# Chapter 16

# Multiparticle Problems in Bounded and Unbounded Domains

In this chapter the work of Power and Miranda is extended in two directions. The completion method is extended to multiparticle problems, also allowing a container surface, in such a way that the direct solution of mobility problems becomes possible. As a preliminary step the properties of the double layer are examined in the multisurface setting, and this will in essence show just that no new null functions arise from the interactions of the individual connected components of the boundary of the fluid domain. Then we show how tractions corresponding to the RBM of particles can be computed from a well-posed formulation.

For mobility problems the real strength of the double layer lies in that it cannot represent a total force or torque; these variables stand separately in the equations. In fact the double layer has other nice properties too, as will be noted when iterative solutions are considered.

## 16.1 The Double Layer on Multiple Surfaces

We shall directly start with a theorem that shows the behavior of the double layer on multiple surfaces. The triviality of the Jordan blocks will be necessary when iterative solutions are considered. Again the boundary surfaces are assumed to be similarly smooth, as in the previous sections.

**Theorem 3 (Particles with a container)** *Let $S_i$, $i = 1, \ldots, M$, be particle surfaces enclosed by a container surface $S_c$, the interior fluid domain being bound between these. Then* $\dim N(1+\mathcal{K}) = 6M+1$, *and double layer densities that coincide with RBM velocities on the particle surfaces are in this null space. Any other null functions in $N(1+\mathcal{K})$ are nonorthogonal to the surface normal on $S_c$, and the null function of the container alone (see Theorem 2) is one of these. All null functions in $N(1+\mathcal{K})$ generate zero velocity field in the interior,* *when used as double layer densities. The null functions of the adjoint second kind operator, forming $N(1+\mathcal{K}^*)$, are single layer densities that generate RBM velocities on the particle surfaces and zero velocity on $S_c$, the resulting tractions in the interior being proportional to these null functions; in particular a constant multiple of the surface normal on the totality of boundary surfaces is such a null function and it also generates zero velocity field. The $6M+1$ Jordan blocks are trivial.*

*Proof.* First consider the interior double layer problem. For each particle surface separately we know six linearly independent null functions, given by RBM-velocities. For the container boundary, if present, we know that there exists a null function. These all generate zero velocity on the other boundary components, and therefore are null functions for the multiboundary case as well. Thus the dimension of the null space is at least $6M + 1$.

Consider now the single layer null functions, generating zero tractions in the exterior. As the tractions vanish outside the container, no work is done on the imaginary fluid there, and due to the decay condition the velocity field must vanish. Similarly inside each particle there is no dissipation, and only RBM is allowed. As the single layer generated velocity field is continuous through the boundary surfaces, in the interior we also have RBM on the particle surfaces and zero velocity on the container. This gives $6M$ degrees of freedom (dof) for the velocity boundary conditions, and due to the indeterminacy in the base pressure the interior tractions have maximally $6M + 1$ dof. Due to the jump condition the single layer densities are proportional to these tractions.

Since the dimensions of the null spaces for the two adjoint problems above must coincide, this common dimension is exactly $6M+1$. Thus the double layer null functions are exactly those that are found for each connected component of the boundary separately (and linear combinations of these), while the single layer null functions follow from the jump condition applied to the analysis carried out above for this adjoint problem.

Showing that the Jordan blocks are trivial is done in very much the same way as with a single boundary surface. As a multiple of the surface normal on all of the bounding surfaces corresponds to constant pressure, it is one of the RBM tractions in $N(1+\mathcal{K}^*)$. To show that the corresponding Jordan block is trivial, again try to find a single layer density that gives exterior tractions equaling this null function. The tractions outside $S_c$ must then be a multiple of the surface normal, and, due to the single layer having no sources, no work is done in that component of the exterior. Thus the velocity field there is RBM, and being a decaying single layer generated velocity it is identically zero. Then the stress field must be just a constant pressure, and again being decaying it must vanish. This gives zero tractions outside $S_c$, showing that a nonzero multiple of the surface normal cannot be generated there. Thus neither the surface normal of the container alone nor the normal on all the boundary surfaces is in the range $R(1+\mathcal{K}^*)$.

Now it shall be proven that the Jordan blocks corresponding to the other single layer null functions than a multiple of the surface normal are also trivial.

Choose one such single layer null function $\psi$ generating RBM velocities on the particle surfaces and zero velocity on the container, the null function being proportional to the tractions in the interior. The flow field $\boldsymbol{V}$ in the interior is then not RBM, and so there is nonzero dissipation; this is the reason why the surface normal had to be inspected separately. The energy equation 14.14 implies that the null function $\psi$, being proportional to the tractions, is not orthogonal to the corresponding surface velocity $\boldsymbol{V}$. But this surface velocity is a null function of the adjoint double layer equation, $\boldsymbol{V} \in N(1+\mathcal{K})$, as shown above. If it were true that $\psi \in R(1+\mathcal{K}^*) = N(1+\mathcal{K})^\perp$, then $\psi$ would have to be orthogonal to $\boldsymbol{V}$, which is a contradiction.

That the double layer null functions generate zero velocity in all of the interior follows from the uniqueness of Stokes flows (with smooth tractions) that satisfy given velocity boundary conditions. $\diamondsuit$

Again corollaries follow. The tractions corresponding to RBMs of the particles with zero velocity on the container are null functions and therefore are continuous. Exactly such continuous velocity BCs that are orthogonal to these RBM tractions and orthogonal to the surface normal can be satisfied by a double layer generated Stokes flow. The former condition can again be interpreted as the total force and torque on each particle having to vanish, and the latter as requiring mass conservation.

A theorem similar to the previous holds for unbounded domains, and since the proof differs little from that above, it shall not be elaborated on.

**Theorem 4 (Particles without a container)** *Let $S_i$, $i = 1, \ldots, M$, be particle surfaces, the interior fluid domain being outside these. Then* $\dim N(1+\mathcal{K}) = 6M$, *and double layer densities that coincide with RBM velocities on the particle surfaces form this null space, giving zero flow fields in the interior. The null functions of the adjoint second kind operator, forming $N(1+\mathcal{K}^*)$, are single layer densities that generate RBM velocities on the particle surfaces, the resulting tractions in the interior being proportional to these null functions. The $6M$ Jordan blocks are trivial.*

The following theorem will be utilized on doing the 'mathematical deflations' in the case of particles with a container, when Wielandt deflations are applied to our integral equations to make iterative solutions converge. Physically the theorem has not much content: since we can deal with different connected components of fluid separately, only connected fluid domains are of interest. The theorem could also be useful on solving a problem with traction BCs, as the adjoint single layer problem deals with boundary conditions in the interior. (The theorem is again labelled according to the double layer BCs.)

