Chapter 4

Solutions in Spherical Coordinates

4.1 Introduction

In this chapter we consider the solution of the Stokes equations in spherical polar coordinates. The slow motion of a single sphere in various flow fields is an old subject, and the earliest solutions have been known for more than a century. However, our aim here is to present the subject in a unified framework and to build an inventory of analytical examples that will serve us well throughout the rest of the book. Indeed, recent work in the numerical analysis for the Stokes equations provides the motivation for new viewpoints for an old subject.

In Section 4.2, we describe Lamb's general solution for the Stokes equation. The connection with the multipole expansion is brought out, and the solutions for the sphere in translation, rotation, and linear fields are derived to illustrate the use of Lamb's solution. The elements of Lamb's solution do not form an orthogonal basis on the sphere surface. Hence, the matching of boundary conditions is far more involved than, for example, the corresponding problems in potential theory. In Sections 4.3 and 4.4, we discuss two variations of Lamb's solution, the so-called *adjoint method* involving the use of a dual basis, and an *orthonormal basis* for the surface vector fields of solutions to the Stokes equations. We conclude this chapter with a brief survey of the Stokes stream-function; again the main emphasis is the connection of this special technique for axisymmetric flows to more general methods.

4.2 Lamb's General Solution

In spherical polar coordinates (r, θ, ϕ) (see Figure 4.1), the Laplace equation $\nabla^2\Phi = 0$, separates to yield

$$\Phi_{nm}(r,\theta,\phi) = r^n P_n^m(\cos\theta)e^{im\phi} ,$$

where P_n^m, $n = 0, 1, 2, \ldots$, $-n \leq m \leq n$ are the associated Legendre functions [1, 42]. Lamb [53] has derived the analogous solutions for the Stokes equations.

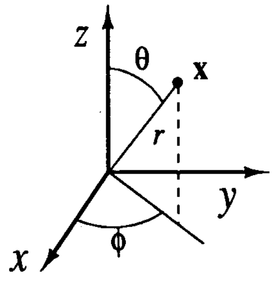

Figure 4.1: Spherical polar coordinates.

Since the pressure satisfies the Laplace equation, we may expand it as above,

$$p = \sum_{n=-\infty}^{\infty} p_n \ ,$$

where p_n is a solid spherical harmonic of order n,

$$p_n = r^n \sum_{m=0}^{n} P_n^m(\cos\theta)\left(a_{mn}\cos m\phi + \tilde{a}_{mn}\sin m\phi\right) \ .$$

The ϕ-dependence is often written in terms of the trigonometric functions as above [35, 53], to avoid the use of complex variables.

We write the momentum balance as

$$\nabla^2 \boldsymbol{v} = \frac{1}{\mu}\nabla p$$

and construct homogeneous and particular solutions of this partial differential equation. The result is *Lamb's general solution* (see also [35]):

$$\begin{aligned} \boldsymbol{v} \;&=\; \sum_{\substack{n=-\infty \\ n\neq 1}}^{\infty} \left[\frac{(n+3)r^2\nabla p_n}{2\mu(n+1)(2n+3)} - \frac{n\boldsymbol{x}p_n}{\mu(n+1)(2n+3)}\right] \\ &\quad + \sum_{n=-\infty}^{\infty} \left[\nabla\Phi_n + \nabla\times(\boldsymbol{x}\chi_n)\right] \ , \end{aligned} \tag{4.1}$$

where Φ_n and χ_n are also solid spherical harmonics of order n. The homogeneous solution is constructed from a potential (the $\nabla\Phi$ term) and a toroidal field (the $\nabla\times(\boldsymbol{x}\chi)$ term). In Exercise 4.1, we verify that the p_n terms form a particular solution.

In essence, Equation 4.1 reduces the task to that of solving the much easier problems, $\nabla^2 p_n = \nabla^2\Phi_n = \nabla^2\chi_n = 0$. The general solution for the harmonics

Φ_n and χ_n are also of the standard form,

$$\Phi_n = r^n \sum_{m=0}^{n} P_n^m(\cos\theta)\left(b_{mn}\cos m\phi + \tilde{b}_{mn}\sin m\phi\right) \tag{4.2}$$

$$\chi_n = r^n \sum_{m=0}^{n} P_n^m(\cos\theta)\left(c_{mn}\cos m\phi + \tilde{c}_{mn}\sin m\phi\right) . \tag{4.3}$$

Axisymmetric flow problems correspond to the case $m = 0$ and the azimuthal angle ϕ does not appear in the solution. The three scalar functions are then written as

$$p_n = a_n r^n P_n(\cos\theta) \tag{4.4}$$

$$\Phi_n = b_n r^n P_n(\cos\theta) \tag{4.5}$$

$$\chi_n = c_n r^n P_n(\cos\theta) , \tag{4.6}$$

where $P_n = P_n^0$ are the Legendre polynomials. Only p_n and Φ_n are needed to describe axisymmetric flows of the type $\boldsymbol{v} = v_r(r,\theta)\boldsymbol{e}_r + v_\theta(r,\theta)\boldsymbol{e}_\theta$, while only χ_n appears for swirling flows, $\boldsymbol{v} = v_\phi(r,\theta)\boldsymbol{e}_\phi$.

For *interior* flows such as that inside a spherical drop, we discard harmonics with $n < 0$ in Equation 4.1, since these fields are singular at the origin, and the expansion becomes

$$\boldsymbol{v} = \sum_{n=1}^{\infty}\left[\frac{(n+3)r^2\nabla p_n}{2\mu(n+1)(2n+3)} - \frac{n\boldsymbol{x}p_n}{\mu(n+1)(2n+3)}\right] + \sum_{n=1}^{\infty}\left[\nabla\Phi_n + \nabla\times(\boldsymbol{x}\chi_n)\right] . \tag{4.7}$$

(The case $n = 0$ gives $\boldsymbol{v}$ identically zero.)

On the other hand, for *exterior* flows such as a disturbance field outside a sphere, we discard the positive harmonics since we expect the disturbance fields to decay as $r \to \infty$. We also find it more convenient to work with positive integers, so we replace n in Equation 4.1 with $-n-1$ and obtain

$$\boldsymbol{v} = \sum_{n=1}^{\infty}\left[-\frac{(n-2)r^2\nabla p_{-n-1}}{2\mu n(2n-1)} + \frac{(n+1)\boldsymbol{x}p_{-n-1}}{\mu n(2n-1)}\right] + \sum_{n=1}^{\infty}\left[\nabla\Phi_{-n-1} + \nabla\times(\boldsymbol{x}\chi_{-n-1})\right] , \tag{4.8}$$

with

$$p_{-n-1} = r^{-n-1}\sum_{m=0}^{n} P_n^m(\cos\theta)\left(A_{mn}\cos m\phi + \tilde{A}_{mn}\sin m\phi\right) \tag{4.9}$$

$$\Phi_{-n-1} = r^{-n-1}\sum_{m=0}^{n} P_n^m(\cos\theta)\left(B_{mn}\cos m\phi + \tilde{B}_{mn}\sin m\phi\right) \tag{4.10}$$

$$\chi_{-n-1} = r^{-n-1}\sum_{m=0}^{n} P_n^m(\cos\theta)\left(C_{mn}\cos m\phi + \tilde{C}_{mn}\sin m\phi\right) . \tag{4.11}$$

The case $n = 0$, *e.g.*, Φ_{-1}, is associated with sources and sinks, and thus is not encountered in the physical description of disturbance flows produced by the motion of rigid particles. However, we will encounter these in the mathematical setting in Part IV.

As a representation in spherical polar coordinates, Lamb's solution is clearly ideal for flow problems involving a single sphere, but the utility of Lamb's solution extends beyond this obvious application. The velocity representation forms a *complete set* and thus provides the basis for numerical solution of multisphere problems in Parts III and IV.

4.2.1 The Connection with the Multipole Expansion

Lamb's original work simply pulls this solution out of thin air; given the expression, it is a simple matter to verify that it satisfies the Stokes equations. However, we may derive Lamb's solution in a systematic fashion starting with the multipole expansion, and work our way towards an expression involving simple combination of the scalar harmonics. In this manner, we show that Lamb's solution is simply a convenient parametrization of the more general multipole expansion in spherical polar coordinates.

We consider axisymmetric flows outside a sphere; the multipole expansion for $\boldsymbol{v}$ takes the form (see Exercise 4.2):

$$v_i = d_j \sum_{n=0}^{\infty} \left\{ a_n \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!} \mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{x}_1) + b_n \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!} \nabla^2 \mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{x}_1) \right\}, \quad (4.12)$$

with $\boldsymbol{d} = -\boldsymbol{e}_z$.

