9
Lorentz Transformations

According to the *Galilean principle of relativity* the laws of classical mechanics are the same for all observers (moving at constant velocity with respect to each other). More precisely, the laws of classical mechanics remain the same under *Galilean transformations*

$$\begin{array}{ll} \underline{\text{direct}} & \underline{\text{inverse}} \\ x' = x - vt & x = x' + vt' \\ t' = t & t = t' \end{array}$$

relating two frames (x, t) and (x', t') moving at relative velocity v. The equations on the left show that the origin of the second frame $x' = 0$ corresponds to uniform motion $x = vt$ in the first frame. There is no privileged inertial frame or absolute rest for moving bodies, but time is preserved, that is, *time is absolute*.

The Galilean principle or invariance does not govern all of physics, most notably electromagnetism and in particular light propagation. For instance, the wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

is not preserved in a Galilean change of variables $(x, t) \to (x', t')$. The wave equation is instead invariant under another transformation, named after H.A. Lorentz. In 1887, Michelson & Morley carried out an experiment which indicated that light travels at the same velocity independent of the motion of the source. In 1905, Einstein took the constancy of the velocity of light as a postulate, and showed that this postulate, together with the principle of relativity, is sufficient for deriving the kinematical formulas of Lorentz. In so doing, Einstein had to revise the notion of time, and abandon the concept of absolute time.

9.1 Lorentz transformations in one space dimension

The simplest modification of the Galilean transformation, preserving linearity and the implication $x' = 0 \Rightarrow x = vt$, is obtained by multiplying with a factor γ:

$$x' = \gamma(x - vt), \qquad x = \gamma(x' + vt')$$

where γ is independent of x and t but may depend on v. We require that γ is the same in both equations since the inverse transformation should be identical to the direct one except for a change of v to $-v$.

In computing γ we use the observation of equal velocity of light. Consider a light-signal travelling at velocity c in both frames, so that $x = ct$ and $x' = ct'$, which substituted into the right-hand side of the previous equations results in

$$x' = \gamma(ct - vt), \qquad x = \gamma(ct' + vt')$$

or, substituting $x' = ct'$ and $x = ct$ also into the left-hand side,

$$ct' = \gamma(c - v)t, \qquad ct = \gamma(c + v)t',$$

a formula admitting explicitly a transformation of time. Divide the two equations

$$\frac{c}{\gamma(c+v)} = \frac{\gamma(c-v)}{c},$$

which gives the factor

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}.$$

Next, compute the transformation of the time coordinate of events. Substitute $x' = \gamma(x - vt)$ into $x = \gamma(x' + vt')$,

$$x = \gamma^2(x - vt) + \gamma v t',$$

use the explicit form of γ, and solve for t',

$$t' = \gamma(t - \frac{v}{c^2}x).$$

Similarly, compute the inverse transformation,

$$t = \gamma(t' + \frac{v}{c^2}x').$$

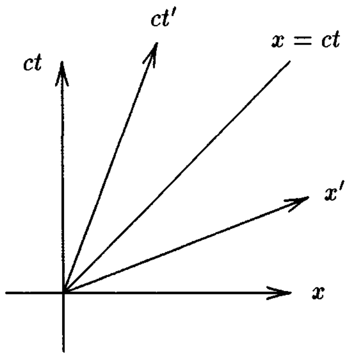
The fact that time is also transformed is referred to as the *relativity of time*.

Summarizing, we have the following transformation laws for the space and time coordinates

$$
\begin{array}{cc}
\underline{\text{direct}} & \underline{\text{inverse}} \\
x' = \dfrac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}} & x = \dfrac{x' + vt'}{\sqrt{1-\frac{v^2}{c^2}}} \\
t' = \dfrac{t - \frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} & t = \dfrac{t' + \frac{v}{c^2}x'}{\sqrt{1-\frac{v^2}{c^2}}}
\end{array}
$$

known as the **Lorentz transformation**. Lorentz transformations preserve the quadratic form $x^2 - c^2t^2 = {x'}^2 - c^2{t'}^2$ and orthogonality of events; two events x_1, ct_1 and x_2, ct_2 are said to be *orthogonal* if $x_1x_2 - c^2t_1t_2 = 0$. In particular, time and space are orthogonal.

It should be noted that time and space do not diverge by 90°, that is, they are not 'perpendicular' or 'rectangular'. If we draw space-time coordinates x, ct on paper so that the time-axis is 'perpendicular' to the space and perform a Lorentz transformation, then the transformed coordinate-axes x', ct' are no longer 'rectangular' (but they are orthogonal, by definition).



Write the direct Lorentz transformation in matrix form:

$$
\begin{pmatrix} x' \\ ct' \end{pmatrix} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}.
$$

This resumes the composition of Lorentz transformations into multiplication of matrices:

$$
L_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix}.
$$

The composition of two Lorentz transformations at velocities v_1 and v_2 results in a Lorentz transformation $L_v = L_{v_2} L_{v_1}$ at velocity

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}},$$

a formula known as the relativistic composition of parallel velocities.

9.2 The Minkowski space-time $\mathbb{R}^{3,1}$

Space-time events can be labelled by points (x^1, x^2, x^3, x^4) or vectors $\mathbf{x} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 + x^4\mathbf{e}_4$ in the Minkowski space-time $\mathbb{R}^{3,1}$. Indices are raised and lowered according to

$$x^1 = x_1, \; x^2 = x_2, \; x^3 = x_3, \; x^4 = ct = -x_4.$$

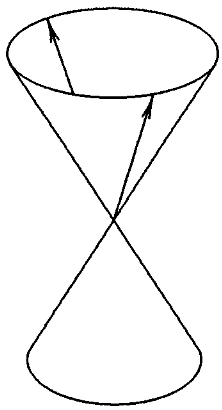
The Minkowski space-time $\mathbb{R}^{3,1}$ has a quadratic form sending a vector $\mathbf{x} \in \mathbb{R}^{3,1}$ to a scalar which we shall denote by $\mathbf{x}^2$,

$$\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2 - x_4^2.$$

Solutions to the equation

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 \quad \text{or} \quad \mathbf{x}^2 = 0$$

form the *null-cone* or *light-cone*.



