4

Pauli Spin Matrices and Spinors

In classical mechanics kinetic energy $\frac{1}{2}mv^2 = \frac{p^2}{2m}$, $\vec{p} = m\vec{v}$, and potential energy $W = W(\vec{r})$ sum up to the total energy [1]

$$E = \frac{p^2}{2m} + W.$$

Inserting differential operators for total energy and momentum,

$$E = i\hbar\frac{\partial}{\partial t} \quad \text{and} \quad \vec{p} = -i\hbar\nabla,$$

into the above equation results in the *Schrödinger equation* [2]

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + W\psi,$$

a quantum mechanical description of the electron. The Schrödinger equation explains all atomic phenomena except those involving magnetism and relativity.

The wave function ψ is complex valued, $\psi(\vec{r}, t) \in \mathbb{C}$. The square norm $|\psi|^2$ integrated over a region in space gives the probability of finding the electron in that region. [3]

The Stern & Gerlach experiment, in 1922, showed that a beam of silver atoms splits in two in a magnetic field [there were two distinct spots on the screen, instead of a smear of silver along a line]. Uhlenbeck & Goudsmit in 1925 proposed that silver atoms and the electron have an intrinsic angular momentum, the *spin*. The spin interacts with the magnetic field, and the electron goes up or down according as the spin is parallel or opposite to the vertical magnetic field.

1 This holds in a conservative system.

2 The Schrödinger equation arose out of the hypothesis that if light has both wave and particle properties, then perhaps particles might have wave properties such as interference and diffraction.

3 This is the Born interpretation.

In an electromagnetic field $\vec{E}, \vec{B}$ with potentials $V, \vec{A}$ the Schrödinger equation becomes [4]

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[(-i\hbar\nabla - e\vec{A})^2]\psi - eV\psi, \tag{1}$$

or after 'squaring'

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[-\hbar^2\nabla^2 + e^2A^2 + i\hbar e(\nabla\cdot\vec{A} + \vec{A}\cdot\nabla)]\psi - eV\psi.$$

This equation does not yet involve the spin of the electron. The differential operator, known as the generalized momentum,

$$\vec{\pi} = \vec{p} - e\vec{A} \quad \text{where} \quad \vec{p} = -i\hbar\nabla$$

is such that its components $\pi_k = p_k - eA_k$ satisfy the commutation relations

$$\pi_1\pi_2 - \pi_2\pi_1 = i\hbar eB_3 \quad \text{(permute 1, 2, 3 cyclically)}.$$

Pauli 1927 introduced the spin into quantum mechanics by adding a new term into the Schrödinger equation. The *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy

$$\sigma_1\sigma_2 = i\sigma_3 \quad \text{(permute 1, 2, 3 cyclically)}$$

and the anticommutation relations

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I.$$

Applying the above commutation and anticommutation relations, and temporarily using the old-fashioned notation

$$\vec{\sigma}\cdot\vec{\pi} = \sigma_1\pi_1 + \sigma_2\pi_2 + \sigma_3\pi_3,$$

we may see that

$$(\vec{\sigma}\cdot\vec{\pi})^2 = \pi^2 - \hbar e(\vec{\sigma}\cdot\vec{B})$$

where

$$\pi^2 = p^2 + e^2A^2 - e(\vec{p}\cdot\vec{A} + \vec{A}\cdot\vec{p}).$$

Pauli replaced π^2 by $(\vec{\sigma}\cdot\vec{\pi})^2$ in equation (1):

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[\pi^2 - \hbar e(\vec{\sigma}\cdot\vec{B})]\psi - eV\psi.$$

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4 A Schrödinger equation with $W = 0$ is brought into this form by a gauge transformation $\psi(\vec{r}, t) \to \varphi(\vec{r}, t)e^{i\alpha(\vec{r}, t)}$, when $eV = \hbar\frac{\partial\alpha}{\partial t}$ and $e\vec{A} = \hbar\nabla\alpha$.

This Schrödinger-Pauli equation describes the spin by virtue of the term

$$\frac{\hbar e}{2m}(\vec{\sigma} \cdot \vec{B}).$$

The matrix $\vec{\sigma} \cdot \vec{B}$ operates on two-component column matrices with entries in $\mathbb{C}$. The wave function sends space-time points to *Pauli spinors*

$$\psi(\vec{r}, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1, \psi_2 \in \mathbb{C},$$

that is, it has values in the complex linear space $\mathbb{C}^2$.

The Schrödinger-Pauli equation in the Clifford algebra $C\ell_3$. The multiplication rules of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Mat}(2, \mathbb{C})$ imply the matrix identity

$$(\vec{\sigma} \cdot \vec{B})^2 = (B_1^2 + B_2^2 + B_3^2)I.$$

Thus, we may regard the set of traceless Hermitian matrices as a Euclidean space $\mathbb{R}^3$ with an orthonormal basis $\{\sigma_1, \sigma_2, \sigma_3\}$.

The length (of the representative) of a vector $\vec{B}$ is preserved under a similarity transformation $U(\vec{\sigma} \cdot \vec{B})U^{-1}$ by a special unitary matrix $U \in SU(2)$,

$$SU(2) = \{U \in \mathrm{Mat}(2, \mathbb{C}) \mid U^\dagger U = I, \ \det U = 1\}.$$

In this way, not only vectors but also rotations become represented within the matrix algebra $\mathrm{Mat}(2, \mathbb{C})$. In fact, each rotation $R \in SO(3)$ becomes represented by two matrices $\pm U \in SU(2)$, and we say that $SU(2)$ is a two-fold covering of $SO(3)$:

$$SO(3) \simeq \frac{SU(2)}{\{\pm I\}}.$$

Pauli spinors could also be replaced by square matrices with only the first column being non-zero,

$$\psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}, \quad \psi_1, \psi_2 \in \mathbb{C}.$$

Such square matrix spinors form a *left ideal* S of the matrix algebra $\mathrm{Mat}(2, \mathbb{C})$, that is, for $U \in \mathrm{Mat}(2, \mathbb{C})$ and $\psi \in S$ we also have $U\psi \in S$. [5]

The matrix algebra $\mathrm{Mat}(2, \mathbb{C})$ is an isomorphic image of the Clifford algebra $C\ell_3$ of the Euclidean space $\mathbb{R}^3$. Thus, not only vectors in $\mathbb{R}$ and rotations in

5 The left ideal can be written as $S = \mathrm{Mat}(2, \mathbb{C})f$, where $f = \frac{1}{2}(I + \sigma_3)$ is an idempotent satisfying $f^2 = f$. The idempotent is primitive and the left ideal is minimal.