First, compare the two pressure representations,

$$p = -2\mu \sum_{n=0}^{\infty} a_n \frac{(\boldsymbol{d}\cdot\nabla)^{n+1}}{n!}\frac{1}{r} = -2\mu \sum_{n=1}^{\infty} n a_{n-1} \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{1}{r},$$

for the multipole expansion and

$$p = \sum_{n=1}^{\infty} A_{0n} r^{-n-1} P_n^0(\cos\theta)$$

for Lamb's general solution. Many useful properties of the Legendre polynomials and associated Legendre functions are found in [1] and [42]. One of the more important result is

$$r^{-n-1} P_n^0(\cos\theta) = \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{1}{r}$$

so that the pressure in Lamb's general solution may be rewritten as

$$p = \sum_{n=1}^{\infty} A_{0n} \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{1}{r}.$$

The two representations for the pressure match if we set $A_{0n} = -2\mu n a_{n-1}$ for $n \geq 1$.

We now consider the full velocity field. We start with

$$\boldsymbol{d}\cdot\mathcal{G} = -r^2\nabla(\boldsymbol{d}\cdot\nabla)\frac{1}{r} - 4\boldsymbol{x}(\boldsymbol{d}\cdot\nabla)\frac{1}{r} .$$

Apply $(\boldsymbol{d}\cdot\nabla)^n$ to both sides and use the Leibniz rule for the n-th derivative of a product. The result is

$$(\boldsymbol{d}\cdot\nabla)^n\boldsymbol{d}\cdot\mathcal{G} = -r^2\nabla\pi_{n+1} - 2n(\boldsymbol{d}\cdot\boldsymbol{x})\nabla\pi_n - n(n-1)\nabla\pi_{n-1} - 4\boldsymbol{x}\pi_{n+1} - 4n\boldsymbol{d}\pi_n .$$

We have introduced the notation $\pi_n = (\boldsymbol{d}\cdot\nabla)^n(1/r)$.

To recover Lamb's solution, we must eliminate terms in $(\boldsymbol{d}\cdot\boldsymbol{x})\nabla\pi_n$ and $\boldsymbol{d}\pi_n$ in the preceding equation. The second is readily accomplished with the aid of the following recursion formula for the Legendre polynomials:

$$\boldsymbol{d}\pi_n = \frac{-r^2\nabla\pi_{n+1}}{(n+1)(2n+1)} + \frac{n}{2n+1}\nabla\pi_{n-1} - \frac{(2n+3)\boldsymbol{x}\pi_{n+1}}{(n+1)(2n+1)} .$$

The other formula can also be derived from this one by applying the inner product with $\boldsymbol{d}$ to both sides, which gives

$$\pi_{n-1} = \frac{-r^2}{n(2n-1)}\pi_{n+1} + \frac{n-1}{2n-1}\pi_{n-1} - \frac{(2n+1)(\boldsymbol{d}\cdot\boldsymbol{x})}{n(2n-1)}\pi_n ,$$

(the index has been lowered by 1) and then taking the gradient, which gives

$$\begin{aligned}\nabla\pi_{n-1} &= \frac{-2\boldsymbol{x}}{n(2n-1)}\pi_{n+1} - \frac{r^2}{n(2n-1)}\nabla\pi_{n+1} + \frac{n-1}{2n-1}\nabla\pi_{n-1} \\ &- \frac{(2n+1)(\boldsymbol{d}\cdot\boldsymbol{x})}{n(2n-1)}\nabla\pi_n - \frac{(2n+1)}{n(2n-1)}\boldsymbol{d}\pi_n .\end{aligned}$$

This may be solved for $(\boldsymbol{d}\cdot\boldsymbol{x})\nabla\pi_n$, with the result,

$$(\boldsymbol{d}\cdot\boldsymbol{x})\nabla\pi_n = \frac{-2\boldsymbol{x}}{2n+1}\pi_{n+1} - \frac{r^2}{2n+1}\nabla\pi_{n+1} - \frac{n^2}{2n+1}\nabla\pi_{n-1} - \boldsymbol{d}\pi_n .$$

Thus, the a_n term in the multipole expansion can be rewritten as

$$\begin{aligned}a_n(\boldsymbol{d}\cdot\nabla)^n\boldsymbol{d}\cdot\mathcal{G} &= a_n\left[\frac{(n-1)r^2\nabla\pi_{n+1}}{(n+1)(2n+1)} - \frac{2(n+2)\,\boldsymbol{x}\pi_{n+1}}{(n+1)(2n+1)}\right] \\ &- a_n\frac{n(n-1)}{(2n+1)}\nabla\pi_{n-1} .\end{aligned}$$

The b_n terms in the multipole expansion are transformed with less effort since $\nabla^2\mathcal{G} = -2\nabla\nabla(1/r)$. Simply re-index the summation, collect terms in π_n to obtain the result:

$$\begin{aligned}\boldsymbol{v} &= \sum_{n=1}^{\infty} a_{n-1}\left[\frac{(n-2)r^2}{(2n-1)}\nabla\frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{1}{r} - \frac{2(n+1)\boldsymbol{x}}{(2n-1)}\frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{1}{r}\right] \\ &- \nabla\sum_{n=1}^{\infty}\left[\frac{n}{2n+3}a_{n+1}\frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{1}{r} + 2nb_{n-1}\right] ,\end{aligned}$$

which is Lamb's solution with

$$A_{0n} = -2\mu n a_{n-1} \qquad B_{0n} = -\frac{n}{2n+3}a_{n+1} - 2nb_{n-1} \ .$$

The relation between the χ_n term and multipoles of the form

$$\epsilon_{jkl}T_j \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{\partial}{\partial x_k}\mathcal{G}_{il} = 2\epsilon_{ijk}T_j\frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\frac{\partial}{\partial x_k}\frac{1}{r}$$

may be established in a similar manner (see Exercise 4.3).

We have successfully derived the coefficients in the p_n terms, although our synthesis applies only for the special case of axisymmetric and exterior flows. It is clear, however, that more general solutions, for both interior as well as exterior flows, are obtained by the use of harmonics of the form $r^n P_n^m(\cos\theta)\sin m\phi$ and $r^n P_n^m(\cos\theta)\cos m\phi$ for all integral n (see also Exercise 4.1).

4.2.2 Force, Torque, and Stresslet

Since the force and torque on any particle are given by the coefficients of the Stokeslet and rotlet in the multipole expansion, we examine the fields that decay as r^{-1} and r^{-2} in the exterior part of Lamb's general solution. We find (see Exercise 4.4) the simple formulae,

$$F_x = -4\pi A_{11} \qquad F_y = -4\pi\tilde{A}_{11} \qquad F_z = -4\pi A_{01}$$

$$T_x = -8\pi C_{11} \qquad T_y = -8\pi\tilde{C}_{11} \qquad T_z = -8\pi C_{01} \ ,$$

for the force and torque on a spherical surface enclosing the origin. A more succinct form is given in [35]

$$\boldsymbol{F} = -4\pi\nabla(r^3 p_{-2}) \qquad (4.13)$$

$$\boldsymbol{T} = -8\pi\mu\nabla(r^3\chi_{-2}) \ . \qquad (4.14)$$

An analogous result for the stresslet,

$$\boldsymbol{S} = -\frac{2\pi}{3}\nabla\nabla(r^5 p_{-3}) \ ,$$

is discussed in Exercise 4.4.

4.2.3 Matching of Boundary Conditions

The demonstration that an arbitrary velocity can be matched at the sphere boundary completes the proof that Lamb's solution is "general." Given boundary conditions for $\boldsymbol{v}$, the r, θ, and ϕ components of $\boldsymbol{v}$ provide three equations for the three unknown coefficients associated with the mode (m,n) (one coefficient per p, Φ, and χ). In the following discussion, we show three different implementations of this idea.

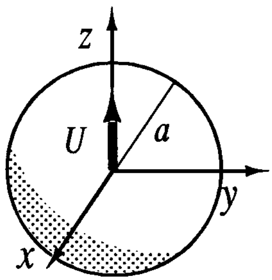

Figure 4.2: The translating sphere.