Light-cone and light-like vectors

The set of non-zero vectors, or *space-time intervals*, $\mathbf{x} \in \mathbb{R}^{3,1}$ can be divided into

$$\begin{aligned} &\mathbf{x}^2 > 0, && \text{space-like vectors,} \\ &\mathbf{x}^2 < 0, && \text{time-like vectors,} \\ &\mathbf{x}^2 = 0, && \text{null vectors or light-like vectors.} \end{aligned}$$

The set of time-like and light-like vectors can be divided into *future oriented* $x^4 > 0$, and *past oriented* $x^4 < 0$ [recall that $x^4 = ct = -x_4$].

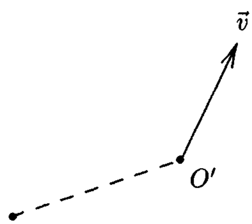
Planes passing through the origin can be divided into time-like, light-like and space-like according as they intersect the light-cone along two, one or zero light-like vectors.

Space-like unit vectors $\mathbf{x}^2 = 1$ form a connected hyperboloid, and time-like unit vectors $\mathbf{x}^2 = -1$ form a two-sheeted hyperboloid. Future oriented time-like unit vectors correspond to *observers*; [1] an observer travelling at velocity $\vec{v} \in \mathbb{R}^3$ is associated to the *time-axis*

$$\mathbf{v} = \frac{\vec{v} + c\mathbf{e}_4}{\sqrt{c^2 - \vec{v}^2}} \quad \text{where} \quad \mathbf{v}^2 = -1.$$

9.3 Lorentz boost at velocity $\vec{v} \in \mathbb{R}^3$

Let us review how a space-time event $(\vec{r}, t)$ of an observer O is seen by another, O', moving at velocity $\vec{v}$ with respect to O.



O = Observer

To do this we first decompose $\vec{r} \in \mathbb{R}^3$ into components $\vec{r} = \vec{r}_{||} + \vec{r}_\perp$ which are parallel $\vec{r}_{||} = (\vec{r} \cdot \vec{v})\dfrac{\vec{v}}{\vec{v}^2}$ and perpendicular $\vec{r}_\perp = \vec{r} - \vec{r}_{||}$ to $\vec{v}$. The transformed space-time event is $(\vec{r}', t')$ where $\vec{r}' = \vec{r}'_{||} + \vec{r}'_\perp$ and

$$\vec{r}'_{||} = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}(\vec{r}_{||} - \vec{v}t), \qquad \vec{r}'_\perp = \vec{r}_\perp,$$

$$t' = \frac{1}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}}\left(t - \frac{\vec{v} \cdot \vec{r}}{c^2}\right).$$

This transformation is called a *boost* at velocity $\vec{v}$.

The scalar $\vec{r}^2 - c^2t^2$, where $\vec{r}^2 = |\vec{r}|^2$, remains invariant under a boost. A boost leaves untouched the perpendicular component $\vec{r}_\perp$, but alters the parallel component $\vec{r}_{||}$.

1 We consider only inertial observers; inertial = free of forces (no acceleration).

A boost at velocity $\vec{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + v_3 \mathbf{e}_3$ can be represented by matrix multiplication $x'^{\alpha} = L^{\alpha}{}_{\beta} x^{\beta}$ or $x' = Lx$ where

$$L_{\vec{v}} = I - \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}\frac{V}{c} + \left(\frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}} - 1\right)\frac{V^2}{\vec{v}^2}$$

and

$$V = \begin{pmatrix} 0 & 0 & 0 & v_1 \\ 0 & 0 & 0 & v_2 \\ 0 & 0 & 0 & v_3 \\ v_1 & v_2 & v_3 & 0 \end{pmatrix}.$$

A boost is a special case of a Lorentz transformation, which can in general also rotate the space $\mathbb{R}^3$.

9.4 Lorentz transformations of the electromagnetic field

The electromagnetic field $\vec{E}, \vec{B}$ experiences a boost at velocity $\vec{v}$ as

$$\vec{E}'_{\perp} = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}(\vec{E}_{\perp} + \vec{v}\times\vec{B}), \qquad \vec{E}'_{\parallel} = \vec{E}_{\parallel},$$

$$\vec{B}'_{\perp} = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}\left(\vec{B}_{\perp} - \frac{\vec{v}\times\vec{E}}{c^2}\right), \quad \vec{B}'_{\parallel} = \vec{B}_{\parallel}.$$

A boost of the electromagnetic field leaves invariant two scalars

$$\vec{E}^2 - c^2\vec{B}^2 \quad \text{and} \quad \vec{E}\cdot\vec{B},$$

called *Lorentz invariants.* [2]

A Lorentz transformation of the electromagnetic field can be written in coordinate form as $F'^{\alpha}{}_{\beta} = L^{\alpha}{}_{\mu} F^{\mu}{}_{\nu} (L^{-1})^{\nu}{}_{\beta}$ or concisely as matrix multiplication $F' = LFL^{-1}$. The matrix $F = (F^{\alpha}{}_{\beta})$ satisfies $gF^{\top}g^{-1} = -F$, where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

F is said to be Minkowski-antisymmetric.

