Minimax Theorems in Hilbert Spaces

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Preliminaries

Let A, B be nonempty sets and $L : A \times B \to \mathbb{R}$ a function. We set

- $\alpha = \inf_{u \in A} \sup_{p \in B} L(u, p)$
- $\beta = \sup_{p \in B} \inf_{u \in A} L(u, p)$
- $F(u) := \sup_{p \in B} L(u, p)$
- $G(p) := \inf_{u \in A} L(u, p)$

Proposition 1

We have the following a priori results:

1. $-\infty \leq \beta \leq \alpha \leq \infty$
2. *For all $u \in A, p \in B$ we have*
$$G(p) \leq \beta \leq \alpha \leq F(u) \tag{1}$$
3. *Suppose that there exist two points $u_0 \in A, p_0 \in B$ such that $G(p_0) \geq F(u_0)$. Then (u_0, p_0) is a saddle point of L.*

Hypotheses

Consider the following hypotheses:

(H1) Suppose that A and B are nonempty, closed and convex subsets of real Hilbert spaces.

(H2) The map $u \mapsto L(u, p)$ is *convex* and lower semi-continuous on A for all $p \in B$.

(H3) The map $p \mapsto L(u, p)$ is concave and upper semi-continuous on B for all $u \in A$.

(H4) The sets A and B are bounded.

Remark

Eventually, we can relax (H4) to (H4'):

- If A is not bounded, then there exists a point $q \in B$ such that $L(u, q) \to \infty$ as $||u|| \to \infty$ in A.
- If B is not bounded, then there exists a point $v \in A$ such that $L(v, p) \to \infty$ for $||p|| \to +\infty$ in B.

Strong Duality Result

Theorem 2

Under the hypotheses (H1) - (H4) we have strong dualty, i.e. $\alpha = \beta$.

Ingredients for the proof

Definition 3

Let $F : M \subset X \to \mathbb{R}$ be a functional on $M \subset X$, where X is a real normed space. Then we say that F is *weakly sequentially lower semi-continuous* if $u_n \rightharpoonup u$ for $u_n, u \in M$ implies that

$$F(u) \leq \liminf_n F(u_n). \tag{2}$$

Furthermore, we say that F is weakly coercive, if

$$F(u) \to \infty \quad \text{as} \quad ||u|| \to \infty \text{ on } M. \tag{3}$$

Theorem 4

Suppose $F : M \to \mathbb{R}$ has the following properties:

1. *M nonempty, closed and convex subset of a real Hilbert space X;*
2. *F weakly sequentially lower semi-continuous;*
3. *if M is unbounded, then suppose that F is weakly coercive.*

Then the minimization problem

$$F(u) = \min!, \quad u \in M \tag{4}$$

has a solution.

Corollary 5

If F is strictly convex, the solution is unique.

Definition 6

Let $F : M \subset X \to \mathbb{R}$ be a functional on $M \subset X$, where X is a real normed space and M is closed and convex. For each $r \in \mathbb{R}$ set

$$\mathcal{M}_r := \{u \in M \mid F(u) \leq r\}. \tag{5}$$

Then we say that

1. F is *lower semi-continuous* on the closed set M if the set $\mathcal{M}_r$ is closed for all $r \in \mathbb{R}$;
2. F is *quasi-convex* on the convex set M if $\mathcal{M}_r$ is convex for all $r \in \mathbb{R}$. Equivalently, we can say that $F(\alpha u + (1-\alpha)v) \leq \max\{F(u), F(v)\}$ for $u, v \in M$ and $\alpha \in [0, 1]$.

Proposition 7

Suppose that $F : M \subset X \to \mathbb{R}$ has the following properties:

1. *M nonempty, closed and convex subset of a real Hilbert space X;*
2. *F quasi-convex and lower semi-continuous;*
3. *if F is unbounded, suppose that F is weakly coercive. Then the minimization problem*

$$F(u) = \min!, \quad u \in M \tag{6}$$

has a solution. This solution is unique if F is strictly convex.

Lemma 8

Let $F : M \subset X \to \mathbb{R}$ be lower semi-continuous and quasi-convex on the nonempty, closed and convex set M. Then F is weakly sequentially lower semi-continuous on M.

Proof of the duality theorem - prologue

Note that by (H2) and Lemma 8, we have that whenever $u_n \rightharpoonup u$ in A, then

$$L(u, p) \leq \liminf_n L(u_n, p), \quad \forall p \in B. \tag{7}$$

Similarly by (H3), $-L$ is convex and lower semi-continuous in p and therefore $p_n \rightharpoonup p$ in B implies

$$L(u, p) \geq \limsup_n L(u, p_n), \quad \forall u \in A. \tag{8}$$

Proof of the duality theorem - step 1

We set

$$G(p) := \min_{u \in A} L(u,p), \qquad p \in B \tag{9}$$

$$F(u) := \max_{p \in B} L(u,p), \qquad u \in A. \tag{10}$$

By Proposition 7, (H2) and (H3), both optimization problems above have a solution, so the definitions make sense. Note that we used the *quasi-convexity* of L in the first argument.

