Preface

This book is intended for people who are not primarily algebraists, but nonetheless get involved in the subject through other areas – having backgrounds, in say, engineering, physics, geometry or analysis. Readers of this book may have different starting levels, backgrounds and goals.

Chapters 1–2 form a unit for an undergraduate course: Vectors, the scalar product; complex numbers, a geometrical interpretation of the imaginary unit $i = \sqrt{-1}$.

Chapters 3–7 guide the reader through bottlenecks and provide necessary building blocks: Bivectors and the exterior product. Pauli spin matrices and Pauli spinors. Quaternions and the fourth dimension. The cross product is generalized to higher dimensions.

Chapters 8–10 aim to serve readers with different backgrounds: Electromagnetism, special relativity, Dirac theory. The Dirac equation is formulated with complex column spinors, spinors in minimal left ideals and considering spinors as operators.

Chapters 11–13 discuss physical applications of spinors: In the case of an electron, Fierz identities are sufficient to reconstruct spinors from their physical observables, but this is not the case for the neutrino. Boomerangs are introduced to handle neutrinos. A new class of spinors is identified by its bilinear observables: the *flag-dipole spinors* which reside between Weyl, Majorana and Dirac spinors.

Chapters 14–15 are more algebraic than the previous chapters. Clifford algebras are defined for the first time. Finite fields. Isometry classes of quadratic forms and their Witt rings. Tensor products of algebras and Brauer groups are discussed.

Chapters 16–19 view Clifford algebras through matrix algebras: Clifford algebras are given isomorphic images as matrix algebras, Cartan's periodicity of 8, spin groups and their matrix images in lower dimensions, scalar products of spinors with a chessboard of their automorphism groups, Möbius transformations represented by Vahlen matrices.

Chapters 20–23 discuss miscellaneous mathematical topics. A one-variable higher-dimensional generalization of complex analysis: Cauchy's integral formula is generalized to higher dimensions. Multiplication rule of standard basis elements of a Clifford algebra and its relation to Walsh functions. Multivector structure of Clifford algebras and the non-existence of k-vectors, $k \geq 2$, in characteristic 2. In the last chapter, we come into contact with final frontiers science: an exceptional phenomenon in dimension 8, triality, which has no counterpart in any other dimension.

The first parts of initial chapters are accessible without knowledge of other parts of the book – thus a teacher may choose his own path for his lectures on Clifford algebras. The latter parts of the chapters are sometimes more advanced, and can be left as independent study for interested students.

Introduction of the Clifford algebra of multivectors and spinors can be motivated in two different ways, in physics and in geometry:

(i) In physics, the concept of Clifford algebra, as such or in a disguise, is a necessity in the description of electron spin: Spinors cannot be constructed by tensorial methods, in terms of exterior powers of the vector space.
(ii) In geometry, information about orientation of subspaces can be encoded in simple multivectors, which can be added and multiplied. Physicists are familiar with this tool in the special case of one-dimensional subspaces of oriented line-segments, which they manipulate by vectors (not by projection operators, which lose information about orientation).

Acknowledgements

In preparing this book I have enjoyed the help of many friends and colleagues: Jerry Segercrantz from Finland, Peter Morgan, Ian Porteous and Ronald Shaw from England, Rafał Abłamowicz, Michael Becker, Geoffrey Dixon, Tevian Dray, Michael Kinyon and Arvind Raja from the USA, Roland Bacher (Switzerland), Helga Baum (Germany), Bernard Jancewicz (Poland), Alphonse Charlier and Yvon Siret (France), Josep Parra (Spain) and Garret Sobczyk (Mexico).

I thank Ross Jones and Göran Långstedt for revising my English. For their assistance with TEX, I am indebted to Timo Korvola and Martti Nikunen. I appreciate feedback from the readers of the first edition: Leo Dorst, Kynn Jones, Joseph Riel and Perttu Puska.

In particular, I would like to thank Jacques Helmstetter for his comments on my DEA lectures in Grenoble as well as Ismo Lindell at the Electromagnetics Laboratory in Helsinki, and Waldyr Rodrigues at IMECC in Campinas. For financial support, I am indebted to the NOKIA Foundation in Helsinki and CNPq in Brazil.

Helsinki, November 2000 PERTTI LOUNESTO