Chapter 11

The Complete Set of Resistance and Mobility Functions for Two Rigid Spheres

11.1 Regimes of Interaction

In this section, we present a summary of all known analytical results for interactions between two rigid spheres. The parameter space for two spheres of radii a and b with $a \geq b$ is shown in Figure 11.1. We delineate three overlapping regions of interest: *widely separated spheres*, *nearly touching spheres*, and *large-small interactions*. For the first case, we have $R \gg a \geq b$, and Jeffrey and Onishi's [38] twin-multipole expansions (Chapter 8) is the method of choice (the results given in the tables in this chapter were obtained by this method). The leading order terms in the far field results may be obtained from the standard method of reflections, but the algebraic manipulations become increasingly difficult for the higher order terms.

For nearly touching spheres, the singular terms may be obtained by lubrication theory (see Chapter 9) and there is a large body of literature on this subject. (Readers interested in the original literature for the various resistance functions are directed to [23, 39, 41, 55, 59, 60, 61].) Unfortunately, at this time we do not have the complete set of analytical results for *all* resistance and mobility functions. A complete set (almost-touching to far field) of information is available for the functions $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ from Jeffrey and Onishi [38] and is reproduced here. Other results in the tables were obtained by combining known lubrication analyses for the resistance functions $\boldsymbol{G}$, $\boldsymbol{H}$, and $\boldsymbol{M}$, as summarized by Jeffrey and coworkers [7, 40, 41] (see also Chapter 9) with boundary collocation results of Kim and coworkers [45, 69] for the $O(1)$ and $o(1)$ terms of the resistance functions in almost-touching geometries.[1] The results for the

[1]Boundary collocation is described in Chapter 13. We thank Mr. Gary Huber for contributing to the numerical analysis of the boundary collocation results. We also thank Dr. David Jeffrey of the University of Western Ontario for providing us with a complete set of results obtained from the twin multipole expansion algorithm.

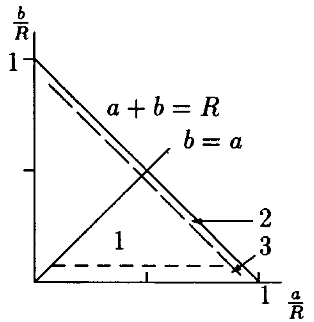

Figure 11.1: Schematic diagram of the parameter space for two spheres. Far field (1); near field (2); disparate spheres (3).

interaction between large and small spheres (disparate-sized interactions) are described in Chapter 10, and thus not repeated here. In the next section, we illustrate the use of these tables.

11.2 Examples of the Usage of Resistance and Mobility Functions

We illustrate how these results can be used to solve a number of resistance and mobility problems. We consider five examples (see Figure 11.2): two spheres moving in tandem along their line of centers; two spheres approaching each other (squeezing flow); sedimentation of a heavy sphere sliding past a neutrally buoyant sphere; two almost-touching spheres undergoing rigid-body rotation about a center located at the midpoint of the gap region; and two force-free and torque-free spheres in a linear field.

11.2.1 Two Spheres Moving in Tandem Along Their Line of Centers

We set $U_2 = U_1 = U$, with U directed along the sphere-sphere axis. The relevant resistance relations for sphere 1 are

$$F = -\mu(X_{11}^A + X_{12}^A)U \ , \qquad S = -\mu(X_{11}^G + X_{12}^G)U \ .$$

At large separations, we simply insert the expansions in inverse powers of R^{-1} from the tables for the resistance functions X_{11}^A, X_{12}^A, X_{11}^G, and X_{12}^G. The result for near-contact is more interesting. We define the small parameter,

$$\xi = \frac{2(R-a-b)}{(a+b)} = \frac{2\epsilon}{1+\beta} \ .$$

This is just the gap divided by a length scale that is symmetric with respect to both sphere labels. The singular terms cancel exactly and we are left with[2]

$$\begin{aligned}
\frac{-F}{6\pi\mu aU} &= A^X_{11}(\beta) + \frac{1}{2}(1+\beta)A^X_{12}(\beta) \\
&\quad + (L^X_{11}(\beta) + \frac{1}{2}(1+\beta)L^X_{12}(\beta))\xi + O(\xi^2 \ln \xi) \\
\frac{-S}{4\pi\mu a^2 U} &= G^X_{11}(\beta) + \frac{1}{2}(1+\beta)G^X_{12}(\beta) + O(\xi) \ .
\end{aligned}$$

The finite terms at $\xi = 0$ correspond to the resistance functions for the "dumbbell" with zero connector length.

11.2.2 Two Spheres Approaching Each Other

Consider two spheres approaching each other, so that $\boldsymbol{U}_2 = -\boldsymbol{U}_1 = \boldsymbol{U}$ with $\boldsymbol{U}$ parallel to the sphere-sphere axis. The force on sphere 1 is then given by

$$F = \mu(X^A_{11} - X^A_{12})U \ ,$$

and at near-contact the singular terms do not cancel. Instead we have

$$\begin{aligned}
\frac{F}{6\pi\mu aU} &= \frac{4\beta^2}{(1+\beta)^3}\xi^{-1} + \frac{2\beta(1+7\beta+\beta^2)}{5(1+\beta)^3}\ln \xi^{-1} \\
&+ A^X_{11}(\beta) - \frac{1}{2}(1+\beta)A^X_{12}(\beta) \\
&+ \frac{1+18\beta-29\beta^2+18\beta^3+\beta^4}{21(1+\beta)^3}\xi \ln \xi^{-1} \\
&+ (L^X_{11}(\beta) - \frac{1}{2}(1+\beta)L^X_{12}(\beta))\xi + O(\xi^2 \ln \xi) \ .
\end{aligned}$$

This result is consistent with that derived in Chapter 9.

11.2.3 Heavy Sphere Falling Past a Neutrally Buoyant Sphere

Consider a heavy sphere falling past a second, neutrally buoyant sphere, at the instant when the line between the sphere centers is horizontal. The drag on sphere 1, $\boldsymbol{F}_1$, is equal and opposite to the known external force (gravity), while $\boldsymbol{F}_2 = 0$. Under these circumstances, the mobility functions y^a_{11} and $y^a_{21} = y^a_{12}$[3] describe the motions of sphere 1 and sphere 2, as given by the following relations:

$$U_1 = y^a_{11}(-F_1/\mu) \ , \qquad U_2 = y^a_{21}(-F_1/\mu) \ .$$

Here, both $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ point in the direction of gravity.

[2] In the tables, we denote the combination $L^X_{11}(\beta) + (1+\beta)L^X_{12}(\beta) + \beta L^X_{22}(\beta)$ as W^X.
[3] Why are these two functions equal?

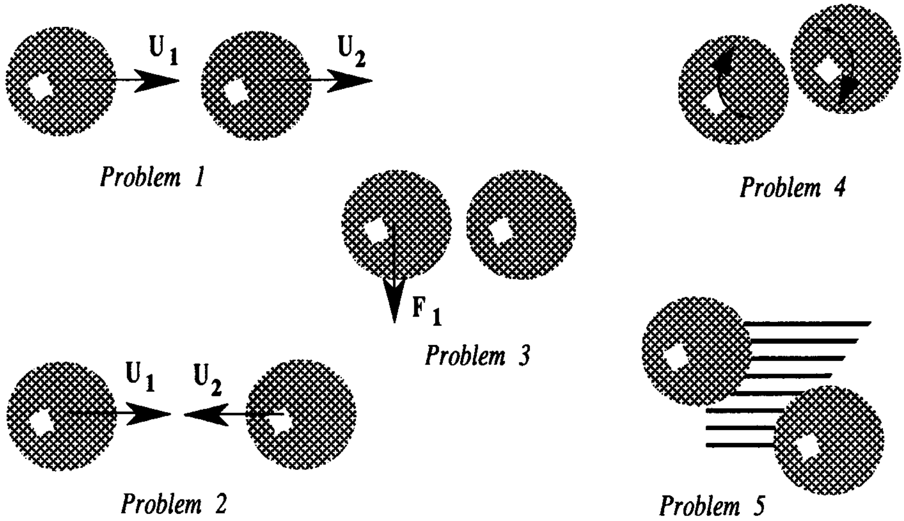

Figure 11.2: Resistance and mobility problems.

