18

Scalar Products of Spinors and the Chessboard

The Euclidean space $\mathbb{R}^3$ has a scalar product $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + x_3 y_3$ with the automorphism group $O(3)$. Pauli spinors of $\mathbb{R}^3$ are of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{where} \quad \psi_1, \psi_2 \in \mathbb{C}$$

and belong to a complex linear space $\mathbb{C}^2$. There are two kinds of scalar products for Pauli spinors $\psi, \varphi \in \mathbb{C}^2$,

$$\psi^{*\top} \varphi = \psi_1^* \varphi_1 + \psi_2^* \varphi_2 \quad \text{and}$$
$$\psi^\top i\sigma_2 \varphi = \psi_1 \varphi_2 - \psi_2 \varphi_1,$$

which have automorphism groups $U(2)$ and $Sp(2, \mathbb{C}) = SL(2, \mathbb{C})$, respectively. The Minkowski space $\mathbb{R}^{1,3}$ has a scalar product

$$\mathbf{x} \cdot \mathbf{y} = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$$

with the automorphism group $O(1, 3)$. Dirac spinors of $\mathbb{R}^{1,3}$ belong to a complex linear space $\mathbb{C}^4$. There is a scalar product of Dirac spinors $\psi, \varphi \in \mathbb{C}^4$,

$$\psi^{*\top} \gamma_0 \varphi = \psi_1^* \varphi_1 + \psi_2^* \varphi_2 - \psi_3^* \varphi_3 - \psi_4^* \varphi_4,$$

with the automorphism group $U(2, 2)$.

One might wonder about the following things:

(i) Why do spinors with complex entries arise in conjunction with the real quadratic spaces $\mathbb{R}^3$ and $\mathbb{R}^{1,3}$?
(ii) If we consider generalizations to arbitrary $\mathbb{R}^{p,q}$, are the scalar products of spinors still Hermitian or antisymmetric?
(iii) Are the scalar products of spinors definite or neutral for all $\mathbb{R}^{p,q}$?
(iv) Is there a general pattern in higher dimensions for the changes from $\mathbb{R}^3$ to $\bar{\mathbb{C}}^2$ or from $\mathbb{R}^{1,3}$ to $\bar{\mathbb{C}}^{2,2}$?

We will answer these questions in the following general form: What is the automorphism group of the scalar product of spinors in the case of the quadratic space $\mathbb{R}^{p,q}$? The scalar products of spinors can be collected into two equivalence classes when p and q are kept fixed in $\mathbb{R}^{p,q}$. There are altogether

$$32 = \frac{8 \times 8}{2}$$

different kinds of scalar products of spinors when we let p and q vary in $\mathbb{R}^{p,q}$.

The situation is much simplified if we consider instead of the real quadratic spaces $\mathbb{R}^{p,q}$ their complexifications $\mathbb{C} \otimes \mathbb{R}^{p,q}$. Then there remain only four different types of scalar products of spinors to be considered.

The reader will notice that the unitary group $U(2,2)$ can be adjoined to the Minkowski space-times $\mathbb{R}^{1,3}$ and $\mathbb{R}^{3,1}$ in two different ways by

– complexifying, or
– adding one extra dimension (of positive signature),

which respectively result in

– $\mathbb{C} \otimes \mathbb{R}^{1,3}$ and $\mathbb{C} \otimes \mathbb{R}^{3,1}$, or
– $\mathbb{R}^{2,3}$ and $\mathbb{R}^{4,1}$.

In both cases $U(2,2)$ is the automorphism group of the scalar product of spinors. The latter case gives a hint of a relation to the conformal group of the Minkowski space. [1]

18.1 Scalar products on spinor spaces

We start with spinors ψ, φ in spinor spaces $S = \mathcal{C}\ell_{p,q}f$ which are linear spaces over division rings $\mathbb{D} = f\mathcal{C}\ell_{p,q}f$. We will consider two cases:

(i) The minimal left ideals $S = \mathcal{C}\ell_{p,q}f$ providing irreducible representations for all $\mathcal{C}\ell_{p,q}$; these representations are also faithful for simple $\mathcal{C}\ell_{p,q}$.
(ii) The left ideals $S \oplus \hat{S} = \mathcal{C}\ell_{p,q}e$, $e = f + \hat{f}$, providing faithful representations for semi-simple $\mathcal{C}\ell_{p,q}$.

