21
Binary Index Sets and Walsh Functions

The present chapter scrutinizes how the sign of the product of two elements in the basis for the Clifford algebra of dimension 2^n can be computed by the Walsh functions of degree less than 2^n. In the multiplication formula the basis elements are labelled by binary n-tuples, which form an abelian group Ω, which in turn gives rise to the maximal grading of the Clifford algebra. The group of binary n-tuples is also employed in the Cayley-Dickson process.

Walsh Functions

Consider n-tuples $\underline{a} = a_1 a_2 \ldots a_n$ of binary digits $a_i = 0, 1$. For two such n-tuples $\underline{a}$ and $\underline{b}$ the sum $\underline{a} \oplus \underline{b} = \underline{c}$ is defined by termwise addition modulo 2, that is,

$$c_i = a_i + b_i \quad \mod 2.$$

These n-tuples form a group so that the group characters are *Walsh functions*

$$w_{\underline{a}}(\underline{b}) = (-1)^{\sum_{i=1}^n a_i b_i}.$$

The Walsh functions have only two values, ± 1, and they satisfy $w_{\underline{k}}(\underline{a} \oplus \underline{b}) = w_{\underline{k}}(\underline{a}) w_{\underline{k}}(\underline{b})$, as group characters, and $w_{\underline{a}}(\underline{b}) = w_{\underline{b}}(\underline{a})$. The Walsh functions $w_{\underline{k}}$, labelled by binary n-tuples $\underline{k} = k_1 k_2 \ldots k_n$, can be ordered by integers $k = \sum_{i=1}^n k_i 2^{n-i}$.

21.1 Sequency order

In applications one often uses the *sequency order* of the Walsh functions,

$$\tilde{w}_{\underline{k}}(\underline{x}) = (-1)^{k_1 x_1 + \sum_{i=2}^n (k_{i-1} + k_i) x_i},$$

for instance, in special analysis of time series, signal processing, communications and filtering, Harmuth 1977 and Maqusi 1981. In the sequency order the index $\underline{k}$ is often replaced by an integer $k = \sum_{i=1}^{n} k_i 2^{n-i}$ and the argument $\underline{x}$ by a real number on the unit interval $x = 2^{-n} \sum_{i=1}^{n} x_i 2^{i-1}$ (Fig. 1 and Fig. 2).

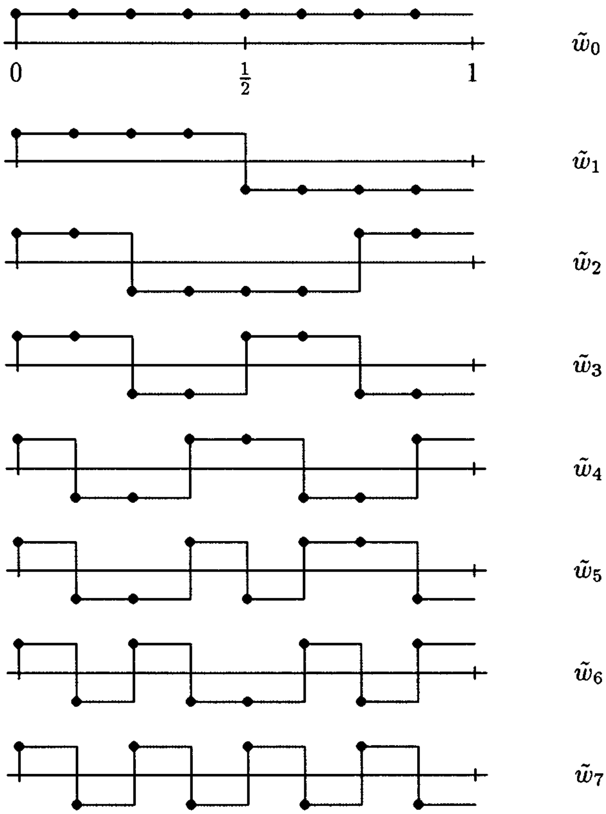

Figure 1. The first eight Walsh functions $\tilde{w}_{\underline{k}}(x)$, $k = 0, 1, \ldots, 7$.