An examination of the results for $y^a_{\alpha\beta}$ given in the tables of this chapter reveal that the far field behavior is as expected: The heavy sphere falls with essentially the Stokes sedimentation velocity, whilst the neutrally buoyant sphere barely moves in reaction to its distant neighbor. Indeed, this far field behavior persists until the partices are almost touching. For the near-contact regime, we again employ the gap parameter, ξ. To capture the sharp transition from the lubrication solution to the far field solution, we use $1/\ln \xi^{-1}$ on the abscissa in Figure 11.3. These results of the continuum theory can be interpreted as follows: Unless the spheres are practically touching, lubrication stresses are too weak to keep the spheres together. The heavier one falls; the neutrally-buoyant one does not. On the other hand, at contact (a heavy and a neutrally-buoyant sphere fused together) the two must fall with a common sedimentation velocity, plus the correction for the rigid-body rotation caused by the torque caused by the asymmetric loading (verify this: use the translation theorems of Chapter 5 and the results in this chapter for the $y^c_{\alpha\beta}$ functions). Note that the transitions occur at gap thicknesses for which the continuum theory may be at best a framework for mathematical model building.

The solid lines in Figure 11.3 correspond to the lubrication solutions, while the discrete points are those obtained from the boundary collocation solutions. The positive role of osculation points at the poles is highlighted in this plot. The solid circles (•) and pluses (+) are 48-point (per sphere) and 96-point (per sphere) boundary collocation solutions with equidistant spacing between

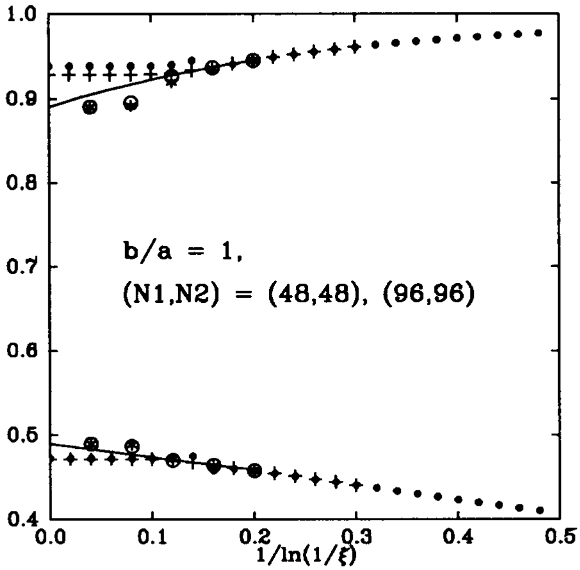

Figure 11.3: Mobility functions y^a_{11} and y^a_{21}.

collocation points, and no points at the poles. The asterisk (*) and open circle (o) are 48-point and 96-point solutions, with equidistant spacing and osculation points at the poles. Without osculation at the poles, the system becomes ill-conditioned and diverges at some critical ξ. Placement of osculation points at the poles circumvents this problem. *Note also that at some small value for ξ, the discrete system (with osculation) cannot resolve the thin film separating the spheres, but this in fact can be used to our advantage: It provides a fairly simple numerical procedure for obtaining values of the mobility functions for touching spheres.*

11.2.4 Two Almost-Touching Spheres in Rigid-Body Rotation

For two almost-touching spheres rotating about the center of sphere 1, the relevant velocities are

$$\omega_1 = \omega_2 = \omega \ ,$$

$$\boldsymbol{U}_1 = 0 \ , \qquad \boldsymbol{U}_2 = a(1+\beta)(1+\xi)\boldsymbol{\omega} \times \boldsymbol{d} \ .$$

The force, torque, and stresslet on sphere 1 are given by

$$\begin{aligned}
\boldsymbol{F}_1 &= -\mu Y^A_{12}\boldsymbol{U}_2 + \mu(Y^B_{11} + Y^B_{21})\boldsymbol{\omega} \times \boldsymbol{d} \\
\boldsymbol{T}_1 &= -\mu Y^B_{12}\boldsymbol{U}_2 \times \boldsymbol{d} - \mu(Y^C_{11} + Y^C_{12})\boldsymbol{\omega} \\
\boldsymbol{S}_1 &= -\mu Y^G_{12}(\boldsymbol{d}\boldsymbol{U}_2 + \boldsymbol{U}_2\boldsymbol{d}) - \mu(Y^H_{11} + Y^H_{12})(\boldsymbol{\omega} \times \boldsymbol{d}\boldsymbol{U}_2 + \boldsymbol{U}_2\boldsymbol{\omega} \times \boldsymbol{d}) \ .
\end{aligned}$$

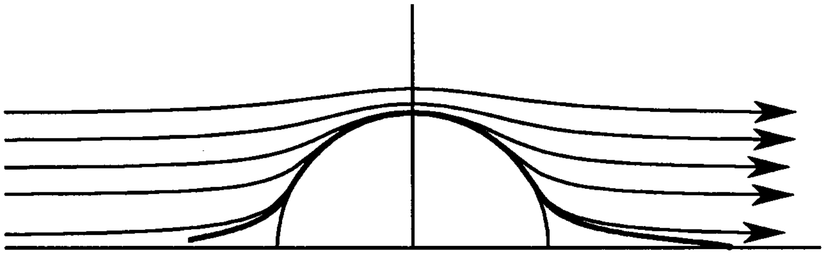

Figure 11.4: Two equal spheres in a shear flow.

The singular terms cancel exactly, and for $\xi = 0$ (touching spheres) we are left with

$$\begin{aligned}
\frac{F_1}{4\pi\mu a^2\omega} &= -\frac{3}{4}(1+\beta)^2 A^Y_{12}(\beta) + B^Y_{11}(\beta) - \frac{1}{4}(1+\beta)^2 B^Y_{12}(\beta^{-1}) \\
\frac{T_1}{8\pi\mu a^3\omega} &= \frac{1}{8}(1+\beta)B^Y_{12} - C^Y_{11} - \frac{1}{8}(1+\beta)^3 C^Y_{12} \\
\frac{S}{8\pi\mu a^3\omega} &= -\frac{1}{8}(1+\beta)^3 G^Y_{12} - H^Y_{11} - \frac{1}{8}(1+\beta)^3 H^Y_{12} \,.
\end{aligned}$$

11.2.5 Two Force-Free and Torque-Free Spheres in a Linear Field

In the ambient field $\boldsymbol{\Omega}^\infty \times \boldsymbol{x} + \boldsymbol{E}^\infty \cdot \boldsymbol{x}$, the translational velocity of spheres 1 and 2 are given by the mobility relations

$$\begin{aligned}
\boldsymbol{\Omega}^\infty \times \boldsymbol{x}_1 + \boldsymbol{E}^\infty \cdot \boldsymbol{x}_1 - \boldsymbol{U}_1 &= \tilde{\boldsymbol{g}}_1 : \boldsymbol{E}^\infty \\
\boldsymbol{\Omega}^\infty \times \boldsymbol{x}_2 + \boldsymbol{E}^\infty \cdot \boldsymbol{x}_2 - \boldsymbol{U}_2 &= \tilde{\boldsymbol{g}}_2 : \boldsymbol{E}^\infty \,.
\end{aligned}$$

