14

Definitions of the Clifford Algebra

In this chapter we shall for the first time give a formal definition of the Clifford algebra. There are several definitions, suitable for different purposes. In mathematics, definitions serve as premises for deductions; in physics, however, definitions are more or less secondary and serve as characterizations. We shall review Clifford's original definition, its basis-free variation given as a deformation of the exterior algebra, definition by the universal property, which does not guarantee existence, and the definition as an ideal of the tensor algebra. The construction of Chevalley, where Clifford algebra is regarded as a subalgebra of the endomorphism algebra of the exterior algebra, is postponed till the discussion on characteristic 2. The definitions by the multiplication table of the basis elements, and by index sets, are postponed till the chapter on the Walsh functions. The definition of Clifford algebras as group algebras of extra-special groups will be omitted.

14.1 Clifford's original definition

Grassmann's exterior algebra $\bigwedge \mathbb{R}^n$ of the linear space $\mathbb{R}^n$ is an associative algebra of dimension 2^n. In terms of a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^n$ the exterior algebra $\bigwedge \mathbb{R}^n$ has a basis

$$
\begin{aligned}
&1\\
&\mathbf{e}_1,\, \mathbf{e}_2, \ldots, \mathbf{e}_n\\
&\mathbf{e}_1 \wedge \mathbf{e}_2,\, \mathbf{e}_1 \wedge \mathbf{e}_3, \ldots, \mathbf{e}_1 \wedge \mathbf{e}_n,\, \mathbf{e}_2 \wedge \mathbf{e}_3, \ldots, \mathbf{e}_{n-1} \wedge \mathbf{e}_n\\
&\vdots\\
&\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_n.
\end{aligned}
$$

The exterior algebra has a unity 1 and satisfies the multiplication rules

$$\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i \quad \text{for} \quad i \neq j,$$
$$\mathbf{e}_i \wedge \mathbf{e}_i = 0.$$

Clifford 1882 kept the first rule but altered the second rule, and arrived at the multiplication rules

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j,$$
$$\mathbf{e}_i\mathbf{e}_i = 1.$$

This time $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for the positive definite Euclidean space $\mathbb{R}^n$. An associative algebra of dimension 2^n so defined is the Clifford algebra $\mathcal{C}\ell_n$.

Clifford had earlier, in 1878, considered the multiplication rules

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j,$$
$$\mathbf{e}_i\mathbf{e}_i = -1$$

of the Clifford algebra $\mathcal{C}\ell_{0,n}$ of the negative definite space $\mathbb{R}^{0,n}$.

14.2 Basis-free version of Clifford's definition

Here we consider as an example the exterior algebra $\bigwedge\mathbb{R}^4$ of the 4-dimensional real linear space $\mathbb{R}^4$. Provide the linear space $\mathbb{R}^4$ with a *quadratic form*

$$Q(\mathbf{x}) = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

and associate to Q the *symmetric* bilinear form

$$<\mathbf{x}, \mathbf{y}> = \frac{1}{2}[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})].$$

This makes $\mathbb{R}^4$ isometric with the Minkowski space-time $\mathbb{R}^{1,3}$. Then define the **left contraction** $u \rfloor v \in \bigwedge\mathbb{R}^{1,3}$ by

(*a*) $\quad \mathbf{x} \rfloor \mathbf{y} = <\mathbf{x}, \mathbf{y}>$

(*b*) $\quad \mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v)$

(*c*) $\quad (u \wedge v) \rfloor w = u \rfloor (v \rfloor w)$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{1,3}$ and $u, v, w \in \bigwedge\mathbb{R}^{1,3}$. [1] The identity (*b*) says that $\mathbf{x}$ operates like a *derivation* and the identity (*c*) makes $\bigwedge\mathbb{R}^{1,3}$ a left module over $\bigwedge\mathbb{R}^{1,3}$. Then introduce the *Clifford product* of $\mathbf{x} \in \mathbb{R}^{1,3}$ and $u \in \bigwedge\mathbb{R}^{1,3}$ by the formula

$$\mathbf{x}u = \mathbf{x} \rfloor u + \mathbf{x} \wedge u$$

1 Recall that $\hat{u}$ is the grade involute of $u \in \bigwedge V$, defined for a k-vector $u \in \bigwedge^k V$ by $\hat{u} = (-1)^k u$.

and extend this product by linearity and associativity to all of $\bigwedge \mathbb{R}^{1,3}$. Provided with the Clifford product (the linear space underlying) the exterior algebra $\bigwedge \mathbb{R}^{1,3}$ becomes the **Clifford algebra** $\mathcal{C}\ell_{1,3}$.