1 The Vahlen matrices of the Minkowski space are such that $\mathrm{Mat}(2, \mathcal{C}\ell_{1,3}) \simeq \mathcal{C}\ell_{2,4}$ and $\mathrm{Mat}(2, \mathcal{C}\ell_{3,1}) \simeq \mathcal{C}\ell_{4,2}$, where the even subalgebras are isomorphic: $\mathcal{C}\ell^+_{2,4} \simeq \mathcal{C}\ell^+_{4,2} \simeq \mathrm{Mat}(4,\mathbb{C})$ or $\mathcal{C}\ell_{2,3} \simeq \mathcal{C}\ell_{4,1} \simeq \mathrm{Mat}(4,\mathbb{C})$. The (connected components of the) conformal groups of $\mathbb{R}^{1,3}$ and $\mathbb{R}^{3,1}$ are isomorphic to

$$\frac{SO_+(2,4)}{\{I,-I\}} \simeq \frac{SO_+(4,2)}{\{I,-I\}} \simeq \frac{SU(2,2)}{\{\pm I, \pm iI\}}.$$

The automorphism group $U(2,2)$ of the scalar product of Dirac spinors contains as a subgroup the universal cover $SU(2,2)$ of the conformal group of the Minkowski space.

As before, let

$$\check{\mathbb{D}} \quad \text{be either} \quad \mathbb{D} \quad \text{or} \quad \mathbb{D} \oplus \hat{\mathbb{D}},$$
$$\check{S} \quad \text{be either} \quad S \quad \text{or} \quad S \oplus \hat{S}$$

according as $\mathcal{C}\ell_{p,q}$ is simple or semi-simple, respectively.

Let β be either of the anti-automorphisms $u \to \tilde{u}$ and $u \to \bar{u}$ of $\mathcal{C}\ell_{p,q}$. The real linear spaces

$$P_+ = \{\psi \in S \mid \beta(\psi) = +\psi\},$$
$$P_- = \{\psi \in S \mid \beta(\psi) = -\psi\}$$

have real dimensions 0, 1, 2 or 3 and

$$P = P_+ \oplus P_- = \{\psi \in S \mid \beta(\psi) \in S\}$$

has real dimension 0, 1, 2 or 4 no matter how large the dimension of S is. To prove this we may use periodicity, $\mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{0,8} \simeq \mathcal{C}\ell_{p,q+8}$, and the fact that for $\mathcal{C}\ell_{0,8}$ the dimension of $P = P_+$ is 1 (over $\mathbb{R}$).

Define the real linear space

$$\check{P} = \{\psi \in \check{S} \mid \beta(\psi) \in \check{S}\}$$

which has real dimension 1, 2, 3 or 4. For all ψ, φ in S or $\check{S}$ we have $\beta(\psi)\varphi$ in P or $\check{P}$. There is an invertible element s in $\mathcal{C}\ell_{p,q}$ with the property $P \subset s^{-1}\mathbb{D}$ and which is, in the case $\dim P \neq 0$, such that for all λ in $\mathbb{D}$ also $\lambda^\sigma = s\beta(\lambda)s^{-1}$ is in $\mathbb{D}$. [2] To prove that such an element s exists in every $\mathcal{C}\ell_{p,q}$ we may first consider the lower-dimensional cases and then proceed by making use of the fact that $\beta(f) = f$ for

$$f = \frac{1}{2}(1 + \mathbf{e}_{1248})\frac{1}{2}(1 + \mathbf{e}_{2358})\frac{1}{2}(1 + \mathbf{e}_{3468})\frac{1}{2}(1 + \mathbf{e}_{4578})$$

in $\mathcal{C}\ell_{0,8}$, and therefore $s = 1$ is such an element in $\mathcal{C}\ell_{0,8}$.