2 The Lorentz invariants remain the same also under rotations of $\mathbb{R}^3$, and therefore under the special Lorentz group $SO(3,1)$. This can be seen by squaring $\mathbf{F} = \vec{E}\mathbf{e}_4 - \vec{B}\mathbf{e}_{123}$: $\mathbf{F}^2 = \mathbf{F} \rfloor \mathbf{F} + \mathbf{F}\wedge\mathbf{F} = \vec{E}^2 - \vec{B}^2 - 2(\vec{E}\cdot\vec{B})\mathbf{e}_{1234}$. The scalar part remains invariant under $L \in O(3,1)$ and the 4-volume part remains invariant under $L \in SL(4,\mathbb{R})$. Note that $O(3,1)\cap SL(4,\mathbb{R}) = SO(3,1)$, $SL(4,\mathbb{R})/\mathbb{Z}_2 \simeq SO_+(3,3)$ and $\bigwedge^2 L \in SO_+(3,3)$ acts on $\bigwedge^2 \mathbb{R}^{3,1} \simeq \mathbb{R}^{3,3}$.

9.5 The Lorentz group $O(3,1)$

A matrix L satisfying $LgL^\top = g$ [3] is said to be Minkowski-orthogonal or a *Lorentz transformation*. The Lorentz transformations form the *Lorentz group*

$$O(3,1) = \{L \in \mathrm{Mat}(4,\mathbb{R}) \mid LgL^\top = g\}.$$

A Lorentz transformation has a unit determinant: $\det L = \pm 1$. The subgroup with positive determinant,

$$SO(3,1) = O(3,1) \cap SL(4,\mathbb{R}),$$

is called the *special Lorentz group*. The special Lorentz group $SO(3,1)$ has two components. The component connected to the identity I is denoted by $SO_+(3,1)$; it preserves orientations of both space and time. The other component $SO(3,1) \setminus SO_+(3,1)$ reverses orientations of both space and time.

The Lorentz group $O(3,1)$ has four components; these form three two-component subgroups preserving space orientation, time orientation or space-time orientation. Time-orientation–preserving Lorentz transformations form the *orthochronous Lorentz group* $O_\uparrow(3,1)$. A *restricted* or special orthochronous Lorentz transformation $L \in SO_+(3,1)$ preserves space-time orientation (orientation of both space and time); its opposite $-L \in SO(3,1) \setminus SO_+(3,1)$ reverses space-time orientation, while gL reverses time orientation, and $-gL \in O_\uparrow(3,1)$ reverses space orientation, $-gL \in O_\uparrow(3,1) \setminus SO_+(3,1)$.

The Lorentz transformations, which stabilize a time-like vector, form a subgroup $O(3)$, the orthogonal group of $\mathbb{R}^3 = \mathbb{R}^{3,0}$. The Lorentz transformations, which stabilize a space-like vector, form a subgroup $O(2,1)$, the small Lorentz group of $\mathbb{R}^{2,1}$. The Lorentz transformations, which stabilize a light-like vector, form a subgroup isomorphic to the group of rigid movements of the Euclidean plane $\mathbb{R}^2$.

Any special orthochronous Lorentz transformation $L \in SO_+(3,1)$ can be written as an exponential $L = e^A$ of a Minkowski-antisymmetric matrix

$$A = \begin{pmatrix} 0 & b_3 & -b_2 & a_1 \\ -b_3 & 0 & b_1 & a_2 \\ b_2 & -b_1 & 0 & a_3 \\ a_1 & a_2 & a_3 & 0 \end{pmatrix},$$

which satisfies $gA^\top g^{-1} = -A$. The matrix A can be characterized by two vectors $\vec{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\vec{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ in $\mathbb{R}^3$. If $\vec{b} = 0$, then L is a boost at velocity

$$|\vec{v}| = c\tanh|\vec{a}|.$$

3 Note the resemblance between $LgL^\top g^{-1} = I$ and the condition of orthogonality $RR^\top = I$ of a matrix R, $R \in O(n)$.

If $\vec{a} = 0$, then $L \in SO(3)$ is a rotation of the Euclidean space $\mathbb{R}^3$ around the axis $\vec{b}$ by the angle $|\vec{b}|$. Boosts and rotations are special cases of simple Lorentz transformations.

9.6 Simple Lorentz transformations

A special orthochronous Lorentz transformation $L \in SO_+(3,1)$, $L \neq I$, has one or two light-like vectors as eigenvectors. If there are two light-like eigenvectors, then they span a time-like eigenplane, which is preserved by the Lorentz transformation; there is also a space-like eigenplane, which is completely orthogonal to the time-like eigenplane. [4] A special orthochronous Lorentz transformation is called *simple*, if it turns vectors only in one eigenplane, leaving the other eigenplane point-wise invariant. Disregarding the case $L = I$, a special orthochronous Lorentz transformation $L = e^A$, where A is characterized as before by $\vec{a}, \vec{b} \in \mathbb{R}^3$, is simple if and only if $\vec{a} \cdot \vec{b} = 0$. A simple Lorentz transformation is called

$$\begin{array}{ll} \textit{hyperbolic}, & |\vec{a}| > |\vec{b}|, \\ \textit{elliptic}, & |\vec{a}| < |\vec{b}|, \\ \textit{parabolic}, & |\vec{a}| = |\vec{b}|. \end{array}$$

A hyperbolic Lorentz transformation is a boost for an observer, whose time-axis is in the time-like eigenplane of the Lorentz transformation. An elliptic Lorentz transformation is a rotation of the Euclidean space $\mathbb{R}^3$, which is orthogonal to an observer, whose time-axis is orthogonal to the space-like eigenplane of the Lorentz transformation. A parabolic Lorentz transformation has only one light-like eigenvector; it is of the form

$$L = I + A + \frac{1}{2}A^2, \quad \text{since} \quad A^3 = 0,$$

and has only one eigenplane, which is light-like and tangent to the light-cone along the light-like eigenvector. A non-parabolic Lorentz transformation can be written as a product of two commuting simple transformations, one hyperbolic and the other elliptic.