In Figure 1 the first eight Walsh functions are given:

$$\tilde{w}_{\underline{k}}(x) = (-1)^{k_1 x_1 + (k_1 + k_2) x_2 + (k_2 + k_3) x_3}$$

with $k = 4k_1 + 2k_2 + k_3$ and $x = \frac{1}{8}(x_1 + 2x_2 + 4x_3)$. Observe that the number of zero crossings per unit interval equals k.

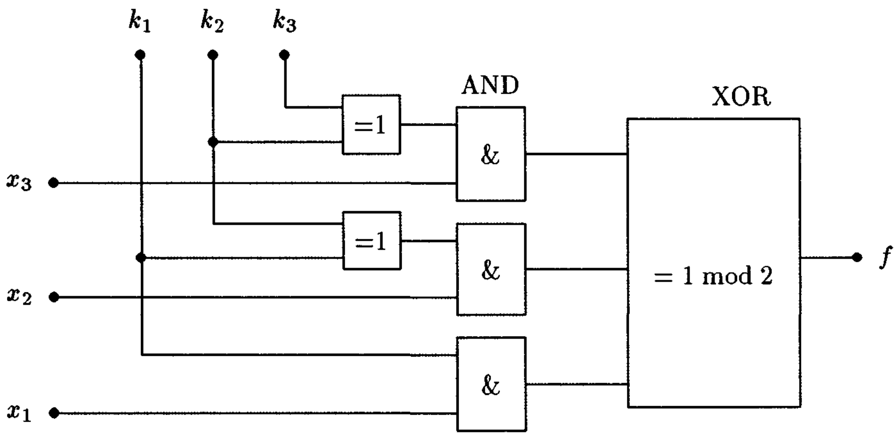

Figure 2. The first eight Walsh functions in hardware, $\tilde{w}_k(x) = (-1)^f$.

21.2 Gray code

The passage to the sequency order is related to the *Gray code* g defined by

$$g(\underline{k})_1 = k_1, \quad g(\underline{k})_i = k_{i-1} + k_i \bmod 2, \quad i = 2, \ldots, n.$$

The formula $\tilde{w}_{\underline{a}}(\underline{x}) = w_{g(\underline{a})}(\underline{x})$ reorders the Walsh functions. The Gray code is a single digit change code, that is, the codes of two consecutive integers differ only in one bit (Table 1).

Table 1. The Gray code for $k < 8$.

k	$\underline{k}$	$g(\underline{k})$
0	000	000
1	001	001
2	010	011
3	011	010
4	100	110
5	101	111
6	110	101
7	111	100

The Gray code is a group isomorphism among the binary n-tuples, that is,

$g(\underline{a} \oplus \underline{b}) = g(\underline{a}) \oplus g(\underline{b})$. The *inverse* h of the Gray code is obtained by

$$h(\underline{a})_i = \sum_{j=1}^{i} a_j \bmod 2.$$

Binary Representations of Clifford Algebras

As a preliminary example, consider the Clifford algebra $\mathcal{C}\ell_{0,2}$, isomorphic to the division ring of quaternions $\mathbb{H}$. Relabel the basis elements of $\mathcal{C}\ell_{0,2}$ by binary 2-tuples

$$\begin{array}{r|l} 1 & \mathbf{e}_{00} \\ \mathbf{e}_1, \mathbf{e}_2 & \mathbf{e}_{10}, \mathbf{e}_{01} \\ \mathbf{e}_{12} & \mathbf{e}_{11} \end{array}$$

and verify the multiplication rule

$$\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = w_{\underline{a}}(h(\underline{b}))\mathbf{e}_{\underline{a}\oplus\underline{b}}.$$

For an alternative representation reorder the basis elements by the formula

$$\tilde{\mathbf{e}}_{\underline{a}} = \mathbf{e}_{g(\underline{a})} \quad \text{or} \quad \mathbf{e}_{\underline{a}} = \tilde{\mathbf{e}}_{h(\underline{a})}$$

to get the correspondences

$$\begin{array}{r|l} 1 & \tilde{\mathbf{e}}_{00} \\ \mathbf{e}_1, \mathbf{e}_2 & \tilde{\mathbf{e}}_{11}, \tilde{\mathbf{e}}_{01} \\ \mathbf{e}_{12} & \tilde{\mathbf{e}}_{10}. \end{array}$$

This yields the multiplication rule

$$\tilde{\mathbf{e}}_{\underline{a}}\tilde{\mathbf{e}}_{\underline{b}} = \tilde{w}_{\underline{a}}(\underline{b})\tilde{\mathbf{e}}_{\underline{a}\oplus\underline{b}}.$$