In the same way, there is an invertible element s in $\mathcal{C}\ell_{p,q}$ with the property $\check{P} = s^{-1}\check{\mathbb{D}}$ and which is moreover such that for all λ in $\check{\mathbb{D}}$ also $\lambda^\sigma = s\beta(\lambda)s^{-1}$ is in $\check{\mathbb{D}}$. Both the maps

$$\check{S} \times \check{S} \to \check{\mathbb{D}}, \ (\psi, \varphi) \to \begin{cases} s\tilde{\psi}\varphi \\ s\bar{\psi}\varphi \end{cases}$$

are scalar products on $\check{S}$. Similarly, we may construct a scalar product on S. The element s can be chosen from the standard basis of $\mathcal{C}\ell_{p,q}$ [when f is constructed by the standard basis of $\mathcal{C}\ell_{p,q}$]. In particular, $\beta(s) = \pm s$, and so the scalar product is symmetric or antisymmetric [on both S and $\check{S}$]. [3] The

2 The mapping $\lambda \to \lambda^\sigma$ is an (anti-)automorphism of the division ring $\mathbb{D}$.

3 More precisely, the scalar product on S is $\mathbb{D}^\sigma$-symmetric or $\mathbb{D}^\sigma$-skew, and the scalar product on $\check{S}$ is $\check{\mathbb{D}}^\sigma$-symmetric or $\check{\mathbb{D}}^\sigma$-skew.

scalar product on $\check{S}$ is more interesting; it is

symmetric or antisymmetric

non-degenerate

positive definite (for the choice $s = 1$) on $\begin{cases} \mathcal{C}\ell_{n,0} & \text{with} \quad s\tilde{\psi}\varphi \\ \mathcal{C}\ell_{0,n} & \text{with} \quad s\bar{\psi}\varphi \end{cases}$

neutral except on $\begin{cases} \mathcal{C}\ell_{n,0},\ \mathcal{C}\ell_{0,1},\ \mathcal{C}\ell_{0,2},\ \mathcal{C}\ell_{0,3} & \text{with} \quad s\tilde{\psi}\varphi \\ \mathcal{C}\ell_{0,n},\ \mathcal{C}\ell_{1,0} \quad \text{with} \quad s\bar{\psi}\varphi. \end{cases}$

The scalar product is definite or neutral except for $\mathcal{C}\ell_{0,1}$, $\mathcal{C}\ell_{0,2}$, $\mathcal{C}\ell_{0,3}$ or $\mathcal{C}\ell_{1,0}$. In these lower-dimensional exceptional cases neutrality is not possible, because the spinor space $\check{S}$ is 1-dimensional over $\check{\mathbb{D}} = \mathbb{C}$, $\mathbb{H}$, ${}^2\mathbb{H}$ or ${}^2\mathbb{R}$, respectively.

For a fixed $\mathcal{C}\ell_{p,q}$, the neutral scalar products on $\check{S}$, induced by arbitrary anti-automorphisms of $\mathcal{C}\ell_{p,q}$, can be collected into *two* equivalence classes, the equivalence relation being

$$\langle\psi,\varphi\rangle_1 \simeq \langle\psi,\varphi\rangle_2 \iff \exists\, U \in \mathrm{End}_{\check{\mathbb{D}}}\check{S},\ \langle U\psi, U\varphi\rangle_1 = \langle\psi,\varphi\rangle_2$$

for all $\psi, \varphi \in \check{S}$. In each class there is a scalar product induced by such an anti-automorphism of $\mathcal{C}\ell_{p,q}$ (extending an orthogonal transformation of $\mathbb{R}^{p,q}$) that does not single out any distinguished direction in $\mathbb{R}^{p,q}$, namely, the reversion $u \to \tilde{u}$ or the Clifford-conjugation $u \to \bar{u}$ of $\mathcal{C}\ell_{p,q}$.

