint_{\partial V} \frac{x-y}{|x-y|^n} dS\, \psi(x). \tag{6.175}$$

This relates the value of a monogenic function at a point to the value of a surface integral over a region surrounding the point.

One consequence of equation (6.175) is that a generalisation of Liouville's theorem applies to monogenic functions in Euclidean spaces. We define the modulus function

$$|M| = \langle M M^\dagger \rangle^{1/2}, \tag{6.176}$$

which is a well-defined positive-definite function for all multivectors M in a Euclidean algebra. The modulus function is easily shown to satisfy Schwarz inequality in the form

$$|A+B| \leq |A| + |B|. \tag{6.177}$$

If we let a denote a unit vector and let ∇_y denote the derivative with respect to the vector y we find that

$$a \cdot \nabla_y \psi(y) = -\frac{1}{IS_n} \oint_{\partial V} \frac{a(x-y)^2 + n a \cdot (x-y)\,(x-y)}{|x-y|^{n+2}} dS\, \psi(x). \tag{6.178}$$

It follows that

$$|a \cdot \nabla_y \psi(y)| \leq \frac{1}{S_n} \oint_{\partial V} \frac{n+1}{|x-y|^n} |dS|\, |\psi(x)|. \tag{6.179}$$

But if ψ is bounded, $|\psi(x)|$ never exceeds some given value. Taking the surface of integration out to large radius $r = |x|$, we find that the right-hand side falls off as $1/r$. This is sufficient to prove that the directional derivative of ψ must vanish in all directions, and the only monogenic function that is bounded over all space is constant ψ.

Equation (6.175) enables us to propagate a function off an initial surface in Euclidean space, subject to suitable boundary conditions. Suppose, for example, that we wish to propagate ψ off the surface of the unit ball, subject to the condition that the function falls to zero at large distance. Much like the two-dimensional case, we can write

$$\psi = \sum_{l=-\infty}^{\infty} \alpha_l \psi_l, \tag{6.180}$$

where the ψ_l are angular monogenics, satisfying

$$x \wedge \nabla \psi = -l\psi. \tag{6.181}$$

Each angular monogenic is multiplied by r^l to yield a full monogenic function, and only the negative powers have their integral closed over the exterior region. The result is the function

$$\psi = \sum_{l=1}^{\infty} \alpha_{-l} r^{-l} \psi_{-l}, \quad r > 1. \tag{6.182}$$

Similarly, the positive powers are picked up if we solve the interior problem.

6.4.8 Spacetime propagators

Propagation in mixed signature spaces is somewhat different to the Euclidean case. There is no analogue of Liouville's theorem to call on, so one can easily construct bounded solutions to the monogenic equation which are non-singular over all space. Plane wave solutions to the massless Dirac equation are an example of such functions. Furthermore, the existence of characteristic surfaces has implications for the how boundary values are specified. To see this, consider a two-dimensional Lorentzian space with basis vectors $\{\gamma_0, \gamma_1\}$, $\gamma_0^2 = -\gamma_1^2 = 1$, and pseudoscalar $I = \gamma_1\gamma_0$. The monogenic equation is $\nabla\psi = 0$, where ψ is an even-grade multivector built from a scalar and pseudoscalar terms. We define the null vectors

$$n_\pm = \gamma_0 \pm \gamma_1. \tag{6.183}$$

Pre-multiplying the monogenic equation by n_+ we find that

$$n_+ \cdot \nabla \psi = -n_+ \wedge \nabla \psi = I\,(n_+ I) \cdot \nabla \psi = -I n_+ \cdot \nabla \psi. \tag{6.184}$$

where we have used the result that $In_+ = n_+$. It follows that

$$(1 + I) n_+ \cdot \nabla \psi = 0, \tag{6.185}$$

and similarly,

$$(1 - I)n_- \cdot \nabla\psi = 0. \tag{6.186}$$

If we take ψ and decompose it into $\psi = \psi_+ + \psi_-$,

$$\psi_\pm = \tfrac{1}{2}(1 \pm I)\psi, \tag{6.187}$$

we see that the values of the separate $\psi_\pm$ components have vanishing derivatives along the respective null vectors $n_\pm$. Propagation of ψ from an initial surface is therefore quite straightforward. The function is split into $\psi_\pm$, and the values of these are transported along the respective null vectors. That is, ψ_+ has the same value along each vector in the n_+ direction, and the same for ψ_-. There is no need for a complicated contour integral.

The fact that the values of ψ are carried along the characteristics illustrates a key point. Any surface on which initial values are specified can cut a characteristic surface only once. Otherwise the initial values are unlikely to be consistent with the differential equation. For the monogenic equation, $\nabla\psi = 0$, suitable initial conditions consist of specifying ψ along the γ_1 axis, for example. But the fundamental theorem involves integrals around closed loops. The theorem is still valid in a Lorentzian space, so it is interesting to see what happens to the boundary data if we attempt to construct an interior solution with arbitrary surface data. The first step is to construct the Lorentzian Green's function. This can be found routinely via its Fourier transformation. With $x = x^0\gamma_0 + x^1\gamma_1$ we find

$$\begin{aligned} G(x) &= i\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\,\frac{\omega\gamma_0 + k\gamma_1}{\omega^2 - k^2}\,\mathrm{e}^{i(kx^1 - \omega x^0)} \\ &= \frac{i}{2}\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\left(\frac{\gamma_0 + \gamma_1}{\omega - k} + \frac{\gamma_0 - \gamma_1}{\omega + k}\right)\mathrm{e}^{i(kx^1 - \omega x^0)} \\ &= \frac{\epsilon(x^0)}{4}\big(\delta(x^1 - x^0)(\gamma_0 + \gamma_1) + \delta(x^1 + x^0)(\gamma_0 - \gamma_1)\big). \end{aligned} \tag{6.188}$$

The function $\epsilon(x^0)$ takes the value $+1$ or -1, depending on whether x^0 is positive or negative respectively.

To apply the fundamental theorem, suppose we take the contour of figure 6.5, which runs along the γ_1 axis for two different times $t_i < t_f$ and is closed at spatial infinity. We assume that the function we are propagating, ψ, falls off at large spatial distance, and write $\psi(x)$ as $\psi(x^0, x^1)$. The fundamental theorem

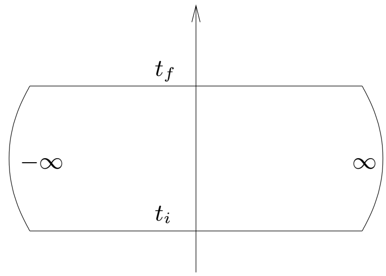

Figure 6.5 *A spacetime contour.* The contour is closed at spatial infinity.

then gives

$$\begin{aligned}\psi(y) =& I\int_{-\infty}^{\infty} d\lambda\, G(t_i\gamma_0 + \lambda\gamma_1 - y)\gamma_1\psi(t_i,\lambda) \\ & - I\int_{-\infty}^{\infty} d\lambda\, G(t_f\gamma_0 + \lambda\gamma_1 - y)\gamma_1\psi(t_f,\lambda) \\ =& \frac{1}{4}(1+I)\big(\psi(t_i, y^1 - y^0 + t_i) + \psi(t_f, y^1 - y^0 + t_f)\big) \\ & - \frac{1}{4}(1-I)\big(\psi(t_i, -y^1 + y^0 + t_i) + \psi(t_f, -y^1 + y^0 + t_f)\big).\end{aligned} \tag{6.189}$$

The construction of $\psi(y)$ in the interior region has a simple interpretation. For the function $\psi_+(y)$, for example, we form the null vector n_+ through y. The value at y is then the average value at the two intersections with the boundary. A similar construction holds for ψ_-. Much like the Euclidean case, only the part of the function on the boundary that is consistent with the monogenic equation is propagated to the interior.

These insights hold in other Lorentzian spaces, such as four-dimensional spacetime. The Green's functions become more complicated, and typically involve derivatives of δ-functions. These are more usefully handled via their Fourier transforms, and are discussed in more detail in section 8.5. In addition, the lack of a Liouville's theorem means that any monogenic function can be added to a Green's function to generate a new Green's function. This has no consequences if one rigorously applies surface integral formulae. In quantum theory, however, this is not usually the case. Rather than a rigorous application of the generalised Green's theorem, it is common instead to talk about propagators which transfer initial data from one timeslice to a later one. Used in this role, the Green's functions we have derived are referred to as *propagators*. As we are not specifying data over a closed surface, adding further terms to our Green's function can have an effect. These effects are related to the desired boundary conditions and are crucial to the formulation of a relativistic quantum field theory. There one is led to employ the complex-valued Feynman propagator, which ensures that positive frequency modes are propagated forwards in time, and negative frequency modes are propagated backwards in time. We will meet this object in greater detail in section 8.5.

6.5 Embedded surfaces and vector manifolds

We now seek a generalisation of the preceding results where the volume integral is taken over a curved surface. We will do this in the setting of the *vector manifold* theory developed by Hestenes and Sobczyk (1984). The essential concept is to treat a manifold as a surface embedded in a larger, flat space. Points in the manifold are then treated as vectors, which simplifies a number of derivations. Furthermore, we can exploit the coordinate freedom of geometric algebra to derive a set of general results without ever needing to specify the dimension of the background space. The price we pay for this approach is that we are working with a more restrictive concept of a manifold than is usually the case in mathematics. For a start, the surface naturally inherits a metric from the embedding space, so we are already restricting to Riemannian manifolds. We will also insist that a pseudoscalar can be uniquely defined throughout the surface, making it orientable.

While this may all appear quite restrictive, in fact these criteria rule out hardly any structures of interest in physics. This approach enables us to quickly prove a number of key results in Riemannian geometry, and to unite these with results for the exterior geometry of the manifold, achieving a richer general theory. We are not prevented from discussing topological features of surfaces either. Rather than build up a theory of topology which makes no reference to the metric, we instead build up results that are unaffected if the embedding is (smoothly) transformed.