**Theorem 5 (Exterior problem for particles with a container.)**
*Let $S_i$, $i = 1, \ldots, M$, be particle surfaces enclosed by a container surface $S_c$, the interior fluid domain being bound between these. Then* $\dim N(-1+\mathcal{K}) = 6+M$, *and double layer densities that coincide with an RBM velocity on the totality*

*of surfaces are in this null space. The null functions of the adjoint second kind operator, forming $N(-1+\mathcal{K}^*)$, are single layer densities on $S_c$ that generate an RBM velocity outside it, the resulting tractions outside $S_c$ being proportional to these null functions, and multiples of the surface normal on each particle surface. The $6+M$ Jordan blocks are trivial.*

*Proof.* It is easy to show that the particle surface normals $\psi_i$, $i = 1, \ldots, M$ are in $N(-1+\mathcal{K}^*)$. Similarly RBMs on the totality of surfaces $\varphi_j$, $j = 1, \ldots, 6$ , are in $N(-1+\mathcal{K})$. Since $\langle \psi_i, \varphi_j \rangle = 0$ for any $i$ and $j$, these null functions do not correspond to each other in the biorthogonal sense, but rather complement each other. Therefore $\dim N(-1+\mathcal{K}) \geq 6+M$.

Consider a single layer null function in $N(-1+\mathcal{K}^*)$, generating zero interior tractions. The interior velocity field must then be RBM, and due to continuity of the single layer generated velocity field the same RBM prevails inside the particles and on $S_c$ viewed from the outside. Now inside each particle the stress field is a constant pressure, and the traction jump over that surface is a constant multiple of the particle surface normal; on the particle surfaces the single layer density must be proportional to these according to the jump condition. The tractions outside $S_c$ correspond to RBM on $S_c$ and vanish at infinity, therefore having six dof. Thus in all $\dim N(-1+\mathcal{K}^*) \leq 6+M$. Together with the previous inequality this implies that $\dim N(-1+\mathcal{K}^*) = 6+M$, and all the single layer densities considered above must be in this null space.

Now we show the triviality of the Jordan blocks corresponding to the $\psi_i$'s. Assume to the contrary, that, for example,

$$\psi_1 = \begin{cases} \hat{\boldsymbol{n}} & \text{on } S_1 \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{16.1}$$

were in $R(-1+\mathcal{K}^*)$, *i.e.*, could be generated as interior tractions by a single layer. Then no work is done on the fluid in the interior, as these tractions are orthogonal to single layer generated flows without sources. Thus there is RBM in the interior, and the stress field is just a constant pressure. But then the tractions are a constant multiple of $\hat{\boldsymbol{n}}$ on the totality of surfaces and cannot coincide with $\psi_1$ — a contradiction.

Finally we verify the triviality of the Jordan blocks corresponding to the $\varphi_j$s. Assume to the contrary that, for example, $\varphi_1 \in R(-1+\mathcal{K}) = N(-1+\mathcal{K}^*)^{\perp}$. Then we could generate a nonzero RBM surface velocity outside $S_c$ with a double layer, corresponding to nonzero dissipation outside $S_c$. But the corresponding tractions outside $S_c$ are in $N(-1+\mathcal{K}^*)$, and due to orthogonality by the energy relation, there should be zero dissipation — a contradiction. $\diamondsuit$

All the previous theorems before this last one have dealt with connected fluid domains for the interior double layer problem. For those the results can be summarized simply by stating that *the interactions of boundary components generate no new double layer null functions.*

The integral equations 15.18 and 15.19 can be called *secondary variable formulations*, since the function solved does not *in general* represent any physically measurable quantity of interest to us, although in the theorems above the null functions were sometimes associated with RBM velocities and tractions. The jump conditions show the physical meanings of these solutions.

## 16.2 The Lyapunov-Smooth Container

The double layer eigenfunction with $\lambda = 1$ is not known explicitly for a container of arbitrary shape. Odqvist's proof that this eigenfunction generates zero flow field inside the container (when used as a double layer density) requires Hölder-continuous second derivatives in the parametric representation of the surface. When there are particles inside, the null function for the container alone will remain a null function for this multisurface problem only if it generates zero flow.

On the other hand, by inspection of the velocity representation 14.15, we see that RBMs as double layer densities generate no flow outside a Lyapunov (surface) particle. This asymmetry is removed in the following stronger version of Theorem 3 involving *Lyapunov-smooth* particle and container surfaces.

**Theorem 6 (Particles with a container)** *Let $S_i$, $i = 1, \ldots, M$, be Lyapunov-smooth particle surfaces enclosed by a Lyapunov-smooth container surface $S_c$, the interior fluid domain being bound between these. Then* $\dim N(1 + \mathcal{K}) = 6M + 1$, *and double layer densities that coincide with RBM velocities on the particle surfaces are in this null space. Any other null functions in $N(1+\mathcal{K})$ are nonorthogonal to the surface normal on $S_c$, and the null function of the container alone (see Theorem 2) is one of these. All null functions in $N(1 + \mathcal{K})$ generate zero velocity field in the interior, when used as double layer densities. The null functions of the adjoint second-kind operator, forming $N(1 + \mathcal{K}^*)$, are single layer densities that generate RBM velocities on the particle surfaces and zero velocity on $S_c$, the resulting tractions in the interior being proportional to these null functions; in particular a constant multiple of the surface normal on the totality of boundary surfaces is such a null function, and it also generates zero velocity field. The* $6M + 1$ *Jordan blocks are trivial.*

*Proof.* Using 14.15 with zero velocity and constant pressure in the interior, we see that a multiple of the surface normal as single layer density gives zero velocity fields in the components of the exterior. The velocity field, being continuous through all the surfaces, vanishes identically everywhere. Therefore in each connected component of the exterior and the interior there is just a constant pressure. Outside $S_c$ the chosen stress field is decaying, and being a constant pressure it must vanish. Due to the jump condition 15.17 the pressure jumps over $S_c$ and over any particle surface are equal in magnitude but opposite in direction, on passing from outside $S_c$ through the interior to inside a particle. Therefore inside the particles the pressure is again zero, and a multiple of the surface normal really is a null function.

To show that the corresponding Jordan block is trivial, again try to find a single layer density that gives exterior tractions equaling this null function.

The tractions outside $S_c$ must then be a multiple of the surface normal, and, due to the single layer having no sources, no work is done in that component of the exterior. Thus the velocity field there is RBM, and being a decaying single layer generated velocity it is identically zero. Then the stress field must be just a constant pressure, and again being decaying it must vanish. This gives zero tractions outside $S_c$, showing that a nonzero multiple of the surface normal cannot be generated there. Thus neither the surface normal of the container alone nor the normal on all the boundary surfaces is in the range $R(1+\mathcal{K}^*)$.

Consider now the single layer null functions in general. Since the tractions are zero inside the particles and outside the container, the energy equation 14.14 implies that dissipation in each connected component of the exterior is zero. Thus in each of these only RBM velocity fields are allowed, and outside the container the velocity field must vanish due to the decay condition. Since the single layer velocities are continuous through the surfaces, in the interior the velocity field has $6M$ dofs, because of the uniqueness of Stokes velocity fields in terms of the velocity boundary conditions. These dofs come from the three components of translation and rotation each, on each particle surface, the velocity vanishing on the container. The corresponding tractions have one more dof, because of the pressure indeterminacy in the interior. Thus the traction jumps giving the single layer densities that are null functions have at most $6M+1$ dofs.

