11

Fierz Identities and Boomerangs

Fierz identities are quadratic relations between the bilinear covariants (or physical observables) of a Dirac spinor. They are used to recover the original Dirac spinor from its bilinear covariants, up to a phase. The Fierz identities are sufficient to examine the non-null case, when either $\psi^\dagger\gamma_0\psi \neq 0$ or $\psi^\dagger\gamma_0\gamma_{0123}\psi \neq 0$. However, they are insufficient for the null case when both $\psi^\dagger\gamma_0\psi = 0$ and $\psi^\dagger\gamma_0\gamma_{0123}\psi = 0$. In this chapter, we introduce a new object called the **boomerang**, which enables us to study also the null case.

11.1 Fierz identities

The bilinear covariants satisfy certain quadratic relations called **Fierz identities** [see Holland 1986 p. 276 (2.8)]

$$\mathbf{J}^2 = \Omega_1^2 + \Omega_2^2, \qquad \mathbf{K}^2 = -\mathbf{J}^2$$
$$\mathbf{J}\cdot\mathbf{K} = 0, \qquad \mathbf{J}\wedge\mathbf{K} = -(\Omega_2 + \Omega_1\gamma_{0123})\mathbf{S}.$$

In coordinate form the Fierz identities are as follows [see Crawford 1985 p. 1439 (1.2)]

$$J_\mu J^\mu = \Omega_1^2 + \Omega_2^2, \qquad J_\mu J^\mu = -K_\mu K^\mu$$
$$J_\mu K^\mu = 0, \qquad J_\mu K_\nu - K_\mu J_\nu = -\Omega_2 S_{\mu\nu} + \Omega_1(\star S)_{\mu\nu}$$

where $(\star S)_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta}$ (with $\varepsilon_{0123} = 1$) or $\star\mathbf{S} = \tilde{\mathbf{S}}\gamma_{0123}$ [in general, $\star v = \tilde{v}\gamma_{0123}$ given by $u\wedge\star v = <u,v>\gamma_0\wedge\gamma_1\wedge\gamma_2\wedge\gamma_3$].

In the non-null case $\Omega \neq 0$ the Fierz identities result in [Crawford 1985 p. 1439 (1.3) and 1986 p. 356 (2.14)]

$$\mathbf{S}\,\llcorner\,\mathbf{J} = \Omega_2\mathbf{K}, \qquad \mathbf{S}\,\llcorner\,\mathbf{K} = \Omega_2\mathbf{J}$$
$$(\gamma_{0123}\mathbf{S})\,\llcorner\,\mathbf{J} = \Omega_1\mathbf{K}, \qquad (\gamma_{0123}\mathbf{S})\,\llcorner\,\mathbf{K} = \Omega_1\mathbf{J}$$

$$\mathbf{S} \,\lrcorner\, \mathbf{S} = \Omega_2^2 - \Omega_1^2, \qquad (\gamma_{0123}\mathbf{S}) \,\lrcorner\, \mathbf{S} = 2\Omega_1\Omega_2$$

and

$$\begin{aligned}
&\mathbf{JS} = -(\Omega_2 + \Omega_1\gamma_{0123})\mathbf{K}, \quad \mathbf{KS} = -(\Omega_2 + \Omega_1\gamma_{0123})\mathbf{J} \\
&\mathbf{SJ} = (\Omega_2 - \Omega_1\gamma_{0123})\mathbf{K}, \qquad \mathbf{SK} = (\Omega_2 - \Omega_1\gamma_{0123})\mathbf{J} \\
&\mathbf{S}^2 = (\Omega_2 - \Omega_1\gamma_{0123})^2 = \Omega_2^2 - \Omega_1^2 - 2\Omega_1\Omega_2\gamma_{0123} \\
&\mathbf{S}^{-1} = -\mathbf{S}(\Omega_1 - \Omega_2\gamma_{0123})^2/(\Omega_1^2 + \Omega_2^2)^2 = \mathbf{KSK}/(\Omega_1^2 + \Omega_2^2)^2.
\end{aligned}$$

In the index-notation some of these identities look like

$$\begin{aligned}
&J_\mu S^{\mu\nu} = -\Omega_2 K^\nu, \qquad J_\mu(\star S)^{\mu\nu} = \Omega_1 K^\nu \\
&\mathbf{S} \,\lrcorner\, \mathbf{S} = -\tfrac{1}{2} S_{\mu\nu} S^{\mu\nu} = \Omega_2^2 - \Omega_1^2 \\
&(\star\mathbf{S}) \,\lrcorner\, \mathbf{S} = -\tfrac{1}{2}(\star S)_{\mu\nu} S^{\mu\nu} = \tfrac{1}{4}\varepsilon_{\mu\nu\alpha\beta} S^{\mu\nu} S^{\alpha\beta} = -2\Omega_1\Omega_2.
\end{aligned}$$