The relative velocity then follows as

$$\begin{aligned}
\boldsymbol{U}_2 - \boldsymbol{U}_1 &= \boldsymbol{\Omega}^\infty \times (\boldsymbol{x}_2 - \boldsymbol{x}_1) + \boldsymbol{E}^\infty \cdot (\boldsymbol{x}_2 - \boldsymbol{x}_1) - (\tilde{\boldsymbol{g}}_2 - \tilde{\boldsymbol{g}}_1) : \boldsymbol{E}^\infty \\
&= \boldsymbol{\Omega}^\infty \times (\boldsymbol{x}_2 - \boldsymbol{x}_1) + \boldsymbol{E}^\infty \cdot (\boldsymbol{x}_2 - \boldsymbol{x}_1) \\
&\quad - [A\boldsymbol{dd} + B(\boldsymbol{\delta} - \boldsymbol{dd})] \cdot \boldsymbol{E}^\infty \cdot (\boldsymbol{x}_2 - \boldsymbol{x}_1) \,,
\end{aligned}$$

where

$$\begin{aligned}
A &= R^{-1}(x^g_{11} + x^g_{21} - x^g_{12} - x^g_{22}) \\
B &= 2R^{-1}(y^g_{11} + y^g_{21} - y^g_{12} - y^g_{22}) \,.
\end{aligned}$$

Using the tables for $x^g_{\alpha\beta}$ and $y^g_{\alpha\beta}$ (read down the column below f_k to pick out the coefficients for the polynomial, $f_k(\beta)$), we will compute A and B accurate to terms of R^{-8}.

$$\begin{aligned}
x^g_{11} &= 2a\left[200\beta^3\left(\frac{a}{2R}\right)^5 + (-1760\beta^3 + 640\beta^5)\left(\frac{a}{2R}\right)^7 + \cdots\right] \\
&= 2a\left[\frac{25a^2b^3}{R^5} + \frac{-55a^4b^3 + 20a^2b^5}{4R^7} + \cdots\right] \\
x^g_{22} &= -2b\left[\frac{25a^3b^2}{R^5} + \frac{-55a^3b^4 + 20a^5b^2}{4R^7} + \cdots\right] ,
\end{aligned}$$

and similarly for x^g_{12}, x^g_{21}, and $y^g_{\alpha\beta}$. Note that the expression for x^g_{22} has been obtained from that for x^g_{11} by switching the labels on the spheres $a \to b$, $b \to a$, and $\boldsymbol{d} \to -\boldsymbol{d}$. The final results for A and B are

$$\begin{aligned}
A &= \frac{5(a^3+b^3)}{2R^3} - \frac{3(a^5+b^5)+5a^2b^2(a+b)}{2R^5} + \frac{25a^3b^3}{R^6} - \frac{35a^3b^3(a^2+b^2)}{2R^8} \\
&\quad + O\left(\left(\frac{a+b}{R}\right)^9\right) , \\
B &= \frac{3(a^5+b^5)+5a^2b^2(a+b)}{3R^5} - \frac{25a^3b^3(a^2+b^2)}{6R^8} + O\left(\left(\frac{a+b}{R}\right)^{10}\right) .
\end{aligned}$$

In Figure 11.4, we show trajectories traced by one of two equal spheres in a shear field, as viewed from the center-of-mass frame. Although all sphere pairs starting from infinity sweep past each other, a critical trajectory (darker solid line) delineates a region with doublets that orbit each other. The hydrodynamic force on its own cannot disrupt the doublet; over a very long time other cumulative effects, such as Brownian motion, may break up the doublet.

11.3 Tables of the Resistance and Mobility Functions

In these tables we present the asymptotic expressions for the resistance and mobility functions. The notation is as follows:

$$\begin{aligned}
R &= |\boldsymbol{x}_2 - \boldsymbol{x}_1| \\
\boldsymbol{d} &= (\boldsymbol{x}_2 - \boldsymbol{x}_1)/R \\
\beta &= b/a \\
\xi &= 2(R-a-b)/(a+b) .
\end{aligned}$$

Only the "11" and "12" functions are provided; the reader should use the following symmetries to obtain the "21" and "22" functions:

$$X^A_{\alpha\beta}\left(\frac{2R}{a+b}, \beta\right) = X^A_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b}, \beta^{-1}\right)$$

$$
\begin{aligned}
Y^A_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^A_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
Y^B_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= -Y^B_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
X^C_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= X^C_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
Y^C_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^C_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
X^G_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= -X^G_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
Y^G_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= -Y^G_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
Y^H_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^H_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
X^M_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= X^M_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
Y^M_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^M_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right)\\
Z^M_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Z^M_{(3-\alpha)(3-\beta)}\left(\frac{2R}{a+b},\beta^{-1}\right) ,
\end{aligned}
$$

with a similar set results for the mobility functions. Also, because of the reciprocal theorem, we have the symmetry relations for functions from the diagonal tensors (Chapter 7):

$$
\begin{aligned}
X^A_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= X^A_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)\\
Y^A_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^A_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)\\
X^C_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= X^C_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)\\
Y^C_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^C_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)\\
X^M_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= X^M_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)\\
Y^M_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Y^M_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)\\
Z^M_{\alpha\beta}\left(\frac{2R}{a+b},\beta\right) &= Z^M_{\beta\alpha}\left(\frac{2R}{a+b},\beta\right)
\end{aligned}
$$

The analogous set for the mobility functions may be deduced from expressions given in Chapter 7.

$$
\begin{aligned}
X_{11}^A &= 6\pi a \left(\frac{2\beta^2}{(1+\beta)^3}\xi^{-1} + \frac{\beta(1+7\beta+\beta^2)}{5(1+\beta)^3} \ln \xi^{-1} + A_{11}^X(\beta) \right. \\
&\left. + \frac{1+18\beta-29\beta^2+18\beta^3+\beta^4}{42(1+\beta)^3} \xi \ln \xi^{-1} + L_{11}^X(\beta)\xi + O(\xi^2 \ln \xi) \right) \\
X_{12}^A &= -6\pi a \left(\frac{2\beta^2}{(1+\beta)^3}\xi^{-1} + \frac{\beta(1+7\beta+\beta^2)}{5(1+\beta)^3} \ln \xi^{-1} - \frac{1}{2}(1+\beta) A_{12}^X(\beta) \right. \\
&+ \frac{1+18\beta-29\beta^2+18\beta^3+\beta^4}{42(1+\beta)^3} \xi \ln \xi^{-1} \\
&\left. - \frac{1}{2}(1+\beta) L_{12}^X(\beta)\xi + O(\xi^2 \ln \xi) \right) \\
Y_{11}^A &= 6\pi a \left(\frac{4\beta(2+\beta+2\beta^2)}{15(1+\beta)^3} \ln \xi^{-1} + A_{11}^Y(\beta) \right. \\
&\left. + \frac{2(16-45\beta+58\beta^2-45\beta^3+16\beta^4)}{375(1+\beta)^3} \xi \ln \xi^{-1} \right) \\
Y_{12}^A &= -6\pi a \left(\frac{4\beta(2+\beta+2\beta^2)}{15(1+\beta)^3} \ln \xi^{-1} - \frac{1}{2}(1+\beta) A_{12}^Y(\beta) \right. \\
&\left. + \frac{2(16-45\beta+58\beta^2-45\beta^3+16\beta^4)}{375(1+\beta)^3} \xi \ln \xi^{-1} \right)
\end{aligned}
$$