$SO(3)$ have representatives in $C\ell_3$, but also spinor spaces or spinor representations of the rotation group $SO(3)$ [6] can be constructed within the Clifford algebra $C\ell_3$. [7]

In the notation of the Clifford algebra $C\ell_3$ we could describe Pauli's achievement by saying that he replaced $\pi^2 = \vec{\pi} \cdot \vec{\pi}$ by $\vec{\pi}^2 = \vec{\pi} \cdot \vec{\pi} + \vec{\pi} \wedge \vec{\pi} = \pi^2 - \hbar e \vec{B}$ and came across the equation

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}[\pi^2 - \hbar e \vec{B}]\psi - eV\psi$$

where $\vec{B} \in \mathbb{R}^3 \subset C\ell_3$ and $\psi(\vec{r}, t) \in S = C\ell_3 f$, $f = \frac{1}{2}(1 + \mathbf{e}_3)$. All the arguments and functions now have values in one algebra, which will facilitate numerical computations.

In this chapter we shall study more closely the Clifford algebra $C\ell_3$ and the spin group **Spin**(3), and reformulate once more the Schrödinger-Pauli equation in terms of $C\ell_3$.

4.1 Orthogonal unit vectors, orthonormal basis

The 3-dimensional Euclidean space $\mathbb{R}^3$ has a basis consisting of three orthogonal unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The *Clifford algebra* $C\ell_3$ of $\mathbb{R}^3$ is the real associative algebra generated by the set $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ satisfying the relations

$$\mathbf{e}_1^2 = 1, \quad \mathbf{e}_2^2 = 1, \quad \mathbf{e}_3^2 = 1,$$
$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1, \quad \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_1, \quad \mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2.$$

The Clifford algebra $C\ell_3$ is 8-dimensional with the following basis:

1	the scalar
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	vectors
$\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3$	bivectors
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	a volume element.

We abbreviate the unit bivectors as $\mathbf{e}_{ij} = \mathbf{e}_i\mathbf{e}_j$, when $i \neq j$, and the unit oriented volume element as $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. An arbitrary element in $C\ell_3$ is a sum of a scalar, a vector, a bivector and a volume element, and can be written as $\alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$, where $\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Example. Compute the product $\mathbf{e}_{12}\mathbf{e}_{13}$. By definition $\mathbf{e}_{12}\mathbf{e}_{13} = (\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_3)$

6 Actually, spinor representations are representations of the universal covering group $SU(2) \simeq$ **Spin**(3) of $SO(3)$. The spinor representations cannot be reached by tensor methods, as irreducible components of tensor products of antisymmetric powers of $\mathbb{R}^3$.

7 The orthogonal group $O(3)$ also has a non-trivial covering group **Pin**(3) residing within $C\ell_3$.

and by associativity $(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1\mathbf{e}_3) = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3$. Use anticommutativity, $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$, and substitute $\mathbf{e}_1^2 = 1$ to get $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_1^2\mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_{23}$. ∎

Imaginary units. The three unit bivectors $\mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_1\mathbf{e}_3$, $\mathbf{e}_2\mathbf{e}_3$ represent unit oriented plane segments as well as generators of rotations in the coordinate planes, and share the basic property of the imaginary unit, $(\mathbf{e}_i\mathbf{e}_j)^2 = -1$ for $i \neq j$. The oriented volume element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ also shares the basic property of the imaginary unit, $(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3)^2 = -1$, and furthermore it commutes with all the elements in $\mathcal{C}\ell_3$. The unit oriented volume element $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ represents the duality operator, which swaps plane segments and line segments orthogonal to the plane segments. ∎

4.2 Matrix representation of $\mathcal{C}\ell_3$

The set of 2×2-matrices with complex numbers as entries is denoted by $\mathrm{Mat}(2,\mathbb{C})$. Mostly we shall regard this set as a *real* algebra with scalar multiplication taken over the real numbers in $\mathbb{R}$ although the matrix entries are in the complex field $\mathbb{C}$. The Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy the multiplication rules

$$\begin{aligned} &\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I \quad \text{and} \\ &\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1, \\ &\sigma_3\sigma_1 = i\sigma_2 = -\sigma_1\sigma_3, \\ &\sigma_2\sigma_3 = i\sigma_1 = -\sigma_3\sigma_2. \end{aligned}$$

They also generate the real algebra $\mathrm{Mat}(2,\mathbb{C})$. The correspondences $\mathbf{e}_1 \simeq \sigma_1$, $\mathbf{e}_2 \simeq \sigma_2$, $\mathbf{e}_3 \simeq \sigma_3$ establish an isomorphism between the real algebras, $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$, with the following correspondences of the basis elements:

$\mathrm{Mat}(2,\mathbb{C})$	$\mathcal{C}\ell_3$
I	1
$\sigma_1,\ \sigma_2,\ \sigma_3$	$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$
$\sigma_1\sigma_2,\ \sigma_1\sigma_3,\ \sigma_2\sigma_3$	$\mathbf{e}_{12},\ \mathbf{e}_{13},\ \mathbf{e}_{23}$
$\sigma_1\sigma_2\sigma_3$	$\mathbf{e}_{123}$

Note that $\mathbf{e}_{ij} = -\mathbf{e}_{ji}$ for $i \neq j$. The essential difference between the Clifford algebra $\mathcal{C}\ell_3$ and its matrix image $\mathrm{Mat}(2,\mathbb{C})$ is that in the Clifford algebra $\mathcal{C}\ell_3$ we will, in its definition, distinguish a particular subspace, the vector space $\mathbb{R}^3$, in which the square of a vector equals its length squared, that is, $\mathbf{r}^2 = |\mathbf{r}|^2$. No such distinguished subspace has been singled out in the definition of the matrix algebra $\mathrm{Mat}(2, \mathbb{C})$. Instead, we have chosen the traceless Hermitian matrices to represent $\mathbb{R}^3$, and thereby added extra structure to $\mathrm{Mat}(2, \mathbb{C})$. [8]

