Final Report - Research Internship

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1 Introduction

2 Modelling of the swimmer as a control problem

We restrict ourselves to considering the swimmer SPr4 proposed in [1]. Let (S_1, S_2, S_3, S_4) be a regular reference tetrahedron centered at $c \in \mathbb{R}^3$ such that $\operatorname{dist}(c, S_i) = 1$ for all $i \in \mathbb{N}_4$. Then the swimmer consists of four balls $(B_i)_{i\in\mathbb{N}_4}$ of $\mathbb{R}^3$ centered at $b_i \in \mathbb{R}^3$, all of radius $a > 0$, such that the ball B_i can move along the ray starting at c and passing through S_i. This reflects the situation where the balls are linked together by think jacks that are able to elongate. However, the viscous resistance of these jacks is neglected and therefore the fluid is assumed to permeate the entire open set $\mathbb{R}^3 \setminus \bigcup_{i=1}^4 \overline{B}_i$. The balls do not rotate around their arms which implies that the shape of the swimmer is completely determined by the four lengths $\xi_1, \xi_2, \xi_3, \xi_4$ of its arms, measured from the c to the center of each ball b_i. However, there are no restrictions for the rotation of the swimmer around the center c, i.e. for fixed arm lengths, the swimmer is considered to be a rigid body in a Stokesian fluid. Hence, the geometrical configuration of the swimmer can be described by two sets of variables:

(i) The vector of *shape variables* $\xi := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{M} := (\sqrt{\frac{3}{2}}a, +\infty)^4 \subseteq \mathbb{R}^4_+$, where the lower bound in the open intervals is chosen such that the balls cannot overlap.

(ii) The vector of *position variables* $p = (c, R) \in \mathcal{P} := \mathbb{R}^3 \times \mathrm{SO}(3)$.

To be more precise, we consider the reference tetrahedron convexly spanned by the four unit vectors $z_1 := (2\sqrt{2}/3, 0, -1/3)$, $z_2 := (-\sqrt{2}/3, -\sqrt{2/3}, -1/3)$, $z_3 := (-\sqrt{2}/3, \sqrt{2/3}, -1/3)$ and $z_4 := (0, 0, 1)$. Position and orientation in $\mathbb{R}^3$ are then described by the coordinates of the center $c \in \mathbb{R}^3$ and the rotation $R \in SO(3)$ of the swimmer with respect to the reference orientation induced by the reference tetrahedron. Thus, we set $b_i := c + \xi_i R z_i$ for the center of the ball B_i.

The swimmer is completely described by the parameters $(\xi, p) \in \mathcal{M} \times \mathcal{P}$. Indeed, if we denote by B_a the ball in $\mathbb{R}^3$ of radius a centered at the origin, then for any $r \in \partial B_a$, the position of the current point on the i-th sphere of the swimmer in the state (ξ, p) is given, for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$, by the function

$$r_i(\xi, p, r) := c + R(\xi_i z_i + r). \tag{1}$$

Note that the functions $(r_i)_{i\in\mathbb{N}_4}$ are analytic in $\mathcal{M} \times \mathcal{P}$ and thus we can use them to calculate the instantaneous velocity on the i-th sphere B_i, which for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$ and every $i \in \mathbb{N}_4$ is given by

$$u_i(\xi, p, r) = \dot{c} + \omega \times (\xi z_i + r) + R z_i \dot{\xi}_i, \tag{2}$$

where ω is the axial vector associated with the skew matrix $\dot{R}R$.

In [1] it is shown that the system SPr4 , i.e. both the shape ξ and the position p, is controllable only using the rate of change $\dot{\xi}$ of the shape. To do so, we have to understand how p changes when we vary $\dot{\xi}$. To that and, the assumptions of *self-propulsion* and negligible inertia of the swimmer are made. They imply that the total viscous force and torque exerted by the surrounding fluid on the swimmer must vanish. More precisely, the system can be written as

$$\dot{p} = F(R, \xi)\dot{\xi} := \begin{pmatrix} F_c(R, \xi) \\ \overline{F_\theta(R, \xi)} \end{pmatrix} \dot{\xi}, \tag{3}$$

where $\dot{c} = F_c(R,\xi)\dot{\xi}$ and $\dot{R} = F_\theta(R,\xi)\dot{\xi}$.