β	A_{11}^X	A_{12}^X	A_{22}^X	W^X	A_{11}^Y	A_{12}^Y	A_{22}^Y
0.1	1.0398	−0.0787	0.4692	0.0011	0.9869	−0.0072	0.2438
0.125	1.0496	−0.0993	0.5001	0.0022	0.9907	−0.0268	0.3637
0.2	1.0730	−0.1556	0.5789	0.0084	1.0015	−0.0844	0.5930
0.25	1.0836	−0.1880	0.6253	0.0146	1.0073	−0.1181	0.6871
0.5	1.0881	−0.2957	0.8083	0.0575	1.0193	−0.2246	0.9009
1.0	0.9954	−0.3502	0.9954	0.1163	0.9983	−0.2737	0.9983

Table 11.1: Near field forms of the $\boldsymbol{A}$ resistance functions.

$$
\begin{aligned}
X_{11}^{A} &= 6\pi a \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k} \\
X_{12}^{A} &= -6\pi a \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}
\end{aligned}
$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		9	−24	16		
β^2			81	108	576	2304
β^3			36	281	4848	20736
β^4				648	5409	42804
β^5				144	4524	115849
β^6					3888	76176
β^7					576	39264
β^8						20736
β^9						2304

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β	3	−4				
β^2		27	72	288	1152	4608
β^3		−4	243	1620	9072	46656
β^4			72	1515	14752	108912
β^5				1620	26163	269100
β^6				288	14752	319899
β^7					9072	269100
β^8					1152	108912
β^9						46656
β^{10}						4608

Table 11.2: Far field forms of the resistance functions $X_{\alpha\beta}^{A}$.

$$Y_{11}^{A} = 6\pi a \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

$$Y_{12}^{A} = -6\pi a \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	$4f_2$	$16f_4$	$64f_6$	$256f_8$	$1024f_{10}$
1	1					
β		9	96	256		
β^2			81	3456	71424	1179648
β^3			288	1241	136352	2011392
β^4				5184	126369	6303168
β^5				4608	−3744	10548393
β^6					165888	8654976
β^7					73728	−179712
β^8						3981312
β^9						1179648

	$2f_1$	$8f_3$	$32f_5$	$128f_7$	$512f_9$	$2048f_{11}$
1	0					
β	3	16				
β^2		27	1008	18432	294912	4718592
β^3		16	243	16848	580608	14598144
β^4			1008	19083	967088	22600704
β^5				16848	766179	43912080
β^6				18432	967088	95203835
β^7					580608	43912080
β^8					294912	22600704
β^9						14598144
β^{10}						4718592

Table 11.3: Far field forms of the resistance functions $Y_{\alpha\beta}^{A}$.

$$Y_{11}^{B} = 4\pi a^2 \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$
$$Y_{12}^{B} = -4\pi a^2 \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

	f_0	f_2	$2f_4$	$8f_6$	$32f_8$	$128f_{10}$
1	0					
β		−6				
β^2			−27	−864	−13824	−221184
β^3				−243	−15552	−497664
β^4				−576	−77451	−1905696
β^5					−12960	−1125603
β^6					−9216	−2816256
β^7						−373248
β^8						−147456

	f_1	f_3	$4f_5$	$16f_7$	$64f_9$	$256f_{11}$
1	0					
β		−9	−48			
β^2			−81	−3024	−55296	−884736
β^3			−144	−8409	−50544	−1741824
β^4				−3888	−283041	−9831696
β^5				−2304	−488400	−4579497
β^6					−103680	−8586864
β^7					−36864	−18941184
β^8						−2322432
β^9						−589824

Table 11.4: Far field forms of the resistance functions $Y_{\alpha\beta}^{B}$.

$$
\begin{aligned}
X_{11}^C &= 8\pi a^3 \left(\frac{\beta^3}{(1+\beta)^3} \zeta(3, \beta/(1+\beta)) - \frac{\beta^2}{4(1+\beta)} \xi \ln \xi^{-1} \right) \\
X_{12}^C &= -8\pi a^3 \left(\frac{\beta^3}{(1+\beta)^3} \zeta(3,1) - \frac{\beta^2}{4(1+\beta)} \xi \ln \xi^{-1} \right) ,
\end{aligned}
$$

where the Riemann zeta function is defined as

$$
\zeta(z,x) = \sum_{k=0}^{\infty} (k+x)^{-z} .
$$

In Jeffrey and Onishi, $\zeta(3,1)$ is denoted by the usual notation as just $\zeta(3)$.

$$
\begin{aligned}
Y_{11}^B &= 4\pi a^2 \left(-\frac{\beta(4+\beta)}{5(1+\beta)^2} \ln \xi^{-1} + B_{11}^Y(\beta) \right. \\
&\quad \left. - \frac{(32 - 33\beta + 83\beta^2 + 43\beta^3)}{250(1+\beta)^2} \xi \ln \xi^{-1} \right) \\
Y_{12}^B &= 4\pi a^2 \left(\frac{\beta(4+\beta)}{5(1+\beta)^2} \ln \xi^{-1} + \frac{1}{4}(1+\beta)^2 B_{12}^Y(\beta) \right. \\
&\quad \left. + \frac{(32 - 33\beta + 83\beta^2 + 43\beta^3)}{250(1+\beta)^2} \xi \ln \xi^{-1} \right) \\
Y_{11}^C &= 8\pi a^3 \left(\frac{2\beta}{5(1+\beta)} \ln \xi^{-1} + C_{11}^Y(\beta) \right. \\
&\quad \left. + \frac{(8 + 6\beta + 33\beta^2)}{125(1+\beta)} \xi \ln \xi^{-1} \right) \\
Y_{12}^C &= 8\pi a^3 \left(\frac{\beta^2}{10(1+\beta)} \ln \xi^{-1} + \frac{1}{8}(1+\beta)^3 C_{12}^Y(\beta) \right. \\
&\quad \left. + \frac{\beta(43 - 24\beta + 43\beta^2)}{250(1+\beta)} \xi \ln \xi^{-1} \right)
\end{aligned}
$$

β	C_{11}^Y	C_{12}^Y	C_{22}^Y	B_{11}^Y	B_{12}^Y	B_{21}^Y	B_{22}^Y
0.1	0.9683	−0.0165	−0.1909	0.0408	−0.0560	0.0214	−0.7559
0.125	0.9625	−0.0210	−0.0918	0.0433	−0.0306	0.0288	−0.7124
0.25	0.9280	−0.0349	0.2097	0.0620	0.0592	0.0594	−0.5664
0.5	0.8489	−0.0349	0.4839	0.1201	0.0817	0.0686	−0.4016
1.0	0.7028	−0.0274	0.7028	0.2390	−0.0017	0.0017	−0.2390

Table 11.5: Near field forms of the $\boldsymbol{B}$ and $\boldsymbol{C}$ resistance functions.

$$X_{11}^C = 8\pi a^3 \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

$$X_{12}^C = -8\pi a^3 \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0				
β^2			0			
β^3				64		
β^4						
β^5					768	
β^6						
β^7						6144

	f_1	f_3	f_5	f_7	f_9	f_{11}
β^3	0	8				
β^4			0			
β^5				0		
β^6					512	6144
β^7						
β^8						6144