4.3 The center of $C\ell_3$

The element $\mathbf{e}_{123}$ commutes with all the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and therefore with every element of $C\ell_3$. In other words, elements of the form

$$x + y\mathbf{e}_{123} \simeq \begin{pmatrix} x+iy & 0 \\ 0 & x+iy \end{pmatrix}$$

commute with all the elements in $C\ell_3$. The subalgebra of scalars and 3-vectors

$$\mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3 = \{x + y\mathbf{e}_{123} \mid x,\, y \in \mathbb{R}\}$$

is the *center* $\mathrm{Cen}(C\ell_3)$ of $C\ell_3$, that is, it consists of those elements of $C\ell_3$ which commute with every element of $C\ell_3$. Note that $\sigma_1\sigma_2\sigma_3 = iI$. Since $\mathbf{e}_{123}^2 = -1$, the center of $C\ell_3$ is isomorphic to the complex field $\mathbb{C}$, that is,

$$\mathrm{Cen}(C\ell_3) = \mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3 \simeq \mathbb{C}.$$

4.4 The even subalgebra $C\ell_3^+$

The elements 1 and $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$, $\mathbf{e}_{13} = \mathbf{e}_1\mathbf{e}_3$, $\mathbf{e}_{23} = \mathbf{e}_2\mathbf{e}_3$ are called *even*, because they are products of an even number of vectors. The even elements are represented by the following matrices:

$$w + x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12} \simeq \begin{pmatrix} w+iz & ix+y \\ ix-y & w-iz \end{pmatrix}.$$

The even elements form a real subspace

$$\begin{aligned} \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3 &= \{w + x\mathbf{e}_{23} + y\mathbf{e}_{31} + z\mathbf{e}_{12} \mid w, x, y, z \in \mathbb{R}\} \\ &\simeq \{wI + xi\sigma_1 + yi\sigma_2 + zi\sigma_3 \mid w, x, y, z \in \mathbb{R}\} \end{aligned}$$

8 We could also have chosen, for the representatives of the anticommuting (and therefore orthogonal) unit vectors in $\mathbb{R}^3$, the following matrices:

$$u_1 = \frac{1}{4}\begin{pmatrix} 3i & 5 \\ 5 & -3i \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_3 = \frac{1}{4}\begin{pmatrix} 5 & -3i \\ -3i & -5 \end{pmatrix},$$

that is, $u_1 = \frac{1}{4}(5\sigma_1 + 3\sigma_1\sigma_2)$, $u_2 = \sigma_2$, $u_3 = \frac{1}{4}(5\sigma_3 - 3\sigma_2\sigma_3)$. These matrices are non-Hermitian and satisfy $u_ju_k + u_ku_j = 2\delta_{jk}I$.

which is closed under multiplication. Thus, the subspace $\mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3$ is a subalgebra, called the *even subalgebra* of $\mathcal{C}\ell_3$. We will denote the even subalgebra by $\text{even}(\mathcal{C}\ell_3)$ or for short by $\mathcal{C}\ell_3^+$. The even subalgebra is isomorphic to the division ring of quaternions $\mathbb{H}$, as can be seen by the following correspondences:

$\mathbb{H}$	$\mathcal{C}\ell_3^+$
i	$-\mathbf{e}_{23}$
j	$-\mathbf{e}_{31}$
k	$-\mathbf{e}_{12}$

Remark. The Clifford algebra $\mathcal{C}\ell_3$ contains two subalgebras, isomorphic to $\mathbb{C}$ [the center] and $\mathbb{H}$ [the even subalgebra], in such a way that [temporarily we denote these subalgebras by their isomorphic images]

1. $ab = ba$ for $a \in \mathbb{C}$ and $b \in \mathbb{H}$,
2. $\mathcal{C}\ell_3$ is generated as a real algebra by $\mathbb{C}$ and $\mathbb{H}$,
3. $(\dim \mathbb{C})(\dim \mathbb{H}) = \dim \mathcal{C}\ell_3$.

These three observations can be expressed as

$$\mathbb{C} \otimes \mathbb{H} \simeq \mathcal{C}\ell_3. \qquad \blacksquare$$

4.5 Involutions of $\mathcal{C}\ell_3$

The Clifford algebra $\mathcal{C}\ell_3$ has three involutions similar to complex conjugation. Take an arbitrary element

$$u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 + \langle u\rangle_3 \quad \text{in} \quad \mathcal{C}\ell_3,$$

written as a sum of a scalar $\langle u\rangle_0$, a vector $\langle u\rangle_1$, a bivector $\langle u\rangle_2$ and a volume element $\langle u\rangle_3$. We introduce the following involutions:

$$\begin{aligned} \hat{u} &= \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2 - \langle u\rangle_3, && \text{grade involution,} \\ \tilde{u} &= \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2 - \langle u\rangle_3, && \text{reversion,} \\ \bar{u} &= \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2 + \langle u\rangle_3, && \text{Clifford-conjugation.} \end{aligned}$$

Clifford-conjugation is a composition of the two other involutions: $\bar{u} = \hat{u}\tilde{\ } = \tilde{u}\hat{\ }$.

The correspondences $\sigma_1 \simeq \mathbf{e}_1$, $\sigma_2 \simeq \mathbf{e}_2$, $\sigma_3 \simeq \mathbf{e}_3$ fix the following representations for the involutions:

$$u \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C},$$

$$\hat{u} \simeq \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix}, \qquad \tilde{u} \simeq \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \qquad \bar{u} \simeq \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where the asterisk denotes complex conjugation. We recognize that the reverse $\tilde{u}$ is represented by the Hermitian conjugate $u^\dagger$ and the Clifford-conjugate $\bar{u}$ by the matrix $u^{-1}\det u \in \mathrm{Mat}(2,\mathbb{R})$ [for an invertible u].