Matching of Radial Velocity, Surface Divergence, and Surface Curl

Given a boundary condition of the form $\boldsymbol{v} = \boldsymbol{V}_s(\theta, \phi)$ at $r = a$, we match the radial velocity, and in place of V_θ and V_ϕ, the surface divergence and radial component of the surface curl. Lamb's coefficients then follow as [11, 35]

$$V_r = \sum_{n=-\infty}^{\infty} \left\{ \frac{na\ p_n|_{r=a}}{2\mu(2n+3)} + \frac{n}{a}\Phi_n|_{r=a} \right\} \quad (4.15)$$

$$-r\nabla_s \cdot \boldsymbol{V}|_{r=a} = \sum_{n=-\infty}^{\infty} \left\{ \frac{n(n+1)a\ p_n|_{r=a}}{2\mu(2n+3)} + \frac{n(n-1)}{a}\Phi_n|_{r=a} \right\} \quad (4.16)$$

$$\boldsymbol{x} \cdot \nabla_s \times \boldsymbol{V} = \sum_{n=-\infty}^{\infty} \left\{ n(n+1)\chi_n|_{r=a} \right\} . \quad (4.17)$$

A complete discussion on the *surface divergence* and *surface curl* operators and their applications in matching interfacial boundary conditions is available in [74]. In spherical polar coordinates, the explicit expressions are

$$-r\nabla_s \cdot \boldsymbol{V} = -2V_r - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) - \frac{1}{\sin\theta}\frac{\partial V_\phi}{\partial\phi} \quad (4.18)$$

$$\boldsymbol{x} \cdot \nabla_s \times \boldsymbol{V} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(V_\phi \sin\theta) - \frac{1}{\sin\theta}\frac{\partial V_\theta}{\partial\phi} . \quad (4.19)$$

In most applications, the matching procedure is further facilitated because modes not present in the surface velocity, $\boldsymbol{V}$, need not be carried along in the analysis, as shown in the examples.

Example 4.1 The Translating Sphere.
We denote the sphere speed and radius by U and a, and take the translation to be in the z-direction, so that the surface velocity field is given by

$$\boldsymbol{V}_s(\theta, \phi) = U\boldsymbol{e}_z = U\cos\theta\ \boldsymbol{e}_r - U\sin\theta\ \boldsymbol{e}_\theta .$$

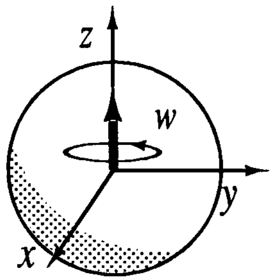

Figure 4.3: The rotating sphere.

From Equations 4.16 to 4.17, the relevant expressions for the boundary conditions are

$$\begin{aligned} V_r &= U\cos\theta \\ -r\nabla_s \cdot \boldsymbol{V}|_{r=a} &= 0 \\ \boldsymbol{x}\cdot\nabla\times\boldsymbol{V} &= 0\,. \end{aligned}$$

These expressions reveal that only $P_1(\cos\theta) = \cos\theta$ is required, the relevant harmonics being p_{-2} and Φ_{-2}, and that χ is not needed. With $n = -2$ in Equations 4.16 to 4.17, we find that $(\mu a U)^{-1}A_1 = 3/2$ and $(Ua^3)^{-1}B_1 = 1/4$, so that

$$p_{-2} = \frac{3}{2}\mu U a r^{-2}\cos\theta\,, \qquad \Phi_{-2} = \frac{1}{4}Ua^3r^{-2}\cos\theta\,,$$

and

$$\begin{aligned} v_r &= U\cos\theta\left[\frac{3}{2}\frac{a}{r} - \frac{1}{2}\left(\frac{a}{r}\right)^3\right] \\ v_\theta &= -U\sin\theta\left[\frac{3}{4}\frac{a}{r} + \frac{1}{4}\left(\frac{a}{r}\right)^3\right]\,. \end{aligned}$$

Using Equation 4.13 we obtain *Stokes law* for the drag exerted by the fluid on the translating sphere,

$$F_z = -4\pi A_1 U = -6\pi\mu a U\,.$$

The negative sign is appropriate since the fluid is resisting the motion. This result can also be obtained by integrating the surface traction, $\boldsymbol{\sigma}\cdot\boldsymbol{n}$, over the sphere surface.◇

Example 4.2 The Rotating Sphere.
Consider a rigid sphere of radius a rotating about the z-axis with angular

velocity, ω, in an otherwise quiescent fluid. At the sphere surface, we have $\boldsymbol{V}(\boldsymbol{x}) = \omega \boldsymbol{e}_z \times \boldsymbol{x}$, or simply

$$\boldsymbol{V} = \omega a \sin\theta\, \boldsymbol{e}_\phi \ .$$

The radial velocity and surface divergence vanish identically. The radial component of the surface vorticity yields the desired solution,

$$\begin{aligned} \boldsymbol{x}\cdot\nabla\times\boldsymbol{V} \;=\; 2\chi_{-2} &= 2C_1 a^{-2}\cos\theta \\ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(V_\phi\sin\theta) - \frac{1}{\sin\theta}\frac{\partial v_\theta}{\partial\phi} &= 2\omega a\cos\theta \ , \end{aligned}$$

so that $C_1 = \omega a^3$ and

$$\boldsymbol{v} = \nabla\times\left(\boldsymbol{x}\left\{C_1 r^{-2}\cos\theta\right\}\right) = \omega\boldsymbol{e}_z\times\boldsymbol{x}\frac{a^3}{r^3} \ .$$

From Equation 4.14 we deduce that the fluid resists this rotation by exerting a hydrodynamic torque, $T_z = -8\pi\mu a^3\omega$.◇

Example 4.3 Motion of a Spherical Drop Revisited (The Hadamard–Rybczynski Problem).
Consider a spherical drop of viscosity μ_i translating in another fluid of viscosity μ_o. Again, we denote the sphere speed and radius by U and a, and take the translation to be in the z-direction. Surface tension acts to retain the spherical shape while fluid motion distorts it. For large surface tensions, we may attempt a perturbation solution, assuming a spherical shape. The kinematic conditions require the velocities normal to the fluid-fluid interface to match the motion of the interface. In addition, we assume a no-slip condition so that the tangential components of the velocities and tractions are continuous across the interface. This leaves just the normal component of the tractions unsatisfied.

By assuming a drop shape, we have arbitrarily set a degree of freedom and, as a consequence, cannot satisfy this condition exactly. For large surface tensions, this small error may be corrected by perturbing the shape of the drop, in effect using this last condition to determine the shape of the drop. For the translation problem, rather fortuitously, the normal component of the tractions is matched properly. However, the more general situation is encountered when the drop is in an extensional flow.

From our experience with the boundary conditions for the rigid sphere of the previous example, we expect only $n = -2$ to appear in the exterior disturbance field (or $n = 1$ in the exterior representation, Equation 4.8). This in turn suggests that only $n = 1$ is used in the interior field. The exterior solution is then of the form,

$$v_r^{(o)} = \left\{A_1\left(\frac{a}{r}\right) - 2B_1\left(\frac{a}{r}\right)^3\right\}U\cos\theta \tag{4.20}$$

$$v_\theta^{(o)} = -\left\{\frac{A_1}{2}\left(\frac{a}{r}\right) + B_1\left(\frac{a}{r}\right)^3\right\}U\sin\theta \ . \tag{4.21}$$

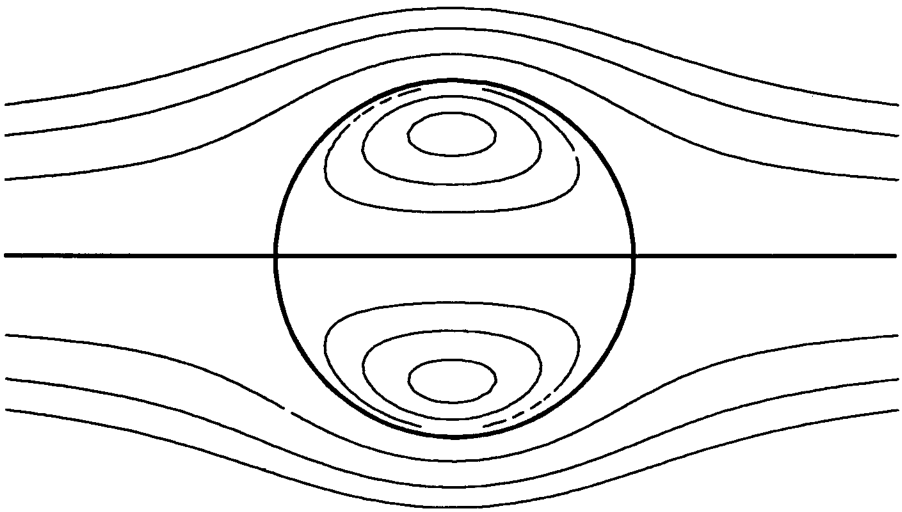

Figure 4.4: The translating drop, for $\mu_i = \mu_o$.