Lorentz Transformations in Clifford Algebras

Lorentz transformations can be described within the Clifford algebras $C\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$, $C\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$ and $C\ell_{1,3} \simeq \mathrm{Mat}(2,\mathbb{H})$.

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4 Completely orthogonal planes have only one point in common, the origin O. For two vectors $\mathbf{x}, \mathbf{y}$ in completely orthogonal planes, the scalar product $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$ vanishes: $\mathbf{x} \cdot \mathbf{y} = 0$.

9.7 In the Clifford algebra $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$

Events in time and space can be labelled by sums of scalars and vectors,

$$x = ct + \vec{x},$$

in $\mathbb{R} \oplus \mathbb{R}^3 \subset \mathcal{C}\ell_3$. A paravector $x = x^0 + x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3$ can be provided with a quadratic form

$$x\bar{x} = c^2t^2 - \vec{x}^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

making $\mathbb{R} \oplus \mathbb{R}^3$ isometric to the Minkowski time-space $\mathbb{R}^{1,3}$. [5] This quadratic form is preserved in a special orthochronous Lorentz transformation $L \in SO_+(1,3)$,

$$L : \mathbb{R} \oplus \mathbb{R}^3 \to \mathbb{R} \oplus \mathbb{R}^3, \quad x \to L(x) = sx\hat{s}^{-1},$$

where s is in the spin group [6]

$$\$pin_+(1,3) = \{s \in \mathcal{C}\ell_3 \mid s\bar{s} = 1\} \simeq SL(2,\mathbb{C}).$$

The time-space event $x = t + \vec{x} \in \mathbb{R} \oplus \mathbb{R}^3$ and the electromagnetic field $F = \vec{E} - \vec{B}\mathbf{e}_{123} \in \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$ behave slightly differently under restricted Lorentz transformations:

$$\begin{aligned} x' &= L(x) = sx\hat{s}^{-1}, \\ F' &= sFs^{-1}. \end{aligned}$$

The spin group $\$pin_+(1,3)$ is a two-fold covering of the special orthochronous Lorentz group $SO_+(1,3)$. In other words, there are two elements $\pm s$ in the group $\$pin_+(1,3)$ inducing the same Lorentz transformation L in $SO_+(1,3)$. This can be expressed by saying that the kernel of the group homomorphism

$$\rho : \$pin_+(1,3) \to SO_+(1,3), \quad s \to L = \rho(s)$$

consists of two elements $\{\pm 1\} \in \$pin_+(1,3)$ [the kernel is the pre-image of the identity element $I \in SO_+(1,3)$].

Every element s in the spin group $\$pin_+(1,3)$ is of the form

$$s = \pm \exp \frac{1}{2}(\vec{a} + \vec{b}\mathbf{e}_{123})$$

where $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$. The minus sign in front of the exponential

5 The raising and lowering conventions are different in $\mathbb{R} \oplus \mathbb{R}^3$ and $\mathbb{R}^{1,3}$. In $\mathbb{R}^{1,3}$ $(x^0, x^1, x^2, x^3) = (x_0, -x_1, -x_2, -x_3)$ whereas in $\mathbb{R}^3 \subset \mathbb{R} \oplus \mathbb{R}^3$ there is a prescribed metric such that $(x^1, x^2, x^3) = (x_1, x_2, x_3)$.

6 The groups $\$pin_+(1,3) \subset \mathcal{C}\ell_3$ and $\mathbf{Spin}_+(1,3) \subset \mathcal{C}\ell_{1,3}$ are isomorphic, and so are their Lie algebras $\mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$ and $\bigwedge^2 \mathbb{R}^{1,3}$.

is needed, [7] because not all the elements in the two-fold cover $\mathbf{Spin}_+(1,3)$ of $SO_+(1,3)$ can be written as exponentials of para-bivectors in $\mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$. In the case $\vec{a} = 0$ the Lorentz transformation is a rotation of $\mathbb{R}^3$, and in the case $\vec{b} = 0$ we have a boost at velocity [8]

$$\vec{v} = \tanh \vec{a} = \frac{\vec{a}}{|\vec{a}|} \tanh|\vec{a}|.$$

For any $s \in \mathbf{Spin}_+(1,3)$ the product $s\tilde{s}$ is a boost, that is, $s\tilde{s} \in \mathbb{R} \oplus \mathbb{R}^3$. Since $\langle s\tilde{s}\rangle_0 > 0$, there is a unique square root of $u = s\tilde{s}$, a boost such that $\sqrt{u} = \alpha(1+u)$, $\alpha > 0$. Squaring both sides and using $u\bar{u} = 1$ results in

$$\sqrt{u} = \frac{1+u}{\sqrt{2(1+\langle u\rangle_0)}}.$$

Write $b_1 = \sqrt{s\tilde{s}}$. The product $r = b_1^{-1}s$ satisfies $\tilde{r}r = 1$ and $\bar{r}r = 1$, and so it is a rotation, $r \in \mathbf{Spin}(3)$. A special orthochronous Lorentz transformation can be uniquely decomposed into a product of a boost and a rotation,

$$s = b_1 r,$$

called the *polar decomposition*. Similarly computing $b_2 = \sqrt{\tilde{s}s}$ and $r = sb_2^{-1}$, we find that $s = rb_2$ with the same rotation r, that is,

$$s = b_1 r = r b_2.$$

Both the decompositions have as a factor the same rotation $r \in \mathbf{Spin}(3) = \{s \in \mathcal{C}\ell_3^+ \mid s\tilde{s} = 1\}$, but the boosts are different: $b_1 \neq b_2$.

