Decomposition Algorithm

Suppose we are given a general bivector $w \in \Lambda^2 \mathbb{R}^4$ written in our basis as

$$w = a_{14} e_{14} + a_{24} e_{24} + a_{34} e_{34} + a_{23} e_{23} + a_{31} e_{31} + a_{12} e_{12}.$$

Then there are two questions:

1) If w is simple, which two vectors $u, v \in \mathbb{R}^4$ realize $w = u \wedge v$?

2) If w is non-simple, which four vectors $u_1, v_1, u_2, v_2 \in \mathbb{R}^4$ yield

$$w = u_1 \wedge v_1 + u_2 \wedge v_2 \ ?$$

It turns out, that one can first answer and then construct an answer to 1) whenever w satisfies the both necessary and sufficient condition $w \wedge w = 0$.

<u>Algorithm 1</u> (bivector $\longrightarrow$ sum of simple bivectors).

Step 1: Rewrite w as

$$w = (a_{14} e_1 + a_{24} e_2 + a_{34} e_3) \wedge e_4 + (a_{23} e_2 - a_{31} e_1) \wedge e_3 + a_{12} e_{12}.$$

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Step 2: If $a_{31} \neq 0$, then set $\alpha := -\frac{a_{12}}{a_{31}} a_{23}$.

Then we have

$$a_{12} e_{12} = (\alpha e_2 + a_{12} e_1) \wedge e_2$$

$$= -\frac{a_{12}}{a_{31}} (a_{23} e_2 - a_{31} e_1) \wedge e_2.$$

Step 3: Rewrite w as

$$w = u \wedge e_4 + (a_{23} e_2 - a_{31} e_1) \wedge \left(e_3 - \frac{a_{12}}{a_{34}} e_2\right)$$

with $u = (a_{14} e_1 + a_{24} e_2 + a_{34} e_3)$.

Step 4: If $a_{31} = 0$, we can simply rewrite w as

$$w = u \wedge e_4 + (a_{12} e_1 - a_{23} e_3) \wedge e_2.$$

Hence, in any case, we find the desired sum of simple bivectors.

To construct an answer to 2) whenever $w \wedge w = 0$, we need the decomposition algorithm for a vector space V with $\dim V = 3$.

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Algorithm 2. (Decomposition in 3D)

Suppose that (u_1, u_2, u_3) is a basis of V. Then, we can write $\omega \in \overset{2}{\Lambda} V$ as

$$\omega = \lambda_{12}\, u_1 \wedge u_2 + \lambda_{13}\, u_1 \wedge u_3 + \lambda_{23}\, u_2 \wedge u_3.$$

Step 1: If $\lambda_{13} = 0$, rewrite ω as

$$\omega = (\lambda_{12} u_1 - \lambda_{23} u_3) \wedge u_2$$

and we are done.

Step 2: If $\lambda_{13} \neq 0$, we write

$$\omega = u_1 \wedge (\lambda_{12} u_2 + \lambda_{13} u_3) + \lambda_{23}\, u_2 \wedge u_3.$$

Similarly to algorithm 1 we have

$$\lambda_{23}\, u_2 \wedge u_3 = \frac{\lambda_{23}}{\lambda_{13}}\, u_2 \wedge (\lambda_{12} u_2 + \lambda_{13} u_3)$$

and thus

$$\omega = \left(u_1 + \frac{\lambda_{23}}{\lambda_{13}} u_2\right) \wedge (\lambda_{12} u_2 + \lambda_{13} u_3).$$

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Algorithm 3. (Simple case in $\mathbb{R}^4$)

Let $\omega \in \Lambda^2 \mathbb{R}^4$ with $\omega \wedge \omega = 0$.

After applying Algorithm 1, we have

$$\omega = u \wedge e_4 + u_1 \wedge u_2,$$

where

$$u = (a_{14} e_1 + a_{24} e_2 + a_{34} e_3)$$

if $a_{31} \neq 0$: $\begin{cases} u_1 = (a_{23} e_2 - a_{31} e_1) \\ u_2 = \left(e_3 - \frac{a_{12}}{a_{31}} e_2\right) \end{cases}$

if $a_{31} = 0$: $\begin{cases} u_1 = (a_{12} e_1 - a_{23} e_3) \\ u_2 = e_2 \end{cases}$

[The assumption $\omega \wedge \omega = 0$ implies that $u \wedge u_1 \wedge u_2 = 0$ and hence u, u_1, u_2 are linearly dependent]

If u_1, u_2 are linearly dependent (Check with Cauchy-Schwarz), then $u_1 \wedge u_2 = 0$ and we are done.

Else $u = \lambda_1 u_1 + \lambda_2 u_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

$\Rightarrow \omega = \lambda_1 (u_1 \wedge e_4) + \lambda_2 (u_2 \wedge e_4) + u_1 \wedge u_2$.
Now apply algorithm 2.

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