The velocity representation inside the drop is obtained from the p_1 and Φ_1 terms. We express these two harmonics as

$$p_1 = a_1 \frac{\mu U}{a} \frac{r}{a} \cos\theta \ , \qquad \Phi_1 = b_1 U r \cos\theta \ .$$

The interior solution simplifies to

$$v_r^{(i)} = \left\{ \frac{a_1}{10} \left(\frac{r}{a}\right)^2 + b_1 \right\} U \cos\theta \tag{4.22}$$

$$v_\theta^{(i)} = -\left\{ \frac{a_1}{5} \left(\frac{r}{a}\right)^2 + b_1 \right\} U \sin\theta \ . \tag{4.23}$$

The boundary conditions at $r = a$ are

1. $v_r^{(o)} = U\cos\theta$ (a kinematic condition)
2. $v_r^{(i)} = U\cos\theta$ (a kinematic condition)
3. v_θ continuous (the no-slip condition)
4. $d_{\theta r}$ continuous (matching of the tangential component of the surface tractions)

This leads to the following system of equations:

$$A_1 - 2B_1 = 1$$

$$
\begin{aligned}
\frac{1}{10}a_1 + b_1 &= 1 \\
\frac{1}{2}A_1 + B_1 - \frac{1}{5}a_1 - b_1 &= 0 \\
6B_1 + \frac{3}{10}\lambda a_1 &= 0 \,.
\end{aligned}
$$

Here, $\lambda = \mu_i/\mu_o$ is the ratio of inner and outer fluid viscosities. To obtain the fourth equation, we have used the following expression for the stress component $d_{\theta r}$:

$$d_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \,.$$

The solution of the preceding system of equations is

$$A_1 = \frac{2+3\lambda}{2(1+\lambda)}\,, \quad B_1 = \frac{\lambda}{4(1+\lambda)}\,, \quad a_1 = -\frac{5}{1+\lambda}\,, \quad b_1 = \frac{3+2\lambda}{2(1+\lambda)}\,.$$

The force exerted by the fluid on the drop is obtained from the exterior field as $-4\pi\mu_o a A_1 U$, or

$$F_z \;=\; -2\pi\mu_o a U\,\frac{2+3\lambda}{1+\lambda}\,.$$

The negative sign indicates that the fluid is resisting the motion. Note the limiting cases:

1. Rigid sphere; as $\lambda \to \infty$, $F_z \to -6\pi\mu_o a U$ and we recover Stokes law.
2. Bubble; for $\lambda = 0$, we obtain $F_z = -4\pi\mu_o a U$.

As an exercise, we ask the reader to examine the jump, $\boldsymbol{\sigma}^{(o)} : \hat{\boldsymbol{n}}\hat{\boldsymbol{n}} - \boldsymbol{\sigma}^{(i)} : \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}$ to check whether the assumed spherical drop shape is consistent with the force balance.◇

4.3 The Adjoint Method

There is another solution method for spherical coordinates that has been developed by Schmitz and Felderhof [73]. It is closely related to Lamb's solution, and we may call it the *adjoint method.* Define the velocity and pressure fields of Lamb's interior solution as

$$
\begin{aligned}
\boldsymbol{v}_{nm}^{(0)}(\boldsymbol{x}) &= \nabla\left\{r^n \tilde{Y}_n^m(\theta,\phi)\right\} \;=\; r^{n-1}\boldsymbol{A}_{nm}(\boldsymbol{x}) && (4.24)\\
\boldsymbol{v}_{nm}^{(1)}(\boldsymbol{x}) &= i\nabla\times\left\{\boldsymbol{x} r^n \tilde{Y}_n^m(\theta,\phi)\right\} \;=\; i r^n \boldsymbol{C}_{nm}(\boldsymbol{x}) && (4.25)\\
\boldsymbol{v}_{nm}^{(2)}(\boldsymbol{x}) &= \left\{\frac{(2n+1)(n+3)}{2n}r^2\nabla - (2n+1)\boldsymbol{x}\right\}\left\{r^n \tilde{Y}_n^m(\theta,\phi)\right\} \\
&= r^{n+1}\left[\frac{(n+1)(2n+3)}{2n}\boldsymbol{A}_{nm}(\boldsymbol{x}) + \boldsymbol{B}_{nm}(\boldsymbol{x})\right] && (4.26)\\
p_{nm}(\boldsymbol{x}) &= p_{nm}^{(2)}(\boldsymbol{x}) \;=\; \mu(n+1)(2n+1)(2n+3) r^n \tilde{Y}_n^m(\theta,\phi)\,. && (4.27)
\end{aligned}
$$

The explicit expressions for the surface vector fields $\boldsymbol{A}_{nm}$, $\boldsymbol{B}_{nm}$ and $\boldsymbol{C}_{nm}$ are obtained readily from the vector operators and are

$$\begin{aligned}
\boldsymbol{A}_{nm} &= n\widetilde{Y}_n^m \boldsymbol{e}_r + \frac{\partial \widetilde{Y}_n^m}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{\sin\theta}\frac{\partial \widetilde{Y}_n^m}{\partial\phi}\boldsymbol{e}_\phi \\
\boldsymbol{B}_{nm} &= -(n+1)\widetilde{Y}_n^m \boldsymbol{e}_r + \frac{\partial \widetilde{Y}_n^m}{\partial\theta}\boldsymbol{e}_\theta + \frac{1}{\sin\theta}\frac{\partial \widetilde{Y}_n^m}{\partial\phi}\boldsymbol{e}_\phi \\
\boldsymbol{C}_{nm} &= \frac{1}{\sin\theta}\frac{\partial \widetilde{Y}_n^m}{\partial\phi}\boldsymbol{e}_\theta - \frac{\partial \widetilde{Y}_n^m}{\partial\theta}\boldsymbol{e}_\phi ,
\end{aligned}$$

where the functions $\widetilde{Y}_n^m(\theta,\phi)$ are related to the normalized spherical harmonics $Y_n^m(\theta,\phi)$ by

$$\widetilde{Y}_n^m(\theta,\phi) = \eta_{nm}Y_n^m(\theta,\phi) = (-1)^m P_n^m(\cos\theta)e^{im\phi}$$

and η_{nm} is the normalization constant,

$$\eta_{nm} = \left[\frac{4\pi}{2n+1}\frac{(n+m)!}{(n-m)!}\right]^{1/2} .$$

The surface integral of products of $\boldsymbol{A}_{nm}$, $\boldsymbol{B}_{nm}$, and $\boldsymbol{C}_{nm}$ are derived in Exercise 4.5 and are summarized below:

$$\begin{aligned}
\oint_S \boldsymbol{A}_{lk}^* \cdot \boldsymbol{A}_{nm}\, dS &= n(2n+1)\eta_{nm}^2 a^2 \delta_{ln}\delta_{km} \\
\oint_S \boldsymbol{B}_{lk}^* \cdot \boldsymbol{B}_{nm}\, dS &= (n+1)(2n+1)\eta_{nm}^2 a^2 \delta_{ln}\delta_{km} \\
\oint_S \boldsymbol{C}_{lk}^* \cdot \boldsymbol{C}_{nm}\, dS &= n(n+1)\eta_{nm}^2 a^2 \delta_{ln}\delta_{km} \\
\oint_S \boldsymbol{A}_{lk}^* \cdot \boldsymbol{B}_{nm}\, dS &= \oint_S \boldsymbol{B}_{lk}^* \cdot \boldsymbol{A}_{nm}\, dS \;=\; 0 \\
\oint_S \boldsymbol{B}_{lk}^* \cdot \boldsymbol{C}_{nm}\, dS &= \oint_S \boldsymbol{C}_{lk}^* \cdot \boldsymbol{B}_{nm}\, dS \;=\; 0 \\
\oint_S \boldsymbol{C}_{lk}^* \cdot \boldsymbol{A}_{nm}\, dS &= \oint_S \boldsymbol{A}_{lk}^* \cdot \boldsymbol{C}_{nm}\, dS \;=\; 0 .
\end{aligned}$$