9.8 In the Clifford algebra $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$

A boost $b \in \mathbb{R} \oplus \mathbb{R}^3\mathbf{e}_4$, at velocity $\vec{v} \in \mathbb{R}^3$, can be computed by

$$b = \exp(\vec{a}\mathbf{e}_4/2), \quad \vec{a} = \operatorname{artanh}(\vec{v}/c),$$

and results in

$$b = \frac{1+\gamma(1+\vec{v}\mathbf{e}_4)}{\sqrt{2(1+\gamma)}}, \quad \gamma = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}},$$

obtained also by taking a square root of $b^2 = \gamma(1+\vec{v}\mathbf{e}_4)$.

The restricted Lorentz group $SO_+(3,1)$ has a double cover

$$\mathbf{Spin}_+(3,1) = \{s \in \mathcal{C}\ell_{3,1}^+ \mid s\tilde{s} = 1\}.$$

7 If $|\vec{a}| = |\vec{b}|$ and $\vec{a}\cdot\vec{b} = 0$ then $(\vec{a}+\vec{b}\mathbf{e}_{123})^2 = 0$, and for a non-zero $F = \vec{a}+\vec{b}\mathbf{e}_{123}$ there is no para-bivector $B \in \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$ such that $e^B = -e^F$.

8 The first tanh-function is evaluated in the Clifford algebra $\mathcal{C}\ell_3$.

Under a Lorentz transformation induced by $s \in \mathbf{Spin}_+(3,1)$ the space-time vector $\mathbf{x}$ transforms according to $\mathbf{x}' = s\mathbf{x}s^{-1}$ and the electromagnetic bivector $\mathbf{F} = \frac{1}{c}\vec{E}\mathbf{e}_4 - \vec{B}\mathbf{e}_{123}$ transforms according to $\mathbf{F}' = s\mathbf{F}s^{-1}$.

9.9 In the Clifford algebra $\mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(2,\mathbb{H})$

Consider the Lorentz group of the Minkowski time-space $\mathbb{R}^{1,3}$ in the Clifford algebra $\mathcal{C}\ell_{1,3}$ which is isomorphic, as an associative algebra, to the real algebra of 2×2-matrices $\mathrm{Mat}(2,\mathbb{H})$ with quaternions as entries. The Clifford algebra $\mathcal{C}\ell_{1,3}$ is generated as a real algebra by the Dirac gamma-matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ satisfying

$$\gamma_0^2 = I,\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I, \quad \text{and} \quad \gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \quad \text{for} \quad \mu \neq \nu.$$

In this case the Lorentz groups $O(1,3)$, $SO(1,3)$, $SO_+(1,3)$ are doubly covered by

$$\begin{aligned}
\mathbf{Pin}(1,3) &= \{s \in \mathcal{C}\ell_{1,3}^+ \cup \mathcal{C}\ell_{1,3}^- \mid s\tilde{s} = \pm 1\},\\
\mathbf{Spin}(1,3) &= \{s \in \mathcal{C}\ell_{1,3}^+ \mid s\tilde{s} = \pm 1\},\\
\mathbf{Spin}_+(1,3) &= \{s \in \mathcal{C}\ell_{1,3}^+ \mid s\tilde{s} = 1\} \simeq SL(2,\mathbb{C}).
\end{aligned}$$

A Lorentz transformation $L \in O(1,3)$ is given by $L(\mathbf{x}) = s\mathbf{x}\hat{s}^{-1}$ in general, but a special Lorentz transformation $L \in SO(1,3)$ corresponds to an even s and can also be written as $L(\mathbf{x}) = s\mathbf{x}s^{-1}$. The group homomorphism $\rho : \mathbf{Pin}(1,3) \to O(1,3)$ is fixed by $L = \rho(s)$, $L(\mathbf{x}) = s\mathbf{x}\hat{s}^{-1}$, and its kernel is $\{\pm 1\}$, that is, each $L \in O(1,3)$ has two pre-images $\pm s$ in $\mathbf{Pin}(1,3)$.

An element $s \in \mathbf{Spin}_+(1,3)$ has a unique polar decomposition

$$s = b_1 r = r b_2,$$

where the boosts are different,

$$b_1 = \sqrt{s\gamma_0\tilde{s}\gamma_0^{-1}} \quad \text{and} \quad b_2 = \sqrt{\gamma_0\tilde{s}\gamma_0^{-1}s},$$

but the rotation is the same, $r = b_1^{-1}s = sb_2^{-1}$.

Penrose & Rindler 1984. On pp. 31-32 the authors give a geometric interpretation for Lorentz transformations, reviewed here in terms of the Clifford algebra $\mathcal{C}\ell_{1,3}$. Take four distinct light-like vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ such that $\mathbf{a}\cdot\mathbf{b} = 1$ and $\mathbf{c}\cdot\mathbf{d} = 1$. The bivector $\mathbf{a}\wedge\mathbf{b}$ represents a time-like plane, since $(\mathbf{a}\wedge\mathbf{b})^2 = (\mathbf{a}\cdot\mathbf{b})^2 - \mathbf{a}^2\mathbf{b}^2 = 1$; the bivector $\mathbf{a}\wedge\mathbf{b}$ belongs to $\mathbf{Spin}(1,3)\setminus\mathbf{Spin}_+(1,3)$ and represents a Lorentz transformation, which reverses the space-time orientation. Therefore, the product