18.2 Automorphism groups of scalar products of spinors

Examples. 1. The Clifford algebra $\mathcal{C}\ell_{2,1}$ is isomorphic to $\mathrm{Mat}(2, {}^2\mathbb{R})$. The idempotent $f = \frac{1}{2}(1 + \mathbf{e}_1)\frac{1}{2}(1 + \mathbf{e}_{23})$ is primitive in $\mathcal{C}\ell_{2,1}$. The subalgebra $\mathbb{D} = f\mathcal{C}\ell_{2,1}f$ is just the line $\{\lambda f \mid \lambda \in \mathbb{R}\}$; with unity f it is isomorphic to the division ring $\mathbb{R}$. The basis elements

$$f_1 = \tfrac{1}{4}(1 + \mathbf{e}_1 + \mathbf{e}_{23} + \mathbf{e}_{123})$$
$$f_2 = \tfrac{1}{4}(\mathbf{e}_2 - \mathbf{e}_{12} + \mathbf{e}_3 - \mathbf{e}_{13})$$

of $S = \mathcal{C}\ell_{2,1}f$ are such that

$$\begin{matrix} \tilde{f}_1 f_1 = 0, & \tilde{f}_1 f_2 = 0 \\ \tilde{f}_2 f_1 = 0, & \tilde{f}_2 f_2 = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} \bar{f}_1 f_1 = 0, & \bar{f}_1 f_2 = f_2 \\ \bar{f}_2 f_1 = -f_2, & \bar{f}_2 f_2 = 0. \end{matrix}$$

The products $s\tilde{\psi}\varphi$, $s = 1$, and $s\bar{\psi}\varphi$, $s = \mathbf{e}_2$, have values in $\mathbb{D}$; they are scalar products on S. The scalar product $\tilde{\psi}\varphi$ vanishes identically; its automorphism group is the full linear group $GL(2, \mathbb{R})$. The scalar product $\mathbf{e}_2\bar{\psi}\varphi$ is antisymmetric; its automorphims group is $Sp(2, \mathbb{R})$. If we consider $\check{S} = S \oplus \hat{S}$ instead of S, then the automorphism group of the scalar product $s\tilde{\psi}\varphi$ becomes non-degenerate (because of the swap) and the automorphism group of the scalar

product $s\bar{\psi}\varphi$ splits: ${}^2Sp(2,\mathbb{R}) = Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R})$.
2. The Clifford algebra $\mathcal{C}\ell_{1,3}$ of the Minkowski space $\mathbb{R}^{1,3}$ is isomorphic to the real matrix algebra $\mathrm{Mat}(2,\mathbb{H})$. Take an orthonormal basis $\{\gamma_0,\gamma_1,\gamma_2,\gamma_3\}$ for $\mathbb{R}^{1,3}$. The idempotent $f = \frac{1}{2}(1+\gamma_0)$ is primitive in $\mathcal{C}\ell_{1,3}$. As a real linear space the minimal left ideal $S = \mathcal{C}\ell_{1,3}f$ is 8-dimensional and the elements

$$\begin{aligned} h_1 &= \tfrac{1}{2}(1+\gamma_0), & h_2 &= \tfrac{1}{2}(-\gamma_{123}+\gamma_{0123}) \\ i_1 &= \tfrac{1}{2}(\gamma_{23}+\gamma_{023}), & i_2 &= \tfrac{1}{2}(\gamma_1-\gamma_{01}) \\ j_1 &= \tfrac{1}{2}(\gamma_{31}+\gamma_{031}), & j_2 &= \tfrac{1}{2}(\gamma_2-\gamma_{02}) \\ k_1 &= \tfrac{1}{2}(\gamma_{12}+\gamma_{012}), & k_2 &= \tfrac{1}{2}(\gamma_3-\gamma_{03}) \end{aligned}$$

form a basis for $S_{\mathbb{R}}$. The set $\{h_1,i_1,j_1,k_1\}$ is a basis for the real linear space $\mathbb{D} = f\mathcal{C}\ell_{1,3}f$. As a ring $\mathbb{D}$ is isomorphic to the quaternion ring $\mathbb{H}$, and the right $\mathbb{D}$-linear module $S_{\mathbb{D}}$ is two-dimensional with basis $\{h_1,h_2\}$. In the basis $\{h_1,h_2\}$ left multiplication by $\gamma_0,\gamma_1,\gamma_2,\gamma_3$ is represented by the following 2×2-matrices with quaternion entries:

$$\gamma_0 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_1 \simeq \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_2 \simeq \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad \gamma_3 \simeq \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