The essential idea behind the method is the existence of an adjoint set (or dual basis) $\{\boldsymbol{w}_{nm}^{(0)}, \boldsymbol{w}_{nm}^{(1)}, \boldsymbol{w}_{nm}^{(2)}\}$,

$$\boldsymbol{w}_{nm}^{(0)} = \frac{r^{-n}}{n(2n+1)\eta_{nm}^2}\left[\boldsymbol{A}_{nm} - \frac{2n+3}{2}\boldsymbol{B}_{nm}\right] \tag{4.28}$$

$$\boldsymbol{w}_{nm}^{(1)} = \frac{ir^{-n-1}}{n(n+1)\eta_{nm}^2}\boldsymbol{C}_{nm} \tag{4.29}$$

$$\boldsymbol{w}_{nm}^{(2)} = \frac{r^{-n-2}}{(n+1)(2n+1)\eta_{nm}^2}\boldsymbol{B}_{nm} , \tag{4.30}$$

that satisfies the orthonormality relations (see Exercise 4.5),

$$\frac{1}{a}\oint_S (\boldsymbol{w}_{lk}^{(i)})^* \cdot \boldsymbol{v}_{nm}^{(j)}\, dS = \delta_{ij}\delta_{ln}\delta_{km} .$$

Here, a is the sphere radius and the asterisk $(*)$ denotes complex conjugation. The surface integration of the vector dot product is in fact a well-known inner product (denoted $\langle \bullet, \bullet \rangle$), which will be encountered again in Part IV.

Suppose that we wish to match the boundary condition, $\boldsymbol{v} = \boldsymbol{V}(\theta, \phi)$, on the sphere surface. We write Lamb's general solution as

$$\boldsymbol{v} = \sum_n \sum_m \left\{ a_{mn}^{(0)} \boldsymbol{v}_{nm}^{(0)} + a_{mn}^{(1)} \boldsymbol{v}_{nm}^{(1)} + a_{mn}^{(2)} \boldsymbol{v}_{nm}^{(2)} \right\}$$

and take the inner product of both sides with fields from the adjoint set. The orthonormal property yields immediately the explicit solution for the coefficients,

$$a_{mn}^{(i)} = \frac{1}{a} \oint_S \boldsymbol{V}^* \cdot \boldsymbol{w}_{nm}^{(i)} \, dS \; .$$

For *exterior* fields, we write Lamb's solution as

$$p_{nm}(\boldsymbol{x}) = p_{nm}^{(0)}(\boldsymbol{x}) = \frac{\mu}{2n+1} r^{-n-1} \tilde{Y}_n^m(\theta, \phi) \tag{4.31}$$

$$\boldsymbol{v}_{nm}^{(0)}(\boldsymbol{x}) = \frac{r^{-n}}{(2n+1)^2} \left[\frac{n+1}{n(2n-1)} \boldsymbol{A}_{nm}(\boldsymbol{x}) - \frac{1}{2} \boldsymbol{B}_{nm}(\boldsymbol{x}) \right] \tag{4.32}$$

$$\boldsymbol{v}_{nm}^{(1)}(\boldsymbol{x}) = \frac{i r^{-n-1}}{n(n+1)(2n+1)} \boldsymbol{C}_{nm}(\boldsymbol{x}) \tag{4.33}$$

$$\boldsymbol{v}_{nm}^{(2)}(\boldsymbol{x}) = \frac{n r^{-n-2}}{(n+1)(2n+1)^2(2n+3)} \boldsymbol{B}_{nm}(\boldsymbol{x}) \; , \tag{4.34}$$

and the adjoint fields are given by the expressions,

$$\boldsymbol{w}_{nm}^{(0)} = \frac{(2n-1)(2n+1)}{(n+1)\eta_{nm}^2} r^{n-1} \boldsymbol{A}_{nm} \tag{4.35}$$

$$\boldsymbol{w}_{nm}^{(1)} = \frac{i(2n+1)}{\eta_{nm}^2} r^n \boldsymbol{C}_{nm} \tag{4.36}$$

$$\boldsymbol{w}_{nm}^{(2)} = \frac{(2n+1)(2n+3)}{n\eta_{nm}^2} r^{n+1} \left[\frac{2n-1}{2} \boldsymbol{A}_{nm} + \boldsymbol{B}_{nm} \right] . \tag{4.37}$$

The problem of the translating sphere can also be solved by the adjoint method. We set $m = 0$, since the problem is axisymmetric and consider the representation for exterior flows. In Exercise 4.7 we ask the reader to show that the inner product, $\langle \boldsymbol{w}_{n0}^{(i)}, U\boldsymbol{e}_z \rangle$, vanishes except for $n = 1$. This then leads to the same solution as before.

4.4 An Orthonormal Basis for Stokes Flow

The solid spherical harmonics, in addition to being solutions of the Laplace equation, form an orthonormal basis for *scalar* functions on the sphere surface. In other words, the solid spherical harmonics $\{\psi_n\}$ form a convenient basis for the expansion of a surface function, $f(\theta,\phi) = \sum_n a_n \psi_n(\theta,\phi)$, because of the orthonormal property satisfied by the ψ_n,

$$\int_0^{2\pi}\int_0^{\pi} \psi_m \psi_n \sin\theta \, d\theta d\phi = \delta_{mn} \; .$$

We have already seen that the elements of Lamb's solution do not form an orthonormal basis for *vector* functions on the sphere surface. However, the solution set is almost decoupled, since the different modes are mutually orthogonal, and since for a given mode the toroidal field is orthogonal to the other two. By taking the appropriate linear combination of the particular and homogeneous solutions, we induce a Gram-Schmidt orthogonalization and obtain an orthonormal basis:

$$\begin{aligned} \varphi^{(1)}_{nm} &= a^{n-1}\sqrt{n(2n+1)}\frac{(2n-1)}{\eta_{nm}(n+1)} \\ &\quad \times\left\{-\frac{(n-2)r^2\nabla}{2n(2n-1)}+\frac{(n+1)\boldsymbol{x}}{n(2n-1)}+\frac{a^2\nabla}{2(2n+1)}\right\}\left\{r^{-n-1}\tilde{Y}^m_n\right\} \\ &= \frac{\sqrt{n(2n+1)}}{a\eta_{nm}}\left[\frac{(a/r)^n\boldsymbol{A}_{nm}}{n(2n+1)}+\frac{(2n-1)}{(2n+1)}\frac{\left((a/r)^2-1\right)(a/r)^n\boldsymbol{B}_{nm}}{2(n+1)}\right] \end{aligned} \tag{4.38}$$

$$\varphi^{(2)}_{nm} = \frac{a^{n+1}\nabla\left\{r^{-n-1}\tilde{Y}^m_n\right\}}{\eta_{nm}\sqrt{(n+1)(2n+1)}} = \frac{(a/r)^{n+2}\boldsymbol{B}_{nm}}{a\eta_{nm}\sqrt{(n+1)(2n+1)}} \tag{4.39}$$

$$\varphi^{(3)}_{nm} = \frac{ia^n}{\eta_{nm}\sqrt{n(n+1)}}\nabla\times\left\{\boldsymbol{x}r^{-n-1}\tilde{Y}^m_n\right\} = \frac{i(a/r)^{n+1}\boldsymbol{C}_{nm}}{a\eta_{nm}\sqrt{n(n+1)}} \tag{4.40}$$

$$n \geq 1, \qquad -n \leq m \leq n \; .$$

The reader should verify that on the sphere surface $r = a$, the orthonormal relations,

$$\oint_S (\varphi^{(\alpha)}_{lk})^* \cdot \varphi^{(\beta)}_{nm} \, dS = \delta_{\alpha\beta}\delta_{ln}\delta_{km} \; ,$$

are satisfied. This basis also has the nice property that $\varphi^{(1)}_{1m}$ and $\varphi^{(3)}_{1m}$, with $(m = -1,0,1)$, correspond to disturbance fields produced by the six independent rigid-body motions of a sphere.

Suppose that we wish to match a general boundary velocity, $\boldsymbol{V}(\theta,\phi)$, on the sphere surface. We may now write the general solution as

$$\boldsymbol{v} = \sum_n\sum_m\left\{a^{(1)}_{mn}\varphi^{(1)}_{nm}+a^{(2)}_{mn}\varphi^{(2)}_{nm}+a^{(3)}_{mn}\varphi^{(3)}_{nm}\right\}$$

and take the inner product of both sides with a particular basis element. The orthonormal property yields immediately the explicit solution for the coefficients,

$$a^{(\alpha)}_{mn} = \oint_S \boldsymbol{V}^* \cdot \varphi^{(\alpha)}_{nm} \, dS \; .$$

In Part IV, we will see that the elements of this basis are in fact eigenfunctions of a self-adjoint operator, and thus obtain a deeper understanding for the orthonormal property.