Using the velocity representation 14.15 within any one of the particle surfaces with an RBM velocity $\boldsymbol{u}$ and zero tractions, it is seen that RBM densities are not only null functions for these individual surfaces, but they generate a zero velocity field everywhere outside the particles. Therefore these RBM densities on particle surfaces are null functions of the multisurface problem also. Here we have $6M$ linearly independent null functions in $N(1+\mathcal{K})$. If these were all the null functions, the surface normal of the container being orthogonal to these would be in the range $R(1+\mathcal{K}^*)$. Since this is not the case, there must be exactly $6M+1$ independent null functions, as it is known that this is the maximal number of them. In addition the null functions in $N(1+\mathcal{K})$ other than RBM on particle surfaces must be nonorthogonal to the surface normal on the container.

Now it shall be proven that the Jordan blocks corresponding to the other single layer null functions than a multiple of the surface normal are also trivial. Choose one such single layer null function $\psi$ generating RBM velocities on the particle surfaces and zero velocity on the container, the null function being proportional to the tractions in the interior. The flow field $\boldsymbol{V}$ in the interior is then not RBM, and so there is nonzero dissipation. The energy equation 14.14 implies that the null function $\psi$ is not orthogonal to the corresponding surface velocity $\boldsymbol{V}$. But this surface velocity is a null function of the adjoint double layer equation, $\boldsymbol{V} \in N(1+\mathcal{K})$, as shown above. If it were true that $\psi \in R(1+\mathcal{K}^*) = N(1+\mathcal{K})^{\perp}$, then $\psi$ would have to be orthogonal to $\boldsymbol{V}$, which is a contradiction.

Choose an orthonormal basis for $N(1+\mathcal{K})$, in which $6M$ of these null functions coincide with RBM velocities on the particle surfaces and vanish on the container. It shall be shown that the remaining basis element $\varphi$ is a null function of the container alone, and generates zero flow field inside the container. First one can show that any such tractions $\boldsymbol{T}$ that are orthogonal to RBM velocities on the particle surfaces and vanish on the container are in the range $\mathcal{R}(1+\mathcal{K}^*)$. If this were not the case, then necessarily $\langle \boldsymbol{T}, \varphi\rangle \neq 0$, since orthogonality to the remaining RBM null functions of the adjoint is satisfied by assumption. But also $\langle \hat{\boldsymbol{n}}, \varphi\rangle \neq 0$, whereas $\hat{\boldsymbol{n}}$ is orthogonal to the RBM null functions. Then there exists a constant $p$ such that $\boldsymbol{T} + p\hat{\boldsymbol{n}}$ is orthogonal to $\mathcal{N}(1+\mathcal{K})$ and therefore is in $\mathcal{R}(1+\mathcal{K}^*)$. But if $p \neq 0$, a constant multiple of the surface normal on the exterior side of the container surface results, and as noted before this is not possible for single layer generated tractions. Therefore $p = 0$ and $\boldsymbol{T} \in \mathcal{R}(1+\mathcal{K}^*)$.

Now $\varphi$ is orthogonal to both the RBM null functions and all $\boldsymbol{T}$ of the type above; these in turn span linearly all surface fields that vanish on the container. Therefore $\varphi$ can attain no nonzero values on the particle surfaces, and so is nonvanishing only on the container. But now $\varphi \in \mathcal{N}(1+\mathcal{K})$ is also in $\mathcal{N}(1+\mathcal{K})$ for the container surface alone, since it gives zero inside velocity on this surface and vanishes itself on the other components of the boundary — so it is the container null function in Theorem 2. As this container null function gives zero velocity on any enclosed particle surfaces, however these are chosen, the corresponding flow field is zero flow inside the container. This completes the proof of this theorem. $\diamondsuit$

Again, we emphasize that even though the container null function generates a Stokes velocity field that vanishes in the limit as $S_c$ is approached, one cannot apply the uniqueness theorem to infer that this velocity field is zero everywhere inside the container. This is because the proof of uniqueness utilized the tractions on the bounding surfaces. With a double layer on a Lyapunov surface there is no guarantee that the tractions exist as finite-valued functions, even if the double layer density is continuous. This situation was remedied in Theorem 3 by requiring the container surface to have Hölder-continuous second derivatives in suitably chosen Cartesian representation of the surface patchwise. Then the corresponding null function and its first derivatives will also be Hölder-continuous (*c.f.*, Odqvist), and the tractions are well defined.

In this discussion, these difficulties have been circumvented, and Odqvist's theory has been extended to Lyapunov surfaces in general, in a multisurface setting. Of course, for a Lyapunov-smooth container without particles we now see the behavior of the container null function: For any point in the fluid, use a virtual particle whose surface runs through this point. This mathematical trickery makes the proofs look fairly complicated, but the general idea of the proofs can be easily summarized. One needs to use the energy relation (which is allowed only with the single layer since the tractions are involved), continuity and jump properties through the surfaces, decay conditions in the unbounded component regions, and the special properties of RBM velocities and constant pressures.

## 16.3 The Canonical Equations

### 16.3.1 The Bordering Method in General

The method of Power and Miranda can be simply visualized in terms of *square* matrices. If a matrix is not of full rank its null space has the same dimension as the "deficiency" of its range space, *i.e.*, the orthogonal complement of its range. The dimension of the orthogonal complement of a subspace (here the range) is also called the *codimension* of the subspace. The result mentioned is often expressed as the row and column ranks of a matrix being equal. Now the matrix can be "completed" to an invertible matrix of full rank by linearly associating the null space with the missing part of the range space. As an example, the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{16.2}$$

has its null space and the orthogonal complement of its range spanned by $(1,0,-1)$ and $(0,0,1)$, respectively. Associating the component in null space with the missing part of the range space gives

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + a \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ a & 0 & -a \end{bmatrix} \quad \text{where } a \neq 0, \tag{16.3}$$

which is invertible. The added part here could be written in inner product notation when operating on a vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \langle \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^t, \begin{bmatrix} x & y & z \end{bmatrix}^t \rangle \tag{16.4}$$

to see the similarity with the equation of Power and Miranda. This association is arbitrary to some degree, as is here demonstrated by the freedom in choosing $a$. Also instead of the null vector, any vector with a nonzero component in the null space could be used, and there is similar freedom in choosing the completing range vector.

In the case at hand the multipliers of the completing vectors, namely, the total force and torque, are exactly those values that are wanted. Therefore we take these as actual new variables, and remove the indeterminacy in the double layer density by requiring it to be orthogonal to the original null space. Thus instead of adding something to the original matrix to complete it, here it is preferred to expand the matrix by bordering it with range vectors corresponding to added variables as new columns and orthogonality conditions to the original null space as added rows. With the example above the bordered matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & -1 & 0 \end{array}\right] \tag{16.5}$$

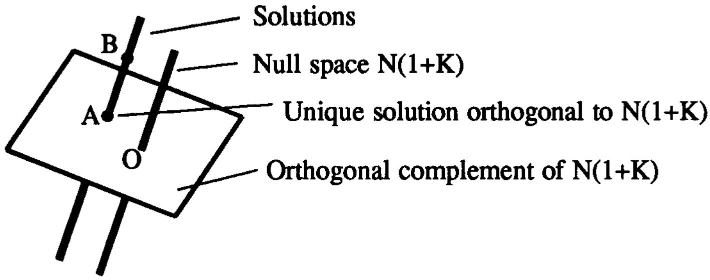


Figure 16.1: Solutions obtained from completion procedures.

and a 0 is added as the last element to the given vector to enforce the orthogonality condition. Since the dimension of the null space equals the codimension of the range (the number of completing range vectors needed), bordering will always keep the matrix square. This situation is also illustrated in Figure 16.1, where point $A$ represents the canonical solution (the solution that is orthogonal to the null space of the matrix operator), point $B$ represents some other solution obtained by the completion procedure, and the distance between $A$ and $B$ corresponds to the free parameter $a$.