The real linear spaces P_+ and P_- have bases

$$\begin{array}{lll} & P_+ & P_- \\ \tilde{u}: & \{h_1\} & \{i_1,j_1,k_1\} \\ \bar{u}: & \{h_2\} & \{i_2,j_2,k_2\}. \end{array}$$

In the scalar products $S\times S \to \mathbb{D}$, $(\psi,\varphi)\to s\beta(\psi)\varphi$ one can take $s=1$ for $s\tilde{\psi}\varphi$ and $s=\gamma_{123}$ for $s\bar{\psi}\varphi$. Direct computation shows that

$$\begin{array}{ll} \tilde{h}_1h_1 = h_1, & \tilde{h}_1h_2 = 0 \\ \tilde{h}_2h_1 = 0, & \tilde{h}_2h_2 = -h_1 \end{array} \quad \text{and} \quad \begin{array}{ll} \bar{h}_1h_1 = 0, & \bar{h}_1h_2 = h_2 \\ \bar{h}_2h_1 = h_2, & \bar{h}_2h_2 = 0. \end{array}$$

Both the scalar products have the automorphism group $Sp(2,2)$. ∎

The Tables 1 and 2 list automorphism groups of the scalar products on $\check{S}$; they are nothing but the groups

$$\{s\in\mathcal{C}\ell_{p,q} \mid s\tilde{s}=1\} \quad \text{and} \quad \{s\in\mathcal{C}\ell_{p,q} \mid s\bar{s}=1\}.$$

If the Clifford algebra $\mathcal{C}\ell_{p,q}$ is semi-simple and if the automorphism group on $\check{S}$ is a direct product ${}^2G = G\times G$, then the automorphism group on S is G.

Table 1. Automorphism Groups of $s\tilde{\psi}\varphi$ on $\check{S}$ in $\mathcal{C}\ell_{p,q}$.

$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$O(1)$							
1							$O(1,\mathbb{C})$		${}^2O(1)$						
2						$SO^*(2)$		$O(1,1)$		$O(2)$					
3					$GL(1,\mathbb{H})$		$U(1,1)$		$GL(2,\mathbb{R})$		$U(2)$				
4				$Sp(2,2)$		$Sp(2,2)$		$Sp(4,\mathbb{R})$		$Sp(4,\mathbb{R})$		$Sp(4)$			
5			$Sp(4,\mathbb{C})$		${}^2Sp(2,2)$		$Sp(4,\mathbb{C})$		${}^2Sp(4,\mathbb{R})$		$Sp(4,\mathbb{C})$		${}^2Sp(4)$		
6		$Sp(8,\mathbb{R})$		$Sp(4,4)$		$Sp(4,4)$		$Sp(8,\mathbb{R})$		$Sp(8,\mathbb{R})$		$Sp(4,4)$		$Sp(8)$	
7	$GL(8,\mathbb{R})$		$U(4,4)$		$GL(4,\mathbb{H})$		$U(4,4)$		$GL(8,\mathbb{R})$		$U(4,4)$		$GL(4,\mathbb{H})$		$U(8)$

Table 2. Automorphism Groups of $s\bar{\psi}\varphi$ on $\check{S}$ in $\mathcal{C}\ell_{p,q}$.

$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$O(1)$							
1							$U(1)$		$GL(1,\mathbb{R})$						
2						$Sp(2)$		$Sp(2,\mathbb{R})$		$Sp(2,\mathbb{R})$					
3					${}^2Sp(2)$		$Sp(2,\mathbb{C})$		${}^2Sp(2,\mathbb{R})$		$Sp(2,\mathbb{C})$				
4				$Sp(4)$		$Sp(2,2)$		$Sp(4,\mathbb{R})$		$Sp(4,\mathbb{R})$		$Sp(2,2)$			
5			$U(4)$		$GL(2,\mathbb{H})$		$U(2,2)$		$GL(4,\mathbb{R})$		$U(2,2)$		$GL(2,\mathbb{H})$		
6		$O(8)$		$SO^*(8)$		$SO^*(8)$		$O(4,4)$		$O(4,4)$		$SO^*(8)$		$SO^*(8)$	
7	${}^2O(8)$		$O(8,\mathbb{C})$		${}^2SO^*(8)$		$O(8,\mathbb{C})$		${}^2O(4,4)$		$O(8,\mathbb{C})$		${}^2SO^*(8)$		$O(8,\mathbb{C})$