Note also that in general $\mathbf{S} \,\llcorner\, \mathbf{K} = -\mathbf{K} \,\lrcorner\, \mathbf{S}$, $\gamma_{0123}(\mathbf{S} \wedge \mathbf{K}) = (\gamma_{0123}\mathbf{S}) \,\llcorner\, \mathbf{K} = -(\star\mathbf{S}) \,\llcorner\, \mathbf{K} = \mathbf{K} \,\lrcorner\, (\star\mathbf{S})$, $\gamma_{0123}(\mathbf{S} \wedge \mathbf{S}) = (\gamma_{0123}\mathbf{S}) \,\lrcorner\, \mathbf{S} = -(\star\mathbf{S}) \,\lrcorner\, \mathbf{S}$ and that $(\mathbf{J} \,\lrcorner\, \mathbf{S}) \wedge \mathbf{S} = \frac{1}{2}\mathbf{J} \,\lrcorner\, (\mathbf{S} \wedge \mathbf{S})$.

Fierz identities via spinor operators. By direct computation we can see that

$$\begin{aligned}
\mathbf{J}^2 &= (\Psi\gamma_0\tilde{\Psi})(\Psi\gamma_0\tilde{\Psi}) = \Psi\gamma_0\Psi\tilde{\Psi}\gamma_0\tilde{\Psi} = \Psi\gamma_0(\Omega_1 + \Omega_2\gamma_{0123})\gamma_0\tilde{\Psi} \\
&= \Psi(\Omega_1 - \Omega_2\gamma_{0123})\gamma_0\gamma_0\tilde{\Psi} = (\Omega_1 - \Omega_2\gamma_{0123})\Psi\tilde{\Psi} \\
&= (\Omega_1 - \Omega_2\gamma_{0123})(\Omega_1 + \Omega_2\gamma_{0123}) = \Omega_1^2 + \Omega_2^2
\end{aligned}$$

which gives one of the Fierz identities. Computing in a similar manner we find

$$\begin{aligned}
\mathbf{JK} &= (\Psi\gamma_0\tilde{\Psi})(\Psi\gamma_3\tilde{\Psi}) = \Psi\gamma_0\Psi\tilde{\Psi}\gamma_3\tilde{\Psi} = \Psi\gamma_0(\Omega_1 + \Omega_2\gamma_{0123})\gamma_3\tilde{\Psi} \\
&= \Psi(\Omega_1 - \Omega_2\gamma_{0123})\gamma_0\gamma_3\tilde{\Psi} = -(\Omega_1 - \Omega_2\gamma_{0123})\Psi\gamma_{0123}\gamma_{12}\tilde{\Psi} \\
&= -(\Omega_1 - \Omega_2\gamma_{0123})\gamma_{0123}\Psi\gamma_{12}\tilde{\Psi} = -(\Omega_2 + \Omega_1\gamma_{0123})\mathbf{S}.
\end{aligned}$$

Since the result is a bivector, we find that $\mathbf{J} \wedge \mathbf{K} = -(\Omega_2 + \Omega_1\gamma_{0123})\mathbf{S}$ and $\mathbf{J} \cdot \mathbf{K} = 0$. ∎

Exercise 1

11.2 Recovering a spinor from its bilinear covariants

Let the spinor ψ have bilinear covariants Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 [a scalar, a vector, a bivector, a vector, a scalar]. Take an arbitrary spinor η such that $\tilde{\eta}^*\psi \neq 0$

in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ or equivalently $\eta^\dagger\gamma_0\psi\neq 0$ in $\mathrm{Mat}(4,\mathbb{C})$. Then the spinor ψ is proportional to

$$\psi\simeq Z\eta \quad \text{where} \quad \boxed{Z=\Omega_1+\mathbf{J}+i\mathbf{S}+i\mathbf{K}\gamma_{0123}+\Omega_2\gamma_{0123}}$$

that is, ψ and $Z\eta$ differ only by a complex factor. The original spinor ψ can be recovered by the algorithm [see Takahashi 1983, and Crawford 1985, who also gave a proof in the non-null case $\Omega\neq 0$]

$$N=\sqrt{\langle\tilde{\eta}^* Z\eta\rangle_0}=\frac{1}{2}\sqrt{\eta^\dagger\gamma_0 Z\eta}$$
$$e^{-i\alpha}=\frac{4}{N}\langle\tilde{\eta}^*\psi\rangle_0=\frac{1}{N}\eta^\dagger\gamma_0\psi$$
$$\psi=\frac{1}{4N}e^{-i\alpha}Z\eta.$$