Proof of the duality theorem - step 2

We show that $F : A \to \mathbb{R}$ is lower semi-continuous and quasi-convex. Put $A_r := \{u \in A \mid F(u) \leq r\}$ for $r \in \mathbb{R}$. Let $v, w \in A_r, \alpha \in [0,1]$. Set $z := \alpha v + (1-\alpha)w$. Then by convexity of L in the first argument, we find

$$L(u,p) \leq \alpha L(v,p) + (1-\alpha)L(w,p) \leq r, \tag{11}$$

for all $p \in B$. This shows the quasi-convexity.
Let $u_n \in A_r, n \geq 1$ such that $u_n \to u$. Then, $L(u_n,p) \leq r$ for all $n \in \mathbb{N}$ and $p \in B$. Since L is lower semi-continuous in the first argument by (H2), we get $L(u,p) \leq r$ for all $p \in B$. This shows that F is lower semi-continuous.
A similar argument shows that G is quasi-concave and upper semi-continuous. Hence, application of Proposition 7 yields solutions $u_\star, p_0$ such that

$$F(u_\star) = \min_{u \in A} F(u)$$
$$G(p_0) = \max_{p \in B} G(p).$$

Proof of the duality theorem - step 3

(H) Suppose that $u \mapsto L(u, p)$ is *strictly convex*.

Under (H), the solution to the minimization problem $G(p) = \min_{u \in A} F(u, p)$ is unique for all $p \in B$. Let us denote it by $u := \phi(p)$, i.e.

$$G(p) = L(\phi(p), p), \quad p \in B, \tag{12}$$

and set $u_0 := \phi(p_0)$. By (12), we have

$$G(p_0) \leq L(u, p_0), \quad \forall u \in A. \tag{13}$$

Now we show the decisive inequality

$$G(p_0) \geq L(u_0, p), \quad \forall p \in B. \tag{14}$$

From inequalities (12) and (14), it then follows that $G(p_0) = L(u_0, p_0)$ and therefore

$$L(u_0, p) \leq L(u_0, p_0) \leq L(u, p_0), \tag{15}$$

for all $u \in A, p \in B$, which is the desired result.

Proof of the duality theorem - step 4

Take $p \in B$, put

$$p_n := (1 - \tfrac{1}{n})p_0 + \tfrac{1}{n}p, \quad u_n := \phi(p_n), \quad n \in \mathbb{N}. \tag{16}$$

By definition of G, we have

$$G(p_0) \geq G(p_n) = L(u_n, p_n), \quad \forall n \in \mathbb{N}. \tag{17}$$

Since $p \mapsto L(u, p)$ is concave, we have

$$G(p_0) \geq (1 - \tfrac{1}{n})L(u_n, p_0) + \tfrac{1}{n}L(u_n, p). \tag{18}$$

By (12), $G(p_0) \leq L(u_n, p_0)$ and thus

$$G(p_0) \geq L(u_n, p), \quad \forall n \in \mathbb{N}. \tag{19}$$

Since $u_n \in A$ for $n \in \mathbb{N}$, the sequence is bounded and thus there exists a subsequence again denoted by u_n such that $u_n \rightharpoonup w$ for some $w \in A$. By (H2), $u \mapsto L(u, p)$ is lower semi-continuous, which implies

$$G(p_0) \geq \liminf_n L(u_n, p) \geq L(w, p). \tag{20}$$

It remains to show that $w = u_0$. By deifnition of the u_n, we have

$$L(u_n, p_n) \leq L(u, p_n), \qquad \forall u \in A, n \in \mathbb{N}.$$

Again, using the concavity of $p \mapsto L(u, p)$, we have

$$(1 - \tfrac{1}{n})L(u_n, p_0) + \tfrac{1}{n}L(u_n, p) \leq L(u, p_n), \qquad \forall u \in A, n \in \mathbb{N}.$$

By (12), we have $G(p) \leq L(u_n, p)$, and therefore

$$(1 - \tfrac{1}{n})L(u_n, p_0) + \tfrac{1}{n}G(p) \leq L(u, p_n), \qquad \forall u \in A, n \in \mathbb{N}. \tag{21}$$

In the limit $n \to \infty$ we find together with (20) that

$$L(w, p_0) \leq \liminf_n L(u, p_n), \qquad \forall u \in A. \tag{22}$$

As $p_n \to p_0$ and by (H3) the map $p \mapsto L(u, p)$ is upper semicontinuous, we have

$$\limsup_n L(u, p_n) \leq L(u, p_0), \qquad \forall u \in A. \tag{23}$$

So finally, we have

$$L(w, p_0) \leq L(u, p_0), \qquad \forall u \in A,$$

so by definition of u_0, we must have $w = u_0$.

