Long Arm Approximation

FLUID MODEL

$$\begin{cases} -\mu \Delta u + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \end{cases} \qquad \Omega := \mathbb{R}^3 \setminus \bigcup_{i=1}^{4} \bar{B}_i .$$

- traction boundary condition: $\quad -\underline{\sigma} n = f \quad$ on $\partial\Omega$

far field condition: $\quad u(x) \in \mathcal{O}(|x|^{-1}), \quad |x| \longrightarrow \infty .$

$$\underline{\sigma} := \mu \nabla^{sym} u - pI$$ (Cauchy stress tensor)

- no-slip boundary condition: $\quad u(x) = 0 \quad$ on $\partial\Omega$

Stokeslet: $\quad G(x) := \dfrac{1}{8\pi\mu} \left(\dfrac{I}{|x|} + \dfrac{x \otimes x}{|x|^3} \right)$

single layer potential sol.: $\quad u(x) = \displaystyle\int_{\partial\Omega} G(x-y) f(y) dy .$

In the paper we have $\tau \in (-b_i + \partial \Omega)$ But why?

For any $i \in \mathbb{N}_4$ and $\tau \in (-b_i + \partial B_a)$ we have

$$u(b_i + \tau) = \int_{\partial B_a} G(\tau - y) f_i(y)\, dy + \sum_{j \neq i \in \mathbb{N}_4} \int_{\partial B_a} G(b_{ij} + \tau - y) f_j(y)\, dy,$$

where $\cdot\ b_{ij} := b_i - b_j$

$\cdot\ f_j := f(b_j + y)$.

Since we have for $\min_{i<j} |b_{ij}| \to +\infty$ at the leading order $G(b_{ij} + \tau - y) \sim G(b_{ij})$, we can write in this limit

$$u_i(\tau) := \frac{1}{|\partial B_a|} \int_{\partial B_a} G(\tau - y) f_i(y)\, dy + \frac{1}{|\partial B_a|} \sum_{j \neq i \in \mathbb{N}_4} G(b_{ij}) \int_{\partial B_a} f_j(y)\, dy.$$

Why do we divide by $|\partial B_a|$?

Stokes law $\Rightarrow$ uniform tractions, i.e. f_j is constant on ∂B_j.

$$\Rightarrow u_i(\tau) := \frac{1}{6\pi\mu a} f_i + \sum_{j \neq i \in \mathbb{N}_4} G(b_{ij}) f_j .$$

- **Assumptions**: · Length of l_i is given by $\xi_0 + \xi_i$, with $\xi_0 \gg a$.

  · W.l.o.g. velocity field u_i applied to the center b_i since due to the constant tractions the u_i are uniform on ~~one~~ the associated boundary.

- **Balance equations**: (Due to negligible inertia)

  Forces : $f_1 + f_2 + f_3 + f_4 = 0.$

  Torques: $\sum_{i \in \mathbb{N}_4} b_i \times f_i = 0.$

⚠ We cannot assume anymore that all geometric and dynamic quantities lie in $\mathbb{R}^2 \times \{0\}$.

In particular, we cannot treat the torques as scalar quantities anymore!

However, for any $k \in \mathbb{N}_3$, the map

$$f \longmapsto w_k(b_i, f) := ((b_i - c) \times f) \cdot \hat{e}_k$$

is a linear form on $\mathbb{R}^3$, where $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ denotes the canonical basis for $\mathbb{R}^3$. Hence, we find $\forall i \in \mathbb{N}_4$ vectors $w_{ki}(b_i) \in \mathbb{R}^3$ such that

$$w_k(b_i, f_i) = w_{ki}(b_i) \cdot f_i .$$

In particular, we find ~~vectors~~ for all $i \in \mathbb{N}_4$ vectors $w_{ki}(\xi_i, R) \in \mathbb{R}^3$ such that

$$w_{ki}(\xi_i, R) \cdot f_i = w_k(b_i, f_i) = ((b_i - c) \times f_i) \cdot \hat{e}_k$$

since $b_i := c + \xi_i R z_i \in \mathbb{R}^3$.