14.3 Definition by generators and relations

The following definition is favored by physicists. It is suitable for non-degenerate quadratic forms, especially the real quadratic spaces $\mathbb{R}^{p,q}$.

Definition. An associative algebra over $\mathbb{F}$ with unity 1 is the Clifford algebra $\mathcal{C}\ell(Q)$ of a non-degenerate Q on V if it contains V and $\mathbb{F} = \mathbb{F} \cdot 1$ as distinct subspaces so that

(1) $\quad \mathbf{x}^2 = Q(\mathbf{x})$ for any $\mathbf{x} \in V$

(2) $\quad V$ generates $\mathcal{C}\ell(Q)$ as an algebra over $\mathbb{F}$

(3) $\quad \mathcal{C}\ell(Q)$ is not generated by any proper subspace of V. ∎

The third condition (3) guarantees the universal property [see below], and dimension 2^n. Using an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ for $\mathbb{R}^{p,q}$, generating $\mathcal{C}\ell_{p,q}$, the condition (1) can be expressed as

$$(1.a) \qquad \mathbf{e}_i^2 = 1,\ 1 \le i \le p, \quad \mathbf{e}_i^2 = -1,\ p < i \le n, \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i,\ i < j,$$

while condition (3) becomes $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n \ne \pm 1$, as in Porteous 1969. Condition (3) is needed only in signatures $p - q = 1 \bmod 4$ where $(\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)^2 = 1$. The relations $(1.a)$ without (3) also generate a lower-dimensional non-universal algebra of dimension 2^{n-1} in any signature $p - q = 1 \bmod 4$ in which all the basis elements $\mathbf{e}_i$ commute with $\mathbf{e}_{12\ldots n} = \mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n$. No similar non-universal algebra exists in even dimensions, and so it is correct to introduce the Clifford algebra of the Minkowski space-time without condition (3). However, in arbitrary dimensions it is controversial to omit condition (3).

The above definition gives a unique algebra only for non-degenerate (non-singular) quadratic forms Q. In particular, the definition is not good for a degenerate Q, for which $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n = 0$, as is shown by the following two counter-examples where $Q = 0$.

1. Define for $\mathbf{x}, \mathbf{y} \in V$, $\dim V = n$, the product $\mathbf{x}\mathbf{y} = 0$. This makes the direct sum $\mathbb{F} \oplus V$ an associative algebra with unity 1. It is of dimension $n + 1$.

2. Introduce a product in $\bigwedge \mathbb{R}^3$ by $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$ for all $i, j = 1, 2, 3$ and $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = 0$. Thus the subspace $\mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$ of $\bigwedge \mathbb{R}^3$ is a 7-dimensional associative algebra with unity, generated by $\mathbb{R}$ and $\mathbb{R}^3$.

This shows that it is not possible to replace condition (3) by the requirement that only parallel vectors commute. We could include arbitrary quadratic forms Q by requiring instead of condition (3) that the product of any set of linearly free vectors in V should not belong to $\mathbb{F}$. However, even this would leave some 'ambiguity' in the definition by generators and relations. The above definition results in a unique algebra only 'up to isomorphism'. Here are two more examples to clarify the meaning of this statement:

3. The multiplication table of the exterior algebra $\bigwedge \mathbb{R}^2$ with respect to the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2\}$ is

$\wedge$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1 \wedge \mathbf{e}_2$
$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{e}_2$	0
$\mathbf{e}_2$	$-\mathbf{e}_1 \wedge \mathbf{e}_2$	0	0
$\mathbf{e}_1 \wedge \mathbf{e}_2$	0	0	0

Introduce a second product on $\bigwedge \mathbb{R}^2$ with multiplication table

$\dot\wedge$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1 \wedge \mathbf{e}_2$
$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{e}_2 + b$	$-b\mathbf{e}_1$
$\mathbf{e}_2$	$-\mathbf{e}_1 \wedge \mathbf{e}_2 - b$	0	$-b\mathbf{e}_2$
$\mathbf{e}_1 \wedge \mathbf{e}_2$	$-b\mathbf{e}_1$	$-b\mathbf{e}_2$	$-b^2 - 2b\mathbf{e}_1 \wedge \mathbf{e}_2$

where $b > 0$. Denote the second product by $u \dot\wedge v$. Rearrange the multiplication table of the second product into the form