We define a vector manifold as a set of points labelled by vectors lying in a geometric algebra of arbitrary dimension and signature. If we consider a path in the surface $x(\lambda)$, the tangent vector is defined in the obvious way by

$$x' = \left.\frac{\partial x(\lambda)}{\partial \lambda}\right|_{\lambda_0} = \lim_{\epsilon \mapsto 0} \frac{x(\lambda_0 + \epsilon) - x(\lambda_0)}{\epsilon}. \tag{6.190}$$

An advantage of the embedding picture is that the meaning of the limit is well defined, since the numerator exists for all ϵ. This is true even if, for finite epsilon, the difference vector does not lie entirely in the tangent space and only becomes a tangent vector in the limit. Standard formulations of differential geometry avoid any mention of an embedding, however, so have to resort to a more abstract definition of a tangent vector.

An immediate consequence of this approach is that we can define the path length as

$$s = \int_{\lambda_1}^{\lambda_2} |x' \cdot x'|^{1/2} \, d\lambda. \tag{6.191}$$

The embedded surface therefore inherits a metric from the 'ambient' background space. All finite-dimensional Riemannian manifolds can be studied in this way since, given a manifold, a natural embedding in a larger flat space can always be found. In applications such as general relativity one is usually not interested in the properties of the embedding, since they are physically unmeasurable. But in many other applications, particularly those involving constrained systems, the embedding arises naturally and useful information is contained in the extrinsic geometry of a manifold.

6.5.1 The pseudoscalar and projection

Suppose that we next introduce a set of paths in the surface all passing through the same point x. The paths define a set of tangent vectors $\{\mathsf{e}_1, \dots, \mathsf{e}_n\}$. We assume that these are independent, so that they form a basis for the n-dimensional tangent space at the point x. The exterior product of the tangent vectors defines the pseudoscalar for the tangent space $I(x)$:

$$I(x) \equiv \mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \cdots \wedge \mathsf{e}_n / |\mathsf{e}_1 \wedge \mathsf{e}_2 \wedge \cdots \wedge \mathsf{e}_n|. \tag{6.192}$$

The modulus in the denominator is taken as a positive number, so that I has the orientation specified by the tangent vectors. The pseudoscalar will satisfy

$$I^2 = \pm 1, \tag{6.193}$$

with the sign depending on dimension and signature. Clearly, to define I in this manner requires that the denominator in (6.192) is non-zero. This provides a restriction on the vector manifolds we consider here, and rules out certain structures in mixed signature spaces. The unit circle in the Lorentzian plane (figure 6.1), for example, falls outside the class of surfaces of studied here, as the tangent space has vanishing norm where the tangent vectors become null. Of course, there is no problem in referring to a closed spacetime curve as a vector manifold. The problem arises when attempting to generalise the integral theorems of the previous sections to such spaces.

The pseudoscalar $I(x)$ contains all of the geometric information about the surface and unites both its intrinsic and extrinsic properties. As well as assuming that $I(x)$ can be defined globally, we will also assume that $I(x)$ is continuous and differentiable over the entire surface, that it has the same grade everywhere, and that it is single-valued. The final assumption implies that the manifold is *orientable*, and rules out objects such as the Mobius strip, where the pseudoscalar is double-valued. Many of the restrictions on the pseudoscalar mentioned above can be relaxed to construct a more general theory, but this is only achieved at some cost to the ease of presentation. We will follow the simpler route, as the results developed here are sufficiently general for our purposes in later chapters.

The pseudoscalar $I(x)$ defines an operator which projects from an arbitrary multivector onto the component that is intrinsic to the manifold. This operator is

$$\mathsf{P}(A_r(x), x) = \begin{cases} A_r(x)\cdot I(x)\, I^{-1}(x) = A_r\cdot I\, I^{-1}, & r \leq n \\ 0 & r > n \end{cases}. \tag{6.194}$$

which defines an operator at every point x on the manifold. It is straightforward to prove that P satisfies the essential requirement of a projection operator, that is,

$$\mathsf{P}^2(A) = \mathsf{P}\big(\mathsf{P}(A)\big) = \mathsf{P}(A). \tag{6.195}$$

The effect of P on a vector a is to project onto the component of a that lies entirely in the tangent space at the point x. Such vectors are said to be *intrinsic* to the manifold. The complement,

$$\mathsf{P}_\perp(a) = a - \mathsf{P}(a), \tag{6.196}$$

lies entirely outside the tangent space, and is said to be extrinsic to the manifold.

Suppose now that $A(x)$ is a multivector field defined over some region of the manifold. We do not assume that A is intrinsic to the manifold. Given a vector a in the tangent space, the directional derivative along a is defined in the obvious manner:

$$a\cdot\nabla A(x) = \lim_{\epsilon\mapsto 0} \frac{A(x+\epsilon a) - A(x)}{\epsilon}. \tag{6.197}$$

Again, the presence of the embedding enables us to write this limit without ambiguity. The derivative operator $a\cdot\nabla$ is therefore simply the vector derivative in the ambient space contracted with a vector in the tangent space. Given a set of linearly independent tangent vectors $\{\mathsf{e}_i\}$, we can now define a vector derivative ∂ intrinsic to the manifold by

$$\partial = \mathsf{e}^i\, \mathsf{e}_i\cdot\nabla = \mathsf{P}(\nabla). \tag{6.198}$$

This is simply the ambient space vector derivative projected onto the tangent space. The use of the ∂ symbol should not cause confusion with the boundary operator introduced in section 6.4.4. The definition of ∂ requires the existence of the reciprocal frame $\{\mathsf{e}^i\}$, which is why we restricted to manifolds over which I is globally defined. The projection of the vector operator ∂ satisfies

$$\mathsf{P}(\partial) = \partial. \tag{6.199}$$

The contraction of ∂ with a tangent vector a satisfies $a\cdot\partial = a\cdot\nabla$, which is simply the directional derivative in the a direction.

6.5.2 Directed integration for embedded surfaces

Now that we have defined the ∂ operator it is a straightforward task to write down a generalized version of the fundamental theorem of calculus appropriate for embedded surfaces. We can essentially follow through the derivation of section 6.4.4 with little modification. The volume to be integrated over is again triangulated into a chain of simplices. The only difference now is that the pseudoscalar for each simplex varies from one simplex to another. This changes very little. For example we still have

$$\oint dS = 0, \tag{6.200}$$

which holds for the directed integral over the closed boundary of any simply-connected vector manifold.

The linear interpolation results used in deriving equation (6.138) are all valid, because we can again fall back on the embedding picture. In addition, the assumption that the pseudoscalar $I(x)$ is globally defined means that the reciprocal frame required in equation (6.138) is well defined. The only change that has to be made is that the ambient derivative ∇ is replaced by its projection into the manifold, because we naturally assemble the inner product of ∇ with the pseudoscalar. The most general statement of the fundamental theorem can now be written as

$$\oint_{\partial V} \mathsf{L}(dS) = \int_V \dot{\mathsf{L}}(\dot{\partial} dX) = \int_V \dot{\mathsf{L}}(\dot{\nabla}\cdot dX). \tag{6.201}$$

The form of the volume integral involving ∂ is generally more useful as it forms a geometric product with the volume element. The function L can be any multivector-valued function in this equation — it is not restricted to lie in the tangent space. An important feature of this more general theorem is that if we write $dX = I|dX|$ we see that the directed element dX is position-dependent. But this position dependence is *not* differentiated in equation (6.201). It is only the integrand that is differentiated.

There are two main applications of the general theorem derived here. The first is a generalisation of the divergence theorem to curved spaces. We again write

$$\mathsf{L}(A) = \langle JAI^{-1}\rangle, \tag{6.202}$$

where J is a vector field in the tangent space, and I is the unit pseudoscalar for the n-dimensional curved space. Equation (6.201) now gives

$$\oint_{\partial V} n\cdot J|dS| = \int_V (\partial\cdot J + \langle J\dot{\partial}\dot{I}^{-1}I\rangle)|dX|, \tag{6.203}$$

where $|dX| = I^{-1}dX$ and $n|dS| = dS\, I^{-1}$. The final term in the integral vanishes, as can be shown by first writing $I^{-1} = \pm I$ and using

$$\langle J\dot{\partial}\dot{I}I\rangle = \tfrac{1}{2}\langle J\dot{\partial}(\dot{I}I + I\dot{I})\rangle = \tfrac{1}{2}\langle J\partial(I^2)\rangle = 0. \tag{6.204}$$

It follows that the divergence theorem in curved space is essentially unchanged from the flat-space version, so

$$\int_V \partial\cdot J\,|dX| = \oint_{\partial V} n\cdot J\,|dS|. \tag{6.205}$$

As a second application we derive Stokes' theorem in three dimensions. Suppose that σ denotes an open, connected surface in three dimensions, with boundary $\partial\sigma$. The linear function L takes a vector as its linear argument and we define

$$\mathsf{L}(\boldsymbol{a}) = \boldsymbol{J}\cdot\boldsymbol{a}. \tag{6.206}$$

Equation (6.201) now gives

$$\oint_{\partial\sigma} \boldsymbol{J}\cdot d\boldsymbol{l} = \int_\sigma \langle \dot{\boldsymbol{J}}\,\dot{\boldsymbol{\nabla}}\cdot dX\rangle = -\int_\sigma (\boldsymbol{\nabla}\wedge\boldsymbol{J})\cdot dX, \tag{6.207}$$

where the line integral is taken around the boundary of the surface, and since the embedding is specified we have chosen a form of the integral theorem involving the three-dimensional derivative $\boldsymbol{\nabla}$. We now define the normal vector to the surface by

$$dX = I\boldsymbol{n}|dX|, \tag{6.208}$$

where I is the three-dimensional (right-handed) pseudoscalar. This equation defines the vector $\boldsymbol{n}$ normal to the surface. The direction in which this points depends on the orientation of dX. Around the boundary, for example, we can denote the tangent vector at the boundary by $\boldsymbol{l}$, and the vector pointing into the surface as $\boldsymbol{m}$. Then dX has the orientation specified by $\boldsymbol{l}\wedge\boldsymbol{m}$, and from equation (6.208) we see that $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}$ must form a right-handed set. This extends inwards to define the normal vector $\boldsymbol{n}$ over the surface (see figure 6.6). We now have

$$\oint_{\partial\sigma} \boldsymbol{J}\cdot d\boldsymbol{l} = \int_\sigma -(I\boldsymbol{\nabla}\wedge\boldsymbol{J})\cdot\boldsymbol{n}\,|dX| = \int_\sigma (\mathrm{curl}\,\boldsymbol{J})\cdot\boldsymbol{n}\,|dX|, \tag{6.209}$$

which is the familiar Stokes' theorem in three dimensions. This is only the scalar part of a more general (and less familiar) theorem which holds in three dimensions. To form this result we remove the projection onto the scalar part, to obtain

$$\oint_{\partial\sigma} d\boldsymbol{l}\,\boldsymbol{J} = -I\int_\sigma \boldsymbol{n}\wedge\boldsymbol{\nabla}\,\boldsymbol{J}\,|dX|. \tag{6.210}$$

A version of this result holds for any open n-dimensional surface embedded in a flat space of dimension $n+1$.