[For the choice $\eta=f$ we get simply

$$N=\sqrt{\langle Zf\rangle_0}=\frac{1}{2}\sqrt{\Omega_1+\mathbf{J}\cdot\gamma_0-\mathbf{S}\,\lrcorner\,\gamma_{12}-\mathbf{K}\cdot\gamma_3}$$
$$e^{-i\alpha}=\frac{\psi_1}{|\psi_1|},$$

which are not the same N, $e^{-i\alpha}$ as those for an arbitrary η.] Once the spinor ψ has been recovered, we may also write

$$N=4|\langle\tilde{\eta}^*\psi\rangle_0|=|\eta^\dagger\gamma_0\psi|$$
$$e^{-i\alpha}=\frac{\langle\tilde{\eta}^*\psi\rangle_0}{|\langle\tilde{\eta}^*\psi\rangle_0|}=\frac{\eta^\dagger\gamma_0\psi}{|\eta^\dagger\gamma_0\psi|}.$$

A spinor ψ is determined by its bilinear covariants Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 up to a phase-factor $e^{-i\alpha}$, and

$$Z=\Omega_1+\mathbf{J}+i\mathbf{S}+i\mathbf{K}\gamma_{0123}+\Omega_2\gamma_{0123}$$

projects/extracts out of η the relevant part parallel to ψ.

Recovery via mother spinors $\Phi\in\mathcal{C}\ell_{1,3}\,\frac{1}{2}(1+\gamma_0)$. Take two arbitrary elements in the real Clifford algebra, $a,b\in\mathcal{C}\ell_{1,3}$, in such a way that $\psi=(a+ib)f$, $f=\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+\gamma_{12})$. Then $\psi\tilde{\psi}^*=0$ and

$$\begin{aligned}\psi\bar{\psi}^* &= (a+ib)f(\tilde{a}-i\tilde{b})=af\tilde{a}+bf\tilde{b}+i(bf\tilde{a}-af\tilde{b})\\ &=\tfrac{1}{2}(ag\tilde{a}+bg\tilde{b}-bg\gamma_{12}\tilde{a}+ag\gamma_{12}\tilde{b}+i(ag\gamma_{12}\tilde{a}+bg\gamma_{12}\tilde{b}+bg\tilde{a}-ag\tilde{b}))\end{aligned}$$

where we have written $g=\frac{1}{2}(1+\gamma_0)$. Next, we introduce a real spinor, called the *mother spinor* [for all real spinors]

$$\Phi=(a-b\gamma_{12})(1+\gamma_0)\in\mathcal{C}\ell_{1,3}\,\frac{1}{2}(1+\gamma_0).$$

Compute $\Phi\bar{\Phi} = 0$ and

$$\Phi\tilde{\Phi} = 4(a - b\gamma_{12})g(\tilde{a} + \gamma_{12}\tilde{b}) = 4(ag\tilde{a} + ag\gamma_{12}\tilde{b} - bg\gamma_{12}\tilde{a} + bg\tilde{b})$$

to find

$$\frac{1}{2}\Phi\tilde{\Phi} = 4\,\mathrm{Re}(\psi\tilde{\psi}^*), \quad \text{and similarly} \quad \frac{1}{2}\Phi\gamma_{12}\tilde{\Phi} = 4\,\mathrm{Im}(\psi\tilde{\psi}^*).$$

Recall that $Z = 4\psi\tilde{\psi}^*$ is sufficient to reconstruct the original Dirac spinor ψ and conclude that the real mother spinor $\Phi \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1+\gamma_0)$ carries all the physically relevant information of the Dirac spinor ψ. In fact,

$$\psi = \Phi\,\frac{1}{4}(1 + i\gamma_{12}) \quad \text{and} \quad \Phi = 4\,\mathrm{Re}(\psi)$$

where the real part is taken in the decomposition $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ [and *not* in the decomposition $\mathbb{C}\otimes\mathrm{Mat}(4,\mathbb{R})$].

Write as before $Z = P + iQ$ where $P = \Omega + \mathbf{J}$ and $Q = \mathbf{S} + \mathbf{K}\gamma_{0123}$. We will show how to recover the real mother spinor Φ from its bilinear covariants $[\tilde{g} = g = \frac{1}{2}(1+\gamma_0)]$:

$$N = \sqrt{\tfrac{1}{2}\langle\tilde{g}(P - Q\gamma_{12})g\rangle_0}, \qquad e^{\gamma_{12}\alpha} = \frac{1}{N}((\tilde{g}\phi)\wedge\gamma_{03})\gamma_{03}^{-1},$$

$$\Phi = \frac{1}{2N}(P - Q\gamma_{12})e^{\gamma_{12}\alpha}g$$

[the same N as for the choice $f \in \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$] or for an arbitrary spinor $\eta \in \mathcal{C}\ell_{1,3}g$, $\tilde{\eta}\Phi \neq 0$,

$$N = \sqrt{\tfrac{1}{2}\langle\tilde{\eta}(P\eta - Q\eta\gamma_{12})\rangle_0}, \qquad e^{\gamma_{12}\alpha} = \frac{1}{N}((\tilde{\eta}\Phi)\wedge\gamma_{03})\gamma_{03}^{-1},$$

$$\Phi = \frac{1}{2N}(P\eta - Q\eta\gamma_{12})e^{\gamma_{12}\alpha}g$$

[$\eta' \in (\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$ and $\eta = 2\,\mathrm{Re}(\eta')$ result in the same numerical value for N]. Note that the role of $i = \sqrt{-1}$ is played by multiplication by $\gamma_2\gamma_1$ on the *right* hand side, that is, $\Phi\gamma_2\gamma_1 = 4\,\mathrm{Re}(i\psi)$.

