Long Arm Approximation

FLUID MODEL

$$\begin{cases} -\mu \Delta u + \nabla p = 0 & \text{in } \Omega \\ \operatorname{div} u = 0 & \text{in } \Omega \end{cases} \qquad \Omega := \mathbb{R}^3 \setminus \bigcup_{i=1}^{4} \bar{B}_i .$$

- traction boundary condition: $-\underline{\sigma}\, n = f$ on $\partial\Omega$

far field condition: $u(x) \in \mathcal{O}(|x|^{-1})$, $|x| \longrightarrow \infty$.

$\underline{\sigma} := \mu \nabla^{sym} u - p I$ (Cauchy stress tensor)

- no-slip boundary condition: $u(x) = 0$ on $\partial\Omega$

Stokeslet: $G(x) := \frac{1}{8\pi\mu} \left(\frac{I}{|x|} + \frac{x \otimes x}{|x|^3} \right)$

single layer potential sol.: $u(x) = \int_{\partial\Omega} G(x-y) f(y) dy.$

In the paper we have $\tau \in (-b_i + \partial\Omega)$ But why?

For any $i \in \mathbb{N}_4$ and $\tau \in (-b_i + \partial B_a)$ we have

$$u(b_i + \tau) = \int_{\partial B_a} G(\tau - y) f_i(y)\, dy + \sum_{j \neq i \in \mathbb{N}_4} \int_{\partial B_a} G(b_{ij} + \tau - y) f_j(y)\, dy,$$

where $\cdot\ b_{ij} := b_i - b_j$

$\cdot\ f_j := f(b_j + y)$.

Since we have for $\min_{i<j} |b_{ij}| \to +\infty$ at the leading order $G(b_{ij} + \tau - y) \sim G(b_{ij})$, we can write in this limit

$$u_i(\tau) := \frac{1}{|\partial B_a|} \int_{\partial B_a} G(\tau - y) f_i(y)\, dy + \frac{1}{|\partial B_a|} \sum_{j \neq i \in \mathbb{N}_4} G(b_{ij}) \int_{\partial B_a} f_j(y)\, dy.$$

Why do we divide by $|\partial B_a|$?

Stokes law $\Rightarrow$ uniform tractions, i.e. f_j is constant on ∂B_j.

$$\Rightarrow u_i(\tau) := \frac{1}{6\pi\mu a} f_i + \sum_{j \neq i \in \mathbb{N}_4} G(b_{ij}) f_j .$$

Assumptions: · Length of l_i is given by $\xi_0 + \xi_i$, with $\xi_0 \gg a$.

· W.l.o.g. velocity field u_i applied to the center b_i since due to the constant tractions the u_i are uniform on ~~one~~ the associated boundary.

Balance equations: (Due to negligible inertia)

Forces: $f_1 + f_2 + f_3 + f_4 = 0.$

Torques: $\sum_{i \in \mathbb{N}_4} b_i \times f_i = 0.$

⚠ We cannot assume anymore that all geometric and dynamic quantities lie in $\mathbb{R}^2 \times \{0\}$.

In particular, we cannot treat the torques as scalar quantities anymore!

However, for any $k \in \mathbb{N}_3$, the map

$$f \longmapsto w_k(b_i, f) := ((b_i - c) \times f) \cdot \hat{e}_k$$

is a linear form on $\mathbb{R}^3$, where $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ denotes the canonical basis for $\mathbb{R}^3$. Hence, we find $\forall i \in \mathbb{N}_4$ vectors $w_{ki}(b_i) \in \mathbb{R}^3$ such that

$$w_k(b_i, f_i) = w_{ki}(b_i) \cdot f_i .$$

In particular, we find ~~vectors~~ $\cancel{f_i}$ for all $i \in \mathbb{N}_4$ vectors $w_{ki}(\xi_i, R) \in \mathbb{R}^3$ such that

$$w_{ki}(\xi_i, R) \cdot f_i = w_k(b_i, f_i) = ((b_i - c) \times f_i) \cdot \hat{e}_k$$

since $b_i := c + \xi_i R z_i \in \mathbb{R}^3$.

Therefore the balance equation for the total torque is equivalent to

$$\forall k \in \mathbb{N}_3 : \sum_{i \in \mathbb{N}_4} w_{ki}(\xi_i, R) \cdot f_i = 0 .$$

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We have $G(b_{ij}) = G(b_{ji})$ and thus we define

$$\ell_i := b_{i,i+1} \;;\; L_i := G(\ell_i), \quad i \in \mathbb{N}_4$$
$$K_1 := G(b_{13}), \; K_2 := G(b_{24})$$

where we take the indices mod 4.

We set

- $\nu := 6\pi\mu a$

- $\mathcal{I} := \mathrm{diag}(\underbrace{I}_{3\times3}, I, I, I) \in \mathcal{M}_{12\times12}(\mathbb{R})$.