The grade involution is an automorphism, that is,

$$\widehat{uv} = \hat{u}\hat{v},$$

while the reversion and the conjugation are anti-automorphisms, that is,

$$\widetilde{uv} = \tilde{v}\tilde{u} \quad \text{and} \quad \overline{uv} = \bar{v}\bar{u}.$$

The grade involution induces the even-odd grading of $\mathcal{C}\ell_3 = \mathcal{C}\ell_3^+ \oplus \mathcal{C}\ell_3^-$.

The reversion can be used to extend the norm from $\mathbb{R}^3$ to all of $\mathcal{C}\ell_3$ by setting

$$|u|^2 = \langle u\tilde{u}\rangle_0.$$

The norm of

$$u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 + u_{12}\mathbf{e}_{12} + u_{13}\mathbf{e}_{13} + u_{23}\mathbf{e}_{23} + u_{123}\mathbf{e}_{123}$$

can be obtained from

$$|u|^2 = |u_0|^2 + |u_1|^2 + |u_2|^2 + |u_3|^2 + |u_{12}|^2 + |u_{13}|^2 + |u_{23}|^2 + |u_{123}|^2.$$

The norm satisfies the inequality

$$|uv| \le \sqrt{2}|u||v| \quad \text{for} \quad u, v \in \mathcal{C}\ell_3.$$

The conjugation can be used to determine the inverse

$$u^{-1} = \frac{\bar{u}}{u\bar{u}}$$

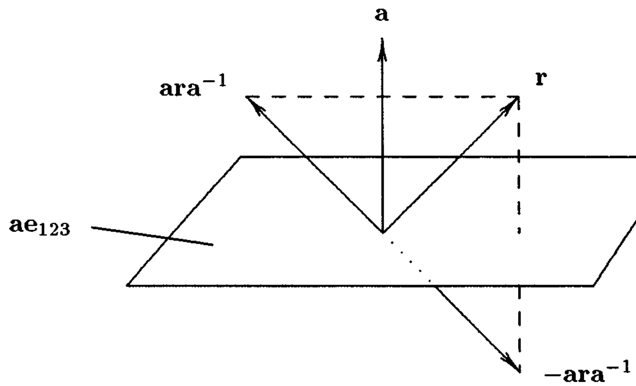
of $u \in \mathcal{C}\ell_3$, $u\bar{u} \neq 0$. The element $u\bar{u} = \bar{u}u$ is in the center $\mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3$ of $\mathcal{C}\ell_3$, so that division by it is unambiguous.

4.6 Reflections and rotations

In the Euclidean space $\mathbb{R}^3$ the vectors $\mathbf{r}$ and $\mathbf{ara}^{-1} = 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a}^{-1} - \mathbf{r}$ are symmetric with respect to the axis $\mathbf{a}$ [use the definition of the Clifford product, $\mathbf{ar} + \mathbf{ra} = 2\mathbf{a}\cdot\mathbf{r}$]. The opposite of $\mathbf{ara}^{-1}$, the vector

$$-\mathbf{ara}^{-1} = \mathbf{r} - 2\frac{\mathbf{a}\cdot\mathbf{r}}{\mathbf{a}^2}\mathbf{a},$$

is obtained by reflecting $\mathbf{r}$ across the mirror perpendicular to $\mathbf{a}$ [reflection across the plane $\mathbf{a}\mathbf{e}_{123}$].



Two successive reflections in planes perpendicular to $\mathbf{a}$ and $\mathbf{b}$ result in a rotation $\mathbf{r} \to \mathbf{bara}^{-1}\mathbf{b}^{-1}$ around the axis which is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Indeed, $\mathbf{r}$ can be decomposed as $\mathbf{r} = \mathbf{r}_{||} + \mathbf{r}_{\perp}$ where $\mathbf{r}_{||}$ and $\mathbf{r}_{\perp}$ are parallel and perpendicular, respectively, to the plane of $\mathbf{a}$ and $\mathbf{b}$. The perpendicular component $\mathbf{r}_{\perp}$ remains invariant under both the reflections while the two successive reflections together rotate the parallel component $\mathbf{r}_{||}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$ by twice the angle between $\mathbf{a}$ and $\mathbf{b}$.

Consider a vector $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and the bivector $\mathbf{ae}_{123} = a_1\mathbf{e}_{23} + a_2\mathbf{e}_{31} + a_3\mathbf{e}_{12}$ dual to $\mathbf{a}$. The vector $\mathbf{a}$ has positive square

$$\mathbf{a}^2 = |\mathbf{a}|^2, \quad \text{where} \quad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

but the bivector $\mathbf{ae}_{123}$ has negative square

$$(\mathbf{ae}_{123})^2 = -|\mathbf{a}|^2.$$

It follows that

$$\exp(\mathbf{ae}_{123}) = \cos\alpha + \mathbf{e}_{123}\frac{\mathbf{a}}{\alpha}\sin\alpha$$

where $\alpha = |\mathbf{a}|$. A spatial rotation of the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ around the axis $\mathbf{a}$ by the angle α is given by

$$\mathbf{r} \to a\mathbf{r}a^{-1}, \qquad a = \exp(\tfrac{1}{2}\mathbf{ae}_{123}).$$

The sense of the rotation is clockwise when regarded from the arrow-head of $\mathbf{a}$. The axis of two consecutive rotations around the axes $\mathbf{a}$ and $\mathbf{b}$ is given by the *Rodrigues formula*

$$\mathbf{c}' = \frac{\mathbf{a}' + \mathbf{b}' + \mathbf{a}' \times \mathbf{b}'}{1 - \mathbf{a}' \cdot \mathbf{b}'} \quad \text{where} \quad \mathbf{a}' = \frac{\mathbf{a}}{\alpha}\tan\frac{\alpha}{2}.$$

This result is obtained by dividing both sides of the formula

$$\exp(\tfrac{1}{2}\mathbf{c}\mathbf{e}_{123}) = \exp(\tfrac{1}{2}\mathbf{b}\mathbf{e}_{123})\exp(\tfrac{1}{2}\mathbf{a}\mathbf{e}_{123})$$

by their scalar parts and then by inspecting the bivector parts.