In preparation for what follows, let us note that we have $F(R,\xi) \in \mathcal{L}(\mathbb{R}^4, T_p\mathcal{P})$ for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$, where $\mathcal{L}(V,W)$ denotes the linear maps between two vector spaces V and W. We quickly recall the fact that at any point $R \in \mathrm{SO}(3)$ we have

$$T_R\,\mathrm{SO}(3) = R^*\,\mathrm{Skew}_3(\mathbb{R}) = \{RM \mid M \in \mathrm{Skew}_3(\mathbb{R})\}, \tag{4}$$

where $\mathrm{Skew}_n(\mathbb{R})$ denotes the set of skew-symmetric real matrices of size $n \times n$. Hence, we have in particular that for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$

$$F_c(R,\xi) \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3) \text{ and } F_\theta(R,\xi) \in \mathcal{L}(\mathbb{R}^4, R^*\,\mathrm{Skew}_3(\mathbb{R})) \tag{5}$$

and therefore we can express both $F_c(R,\xi)$ and $F_\theta(R,\xi)$ as real matrices of size 3×4 once we have chosen a basis for the tangent corresponding spaces.

In analogy to [2], it is important to note here that the control system F is independent of c due to the translational invariance of the Stokes equations. However, the translational invariance is not the only symmetry that SPr4 satisfies. The goal of the following section is to examine the structure of the control system F in consequence of the symmetries it must satisfy being driven by the Stokes equations

3 Symmetries

For any initial condition $p_0 = (c_0, R_0) \in \mathcal{P}$ and any control curve $\xi : I \subseteq \mathbb{R} \to \mathcal{M}$, with I a neighborhood of zero, we denote $\gamma(p_0,\xi) : I \to \mathcal{P}$ the solution associated to the dynamical system

$$\dot{p} = F(R,\xi)\dot{\xi}, \quad p(0) := p_0, \tag{6}$$

as well as $\gamma_c(c_0,R_0,\xi)$ and $\gamma_\theta(c_0,R_0,\xi)$ its projections on $\mathbb{R}^3$ and $\mathrm{SO}(3)$, respectively, such that for any $t \in I$

$$\dot{\gamma}(c_0,R_0,\xi)(t) = F(\gamma_\theta(c_0,R_0,\xi)(t), \xi(t))\dot{\xi}(t), \tag{7}$$

and similarly for the projections $\gamma_c(c_0,R_0,\xi)$ and $\gamma_\theta(c_0,R_0,\xi)$.

3.1 Rotational invariance

Rotational invariance of the Stokes equations expresses the fact that the solution of the dynamical system (6) is invariant under rotations, i.e. that for any rotation $R \in \mathrm{SO}(3)$ we have for the spatial part of the solution

$$\gamma_c(c_0, RR_0, \xi)(t) = R\gamma_c(c_0,R_0,\xi)(t) + (I - R)c_0 \tag{8}$$

and for the angular part of the solution

$$\gamma_\theta(c_0, RR_0, \xi)(t) = R\gamma_\theta(c_0,R_0,\xi)(t) \tag{9}$$

at any point in time $t \in I$. Eventually, we can rigorously state the following symmetry property of the control system (6) with respect to rotations:

Condition 1 (Rotational invariance)**.** *If $\gamma(c_0,R_0,\xi)$ is a solution of the control system (6) then so is $\gamma(c_0,RR_0,\xi)$ and (8) and (9) hold.*

Remark. To follow the reasoning of [2], the symmetry relations satisfied by SPr4 are stated as hypotheses on the solution γ. In so doing, the results work for any control system of the form (3) and satisfying the hypotheses we state, independently of these hypotheses being guaranteed by the invariance of the Stokes equations under a certain group of transformations.