- $\underline{u} := (u_1, u_2, u_3, u_4) \in \mathbb{R}^{12}$

- $\underline{f} := (f_1, f_2, f_3, f_4) \in \mathbb{R}^{12}$.

- $$\mathcal{L} := \begin{pmatrix} 0 & L_1 & K_1 & L_4 \\ L_1 & 0 & L_2 & K_2 \\ K_1 & L_2 & 0 & L_3 \\ L_4 & K_2 & L_3 & 0 \end{pmatrix}.$$

Then we can write

$$\underline{u} = \left(\frac{1}{\nu}\mathcal{I} + \mathcal{L}\right)\underline{f}.$$

For the velocities at the centers of the balls, we have for $i \in \mathbb{N}_4$

$$u_i = \dot{c} + \dot{\xi}_i R z_i + \dot{R} R^T (\xi_0 + \xi_i) z_i .$$

In particular, at $R_0 = I$, we have

$$u_i = \dot{c} + \dot{\xi}_i z_i + \dot{R}_0 (\xi_0 + \xi_i) z_i .$$

$$= \dot{c} + \dot{\xi}_i z_i + \omega \times (\xi_0 + \xi_i) z_i$$

$$= \dot{c} + \dot{\xi}_i z_i + (\xi_0 + \xi_i) [z_i]_\times^T \omega ,$$

Is there a better way to do that ?

where $\omega = Ax(\dot{R}_0)$ and $[z_i]_\times$ the skew symmetric matrix such that

$$[z_i]_\times \omega = z_i \times \omega .$$

Then we have

$$\underline{u}(\xi, R_0) = \mathrm{diag}(z_i)\, \dot{\xi} + \begin{pmatrix} I_{3\times 3} & (\xi_0 + \xi_1)[z_1]_\times^T \\ I_{3\times 3} & (\xi_0 + \xi_2)[z_2]_\times^T \\ I_{3\times 3} & (\xi_0 + \xi_3)[z_3]_\times^T \\ I_{3\times 3} & (\xi_0 + \xi_4)[z_4]_\times^T \end{pmatrix} \begin{pmatrix} \dot{c} \\ \omega \end{pmatrix}$$

$$=: \mathcal{X}_0 \dot{\xi} + \mathcal{Y}(\xi) \dot{p} .$$

Note that $\omega = [\dot{R}_0]_{\mathcal{L}}$, where $\mathcal{L} = (L_1, L_2, L_3)$ is the ordered basis in the report. Hence, we have

$$\underline{0} = \frac{1}{v} W \underline{f}(\underline{u}(\underline{\xi}, R_0)) = W \mathcal{L}_v \, \underline{u}(\xi, I)$$

$$= W \mathcal{L}_v [X_0 \dot{\xi} + Y(\xi) \dot{p}]$$

$$= \underbrace{W \mathcal{L}_v}_{=: V_v} X_0 \dot{\xi} + W \mathcal{L}_v Y(\xi) \dot{p} = \underbrace{V_v X_0}_{\text{invertible?}} \dot{\xi} + \underbrace{V_v Y(\xi)}_{\text{invertible?}} \dot{p}$$

$$\Leftrightarrow \quad \dot{p} = - \frac{V_v X_0}{V_v Y(\xi)} \dot{\xi} \overset{!}{=} F(\xi, I) \dot{\xi}.$$

We know that in general

$$\dot{p} = \mathrm{diag}(R) F(\xi, I) \dot{\xi},$$

so we are done?

~~OPTIMAL~~

OPTIMAL SWIMMING

The instantaneous power dissipated is given by

$$\mathcal{P}(\underline{u}) = \frac{1}{\nu} f(\underline{u}) \cdot \underline{u} .$$

Since it is independent of the position $p = (c, R)$, we can calculate it at $R = I$, which allows us to express it as

$$\mathcal{P}(\underline{u}(\xi, I)) = \frac{1}{\nu} f(\underline{u}(\xi, I)) \cdot \underline{u}(\xi, I)$$

$$= g(\xi) \dot{\xi} \cdot \dot{\xi} ,$$

with the positive definite matrix $g(\xi)$ given by

$$g(\xi) = \mathcal{X}_0^T \left(\mathcal{I} - y(\xi) \frac{\mathcal{V}_\nu}{\mathcal{V}_\nu y(\xi)}\right)^T \mathcal{L}_\nu \left(\mathcal{I} - y(\xi) \frac{\mathcal{V}_\nu}{\mathcal{V}_\nu y(\xi)}\right) \times \mathcal{X}_0$$

In the limit of small strokes, we can assume that $\mathcal{P}(\underline{u}(t)) = G \dot{\xi}(t) \cdot \dot{\xi}(t)$ with $G := g(0, I)$ and the total energy dissipation associated to a stroke $\xi \xrightarrow{\mathcal{V}} \mathcal{U}$ is given by

$$G(\xi) := \int_J G\,\dot{\xi}(t)\cdot\dot{\xi}(t)\,dt.$$

We have already seen that G must be of the form

$$G = \begin{pmatrix} K & h & h & h \\ h & K & h & h \\ h & h & K & h \\ h & h & h & K \end{pmatrix} \qquad K > \max(h, -3h)$$