$\dot\wedge$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1 \wedge \mathbf{e}_2 + b$
$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{e}_2 + b$	0
$\mathbf{e}_2$	$-\mathbf{e}_1 \wedge \mathbf{e}_2 - b$	0	0
$\mathbf{e}_1 \wedge \mathbf{e}_2 + b$	0	0	0

which shows that we have generated a new exterior algebra $\dot\bigwedge \mathbb{R}^2$ on $\mathbb{R}^2$, different from $\bigwedge \mathbb{R}^2$ but isomorphic with $\bigwedge \mathbb{R}^2$. In other words, we have introduced a linear mapping $\alpha : \bigwedge \mathbb{R}^2 \to \dot\bigwedge \mathbb{R}^2$ for which $\alpha(\mathbf{e}_i) = \mathbf{e}_i$, $i = 1, 2$, and $\alpha(\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_1 \dot\wedge \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 + b$ so that it is the identity on $\mathbb{R}^2$, preserves even-odd grading and gives an isomorphism between the two products, $\alpha(u \wedge v) = \alpha(u) \dot\wedge \alpha(v)$.

4. An orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ for $\mathbb{R}^2$ satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{ij}$ generates the Clifford algebra $\mathcal{C}\ell_2 = \mathcal{C}\ell_{2,0}$ with basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_{12}\}$ where $\mathbf{e}_{12} =$

$\mathbf{e}_1\mathbf{e}_2\,(= \mathbf{e}_1 \wedge \mathbf{e}_2)$. We have the following multiplication table for $C\ell_2$:

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

Introduce a second product on $C\ell_2$ with multiplication table

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}+b$	$\mathbf{e}_2 - b\mathbf{e}_1$
$\mathbf{e}_2$	$-\mathbf{e}_{12}-b$	1	$-\mathbf{e}_1 - b\mathbf{e}_2$
$\mathbf{e}_{12}$	$-\mathbf{e}_2 - b\mathbf{e}_1$	$\mathbf{e}_1 - b\mathbf{e}_2$	$-1 - b^2 - 2b\mathbf{e}_{12}$

The anticommutation relations $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{ij}$ are also satisfied by the new product, and one may directly verify associativity. As the real number b varies we have a family of different but isomorphic Clifford algebras on $\mathbb{R}^2$.

14.4 Universal object of quadratic algebras

The Clifford algebra $C\ell(Q)$ is the universal associative algebra over $\mathbb{F}$ generated by V with the relations $\mathbf{x}^2 = Q(\mathbf{x}),\ \mathbf{x} \in V$.

Let Q be the quadratic form on a linear space V over a field $\mathbb{F}$, and let A be an associative algebra over $\mathbb{F}$ with unity 1_A. A linear mapping $V \to A$, $\mathbf{x} \to \varphi_{\mathbf{x}}$ such that

$$(\varphi_{\mathbf{x}})^2 = Q(\mathbf{x}) \cdot 1_A \quad \text{for all} \quad \mathbf{x} \in V$$

is called a *Clifford map*. The subalgebra of A generated by $\mathbb{F} = \mathbb{F} \cdot 1_A$ and V (or more precisely by the images of $\mathbb{F}$ and V in A) will be called a *quadratic algebra*. [2] The **Clifford algebra** $C\ell(Q)$ is a quadratic algebra with a Clifford map $V \to C\ell(Q)$, $\mathbf{x} \to \gamma_{\mathbf{x}}$ such that for any Clifford map $\varphi : V \to A$ there exists a unique algebra homomorphism $\psi : C\ell(Q) \to A$ making the following diagram commutative:

$$\begin{tikzcd} V \arrow[r, "\gamma"] \arrow[dr, "\varphi"'] & C\ell(Q) \arrow[d, "\psi"] \\ & A \end{tikzcd} \qquad\qquad \varphi_{\mathbf{x}} = \psi(\gamma_{\mathbf{x}})$$

This definition says that **all** Clifford maps may be obtained from $\gamma : V \to C\ell(Q)$ which is thereby **universal**.

2 The term quadratic algebra is commonly used for something else: in a quadratic algebra each square x^2 is linearly dependent on x and 1.

The definition by the universal property is meaningful for an algebraist who knows categories and morphisms up to the theory of universal objects. A category contains objects and morphisms between the objects. Invertible morphisms are called isomorphisms. In a category there is an initial (resp. final) universal object U, if for any object A, there is a unique morphism $\alpha : U \to A$ (resp. $A \to U$). The universal objects are unique up to isomorphism. In many categories there exists trivially the final universal object, which often reduces to 0. The Clifford algebra is the initial universal object in the category of quadratic algebras.