4.7 The group Spin(3)

The Clifford algebra $\mathcal{C}\ell_3$ of $\mathbb{R}^3$ can be employed to construct the universal covering group for the rotation group $SO(3)$ of $\mathbb{R}^3$. A vector $\mathbf{x} \in \mathbb{R}^3$ can be rotated by the formula

$$\mathbb{R}^3 \to \mathbb{R}^3, \ \mathbf{x} \to \rho(s)\mathbf{x} = s\mathbf{x}s^{-1}$$

where s is an element of the group

$$\mathbf{Spin}(3) = \{s \in \mathcal{C}\ell_3 \mid \tilde{s}s = 1, \ \bar{s}s = 1\}.$$

The group **Spin**(3), called the *spin group*, is a two-fold covering group of the rotation group $SO(3)$.

In the matrix formulation provided by the Pauli spin matrices, the spin group **Spin**(3) has an isomorphic image, the special unitary group

$$SU(2) = \{s \in \mathrm{Mat}(2,\mathbb{C}) \mid s^\dagger s = I, \ \det s = 1\}.$$

For an element $s \in SU(2)$ the function $\mathbf{x} \to \rho(s)\mathbf{x} = s\mathbf{x}s^\dagger$ is a rotation of the Euclidean space of traceless Hermitian matrices,

$$\{\mathbf{x} \in \mathrm{Mat}(2,\mathbb{C}) \mid \mathrm{trace}(\mathbf{x}) = 0, \ \mathbf{x}^\dagger = \mathbf{x}\} \simeq \mathbb{R}^3.$$

Every element in $SO(3)$ can be represented by a matrix in $SU(2)$. There are two matrices s and $-s$ in $SU(2)$ representing the same rotation $R = \rho(\pm s) \in SO(3)$. In other words, the group homomorphism $\rho : \mathbf{Spin}(3) \to SO(3)$ is surjective with kernel $\{\pm 1\}$. This can be depicted by a sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{Spin}(3) \xrightarrow{\rho} SO(3) \longrightarrow 1$$

which is exact, that is, the image of a homomorphism coincides with the kernel of the successive homomorphism.

The spin group **Spin**(3) is a universal cover of the rotation group $SO(3)$, that is, the Lie group **Spin**(3) is simply connected. [9] The group $SO(3)$ is doubly connected. [10]

9 A Lie group is simply connected if it is connected and every loop in the group can be shrunk to a point.

10 Rotations in $SO(3)$ can be represented by vectors $\mathbf{a} \in \mathbb{R}^3$, $|\mathbf{a}| \leq \pi$. Each rotation, $|\mathbf{a}| < \pi$, has a unique representative, and each half-turn, $|\mathbf{a}| = \pi$, is represented twice, $\pm\mathbf{a}$. A loop connecting the identity and a half-turn does not shrink to a point.

4.8 Pauli spinors

In the non-relativistic theory of the spinning electron one considers column matrices, the *Pauli spinors*

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathbb{C}^2 \quad \text{where} \quad \psi_1, \psi_2 \in \mathbb{C}.$$

An isomorphic complex linear space is obtained if one replaces Pauli spinors by the *square matrix spinors*

$$\psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix}$$

where only the first column is non-zero. The fact that only the first column is non-zero can be expressed as

$$\psi \in \mathrm{Mat}(2,\mathbb{C})f \quad \text{where} \quad f = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We shall regard the correspondences $\mathbf{e}_1 \simeq \sigma_1$, $\mathbf{e}_2 \simeq \sigma_2$, $\mathbf{e}_3 \simeq \sigma_3$ as an identification between $\mathcal{C}\ell_3$ and $\mathrm{Mat}(2,\mathbb{C})$. If we multiply $\psi \in \mathrm{Mat}(2,\mathbb{C})f$ on the left by an arbitrary element $u \in \mathcal{C}\ell_3 = \mathrm{Mat}(2,\mathbb{C})$, then the result is also of the same type:

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix}.$$

Such matrices, with only the first column being non-zero, form a *left ideal* S of $\mathcal{C}\ell_3$, that is,

$$u\psi \in S \quad \text{for all} \quad u \in \mathcal{C}\ell_3 \quad \text{and} \quad \psi \in S \subset \mathcal{C}\ell_3.$$

This left ideal S of $\mathcal{C}\ell_3$ contains no left ideal other than S itself and the zero ideal $\{0\}$. Such a left ideal is called *minimal* in $\mathcal{C}\ell_3$.

As a real linear space, S has a basis $\{f_0, f_1, f_2, f_3\}$ where

$$\begin{aligned}
f_0 &= \tfrac{1}{2}(1 + \mathbf{e}_3) &&\simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
f_1 &= \tfrac{1}{2}(\mathbf{e}_{23} + \mathbf{e}_2) &&\simeq \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \\
f_2 &= \tfrac{1}{2}(\mathbf{e}_{31} - \mathbf{e}_1) &&\simeq \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \\
f_3 &= \tfrac{1}{2}(\mathbf{e}_{12} + \mathbf{e}_{123}) &&\simeq \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

The element $f = f_0$ is an *idempotent*, that is, $f^2 = f$.

The subset

$$\mathbb{F} = f\mathcal{C}\ell_3 f \simeq \left\{ \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \middle| \, c \in \mathbb{C} \right\}$$

of $\mathcal{C}\ell_3$ is a subring with unity f, that is, $af = fa$ for $a \in \mathbb{F}$. None of the elements of $\mathbb{F}$ is invertible as an element of $\mathcal{C}\ell_3$, but for each non-zero $a \in \mathbb{F}$ there is a unique $b \in \mathbb{F}$ such that $ab = f$. Thus, $\mathbb{F}$ is a *division ring* with unity f [this follows from the idempotent f being *primitive* in $\mathcal{C}\ell_3$]. As a 2-dimensional real division algebra $\mathbb{F}$ must be isomorphic to $\mathbb{C}$. The isomorphism $\mathbb{F} \simeq \mathbb{C}$ is seen by the equation $f_3^2 = -f_0$ relating the basis elements $\{f_0, f_3\}$ of the real algebra $\mathbb{F}$.