$$s = (\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d})$$

is in $\mathbf{Spin}_+(1,3)$. Let the light-like eigenvectors of the corresponding Lorentz transformation be $\mathbf{l}_1$ and $\mathbf{l}_2$, and choose $\mathbf{l}_1 \cdot \mathbf{l}_2 = 1$ so that $(\mathbf{l}_1 \wedge \mathbf{l}_2)^2 = 1$. The bivector $\mathbf{l}_1 \wedge \mathbf{l}_2$ anticommutes with $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c} \wedge \mathbf{d}$, that is, it is the unique 'normal' to $\mathbf{a}\wedge\mathbf{b}$ and $\mathbf{c}\wedge\mathbf{d}$. The bivector $\mathbf{F} = \log(s)$ in $\bigwedge^2 \mathbb{R}^{1,3}$ is determined up to a multiple of $2\pi\gamma_{0123}(\mathbf{l}_1 \wedge \mathbf{l}_2)$. The square root $\phi + \psi\gamma_{0123} = \sqrt{\mathbf{F}^2}$ is such that $\mathbf{F} = \pm(\phi + \psi\gamma_{0123})\mathbf{l}_1 \wedge \mathbf{l}_2$; it is determined up to a sign; choosing $\phi \geq 0$, the Lorentz transformation $L = \rho(s)$ has velocity $v = \tanh(2\phi)$ and eigenvalues

$$e^{\pm 2\phi} = \sqrt{\frac{1 \pm v}{1 \mp v}}.$$

The planes $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{c} \wedge \mathbf{d}$ 'differ' in the sense that $(\mathbf{a} \wedge \mathbf{b})s = \mathbf{c} \wedge \mathbf{d}$ and $s(\mathbf{c} \wedge \mathbf{d}) = \mathbf{a} \wedge \mathbf{b}$ by a sum of an elliptic angle ψ about the plane $\mathbf{l}_1 \wedge \mathbf{l}_2$ and a hyperbolic angle ϕ in the plane $\mathbf{l}_1 \wedge \mathbf{l}_2$. Indicating the transformed light-like vectors by primes,

$$\mathbf{a}' = s\mathbf{a}s^{-1}, \quad \mathbf{b}' = s\mathbf{b}s^{-1}, \quad \mathbf{c}' = s\mathbf{c}s^{-1}, \quad \mathbf{d}' = s\mathbf{d}s^{-1},$$

we find that $\mathbf{c}' \wedge \mathbf{d}' = s(\mathbf{c} \wedge \mathbf{d})s^{-1} = (\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d})(\mathbf{a} \wedge \mathbf{b})^{-1}$, that is, the Lorentz transformation reflects the plane $\mathbf{c} \wedge \mathbf{d}$ across the plane $\mathbf{a} \wedge \mathbf{b}$. But, $s(\mathbf{a}\wedge\mathbf{b})s^{-1} = (\mathbf{c}'\wedge\mathbf{d}')(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}'\wedge\mathbf{d}')^{-1}$ and $(\mathbf{c}\wedge\mathbf{d})(\mathbf{a}\wedge\mathbf{b})(\mathbf{c}\wedge\mathbf{d})^{-1} = s^{-1}(\mathbf{a}\wedge\mathbf{b})s$, that is, the inverse of $s(\mathbf{a} \wedge \mathbf{b})s^{-1}$. Take a square root of the inverse of s,

$$u = \pm\frac{1}{\sqrt{s}},$$

within $\mathbf{Spin}_+(1,3)$, and find that

$$u\mathbf{a}u^{-1} = \mathbf{c} \quad \text{and} \quad u\mathbf{b}u^{-1} = \mathbf{d},$$

a kind of 'half' of the reflection above. ∎

Jancewicz 1988. On pp. 252-256 the author shows how to decompose a non-simple bivector $\mathbf{F}$ into simple components. He defines

$$\alpha + \beta\gamma_{0123} = (\phi + \psi\gamma_{0123})^2 = \mathbf{F}^2$$

and sets $\delta^2 = \alpha^2 + \beta^2$. Then he gives the simple components

$$\mathbf{F}_{1,2} = \frac{\mathbf{F}}{2\delta}(\delta \pm \alpha \mp \beta\gamma_{0123})$$

for $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ so that $\mathbf{F}_1^2 > 0$ and $\mathbf{F}_2^2 < 0$, that is, $\mathbf{F}_1$ is hyperbolic or time-like and $\mathbf{F}_2$ is elliptic or space-like. Observing that

$$\left(\frac{\mathbf{F}}{\phi + \psi\gamma_{0123}}\right)^2 = 1$$

enables us to work out the decomposition in another way:

$$\mathbf{F}_1 = \frac{\mathbf{F}\phi}{\phi + \psi\gamma_{0123}} \quad \text{and} \quad \mathbf{F}_2 = \frac{\mathbf{F}\psi\gamma_{0123}}{\phi + \psi\gamma_{0123}}.$$

Hestenes & Sobczyk 1984 p. 81 note that $\phi^2 = \mathbf{F}_1^2$, $\psi^2 = -\mathbf{F}_2^2$ and

$$\mathbf{F}_{1,2}^2 = \frac{1}{2}[\mathbf{F} \rfloor \mathbf{F} \pm \sqrt{(\mathbf{F} \rfloor \mathbf{F})^2 - (\mathbf{F} \wedge \mathbf{F})^2}]$$

(their formula 4.16 concerns only the positive definite case). ∎

Hestenes 1966. The author gives on pp. 52-53 a method to find out $s \in \mathbf{Spin}_+(1,3)$ from the coordinates $L^\mu{}_\nu$ of a special orthochronous Lorentz transformation $L(\mathbf{x}) = s\mathbf{x}s^{-1}$, $L \in SO_+(1,3)$. Recall that $L(\gamma_\nu) = \gamma_\mu L^\mu{}_\nu$, and deduce

$$L^\mu{}_\nu = \gamma^\mu \cdot L(\gamma_\nu) = \langle \gamma^\mu s \gamma_\nu \tilde{s} \rangle_0.$$