We illustrate this method by considering again the problem of the translating sphere. The element $\varphi_{10}^{(1)}$ is the disturbance velocity field produced by a constant translational velocity in the z-direction, so on the sphere surface it becomes a constant vector in the z-direction. The explicit results for the surface vector fields and the basis element are

$$\begin{aligned}
\boldsymbol{A}_{10} &= \cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta = \boldsymbol{e}_z \\
\boldsymbol{B}_{10} &= -2\cos\theta \boldsymbol{e}_r - \sin\theta \boldsymbol{e}_\theta \\
\varphi_{10}^{(1)} &= \frac{(a/r)}{a\sqrt{4\pi}}\left[\boldsymbol{A}_{10} + \frac{1}{4}\left(\frac{a^2}{r^2} - 1\right)\boldsymbol{B}_{10}\right] \\
&= \frac{1}{a\sqrt{4\pi}}\left\{\boldsymbol{e}_r \cos\theta\left[\frac{3}{2}\frac{a}{r} - \frac{1}{2}\left(\frac{a}{r}\right)^3\right] - \boldsymbol{e}_\theta\left[\frac{3}{4}\frac{a}{r} + \frac{1}{4}\left(\frac{a}{r}\right)^3\right]\right\} .
\end{aligned}$$

Since $a_{01}^{(1)} = \langle U\boldsymbol{e}_z, \varphi_{10}^{(1)}\rangle = \sqrt{4\pi}aU$, we recover the same solution as before.

4.5 The Stokes Streamfunction

Although the emphasis in this book is on general solution methods needed to tackle the problems of microhydrodynamics, we present here a brief overview of the Stokes' streamfunction method for axisymmetric flows. The student will encounter it frequently in the literature, and we shall call upon it occasionally from our repertoire of solution methods to check more general numerical and analytic methods. For a more complete and self-contained treatment of the method, especially for axisymmetric curvilinear coordinates, we direct the reader to Chapter 4 of Happel and Brenner [35].

4.5.1 Relation to the Vector Potential

Consider an axisymmetric Stokes flow field given by $\boldsymbol{v} = v_r(r,\theta) + v_\theta(r,\theta)$. A solenoidal velocity field must be the curl of a vector potential, *i.e.*, $\boldsymbol{v} = \nabla \times \boldsymbol{A}$. Only one component of the vector potential appears in two-dimensional flows, and for our axisymmetric flow it is the ϕ-component, so that $\boldsymbol{A} = A_\phi(r,\theta)\boldsymbol{e}_\phi$. As we shall show presently, it is convenient to express this one component of the vector potential as $A_\phi(r,\theta) = -\psi(r,\theta)/r\sin\theta$, where ψ is the Stokes streamfunction. In terms of this streamfunction, the two velocity components are

$$v_r = \frac{-1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} , \qquad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} . \tag{4.41}$$

By the use of the Stokes (curl) theorem, we can show that the difference $\psi(r_2,\theta_2) - \psi(r_1,\theta_1)$ is related to the flow through an axisymmetric surface generated by rotating (about the axis of symmetry) any simple curve connecting the points (see Figure 4.5). The flow rate through this surface of revolution is given by

$$Q = \int_S \boldsymbol{v}\cdot\boldsymbol{n}\, dS = \int_S (\nabla\times\boldsymbol{A})\cdot\boldsymbol{n}\, dS$$

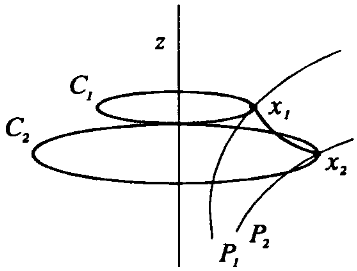

Figure 4.5: The volumetric flow and the streamfunction.

$$
\begin{aligned}
&= \int_{C1} \boldsymbol{A} \cdot \boldsymbol{e}_\phi \, r \sin\theta \, d\phi - \int_{C2} \boldsymbol{A} \cdot \boldsymbol{e}_\phi \, r \sin\theta \, d\phi \\
&= 2\pi\psi_2 - 2\pi\psi_1 \,.
\end{aligned}
$$

The Stokes theorem was used to convert the surface integrals into line integrals along the edges of the surface, with the directed differential arc lengths parametrized as $d\boldsymbol{s} = \boldsymbol{e}_\phi r \sin\theta d\phi$. We have thus derived an important property of the Stokes streamfunction: *Lines with ψ constant are the streamlines.*

Another interesting property of the streamfunction is uncovered upon two successive applications of the curl operator:

$$
\begin{aligned}
\nabla \times \frac{\boldsymbol{e}_\phi \psi}{r \sin\theta} &= \frac{\boldsymbol{e}_r}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} - \frac{\boldsymbol{e}_\theta}{r \sin\theta} \frac{\partial \psi}{\partial r} \\
\nabla \times \nabla \times \frac{\boldsymbol{e}_\phi \psi}{r \sin\theta} &= \frac{\boldsymbol{e}_\phi}{r} \left[\frac{\partial}{\partial r} \left(\frac{-1}{\sin\theta} \frac{\partial \psi}{\partial r} \right) - \frac{\partial}{\partial \theta} \left(\frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \right) \right] \\
&= \frac{-\boldsymbol{e}_\phi E^2 \psi}{r \sin\theta} \,,
\end{aligned}
$$

where the streamfunction operator E^2 is defined by

$$
E^2 \psi = \frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \psi}{\partial \theta} \right) \,. \tag{4.42}
$$

A simple mnemonic for this result is: Commute $\nabla \times \nabla \times$ with $\boldsymbol{e}_\phi / r \sin\theta$, and convert the differential operator to $-E^2$. Note that we have derived an expression for the vorticity in terms of the streamfunction,

$$
\nabla \times \boldsymbol{v} = -\nabla \times \nabla \times \frac{\boldsymbol{e}_\phi \psi}{r \sin\theta} = \frac{\boldsymbol{e}_\phi E^2 \psi}{r \sin\theta} \,.
$$

4.5.2 The Stokes Equations and the Streamfunction

We may write the governing Stokes equations in terms of the streamfunction. The vector Laplacian may be rewritten as

$$
\nabla^2 \boldsymbol{v} = \nabla(\nabla \cdot \boldsymbol{v}) - \nabla \times \nabla \times \boldsymbol{v} = -\nabla \times \nabla \times \boldsymbol{v} \,,
$$

since $\nabla \cdot \boldsymbol{v} = 0$. Taking the curl of the Stokes equation, we eliminate the pressure gradient and arrive at

$$-\mu\nabla \times (\nabla \times \nabla \times \boldsymbol{v}) = -\mu\nabla \times \nabla \times \frac{\boldsymbol{e}_\phi E^2\psi}{r\sin\theta} = \mu\frac{\boldsymbol{e}_\phi E^2(E^2\psi)}{r\sin\theta} = 0 \ ,$$

so that the governing equations reduce to $E^4\psi = 0$.

4.5.3 Boundary Conditions for the Streamfunction

The most common axisymmetric flow problems is that of a fixed axisymmetric body in a uniform stream flowing in the axial direction, or the closely related problem of a rigid body translating along the direction of its axis of symmetry. For the first problem, we set ψ to a constant along the particle profile, to satisfy the kinematic condition of no flow into the body. If the surface of the body intersects the axis of symmetry (this condition would exclude the torus, for example), it is convenient to set this constant to zero, so that with $\psi = 0$ also along the axis of symmetry, we obtain the desired condition of no flow through the particle surface. The no-slip boundary condition requires the vanishing of the normal derivative of ψ, *i.e.*, $\partial\psi/\partial n = 0$. Far away from the particle, the disturbance field decays and we should have just the uniform stream, $\boldsymbol{v}^\infty = U^\infty \boldsymbol{e}_z$. The streamfunction for this flow is readily derived as

$$\psi^\infty = -\frac{1}{2}U^\infty r^2 \sin^2\theta \ .$$

For a particle in uniform translation, we must have $r^{-2}\psi \to 0$ as $r \to \infty$, to get vanishing velocity fields, while on the surface of the particle, the boundary conditions to match the rigid body translation become

$$\psi + \frac{1}{2}Ur^2\sin^2\theta = C \ , \qquad \frac{\partial}{\partial n}(\psi + \frac{1}{2}Ur^2\sin^2\theta) = 0 \ . \tag{4.43}$$

Here again, the constant is set at $C = 0$, the axial value if the body intersects the axis of symmetry.