Examples. 1. $\mathcal{C}\ell_{0,2}$, $s\tilde{\psi}\varphi$: $SO^*(2) = \{U \in SO(2,\mathbb{C}) \mid U^*J = JU\} \simeq SO(2)$.
2. $\mathcal{C}\ell_2$, $s\bar{\psi}\varphi$: $Sp(2,\mathbb{R}) = \{U \in \mathrm{Mat}(2,\mathbb{R}) \mid U^\top JU = U\} \simeq SL(2,\mathbb{R})$.
3. $\mathcal{C}\ell_5$, $s\bar{\psi}\varphi$: ${}^2Sp(4) = Sp(4) \times Sp(4)$, $Sp(4)/\{\pm I\} \simeq SO(5)$.
4. $\mathcal{C}\ell_{1,3}$, $Sp(2,2) = U(2,2) \cap Sp(4,\mathbb{C})$, $Sp(2,2)/\{\pm I\} \simeq SO_+(4,1)$. ∎

Note that the group $U(2,2)$ appears as an automorphism group of the scalar product $s\bar{\psi}\varphi$ for $\mathcal{C}\ell_{2,3}$ and $\mathcal{C}\ell_{4,1}$. To explain the presence of $U(2,2)$ in the Dirac theory by the real Clifford algebras $\mathcal{C}\ell_{p,q}$, we must add one dimension of positive square to the Minkowski spaces $\mathbb{R}^{1,3}$ and $\mathbb{R}^{3,1}$.

There is another explanation: use complexifications $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$. For a fixed $n = p + q$ we have the isomorphisms of algebras $\mathbb{C} \otimes \mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell(\mathbb{C}^n)$. Although the complex linear space $\mathbb{C}^n$ has a symmetric (= not sesquilinear) bilinear form on itself, we may equip the spinor spaces of $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ with sesquilinear forms

$s\tilde{\psi}^*\varphi$ and $s\bar{\psi}^*\varphi$. These sesquilinear products have automorphism groups

$$\{s \in \mathbb{C} \otimes \mathcal{C}\ell_{p,q} \mid s\tilde{s}^* = 1\} \quad \text{and} \quad \{s \in \mathbb{C} \otimes \mathcal{C}\ell_{p,q} \mid s\bar{s}^* = 1\}.$$

For a fixed $n = p + q$ these groups depend on the values of p and q [although the algebra $\mathbb{C} \otimes \mathcal{C}\ell_{p,q}$ is independent of p and q].

Table 3. Automorphism Groups of $s\tilde{\psi}^*\varphi$ in $\mathbb{C}^* \otimes \mathcal{C}\ell_{p,q}$.

$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$U(1)$							
1							$GL(1,\mathbb{C})$		${}^2U(1)$						
2						$U(1,1)$		$U(1,1)$		$U(2)$					
3					$GL(2,\mathbb{C})$		${}^2U(1,1)$		$GL(2,\mathbb{C})$		${}^2U(2)$				
4				$U(2,2)$		$U(2,2)$		$U(2,2)$		$U(2,2)$		$U(4)$			
5			$GL(4,\mathbb{C})$		${}^2U(2,2)$		$GL(4,\mathbb{C})$		${}^2U(2,2)$		$GL(4,\mathbb{C})$		${}^2U(4)$		
6		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(8)$	
7	$GL(8,\mathbb{C})$		${}^2U(4,4)$		$GL(8,\mathbb{C})$		${}^2U(4,4)$		$GL(8,\mathbb{C})$		${}^2U(4,4)$		$GL(8,\mathbb{C})$		${}^2U(8)$

Table 4. Automorphism Groups of $s\bar{\psi}^*\varphi$ in $\mathbb{C}^* \otimes \mathcal{C}\ell_{p,q}$.