*Second-kind operators with compact integral operators behave analogously. The dimension of the null space coincides with the codimension of the range, and the bordering procedure can be applied.* In the case of a single particle in infinite fluid the equations become

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{\eta}) &= \boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) + (\boldsymbol{F} - \frac{1}{2}\boldsymbol{T} \times \nabla) \cdot \mathcal{G}(\boldsymbol{\eta})/(8\pi\mu) \\ & \quad \text{for } \boldsymbol{\eta} \in S \qquad (16.6) \\ \langle \boldsymbol{\varphi}, \boldsymbol{\varphi}_i \rangle &= 0 \text{ for } i = 1, \ldots, 6 \;, \qquad (16.7) \end{aligned}$$

from which $\boldsymbol{F}$ and $\boldsymbol{T}$ can be directly solved. Note again that the origin can be freely chosen within the particle as we please, or actually the Stokeslet can be replaced with a cluster of Stokeslets whose contribution to the total torque must be separately considered. The equation system above is again of the second kind, although this is not obvious from the way the equations are written above, the operator now acting on the product space $L_2(S) \times R^6$.

### 16.3.2 On Removing the Null Functions

Admittedly the choice of the orthogonality conditions is again somewhat arbitrary. Actually the use of arbitrarily chosen orthogonality conditions—just having the right number of them—would almost surely remove the indeterminacy. This happens if the projections of the chosen vectors to the null space are linearly independent, *i.e.*, if the volume of the corresponding parallelepiped in the null space is nonzero. This volume being small although nonzero can, however, lead to numerical difficulties through an ill-conditioned linear system. Therefore it is better to use the null functions when they are known, or otherwise try to make a good choice for removing the indeterminacy caused by nontrivial null space.

The double layer representation can now be made fully determined for a multiparticle system within a container, since the exact number of null densities is known. In summary, the complete set of null functions is those given by RBMs on the particle surfaces plus the extra one for the container (if present). The last one, although not explicitly known, is denied by requiring orthogonality to the surface normal of the container. These are immediate consequences of Theorems 3 and 4.

### 16.3.3 Additions to the Range Space

Consider again a multiparticle system with or without a container. A double layer on the totality of surfaces can induce no total force or torque on any of the particles. Therefore one can add a Stokeslet and rotlet (or some fixed distribution of them) to the inside of each particle in sequence, knowing that from within one particle the flow fields of these will induce no force or torque on the surface of another — this argument shows that the codimension of the range is reduced at each step by the number of the functions added. For the container add a multiple of its surface normal (zero on the particle surfaces). According to Theorem 3 the range $R(1+\mathcal{K})$ is orthogonal to the surface normal on the totality of surfaces $S$, and so are the previous additions to the range, whereas this last one is not; therefore the codimension of the range is reduced again. Now the number of these additions to the range exactly matches the number of the original linearly independent null densities, which in turn equals the "deficiency" of the range, so it is known that the range is fully completed. *Thus a simple counting argument is used, such that it easily allows the treatment of multiparticle systems, even with a container.*

### 16.3.4 Summary of the Canonical Equations for Resistance and Mobility Problems

The following notation shall be used. There are $n$ particles with surfaces $S_i$, $i = 1, \ldots, n$, with or without a containing surface $S_c$. Each solid particle is cancelling the ambient velocity field $\boldsymbol{u}$ and in addition moving with some RBM velocity $\boldsymbol{U}_i + \boldsymbol{\omega}_i \times \boldsymbol{\eta}$, exerting force $\boldsymbol{F}_i$ and torque $\boldsymbol{T}_i$ on the fluid. On the containing surface, if present, some velocity boundary conditions are imposed for the flow field, given by $\boldsymbol{u}_c$; in this case the ambient velocity field $\boldsymbol{u}$ on the particle surfaces is set to zero. The whole equation system is

$$-\boldsymbol{u}(\boldsymbol{\eta}) + \boldsymbol{U}_i + \boldsymbol{\omega}_i \times \boldsymbol{\eta} =$$
$$\boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) + \sum_{j=1}^{n} \left\{ (\boldsymbol{F}_j - \frac{1}{2}\boldsymbol{T}_j \times \nabla) \cdot \mathcal{G}(\boldsymbol{\eta} - \boldsymbol{c}_j)/(8\pi\mu) \right\}$$

$$\text{for } \boldsymbol{\eta} \in S_i,\ i = 1, \ldots, n \tag{16.8}$$

$$\oint_{S_i} \boldsymbol{\varphi}(\boldsymbol{\xi}) \cdot \boldsymbol{\varphi}_{i,k}(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) = 0 \quad \text{for } i = 1, \ldots, n \text{ and } k = 1, \ldots, 6 \tag{16.9}$$

$$\begin{aligned} \boldsymbol{u}_c(\boldsymbol{\eta}) &= \boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) + \sum_{j=1}^{n} \left\{ (\boldsymbol{F}_j - \frac{1}{2}\boldsymbol{T}_j \times \nabla) \cdot \mathcal{G}(\boldsymbol{\eta} - \boldsymbol{c}_j)/(8\pi\mu) \right\} \\ &+ \ V\hat{\boldsymbol{n}} \qquad \text{for } \boldsymbol{\eta} \in S_c \end{aligned} \tag{16.10}$$

$$\oint_{S_c} \boldsymbol{\varphi}(\boldsymbol{\xi}) \cdot \hat{\boldsymbol{n}}(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) = 0\ . \tag{16.11}$$

Here $\boldsymbol{\varphi}_{i,k}$ are the RBM null densities for each of the particle surfaces, $\boldsymbol{c}_i$ the points inside each particle where the inside singularities are placed, and $V$ the added variable for the container surface (usually zero). These shall be called the *canonical equations* for resistance and mobility problems, because the boundary conditions (for resistance problems) or force/torque constraints (for mobility problems) can be directly imposed and the results of interest directly solved numerically. In a resistance problem $\boldsymbol{F}_i$ and $\boldsymbol{T}_i$ are solved for, whereas these are given in a mobility problem and one solves for $\boldsymbol{U}_i$ and $\boldsymbol{\omega}_i$. With most methods the total force and torque are not explicit variables in the equations and so mobility problems cannot be directly attacked.

Several exceptions are listed below. The exterior singularity distribution method of Dabros [12] provides an approximate method, but currently there is no proof that it would always allow the pursuit of high accuracy and lead to well-posed systems. The method of Dabros is actually a variant of the general method explained by Mathon and Johnston [48] (he does not refer to this earlier work), and a more recent application to elasticity problems is given by Han and Olson [28]. For two equal spheres Yoon and Kim [71] have examined the direct solution of mobility problems, based on boundary collocation with an expansion of the velocity field in terms of Lamb's general solution for each sphere individually. The results indicate that the direct solution of mobility problems provides us with accurate results over a wide range of surface separations, including surfaces almost touching. Jeffrey and Onishi [35] have considered both resistance and mobility problems for two unequal spheres, and Ganatos *et al.* [22] for multiple spheres. With most methods it is necessary to first solve a general resistance problem and then to invert the linear relationship connecting all the forces and torques with all the RBM velocities of the particles. Especially in problems where the surfaces are almost touching, this inversion may be ill-conditioned.