Exercises 2,3,4

11.3 Fierz identities and the recovery of spinors

It might be interesting to know if given multivectors Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 [a scalar, a vector, a bivector, a vector, a scalar] are bilinear covariants for some spinor. The answer is postponed to the next section in the case $\Omega_1 = 0 = \Omega_2$. Writing $\Omega = \Omega_1 + \Omega_2\gamma_{0123}$, we are left with the remaining case $\Omega \neq 0$, in which we can say that the multivectors are bilinear covariants essentially if they satisfy the Fierz identities.

If the multivectors Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 satisfy the Fierz identities, then their aggregate

$$Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$$

can be factored as [see Crawford 1985 p. 1439 (2.2)] [1]

$$Z = (\Omega_1 + \mathbf{J} + \Omega_2\gamma_{0123})(1 + i(\Omega_1 + \Omega_2\gamma_{0123})^{-1}\mathbf{K}\gamma_{0123}).$$

This factorization is valid only in the non-null case $\Omega \neq 0$. Using this factorization Crawford proved that if the multivectors Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 satisfy the Fierz identities [and $J^0 > 0$ with $4\langle\tilde{\eta}^* Z\eta\rangle_0 = \eta^\dagger\gamma_0 Z\eta > 0$ for all non-zero spinors η], then Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 are bilinear covariants for some spinor ψ, for instance,

$$\psi = \frac{1}{4N}Z\eta, \qquad N = \sqrt{\langle\tilde{\eta}^* Z\eta\rangle_0} = \frac{1}{2}\sqrt{\eta^\dagger\gamma_0 Z\eta}$$

[and two such spinors ψ obtained by distinct choices of η differ only in their phases].

Hamilton 1984 p. 1827 (4.2) mentioned how ψ determines $Z = 4\psi\psi^\dagger\gamma_0$, see also Holland 1986 p. 276 (2.9), Keller & Rodríguez-Romo 1990 p. 2502 (2.3b) and Hestenes 1986 p. 334 (2.28).

11.4 Boomerangs

Definition. If the multivectors Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 [a scalar, a vector, a bivector, a vector, a scalar] satisfy the Fierz identities, then their aggregate $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$ is called a *Fierz aggregate*. ∎

Definition. A multivector $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$, which is Dirac self-adjoint $\tilde{Z}^* = Z$, is called a *boomerang*, if its components Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 are bilinear covariants for some spinor $\psi \in \mathbb{C}^4$. ∎

Both in the non-null case $\Omega \neq 0$ and in the null case $\Omega = 0$ a spinor ψ is determined up to a phase-factor by its aggregate of bilinear covariants $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$ [as $\psi = \frac{1}{4N}e^{-i\alpha}Z\eta$], which in turn is determined by the original spinor ψ via the formula $Z = 4\psi\tilde{\psi}^* = 4\psi\psi^\dagger\gamma_0$ [thus we have a boomerang, which comes back].

If Z is a boomerang so that $Z = 4\psi\psi^\dagger\gamma_0$ then $Z^2 = 4\Omega_1 Z$ where $\Omega_1 = \langle Z\rangle_0$, because

1 In Crawford's factorization $Z = (\Omega + \mathbf{J})(1 + i\Omega^{-1}\mathbf{K}\gamma_{0123})$ the first factor $P = \Omega + \mathbf{J}$ is Dirac self-adjoint, $\tilde{P}^* = P$. Writing $\Gamma = 1 + i\mathbf{K}\Omega^{-1}\gamma_{0123}$, we can write Crawford's factorization as $Z = P\tilde{\Gamma}^*$, and note that $P\tilde{\Gamma}^* = \Gamma P \neq P\Gamma$. Crawford 1985 posed an open problem of decomposing Z into a product of two commuting Dirac self-adjoint factors. This problem is solved at the end of this chapter.

$$\begin{aligned} Z^2 &= 4\psi\psi^\dagger\gamma_0\, 4\psi\psi^\dagger\gamma_0 = 16\psi(\psi^\dagger\gamma_0\psi)\psi^\dagger\gamma_0 \\ &= 16\,\mathrm{trace}(\psi^\dagger\gamma_0\psi)\psi\psi^\dagger\gamma_0 \qquad [\text{since } \psi^\dagger\gamma_0\psi = \mathrm{trace}(\psi^\dagger\gamma_0\psi)f] \\ &= 16\,\mathrm{trace}(\psi\psi^\dagger\gamma_0)\psi\psi^\dagger\gamma_0 = \mathrm{trace}(4\psi\psi^\dagger\gamma_0)\, 4\psi\psi^\dagger\gamma_0. \end{aligned}$$