Therefore the balance equation for the total torque is equivalent to

$$\forall k \in \mathbb{N}_3 : \sum_{i \in \mathbb{N}_4} w_{ki}(\xi_i, R) \cdot f_i = 0 .$$

We have $G(b_{ij}) = G(b_{ji})$ and thus we define

$$l_i := b_{i,i+1} \; ; \; L_i := G(l_i), \quad i \in \mathbb{N}_4$$
$$K_1 := G(b_{13}), \; K_2 := G(b_{24})$$

where we take the indices mod 4.

We set

- $\nu := 6\pi\mu a$
- $\mathcal{I} := \mathrm{diag}(\underbrace{I}_{3\times 3}, I, I, I) \in \mathcal{M}_{12\times 12}(\mathbb{R})$.
- $\underline{u} := (u_1, u_2, u_3, u_4) \in \mathbb{R}^{12}$
- $\underline{f} := (f_1, f_2, f_3, f_4) \in \mathbb{R}^{12}$.
- $$\mathcal{L} := \begin{pmatrix} 0 & L_1 & K_1 & L_4 \\ L_1 & 0 & L_2 & K_2 \\ K_1 & L_2 & 0 & L_3 \\ L_4 & K_2 & L_3 & 0 \end{pmatrix}.$$

Then we can write

$$\underline{u} = \left(\frac{1}{\nu} \mathcal{I} + \mathcal{L} \right) \underline{f}.$$

For the velocities at the centers of the balls, we have for $i \in \mathbb{N}_4$

$$u_i = \dot{c} + \dot{\xi}_i R z_i + \dot{R} R^T (\xi_0 + \xi_i) z_i .$$

In particular, at $R_0 = I$, we have

$$u_i = \dot{c} + \dot{\xi}_i z_i + \dot{R}_0 (\xi_0 + \xi_i) z_i .$$

$$= \dot{c} + \dot{\xi}_i z_i + \omega \times (\xi_0 + \xi_i) z_i$$

$$= \dot{c} + \dot{\xi}_i z_i + (\xi_0 + \xi_i) [z_i]_\times^T \omega ,$$

Is there a better way to do that ?

where $\omega = Ax(\dot{R}_0)$ and $[z_i]_\times$ the skew symmetric matrix such that

$$[z_i]_\times \omega = z_i \times \omega .$$

Then we have

$$\underline{u}(\underline{\xi}, R_0) = \operatorname{diag}(z_i)\, \dot{\xi} + \begin{pmatrix} I_{3\times 3} & (\xi_0 + \xi_1)[z_1]_\times^T \\ I_{3\times 3} & (\xi_0 + \xi_2)[z_2]_\times^T \\ I_{3\times 3} & (\xi_0 + \xi_3)[z_3]_\times^T \\ I_{3\times 3} & (\xi_0 + \xi_4)[z_4]_\times^T \end{pmatrix} \begin{pmatrix} \dot{c} \\ \omega \end{pmatrix}$$

$$=: \mathcal{X}_0 \dot{\xi} + \mathcal{Y}(\xi)\, \dot{p} .$$

Note that $\omega = [\dot{R}_0]_{\mathcal{L}}$, where $\mathcal{L} = (L_1, L_2, L_3)$ is the ordered basis in the report. Hence, we have

$$\underline{0} = \frac{1}{\nu} W \underline{f}(\underline{u}(\underline{\xi}, R_0)) = W \mathcal{L}_\nu \underline{u}(\xi, I)$$

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$$= W \mathcal{L}_\nu [\mathcal{X}_0 \dot{\xi} + \mathcal{Y}(\xi)\dot{p}]$$

$$= \underbrace{W \mathcal{L}_\nu}_{=: V_\nu} \mathcal{X}_0 \dot{\xi} + W\mathcal{L}_\nu \mathcal{Y}(\xi)\dot{p} = \underbrace{V_\nu \mathcal{X}_0}_{\text{invertible?}} \dot{\xi} + \underbrace{V_\nu \mathcal{Y}(\xi)}_{\text{invertible?}} \dot{p}$$

- $$\Leftrightarrow \quad \dot{p} = - \frac{V_\nu \mathcal{X}_0}{V_\nu \mathcal{Y}(\xi)} \dot{\xi} \overset{!}{=} F(\xi, I)\dot{\xi}.$$

We know that in general

$$\dot{p} = \mathrm{diag}(R) F(\xi, I)\,\dot{\xi},$$

so we are done?