$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$U(1)$							
1							${}^2U(1)$		$GL(1,\mathbb{C})$						
2						$U(2)$		$U(1,1)$		$U(1,1)$					
3					${}^2U(2)$		$GL(2,\mathbb{C})$		${}^2U(1,1)$		$GL(2,\mathbb{C})$				
4				$U(4)$		$U(2,2)$		$U(2,2)$		$U(2,2)$		$U(2,2)$			
5			${}^2U(4)$		$GL(4,\mathbb{C})$		${}^2U(2,2)$		$GL(4,\mathbb{C})$		${}^2U(2,2)$		$GL(4,\mathbb{C})$		
6		$U(8)$		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(4,4)$		$U(4,4)$	
7	${}^2U(8)$		$GL(8,\mathbb{C})$		${}^2U(4,4)$		$GL(8,\mathbb{C})$		${}^2U(4,4)$		$GL(8,\mathbb{C})$		${}^2U(4,4)$		$GL(8,\mathbb{C})$

See Porteous 1969 p. 271 ll. 1-8. Note that complexification explains the occurrence of $U(2,2)$ in conjunction with the Minkowski spaces.

In complexifications of real algebras we replaced the ground field $\mathbb{R}$ by $\mathbb{C}$, a field extension with an involution, the complex conjugation [to emphasize that $\mathbb{C}$ comes with a complex conjugation we denote $\bar{\mathbb{C}}$ or $\mathbb{C}^*$].

We could also tensor $\mathcal{C}\ell_{p,q}$ by the real algebra ${}^2\mathbb{R}$, a commutative ring with an irreducible involution, the swap. See Porteous 1969 pp. 193, 251. This leads to the automorphism groups shown in Table 5 [isomorphic to the subgroup of invertible elements in $\mathcal{C}\ell_{p,q}$].

Table 5. Automorphism Groups for ${}^2\mathbb{R} \otimes \mathcal{C}\ell_{p,q}$.

$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$GL(1,\mathbb{R})$							
1							$GL(1,\mathbb{C})$		${}^2GL(1,\mathbb{R})$						
2						$GL(1,\mathbb{H})$		$GL(2,\mathbb{R})$		$GL(2,\mathbb{R})$					
3					${}^2GL(1,\mathbb{H})$		$GL(2,\mathbb{C})$		${}^2GL(2,\mathbb{R})$		$GL(2,\mathbb{C})$				
4				$GL(2,\mathbb{H})$		$GL(2,\mathbb{H})$		$GL(4,\mathbb{R})$		$GL(4,\mathbb{R})$		$GL(2,\mathbb{H})$			
5			$GL(4,\mathbb{C})$		${}^2GL(2,\mathbb{H})$		$GL(4,\mathbb{C})$		${}^2GL(4,\mathbb{R})$		$GL(4,\mathbb{C})$		${}^2GL(2,\mathbb{H})$		
6		$GL(8,\mathbb{R})$		$GL(4,\mathbb{H})$		$GL(4,\mathbb{H})$		$GL(8,\mathbb{R})$		$GL(8,\mathbb{H})$		$GL(4,\mathbb{H})$		$GL(4,\mathbb{H})$	
7	${}^2GL(8,\mathbb{R})$		$GL(8,\mathbb{C})$		${}^2GL(4,\mathbb{H})$		$GL(8,\mathbb{C})$		${}^2GL(8,\mathbb{R})$		$GL(8,\mathbb{C})$		${}^2GL(4,\mathbb{H})$		$GL(8,\mathbb{C})$

See Porteous 1969 p. 271 ll. 11-18.

In the case of the complex Clifford algebras $\mathcal{C}\ell(\mathbb{C}^n)$ we may further equip the spinor space with a symmetric (= not sesquilinear) form on itself, sending (ψ,φ) to $s\tilde{\psi}\varphi$ or $s\bar{\psi}\varphi$, see Table 6.