Conversely if $\Omega_1 \neq 0$ then $Z^2 = 4\Omega_1 Z$ ensures a boomeranging Z. If Z is a Fierz aggregate and $\Omega \neq 0$, then it boomerangs back to Z. Crawford's results say that in the non-null case $\Omega \neq 0$ we have a boomeranging Z if and only if Z is a Fierz aggregate. However, in the null case $\Omega = 0$, there are such Z which are Fierz aggregates but still do not boomerang [for instance $Z = \mathbf{J}$, $\mathbf{J}^2 = 0$, $\mathbf{J} \neq 0$].

If $\Omega = 0$ and $\mathbf{J}, \mathbf{S}, \mathbf{K}$ satisfy the Fierz identities, then for a spinor constructed by

$$\psi = \frac{1}{4N} Z\eta \quad \text{where} \quad Z = \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123}$$

we have in general $Z \neq 4\psi\tilde{\psi}^*$ (the Fierz identities are reduced to $\mathbf{J}^2 = \mathbf{K}^2 = 0$, $\mathbf{J}\cdot\mathbf{K} = \mathbf{J}\wedge\mathbf{K} = 0$ which impose no restriction on $\mathbf{S}$). Even if the Fierz identities were supplemented by all the conditions presented in section 12.2 (in the non-null case these conditions are consequences of the Fierz identities) these extended identities would not result in a boomeranging Z. To handle also the null case $\Omega = 0$ we could replace the Fierz identities by the more restrictive conditions

$$\begin{aligned} &Z^2 = 4\Omega_1 Z, \qquad Z\gamma_\mu Z = 4J_\mu Z, \qquad Zi\gamma_{\mu\nu}Z = 4S_{\mu\nu}Z, \\ &Zi\gamma_{0123}\gamma_\mu Z = 4K_\mu Z, \qquad Z\gamma_{0123}Z = -4\Omega_2 Z \end{aligned}$$

[see Crawford 1986 p. 357 (2.16)], but this would result in a tedious checking process. If $Z = \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123}$ is a boomerang, then $Z^2 = 0$, and so each dimension degree vanishes,

$$\begin{aligned} &\langle Z^2\rangle_0 = \mathbf{J}^2 - \mathbf{S} \,\lrcorner\, \mathbf{S} - \mathbf{K}^2 && \\ &\langle Z^2\rangle_1 = +2\gamma_{0123}(\mathbf{S}\wedge\mathbf{K}) && \mathbf{K} \text{ in the plane of } \mathbf{S} \\ &\langle Z^2\rangle_2 = +i2\gamma_{0123}(\mathbf{J}\wedge\mathbf{K}) && \mathbf{J} \text{ and } \mathbf{K} \text{ are parallel} \\ &\langle Z^2\rangle_3 = +i2\mathbf{J}\wedge\mathbf{S} && \mathbf{J} \text{ in the plane of } \mathbf{S} \\ &\langle Z^2\rangle_4 = -\mathbf{S}\wedge\mathbf{S} && \mathbf{S} \text{ is simple.} \end{aligned}$$

The bivector part implies that $\mathbf{J}$ and $\mathbf{K}$ are parallel, the 4-vector part implies that $\mathbf{S}$ is simple, and the vector and 3-vector parts imply that $\mathbf{J}$ and $\mathbf{K}$ are in the plane of $\mathbf{S}$. Altogether we must have

$$\boxed{Z = \mathbf{J}(1 + is + ih\gamma_{0123})}$$

where h is a real number and $\mathbf{s}$ is a space-like vector orthogonal to $\mathbf{J}$, $\mathbf{J}\cdot\mathbf{s}=0$. We again compute $Z^2=\mathbf{J}^2(1+(\mathbf{s}+h\gamma_{0123})^2)=0$ and conclude that either

1. $\mathbf{J}^2=0$ or else
2. $(\mathbf{s}+h\gamma_{0123})^2=-1$.

Neither condition alone is sufficient to force Z to become a boomerang [Z is not even a Fierz aggregate if $\mathbf{J}^2\neq 0$]. However, such a Z is a boomerang if both conditions are satisfied simultaneously.

Counter-examples. 1. In the case $\Omega_1=0$, the element $Z=\mathbf{J}-\Omega_2\gamma_{0123}$, $\mathbf{J}^2=\Omega_2^2>0$, is such that $Z^2=0$, but Z is not a Fierz aggregate.