Comment. The multiplication of an element ψ of the real linear space S on the left by an arbitrary even element $u \in \mathcal{C}\ell_3^+$, expressed in coordinate form in the basis $\{f_0, f_1, f_2, f_3\}$,

$$u\psi = (u_0 + u_1\mathbf{e}_{23} + u_2\mathbf{e}_{31} + u_3\mathbf{e}_{23})(\psi_0 f_0 + \psi_1 f_1 + \psi_2 f_2 + \psi_3 f_3),$$

corresponds to the matrix multiplication

$$u\psi \simeq \begin{pmatrix} u_0 & -u_1 & -u_2 & -u_3 \\ u_1 & u_0 & u_3 & -u_2 \\ u_2 & -u_3 & u_0 & u_1 \\ u_3 & u_2 & -u_1 & u_0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}.$$

The square matrices corresponding to the left multiplication by even elements constitute a subring of $\mathrm{Mat}(4, \mathbb{R})$; this subring is an isomorphic image of the quaternion ring $\mathbb{H}$. ∎

The minimal left ideal

$$S = \mathcal{C}\ell_3 f \simeq \left\{ \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \middle| \, \psi_1, \psi_2 \in \mathbb{C} \right\}$$

has a natural right $\mathbb{F}$-linear structure defined by

$$S \times \mathbb{F} \to S, \ (\psi, \lambda) \to \psi\lambda.$$

We shall provide the minimal left ideal S with this right $\mathbb{F}$-linear structure, and call it a *spinor space*. [11]

The map $\mathcal{C}\ell_3 \to \mathrm{End}_{\mathbb{F}} S$, $u \to \tau(u)$, where $\tau(u)$ is defined by the relation $\tau(u)\psi = u\psi$, is a real algebra isomorphism. Employing the basis $\{f_0, -f_2\}$ for the $\mathbb{F}$-linear space S, the elements $\tau(\mathbf{e}_1), \tau(\mathbf{e}_2), \tau(\mathbf{e}_3)$ will be represented by the matrices $\sigma_1, \sigma_2, \sigma_3$. In this way the Pauli matrices are reproduced.

11 Note that multiplying a matrix ψ in S, a left ideal, on the left by $\lambda \in \mathbb{F}$ does not result in a left $\mathbb{F}$-linear structure.

There is a natural way to introduce scalar products on the spinor space $S \subset \mathcal{C}\ell_3$. First, note that for all $\psi, \varphi \in S$ the product

$$\tilde{\psi}\varphi \simeq \begin{pmatrix} \psi_1^* & \psi_2^* \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1^*\varphi_1 + \psi_2^*\varphi_2 & 0 \\ 0 & 0 \end{pmatrix}$$

falls in the division ring $\mathbb{F}$ ($z \to z^*$ means complex conjugation). To show that the map

$$S \times S \to \mathbb{F},\ (\psi, \varphi) \to \tilde{\psi}\varphi$$

defines a scalar product we only have to verify that the reversion $\psi \to \tilde{\psi}$ is a right-to-left $\mathbb{F}$-semilinear map. For all $\psi \in S$, $\lambda \in \mathbb{F}$ we have $(\psi\lambda)\tilde{} = \tilde{\lambda}\tilde{\psi}$ where the map $\lambda \to \tilde{\lambda}$ is an anti-involution of the division algebra $\mathbb{F}$ (actually complex conjugation).

Multiplying a spinor $\psi \in S \subset \mathcal{C}\ell_3$ by an element $s \in \mathcal{C}\ell_3$ is a right $\mathbb{F}$-linear transformation $S \to S$, $\psi \to s\psi$. The automorphism group of the scalar product is formed by those right $\mathbb{F}$-linear transformations which preserve the scalar product, that is,

$$(s\psi)\tilde{}(s\varphi) = \tilde{\psi}\varphi \quad \text{for all} \quad \psi, \varphi \in S.$$

The automorphism group of the scalar product $\tilde{\psi}\varphi$ is seen to be the group $\{s \in \mathcal{C}\ell_3 \mid \tilde{s}s = 1\}$ which is isomorphic to the group of unitary 2×2-matrices,

$$U(2) = \{s \in \mathrm{Mat}(2, \mathbb{C}) \mid s^\dagger s = I\}.$$

We can also use the Clifford conjugate $u \to \bar{u}$ of $\mathcal{C}\ell_3$ to introduce a scalar product for spinors. In this case, the element

$$\bar{\psi}\varphi \simeq \begin{pmatrix} 0 & 0 \\ -\psi_2 & \psi_1 \end{pmatrix} \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \psi_1\varphi_2 - \psi_2\varphi_1 & 0 \end{pmatrix}$$

does not appear in the division ring $\mathbb{F} = f\mathcal{C}\ell_3 f$. However, we can find an invertible element $a \in \mathcal{C}\ell_3$ so that $a\bar{\psi}\varphi \in \mathbb{F}$, e.g. $a = \mathbf{e}_1$ or $a = \mathbf{e}_{31}$. The map

$$S \times S \to \mathbb{F},\ (\psi, \varphi) \to a\bar{\psi}\varphi$$

defines a scalar product. Writing

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we find that $a\bar{\psi}\varphi \simeq \tau(\psi)^\top J \tau(\varphi)$. Hence, the automorphism group $\{s \in \mathcal{C}\ell_3 \mid \bar{s}s = 1\}$ of the scalar product $a\bar{\psi}\varphi$ is the group of symplectic 2×2-matrices,

$$Sp(2, \mathbb{C}) = \{s \in \mathrm{Mat}(2, \mathbb{C}) \mid s^\top J s = J\}.$$

4.9 Spinor operators

Up till now spinors have been objects which have been operated upon. Next we will replace such passive spinors by active spinor operators. Instead of spinors

$$\psi = \begin{pmatrix} \psi_1 & 0 \\ \psi_2 & 0 \end{pmatrix} \in \mathcal{C}\ell_3 f$$

in minimal left ideals we will consider the following even elements:

$$\Psi = 2\,\mathrm{even}(\psi) = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \end{pmatrix} \in \mathcal{C}\ell_3^+,$$

also computed as $\Psi = \psi + \hat{\psi}$ for $\psi \in \mathcal{C}\ell_3 f$. Classically, the expectation values of the components of the spin have been determined in terms of the column spinor $\psi \in \mathbb{C}^2$ by computing the following three real numbers:

$$s_1 = \psi^\dagger \sigma_1 \psi, \quad s_2 = \psi^\dagger \sigma_2 \psi, \quad s_3 = \psi^\dagger \sigma_3 \psi.$$