Table 11.6: Far field forms of the resistance functions $X_{\alpha\beta}^C$.

$$Y_{11}^C = 8\pi a^3 \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

$$Y_{12}^C = 8\pi a^3 \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	$4f_8$	$16f_{10}$
1	1					
β		0	12			
β^2				27	864	13824
β^3				256	243	15552
β^4					864	151179
β^5					9984	15552
β^6						20736
β^7						294912

	f_1	f_3	f_5	$2f_7$	$8f_9$	$32f_{11}$
β^2	0					
β^3		4				
β^4			18	144	2304	36864
β^5				81	3888	103680
β^6				144	−6439	−175152
β^7					3888	518049
β^8					2304	−175152
β^9						103680
β^{10}						36864

Table 11.7: Far field forms of the resistance functions $Y_{\alpha\beta}^C$.

$$
\begin{aligned}
X_{11}^G &= 4\pi a^2\left(\frac{3\beta^2}{(1+\beta)^3}\xi^{-1}+\frac{3\beta(1+12\beta-4\beta^2)}{10(1+\beta)^3}\ln\xi^{-1}+G_{11}^X(\beta)\right.\\
&\quad\left.+\frac{5+181\beta-453\beta^2+566\beta^3-65\beta^4}{140(1+\beta)^3}\xi\ln\xi^{-1}+O(\xi)\right)\\
X_{12}^G &= -4\pi a^2\left(\frac{3\beta^2}{(1+\beta)^3}\xi^{-1}+\frac{3\beta(1+12\beta-4\beta^2)}{10(1+\beta)^3}\ln\xi^{-1}-\frac{1}{4}(1+\beta)^2G_{12}^X(\beta)\right.\\
&\quad\left.+\frac{5+181\beta-453\beta^2+566\beta^3-65\beta^4}{140(1+\beta)^3}\xi\ln\xi^{-1}+O(\xi)\right)\\
Y_{11}^G &= 4\pi a^2\left(\frac{\beta(4-\beta+7\beta^2)}{10(1+\beta)^3}\ln\xi^{-1}+G_{11}^Y(\beta)\right.\\
&\quad\left.+\frac{32-179\beta+532\beta^2-356\beta^3+221\beta^4}{500(1+\beta)^3}\xi\ln\xi^{-1}+O(\xi)\right)\\
Y_{12}^G &= -4\pi a^2\left(\frac{\beta(4-\beta+7\beta^2)}{10(1+\beta)^3}\ln\xi^{-1}-\frac{1}{4}(1+\beta)^2G_{12}^Y(\beta)\right.\\
&\quad\left.+\frac{32-179\beta+532\beta^2-356\beta^3+221\beta^4}{500(1+\beta)^3}\xi\ln\xi^{-1}+O(\xi)\right)\\
Y_{11}^H &= 8\pi a^3\left(\frac{\beta(2-\beta)}{10(1+\beta)^2}\ln\xi^{-1}+H_{11}^Y(\beta)\right.\\
&\quad\left.+\frac{16-61\beta+180\beta^2+2\beta^3}{500(1+\beta)^2}\xi\ln\xi^{-1}+O(\xi)\right)\\
Y_{12}^H &= 8\pi a^3\left(\frac{\beta^2(1+7\beta)}{20(1+\beta)^2}\ln\xi^{-1}+\frac{1}{8}(1+\beta)^3H_{12}^Y(\beta)\right.\\
&\quad\left.+\frac{\beta(43+147\beta-185\beta^2+221\beta^3)}{1000(1+\beta)^2}\xi\ln\xi^{-1}+O(\xi)\right)
\end{aligned}
$$

β	G_{11}^X	G_{12}^X	G_{11}^Y	G_{12}^Y	H_{11}^Y	H_{12}^Y
0.125	0.059	−0.217	−0.025	0.040	−0.018	−0.010
0.25	0.065	−0.292	−0.040	0.039	−0.032	−0.015
0.5	−0.078	−0.137	−0.072	0.068	−0.056	−0.011
1.0	−0.469	0.195	−0.142	0.103	−0.074	−0.030
2.0	−0.780	0.229	−0.331	0.125	−0.056	−0.086
4.0	−0.472	0.051	−0.736	0.108	0.013	−0.098
8.0	0.487	−0.027	−1.318	0.063	0.116	−0.061

Table 11.8: Near field forms of the $\boldsymbol{G}$ and $\boldsymbol{H}$ resistance functions.

$$X_{11}^{G} = 4\pi a^2 \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

$$X_{12}^{G} = -4\pi a^2 \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0					
β		15	−36			
β^2			135	216	864	3456
β^3			−60	1215	6804	34992
β^4				900	8679	66384
β^5					12960	155007
β^6					5040	75900
β^7						97200
β^8						25920
β^9						

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β		45	−168	144		
β^2			405	108	1728	6912
β^3			360	−1251	27072	77760
β^4				4860	25893	178260
β^5				2160	−3888	790845
β^6					38880	500580
β^7					11520	14880
β^8						259200
β^9						57600
β^{10}						

Table 11.9: Far field forms of the resistance functions $X_{\alpha\beta}^{G}$.

$$Y^G_{11} = 4\pi a^2 \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

$$Y^G_{12} = -4\pi a^2 \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0					
β		0	12			
β^2				27	216	864
β^3			20		243/4	972
β^4				135	−1008	42123/16
β^5					1215/4	648
β^6					1080	−96073/16
β^7						4860
β^8						6240
β^9						

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β		0	18	24		
β^2				81/2	378	1728
β^3			90	−336	21209/8	3158/2
β^4				405/2	−972	103329/32
β^5				600	−62147/8	69387
β^6					2970	763173/32
β^7					3360	−166737/2
β^8						24840
β^9						17280
β^{10}						

Table 11.10: Far field forms of the resistance functions $Y^G_{\alpha\beta}$.

$$
\begin{aligned}
Y_{11}^{H} &= 8\pi a^{3}\sum_{k=0}^{\infty} f_{2k}(\beta)\left(\frac{a}{2R}\right)^{2k} \\
Y_{12}^{H} &= 8\pi a^{3}\sum_{k=0}^{\infty} f_{2k+1}(\beta)\left(\frac{a}{2R}\right)^{2k+1}
\end{aligned}
$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0					
β		0	0	24		
β^2					54	432
β^3				−120	1280	243/2
β^4					270	−1872
β^5					−1280	46015/2
β^6						2880
β^7						−9600
β^8						
β^9						

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β						
β^2						
β^3		10	0			
β^4				36	144	576
β^5					81	972
β^6				180	5248	153049/4
β^7					405	−864
β^8					1200	186173/4
β^9						5940
β^{10}						6720

Table 11.11: Far field forms of the resistance functions $Y_{\alpha\beta}^{H}$.