We then have

Proposition 1. *Let $\xi_0 := \xi(0) \in \mathcal{M}$ denote the initial state of the control parameters and by $T_\xi \mathcal{M}$ the tangent space of $\mathcal{M}$ at ξ. If the control system (6) is invariant under rotations and for every $\xi \in \mathcal{M}$ it holds that $T_\xi \mathcal{M} \simeq \mathbb{R}^4$, then*

$$F_c(R,\xi) = RF_c(\xi) \text{ and } F_\theta(R,\xi) = RF_\theta(R,\xi), \tag{10}$$

for every $(R,\xi) \in \mathrm{SO}(3) \times \mathcal{M}$, where $F_c(\xi) := F_c(I,\xi)$ and $F_\theta(\xi) := F_\theta(I,\xi)$.

Proof. On one hand, we have by definition of the dynamical system (6) that

$$\dot{\gamma}_c(c_0,R,\xi) = F_c(\gamma_\theta(c_0,R,\xi),\xi)\dot{\xi}. \tag{11}$$

On the other hand, using equation (8) and once more the definition of the dynamical system (3), we obtain

$$\dot{\gamma}_c(c_0,R\xi) = R\dot{\gamma}_c(c_0,I,\xi) = RF_c(\gamma_\theta(c_0,I,\xi),\xi)\dot{\xi}. \tag{12}$$

Therefore, $F_c(\gamma_\theta(c_0,R,\xi),\xi)\dot{\xi} = RF_c(\gamma_\theta(c_0,I,\xi),\xi)\dot{\xi}$ for every $R \in \mathrm{SO}(3)$. Since $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$, evaluation of the preceding expression at $t = 0$ yields $F_c(R,\xi) = RF_c(I,\xi)$, as desired. The proof for F_θ is completely analogous. □

3.2 Permutation of two arms

In this section, we investigate the effect of a swap of two arms on the generic solution of the dynamical system (6). To that end, let $P_{ij} \in M_{4\times4}(\mathbb{R})$ denote the permutation matrix that interchanges the i-th and j-th index, which corresponds to the swap of the arms i and j, denoted by $(||i \leftrightsquigarrow ||j)$, if applied to the shape space $\mathcal{M}$. In addition, let S_{ij} denote the reflection of $\mathbb{R}^3$ sending arm $||i$ onto arm $||j$ in the reference orientation I. Geometrical inspection of the reference tetrahedron shows that S_{ij} is always a reflection at a plane containing the remaining arms $||k$ and $||l$.

Before we formulate the symmetry conditions for the interchanging of two arms, we recall some results about how rotations behave under reflections. So far, we have only regarded the orientation of SPr4 as a rotation matrix in SO(3). However, by Euler's rotation theorem to every such rotation matrix $R \in \mathrm{SO}(3)$ there exists a corresponding rotation vector $\omega \in \mathbb{R}^3$ which is collinear to the unique axis of rotation defined by R, i.e. ω is an eigenvector associated to the eigenvalue 1 of R. It's length is then given by the angle of rotation around this axis. The rotation vector ω is then directly related to the rotation matrix R via the map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, where $\mathfrak{so}(3) = T_I\,\mathrm{SO}(3) = \mathrm{Skew}_3(\mathbb{R})$ denotes the Lie algebra over SO(3), which we will illustrate in the following paragraphs.

It is clear that $\dim \mathrm{Skew}_3(\mathbb{R}) = 3$. In particular, if $R_1(\theta), R_2(\theta)$ and $R_3(\theta)$ denote the simple rotations around the $\hat{e}_1$-, $\hat{e}_2$- and $\hat{e}_3$-axis, where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the canonical basis vectors of $\mathbb{R}^3$, then the matrices

$$L_1 = \frac{d}{d\theta} R_1(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{13}$$

$$L_2 = \frac{d}{d\theta} R_2(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{14}$$

$$L_3 = \frac{d}{d\theta} R_3(\theta)_{|\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

form a basis of $\mathfrak{so}(3)$ consisting of the infinitesimal rotations around the corresponding axes. If we now write $\mathbf{L} := (L_1, L_2, L_3)^T$ and allow the slight abuse of notation

$$\omega \cdot \mathbf{L} = \omega_1 L_1 + \omega_2 L_2 + \omega_3 L_3, \tag{16}$$

we find by direct computation that $\exp(\omega \cdot \mathbf{L}) = R$. This relationship allows us to formulate the behavior of the orientation of SPr4 under reflection and thus under permutation of two arms as we shall see later. Indeed, we have