6.5.3 Intrinsic and extrinsic geometry

Suppose now that the directional derivative $a\cdot\partial$ acts on a tangent vector field $b(x) = \mathsf{P}(b(x))$. There is no guarantee that the resulting vector also lies entirely

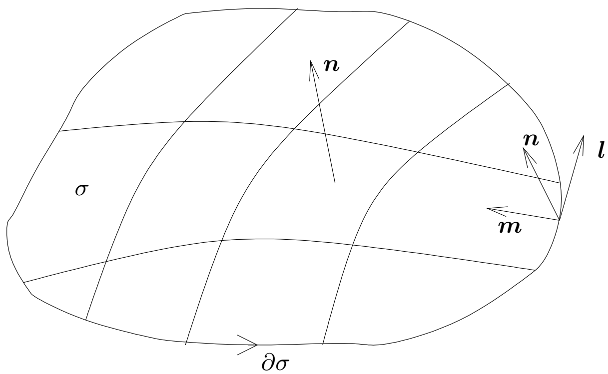

Figure 6.6 *Orientations for Stokes' theorem.* The bivector measure dX defines an orientation over the surface and at the boundary. With $\boldsymbol{l}$ and $\boldsymbol{m}$ the tangent and inward directions at the boundary, the normal $\boldsymbol{n}$ is defined so that $\boldsymbol{l}, \boldsymbol{m}, \boldsymbol{n}$ form a right-handed set.

in the tangent space, even if a does. For example, consider the simple case of a circle in the plane. The derivative of the tangent vector around the circle is a radial vector, which is entirely extrinsic to the manifold. In order to restrict to quantities intrinsic to the manifold we define a new derivative — the covariant derivative D — as follows:

$$a\cdot DA(x) = \mathsf{P}(a\cdot\partial A(x)). \tag{6.211}$$

The operator $a\cdot D$ acts on multivectors in the tangent space, returning a new multivector field in the tangent space. Since the $a\cdot\partial$ operator satisfies Leibniz's rule, the covariant derivative $a\cdot D$ must as well,

$$a\cdot D(AB) = \mathsf{P}\big(a\cdot\partial(AB)\big) = (a\cdot DA)B + A\,a\cdot DB. \tag{6.212}$$

The vector operator D is then defined in the obvious way from the covariant directional derivatives,

$$D = \mathsf{e}^i\,\mathsf{e}_i\cdot D. \tag{6.213}$$

So, for example, we can write

$$DA_r = \mathsf{e}^i(\mathsf{e}_i\cdot DA_r) = \mathsf{P}(\partial A_r). \tag{6.214}$$

The result decomposes into grade-raising and grade-lowering terms, so we write

$$\begin{aligned} D\cdot A_r &= \langle DA_r\rangle_{r-1}, \\ D\wedge A_r &= \langle DA_r\rangle_{r+1}. \end{aligned} \tag{6.215}$$

So, like ∂, D has the algebraic properties of a vector in the tangent space. Acting on a scalar function $\alpha(x)$ defined over the manifold the two derivatives coincide, so

$$\partial\alpha(x) = D\alpha(x). \tag{6.216}$$

Suppose now that a is a tangent vector to the manifold, and we look at how the pseudoscalar changes along the a direction. It should be obvious, from considering a 2-sphere for example, that the resulting quantity must lie at least partly outside the manifold. We let $\{\mathsf{e}_i\}$ denote an orthonormal frame, so

$$I = \mathsf{e}_1\mathsf{e}_2\cdots\mathsf{e}_n. \tag{6.217}$$

It follows that

$$\begin{aligned} a\cdot\partial\, I\, I^{-1} &= \sum_{i=1}^{n} \mathsf{e}_1\cdots\big(a\cdot D\mathsf{e}_i + \mathsf{P}_\perp(a\cdot\partial\,\mathsf{e}_i)\big)\cdots\mathsf{e}_n\, I^{-1} \\ &= a\cdot D\, I\, I^{-1} + \mathsf{P}_\perp(a\cdot\partial\,\mathsf{e}_i)\wedge\mathsf{e}^i. \end{aligned} \tag{6.218}$$

The final term is easily shown to be independent of the choice of frame. But $a\cdot DI$ must remain in the tangent space, so it can only be a multiple of the pseudoscalar I. It follows that

$$(a\cdot D\, I)I = \langle(a\cdot D\, I)I\rangle = \tfrac{1}{2}\langle a\cdot D(I^2)\rangle = 0, \tag{6.219}$$

so

$$a\cdot D\, I = 0. \tag{6.220}$$

That is, the (unit) pseudoscalar is a covariant constant over the manifold. Equation (6.218) now simplifies to give

$$a\cdot\partial\, I = \mathsf{P}_\perp(a\cdot\partial\,\mathsf{e}_i)\wedge\mathsf{e}^i\, I = -S(a)I, \tag{6.221}$$

which defines the *shape tensor* $S(a)$. This is a bivector-valued, linear function of its vector argument a, where a is a tangent vector. Since the result of $a\cdot\partial\, I$ has the same grade as I, we can write

$$a\cdot\partial I = I\times S(a) \tag{6.222}$$

with

$$S(a)\cdot I = S(a)\wedge I = 0. \tag{6.223}$$

The fact that $S(a)\cdot I = 0$ confirms that $S(a)$ lies partly outside the manifold, so that $\mathsf{P}(S(a)) = 0$.

The shape tensor $S(a)$ unites the intrinsic and extrinsic geometry of the manifold in a single quantity. It can be thought of as the 'angular momentum' of $I(x)$ as it slides over the manifold. The shape tensor provides a compact relation between directional and covariant derivatives. We first form

$$b\cdot S(a) = b^i\mathsf{P}_\perp(a\cdot\partial\,\mathsf{e}_i) = \mathsf{P}_\perp(a\cdot\partial\, b), \tag{6.224}$$

where a and b are tangent vectors. It follows that

$$a\cdot\partial\, b = \mathsf{P}(a\cdot\partial\, b) + \mathsf{P}_\perp(a\cdot\partial\, b) = a\cdot D\, b + b\cdot S(a), \tag{6.225}$$

which we can rearrange to give the neat result

$$a\cdot D\, b = a\cdot\partial\, b + S(a)\cdot b. \tag{6.226}$$

Applying this result to the geometric product bc we find that

$$\begin{aligned} a\cdot D(bc) &= (a\cdot\partial\, b)c + S(a)\cdot b\, c + b(a\cdot\partial\, c) + b\, S(a)\cdot c \\ &= a\cdot\partial(bc) + S(a)\times(bc), \end{aligned} \tag{6.227}$$

where $\times$ is the commutator product, $A\times B = (AB - BA)/2$. It follows that for any multivector field A taking its values in the tangent space we have

$$a\cdot DA = a\cdot\partial A + S(a)\times A. \tag{6.228}$$

The fact that $S(a)$ is bivector-valued ensures that $S(a)\times A$ does not alter the grade of A. As a check, setting $A = I$ recovers equation (6.222). If we now write

$$a\cdot\partial\, b = a\cdot\partial\,\mathsf{P}(b) = a\cdot\dot{\partial}\,\dot{\mathsf{P}}(b) + \mathsf{P}(a\cdot\partial b) = a\cdot\dot{\partial}\,\dot{\mathsf{P}}(b) + a\cdot Db \tag{6.229}$$

we establish the further relation

$$a\cdot\dot{\partial}\,\dot{\mathsf{P}}(b) = b\cdot S(a). \tag{6.230}$$

This holds for any pair of tangent vectors a and b.

6.5.4 Coordinates and derivatives

A number of important results can be derived most simply by introducing a coordinate frame. In a region of the manifold we introduce local coordinates x^i and define the frame vectors

$$\mathsf{e}_i = \frac{\partial x}{\partial x^i}. \tag{6.231}$$

From the definition of ∂ it follows that $\mathsf{e}^i = \partial x^i$. The $\{\mathsf{e}_i\}$ are usually referred to as *tangent* vectors and the reciprocal frame $\{\mathsf{e}^i\}$ as *cotangent* vectors (or 1-forms). The fact that the space is curved implies that it may not be possible to construct a global coordinate system. The 2-sphere is the simplest example of this. In this case we simply patch together a series of local coordinate systems. The covariant derivative along a coordinate vector, $\mathsf{e}_i\cdot D$, satisfies

$$\mathsf{e}_i\cdot DA = D_iA = \mathsf{e}_i\cdot\partial A + S(\mathsf{e}_i)\times A = \partial_i A + S_i\times A, \tag{6.232}$$

which defines the D_i and S_i symbols.