$$X_{11}^M = \frac{20}{3}\pi a^3\left(\frac{6\beta^2}{5(1+\beta)^3}\xi^{-1} + \frac{3\beta(1+17\beta-9\beta^2)}{25(1+\beta)^3}\ln\xi^{-1} + M_{11}^X(\beta)\right.$$
$$\left. + \frac{5+272\beta-831\beta^2+1322\beta^3-415\beta^4}{350(1+\beta)^3}\xi\ln\xi^{-1} + O(\xi)\right)$$
$$X_{12}^M = \frac{20}{3}\pi a^3\left(\frac{6\beta^3}{5(1+\beta)^3}\xi^{-1} - \frac{3\beta^2(4-17\beta+4\beta^2)}{25(1+\beta)^3}\ln\xi^{-1} + \frac{1}{8}(1+\beta)^3 M_{12}^X(\beta)\right.$$
$$\left. - \frac{\beta(65-832\beta+1041\beta^2-832\beta^3+65\beta^4)}{350(1+\beta)^3}\xi\ln\xi^{-1} + O(\xi)\right)$$

$$Y_{11}^M = \frac{20}{3}\pi a^3\left(\frac{6\beta(1-\beta+4\beta^2)}{25(1+\beta)^3}\ln\xi^{-1} + M_{11}^Y(\beta)\right.$$
$$\left. + \frac{3(8-67\beta+294\beta^2-394\beta^3+197\beta^4)}{625(1+\beta)^3}\xi\ln\xi^{-1} + O(\xi)\right)$$
$$Y_{12}^M = \frac{20}{3}\pi a^3\left(\frac{3\beta^2(7-10\beta+7\beta^2)}{50(1+\beta)^3}\ln\xi^{-1} + \frac{1}{8}(1+\beta)^3 M_{12}^Y(\beta)\right.$$
$$\left. - \frac{3\beta(221-748\beta+1902\beta^2-748\beta^3+221\beta^4)}{2500(1+\beta)^3}\xi\ln\xi^{-1} + O(\xi)\right)$$

Z^M is also a mobility function, and therefore displayed as z^m.

β	$M_{11}^X + \frac{1}{8}(1+\beta)^3 M_{12}^X$	$M_{11}^Y + \frac{1}{8}(1+\beta)^3 M_{12}^Y$
0.125	1.0194	0.9679
0.25	0.9908	0.9414
0.5	0.8438	0.8819
1.0	0.5712	0.6760
2.0	0.3219	0.0577
4.0	0.5503	−1.6130
8.0	3.1595	−6.3412

Table 11.12: Near field forms of the $\boldsymbol{M}$ resistance functions.

$$X_{11}^M = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_{2k}(\beta)\left(\frac{a}{2R}\right)^{2k}$$

$$X_{12}^M = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_{2k+1}(\beta)\left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0	60	−288	345.6	
β^2				540	−432.0	1382.4
β^3				1120	−9348.0	52416.0
β^4					8640.0	28380.0
β^5					10560.0	−151632.0
β^6						86400.0
β^7						76800.0

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β						
β^2						
β^3		40	−192			
β^4			180	1008	5184.0	25344.0
β^5			−192	1620	15552.0	108864.0
β^6				1008	24128.8	169257.6
β^7					15552.0	266695.2
β^8					5184.0	169257.6
β^9						108864.0
β^{10}						25344.0

Table 11.13: Far field forms of the resistance functions $X_{\alpha\beta}^M$.

$$Y_{11}^{M} = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_{2k}(\beta)\left(\frac{a}{2R}\right)^{2k}$$

$$Y_{12}^{M} = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_{2k+1}(\beta)\left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0			115.2	
β^2			0			259.2
β^3				640	−4736.0	16384.0
β^4						2592.0
β^5					6720.0	−110592.0
β^6						6480.0
β^7						51200.0

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β						
β^2						
β^3		−20	128			
β^4				0	864.0	5760.0
β^5			128			1944.0
β^6					−14067.2	−75825.6
β^7						10108.8
β^8					864.0	−75825.6
β^9						1944.0
β^{10}						5760.0

Table 11.14: Far field forms of the resistance functions $Y_{\alpha\beta}^{M}$.

$$3\pi(a_\alpha + a_\beta)x^a_{\alpha\beta} = d^{(1)}_{\alpha\beta}(\beta) + d^{(2)}_{\alpha\beta}(\beta)\xi + d^{(3)}_{\alpha\beta}(\beta)\xi^2 \ln\xi + d^{(4)}_{\alpha\beta}(\beta)\xi^2$$

β	$d^{(1)}_{11}$	$d^{(2)}_{11}$	$d^{(3)}_{11}$	$d^{(4)}_{11}$
0.125	0.9997	$* -0.0002$	0.003	0.008
0.25	0.9951	0.009	0.026	0.013
0.5	0.9537	0.152	0.194	-0.322
1.0	0.7750	0.930	0.900	$* -2.685$
2.0	0.4768	2.277	2.188	$* -6.236$
4.0	0.2488	3.610	4.061	$* -9.165$
8.0	0.1250	5.620	8.500	$* -16.26$

β	$d^{(1)}_{12}$	$d^{(2)}_{12}$	$d^{(3)}_{12}$	$d^{(4)}_{12}$
0.125	0.5623	-0.170	-0.256	$* -0.114$
0.25	0.6219	-0.372	-0.408	$*0.360$
0.5	0.7152	-0.766	-0.691	$*1.566$
1.0	0.7750	-1.070	-0.900	$*2.697$

* These entries differ from the original work of Jeffrey and Onishi.

Table 11.15: Near field forms of the mobility functions $x^a_{\alpha\beta}$.

$$x_{11}^a = (6\pi a)^{-1} \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

$$x_{12}^a = -(6\pi a)^{-1} \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0				
β^2						
β^3			−60	480	−960	
β^4						
β^5				−128	4224	−17920
β^6						−96000
β^7					−576	30720
β^8						
β^9						−2304

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	−3	4				
β						
β^2		4	0			
β^3				−2400	1920	−15360
β^4						
β^5					1920	231936
β^6						
β^7						−15360

Table 11.16: Far field forms of the mobility functions $x_{\alpha\beta}^a$.

$$3\pi(a_\alpha + a_\beta)y^a_{\alpha\beta} = \frac{a^{(1)}_{\alpha\beta}(\ln\xi^{-1})^2 + a^{(2)}_{\alpha\beta}\ln\xi^{-1} + a^{(3)}_{\alpha\beta}}{(\ln\xi^{-1})^2 + e^{(1)}\ln\xi^{-1} + e^{(2)}} + O(\xi(\ln\xi)^3)$$

$$\pi(a_\alpha + a_\beta)^2 y^b_{\alpha\beta} = \frac{b^{(1)}_{\alpha\beta}(\ln\xi^{-1})^2 + b^{(2)}_{\alpha\beta}\ln\xi^{-1} + b^{(3)}_{\alpha\beta}}{(\ln\xi^{-1})^2 + e_{(1)}\ln\xi^{-1} + e^{(2)}} + O(\xi\ln\xi) \ .$$

β	$a^{(1)}_{11}$	$a^{(2)}_{11}$	$a^{(3)}_{11}$	$a^{(1)}_{12}$	$a^{(2)}_{12}$	$a^{(3)}_{12}$
0.125	0.99415	1.53362	−1.54846	0.55315	0.45721	−0.64730
0.25	0.97292	3.84204	0.33945	0.57085	1.53280	−0.06410
0.5	0.92729	5.61052	4.40223	0.53482	2.50225	1.23963
1.0	0.89056	5.77196	7.06897	0.48951	2.80545	1.98174
2.0	0.76425	5.01983	5.60143	0.53482	2.50225	1.23963
4.0	0.47343	3.71016	1.88922	0.57085	1.53280	−0.06410
8.0	0.23775	2.70340	−0.84740	0.55315	0.45721	−0.64730

β	$b^{(1)}_{11}$	$b^{(2)}_{11}$	$b^{(3)}_{11}$	$b^{(1)}_{12}$	$b^{(2)}_{12}$	$b^{(3)}_{12}$
0.125	0.00638	−0.01937	0.00922	−0.17225	−0.02980	0.13570
0.25	0.03176	−0.05584	−0.01788	−0.20424	−0.31661	0.07668
0.5	0.09519	−0.03922	−0.23881	−0.20385	−0.68106	−0.10094
1.0	0.13368	0.19945	−0.79238	−0.13368	−0.92720	−0.18805
2.0	0.09060	0.52663	−1.45080	−0.05355	−0.99178	0.04487
4.0	0.03268	0.61847	−1.74796	−0.01241	* − 0.86793	*0.47544
8.0	0.00851	0.53163	−1.74774	−0.00202	−0.71569	0.82610

β, β^{-1}	$e^{(1)}$	$e^{(2)}$
1.0	6.04250	6.32549
2.0	5.59906	4.17702
4.0	3.79489	0.32014
8.0	1.51572	−1.54010

* These entries differ from the original work of Jeffrey and Onishi.