Example. Consider the category of quadratic algebras on $\mathbb{R}^{p,q}$. In this category the initial universal object is the Clifford algebra $\mathcal{C}\ell_{p,q}$ of dimension 2^n and the final universal object is 0. Between these two objects there are no other objects, when $p - q \neq 1 \bmod 4$. However, there are four objects in this category, when $p - q = 1 \bmod 4$; between $\mathcal{C}\ell_{p,q}$ and 0 there are two algebras both of dimension 2^{n-1}; in one we have the relation $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n = 1$ and in the other $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n = -1$; these two algebras are not isomorphic in the category of quadratic algebras (the identity mapping on $\mathbb{R}^{p,q}$ does not extend to an isomorphism from one algebra to the other); however, they are isomorphic as associative algebras (in the category of all real algebras). ∎

The above definition of Clifford algebras is most suitable for an algebraist who wants to study Clifford algebras over commutative rings (and who does not insist on injectivity of mappings $\mathbb{F} \to A$ and $V \to A$). However, this approach does not guarantee existence, which is given by constructing the Clifford algebra as the quotient algebra of the tensor algebra (which in turn is regarded by algebraists as the mother of all algebras).

14.5 Clifford algebra as a quotient of the tensor algebra

Chevalley 1954 p. 37 constructs the Clifford algebra $\mathcal{C}\ell(Q)$ as the quotient algebra $\otimes V/I(Q)$ of the tensor algebra $\otimes V$ with respect to the two-sided ideal $I(Q)$ generated by the elements $\mathbf{x}\otimes\mathbf{x}-Q(\mathbf{x})$ where $\mathbf{x} \in V$. See also N. Bourbaki 1959 p. 139 and T.Y. Lam 1973 p. 103. The tensor algebra approach gives a proof of existence by construction – suitable for an algebraist who is interested in rapid access to the main properties of Clifford algebras over commutative rings.

In characteristic zero we may avoid quotient structures by making the exterior algebra $\bigwedge V$ concrete as the subspace of antisymmetric tensors in $\otimes V$. For example, if $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x}) \in \bigwedge^2 V$. More generally, a simple k-vector $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k$ is identified with [3]

$$Alt(\mathbf{x}_1 \otimes \mathbf{x}_2 \otimes \ldots \otimes \mathbf{x}_k) = \frac{1}{k!} \sum_{\pi} \operatorname{sign}(\pi)\, \mathbf{x}_{\pi(1)} \otimes \mathbf{x}_{\pi(2)} \otimes \ldots \otimes \mathbf{x}_{\pi(k)},$$

where the linear operator $Alt : \otimes V \to \bigwedge V$, called **alternation**, is a projection operator $Alt(\otimes V) = \bigwedge V$ satisfying $u \wedge v = Alt(u \otimes v)$.

Similarly, we may obtain an isomorphism of linear spaces $\bigwedge V \to C\ell(Q)$ by identifying simple k-vectors with antisymmetrized Clifford products

$$\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k \to \mathbf{x}_1 \dot{\wedge} \mathbf{x}_2 \dot{\wedge} \ldots \dot{\wedge} \mathbf{x}_k = \frac{1}{k!} \sum_{\pi} \operatorname{sign}(\pi)\, \mathbf{x}_{\pi(1)} \mathbf{x}_{\pi(2)} \ldots \mathbf{x}_{\pi(k)}$$

thus splitting the Clifford algebra $C\ell(Q)$ into fixed subspaces of k-vectors $\bigwedge^k V \subset C\ell(Q)$. Any orthogonal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of V gives a correspondence

$$\mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \ldots \wedge \mathbf{e}_{i_k} \to \mathbf{e}_{i_1} \dot{\wedge} \mathbf{e}_{i_2} \dot{\wedge} \ldots \dot{\wedge} \mathbf{e}_{i_k} = \mathbf{e}_{i_1} \mathbf{e}_{i_2} \ldots \mathbf{e}_{i_k}$$

of bases for $\bigwedge V$ and $C\ell(Q)$.

Exercises

1. Show that the subspace $Alt(\otimes V)$ of $\otimes V$ is not closed under the tensor product.
2. Show that $\mathbf{A} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{A} = \frac{1}{2}(\mathbf{AB} - \mathbf{BA})$ for bivectors $\mathbf{A}, \mathbf{B} \in \bigwedge^2 V$.

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3 Another alternative is to omit the factor $\frac{1}{k!}$. This gives in all characteristics a correspondence between the exterior product and the antisymmetrized tensor product.