The traction surface field is almost never known *a priori* for a physical flow problem. An exception here is the drop deformation problem studied by Rallison and Acrivos [59], where surface tension together with surface shape gives a boundary condition for the tractions, and the IE in primary variables is of the second kind. Some problems with velocity BCs can be transformed to *exterior* problems with traction BCs, but this transformation is mainly of theoretical interest [37]. The remaining problems that are well suited for BIE are either resistance or mobility problems for a collection of rigid particles. For these problems the canonical equations provide a natural setting, with full mathematical rigor for any smooth surface shapes and guaranteed well-posedness of the discretized linear systems. Corners and edges change the IE so that the Fredholm–Riesz–Schauder theory seems not to be applicable any more. Through numerical examples the complications caused by edges on the surface of a single particle in infinite fluid shall be studied: It seems that the smoothness restriction for surfaces, although necessary for the mathematical development here, can be neglected in numerical applications to some extent.

The bordered double layer equations can naturally be applied to any Stokes flow problems with velocity boundary conditions, not just those involving solid particles. The integral representation for the disturbance flow field in the interior fluid domain is found from the solved double layer density and the total forces and torques by dropping the $\varphi$-terms and the $V\hat{\boldsymbol{n}}$-term from the RHSs of the canonical equations and substituting $\boldsymbol{\eta} \leftarrow \boldsymbol{y} \in Q_{(i)}$. Recall that the $\varphi$-term — giving a second-kind instead of a first-kind equation — came from a limiting procedure, when the double layer integral was evaluated *on the boundary surface* and this was used to replace the double layer representation as the surface is approached. In the interior fluid domain the double layer generated velocity field is $\boldsymbol{W}(\boldsymbol{y})$, as in Equation 15.6, and may also be denoted by $\mathcal{K}\varphi(\boldsymbol{y})$. (This notation is *not* consistent with the interpretation of $\mathcal{K}$ as a linear operator mapping surface fields to *surface* fields.)

The completed double layer representation finds greatest utility in numerical applications, but the following analytic examples for a single sphere illustrate the general properties of the double layer representation. Note that the double layer density $\varphi(\boldsymbol{\xi})$ can be determined from the jump properties of the double layer operator.

**Example 16.1** The Translating Sphere

For translation induced by an external force, $\boldsymbol{F}$, we know that the Stokes solution may be written in the singularity form,

$$\boldsymbol{v}(\boldsymbol{y}) = \boldsymbol{F}\cdot\frac{\mathcal{G}(\boldsymbol{y})}{8\pi\mu} + \frac{a^2}{6}\boldsymbol{F}\cdot\frac{\nabla^2\mathcal{G}(\boldsymbol{y})}{8\pi\mu} .$$

Thus the problem reduces to finding a double layer density $\varphi(\boldsymbol{\xi})$ that satisfies

$$\frac{a^2}{6}\boldsymbol{F}\cdot\nabla^2\frac{\mathcal{G}(\boldsymbol{y})}{8\pi\mu} = \oint_S \boldsymbol{K}(\boldsymbol{y},\boldsymbol{\xi})\cdot\varphi(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) .$$

We know that the velocity field induced by a double layer has a traction field that is continuous across the surface, while the velocity itself has a jump equal to twice the double layer density. So the solution strategy is to derive first the expression for the traction field of the degenerate Stokes quadrupole, and then find an interior solution that matches this traction. The inner and outer velocity fields will differ at the surface, this difference being exactly twice the desired density.

The degenerate quadrupole $(a^2/48\pi\mu)\boldsymbol{F}\cdot\nabla^2\mathcal{G}(\boldsymbol{y})$ has the stress field

$$\boldsymbol{\sigma} \;=\; -a^2|4\pi y|^5(\boldsymbol{F}\boldsymbol{y}+\boldsymbol{y}\boldsymbol{F}+\boldsymbol{F}\cdot\boldsymbol{y}\boldsymbol{\delta}) \;+\; \frac{5a^2}{4\pi|y|^7}\boldsymbol{F}\cdot\boldsymbol{y}\boldsymbol{y}\boldsymbol{y}\;,$$

so at the sphere surface $|\boldsymbol{y}| = a$ the traction is

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} \;=\; -\frac{\boldsymbol{F}}{4\pi a^4}(\boldsymbol{\delta}-3\hat{\boldsymbol{n}}\hat{\boldsymbol{n}})\;.$$

The interior velocity fields may be constructed from Lamb's general solution. However, from our experience with the Hadamard–Rybczynski drop problem, we recognize that the above matches the traction field of the Stokeson,

$$\begin{aligned}
\boldsymbol{v} &= 2|\boldsymbol{y}|^2\boldsymbol{U} - \boldsymbol{U}\cdot\boldsymbol{y}\boldsymbol{y}\\
p &= 10\mu\boldsymbol{U}\cdot\boldsymbol{y}\\
\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} &= 3\mu a\boldsymbol{U}(\boldsymbol{\delta}-3\hat{\boldsymbol{n}}\hat{\boldsymbol{n}})\;.
\end{aligned}$$

From a comparison of the two traction fields, we see that the required velocity field inside the sphere is

$$\boldsymbol{v}^{(i)} \;=\; \frac{\boldsymbol{F}}{12\pi\mu a^3}(\boldsymbol{y}\boldsymbol{y}-2|\boldsymbol{y}|^2\boldsymbol{\delta})\;.$$

The desired solution for the double layer density follows:

$$\begin{aligned}
\boldsymbol{\varphi}(\boldsymbol{\xi}) &= \frac{1}{2}(\boldsymbol{v}^{(o)}-\boldsymbol{v}^{(i)})\\
&= \frac{a^2}{96\pi\mu}\boldsymbol{F}\cdot\nabla^2\mathcal{G}(\boldsymbol{y})|_{y=a} \;-\; \frac{1}{24\pi\mu a}\boldsymbol{F}\cdot(\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}-2\boldsymbol{\delta})\\
&= \frac{1}{48\pi\mu a}\boldsymbol{F}\cdot(\boldsymbol{\delta}-3\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}) \;-\; \frac{1}{24\pi\mu a}\boldsymbol{F}\cdot(\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}-2\boldsymbol{\delta})\\
&= -\frac{5\boldsymbol{F}\cdot\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}}{48\pi\mu a} + \frac{5\boldsymbol{F}}{48\pi\mu a}\;.
\end{aligned}$$

To this particular solution, we may add any null function. The solutions of interest are all of the form

$$\boldsymbol{\varphi} \;=\; -\frac{5\boldsymbol{F}\cdot\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}}{48\pi\mu a} + C\frac{\boldsymbol{F}}{8\pi\mu a}\;.$$

We determine $C$ of the *canonical solution* from the orthogonality condition, $\langle\boldsymbol{\varphi},\boldsymbol{F}\rangle = 0$ as $C = 5/18$ so that

$$\boldsymbol{\varphi}(\boldsymbol{\xi}) \;=\; -\frac{5\boldsymbol{F}}{48\pi\mu a}\cdot(\hat{\boldsymbol{n}}\hat{\boldsymbol{n}}-\frac{1}{3}\boldsymbol{\delta}) \quad \text{(Canonical solution)}$$

The solution approach of Power and Miranda, with the force set proportional to the projection on the null space,

$$\boldsymbol{F}\cdot\boldsymbol{U} \;=\; \frac{\mu}{a} < \boldsymbol{U},\boldsymbol{\varphi} > \;,$$

leads to $C = 2 + 5/18$ and the solution,

$$\varphi(\xi) = -\frac{5\boldsymbol{F}}{48\pi\mu a}\cdot(\hat{\boldsymbol{n}}\hat{\boldsymbol{n}} - \frac{1}{3}\boldsymbol{\delta}) + \frac{\boldsymbol{F}}{4\pi\mu a} .$$

Note that for the canonical solution the density is proportional to the degenerate quadrupole. We will have more to say on this matter when we examine the spectrum of $\mathcal{K}$ acting on a single sphere in Section 17.2.