21.3 Clifford multiplication

In general, consider the Clifford algebra $\mathcal{C}\ell_{0,n}$ with n generators $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ such that

$$\begin{aligned} \mathbf{e}_i^2 &= -1 \quad \text{for} \quad i = 1, 2, \ldots, n, \\ \mathbf{e}_i\mathbf{e}_j &= -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j. \end{aligned}$$

Theorem 1. If a real 2^n-dimensional algebra A has the multiplication rule

$$\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = w_{\underline{a}}(h(b))\mathbf{e}_{\underline{a}\oplus\underline{b}}$$

between the basis elements labelled by the binary n-tuples, then A is isomorphic to the Clifford algebra $\mathcal{C}\ell_{0,n}$.

Proof. It is sufficient to show that A is associative, has a unit element and is generated by n anticommuting elements with square -1.

The element $\mathrm{e}_{\underline{0}} = \mathrm{e}_{000\ldots00}$ is the unit, since $\mathrm{e}_{\underline{a}}\mathrm{e}_{\underline{0}} = w_{\underline{a}}(h(\underline{0}))\mathrm{e}_{\underline{a}\oplus\underline{0}} = w_{\underline{0}}(\underline{0})\mathrm{e}_{\underline{a}} = +\mathrm{e}_{\underline{a}}$ and similarly $\mathrm{e}_{\underline{0}}\mathrm{e}_{\underline{a}} = +\mathrm{e}_{\underline{a}}$. The n basis elements

$$\mathrm{e}_{100\ldots00},\ \mathrm{e}_{010\ldots00},\ \ldots,\ \mathrm{e}_{000\ldots01}$$

generate by definition all of A. Each generator has square $-\mathrm{e}_{\underline{0}}$; in particular for the i:th generator $\mathrm{e}_{\underline{a}}$

$$\underline{a} = \underset{1}{0}0\ldots0\underset{i}{1}0\ldots0\underset{n}{0}, \qquad h(\underline{a}) = \underset{1}{0}0\ldots0\underset{i}{1}1\ldots1\underset{n}{1}$$

and so $w_{\underline{a}}(h(\underline{a})) = -1$, from which one concludes that $\mathrm{e}_{\underline{a}}\mathrm{e}_{\underline{a}} = w_{\underline{a}}(h(\underline{a}))\mathrm{e}_{\underline{a}\oplus\underline{a}} = -\mathrm{e}_{\underline{0}}$. In a similar manner one finds that generators anticommute with each other.

Finally, A is associative, since for three arbitrary basis elements $\mathrm{e}_{\underline{a}}, \mathrm{e}_{\underline{b}}, \mathrm{e}_{\underline{c}}$ the condition $(\mathrm{e}_{\underline{a}}\mathrm{e}_{\underline{b}})\mathrm{e}_{\underline{c}} = \mathrm{e}_{\underline{a}}(\mathrm{e}_{\underline{b}}\mathrm{e}_{\underline{c}})$ is equivalent to

$$w_{\underline{a}}(h(\underline{b}))w_{\underline{a}\oplus\underline{b}}(h(\underline{c})) = w_{\underline{a}}(h(\underline{b}\oplus\underline{c}))w_{\underline{b}}(h(\underline{c})),$$

which is a consequence of $w_{\underline{a}\oplus\underline{b}}(\underline{x}) = w_{\underline{a}}(\underline{x})w_{\underline{b}}(\underline{x})$ and $w_{\underline{a}}(\underline{x}\oplus\underline{y}) = w_{\underline{a}}(\underline{x})w_{\underline{a}}(\underline{y})$ and h being a group isomorphism. ∎

It is convenient to assume the correspondences

$$\mathrm{e}_i = \mathrm{e}_{\underset{1}{0}0\ldots0\underset{i}{1}0\ldots0\underset{n}{0}} \quad \text{for} \quad i = 1, 2, \ldots, n$$

between the ordinary and binary representations of the generators of the Clifford algebra $\mathcal{C}\ell_{0,n}$. Then the basis elements of $\mathcal{C}\ell_{0,n}$ are labelled by the binary n-tuples $\underline{a} = a_1a_2\ldots a_n$ as follows:

$$\mathrm{e}_{\underline{a}} = \mathrm{e}_1^{a_1}\mathrm{e}_2^{a_2}\ldots\mathrm{e}_n^{a_n}, \quad a_i = 0, 1.$$