The tangent frame vectors satisfy

$$\partial_i\mathsf{e}_j - \partial_j\mathsf{e}_i = (\partial_i\partial_j - \partial_j\partial_i)x = 0. \tag{6.233}$$

Projecting this result into the manifold establishes that

$$D_i \mathsf{e}_j - D_j \mathsf{e}_i = 0. \tag{6.234}$$

Projecting out of the manifold we similarly establish the result

$$\mathsf{e}_i \cdot S_j = \mathsf{e}_j \cdot S_i. \tag{6.235}$$

In terms of arbitrary tangent vectors a and b this can be written as

$$a \cdot S(b) = b \cdot S(a). \tag{6.236}$$

The shape tensor can be written in terms of the coordinate vectors as

$$S(a) = \mathsf{e}^k \wedge \mathsf{P}_\perp(a \cdot \partial \mathsf{e}_k). \tag{6.237}$$

It follows that

$$S_i = \mathsf{e}^k \wedge \mathsf{P}_\perp(\partial_i \mathsf{e}_k) = \mathsf{e}^k \wedge \mathsf{P}_\perp(\partial_k \mathsf{e}_i). \tag{6.238}$$

The tangent vectors therefore satisfy

$$\partial \wedge \mathsf{e}_i = \mathsf{e}^k \wedge \big(\mathsf{P}(\partial_k \mathsf{e}_i) + \mathsf{P}_\perp(\partial_k \mathsf{e}_i)\big) = D \wedge \mathsf{e}_i + S_i. \tag{6.239}$$

If we decompose a vector in the tangent space as $a = a^i \mathsf{e}_i$ we establish the general result that

$$\partial \wedge a = D \wedge a + S(a). \tag{6.240}$$

This gives a further interpretation to the shape tensor. It is the object which picks up the component of the curl of a tangent vector which lies outside the tangent space. As we can write

$$\partial \wedge a = \partial \wedge \big(\mathsf{P}(a)\big) = \dot{\partial} \wedge \dot{\mathsf{P}}(a) + \mathsf{P}(\partial \wedge a) = D \wedge a + \dot{\partial} \wedge \dot{\mathsf{P}}(a), \tag{6.241}$$

we establish the further result

$$\dot{\partial} \wedge \dot{\mathsf{P}}(a) = S(a). \tag{6.242}$$

This is easily seen to be consistent with the definition of the shape tensor in terms of the derivative of pseudoscalar.

If we now apply the preceding to the case of the curl of a gradient of a scalar, we find that

$$\partial \wedge \partial \phi = \mathsf{P}(\nabla) \wedge \mathsf{P}(\nabla \phi) = \mathsf{P}(\nabla \wedge \nabla \phi) + \dot{\partial} \wedge \dot{\mathsf{P}}(\nabla \phi). \tag{6.243}$$

But the ambient derivative satisfies the integrability condition $\nabla \wedge \nabla = 0$. It follows that we have

$$\partial \wedge \partial \phi = S(\nabla \phi), \tag{6.244}$$

which lies outside the manifold. The covariant derivative therefore satisfies

$$D \wedge (D\phi) = 0. \tag{6.245}$$

An important application of this result is to the coordinate scalars themselves. We find that

$$D\wedge(Dx^i) = D\wedge \mathsf{e}^i = 0, \tag{6.246}$$

which can also be proved directly from equation (6.234). Applying this result to an arbitrary vector $a = a_i \mathsf{e}^i$ we find that

$$D\wedge a = D\wedge(a_j \mathsf{e}^j) = \mathsf{e}^i\wedge \mathsf{e}^j(\partial_i a_j) = \tfrac{1}{2}\mathsf{e}^i\wedge \mathsf{e}^j(\partial_i a_j - \partial_j a_i). \tag{6.247}$$

This demonstrates that the $D\wedge$ operator is precisely the *exterior derivative* of differential geometry.

6.5.5 Riemannian geometry

To understand further how the shape tensor can specify the intrinsic geometry of a surface, we now make contact with Riemannian geometry. In Riemannian geometry one focuses entirely on the intrinsic properties of a manifold. It is customary to formulate the subject using the metric tensor as the starting point. In terms of the $\{\mathsf{e}_i\}$ coordinate frame the metric tensor is defined in the expected manner:

$$g_{ij} = \mathsf{e}_i \cdot \mathsf{e}_j. \tag{6.248}$$

In what follows we will not place any restriction on the signature of the tangent space. Some texts prefer to use the adjective 'Riemannian' to refer to extensions of Euclidean geometry to curved spaces (as Riemann originally intended). But in the physics literature it is quite standard now to refer to general relativity as a theory of Riemannian geometry, despite the Lorentzian signature.

After the metric, the next main object in Riemannian geometry is the Christoffel connection. The directional covariant derivative, D_i, restricts the result of its action to the tangent space. The result of its action on one of the $\{\mathsf{e}_i\}$ vectors can therefore be decomposed uniquely in the $\{\mathsf{e}_i\}$ frame. The coefficients of this define the Christoffel connection by

$$\Gamma^i_{jk} = (D_j \mathsf{e}_k)\cdot \mathsf{e}^i. \tag{6.249}$$

The components of the connection are clearly dependent on the choice of coordinate system, as well as the underlying geometry. It follows that a connection is necessary even when working in a curvilinear coordinate system in a flat space. A connection on its own does not imply that a space is curved. A typical use of the Christoffel connection is in finding the components in the $\{\mathsf{e}^i\}$ frame of a covariant derivative $a\cdot D\, b$, for example. We form

$$(a\cdot D\, b)\cdot \mathsf{e}^i = a^j\big(D_j(b^k \mathsf{e}_k)\big)\cdot \mathsf{e}^i = a^j(\partial_j b^i + \Gamma^i_{jk} b^k), \tag{6.250}$$

which shows how the connection accounts for the position dependence in the coordinate frame.

The components of the Christoffel connection can be found directly from the metric without referring to the frame vectors themselves. To achieve this we first establish a pair of results. The first is that the connection Γ^i_{jk} is symmetric on the jk indices. This follows from

$$\Gamma^i_{jk} - \Gamma^i_{kj} = (D_j \mathsf{e}_k - D_k \mathsf{e}_j)\cdot \mathsf{e}^i = 0, \tag{6.251}$$

where we have used equation (6.234). The second result is for the curl of a frame vector,

$$D\wedge \mathsf{e}_i = D\wedge (g_{ij}\mathsf{e}^j) = (D g_{ij})\wedge \mathsf{e}^j. \tag{6.252}$$

We can now write

$$\begin{aligned}
\Gamma^i_{jk} &= \tfrac{1}{2}\mathsf{e}^i\cdot(D_j\mathsf{e}_k + D_k\mathsf{e}_j)\\
&= \tfrac{1}{2}\mathsf{e}^i\cdot\big(\mathsf{e}_j\cdot(Dg_{kl}\wedge\mathsf{e}^l) + \mathsf{e}_k\cdot(Dg_{jl}\wedge\mathsf{e}^l) + Dg_{jk}\big)\\
&= \tfrac{1}{2}\mathsf{e}^i\cdot(\partial_j g_{kl}\mathsf{e}^l + \partial_k g_{jl}\mathsf{e}^l - Dg_{jk})\\
&= \tfrac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}),
\end{aligned} \tag{6.253}$$

which recovers the familiar definition of the Christoffel connection.

We now seek a method of encoding the intrinsic curvature of a Riemannian manifold. Suppose we form the commutator of two covariant derivatives

$$\begin{aligned}
[D_i, D_j]A &= \partial_i(\partial_j A + S_j\times A) + S_i\times(\partial_j A + S_j\times A)\\
&\quad -\partial_j(\partial_i A + S_i\times A) - S_j\times(\partial_i A + S_i\times A)\\
&= (\partial_i S_j - \partial_j S_i)\times A + (S_i\times S_j)\times A,
\end{aligned} \tag{6.254}$$

where we have used the Jacobi identity of section 4.1.3. Remarkably, all derivatives of the multivector A have cancelled out and what remains is a commutator with a bivector. To simplify this we form

$$\begin{aligned}
\partial_i S_j - \partial_j S_i &= -\partial_i(\partial_j I\, I^{-1}) + \partial_j(\partial_i I\, I^{-1})\\
&= -S_j I S_i I^{-1} + S_i I S_j I^{-1}\\
&= -2S_i\times S_j,
\end{aligned} \tag{6.255}$$

where we have used the fact that $S(a)$ anticommutes with I. On substituting this result in equation (6.254) we obtain the simple result

$$[D_i, D_j]A = -(S_i\times S_j)\times A. \tag{6.256}$$

The commutator of covariant derivatives defines the *Riemann tensor*. We denote this by $\mathsf{R}(a\wedge b)$, where

$$\mathsf{R}(\mathsf{e}_i\wedge\mathsf{e}_j)\times A = [D_i, D_j]A. \tag{6.257}$$

$\mathsf{R}(a\wedge b)$ is a bivector-valued linear function of its bivector argument. In terms of the shape tensor we have

$$\mathsf{R}(a\wedge b) = \mathsf{P}\big(S(b)\wedge S(a)\big). \tag{6.258}$$

The projection is required here because the Riemann tensor is defined to be entirely intrinsic to the manifold. The Riemann tensor (and its derivatives) fully encodes all of the local intrinsic geometry of a manifold. Since it can be derived easily from the shape tensor, it follows that the shape tensor also captures all of the intrinsic geometry. In addition to this, the shape tensor tells us about the extrinsic geometry — how the manifold is embedded in the larger ambient space.