We conclude this example with a comment on the use of an off-center Stokeslet, thus illustrating the point that range completion is possible with any vector that contains a nonzero projection in the missing part of the range. If we place the Stokeslet at a distance $R < a$ from the origin, or $R\boldsymbol{F}/F$, instead of at the origin, we must add the following terms:

$$\begin{aligned}\tilde{\varphi}(\xi) &= \frac{F}{8\pi\mu a}\sum_{n=2}^{\infty}\left\{\frac{(R/a)^{n-1}}{2(n-1)}\left[\boldsymbol{e}_r n P_n(\cos\theta) + \boldsymbol{e}_\theta\frac{\partial P_n}{\partial\theta}\right]\right.\\ &\quad + \left(\frac{(R/a)^{n-1}}{2n+1} - \frac{(R/a)^{n+1}}{2n+3}\right)\frac{n(2n+1)(2n+3)}{2(2n^2+4n+3)}\\ &\quad \left.\times\left[-\boldsymbol{e}_r(n+1)P_n(\cos\theta) + \boldsymbol{e}_\theta\frac{\partial P_n}{\partial\theta}\right]\right\}\end{aligned}$$

to the double layer density. The techniques for deriving this expression are described in Chapter 4. For small $R/a$, the dominant, $O(R/a)$, part of the correction is that due to a double layer representation for a Stokes dipole. As $R/a \to 1$, the contributions from the higher order terms become significant and the highly oscillatory nature of $\varphi$ is the price paid for the poor choice of range completion.◇

**Example 16.2** The Rotating Sphere

This is a trivial example, because a rotlet placed at the sphere center produces the exact solution for the rotating sphere. Therefore, the canonical solution is $\varphi(\xi) = 0$.◇

**Example 16.3** Fixed Sphere in a Constant Rate-of-Strain Field

The disturbance field is

$$v_i(\boldsymbol{y}) = S_{jk}\mathcal{G}_{ij,k}(\boldsymbol{y})/8\pi\mu + \frac{a^2}{10}S_{jk}\frac{\nabla^2\mathcal{G}_{ij,k}(\boldsymbol{y})}{8\pi\mu} ,$$

and the double layer distribution must represent both the dipole and degenerate octupole terms. At $r = a$, the disturbance field generates the traction

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = \frac{9\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}{20\pi a^3} .$$

This traction is also generated by the linear interior velocity field,

$$\boldsymbol{v}^{(i)} = \frac{9\boldsymbol{S}\cdot\boldsymbol{y}}{40\pi\mu a^3},$$

and the double layer density follows as

$$\varphi(\boldsymbol{\xi}) = \frac{1}{2}(\boldsymbol{v}^{(o)} - \boldsymbol{v}^{(i)}) = \frac{-3\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}{40\pi\mu a^2} - \frac{9\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}{80\pi\mu a^2} = \frac{-3\boldsymbol{S}\cdot\hat{\boldsymbol{n}}}{16\pi\mu a^2}. \diamond$$

# 16.4 RBM-Tractions from the Riesz Representation Theorem

## 16.4.1 The Riesz Representation Theorem

In finite dimensional inner product spaces it is easy to see that any linear functional $L$ can be uniquely represented by a corresponding vector $\boldsymbol{v}$, the functional being the inner product (dot product) with this vector. For this purpose let $\boldsymbol{x} = \sum_{i=1}^{m} x_i\hat{e}_i$ be a general vector expressed in terms of the orthonormal basis vectors. Then

$$\begin{aligned} L(\boldsymbol{x}) &= \sum_{i=1}^{m} x_i L(\hat{e}_i) \\ &= \sum_{i=1}^{m} \langle \boldsymbol{x}, \hat{e}_i\rangle\ L(\hat{e}_i) \\ &= \langle\ \boldsymbol{x}, \sum_{i=1}^{m} \hat{e}_i L(\hat{e}_i)\ \rangle \\ &= \langle \boldsymbol{x}, \boldsymbol{v}\rangle \end{aligned}$$

shows the existence of $\boldsymbol{v}$. (The equations above require complex conjugation of the scalars absorbed into the second arguments of the inner products, in case the inner product is complex.) The uniqueness follows on substituting $\boldsymbol{x} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ into $\langle \boldsymbol{x}, \boldsymbol{v}_1\rangle = \langle \boldsymbol{x}, \boldsymbol{v}_2\rangle$. This unique representation of bounded linear functionals as inner products is extended to certain infinite dimensional spaces, the *Hilbert spaces*, by the *Riesz representation theorem* (see any of the references cited when discussing compact operators in the beginning of this part). For our purposes it suffices to know that the set of infinitely differentiable functions, and therefore also the larger class of functions with Hölder-continuous first derivatives, is a dense subspace of the Hilbert space whose elements are quadratically summable functions. Then any bounded linear functional on either of these subspaces is also uniquely representable by a quadratically summable function (which may or may not be continuous). This is because such a functional can uniquely be extended to a bounded functional on the whole Hilbert space of functions.

### 16.4.2 Force and Torque from Lorentz Reciprocal Theorem

Let again $S$ be the totality of bounding surfaces for a fluid volume. According to Lorentz reciprocal theorem

$$\oint_S \boldsymbol{u}(\boldsymbol{\xi}) \cdot \boldsymbol{\sigma}(\boldsymbol{\xi}; \boldsymbol{v}) \cdot \hat{\boldsymbol{n}}(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) = \oint_S \boldsymbol{v}(\boldsymbol{\xi}) \cdot \boldsymbol{\sigma}(\boldsymbol{\xi}; \boldsymbol{u}) \cdot \hat{\boldsymbol{n}}(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) \ , \tag{16.12}$$

or in inner product notation,

$$\langle \boldsymbol{u}, \boldsymbol{\sigma}(; \boldsymbol{v}) \cdot \hat{\boldsymbol{n}} \rangle = \langle \boldsymbol{v}, \boldsymbol{\sigma}(; \boldsymbol{u}) \cdot \hat{\boldsymbol{n}} \rangle \tag{16.13}$$

for any Stokes velocity fields $\boldsymbol{u}$ and $\boldsymbol{v}$ in the fluid, with smooth tractions. Choose $\boldsymbol{v}$ so that on one particle surface it equals $\boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{\eta}$ and is zero on the rest of $S$. This corresponds to moving one particle while holding the rest of the surfaces fixed. Then the equations above become