2. $Z=\mathbf{J}+i\mathbf{S}$ with $\mathbf{J}^2>0$, $\mathbf{S}=\gamma_{0123}\mathbf{J}\mathbf{s}$, $\mathbf{J}\cdot\mathbf{s}=0$, $\mathbf{s}^2=-1$, is not aFierz aggregate, and $Z^2\neq 0$, but we have $Z\gamma_{0123}Z=0$.

3. $Z=\mathbf{J}+i\mathbf{S}+i\mathbf{K}\gamma_{0123}$ where $\mathbf{J}^2=\mathbf{K}^2=0$, $\mathbf{J}\cdot\mathbf{K}=0$, $\mathbf{J}\wedge\mathbf{K}=0$, $\mathbf{S}\wedge\mathbf{S}\neq 0$, is a Fierz aggregate but does not satisfy $Z^2=0$, $Z\gamma_{0123}Z=0$.

4. $Z=\mathbf{J}(1+i\mathbf{s}+ih\gamma_{0123})$ with $\mathbf{J}^2=0$, $\mathbf{J}\cdot\mathbf{s}=0$, $(\mathbf{s}+h\gamma_{0123})^2\neq -1$, is a Fierz aggregate and satisfies $Z^2=0$ and $Z\gamma_{0123}Z=0$, but still we do not have a boomeranging Z. ∎

Throughout this chapter we assume that Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2 are real multivectors or equivalently that $Z=\Omega_1+\mathbf{J}+i\mathbf{S}+i\mathbf{K}\gamma_{0123}+\Omega_2\gamma_{0123}$ is Dirac self-adjoint [$\tilde{Z}^*=Z$ or in matrix notation $\gamma_0 Z^\dagger\gamma_0=Z$]. This implies that $\eta^\dagger\gamma_0 Z\eta$ [$=4\langle\tilde{\eta}^* Z\eta\rangle_0$] is a real number for all spinors η.

For a boomerang Z we have $\eta^\dagger\gamma_0 Z\eta\geq 0$, for all spinors η, and also $J^0>0$ [the grade involute $\hat{Z}$ of Z is such that $\langle\hat{Z}\rangle_0\cdot\gamma_0<0$ and $4\langle\tilde{\eta}^*\hat{Z}\eta\rangle_0=\eta^\dagger\gamma_0\hat{Z}\eta\leq 0$].

Theorem. Let Z be such that $\eta^\dagger\gamma_0 Z\eta\geq 0$ for all spinors η, and that $J^0>0$. Then the following statements hold.

1. Z is a boomerang if and only if $Z\gamma^0\tilde{Z}^*=4J^0Z$ or equivalently $ZZ^\dagger\gamma^0=4J^0Z$.
2. In the non-null case $\Omega\neq 0$, Z is a boomerang if and only if it is a Fierz aggregate.
3. In the null case $\Omega=0$, Z **is a boomerang if and only if** $Z=\mathbf{J}(1+i\mathbf{s}+ih\gamma_{0123})$ where $\mathbf{J}$ is a null-vector, $\mathbf{J}^2=0$, $\mathbf{s}$ is a space-like vector, $\mathbf{s}^2<0$ or $\mathbf{s}=0$, orthogonal to $\mathbf{J}$, $\mathbf{J}\cdot\mathbf{s}=0$, and h is a real number such that $h=\pm\sqrt{1+\mathbf{s}^2}$, $|h|\leq 1$. ∎

The condition $Z\gamma^0\tilde{Z}^*=4J^0Z$ could also be written with an arbitrary time-like vector $\mathbf{v}$ as follows: $Z\mathbf{v}\tilde{Z}^*=4(\mathbf{v}\cdot\mathbf{J})Z$.

11.5 Decomposition and factorization of boomerangs

Write

$$P = \Omega_1 + \mathbf{J} + \Omega_2\gamma_{0123}, \qquad Q = \mathbf{S} + \mathbf{K}\gamma_{0123} = \mathbf{S} - \gamma_{0123}\mathbf{K}$$
$$\Sigma = 1 - i\mathbf{J}\mathbf{K}^{-1}\gamma_{0123} = 1 - i\mathbf{K}\mathbf{J}^{-1}\gamma_{0123} \qquad [\text{when } \Omega \neq 0]$$

so that $Z = P + iQ = P\Sigma$. Then $P\Sigma = \Sigma P$ and we have found a solution to the open problem posed by Crawford 1985 p. 1441 ref. (10). [Crawford's second factor in (4.1),

$$\begin{aligned} 1 + i(\Omega_1 + \Omega_2\gamma_{0123})^{-1}\mathbf{K}\gamma_{0123} &= 1 - i(\Omega_2 - \Omega_1\gamma_{0123})^{-1}\mathbf{K} \\ &= 1 - i\mathbf{S}^{-1}\mathbf{J} = 1 + i\mathbf{J}^{-1}\mathbf{S}, \end{aligned}$$

did not commute with P unless $\Omega_2 = 0$.] In the case $\Omega_1 \neq 0$ there is another factorization

$$Z = P\Big(1 + i\frac{1}{2\Omega_1}Q\Big) = P\,\frac{1}{2}\Big(1 + i\frac{1}{2\Omega_1}Q\Big)^2$$

where the factors commute and are Dirac self-adjoint, but in this factorization the second factor is not an idempotent (even though it behaves like one when multiplied by P).