The Riemann tensor can also be expressed entirely in terms of intrinsic quantities. To achieve this we first write

$$\mathsf{R}(\mathsf{e}_i\wedge\mathsf{e}_j)\cdot\mathsf{e}_k = [D_i, D_j]\mathsf{e}_k = D_i(\Gamma^a_{jk}\mathsf{e}_a) - D_j(\Gamma^a_{ik}\mathsf{e}_a). \tag{6.259}$$

It follows that

$$\begin{aligned}\mathsf{R}_{ijk}{}^l &= \mathsf{R}(\mathsf{e}_i\wedge\mathsf{e}_j)\cdot(\mathsf{e}_k\wedge\mathsf{e}^l)\\ &= \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^a_{jk}\Gamma^l_{ia} - \Gamma^a_{ik}\Gamma^l_{ja},\end{aligned} \tag{6.260}$$

recovering the standard definition of Riemannian geometry. An immediate advantage of the geometric algebra route is that many of the symmetry properties of $\mathsf{R}_{ijk}{}^l$ follow immediately from the fact that $\mathsf{R}(a\wedge b)$ is a bivector-valued linear function of a bivector. This immediately reduces the number of degrees of freedom to $n^2(n-1)^2/4$.

A further symmetry of the Riemann tensor can be found as follows:

$$\begin{aligned}\mathsf{R}(\mathsf{e}_i\wedge\mathsf{e}_j)\cdot\mathsf{e}_k &= D_iD_j\mathsf{e}_k - D_jD_i\mathsf{e}_k\\ &= D_iD_k\mathsf{e}_j - D_jD_k\mathsf{e}_i\\ &= [D_i, D_k]\mathsf{e}_j - [D_j, D_k]\mathsf{e}_i + D_k(D_i\mathsf{e}_j - D_j\mathsf{e}_i)\\ &= \mathsf{R}(\mathsf{e}_i\wedge\mathsf{e}_k)\cdot\mathsf{e}_j - \mathsf{R}(\mathsf{e}_j\wedge\mathsf{e}_k)\cdot\mathsf{e}_i.\end{aligned} \tag{6.261}$$

It follows that

$$a\cdot\mathsf{R}(b\wedge c) + c\cdot\mathsf{R}(a\wedge b) + b\cdot\mathsf{R}(c\wedge a) = 0, \tag{6.262}$$

for any three vectors a, b, c in the tangent space. This equation tells us that a vector quantity vanishes for all trivectors $a\wedge b\wedge c$, which provides a set of $n^2(n-1)(n-2)/6$ scalar equations. The number of independent degrees of freedom in the Riemann tensor is therefore reduced to

$$\frac{1}{4}n^2(n-1)^2 - \frac{1}{6}n^2(n-1)(n-2) = \frac{1}{12}n^2(n^2-1). \tag{6.263}$$

This gives the values 1, 6 and 20 for two, three and four dimensions respectively. Further properties of the Riemann tensor are covered in more detail in later chapters, where in particular we are interested in its relevance to gravitation.

The fact that Riemannian geometry is founded on the covariant derivative D, as opposed to the projected vector derivative ∂ limits the application of the integral theorem of equation (6.201). If one attempts to add multivectors from different points in the surface, there is no guarantee that the result remains intrinsic. The only quantities that can be combined from different points on the surface are scalars, or functions taking their values in a different space (such as a Lie group). The most significant integral theorem that remains is a generalization of Stokes' theorem, applicable to a grade-r multivector A_r and an open surface σ of dimension $r+1$. For this case we have

$$\oint_{\partial\sigma} A_r \cdot dS = \int_\sigma (\dot{A}_r \wedge \dot{\partial}) \cdot dX = (-1)^r \int_\sigma (D \wedge A_r) \cdot dX, \tag{6.264}$$

which only features intrinsic quantities. A particular case of this is when $r = n-1$, which recovers the divergence theorem. This is important for constructing conservation theorems in curved spaces.

6.5.6 Transformations and maps

The study of maps between vector manifolds helps to clarify some of the relationships between the structures defined in this chapter and more standard formulations of differential geometry. Suppose that $f(x)$ defines a map from one vector manifold to another. We denote these $\mathcal{M}$ and $\mathcal{M}'$, so that

$$x' = f(x) \tag{6.265}$$

associates a point in the manifold $\mathcal{M}'$ with one in $\mathcal{M}$. We will only consider smooth, differentiable, invertible maps between manifolds. In the mathematics literature these are known as *diffeomorphisms*. These are a subset of the more general concept of a *homeomorphism*, which maps continuously between spaces without the restriction of smoothness. Somewhat surprisingly, these two concepts are not equivalent. It is possible for two manifolds to be homeomorphic, but not admit a diffeomorphism between them. This implies that it is possible for a single topological space to admit more than one differentiable structure. The first example of this to be discovered was the sphere S^7, which admits 28 distinct differentiable structures! In 1983 Donaldson proved the even more striking result that four-dimensional space R^4 admits an infinite number of differentiable structures.

A path in $\mathcal{M}$, $x(\lambda)$, maps directly to a path in $\mathcal{M}'$. The map accordingly induces a map between tangent vectors, as seen by forming

$$\frac{\partial x'(\lambda)}{\partial \lambda} = \frac{\partial f\big(x(\lambda)\big)}{\partial \lambda} = \mathsf{f}(v), \tag{6.266}$$

where v is the tangent vector in $\mathcal{M}$, $v = \partial_\lambda x(\lambda)$ and the linear function f is defined by

$$\mathsf{f}(a) = a \cdot \partial f(x) = \mathsf{f}(a; x). \tag{6.267}$$

The function $\mathsf{f}(a)$ takes a tangent vector in $\mathcal{M}$ as its linear argument, and returns the image tangent vector in $\mathcal{M}'$. If we denote the latter by a', and write out the position dependence explicitly, we have

$$a'(x') = \mathsf{f}(a(x); x). \tag{6.268}$$

This map is appropriate for tangent vectors, so applies to the coordinate frame vectors $\{\mathsf{e}_i\}$. These map to an equivalent frame for the tangent space to $\mathcal{M}'$,

$$\mathsf{e}'_i = \mathsf{f}(\mathsf{e}_i). \tag{6.269}$$

The reciprocal frame in the transformed space is therefore given by

$$\mathsf{e}^{i\prime} = \bar{\mathsf{f}}^{-1}(\mathsf{e}^i). \tag{6.270}$$

The fact that the map $x \mapsto f(x)$ is assumed to be invertible ensures that the adjoint function $\bar{\mathsf{f}}(a)$ is also invertible.

Under transformations, therefore, vectors in one space can transform in two different ways. If they are tangent vectors they transform under the action of $\mathsf{f}(a)$. If they are cotangent vectors they transform under action of $\bar{\mathsf{f}}^{-1}(a)$. In differential geometry it is standard practice to maintain a clear distinction between these types of vectors, so one usually thinks of tangent and cotangent vectors as lying in separate linear spaces. The contraction relation $\mathsf{e}^i \cdot \mathsf{e}_j = \delta^i_j$ identifies the spaces as dual to each other. This relation is metric-independent and is preserved by arbitrary diffeomorphisms. These maps relate differentiable manifolds, and two diffeomorphic spaces are usually viewed as the same manifold.

A metric is regarded as an additional construct on a differentiable manifold, which maps between the tangent and cotangent spaces. In the vector manifold picture this map is achieved by constructing the reciprocal frame using equation (4.94). In using this relation we are implicitly employing a metric in the contraction with the pseudoscalar. For the theory of vector manifolds it is therefore useful to distinguish objects and operations that transform simply under diffeomorphisms. These will define the metric-independent features of a vector manifold. Metric-dependent quantities, like the Riemann tensor, invariably have more complicated transformation laws.

The exterior product of a pair of tangent vectors transforms as

$$\mathsf{e}_i \wedge \mathsf{e}_j \mapsto \mathsf{f}(\mathsf{e}_i) \wedge \mathsf{f}(\mathsf{e}_j) = \mathsf{f}(\mathsf{e}_i \wedge \mathsf{e}_j). \tag{6.271}$$

For example, if I' is the unit pseudoscalar for $\mathcal{M}'$ we have

$$\mathsf{f}(I) = \det(\mathsf{f}) I' \tag{6.272}$$

and for invertible maps we must have $\det(\mathsf{f}) \neq 0$. Similarly, for cotangent vectors we see that

$$\mathsf{e}^i \wedge \mathsf{e}^j \mapsto \bar{\mathsf{f}}^{-1}(\mathsf{e}^i) \wedge \bar{\mathsf{f}}^{-1}(\mathsf{e}^j) = \bar{\mathsf{f}}^{-1}(\mathsf{e}^i \wedge \mathsf{e}^j). \tag{6.273}$$

So exterior products of like vectors give rise to higher grade objects in a manner that is unchanged by diffeomorphisms. Metric invariants are constructed from inner products between tangent and cotangent vectors. Since the derivative of a scalar field is

$$\partial\phi = \mathsf{e}^i\partial_i\phi, \tag{6.274}$$

we see that $\partial\phi$ is a cotangent vector, and we can write

$$\partial' = \bar{\mathsf{f}}^{-1}(\partial). \tag{6.275}$$

A similar result holds for the covariant derivative D. If a is a tangent vector the directional derivative of a scalar field $a\cdot\partial\phi$ is therefore an invariant,

$$a'\cdot\partial'\phi' = \mathsf{f}(a)\cdot\bar{\mathsf{f}}^{-1}(\partial)\phi = a\cdot\partial\phi, \tag{6.276}$$

where $\phi'(x') = \phi(x)$.