Table 11.17: Near field forms of the mobility functions $y^a_{\alpha\beta}$ and $y^b_{\alpha\beta}$.

$$y_{11}^a = (6\pi a)^{-1} \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

$$y_{12}^a = (6\pi a)^{-1} \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0	0			
β^2						
β^3					−320	
β^4						
β^5				−68	288	−6720
β^6						
β^7					−288	3456
β^8						
β^9						−1152

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	3/2	2				
β						
β^2		2	0			
β^3				0		8960
β^4					0	
β^5						−8848
β^6						
β^7						8960

Table 11.18: Far field forms of the mobility functions $y_{\alpha\beta}^a$.

$$y_{11}^b = (4\pi a^2)^{-1} \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

$$y_{12}^b = (4\pi a^2)^{-1} \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0	−2				
β			0			
β^2				0		
β^3					0	−4480
β^4						
β^5						−3200

	f_1	f_3	f_5	f_7	f_9	f_{11}
β^3	0	0	0	160		
β^4						
β^5				48	2240	
β^6						
β^7					192	21504
β^8						
β^9						768

Table 11.19: Far field forms of the mobility functions $y^b_{\alpha\beta}$.

(Near field forms easily obtained by inversion of X^C functions)

$$x_{11}^c = (8\pi a^3)^{-1}\sum_{k=0}^{\infty} f_{2k}(\beta)\left(\frac{a}{R}\right)^{2k}$$

$$x_{12}^c = -(8\pi a^3)^{-1}\sum_{k=0}^{\infty} f_{2k+1}(\beta)\left(\frac{a}{R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1	0	0	0		
β						
β^2						
β^3						
β^4						
β^5					-3	
β^6						
β^7						-6

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0	1	0	0	0	0
β						
β^2						
β^3						
β^4						
β^5						
β^6						
β^7						

Table 11.20: Far field forms of the mobility functions $x_{\alpha\beta}^c$.

$$\pi(a_\alpha + a_\beta)^3 y^c_{\alpha\beta} = \frac{c^{(1)}_{\alpha\beta}(\ln\xi^{-1})^2 + c^{(2)}_{\alpha\beta}\ln\xi^{-1} + c^{(3)}_{\alpha\beta}}{(\ln\xi^{-1})^2 + e_{(1)}\ln\xi^{-1} + e^{(2)}}\ .$$

β	$c^{(1)}_{11}$	$c^{(2)}_{11}$	$c^{(3)}_{11}$	$c^{(1)}_{12}$	$c^{(2)}_{12}$	$c^{(3)}_{12}$
0.125	0.97916	1.57872	−1.57064	0.17427	−1.07599	1.17174
0.25	0.88738	3.99132	0.37078	0.21664	−1.09737	*0.78409
0.5	0.61012	5.80416	4.98252	0.25740	−1.09618	0.45026
1.0	0.26736	5.60896	9.28111	0.26736	−1.05770	0.29981
2.0	0.07626	4.62489	9.87487	0.25740	* − 1.09618	*0.45026
4.0	0.01386	3.75550	6.51310	0.21664	* − 1.09737	*0.78409
8.0	0.00191	3.20768	2.68635	0.17427	−1.07599	1.17174

$\beta,\ \beta^{-1}$	$e^{(1)}$	$e^{(2)}$
1.0	6.04250	6.32549
2.0	5.59906	4.17702
4.0	3.79489	0.32014
8.0	1.51572	−1.54010

* These entries differ from the original work of Jeffrey and Onishi.

Table 11.21: Near field forms of the mobility functions $y^c_{\alpha\beta}$.

$$y^c_{11} = (8\pi a^3)^{-1} \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$

$$y^c_{12} = (8\pi a^3)^{-1} \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0	0			
β^2						
β^3				−240		
β^4						
β^5					−2496	
β^6						
β^7						−18432

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0	−4				
β			0			
β^2				0		
β^3					4800	30720
β^4						
β^5						30720

Table 11.22: Far field forms of the mobility functions $y^c_{\alpha\beta}$.

$$x^g_{\alpha\beta} = (a_\alpha + a_\beta)\left(g^{(1)}_{\alpha\beta}(\beta) + g^{(2)}_{\alpha\beta}(\beta)\xi \ln \xi^{-1} + g^{(3)}_{\alpha\beta}(\beta)\xi\right)$$

β	$g^{(1)}_{11}$	$g^{(2)}_{11}$	$g^{(3)}_{11}$	$g^{(1)}_{12}$	$g^{(2)}_{12}$	$g^{(3)}_{12}$
0.125	2.0e − 05	0.0014	−0.0016	−0.8889	−0.3863	0.6726
0.25	0.0031	0.0072	−0.0128	−0.7950	−0.2885	0.9797
0.5	0.0379	0.0217	−0.1598	−0.6162	−0.1379	1.3129
1.0	0.1792	0.0000	−0.8703	−0.3208	0.0000	0.9184
2.0	0.3709	−0.2065	−1.7555	−0.0861	0.0290	0.2184
4.0	0.4679	−0.7210	−1.7835	−0.0128	0.0116	0.0132
8.0	0.4941	−1.7384	−0.5125	−0.0013	0.0026	−0.0032

Table 11.23: Near field forms of the mobility functions $x^g_{\alpha\beta}$.

$$x^m_1 = \frac{20}{3}\pi a^3\left(1 + m^{(1)}_1(\beta) + m^{(2)}_1(\beta)\xi \ln \xi^{-1} + m^{(3)}_1(\beta)\xi\right)$$

β	$m^{(1)}_1$	$m^{(2)}_1$	$m^{(3)}_1$
0.125	0.00011	0.0067	−0.0026
0.25	0.017	0.034	−0.046
0.5	0.199	0.115	−0.810
1.0	0.910	0.0000	−3.85
2.0	1.77	−0.720	−7.04
4.0	1.81	−1.27	−8.59
8.0	1.14	−2.44	−2.23

Table 11.24: Near field forms of the mobility functions x^m_1.

$$x_{11}^g = (2a)\sum_{k=0}^{\infty} f_{2k+1}(\beta)\left(\frac{a}{2R}\right)^{2k+1}$$
$$x_{12}^g = -(2a)\sum_{k=0}^{\infty} f_{2k}(\beta)\left(\frac{a}{2R}\right)^{2k}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0	5	−12			
β						
β^2			−20	0		
β^3					8000	−12800
β^4						
β^5						16000

	f_1	f_3	f_5	f_7	f_9	f_{11}
β^3	0	0	200	−1760	3840	−19200
β^4						
β^5				640	−20736	89600
β^6						320000
β^7					3840	−189440
β^8						
β^9						19200

Table 11.25: Far field forms of the mobility functions $x_{\alpha\beta}^g$.