$$\langle \boldsymbol{u}, \boldsymbol{\sigma}(; \boldsymbol{v}) \cdot \hat{\boldsymbol{n}} \rangle = -\left( \boldsymbol{U} \cdot \boldsymbol{F} + \boldsymbol{\omega} \cdot \boldsymbol{T} \right) \ , \tag{16.14}$$

where $\boldsymbol{F}$ and $\boldsymbol{T}$ are the force and torque exerted by the particle surface on the fluid with flow field $\boldsymbol{u}$. From the theorems proven in this chapter we know that the tractions corresponding to $\boldsymbol{v}$ are single layer null functions and therefore are smooth. The same holds for $\boldsymbol{u}$ at least if this has Hölder-continuous first derivatives, since then the bordered double layer representation can be applied with a density of similar smoothness, and this results in smooth tractions. By choosing $\boldsymbol{v}$ suitably we can pick out components of $\boldsymbol{F}$ and $\boldsymbol{T}$ in the form of an inner product acting on the given field $\boldsymbol{u}$, *i.e.*, here we have the linear functionals that associate the surface velocity field to a corresponding force and torque, on a dense subspace of the underlying Hilbert space. These functionals are bounded since the tractions are smooth and thus quadratically summable, and we already have their representations in the form given by the Riesz representation theorem. This connection of RBM tractions to force and torque in a general flow field has been noted by Brenner (1964), but the *uniqueness* seems to have escaped notice so far. The uniqueness means in practice that these equations can be used to solve for the RBM tractions, provided that we already have a linear functional that maps a surface velocity field to a component of corresponding force or torque. Such functionals are supplied (at least as numerical approximations) by any method that can solve the general resistance problem, in particular by a numerical implementation of the canonical equations.

On applying the Lorentz reciprocal theorem to infinite fluid domains, we must again assume that the fields obey a decay condition enabling an "enclosing sphere expanded to infinity" argument. Then the pressure field corresponding to the RBM tractions is uniquely determined by this decay condition, and no undetermined constants arise. The case of a finite fluid domain with a containing surface is discussed below and in the next section.

We shall also present here a somewhat different view on applying the Lorentz and Riesz theorems to get RBM tractions. Consider the canonical equations, which are well-posed equations of the second kind. The linear functionals mapping the data to components of the solution are bounded, which is just another way of expressing the well-posedness. The Riesz representation theorem can be applied to the bounded linear functionals mapping the LHSs of the equation system to components of the forces and torques. Let $\boldsymbol{v}$ be the surface field that represents, in the sense of Riesz theorem, the mapping to the component $F_1$ acting from within some specified particle. If the data is in $R(1+\mathcal{K})$, *i.e.*, representable by a double layer alone, the total forces and torques must vanish. Thus $\boldsymbol{v}$ must be orthogonal to the range above, or equivalently it must be in $N(1+\mathcal{K}^*)$, and so it is the sum of an RBM traction field and the tractions due to constant pressure $p$ in the interior. With the Lorentz reciprocal theorem it is found that the RBM traction field must be that corresponding to translation of the particle in question in the "1-direction," while on the other boundaries this velocity field vanishes. This is most easily accomplished by choosing the "test field" $\boldsymbol{u}$ to be that due to the inside singularities (point forces and torques). The surface normal on the container alone corresponds to a solution where only $V$ is nonzero of all the solved quantities, and so must be orthogonal to $\boldsymbol{v}$; this condition determines the constant multiple of the surface normal in the RBM tractions, due to base pressure. Therefore again the RBM tractions correspond through the Riesz theorem to the functionals mapping velocity boundary conditions to components of forces and torques. If there is no container surface, the tractions correspond to a decaying stress field, as in Equation 15.19. If a container is present, RBM tractions are indeterminate up to a constant pressure, but on using Riesz theorem with the canonical equations this constant is determined, as was just shown. Later on, this connection of the RBM tractions to the functionals giving the total force and torque shall be applied in the numerical computations.

### 16.4.3 Tractions for Rigid-Body Motion

We now show how the existence and uniqueness considerations above can be utilized in practice. On solving our canonical equations we find the double layer density and the strengths of the inside singularities. Getting the surface tractions from these requires, in general, elaborate integrations. We should take the stress of the bordered double layer representation, and use this representation of the stress field in the fluid with the known double layer density distribution. However, for the particular case of RBM tractions on the particle surfaces, we can use the theory above to create an efficient numerically applicable procedure. On discretizing the IE we replace the infinite process of integration with some finite approximation to get a numerically solvable linear system. This same discretization replaces our original natural inner product with a discrete version $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \approx [\boldsymbol{A}, \boldsymbol{B}]$. Let the discrete linear system be $\boldsymbol{U} = \boldsymbol{K}\boldsymbol{X}$, where $\boldsymbol{U}$ contains the discretized velocity surface field and the zeroes for orthogonality conditions. Consider one component $F_1$ of one of the forces, a scalar that is also one of the variables in $\boldsymbol{X}$. We first extend the discrete inner product so that the added variables are accounted for. Choosing $\boldsymbol{C}$ properly we have

$$\begin{aligned} F_1 &= [\boldsymbol{C}, \boldsymbol{X}] \\ &= [\boldsymbol{C}, \boldsymbol{K}^{-1}\boldsymbol{U}] \\ &= [\boldsymbol{K}^{-t}\boldsymbol{C}, \boldsymbol{U}] \\ &= [\boldsymbol{D}, \boldsymbol{U}], \end{aligned}$$

where $\boldsymbol{D}$ is found by solving

$$\boldsymbol{K}^t \boldsymbol{D} = \boldsymbol{C}. \tag{16.15}$$

Because the extended part of $\boldsymbol{U}$ contains only zeroes for the orthogonality conditions, we can drop the extension also from $\boldsymbol{D}$ and consider the inner product with $\boldsymbol{U}$ as the discretized approximation alone. Since for a suitable RBM traction field $\boldsymbol{d}$, *and only for this*, we have

$$F_1 = \langle \boldsymbol{d}, \boldsymbol{u} \rangle, \text{ for any smooth } \boldsymbol{u}, \tag{16.16}$$

we must have a discrete approximation of this traction surface field in the current form of $\boldsymbol{D}$. *So we use the reciprocal theorem in exactly the opposite direction from how it conventionally is used and justify this with Riesz theorem.* Thus the discretized RBM tractions can be found by solving the transposed linear system, and after this computing the strengths of singularities reduce to computing dot products. If the forces and torques are to be found for several given flow fields, we save some time by just once solving for the vectors $\boldsymbol{D}$ and applying these to all of the flow fields — this being just the conventional way of applying Brenner's result. The line of reasoning above shows that any method that provides the forces and torques in *arbitrary* flow fields necessarily contains all the information about the RBM traction fields. The uniqueness part of Riesz representation theorem is essential to this deduction.