4.5.4 The Axisymmetric Stokeslet and the Drag on a Body

The velocity field produced by a Stokeslet of strength F directed along the z-axis, $F\boldsymbol{e}_z \cdot \mathcal{G}(\boldsymbol{x})/8\pi\mu$, generates an axisymmetric flow field. It is easy to show that the streamfunction for this flow field is (see Exercise 4.10)

$$\psi = \frac{-F}{8\pi\mu} r\sin^2\theta \ . \tag{4.44}$$

Note that the corresponding velocity field decays as r^{-1}.

The hydrodynamic drag exerted by the fluid on a particle follows from this result as

$$\lim_{r\to\infty} \frac{8\pi\mu\psi}{r\sin^2\theta} ,$$

where ψ is the streamfunction for the disturbance field, since far away from the particle the velocity field must be dominated by the Stokeslet field. More explicitly, all other terms in the multipole expansion (for the streamfunction) decay to zero after multiplication with r. An alternate derivation of this result is given in Payne and Pell [65].

A different expression for the drag on the body was developed by Stimson and Jeffery [80] starting from the surface tractions on the particle surface. The expressions for the relevant components of $\sigma \cdot \hat{n}$ in terms of the streamfunction and the key steps in the derivation are given in Happel and Brenner [35]. The final result for the drag is

$$\pi\mu \int (r\sin\theta)^3 \frac{\partial}{\partial n}\left(\frac{E^2\psi}{(r\sin\theta)^2}\right) ds ,$$

where the line integral is along a meridian of the body.

4.5.5 Separation in Spherical Coordinates

The separability of the operator E^2 is closely related to the separability of the Laplacian operator ∇^2 and the solutions of $E^2\psi = 0$ in a number of coordinate systems are given in [35, 53, 60]. The problem of finding solutions to $E^4\psi = 0$ is usually split into the homogeneous solution ψ^h and the particular solution ψ^p, $\psi = \psi^h + \psi^p$, with

$$\begin{aligned} E^2\psi^h &= 0 , \\ E^2\psi^p &\neq 0 , \qquad \text{but} \quad E^2(E^2\psi^p) = 0 . \end{aligned}$$

The first general solution of this problem (in spherical coordinates) appears in Sampson [71]. We write

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\xi^2}{r^2}\frac{\partial^2}{\partial \xi^2}$$

with $\xi = \cos\theta$, and look for a solution to $E^2\psi^h = 0$ of the form $\psi^h = R(r)X(\xi)$. The equation separates to yield

$$\frac{r^2}{R}\frac{d^2R}{dr^2} = -\frac{1-\xi^2}{X}\frac{d^2X}{d\xi^2} = n(n-1) .$$

As with the Laplace equation, the separation constant must be of a special form, in this case $n(n-1)$ with integral n, to keep ψ bounded at both poles. The solutions for $R(r)$ are simply r^n and r^{-n+1}, while for $X(\xi)$, we have Gegenbauer's equation, whose solutions are the Gegenbauer functions of degree $-1/2$, which, following [35], we denote by $\mathcal{G}_n$ and $\mathcal{H}_n$ (see also [1]). These functions are related to the Legendre functions of the first and second kind by

$$\begin{aligned}
\mathcal{G}_n(\xi) &= \frac{P_{n-2} - P_n}{2n-1}, \qquad n \geq 2 \\
\mathcal{H}_n(\xi) &= \frac{Q_{n-2} - Q_n}{2n-1}, \qquad n \geq 2
\end{aligned}$$

The first two functions may be *defined* as $\mathcal{G}_0(\xi) = 1$, $\mathcal{G}_1(\xi) = -\xi$, $\mathcal{H}_0(\xi) = -\xi$, and $\mathcal{H}_1(\xi) = -1$.

The functions of the first kind are polynomials in ξ, since they are simple combinations of the Legendre polynomials, and are therefore regular everywhere in $-1 \leq \xi \leq 1$. The explicit formula of the Rodriguez type is

$$\mathcal{G}_n(\xi) = \frac{-1}{(n-1)!}\frac{d^{n-2}}{d\xi^{n-2}}\left(\frac{\xi^2-1}{2}\right)^{n-1},$$

and the first few are

$$\mathcal{G}_2(\xi) = \frac{1}{2}(1-\xi^2), \qquad \mathcal{G}_3(\xi) = \frac{1}{2}\xi(1-\xi^2),$$

$$\mathcal{G}_4(\xi) = \frac{1}{8}(1-\xi^2)(5\xi^2-1), \qquad \mathcal{G}_5(\xi) = \frac{1}{8}\xi(1-\xi^2)(7\xi^2-3).$$

The functions of the second kind follow the form usually seen in "second" solutions to second-order ordinary differential equations obtained by the method of Fröbenius,

$$\mathcal{H}_n(\xi) = \frac{1}{2}\mathcal{G}_n(\xi)\ln\frac{1+\xi}{1-\xi} + \mathcal{K}_n(\xi),$$

where

$$\mathcal{K}_n(\xi) = -\sum_{k}^{[(n+1)/2]} \frac{(2n-4k+1)}{(2k-1)(n-k)}\left[1 - \frac{(2k-1)(n-k)}{n(n-1)}\right]\mathcal{G}_{n+1-2k}(\xi).$$

Since $\mathcal{G}_n$ is regular, $\mathcal{H}_n$ is unbounded at both poles, as required by the theory concerning singular points of linear second-order ordinary differential equations. Thus, in most applications of interest, we disregard the second solutions, and write the solution to $E^2\psi^h = 0$ as

$$\psi^h(r,\xi) = \sum_{n=0}^{\infty}(A_n r^n + B_n r^{-n+1})\mathcal{G}_n(\xi).$$

For ψ_n^p, we try solutions of the form $r^\alpha \mathcal{G}_n(\xi)$, such that $E^2\psi_n^h$ is of the form given above. This immediately leads to the result that α can be either $n+2$ or $-n+3$. Thus the general solution for the streamfunction is

$$\psi = \sum_{n=2}^{\infty}(A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3})\mathcal{G}_n(\xi).$$

Geometry	Reference
Conical diffuser	Happel and Brenner [35]
Lens, spherical cap	Payne and Pell [65], Collins [21]
Multiple spheres/spheroids	Gluckman *et al.* [32]
Sphere near a plane wall	Brenner [9]
Spheroid (disk)	Lamb [53], Payne and Pell [65], Sampson [71]
Torus	Pell and Payne [66]
Two spheres (translation)	Stimson and Jeffery [80]
Two spheres (linear field)	Green (Ph.D. Dissertation, Cambridge)
Two spherical drops (transl.)	Haber *et al.* [33]
Two spheroids	Liao and Krueger [57]
Venturi tube	Happel and Brenner [35]

Table 4.1: Streamfunction solutions for various axisymmetric geometries.

For the disturbance flow produced by a translating sphere, we need only $n = 2$ for which $\mathcal{G}_2(\xi) = (1-\xi^2)/2 = \frac{1}{2}\sin^2\theta$, so that

$$\psi = \frac{1}{2}(A_2 r^2 + B_2 r^{-1} + C_2 r^4 + D_2 r)\mathcal{G}_2(\xi) \ .$$

We must set $A_2 = C_2 = 0$, since the disturbance velocity vanishes at infinity. The boundary conditions at the sphere surface, Equation 4.43, yield two equations for the remaining two unknowns, and the final result for the streamfunction for the sphere (translating in the positive z-direction) is

$$\psi = \frac{1}{4}Ur^2\sin^2\theta\left[-3\left(\frac{a}{r}\right) + \left(\frac{a}{r}\right)^3\right] \ .$$

This concludes our brief overview of the streamfunction. Table 4.1 lists solutions available in the literature for various geometries. The solution for the torus [66] is particularly noteworthy, since $\psi \neq 0$ on the surface of the torus. In fact, this (initially unknown) constant corresponds to the net flow through the hole of the torus. The constant must be determined by solution of an implicit transcendental equation for the one separating streamline coming from infinity that hits the surface of the torus. The works on flow past multiple spheres and spheroids [32, 57] use the streamfunction, but the equations are solved numerically by the boundary collocation technique of Chapter 13.