To compute s in terms of $L^\mu{}_\nu$ define first

$$\mathcal{L} \equiv L^\mu{}_\nu \gamma_\mu \gamma^\nu = L^\mu{}_\mu + L^\mu{}_\nu \gamma_\mu \wedge \gamma^\nu \in \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^{1,3}. \tag{1}$$

It follows that

$$\mathcal{L} = L(\gamma_\nu)\gamma^\nu = s\gamma_\nu \tilde{s} \gamma^\nu.$$

In computing $\gamma_\nu \tilde{s} \gamma^\nu$, note that in general $\mathbf{e}_\nu u \mathbf{e}^\nu = (n-2k)\hat{u}$ for $u \in \bigwedge^k \mathbb{R}^n$, and deduce that for $s = \langle s \rangle_0 + \langle s \rangle_2 + \langle s \rangle_4$

$$\gamma_\nu \tilde{s} \gamma^\nu = 4[\langle s \rangle_0 - \langle s \rangle_4].$$

Therefore,

$$\mathcal{L} = 4s[\langle s \rangle_0 - \langle s \rangle_4].$$

Since $\tilde{s}s = 1$, $\tilde{\mathcal{L}}\mathcal{L} = 16[\langle s \rangle_0 - \langle s \rangle_4]^2$, and

$$s = \pm \frac{\mathcal{L}}{\sqrt{\tilde{\mathcal{L}}\mathcal{L}}}.$$

Substituting (1) this gives s explicitly as a function of $L^\mu{}_\nu$. This construction is an accident in dimension $n = 4$, because only then does the sum $\mathbf{e}_\nu u \mathbf{e}^\nu$ vanish for a bivector $u \in \bigwedge^2 \mathbb{R}^n$. ∎

Historical survey

In 1881, A.A. Michelson carried out, for the first time, measurements intended to determine the motion of the Earth relative to an absolute, imaginary 'light medium'. For this purpose he measured the velocity of light in different directions. Michelson & Morley repeated the experiment in 1887 and came to the conclusion that light travels at the same velocity independent of the motion of the source with respect to the 'light medium'.

Voigt in 1887 was the first to recognize that the wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} = 0$$

is invariant with respect to the change of variables

$$x' = x - vt,$$
$$t' = t - \tfrac{vx}{c^2},$$

where also time is transformed. Voigt's formulas are not identical for direct and inverse transformations; symmetry was restored later by introducing the factor $\sqrt{1 - v^2/c^2}$. This factor was first encountered in another connection: FitzGerald and Lorentz 1892 gave independently an explanation of the Michelson & Morley experiment by suggesting that moving bodies are contracted in the direction of motion by the ratio $\sqrt{1 - v^2/c^2}$.

The Lorentz transformations of space-time events were introduced by Larmor in 1900, while the relativistic covariance of the Maxwell equations was demonstrated by H.A. Lorentz 1903 [9] (and conformal covariance by Cunningham 1909/1910 and Bateman 1910).

In 1905 [10] Einstein supplemented the principle of relativity by postulating the principle of independence of the velocity of light (of the motion of the source). These two principles led Einstein to a revision of the notion of time and enabled him to deduce the kinematical transformation laws of Lorentz; his predecessors had obtained the transformation laws by considering transformations which do not change the form of the Maxwell equations.

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9 Poincaré noticed that restricted Lorentz transformations of space-dimension 1 form a group $SO_+(1,1)$ consisting of the elements

$$\begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}$$

where $\chi \in \mathbb{R}$.

10 A. Einstein: Zur Elektrodynamik bewegter Körper. *Ann. Physik* **17** (1905), 891-921. In this paper Einstein compared the same phenomenon when observed in two different frames: a magnet moving near a closed conductor and a closed conductor moving near a magnet. In another paper of 1905 Einstein gave a relation between mass and energy, which was later popularized as the formula $E = mc^2$, written today as $E = mc^2/\sqrt{1 - v^2/c^2}$ or $E^2 = m^2c^4 + p^2c^2$.

Later Einstein reformulated the principle of relativity so that it embraces not only mechanical but also electromagnetic phenomena:

$$\boxed{\text{the laws of physics have the same form in all reference frames}}$$

When this *Einsteinian principle of relativity* is applied to the Maxwell equations, one is compelled to conclude that the velocity of light is the same in all reference frames. In other words, the principle of constancy of the velocity of light becomes superfluous as an amendment to the principle of relativity. The principle of relativity and knowledge of the Maxwell equations are enough to deduce the transformation laws of Lorentz.

Nowadays the terms 'relativistic' and 'relativity' almost invariably refer to the Einsteinian principle.

Questions

1. How many light-like eigenvectors does a Lorentz transformation have?
2. Are all $L \in SO_+(3,1)$ of the form $L = \exp(A)$, $gA^\top g^{-1} = -A$?
3. Are all $s \in \mathbf{Spin}_+(3,1)$ of the form $s = \exp(\mathbf{B}/2)$, $\mathbf{B} \in \bigwedge^2 \mathbb{R}^{3,1}$?
4. A special orthochronous Lorentz transformation can be written as a product of a boost and a rotation, in two different orders. In the two expressions, which factor is the same: the boost or the rotation?
5. Are all the special orthochronous Lorentz transformations products of two commuting transformations, one hyperbolic and one elliptic?

Let $\mathbf{B} \in \bigwedge^2 \mathbb{R}^{3,1}$.

6. Is $\mathbb{R}^{3,1} \ni \mathbf{x} \to u\mathbf{x}u^{-1}$, $u = 1 + \mathbf{B} + \frac{1}{2}\mathbf{B} \wedge \mathbf{B}$, a Lorentz transformation?
7. Do the Lorentz transformations induced by $\exp(\mathbf{B}/2)$ and $1 + \mathbf{B} + \frac{1}{2}\mathbf{B} \wedge \mathbf{B}$, $\mathbf{B}^2 \neq 1$, have the same eigenvectors?
8. Does $(1 + \mathbf{B})(1 - \mathbf{B})^{-1}$ represent a Lorentz transformation?
9. Do the Lorentz transformations induced by $\exp(\mathbf{B}/2)$ and $(1 + \mathbf{B})(1 - \mathbf{B})^{-1}$, $\mathbf{B}^2 \neq 1$, have the same eigenvectors?