$$y^g_{\alpha\beta} = (a_\alpha + a_\beta)\frac{g^{(1)}_{\alpha\beta}(\ln\xi^{-1})^2 + g^{(2)}_{\alpha\beta}\ln\xi^{-1} + g^{(3)}_{\alpha\beta}}{(\ln\xi^{-1})^2 + e_{(1)}\ln\xi^{-1} + e^{(2)}} + O(\xi\ln\xi)\ .$$

$$y^h_{\alpha\beta} = \frac{h^{(1)}_{\alpha\beta}(\ln\xi^{-1})^2 + h^{(2)}_{\alpha\beta}\ln\xi^{-1} + h^{(3)}_{\alpha\beta}}{(\ln\xi^{-1})^2 + e_{(1)}\ln\xi^{-1} + e^{(2)}} + O(\xi\ln\xi)\ .$$

β	$g^{(1)}_{11}$	$g^{(2)}_{11}$	$g^{(3)}_{11}$	$g^{(1)}_{12}$	$g^{(2)}_{12}$	$g^{(3)}_{12}$
0.125	0.0021	−0.0074	0.0035	−0.3784	−0.0937	0.3488
0.25	0.0070	−0.0176	−0.0100	−0.2650	−0.5043	0.2345
0.5	0.0086	−0.0059	−0.0988	−0.1337	−0.5547	0.1588
1.0	0.0145	0.0786	−0.3193	−0.0869	−0.2956	0.1584
2.0	0.1278	0.0586	−0.6096	−0.0611	−0.0435	0.0888
4.0	0.3281	−0.5175	−0.4652	−0.0213	0.0248	0.0105
8.0	0.4446	−1.4007	0.4763	−0.0043	0.0118	−0.0054

β	$h^{(1)}_{11}$	$h^{(2)}_{11}$	$h^{(3)}_{11}$	$h^{(1)}_{12}$	$h^{(2)}_{12}$	$h^{(3)}_{12}$
0.125	0.0068	−0.0239	0.0116	0.5068	3.2733	−4.1376
0.25	0.0239	−0.0635	−0.0272	0.5239	3.4701	−2.0760
0.5	0.0214	−0.0713	−0.2932	0.5214	2.6560	−0.6409
1.0	−0.1014	0.0764	−0.7905	0.3986	1.0762	−0.3510
2.0	−0.3130	0.5024	−1.0825	0.1870	0.1161	−0.1902
4.0	−0.4525	1.1984	−1.3297	0.0475	−0.0533	−0.0180
8.0	−0.4920	1.9231	−2.1161	0.0080	−0.0214	0.0099

β, β^{-1}	$e^{(1)}$	$e^{(2)}$
1.0	6.04250	6.32549
2.0	5.59906	4.17702
4.0	3.79489	0.32014
8.0	1.51572	−1.54010

Table 11.26: Near field forms of the mobility functions $y^g_{\alpha\beta}$ and $y^h_{\alpha\beta}$.

$$y_{11}^g = (2a)\sum_{k=0}^{\infty} f_{2k+1}(\beta)\left(\frac{a}{2R}\right)^{2k+1}$$

$$y_{12}^g = -(2a)\sum_{k=0}^{\infty} f_{2k}(\beta)\left(\frac{a}{2R}\right)^{2k}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0	0	4			
β						
β^2			20/3	0		
β^3					0	−11200/3
β^4						
β^5						8000/3

	f_1	f_3	f_5	f_7	f_9	f_{11}
β^3	0	0	0	−400/3	2560/3	
β^4						
β^5				560/3	−8224/3	22400
β^6						
β^7					1120	−31232
β^8						
β^9						5760

Table 11.27: Far field forms of the mobility functions $y_{\alpha\beta}^g$.

$$y_{11}^h = \sum_{k=0}^{\infty} f_{2k}(\beta) \left(\frac{a}{2R}\right)^{2k}$$
$$y_{12}^h = \sum_{k=0}^{\infty} f_{2k+1}(\beta) \left(\frac{a}{2R}\right)^{2k+1}$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	0	0	0			
β						
β^2						
β^3				−200	1280	
β^4						
β^5					−1280	22400
β^6						
β^7						−9600

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0	10				
β			0			
β^2				0		
β^3					4000	
β^4						
β^5						64000
β^6						

Table 11.28: Far field forms of the mobility functions $y_{\alpha\beta}^h$.

$$x_1^m = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_k(\beta)\left(\frac{a}{2R}\right)^k$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0				
β^2			0			
β^3				1600	−15360	36864
β^4						
β^5					9600	−248832
β^6						
β^7						76800

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β						
β^2						
β^3		40	−192			
β^4						
β^5			−192	0		
β^6					64000	76800
β^7						
β^8						76800
β^9						
β^{10}						

Table 11.29: Far field forms of the mobility function x_1^m.

$$y_1^m = \frac{20}{3}\pi a^3 \frac{m_1^{(1)}(\ln\xi^{-1})^2 + m_1^{(2)}\ln\xi^{-1} + m_1^{(3)}}{(\ln\xi^{-1})^2 + e^{(1)}\ln\xi^{-1} + e^{(2)}} + O(\xi(\ln\xi)^3)$$

β	$m_1^{(1)}$	$m_1^{(2)}$	$m_1^{(3)}$
0.125	1.0118	1.4746	−1.5212
0.25	1.0377	3.7041	0.2573
0.5	1.0473	5.5678	3.5328
1.0	1.1456	6.1694	3.7112
2.0	2.2657	4.2124	−1.2260
4.0	4.7033	−4.4574	−2.5413
8.0	7.0021	−18.9350	8.4871

β, β^{-1}	$e^{(1)}$	$e^{(2)}$
1.0	6.04250	6.32549
2.0	5.59906	4.17702
4.0	3.79489	0.32014
8.0	1.51572	−1.54010

Table 11.30: Near field forms of the mobility function y_1^m.

$$z_1^m = \frac{20}{3}\pi a^3 \left(m_1^{(1)}(\beta) + m_1^{(2)}(\beta)\xi + m_1^{(3)}(\beta)\xi^2\right)$$

β	$m_1^{(1)}$	$m_1^{(2)}$	$m_1^{(3)}$
0.125	0.9995	0.00008	0.0017
0.25	0.9967	0.00385	0.0014
0.5	0.9851	0.02584	−0.0156
1.0	0.9527	0.0914	−0.081
2.0	0.8689	0.266	−0.26
4.0	0.6979	0.632	−0.57
8.0	0.4783	1.12	−0.94

Table 11.31: Near field forms of the mobility function z_1^m.

$$y_1^m = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_k(\beta) \left(\frac{a}{2R}\right)^k$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0				
β^2			0			
β^3				400	−5120	16384
β^4						
β^5					6400	−110592
β^6						
β^7						51200

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β						
β^2						
β^3		−20	128			
β^4						
β^5			128	0		
β^6					−39998/5	−383994/5
β^7						
β^8						−383994/5

Table 11.32: Far field forms of the mobility functions y_1^m.

$$z_1^m = \frac{20}{3}\pi a^3 \sum_{k=0}^{\infty} f_k(\beta) \left(\frac{a}{2R}\right)^k$$

	f_0	f_2	f_4	f_6	f_8	f_{10}
1	1					
β		0				
β^2			0			
β^3				0		1024
β^4						
β^5					800	−7040
β^6						
β^7						9088

	f_1	f_3	f_5	f_7	f_9	f_{11}
1	0					
β		0				
β^2						
β^3			−32			
β^4				0		
β^5			−32		0	
β^6						0
β^7						

Table 11.33: Far field forms of the mobility functions z_1^m.