In terms of $\psi \in \mathcal{C}\ell_3 f$ this computation could be repeated as

$$s_1 = 2\langle \psi \mathbf{e}_1 \tilde{\psi} \rangle_0, \quad s_2 = 2\langle \psi \mathbf{e}_2 \tilde{\psi} \rangle_0, \quad s_3 = 2\langle \psi \mathbf{e}_3 \tilde{\psi} \rangle_0.$$

However, in terms of $\Psi \in \mathcal{C}\ell_3^+$ we may compute $\mathbf{s} = s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3$ directly as

$$\mathbf{s} = \Psi \mathbf{e}_3 \tilde{\Psi}.$$

Since Ψ acts here like an operator, we call it a *spinor operator*. It should be emphasized that not only did we get all the components of the spin vector $\mathbf{s}$ at one stroke, but we also got the entity $\mathbf{s}$ as a whole.

Remark. The mapping $\mathcal{C}\ell_3^+ \to \mathbb{R}^3$, $\Psi \to \Psi\sigma_3\Psi^\dagger = \Psi\mathbf{e}_3\tilde{\Psi}$ is the KS-transformation (introduced by Kustaanheimo & Stiefel 1965) for spinor regularization of Kepler motion, and its restriction to norm-one spinor operators Ψ satisfying $\Psi\tilde{\Psi} = 1$ (or equivalently $\Psi\Psi^\dagger = I$) results in a Hopf fibration $S^3 \to S^2$ (the matrix $\Psi\sigma_3\Psi^\dagger$ is both unitary and involutory and represents a reflection of the spinor space with axis ψ).

The above mapping should not be confused with the 'Cartan map', see Cartan 1966 p. 41 and Keller & Rodríguez-Romo 1991 p. 1591. A 'Cartan map' $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathcal{C}\ell_3$, $(\psi, \varphi) \to 2\psi\mathbf{e}_1\bar{\varphi}$, where $\mathbb{C}^2 = \mathcal{C}\ell_3 f$, sends a pair of square matrix spinors to a complex 4-vector $x_0 + \mathbf{x}$,

$$x_0 = -(\psi_1\varphi_2 - \psi_2\varphi_1), \quad \mathbf{x} = \begin{pmatrix} \psi_1\varphi_1 - \psi_2\varphi_2 \\ i(\psi_1\varphi_1 + \psi_2\varphi_2) \\ -(\psi_1\varphi_2 + \psi_2\varphi_1) \end{pmatrix}.$$

When $\psi = \varphi$, $\mathbf{x}^2 = 0$. ∎

Note also that $\mathrm{trace}(\psi\psi^\dagger) = 2\langle \psi\tilde{\psi} \rangle_0 = \Psi\tilde{\Psi}$ which equals $\Psi\bar{\Psi} = \det(\Psi)$.

In operator form the Schrödinger-Pauli equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\pi^2\Psi - \frac{\hbar e}{2m}\vec{B}\Psi\mathbf{e}_3 - eV\Psi$$

shows explicitly the quantization direction $\mathbf{e}_3$ of the spin. The explicit occurrence of $\mathbf{e}_3$ is due to the injection $\mathbb{C}^2 \to \mathcal{C}\ell_3 f,\ \psi \to \Psi$; technically $2\,\mathrm{even}(\vec{B}\psi) = \vec{B}\Psi\mathbf{e}_3$. If we rotate the system $90°$ around the y-axis, counter-clockwise as seen from the positive y-axis, then vectors and spinors transform to

$$\vec{B}' = u\vec{B}u^{-1} \quad \text{and} \quad \Psi' = u\Psi \quad \text{where} \quad u = \exp(\frac{\pi}{4}\mathbf{e}_{13}),$$

and the Pauli equation transforms to

$$i\hbar\frac{\partial\Psi'}{\partial t} = \frac{1}{2m}\pi'^2\Psi' - \frac{\hbar e}{2m}\vec{B}'\Psi'\mathbf{e}_3 - eV\Psi'.$$

If this equation is multiplied on the right by u^{-1}, then $\mathbf{e}_3$ goes to $\mathbf{e}_1 = u\mathbf{e}_3u^{-1}$, and the equation looks like

$$i\hbar\frac{\partial\Psi''}{\partial t} = \frac{1}{2m}\pi'^2\Psi'' - \frac{\hbar e}{2m}\vec{B}'\Psi''\mathbf{e}_1 - eV\Psi'',$$

where $\Psi'' = u\Psi u^{-1}$. Both the transformation laws give the same values for observables, that is, $\Psi'\mathbf{e}_3\tilde{\Psi}' = \Psi''\mathbf{e}_1\tilde{\Psi}''$.

Exercises

1. Compute the square of $\mathbf{a} + \mathbf{b}\mathbf{e}_{123}$ where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.
2. Compute p^2, q^2 and pq for $p = \frac{1}{2}(1 + \mathbf{e}_3)$ and $q = \frac{1}{2}(1 - \mathbf{e}_3)$.
3. Compute the squares of $\frac{1}{2}(1 + \mathbf{e}_3) \pm \frac{1}{2}(1 - \mathbf{e}_3)\mathbf{e}_{12}$.
4. Find all the four square roots of $\cos\varphi + \mathbf{e}_{12}\sin\varphi$. Hint: $\mathbf{e}_{12}\mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_{12}$.
5. Find the exponentials of $\pm\frac{\pi}{2}(1 - \mathbf{e}_3)\mathbf{e}_{12}$. Hint: $\mathbf{e}_{12}$ and $\mathbf{e}_{123}$ commute [or $q = \frac{1}{2}(1 - \mathbf{e}_3)$ is an idempotent satisfying $q^2 = q$].
6. Let $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$ [$\alpha, \beta \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$]. Compute $u\bar{u}$.
7. Find the inverse of $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$. Hint: $u\bar{u}$ is of the form $x + y\mathbf{e}_{123},\ x, y \in \mathbb{R}$.
8. Find the exponential of $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$. Hint: compute $(\mathbf{a} + \mathbf{b}\mathbf{e}_{123})^2$.
9. Show that each non-zero even element in $\mathcal{C}\ell_3^+$ is invertible.
10. Show that $u\tilde{u} \in \mathbb{R} \oplus \mathbb{R}^3$ for all $u \in \mathcal{C}\ell_3$.
11. Show that $|u\mathbf{a}\tilde{u}| = |u|^2|\mathbf{a}|$ for $\mathbf{a} \in \mathbb{R}^3,\ u \in \mathbb{R} \oplus \bigwedge^2\mathbb{R}^3$.
12. Show that the norm on $\mathcal{C}\ell_3$, defined by $|u|^2 = \langle u\tilde{u}\rangle_0$, agrees with the