Answers

1. In general two, parabolic has one, $\pm I$ have all of them.
2. Yes. 3. No. 4. Rotation. 5. No (parabolic are not).
6. Yes, if $\mathbf{B}^2 \neq 1$.
7. Yes, because both the Lorentz transformations are functions (= power series with real coefficients) of A, $A(\mathbf{x}) = \mathbf{B} \llcorner \mathbf{x}$; namely e^A and $(I + A)(I - A)^{-1}$, respectively.

8. Yes, if $\mathbf{B}^2 \neq 1$ (but this is no longer true in dimension 6).
9. No, because the latter Lorentz transformation is not a rational function of A alone (but also of $A^\top$).

Exercises

1. Derive the composition rule for non-parallel velocities,

$$\vec{v}_2' = \frac{1}{1+\frac{\vec{v}_1\cdot\vec{v}_2}{c^2}}[\vec{v}_1 + \vec{v}_{2\|} + \sqrt{1-\tfrac{\vec{v}_1^2}{c^2}}\,\vec{v}_{2\perp}].$$

Hint: use the inverse Lorentz transformation

$$\vec{r}_2' = \frac{1}{\sqrt{1-\frac{\vec{v}_1^2}{c^2}}}(\vec{r}_{2\|} + \vec{v}_1 t_2) + \vec{r}_{2\perp},$$
$$t_2' = \frac{1}{\sqrt{1-\frac{\vec{v}_1^2}{c^2}}}(t_2 + \tfrac{\vec{v}_1\cdot\vec{r}_2}{c^2}).$$

2. Show that $\vec{v}_2' = \tanh(2\log(b_2'))$ where $b_2' = \sqrt{s_2'\tilde{s}_2'}$, $s_2' = s_1 s_2 \in \mathbb{R}\oplus\mathbb{R}^3$ and $s_1 = \exp(\frac{1}{2}\vec{a}_1)$, $\vec{v}_1 = \tanh(\vec{a}_1)$.
3. Show that the composite of two boosts is a hyperbolic transformation.
4. Consider a time-space event $x = ct + \vec{x}$ in $\mathbb{R}\oplus\mathbb{R}^3$ corresponding to $y = \vec{x} + ct\mathbf{e}_4$ in $\mathbb{R}^{3,1}$. Define $s = \exp(\vec{a}/2)$ and $u = \exp(\vec{a}\mathbf{e}_4/2)$ for $\vec{a}\in\mathbb{R}^3$. Show that the boost $\hat{s}xs^{-1} = s^{-1}x\hat{s}$ corresponds to the boost uyu^{-1}.
5. Show that for $u \in \mathbf{Spin}_+(3,1)$, when decomposed into a product of a boost and a rotation, $u = b_1 r = r b_2$, the rotation-factor $r \in \mathbf{Spin}(3)$ can be obtained by normalizing $(u\wedge\mathbf{e}_4)\mathbf{e}_4^{-1}$.
6. Take a bivector $\mathbf{F} = \vec{a}\mathbf{e}_4 + \vec{b}\mathbf{e}_{123} \in \bigwedge^2\mathbb{R}^{3,1}$ such that $|\vec{a}| = |\vec{b}|$. Consider the antisymmetric linear transformation $\mathbb{R}^{3,1}\to\mathbb{R}^{3,1}$, $\mathbf{x}\to A\mathbf{x} = \langle\mathbf{F}\mathbf{x}\rangle_1$. Show that $(A^3\mathbf{x}) \parallel \mathbf{x}$.
7. Take a non-simple bivector $\mathbf{F}\in\bigwedge^2\mathbb{R}^{3,1}$ with simple components $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$, $\mathbf{F}_1^2 > 0$, $\mathbf{F}_2^2 < 0$. Show that

$$\mathbb{R}^{3,1}\to\mathbb{R}^{3,1}, \quad \mathbf{x}\to\frac{\mathbf{F}\mathbf{x}\mathbf{F}}{\mathbf{F}_1^2 - \mathbf{F}_2^2}$$

is a Lorentz transformation, a reflection across the plane of $\mathbf{F}_1$.
8. Show that $(\phi + \psi\gamma_{0123})(\delta + \alpha - \beta\gamma_{0123}) = 2\delta\phi$.
9. Show that as topological spaces $\mathbf{Spin}_+(1,3) \simeq \mathbb{R}^3\times S^3$.
10. Show that as groups $\$pin_+(3,1) \simeq \mathbf{Spin}_+(3,1) \simeq SL(2,\mathbb{C})$ and $SO_+(3,1)\simeq SO(3,\mathbb{C}) = \{R\in\mathrm{Mat}(3,\mathbb{C}) \mid RR^\top = I,\ \det R = 1\}$.
11. Show that for $u\in\mathbf{Spin}_+(3,1)$ there is a square root in $\mathbf{Spin}_+(3,1)$ given

by

$$\sqrt{u} = \frac{u+1}{\sqrt{2(1+\langle u\rangle_0+\langle u\rangle_4)}}.$$

Hint: for $s \in \mathbf{Spin}_+(3,1)$, $s^2+1 = s^2+\tilde{s}s = (s+\tilde{s})s = 2(\langle s\rangle_0+\langle s\rangle_4)s$. Therefore, $(\alpha+\beta\mathbf{e}_{1234})\sqrt{u} = u+1$ with $\alpha,\beta \in \mathbb{R}$.

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