Proof of the duality theorem - step 5

Eventually, we have to discard the additional assumption (H). Consider the regularized functions

$$L_n(u,p) := L(u,p) + \tfrac{1}{n}||u||, \qquad n \in \mathbb{N}. \tag{24}$$

Since X is a real Hilbert space, $u \mapsto ||u||$ is strictly convex, which carries over to L_n. Therefore, we have (H) for every such L_n.

Remark

Note that the sum of convex functions always stays convex. However, the sum of quasi-convex functions need not be quasi-convex.

By the preceding arguments, there exists a saddle point (u_n, p_n) for every L_n in $A \times B$. Hence,

$$L(u_n, p) + \frac{1}{n}||u_n|| \leq L(u_n, p_n) + \frac{1}{n}||u_n|| \leq L(u, p_n) + \frac{1}{n}||u||, \tag{25}$$

for all $u \in A, p \in B, n \in \mathbb{N}$. The sequences $(u_n)_n, (p_n)_n$ are bounded and therefore we can extract subsequences again denoted by $(u_n)_n$ and $(p_n)_n$ such that

$$u_n \rightharpoonup u_0 \quad \text{and} \quad p_n \rightharpoonup p_0 \tag{26}$$

for some $u_0 \in A$ and $p_0 \in B$, since the latter two sets are closed and convex. In particular, in the limit $n \to \infty$, we have

$$L(u_0, p) \leq \liminf_n L(u_n, p) \leq \limsup_n L(u, p_n) \leq L(u, p_0), \qquad \forall u \in A, p \in B. \tag{27}$$

Hence, we have

$$L(u_0, p) \leq L(u_0, p_0) \leq L(u, p_0), \qquad \forall u \in A, p \in B, \tag{28}$$

as desired.

Generalized duality theorem

The following generalization can be found in Zeidler 1986, Vol. I., p. 458:

Theorem 9

Suppose that A and B are nonempty, closed, bounded convex subsets in reflexive Banach spaces X and Y, respectively. Let $L : A \times B \to \mathbb{R}$ be a function such that

1. $u \mapsto L(u, p)$ *is lower semi-continuous and quasi-convex on A for all $p \in B$;*
2. $p \mapsto L(u, p)$ *is upper semi-continuous and quasi-concave on B for all $u \in A$.*

Then L has a saddle point and we have strong duality.

Ingredients for the proof

Proposition 10 (Fixed point theorem)

A mapping $T : K \to 2^K$, where $K \subset X$, has a fixed point if the following conditions hold:

1. *X is locally convex, K is nonempty, compact and convex;*
2. *the set $T(x)$ is nonempty and convex for all $x \in K$, and the preimages $T^{-1}(\{y\})$ are relatively open with respect to K for all $y \in K$.*

Proof of the generalized duality theorem

We set again $\alpha = \min_{u\in A} \max_{p\in B} L(u,p)$ and $\beta = \max_{p\in B} \min_{u\in A} L(u,p)$. The well-definition of the minimax problem follows very similarly to steps 1 and 2 of the previous proof.
Futhermore, from Proposition 1 it follows already that $\beta \leq \alpha$. So it only remains to show that $\alpha \leq \beta$.
Let $s = \alpha - \varepsilon, t = \beta + \varepsilon$ for $\varepsilon > 0$. We construct the map $T : A \times B \to 2^{A\times B}$ by setting

$$T(u,p) = \{(v,q) \in A \times B \mid L(v,p) < t, L(u,q) > s\}. \tag{29}$$

Note that

1. $T(u,p) \neq \emptyset$ follows from the definition of α and β;
2. the set $T(u,p)$ is convex since L is *quasi-convex* in u and quasi-concave in p;
3. the preimage

$$T^{-1}(\{(u,p)\}) = \{(v,q) \in A \times B \mid L(u,q) < t, L(v,p) > s\} \quad (30)$$

is weakly relatively open in $A \times B$. For the sets

$$\{v \in A \mid L(v,p) \leq s\} \quad \{and\} \quad \{q \in B \mid L(u,q) \geq t\}$$

are closed and convex by assumption on L and therefore are weakly closed with respect to $A \times B$.
Thus, Proposition 10 applies and we find $(u_0, p_0) \in A \times B$ such that

$$\alpha - \varepsilon = s < L(u_0, p_0) < t = \beta + \varepsilon, \quad (31)$$

and since $\varepsilon > 0$ was arbitrary, this proves the theorem.