norm given by $|u|^2 = <u, u>$ where the symmetric bilinear product is determined by

$$\begin{aligned} <\alpha, \beta> &= \alpha\beta \quad \text{for} \quad \alpha, \beta \in \mathbb{R}, \\ <\mathbf{a}, \mathbf{b}> &= \mathbf{a} \cdot \mathbf{b} \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \end{aligned}$$

and by

$$<\mathbf{x}_1 \wedge \ldots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_k> = \begin{vmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \ldots & \mathbf{x}_1 \cdot \mathbf{y}_k \\ \vdots & \ddots & \vdots \\ \mathbf{x}_k \cdot \mathbf{y}_1 & \ldots & \mathbf{x}_k \cdot \mathbf{y}_k \end{vmatrix}$$

in $\bigwedge^k \mathbb{R}^3$, $k \geq 2$. [One also needs to assume orthogonality of the components in $\mathcal{C}\ell_3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3$.]

13. Show that the reflection across the plane of the bivector $\mathbf{A}$ is obtained by $\mathbf{r} \to \mathbf{r}' = -\mathbf{A}\mathbf{r}\mathbf{A}^{-1}$.
14. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$. Compute $\langle \mathbf{xyz} \rangle_1$ and $\langle \mathbf{xyz} \rangle_3$. Hint: use reversion.

Solutions

1. $(\mathbf{a} + \mathbf{b}\mathbf{e}_{123})^2 = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} + 2(\mathbf{a} \cdot \mathbf{b})\mathbf{e}_{123}$.
2. $p^2 = p$ and $q^2 = q$, that is, p and q are idempotents; and $pq = 0$ [and so there are zero-divisors in the Clifford algebra $\mathcal{C}\ell_3$].
3. $\mathbf{e}_3$ [this shows that vectors can have square roots].
4. $\pm(\cos \frac{\varphi}{2} + \mathbf{e}_{12} \sin \frac{\varphi}{2})$, $\pm\mathbf{e}_3(\cos \frac{\varphi}{2} + \mathbf{e}_{12} \sin \frac{\varphi}{2})$.
5. $\mathbf{e}_3$ [this shows that vectors also have logarithms].
6. $\alpha^2 - \beta^2 - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2(\alpha\beta - \mathbf{a} \cdot \mathbf{b})\mathbf{e}_{123}$.
8. Denote $r = \sqrt{(\mathbf{a} + \mathbf{b}\mathbf{e}_{123})^2} \in \mathbb{R} \oplus \bigwedge^3 \mathbb{R}^3$, $v = (\mathbf{a} + \mathbf{b}\mathbf{e}_{123})/r$, $v^2 = 1$. Then $\exp(u) = \exp(\alpha + \beta\mathbf{e}_{123})[\frac{1}{2}(1 + v)\exp(r) + \frac{1}{2}(1 - v)\exp(-r)]$ when $r \neq 0$. When $r = 0$: $\exp(u) = \exp(\alpha + \beta\mathbf{e}_{123})(1 + \mathbf{a} + \mathbf{b}\mathbf{e}_{123})$.
10. $u = \alpha + \mathbf{a} + \mathbf{b}\mathbf{e}_{123} + \beta\mathbf{e}_{123}$, $u\tilde{u} = \alpha^2 + \beta^2 + \mathbf{a}^2 + \mathbf{b}^2 + 2(\alpha\mathbf{a} + \beta\mathbf{b} + \mathbf{a} \times \mathbf{b})$ which is in $\mathbb{R} \oplus \mathbb{R}^3$. Direct proof:

$$(u\tilde{u})\tilde{} = \tilde{\tilde{u}}\tilde{u} = u\tilde{u}$$

which implies $u\tilde{u} \in \mathbb{R} \oplus \mathbb{R}^3$, since the reversion sends bivectors and 3-vectors to their opposites.

13. Decompose $\mathbf{r}$ into components parallel, $\mathbf{r}_{||}$, and perpendicular, $\mathbf{r}_{\perp}$, to $\mathbf{A}$, and note that $\mathbf{A}$ anticommutes with vectors in its plane, $\mathbf{A}(\mathbf{r}_{||} + \mathbf{r}_{\perp}) = (-\mathbf{r}_{||} + \mathbf{r}_{\perp})\mathbf{A}$. Then $\mathbf{A}(\mathbf{r}_{||} + \mathbf{r}_{\perp})\mathbf{A}^{-1} = (-\mathbf{r}_{||} + \mathbf{r}_{\perp})\mathbf{A}\mathbf{A}^{-1} = -\mathbf{r}'$.
14. First, $(\mathbf{xyz})\tilde{} = \mathbf{zyx}$ and $(\mathbf{xyz})\tilde{} = \langle \mathbf{xyz} \rangle_1 - \langle \mathbf{xyz} \rangle_3$. Therefore,

$\langle\mathbf{xyz}\rangle_1 = \frac{1}{2}(\mathbf{xyz}+\mathbf{zyx})$ and $\langle\mathbf{xyz}\rangle_3 = \frac{1}{2}(\mathbf{xyz}-\mathbf{zyx})$, and also
$\langle\mathbf{xyz}\rangle_1 = (\mathbf{y}\cdot\mathbf{z})\mathbf{x} - (\mathbf{z}\cdot\mathbf{x})\mathbf{y} + (\mathbf{x}\cdot\mathbf{y})\mathbf{z}$ and $\langle\mathbf{xyz}\rangle_3 = \mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z}$.

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