The particular case where we have a container deserves a further comment. Since the Stokes velocity field in a bounded volume is mass conserving on the totality of surfaces, the velocity surface fields $\boldsymbol{u}$ used with the reciprocal theorem are restricted by this one condition $\langle \boldsymbol{u}, \hat{\boldsymbol{n}} \rangle = 0$. Then the vector $\boldsymbol{d}$ is also unique only up to a constant multiple of $\hat{\boldsymbol{n}}$. This is in complete agreement with the fact that in a bounded volume the flow field determines the pressure only up to a constant. We have, however, added the new variable $V$ to complete the range of the container double layer, and this makes $F_1$ fully determined for *any* surface field $\boldsymbol{u}$ (given velocity BCs), including those that are not mass conserving. Therefore there is no further indeterminacy in our equations, but the pressure is automatically set to some fixed value.

Later on a modification of the canonical equations will be presented, which allows the efficient solution of mobility problems. For that case it can be shown, quite similarly as has been done above for resistance problems, that the functionals mapping ambient-velocity boundary conditions to translation and rotation velocities that keep the particles force- and torque-free are again represented by RBM tractions. The difference is that now the tractions are "mobility-based," corresponding to given forces and torques, instead of the "resistance-based" tractions corresponding to given translation and rotation velocities. These ideas are also brought out in the exercises.

## 16.5 The Stresslet

Above it was shown how the tractions corresponding to rigid-body motion *without any ambient field* can be numerically found from a well-posed formulation. Naturally for this case other weighted averages of the tractions can then be computed also, such as the stresslet. However, it turns out that the double layer singularity is closely related to the stresslet, so that for *solid particles* a completely general formula can be derived that relates the stresslet to the double layer density. This of course implies that, in complete analogy to the previous section, the surface tractions for a particle in a rate-of-strain field can be obtained directly by the combined use of the Riesz and Lorentz theorems.

As the reader may recall from Part II, the first terms in the multipole expansion, before the stresslet, involve the total force and torque, and these are directly solvable from the canonical equations. As these terms do not interact with the stresslet, we shall drop them from the completed (bordered) double layer representation, and inspect the double layer term alone.

Consider the multipole expansion of the double layer term:

$$\begin{aligned}\int_S K_{ij}(\boldsymbol{y},\boldsymbol{\xi})\varphi_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) &= \frac{\mathcal{P}_i(\boldsymbol{y})}{4\pi\mu}\oint_S \boldsymbol{\varphi}(\boldsymbol{\xi})\cdot\hat{\boldsymbol{n}}\, dS(\boldsymbol{\xi}) \\ &- \frac{1}{4\pi}\oint_S \hat{n}_k\varphi_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\ \left(\mathcal{G}_{ij,k}(\boldsymbol{y})+\mathcal{G}_{ik,j}(\boldsymbol{y})\right)\ +\ \cdots\ .\end{aligned}$$

The first term on the RHS vanishes because $2\boldsymbol{\varphi} = \boldsymbol{v}^{(o)} - \boldsymbol{v}^{(i)}$ and both velocity fields have no net flux through the particle surface. Since the stresslet is the coefficient of the symmetric Stokes dipole, we conclude that the stresslet is given exactly by the expression

$$\boldsymbol{S} = -2\mu\oint_S (\hat{\boldsymbol{n}}\boldsymbol{\varphi}+\boldsymbol{\varphi}\hat{\boldsymbol{n}})\ dS(\boldsymbol{\xi})\ .$$

This result for the stresslet should be compared with that obtained in Part II for the mobile interface; there the surface velocity played the same role as the double layer density (or more precisely, the density $2\boldsymbol{\varphi}$). Finally, we note that this more general result concerning the stresslet is consistent with the result for the sphere in a rate-of-strain field (Example 16.3).

# Exercises

**Exercise 16.1 The Adjoint of a Matrix.**

**Note:** In this exercise we exceptionally denote the complex conjugation of a scalar with the superscript $c$ for clarity.

Let $\boldsymbol{A} = (a_{ij})_{N\times N}$ be a complex matrix, and let the inner product between two complex $N$-vectors be $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_i y_i^c$. Show that the matrix $\boldsymbol{A}^*$ satisfying $\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{A}^* \boldsymbol{y} \rangle$, for all $\boldsymbol{x}$ and $\boldsymbol{y}$, is given by $\boldsymbol{A}^* = (a_{ij}^*)_{N \times N}$ with $a_{ij}^* = a_{ji}^c$. Deduce that $(\boldsymbol{A}^*)^* = \boldsymbol{A}$.

**Exercise 16.2 The Relation Between Null Space and Range.**
Show that for an ordinary square matrix $\boldsymbol{A}$ the relation $R(\boldsymbol{A}) = N(\boldsymbol{A}^*)^\perp$ holds. By substituting $\boldsymbol{A}^*$ for $\boldsymbol{A}$ derive the complementary relation $N(\boldsymbol{A}) = R(\boldsymbol{A}^*)^\perp$.
**Hint**: See the previous exercise.

**Exercise 16.3 The RBM Tractions for a Mobility Problem.**
Let $\boldsymbol{u}$ be a given velocity surface field, $\boldsymbol{v}_{RBM}$ the velocity field corresponding to total force $\boldsymbol{F}$ and torque $\boldsymbol{T}$ (mobility solution), and $\boldsymbol{u}_{RBM} = \boldsymbol{U} + \boldsymbol{\omega} \times \boldsymbol{r}$ such RBM velocity that $\boldsymbol{u} - \boldsymbol{u}_{RBM}$ is force- and torque-free. Show that

$$\boldsymbol{U} \cdot \boldsymbol{F} + \boldsymbol{\omega} \cdot \boldsymbol{T} = -\langle \boldsymbol{u}, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(; \boldsymbol{v}_{RBM}) \rangle \ .$$

**Hint:** Explain the following steps:

$$\begin{aligned} 0 &= \langle \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(; \boldsymbol{u} - \boldsymbol{u}_{RBM}), \boldsymbol{v}_{RBM} \rangle \\ &= -\langle \boldsymbol{u}_{RBM}, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(; \boldsymbol{v}_{RBM}) \rangle + \langle \boldsymbol{u}, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(; \boldsymbol{v}_{RBM}) \rangle \\ &= (\boldsymbol{U} \cdot \boldsymbol{F} + \boldsymbol{\omega} \cdot \boldsymbol{T}) + \langle \boldsymbol{u}, \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(; \boldsymbol{v}_{RBM}) \rangle \ . \end{aligned}$$

**Note:** This shows the physical significance of mobility-based tractions; they map a given velocity field to (components of) such an RBM that absorbs the total force and torque. Another view of the same fact is presented in the section on iterative solution of RBM tractions, in the next chapter, using the completed (deflated) double layer representation.

**Exercise 16.4 Unique Decomposition of Disturbance Fields.**
The integral representation of Stokes fields, involving both single and double layers, can be utilized to show that the disturbance field of a multisurface problem can be decomposed to the sum of disturbance fields of each individual surface. Show that such a decomposition is unique in case we have no container. State the smoothness conditions involved.
**Hint:** The integral over totality of surfaces is the sum of integrals over individual surfaces. For uniqueness use the completed double layer representation, known to have a unique solution.
**Note:** The significance of this result is that multiparticle problems are sometimes handled by matching single-particle disturbance fields, for example, a two-sphere problem can be handled by superposing Lamb's general solution for each sphere (see Chapter 13). Now we know *a priori* that the resulting system has a unique solution; we have not introduced any indeterminacies by artificial decomposition of the problem.