Since the Gray code is a group isomorphism among the binary n-tuples, we can reorder the basis of the Clifford algebra $\mathcal{C}\ell_{0,n}$ by

$$\tilde{\mathrm{e}}_{\underline{a}} = \mathrm{e}_{g(\underline{a})}.$$

This reordering results in a simple multiplication formula:

Corollary. The product of the basis elements of the Clifford algebra $\mathcal{C}\ell_{0,n}$ is given by

$$\tilde{\mathrm{e}}_{\underline{a}}\tilde{\mathrm{e}}_{\underline{b}} = \tilde{w}_{\underline{a}}(\underline{b})\tilde{\mathrm{e}}_{\underline{a}\oplus\underline{b}}.$$

Proof.

$$\tilde{\mathrm{e}}_{\underline{a}}\tilde{\mathrm{e}}_{\underline{b}} = \mathrm{e}_{g(\underline{a})}\mathrm{e}_{g(\underline{b})} = w_{g(\underline{a})}(h(g(\underline{b})))\mathrm{e}_{g(\underline{a})\oplus g(\underline{b})}$$

$$= w_{g(\underline{a})}(\underline{b})\mathrm{e}_{g(\underline{a}\oplus\underline{b})} = \tilde{w}_{\underline{a}}(\underline{b})\tilde{\mathrm{e}}_{\underline{a}\oplus\underline{b}}. \qquad \blacksquare$$

If you choose the signs in $\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = \pm\mathbf{e}_{\underline{a}\oplus\underline{b}}$ in some other way, you get other algebras than $\mathcal{C}\ell_{0,n}$. For instance, the Clifford algebra $\mathcal{C}\ell_{p,q}$ over the quadratic form $x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$ has the multiplication formula

$$\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = (-1)^{\sum_{i=1}^{p} a_i b_i} w_{\underline{a}}(h(\underline{b}))\mathbf{e}_{\underline{a}\oplus\underline{b}}.$$

Of course, this might also be written without Walsh functions:

$$\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = (-1)^{\sum_{i=p+1}^{n} a_i b_i}(-1)^{\sum_{i>j} a_i b_j}\mathbf{e}_{\underline{a}\oplus\underline{b}},$$

a formula essentially obtained by Brauer & Weyl 1935. See also Artin 1957 and Delanghe & Brackx 1978 for a related definition of the product on the Clifford algebras (based on sums of multi-indices).

21.4 An iterative process to form Clifford algebras

Clifford algebras can be obtained by a method analogous to the Cayley-Dickson process. Consider pairs (u, v) of elements u and v in the Clifford algebra $\mathcal{C}\ell_{p,q}$. Define a product for two such pairs,

$$(u_1, v_1)(u_2, v_2) = (u_1 u_2 \pm v_1 \hat{v}_2, u_1 v_2 + v_1 \hat{u}_2),$$

where $u \to \hat{u}$ is the grade involution of $\mathcal{C}\ell_{p,q}$. This results in an algebra isomorphic to the Clifford algebra

$$\mathcal{C}\ell_{p+1,q}$$

or

$$\mathcal{C}\ell_{p,q+1}$$

according to the $\pm$ sign. This iterative process could be repeated by noting that $(u, v)\hat{} = (\hat{u}, -\hat{v})$.

For more details on the Clifford algebras see Micali & Revoy 1977 and Porteous 1969, 1981.

Some Clifford-like Algebras

All the above algebras are special cases of the following. Let A be a real linear space of dimension 2^n. Label a basis for A by binary n-tuples $\underline{a}$ to get the basis elements $\mathbf{e}_{\underline{a}}$. Then define a multiplication between the basis elements $\mathbf{e}_{\underline{a}}$ and extend it to all of A by linearity. The definition is of the form

$$\mathbf{e}_{\underline{a}}\mathbf{e}_{\underline{b}} = \pm\mathbf{e}_{\underline{a}\oplus\underline{b}}$$

for a certain choice of signs. Then the algebra A is a direct sum of the 1-dimensional subspaces $U_{\underline{a}}$, spanned by $\mathbf{e}_{\underline{a}}$, satisfying

$$U_{\underline{a}} U_{\underline{b}} \subset U_{\underline{a} \oplus \underline{b}}.$$

In other words A is an algebra graded by the abelian group of binary n-tuples Ω. This grading is maximal (Kwasniewski 1985), and these algebras will be called Clifford-like algebras. Next we shall study some Clifford-like algebras.