In constructing the covariant derivative in section 6.5.3, we made use of the projection operation $\mathsf{P}(a)$. This *is* a metric operation, as it relies on a contraction with I. Hence the covariant derivatives $D_i\mathsf{e}_j$ do depend on the metric (via the connection). To establish a metric-independent operation we let a and b represent tangent vectors and form

$$\begin{aligned} a\cdot\partial b - b\cdot\partial a &= a\cdot Db - b\cdot Da + a\cdot S(b) - b\cdot S(a) \\ &= a\cdot Db - b\cdot Da. \end{aligned} \tag{6.277}$$

The shape terms cancel, so the result is intrinsic to the manifold. Under a diffeomorphism the result transforms to

$$a\cdot\partial\mathsf{f}(b) - b\cdot\partial\mathsf{f}(a) = \mathsf{f}(a\cdot\partial b - b\cdot\partial a) + a\cdot\dot{\partial}\dot{\mathsf{f}}(b) - b\cdot\dot{\partial}\dot{\mathsf{f}}(a). \tag{6.278}$$

But $\mathsf{f}(a)$ is the differential of the map $f(x)$, so we have

$$(\partial_i\partial_j - \partial_j\partial_i)f(x) = \partial_i\mathsf{f}(\mathsf{e}_j) - \partial_j\mathsf{f}(\mathsf{e}_i) = \dot{\partial}_i\dot{\mathsf{f}}(\mathsf{e}_j) - \dot{\partial}_j\dot{\mathsf{f}}(\mathsf{e}_i) = 0. \tag{6.279}$$

It follows that, for tangent vectors a and b,

$$a\cdot\dot{\partial}\dot{\mathsf{f}}(b) - b\cdot\dot{\partial}\dot{\mathsf{f}}(a) = 0. \tag{6.280}$$

We therefore define the *Lie derivative* $\mathcal{L}_a b$ by

$$\mathcal{L}_a b = a\cdot\partial b - b\cdot\partial a. \tag{6.281}$$

This results in a new tangent vector, and transforms under diffeomorphisms as

$$\mathcal{L}_a b \mapsto \mathcal{L}'_{a'}b' = \mathsf{f}(\mathcal{L}_a b). \tag{6.282}$$

Relations between tangent vectors constructed from the Lie derivative will therefore be unchanged by diffeomorphisms.

A similar construction is possible for cotangent vectors. If we contract equation (6.279) with $\bar{\mathsf{f}}^{-1}(\mathsf{e}^k)$ we obtain

$$\mathsf{f}(\mathsf{e}_j)\cdot\big(\partial_j\bar{\mathsf{f}}^{-1}(\mathsf{e}^k)\big) - \mathsf{f}(\mathsf{e}_i)\cdot\big(\partial_i\bar{\mathsf{f}}^{-1}(\mathsf{e}^k)\big) = 0. \tag{6.283}$$

Now multiplying by $\bar{\mathsf{f}}^{-1}(\mathsf{e}^i\wedge\mathsf{e}^j)$ and summing we find that

$$\mathsf{P}'\big(\bar{\mathsf{f}}^{-1}(\partial)\wedge\bar{\mathsf{f}}^{-1}(\mathsf{e}^k)\big) = 0. \tag{6.284}$$

This result can be summarised simply as

$$D'\wedge\mathsf{e}^{k\prime} = D'\wedge\bar{\mathsf{f}}^{-1}(\mathsf{e}^k) = 0. \tag{6.285}$$

This is sufficient to establish that the exterior derivative of a cotangent vector results in a cotangent bivector (equivalent to a 2-form). The result transforms in the required manner:

$$D\wedge A \mapsto D'\wedge A' = \bar{\mathsf{f}}^{-1}(D\wedge A). \tag{6.286}$$

This is the result that makes the exterior algebra of cotangent vectors so powerful for studying the topological features of manifolds. This algebra is essentially that of differential forms, as is explained in section 6.5.7. For example, a form is said to be closed if its exterior derivative is zero, and to be exact if it can be written as the exterior derivative of a form of one degree lower. Both of these properties are unchanged by diffeomorphisms, so the size of the space of functions that are closed but not exact is a topological feature of a space. This is the basis of de Rham cohomology.

It is somewhat less common to see diffeomorphisms discussed when studying Riemannian geometry. More usually one focuses attention on the restricted class of *isometries*, which are diffeomorphisms that preserve the metric. These define symmetries of a Riemannian space. In the vector manifold setting, however, it is natural to study the effect of maps on metric-dependent quantities. The reason being that vector manifolds inherit their metric structure from the embedding, and if the embedding is changed by a diffeomorphism, the natural metric is changed as well. One does not have to inherit the metric from an embedding. One can easily impose a metric on a vector manifold by defining a linear transformation over the manifold. This takes us into the subject of induced geometries, which is closer to the spirit of the approach to gravity adopted in chapter 14. Similarly, when transforming a vector manifold, one need not insist that the transformed metric is that inherited by the new embedding. One can instead simply define a new metric on the transformed space directly from the original one.

The simplest example of a diffeomorphism inducing a new geometry is to consider a flat plane in three dimensions. If the plane is distorted in the third direction, and the new metric taken as that implied by the embedding, the surface clearly becomes curved. Formulae for the effects of such transformations are generally quite complex. Most can be derived from the transformation properties of the projection operation,

$$\mathsf{P}' = \mathsf{f}\mathsf{P}\mathsf{f}^{-1}. \tag{6.287}$$

This identity ensures that the projection and transformation formulae can be applied in either order. If we now form

$$\begin{aligned} \mathsf{e}'_i \cdot S'_j &= \mathsf{P}'_\perp\big(\partial_j \mathsf{f}(\mathsf{e}_i)\big) \\ &= \mathsf{f}(\mathsf{e}_i \cdot S_j) + \mathsf{P}'_\perp\big(\dot{\partial}_j \dot{\mathsf{f}}(\mathsf{e}_i)\big), \end{aligned} \tag{6.288}$$

we see that the shape tensor transforms according to

$$a' \cdot S'(b') = \mathsf{f}(a \cdot S(b)) + \mathsf{P}'_\perp\big(b \cdot \dot{\partial} \dot{\mathsf{f}}(a)\big). \tag{6.289}$$

Further results can be built up from this. For example, the new Riemann tensor is constructed from the commutator of the transformed shape tensor.

6.5.7 Differential geometry and forms

So far we have been deliberately loose in relating objects in vector manifold theory to those of modern differential geometry texts. In this section we clarify the relations and distinctions between the viewpoints. In the subject of differential geometry it is now common practice to identify directional derivatives as tangent vectors, so that the tangent vector a is the scalar operator

$$a = a^i \frac{\partial}{\partial x^i}. \tag{6.290}$$

Tangent vectors form a linear space, denoted $T_x\mathcal{M}$, where x labels a point in the manifold $\mathcal{M}$. This notion of a tangent vector is slightly different from that adopted in the vector manifold theory, where we explicitly let the directional derivative act on the vector x. As explained earlier, the limit implied in writing $\partial x/\partial x^i$ is only well defined if an embedding picture is assumed. The reason for the more abstract definition of a tangent vector in the differential geometry literature is to remove the need for an embedding, so that a topological space can be viewed as a single distinct entity. There are arguments in favour, and against, both viewpoints. For all practical purposes, however, the philosophies behind the two viewpoints are largely irrelevant, and calculations performed in either scheme will return the same results.

The dual space to $T_x\mathcal{M}$ is called the cotangent space and is denoted $T_x^*\mathcal{M}$. Elements of $T_x^*\mathcal{M}$ are called cotangent vectors, or 1-forms. The inner product between a tangent and cotangent vector can be written as $\langle \omega, a \rangle$. A basis for the dual space is defined by the coordinate differentials dx^i, so that

$$\langle dx^i, \partial/\partial x^j \rangle = \delta^i_j. \tag{6.291}$$

A 1-form therefore implicitly contains a directed measure on a manifold. So, if α is a 1-form we have

$$\alpha = \alpha_i dx^i = A \cdot (dx), \tag{6.292}$$

where A is a grade-1 multivector in the vector manifold sense. Similarly, if dX is a directed measure over a two-dimensional surface, we have

$$dX = e_i \wedge e_j \, dx^i \, dx^j, \tag{6.293}$$

so that

$$(e^j \wedge e^i) \cdot dX = dx^i \, dx^j - dx^j \, dx^i. \tag{6.294}$$

An arbitrary 2-form can be written as

$$\alpha_2 = \frac{1}{2!} \alpha_{ij} (dx^i \, dx^j - dx^j \, dx^i) = A_2^\dagger \cdot dX. \tag{6.295}$$

Here A_2 is the multivector

$$A_2 = \frac{1}{2!} \alpha_{ij} \, e^i \wedge e^j, \tag{6.296}$$

which has the same components as the differential form. More generally, an r-form α_r can be written as

$$\alpha_r = A_r^\dagger \cdot dX_r = A_r \cdot dX_r^\dagger. \tag{6.297}$$

Clearly there is little difference in working with the r-form α_r or the equivalent multivector A_r. So, for example, the outer product of two 1-forms results in the 2-form

$$\alpha_1 \wedge \beta_1 = \alpha_i \beta_i (e^i \wedge e^j) \cdot dX_2^\dagger = (A_1 \wedge B_1) \cdot dX_2^\dagger, \tag{6.298}$$

where dX_2 is a two-dimensional surface measure and A_1, B_1 are the grade-1 multivectors with components α_i and β_i respectively. Similarly, the exterior derivative of an r-form is given by

$$d\alpha_r = (D \wedge A_r) \cdot dX_{r+1}^\dagger. \tag{6.299}$$

The fact that forms come packaged with an implicit measure allows for a highly compact statement of Stokes' theorem, as given in equation (6.264). In ultra-compact notation this says that

$$\int_{\sigma_r} d\alpha = \oint_{\partial \sigma_r} \alpha, \tag{6.300}$$

where α is an $(r-1)$-form integrated over an open r-surface σ_r. This is entirely equivalent to equation (6.264), as can be seen by writing

$$\int_{\sigma_r} d\alpha = \int_{\sigma_r} (\dot{A}_{r-1}^\dagger \wedge \dot{D}) \cdot dX_r = \oint_{\partial \sigma_r} (A_{r-1}^\dagger) \cdot dS_{r-1} = \oint_{\partial \sigma_r} \alpha. \tag{6.301}$$

One can proceed in this manner to establish a direct translation scheme between the languages of differential forms and vector manifolds. Many of the expressions are so similar that there is frequently little point in maintaining a distinction.