Exercises

Exercise 4.1 Derivation of Lamb's General Solution.
Verify that if Φ and χ are harmonics, then $\nabla\Phi$ and $\nabla \times (\boldsymbol{x}\chi)$ satisfy $\nabla^2\boldsymbol{v} = \boldsymbol{0}$ and $\nabla \cdot \boldsymbol{v} = 0$.

Also show that if $\mu\boldsymbol{v} = \alpha_n r^2 \nabla p_n + \beta_n \boldsymbol{x} p_n$, then

1. $\nabla \cdot \boldsymbol{v} = 0$ only if $2n\alpha_n + (n+3)\beta_n = 0$.
2. $\mu \nabla^2 \boldsymbol{v} = \nabla p$ only if $2(2n+1)\alpha_n + 2\beta_n = 1$.

Solve these two equations for α_n and β_n and show that this gives Lamb's general solution.
Hint: $\boldsymbol{x} \cdot \nabla p_n = np_n$ and $\boldsymbol{x} \cdot \nabla\nabla p_n = (n-1)\nabla p_n$.

Exercise 4.2 The Multipole Expansion for Axisymmetric Flows.
Show that for axisymmetric flows, with $\boldsymbol{v} = v_r(r,\theta)\boldsymbol{e}_r + v_\theta(r,\theta)\boldsymbol{e}_\theta$, the multipole expansion for the velocity field can be reduced to the form

$$\boldsymbol{v} = \boldsymbol{d}\sum_{n=0}^{\infty}\left\{a_n \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{x}_c) \; + \; b_n \frac{(\boldsymbol{d}\cdot\nabla)^n}{n!}\nabla^2\mathcal{G}_{ij}(\boldsymbol{x}-\boldsymbol{x}_c)\right\} .$$

Here, $\boldsymbol{d}$ is a unit vector parallel to the symmetry axis and reference point $\boldsymbol{x}_c$ lies inside the body on the axis of symmetry.
Hint: Start with the nonvanishing components of $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}$ on the particle surface, and expand $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\cdot\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})$ in a Taylor series in $\boldsymbol{\xi}$ about $\boldsymbol{\xi} = 0$. Consider the surface integral as a series of rings around the z-axis (the axis of symmetry). Symmetry arguments should convince you that the differential operator $\nabla_\xi\nabla_\xi$ has only terms of the type

$$\frac{\partial^m}{\partial\xi_3^m}\left(\frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2} - C\frac{\partial^n}{\partial\xi_3^n}\right) = \frac{\partial^m}{\partial\xi_3^m}\left(\nabla_\xi^2 - \frac{\partial^2}{\partial\xi_3^2} - C\frac{\partial^n}{\partial\xi_3^n}\right)$$

(non-negative integers m, n with $0 \le m+n \le 2$) left over after integration around any ring; now successive applications of this differential operator gives the desired velocity representation.

Exercise 4.3 The Toroidal Fields.
Derive the relation between the toroidal fields $\nabla \times (\boldsymbol{x}\chi_{-n-1})$ and the derivatives of the rotlet field.

Exercise 4.4 Force, Torque, and Stresslet from Lamb's Solution.
Examine Lamb's general solution for $n = -2$, rewrite these fields as Stokes singularities, and then derive the following expressions for the force, torque, and stresslet:

$$\boldsymbol{F} = -4\pi\nabla(r^3 p_{-2}) \; , \quad \boldsymbol{T} = -8\pi\mu\nabla(r^3\chi_{-2}) \; , \quad \boldsymbol{S} = -\frac{2}{3}\pi\nabla\nabla(r^5 p_{-3}) \; .$$

Exercise 4.5 Inner Products for the Adjoint Method.
Derive expressions for the inner products between the surface vector fields $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ of the adjoint method, and verify that the elements in the dual basis are orthogonal to the basis elements in Lamb's general solution.
Hint: First verify that

$$\begin{aligned}\int_0^\pi \frac{P_n^m P_n^k}{\sin^2\theta}\sin\theta d\theta &= \frac{(n+m)!}{m(n-m)!}\delta_{km} \\ \int_0^\pi\left[\frac{m^2 P_n^m P_l^m}{\sin^2\theta} + \frac{dP_n^m}{d\theta}\frac{dP_l^m}{d\theta}\right]\sin\theta d\theta &= \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{m(n-m)!}\delta_{ln} \; .\end{aligned}$$

Exercise 4.6 Surface Tractions from Lamb's Solution.
Show that the traction fields for Lamb's general solution are

$$\begin{aligned}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} &= \frac{\mu}{r}\sum_{n=-\infty}^{\infty}\left[(n-1)\nabla\times(\boldsymbol{x}\chi_n)+2(n-1)\nabla\Phi_n\right] \\ &\quad + \frac{1}{r}\sum_{n=-\infty}^{\infty}\left[\frac{n(n+2)r^2\nabla p_n}{(n+1)(2n+3)}-\frac{(2n^2+4n+3)\boldsymbol{x}p_n}{(n+1)(2n+3)}\right].\end{aligned}$$

Use this result to derive the expressions for the force, torque, and stresslet on a sphere.

Exercise 4.7 The Translating Sphere by the Adjoint Method.
Show that $\langle \boldsymbol{w}_n^{(i)}, \boldsymbol{e}_z\rangle = 0$ for all $n \neq 1$, by writing $\boldsymbol{e}_z$ as a linear combination of $\boldsymbol{A}_{10}(\theta,\phi)$ and $\boldsymbol{B}_{10}(\theta,\phi)$.

Exercise 4.8 The Sphere in a Rate-of-Strain Field.
Use Lamb's general solution to derive the solution for the disturbance field for a fixed sphere of radius a in a rate-of-strain field $\boldsymbol{E}\cdot\boldsymbol{x}$. Note that there are five cases:

$$\boldsymbol{E}^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \boldsymbol{E}^{(2)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{E}^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \boldsymbol{E}^{(4)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{E}^{(5)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The general field may be reconstructed as

$$\boldsymbol{E} = \frac{1}{2}E_{zz}\boldsymbol{E}^{(1)} + \frac{1}{2}(E_{xx}-E_{yy})\boldsymbol{E}^{(2)} + E_{xy}\boldsymbol{E}^{(3)} + E_{xz}\boldsymbol{E}^{(4)} + E_{yz}\boldsymbol{E}^{(5)}.$$

Write the surface velocity $\boldsymbol{V}(\theta,\phi) = -\boldsymbol{E}\cdot\boldsymbol{x}$ and identify $m=0$ (degeneracy 1), $m=1$ (degeneracy 2), and $m=2$ (degeneracy 2). The multiple solutions at $m\neq 0$ correspond to symmetry about the z-axis. Show that, in all cases, the stresslet is given by $\boldsymbol{S} = (20/3)\pi\mu a^3\boldsymbol{E}$.

Exercise 4.9 The Translating Spherical Drop: Jump in the Normal Stresses.
Evaluate $\boldsymbol{\sigma}^{(i)} : \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}$ and $\boldsymbol{\sigma}^{(o)} : \hat{\boldsymbol{n}}\hat{\boldsymbol{n}}$ for the translating spherical drop. Then show that the force balance at the drop surface is satisfied exactly.

Exercise 4.10 Streamfunctions for Stokes Singularities.
Derive the streamfunction ψ for the following Stokes singularities:

1. *The Stokeslet*, $F\boldsymbol{e}_z \cdot \mathcal{G}(\boldsymbol{x})$. Plot the streamfunction and show that one streamline profiles a sphere (the radius of this sphere depends on the strength of the Stokeslet). So the Stokeslet alone is an exact solution for a flow past a spherical object. Clearly, it is not the rigid sphere, so what is this spherical object?
2. *The Stresslet*, $[S(\boldsymbol{e}_z\boldsymbol{e}_z - \frac{1}{3}\boldsymbol{\delta})\cdot\nabla]\cdot\mathcal{G}(\boldsymbol{x})$. Derive an expression for the stresslet in terms of ψ, analogous to the result of Payne and Pell for the drag.
3. *The Axisymmetric Stokeson*. Combine the solution elements for the rigid sphere with the Stokeson and the uniform stream to construct the solution for a viscous drop. Plot the streamlines inside and outside the drop.

Exercise 4.11 The Streamfunction and Lamb's General Solution.
By rewriting the Gegenbauer functions in terms of the Legendre polynomials, and by comparing expressions for the radial and tangential velocity components, show that the Sampson's expansion for the streamfunction in spherical coordinates and the axisymmetric version of Lamb's general solution are equivalent. Find the relation between the two sets of coefficients.