Lemma 1. *For any orientation of a rigid body characterized by $R \in \mathrm{SO}(3)$, the orientation of its mirror image under a reflection S is characterized by*

$$\tilde{R} = SRS. \tag{17}$$

Proof. Let us first consider the simple case where S is the reflection of the $\hat{e}_1$-axis. Let ω and $\tilde{\omega}$ be the rotation vectors corresponding to R and $\tilde{R}$, respectively. They are related by $\tilde{\omega} = -S\omega$, where the gain of the minus sign stems from the fact that rotation vectors are in fact pseudovectors. In other words, we not only reflect the axis of rotation but we also reverse the sense of rotation around the axis. It follows then from direct computation that

$$\tilde{\omega} \cdot \mathbf{L} = (-S\omega) \cdot \mathbf{L} = S(\omega \cdot \mathbf{L})S \tag{18}$$

and thus we have

$$\tilde{R} = \exp(\tilde{\omega} \cdot \mathbf{L}) = \exp(S(\omega \cdot \mathbf{L})S) = SRS, \tag{19}$$

as $S^{-1} = S$.

If now S' is now an arbitrary reflection, we always find a rotation $Q \in \mathrm{SO}(3)$ such that $S' = QSQ^T$. Moreover, for any rotation $R' \in \mathrm{SO}(3)$ we find another $R \in \mathrm{SO}(3)$ such that $R' = QRQ^T$. In particular, we have

$$\tilde{R}' = Q\tilde{R}Q^T = QSRSQ^T = S'R'^T S', \tag{20}$$

as desired. □

With this lemma at hand, we can now finally state the following

Condition 2 (Swap $(||i \leftrightsquigarrow ||j)$)**.** *Let the initial position be $p_0 := (c_0, I)$. If $\gamma(c_0, I, P_{ij}\xi)$ is a solution of the control system (3), then so is $\gamma(S_{ij}c_0, I, \xi)$ and the following relations hold*

$$\gamma_c(c_0, I, P_{ij}\xi) = S_{ij}\gamma_c(S_{ij}c_0, I, \xi) \tag{21}$$

and

$$\gamma_\theta(c_0, I, P_{ij}\xi) = S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{22}$$

To avoid chaos in our notation, we treat the the spatial and angular parts now separately. For the spatial part, we find

Proposition 2. *If the control system (3) is invariant under the swap $(||i \leftrightsquigarrow ||j)$ and $T_\xi\mathcal{M} \simeq \mathbb{R}^4$ for all $\xi \in \mathcal{M}$, then for all $\xi \in \mathcal{M}$*

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij}. \tag{23}$$

Proof. Let $\gamma_c(c_0, R_0, P_{ij}\xi)$ be the spatial part of any solution of the control problem (3). The hypothesis of rotational invariance, i.e. (8), implies that

$$\gamma(c_0, R_0, P_{ij}\xi) = R_0^T\gamma(c_0, I, P_{ij}\xi) - (R_0^T - I)c_0. \tag{24}$$

□

4 The control problem in the regime of small strokes

5 Energy optimizing strokes

References

[1] F. Alouges, A. DeSimone, L. Heltai, A. Lefebvre-Lepot, and B. M. and, "Optimally swimming stokesian robots," *Discrete & Continuous Dynamical Systems - B*, vol. 18, no. 5, pp. 1189–1215, 2013. DOI: `10.3934/dcdsb.2013.18.1189`.

[2] F. Alouges and G. D. Fratta, "Parking 3-sphere swimmer. i. energy minimizing strokes," Sep. 2017. DOI: `10.31219/osf.io/7sfbj`.