21.5 Cayley-Dickson process

Consider a generalized quaternion ring Q with $i^2 = \gamma_1$, $j^2 = \gamma_2$ and $k^2 = \gamma_1 \gamma_2$, where $\gamma_1, \gamma_2 = \pm 1$. The conjugation-involution $u \to u^L$ of Q is given by

$$i^L = -i, \quad j^L = -j, \quad k^L = -k.$$

Introduce a multiplication in the 8-dimensional real linear space $Q \times Q$ by the formula

$$(u_1, v_1) \circ (u_2, v_2) = (u_1 u_2 + \gamma_3 v_2^L v_1, v_2 u_1 + v_1 u_2^L)$$

where $\gamma_3 = \pm 1$. Inducing an anti-involution $(u, v)^L = (u^L, -v)$ of $Q \times Q = CD(\gamma_1, \gamma_2, \gamma_3)$ makes it possible to repeat this *Cayley-Dickson process* to get an algebra $CD(\gamma_1, \gamma_2, \ldots, \gamma_n)$, where $\gamma_i = \pm 1$. In fact, the Cayley-Dickson process could be started with $\mathbb{R}$ to give $CD(\gamma_1)$ and $Q = CD(\gamma_1, \gamma_2)$.

Example. $CD(-1) \simeq \mathbb{C}$, $CD(-1,-1) \simeq \mathbb{H}$, and $CD(-1,-1,-1) \simeq \mathbb{O}$, the real 8-dimensional alternative division algebra of octonions (Porteous 1969, 1981). ▮

The algebras $CD(\gamma_1, \gamma_2, \ldots, \gamma_n)$ obtained by the Cayley-Dickson process are simple flexible algebras of dimension 2^n (Schafer 1954). Every element of such an algebra satisfies a quadratic equation with real coefficients.

21.6 Binary representation of the Cayley-Dickson process

The algebras formed by the Cayley-Dickson process are Clifford-like algebras. For instance, choose a basis of $CD(\gamma_1) = \mathbb{R} \times \mathbb{R}$,

$$\mathbf{e}_0 = (1, 0), \quad \mathbf{e}_1 = (0, 1),$$

and introduce the multiplication rule

$$\mathbf{e}_{\underline{a}} \mathbf{e}_{\underline{b}} = \gamma_1^{a_1 b_1} \mathbf{e}_{\underline{a} \oplus \underline{b}} \quad (\underline{a} = a_1, \underline{b} = b_1).$$

The involution is given by

$$e_0^L = (1,0) = e_0, \quad e_1^L = (0,-1) = -e_1$$

or in a condensed form $e_{\underline{a}}^L = (-1)^{a_1} e_{\underline{a}}$.

Theorem 2. A Clifford-like algebra A, $\dim A = 2^n$, with multiplication rule

$$\begin{aligned} &e_{\underline{a}} e_{\underline{b}} = f(\underline{a}, \underline{b}) e_{\underline{a} \oplus \underline{b}} \\ &f(\underline{a}, \underline{b}) = (-1)^{\sum_{i=1}^{n-1} ((S_i(\underline{a}) + S_i(\underline{b}) + S_i(\underline{a}\oplus\underline{b})) b_{i+1} + S_i(\underline{b}) a_{i+1})} \times \prod_{i=1}^{n} \gamma_i^{a_i b_i}, \end{aligned}$$

where $S_i(\underline{a})$ is the maximum of a_j for $1 \le j \le i$, is isomorphic to the Cayley-Dickson algebra $CD(\gamma_1, \gamma_2, \ldots, \gamma_n)$. The anti-involution is

$$e_{\underline{a}}^L = (-1)^{S_n(\underline{a})} e_{\underline{a}}.$$

Proof. The first case of the mathematical induction is proved in the example above.