11.6 Multiplication by the imaginary unit $i = \sqrt{-1}$

We have found that $i\psi = \psi\gamma_2\gamma_1$ $(\neq \psi\gamma_{0123})$ corresponds to $\Phi\gamma_2\gamma_1 = 4\,\mathrm{Re}(i\psi)$ and further to $\phi\gamma_2\gamma_1 = \Phi\gamma_2\gamma_1\frac{1}{2}(1-\gamma_{03})$. In the non-null case $\Omega \neq 0$ write

$$s = \gamma_{0123}\mathbf{J}\mathbf{K}^{-1} = \mathbf{J}(\mathbf{K}\gamma_{0123})^{-1} = -\Omega^{-1}\mathbf{S} = \Omega\mathbf{S}^{-1}$$
$$k = -(\Omega_2 + \Omega_1\gamma_{0123})^{-1}\mathbf{K} = \mathbf{J}\mathbf{S}^{-1} = -\mathbf{S}\mathbf{J}^{-1}.$$

[Using $\rho e_\mu = \Psi\gamma_\mu\tilde{\Psi}$, $\rho^2 = \Omega_1^2 + \Omega_2^2$ Hestenes 1986 p. 333 (2.23) gives $s = \gamma_{0123}e_3e_0$, Boudet 1985 p. 719 (2.6) gives $-s = e_1e_2$.] Note that

$$s = \Psi(-\gamma_{12})\Psi^{-1} = \Psi\gamma_2\gamma_1\Psi^{-1} \in \textstyle\bigwedge^2\mathbb{R}^{1,3} \qquad [\text{simple bivector}]$$
$$k = \Psi(-\gamma_{012})\Psi^{-1} = \Psi\gamma_{0123}\gamma_3\Psi^{-1} \in \mathbb{R}^{1,3} \oplus \textstyle\bigwedge^3\mathbb{R}^{1,3}$$

and $s^2 = -1$, $k^2 = -1$ and $sk = ks$. Both s and k play the role of the imaginary unit (multiplication on the left side):

$i\psi = s\psi = k\psi = \psi\gamma_2\gamma_1 \neq \psi\gamma_{0123}$	Dirac spinor
$s\Phi = k\Phi = \Phi\gamma_2\gamma_1$	mother spinor
$s\phi = k\phi = \phi\gamma_2\gamma_1 = \phi\gamma_{0123} \neq \phi(-\gamma_{012})$	ideal spinor
$s\Psi = k\Psi\gamma_0 = \Psi\gamma_2\gamma_1$	spinor operator

and $P = sQ = kQ$. [Hestenes 1986 p. 334 (2.24) reports $s\Psi = \Psi\gamma_2\gamma_1$ and also $s\phi = \phi\gamma_2\gamma_1 = \phi\gamma_{0123}$.]

Question

Do the conditions $Z^2 = 0$ and $Z\gamma_{0123}Z = 0$ imply that Z is a Fierz aggregate?

Exercises

1. Compute $\mathbf{K}^2$, when $\mathbf{K} = \Psi\gamma_3\tilde{\Psi}$.

Show that

2. $\mathrm{Im}(\psi) = \mathrm{Re}(\psi)\gamma_{12}$ in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$.
3. $\frac{1}{2}\Phi\gamma_1\bar{\Phi} = 4\,\mathrm{Re}(\psi\gamma_1\bar{\psi}) = -4\,\mathrm{Im}(\psi\gamma_2\bar{\psi})$ [no complex conjugation]
   $\frac{1}{2}\Phi\gamma_2\bar{\Phi} = 4\,\mathrm{Re}(\psi\gamma_2\bar{\psi}) = 4\,\mathrm{Im}(\psi\gamma_1\bar{\psi})$.
4. $Q_k^2 = -2\Omega_1 P,\ Q_iQ_j = 2\Omega_1 Q_k$ (ijk cycl.) for $Q_k = \frac{1}{2}(\Phi\gamma_k\bar{\Phi})\gamma_{0123}$
   [$P, Q_1, Q_2, Q_3 = Q$ span a quaternion algebra when $\Omega_1 \neq 0$].