If the language of differential forms is applied in a metric setting, an important additional concept is that of a duality transformation, also known as the *Hodge* $*$ (star) operation. To define this we first introduce the volume form

$$\Omega = \sqrt{|g|} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = \sqrt{|g|} (e^n \wedge e^{n-1} \wedge \cdots \wedge e^1) \cdot dX. \tag{6.302}$$

The pseudoscalar for a vector manifold, given a coordinate frame with the specified orientation, is given by

$$I = \frac{1}{\sqrt{|g|}} (e_1 \wedge e_2 \wedge \cdots \wedge e_n). \tag{6.303}$$

This definition was chosen earlier to ensure that $I^2 = \pm 1$ and that I keeps the orientation specified by the frame. It follows that

$$\Omega = I^{-1} \cdot dX, \tag{6.304}$$

so that the equivalent multivector is $I^{-1\dagger}$. This will equal $\pm I$, depending on signature. The Hodge $*$ of an r-form α_r is the $(n-r)$-form

$$*\alpha_r = \frac{\sqrt{|g|}}{r!(m-r)!} \omega_{i_1,\ldots,i_r} \epsilon^{i_1,\ldots,i_r}{}_{j_{r+1},\ldots,j_n} \, dx^{j_{r+1}} \wedge \cdots \wedge dx^{j_n}, \tag{6.305}$$

where $\epsilon_{i_1,\ldots,i_n}$ denotes the alternating tensor. If A_r is the multivector equivalent of α_r, the Hodge $*$ takes on the rather simpler expression

$$*A_r = (I^{-1} A_r)^\dagger = (I^{-1} \cdot A_r)^\dagger. \tag{6.306}$$

In effect, we are multiplying by the pseudoscalar, as one would expect for a duality relation. Applied twice we find that

$$**A_r = \big(I^{-1} (I^{-1} \cdot A_r)^\dagger\big)^\dagger = (-1)^{r(m-r)} A_r (I^\dagger I). \tag{6.307}$$

In spaces with Euclidean signature, $I^\dagger I = +1$. In spaces of mixed signature the sign depends on whether there are an even or odd number of basis vectors with negative norm. It is a straightforward exercise to prove the main results for the Hodge $*$ operation, given equation (6.307) and the fact that I is covariantly conserved.

6.6 Elasticity

As a more extended application of some of the ideas developed in this chapter, we discuss the foundations of the subject of elasticity. The behaviour of a solid object is modelled by treating the object as a continuum. Locally, the strains in the object will tend to be small, but these can build up to give large global displacements. As such, it is important to treat the full, non-linear theory of elasticity. Only then can one be sure about the validity of various approximation schemes, such as assuming small deflections.

Our discussion is based on a generalisation of the ideas employed in the treatment of a rigid body. We first introduce an undeformed, *reference* configuration, with points in this labelled with the vector x. This is sometimes referred to as the material configuration. Points in the *spatial* configuration, y, are obtained by a non-linear displacement f of the reference configuration, so that

$$y = y(x,t) = f(x,t). \tag{6.308}$$

We use non-bold vectors to label points in the body, and bold to label tangent vectors in either the reference or spatial body. We assume that the background space is flat, three-dimensional Euclidean space.

6.6.1 Body strains

To calculate the strains in the body, consider the image of the vector between two nearby points in the reference configuration,

$$(x+\epsilon \boldsymbol{a}) - x \mapsto y(x+\epsilon \boldsymbol{a}) - y(x) = \epsilon \mathsf{f}(\boldsymbol{a}) + O(\epsilon^2), \tag{6.309}$$

where f is the deformation gradient,

$$\mathsf{f}(\boldsymbol{a}) = \boldsymbol{a}\cdot\boldsymbol{\nabla} y = \boldsymbol{a}\cdot\boldsymbol{\nabla} f(x,t). \tag{6.310}$$

The function f maps a tangent vector in the reference configuration to the equivalent vector in the spatial configuration. That is, if $x(\lambda)$ is a curve in the reference configuration with tangent vector

$$\boldsymbol{x}' = \frac{\partial x(\lambda)}{\partial \lambda}, \tag{6.311}$$

then the spatial curve has tangent vector $\mathsf{f}(\boldsymbol{v})$. The length of the curve $\boldsymbol{x}(\lambda)$ in the reference configuration is

$$\int \left| \frac{\partial x}{\partial \lambda} \right| d\lambda = \int |\boldsymbol{x}'|\, d\lambda. \tag{6.312}$$

The length of the induced curve in the spatial configuration is therefore

$$\int d\lambda \big(\mathsf{f}(\boldsymbol{x}')^2\big)^{1/2} = \int d\lambda \big(\boldsymbol{x}'\cdot\bar{\mathsf{f}}\mathsf{f}(\boldsymbol{x}')\big)^{1/2}. \tag{6.313}$$

We define the (right) *Cauchy–Green tensor* C, by

$$\mathsf{C}(\boldsymbol{a}) = \bar{\mathsf{f}}\mathsf{f}(\boldsymbol{a}). \tag{6.314}$$

This tensor is a symmetric, positive-definite map between vectors in the reference configuration. It describes a set of positive dilations along the principal directions in the reference configuration. The eigenvalues of C can be written as $(\lambda_1^2, \lambda_2^2, \lambda_3^2)$, where the λ_i define the *principal stretches*. The deviations of these from unity measure the strains in the material.

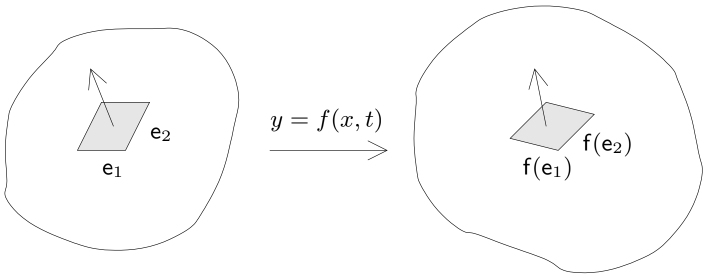

Figure 6.7 *An elastic body.* The function $f(x,t)$ maps points in the reference configuration to points in the spatial configuration. Coordinate curves e_1 and e_2 map to $\mathsf{f}(\mathsf{e}_1)$ and $\mathsf{f}(\mathsf{e}_2)$. The normal vector in the spatial configuration therefore lies in the $\bar{\mathsf{f}}^{-1}(\mathsf{e}^3)$ direction.

6.6.2 Body stresses

If we take a cut through the body then the contact force between the surfaces will be a function of the normal to the surface (and position in the body). Cauchy showed that, under reasonable continuity conditions, this force must be a linear function of the normal, which we write $\boldsymbol{\sigma}(\boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{n}; x)$. The tensor $\boldsymbol{\sigma}(\boldsymbol{n})$ maps a vector normal to a surface in the spatial configuration onto the force vector, also in the spatial configuration. We will verify shortly that $\boldsymbol{\sigma}$ is symmetric.

The total force on a volume segment in the body involves integrating $\boldsymbol{\sigma}(\boldsymbol{n})$ over the surface of the volume. But, as with the rigid body, it is simpler to perform all calculations back in the reference copy. To this end we let x^i denote a set of coordinates for position in the reference body. The associated coordinate frame is $\{\mathsf{e}_i\}$, with reciprocal frame $\{\mathsf{e}^i\}$. Suppose now that x^1 and x^2 are coordinates for a surface in the reference configuration. The equivalent normal in the spatial configuration is (see figure 6.7)

$$\boldsymbol{n} = \mathsf{f}(\mathsf{e}_1)\wedge\mathsf{f}(\mathsf{e}_2)\, I^{-1} = \det\,(\mathsf{f})\,\bar{\mathsf{f}}^{-1}(\mathsf{e}^3). \tag{6.315}$$

The force over this surface is found by integrating the quantity

$$\boldsymbol{\sigma}\big(\mathsf{f}(\boldsymbol{e}_1\wedge\boldsymbol{e}_2)I^{-1}\big)dx^1\,dx^2 = \det\,(\mathsf{f})\boldsymbol{\sigma}(\bar{\mathsf{f}}^{-1}(\boldsymbol{e}^3))dx^1\,dx^2. \tag{6.316}$$

We therefore define the *first Piola–Kirchoff stress tensor* T by

$$\mathsf{T}(\boldsymbol{a}) = \det\,(\mathsf{f})\boldsymbol{\sigma}\bar{\mathsf{f}}^{-1}(\boldsymbol{a}). \tag{6.317}$$

The stress tensor T takes as its argument a vector normal to a surface in the reference configuration, and returns the contact force in the spatial body. The force balance equation tells us that, for any sub-body, we have

$$\frac{d}{dt}\int d^3x\, \rho \boldsymbol{v} = \oint \mathsf{T}(d\boldsymbol{s}) + \int d^3x\, \rho \boldsymbol{b}, \tag{6.318}$$

where ρ is the density in the reference configuration, $\boldsymbol{v} = \dot{y}$ is the spatial velocity, and $\boldsymbol{b}$ is the applied *body force*. The fundamental theorem immediately converts this to the local equation

$$\rho \dot{\boldsymbol{v}} = \check{\mathsf{T}}(\check{\boldsymbol{\nabla}}) + \rho \boldsymbol{b}. \tag{6.319}$$

The check symbol is used for the scope of the derivative, to avoid confusion with time derivatives (denoted with an overdot). This equation is sensible as $\boldsymbol{\nabla}$ is the vector derivative in the reference configuration, and $\check{\mathsf{T}}(\check{\boldsymbol{\nabla}})$ is a vector in the spation configuration.