Assume that the statement holds up to the n:th step, and apply the Cayley-Dickson process. If the new basis elements are denoted by

$$e_{a_1 a_2 \ldots a_n a_{n+1}} = \begin{cases} (e_{\underline{a}}, 0), & a_{n+1} = 0 \\ (0, e_{\underline{a}}), & a_{n+1} = 1 \end{cases}$$

or $e_{\underline{a} a_{n+1}} = e_{a_1 a_2 \ldots a_n a_{n+1}}$ for short, then

$$\begin{aligned} e_{\underline{a}0} e_{\underline{b}0} &= (e_{\underline{a}}, 0)(e_{\underline{b}}, 0) = (e_{\underline{a}} e_{\underline{b}}, 0) = f(\underline{a}, \underline{b})(e_{\underline{a}\oplus\underline{b}}, 0) = f(\underline{a}, \underline{b}) e_{\underline{a}\oplus\underline{b}0} \\ e_{\underline{a}1} e_{\underline{b}0} &= (0, e_{\underline{a}})(e_{\underline{b}}, 0) = (0, e_{\underline{a}} e_{\underline{b}}^L) = (-1)^{S_n(\underline{b})} f(\underline{a}, \underline{b}) e_{\underline{a}\oplus\underline{b}1} \\ e_{\underline{a}0} e_{\underline{b}1} &= (e_{\underline{a}}, 0)(0, e_{\underline{b}}) = (0, e_{\underline{b}} e_{\underline{a}}) = f(\underline{b}, \underline{a}) e_{\underline{a}\oplus\underline{b}1} \\ e_{\underline{a}1} e_{\underline{b}1} &= (0, e_{\underline{a}})(0, e_{\underline{b}}) = (\gamma_{n+1} e_{\underline{b}}^L e_{\underline{a}}, 0) = \gamma_{n+1} (-1)^{S_n(\underline{b})} f(\underline{b}, \underline{a}) e_{\underline{a}\oplus\underline{b}0}. \end{aligned}$$

These four equations can be condensed into one equation

$$\begin{aligned} &e_{\underline{a} a_{n+1}} e_{\underline{b} b_{n+1}} \\ &\quad = f(\underline{a}, \underline{b})^{1-b_{n+1}} f(\underline{b}, \underline{a})^{b_{n+1}} \times \gamma_{n+1}^{a_{n+1} b_{n+1}} (-1)^{a_{n+1} S_n(\underline{b})} e_{\underline{a}\oplus\underline{b}(a_{n+1}\oplus b_{n+1})}, \end{aligned}$$

where

$$f(\underline{b}, \underline{a}) = f(\underline{a}, \underline{b})(-1)^{S_n(\underline{a}) + S_n(\underline{b}) + S_n(\underline{a}\oplus\underline{b})},$$

which is a consequence of $(e_{\underline{a}} e_{\underline{b}})^L = e_{\underline{b}}^L e_{\underline{a}}^L$. Thus we have proved the desired multiplication rule in the case $n+1$. The induced anti-involution is also of the assumed type:

$$\begin{aligned} e_{\underline{a}0}^L &= (e_{\underline{a}}^L, 0) = (-1)^{S_n(\underline{a})} e_{\underline{a}0} \\ e_{\underline{a}1}^L &= (0, -e_{\underline{a}}) = -e_{\underline{a}1} \end{aligned}$$

or in a condensed form

$$\mathrm{e}^L_{\underline{a}a_{n+1}} = (-1)^{\max\,(S_n(\underline{a}),a_{n+1})}\mathrm{e}_{\underline{a}a_{n+1}}. \qquad \blacksquare$$

The algebra $CD(\gamma_1, \gamma_2, \ldots, \gamma_n)$ is generated by an n-dimensional vector space, whose elements

$$x_1\mathrm{e}_{100\ldots00} + x_2\mathrm{e}_{010\ldots00} + \ldots + x_n\mathrm{e}_{000\ldots01}$$

have squares $(\gamma_1 x_1^2 + \gamma_2 x_2^2 + \ldots + \gamma_n x_n^2)\mathrm{e}_{\underline{0}}$. In contrast to the Clifford algebras, different orderings of the parameters γ_i in $CD(\gamma_1, \gamma_2, \ldots, \gamma_n)$ may result in non-isomorphic algebras in the case where $n > 3$.

Another construction relating Clifford algebras and Cayley-Dickson algebras is found in Wene 1984.

For more details of the algebraic extensions of the group of binary n-tuples Ω see Hagmark 1980.

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