Show that for $Z = 4\psi\tilde{\psi}^*$, where $\psi \in (\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$, the following hold:

5. $Z\gamma^0 Z = 4J^0 Z$ ($\neq 0$ for $\psi \neq 0$).
6. $P\gamma^0 P = 2J^0 P = -Q\gamma^0 Q$.
7. $P\gamma_0 Q = Q\gamma_0 P$.
8. $\mathrm{even}(\tilde{Z})Z = 0,\ \mathrm{odd}(\tilde{Z})Z = 0$.
9. $\tilde{Z}Z = 0\ \Rightarrow P^2 = -Q^2,\ PQ = QP$ [no complex conjugation].
10. $\tilde{Z}Z = 0,\ Z^2 = 4\Omega_1 Z\ \Rightarrow P^2 = 2\Omega_1 P,\ PQ = 2\Omega_1 Q$.
11. $\bar{Z}Z = 0\ \Rightarrow \bar{P}P = \bar{Q}Q,\ \bar{P}Q = -\bar{Q}P$.
12. $\bar{Z}Z = 0,\ \bar{Z}^*Z = 4\Omega_2\gamma_{0123}Z\ \Rightarrow \bar{P}P = 2\Omega_2\gamma_{0123}P,\ \bar{P}Q = 2\Omega_2\gamma_{0123}Q$.

Write

$$K = \Omega_1 + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}, \quad S = \Omega_1 + i\mathbf{S} + \Omega_2\gamma_{0123}$$
$$\Pi = P(\Omega_1 + \Omega_2\gamma_{0123})^{-1}, \quad \Gamma = K(\Omega_1 + \Omega_2\gamma_{0123})^{-1}$$

and show that:

13. $\Sigma = S(\Omega_1 + \Omega_2\gamma_{0123})^{-1} = 1 - i\mathbf{s}$.
14. $\Gamma = 1 - i\mathbf{k},\ \tilde{\Gamma}^* = 1 + i\tilde{\mathbf{k}} = 1 - i(\Omega_1 + \Omega_2\gamma_{0123})^{-1}\gamma_{0123}\mathbf{K}$.
15. $Z = K\Sigma = \Sigma K = \Gamma S = S\tilde{\Gamma}^*$.
16. $Z = \Pi S = S\tilde{\Pi} = \Pi K = K\tilde{\Pi}$ [no complex conjugation needed]
    $= \Gamma P = P\tilde{\Gamma}^*$ [this is Crawford's factorization]
    [$= P\Gamma$ only if $\Omega_2 = 0$ since then $\Gamma = \tilde{\Gamma}^*$].

Bibliography

R. Boudet: Conservation laws in the Dirac theory. *J. Math. Phys.* **26** (1985), 718-724.

R. Boudet: Les algèbres de Clifford et les transformations des multivecteurs; pp. 343-352 in A. Micali et al. (eds.): *Proceedings of the Second Workshop on 'Clifford Algebras and their Applications in Mathematical Physics' (Montpellier, 1989).* Kluwer, Dordrecht, The Netherlands, 1992.

J.P. Crawford: On the algebra of Dirac bispinor densities: Factorization and inversion theorems. *J. Math. Phys.* **26** (1985), 1439-1441.

J.P. Crawford: Dirac equation for bispinor densities; pp. 353-361 in J.S.R. Chisholm, A.K. Common (eds.): *Proceedings of the Workshop on 'Clifford Algebras and their Applications in Mathematical Physics' (Canterbury 1985).* Reidel, Dordrecht, 1986.

J.D. Hamilton: The Dirac equation and Hestenes' geometric algebra. *J. Math. Phys.* **25** (1984), 1823-1832.

D. Hestenes: Clifford algebra and the interpretation of quantum mechanics; pp. 321-346 in J.S.R. Chisholm, A.K. Common (eds.): *Proceedings of the Workshop on 'Clifford Algebras and their Applications in Mathematical Physics' (Canterbury 1985).* Reidel, Dordrecht, The Netherlands, 1986.

P.R. Holland: Minimal ideals and Clifford algebras in the phase space representation of spin-$\frac{1}{2}$ fields; pp. 273-283 in J.S.R. Chisholm, A.K. Common (eds.): *Proceedings of the Workshop on 'Clifford Algebras and their Applications in Mathematical Physics' (Canterbury 1985).* Reidel, Dordrecht, 1986.

P.R. Holland: Relativistic algebraic spinors and quantum motions in phase space. *Found. Phys.* **16** (1986), 708-709.

J. Keller, S. Rodríguez-Romo: A multivectorial Dirac equation. *J. Math. Phys.* **31** (1990), 2501-2510.

J. Keller, F. Viniegra: The multivector structure of the matter and interaction field theories; pp. 437-445 in A. Micali et al. (eds.): *Proceedings of the Second Workshop on 'Clifford Algebras and their Applications in Mathematical Physics' (Montpellier 1985).* Reidel, Dordrecht, The Netherlands, 1992.

Y. Takahashi: A passage between spinors and tensors. *J. Math. Phys.* **24** (1983), 1783-1790.