The total torque on a volume element, centred on y_0, is (ignoring body forces)

$$M = \oint (y - y_0)\wedge \mathsf{T}(d\boldsymbol{s}). \tag{6.320}$$

This integral runs over the reference body, and returns a torque in the spatial configuration. This must be equated with the rate of change of angular momentum, which is

$$\begin{aligned}\frac{d}{dt}\int d^3x\, \rho (y - y_0)\wedge \dot{y} &= \int d^3x\, (y - y_0)\wedge \check{\mathsf{T}}(\check{\boldsymbol{\nabla}}) \\ &= \oint (y - y_0)\wedge \mathsf{T}(d\boldsymbol{s}) - \int d^3x\, \check{y}\wedge \mathsf{T}(\check{\boldsymbol{\nabla}}).\end{aligned} \tag{6.321}$$

Equating this with M we see that

$$\check{y}\wedge \mathsf{T}(\check{\boldsymbol{\nabla}}) = (\partial_i f(x))\wedge \mathsf{T}(\mathsf{e}^i) = \mathsf{f}(\mathsf{e}_i)\wedge \mathsf{T}(\mathsf{e}^i) = 0. \tag{6.322}$$

It follows that

$$\mathsf{f}(\mathsf{e}_i)\wedge \mathsf{T}(\mathsf{e}^i) = \det\,(\mathsf{f})\, \mathsf{f}(\mathsf{e}_i)\wedge \boldsymbol{\sigma} \bar{\mathsf{f}}^{-1}(\mathsf{e}^i) = 0, \tag{6.323}$$

and we see that $\boldsymbol{\sigma}$ must be a symmetric tensor in order for angular momentum to be conserved.

It is often convenient to work with a version of T that is symmetric and defined entirely in the material frame. We therefore define the *second Piola–Kirchoff stress tensor* $\mathcal{T}$ by

$$\mathcal{T}(\boldsymbol{a}) = \mathsf{f}^{-1}\mathsf{T}(\boldsymbol{a}). \tag{6.324}$$

It is meaningless to talk about symmetries of T, since it maps between different spaces, whereas $\mathcal{T}$ is defined entirely in the reference configuration and, by construction, is symmetric.

The equations of motion for an elastic material are completed by defining a constitutive relation. This relates the stresses to the strains in the body. These relations are most easily expressed in the reference copy as a relationship between

$\mathcal{T}$ and C. There is no universal definition of the strain tensor $\mathcal{E}$, though for certain applications a useful definition is

$$\mathcal{E}(\boldsymbol{a}) = \mathsf{C}^{1/2}(\boldsymbol{a}) - \boldsymbol{a}. \tag{6.325}$$

This tensor is zero if the material is undeformed. Linear materials have the property that $\mathcal{T}$ and $\mathcal{E}$ are linearly related by a rank-4 tensor. This can, in principle, have 36 independent degrees of freedom, all of which may need to be determined experimentally. If the material is homogeneous then the components of the rank-4 tensor are constants. If the material is also isotropic then the 36 degrees of freedom reduce to two. These are usually given in terms of the bulk modulus B and shear modulus G, with $\mathcal{T}$ and $\mathcal{E}$ related by an expression of the form

$$\mathcal{T}(\boldsymbol{a}) = 2G\mathcal{E}(\boldsymbol{a}) + (B - \tfrac{2}{3}G)\mathrm{tr}(\mathcal{E})\boldsymbol{a}. \tag{6.326}$$

In many respects this is the simplest material one can consider, though even in this case the non-linearity of the force law makes the full equations very hard to analyse. The analysis can be aided by the fact that these materials are described by an action principle, as discussed in section 12.4.1.

6.7 Notes

The treatment of vector manifolds presented here is a condensed version of the theory developed by Hestenes & Sobczyk in the book *Clifford Algebra to Geometric Calculus* (1984) and in a series of papers by Garret Sobczyk. There are a number of differences in our presentation, however. Most significant is our definition of the orientations in the fundamental theorem of integral calculus. Our definition of the boundary operator ensures that a boundary inherits its orientation from the directed volume measure. Hestenes & Sobczyk used the opposite specification for their boundary operator, which gives rise to a number of (fairly trivial) differences. A significant advantage of our conventions is that in two dimensions the pseudoscalar has the correct orientation implied by the imaginary in the Cauchy integral formula.

A further difference is that from the outset we have emphasised both the implied embedding of a vector manifold, and the fact that this gives rise to a metric. A vector manifold thus has greater structure than a differentiable manifold in the sense of differential geometry. For applications to finite-dimensional Riemannian geometry the different approaches are entirely equivalent, as any finite-dimensional Riemannian manifold can be embedded in a larger dimensional flat space in such a way that the metric is generated by the embedding. This result was proved by John Nash in 1956. His remarkable story is the subject of the book *A Beautiful Mind* by Sylvia Nasar (1998) and, more recently, a film of the same name. In other applications of differential geometry the full range of validity of the vector manifold approach has yet to be fully established. The approach certainly does give streamlined proofs of a number of key results. But whether this comes with some loss of generality is an open question.

A final, small difference in our approach here to the original one of Hestenes & Sobczyk is our definition of the shape tensor. We have only considered the shape tensor $S(a)$ taking intrinsic vectors as its linear argument. This concept can be generalised to define a function that can act linearly on general vectors. One of the most interesting properties of this generalized version of the shape tensor is that it provides a natural square root of the Ricci tensor. This theory is developed in detail in chapter 5 of *Clifford Algebra to Geometric Calculus*, to which readers are referred for further information. There is no shortage of good textbooks on modern differential geometry. The books by Nakahara (1990), Schutz (1980) and Göckeler & Schucker (1987) are particularly strong on emphasising physical applications. Elasticity is described in the books by Marsden & Hughes (1994) and Antman (1995).

6.8 Exercises

6.1 Confirm that the vector derivative is independent of the choice of coordinate system.

6.2 If we denote the curl of a vector field $\boldsymbol{J}$ in three dimensions by $\boldsymbol{\nabla}\times\boldsymbol{J}$, show that

$$\boldsymbol{\nabla}\times\boldsymbol{J} = -I\nabla\wedge J.$$

Hence prove that

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{J}) &= 0,\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{J}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{J}) - \boldsymbol{\nabla}^2\boldsymbol{J}.\end{aligned}$$

6.3 An oblate spheroidal coordinate system can be defined by

$$\begin{aligned}a\cosh(u)\sin(v) &= \sqrt{(x^2+y^2)},\\ a\sinh(u)\cos(v) &= z,\\ \tan(\phi) &= y/x,\end{aligned}$$

where (x, y, z) denote standard Cartesian coordinates and a is a scalar. Prove that

$$\mathsf{e}_u^2 = \mathsf{e}_v^2 = a^2\big(\sinh^2(u) + \cos^2(v)\big) = \rho^2,$$

which defines the quantity ρ. Hence prove that the Laplacian becomes

$$\begin{aligned}\boldsymbol{\nabla}^2\psi = &\frac{1}{\rho^2\cosh(u)}\frac{\partial}{\partial u}\left(\cosh(u)\frac{\partial\psi}{\partial u}\right) + \frac{1}{\rho^2\sin(v)}\frac{\partial}{\partial v}\left(\sin(v)\frac{\partial\psi}{\partial v}\right)\\ &+ \frac{1}{a^2\cosh^2(u)\,\sin^2(v)}\frac{\partial^2\psi}{\partial\phi^2},\end{aligned}$$

and investigate the properties of separable solutions in oblate spheroidal coordinates.

6.4 Prove that over the surface of a tetrahedron the directed surface integral satisfies

$$\oint dS = 0.$$

By considering pairs of adjacent tetrahedra, prove that this integral vanishes for all orientable, connected closed surfaces.

6.5 For a circle in a plane confirm that the line integral around the perimeter satisfies

$$\oint b\cdot x\, dl = b\cdot A,$$

where A is the oriented area of the circle.

6.6 Prove that

$$\sum_{i=0}^{k}(-1)^i b\cdot(x_0 + \cdots \check{x}_i \cdots + x_n)\Delta(\check{x}_i)_{(k-1)} = \frac{1}{k!} b\cdot(e_1\wedge\cdots\wedge e_n),$$

where the notation follows section 6.4.4.

6.7 Suppose that σ is an n-dimensional surface embedded in a flat space of dimensions $n+1$ with (constant) unit pseudoscalar I. Prove that

$$\oint_{\partial\sigma} dS J = -I \int_{\sigma} l\wedge\nabla\, J\,|dX|,$$

where the normal l is defined by $dX = Il\,|dX|$.

6.8 The shape tensor is defined by

$$a\cdot\partial I = IS(a) = I\times S(a).$$

Prove that the shape tensor satisfies

$$a\cdot S(b) = b\cdot S(a)$$

and

$$\dot{\partial}\wedge\dot{\mathsf{P}}(a) = S(a),$$

where P projects into the tangent space, and a and b are tangent vectors.

6.9 An open two-dimensional surface in three-dimensional space is defined by

$$\boldsymbol{r}(x,y) = x\mathsf{e}_1 + y\mathsf{e}_2 + \alpha(r)\mathsf{e}_3,$$

where $r = (x^2+y^2)^{1/2}$ and the $\{\mathsf{e}_i\}$ are a standard Cartesian frame. Prove that the Riemann tensor can be written

$$\mathsf{R}(a\wedge b) = \frac{\alpha'\alpha''}{r(1+\alpha'^2)^2}\, a\wedge b,$$

where the primes denote differentiation with respect to r. The scalar factor κ in $\mathsf{R}(a\wedge b) = \kappa a\wedge b$ is called the *Gaussian curvature*.

6.10 A linear, isotropic, homogeneous material is described by a bulk modulus B and shear modulus G. By linearising the elasticity equations, show that the longitudinal and transverse sound speeds v_l and v_t are given by

$$v_l^2 = \frac{1}{3\rho}\big(3B + 4G\big), \qquad v_t^2 = \frac{G}{\rho}.$$

6.11 Consider an infinite linear, isotropic, homogeneous material containing a spherical hole into which air is pumped. Show that, in the linearised theory, the radial stress τ_r is related to the radius of the hole r by $\tau_r \propto r^{-3}$. Discuss how the full non-linear theory might modify this result.