By comparison of the terms, we find that in terms of a and ξ_0 we have

$$K(a,\xi_0) = \frac{3}{256}\left(64 + \frac{25\sqrt{6}\,a}{\xi_0} + \frac{12a}{9a - 2\sqrt{6}\,\xi_0}\right)$$

$$h(a,\xi_0) = \frac{1}{768}\left(60 + \frac{69\sqrt{6}\,a}{\xi_0} - \frac{16\xi_0}{3\sqrt{6}\,a - 4\xi_0}\right)$$

$$g_c(a,\xi_0) = \frac{1}{192}\left(129 + \frac{39\sqrt{6}\,a}{\xi_0} + \frac{4\xi_0}{3\sqrt{6}\,a - 4\xi_0} + \frac{27a}{9a - 2\sqrt{6}\,\xi_0}\right)$$

$$g_\theta(a,\xi_0) = 1 + \frac{9}{8}\sqrt{\frac{3}{2}}\,\frac{a}{\xi_0}.$$

$$a(a,\xi_0) = -\frac{5}{16} + \frac{\xi_0}{16\xi_0 - 12\sqrt{6}\,a}.$$

To determine the remaining parameters α, β, λ and δ in terms of a and ξ_0, we can follow the same approach as for SPr3. For any $i, j \in \mathbb{N}_4$ the (ij)-entry of the matrix $A_k B_k$ is given by $\dot{\varphi}_{e_i}(0) e_j \cdot f_k$ with $\varphi_{e_i}(t) := F(t e_i)$.

Recall that if $\mathcal{A}, \mathcal{B} : t \in \mathbb{R} \longmapsto \mathcal{A}(t), \mathcal{B}(t) \in \mathbb{R}^{N \times N}$ are smooth matrix valued functions and $\mathcal{A}(0)$ is invertible, then

$$\partial_t [\mathcal{A}^{-1}(t) \mathcal{B}(t)] \Big|_{t=0} = \frac{\mathcal{I}}{\mathcal{A}(0)} \left(\dot{\mathcal{B}}(0) - \dot{\mathcal{A}}(0) \frac{\mathcal{B}(0)}{\mathcal{A}(0)} \right)$$

Set $\mathcal{A}(t) := -V_v(t\xi) y(t\xi)$, $\mathcal{B}(t) := V_v(t\xi) x_0$ to get

$$\dot{\varphi}_{e_i}(0) e_j = \frac{\mathcal{I}}{V_v(0) y(0)} \Big(\partial_t [V_v(t e_i) y(t e_i)]_{t=0} \frac{V_v(0)}{V_v(0) y(0)}$$

$$- \partial_t [V_v(t e_i)]_{t=0} \Big) x_0 e_j .$$

We have

$$\alpha = \tfrac{1}{2} \left(A_{1,41} - A_{1,14} \right) \qquad \lambda = -3 \, A_{3,11} .$$

$$\beta = \tfrac{3}{2\sqrt{2}} \left(A_{1,41} + A_{1,14} \right)$$

$$\delta = \tfrac{1}{2} \left(B_{1,12} - B_{1,12} \right)$$

Some pretty rough caclulations with the help of Mathematica yield

$$\alpha(a,\xi_0) = -\frac{\sqrt{3}\,a}{8\left(3\sqrt{6}\,a - 4\xi_0\right)^2}$$

$$\beta(a,\xi_0) = -\frac{3a\left(81\sqrt{2}\,a - 16\sqrt{3}\,\xi_0\right)}{128\sqrt{2}\left(3\sqrt{6}\,a - 4\xi_0\right)^2\xi_0}$$

$$\lambda(a,\xi_0) = \frac{9a\left(27a - 8\sqrt{6}\,\xi_0\right)}{128\,\xi_0\left(3\sqrt{6}\,a - 4\xi_0\right)^2}$$

$$\delta(a,\xi_0) = \frac{81a^2 - 240\sqrt{6}\,a\xi_0 + 256\,\xi_0^2}{64\,\xi_0^2\left(-27\sqrt{6}\,a^2 - 252\,a\xi_0 + 64\sqrt{6}\,\xi_0^2\right)}$$

Let us note that

$$\lim_{a\to 0}\alpha(a,\xi_0) = \lim_{\xi_0\to+\infty}\alpha(a,\xi_0) = 0$$

and the same for β and λ but

$$\lim_{a\to 0}\delta(a,\xi_0) = \frac{1}{16\sqrt{6}\,\xi_0^2} \neq 0 = \lim_{\xi_0\to+\infty}\delta(a,\xi_0)$$

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