Table 6. Automorphism Groups for $\mathbb{C}^n$.

n	$s\tilde{\psi}\varphi$	n	$s\bar{\psi}\varphi$
0	$O(1,\mathbb{C})$	0	$O(1,\mathbb{C})$
1	${}^2O(1,\mathbb{C})$	1	$GL(1,\mathbb{C})$
2	$O(2,\mathbb{C})$	2	$Sp(2,\mathbb{C})$
3	$GL(2,\mathbb{C})$	3	${}^2Sp(2,\mathbb{C})$
4	$Sp(4,\mathbb{C})$	4	$Sp(4,\mathbb{C})$
5	${}^2Sp(4,\mathbb{C})$	5	$GL(4,\mathbb{C})$
6	$Sp(8,\mathbb{C})$	6	$O(8,\mathbb{C})$
7	$GL(8,\mathbb{C})$	7	${}^2O(8,\mathbb{C})$

See Porteous 1969 p. 271 l. 9.

As the last extension we consider the tensor product ${}^2\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}^n$. The scalar products of spinors are formed by reversion or Clifford-conjugation composed with swap (no complex conjugation), see Table 7.

Table 7. Automorphism Groups for ${}^2\mathbb{C} \otimes \mathbb{C}^n$.

n	$s\tilde{\psi}\varphi$ or $s\bar{\psi}\varphi$
0	$GL(1,\mathbb{C})$
1	${}^2GL(1,\mathbb{C})$
2	$GL(2,\mathbb{C})$
3	${}^2GL(2,\mathbb{C})$
4	$GL(4,\mathbb{C})$
5	${}^2GL(4,\mathbb{C})$
6	$GL(8,\mathbb{C})$
7	${}^2GL(8,\mathbb{C})$

See Porteous 1969 p. 271 l. 10.

18.3 Brauer-Wall-Porteous groups

As before, we consider only finite-dimensional associative algebras.

Central simple algebras over $\mathbb{R}$ are isomorphic to the real matrix algebras $\mathrm{Mat}(d,\mathbb{R})$ and $\mathrm{Mat}(d,\mathbb{H})$. A tensor product of two matrix algebras with entries in $\mathbb{H}$ is a matrix algebra with entries in $\mathbb{R}$. This can be expressed by saying that the Brauer group $Br(\mathbb{R})$ of $\mathbb{R}$ is a two-element group $\{\mathbb{R},\mathbb{H}\}$.

Tensor products of graded central simple algebras over $\mathbb{R}$ lead to the Brauer-Wall group $BW(\mathbb{R})$ of $\mathbb{R}$; this is a cyclic group of eight elements,

$$\left\{\frac{\mathbb{R}(2\nu)}{{}^2\mathbb{R}(\nu)}, \frac{{}^2\mathbb{R}(\nu)}{\mathbb{R}(\nu)}, \frac{\mathbb{R}(2\nu)}{\mathbb{C}(\nu)}, \frac{\mathbb{C}(2\nu)}{\mathbb{H}(\nu)}, \frac{\mathbb{H}(2\nu)}{{}^2\mathbb{H}(\nu)}, \frac{{}^2\mathbb{H}(\nu)}{\mathbb{H}(\nu)}, \frac{\mathbb{H}(\nu)}{\mathbb{C}(\nu)}, \frac{\mathbb{C}(\nu)}{\mathbb{R}(\nu)}\right\}.$$

Here we use the abbreviation $\mathbb{A}(\nu) = \mathrm{Mat}(\nu,\mathbb{A})$; the notation

$$\frac{A}{B}$$

means that B is the even subalgebra of A. The elements of $BW(\mathbb{R})$ can be represented by the graded algebras

$$\frac{\mathcal{C}\ell_{0,n}}{\mathcal{C}\ell^+_{0,n}}$$

where n is taken modulo 8. This is just another way of expressing Cartan's periodicity of 8.

Graded algebras are algebras with an involution (= involutory automorphism). We could further consider tensor products in graded central simple algebras with an anti-involution (= involutory anti-automorphism). When the involution and the anti-involution commute, this leads to the Brauer-Wall-Porteous group $BWP(\mathbb{R})$ of $\mathbb{R}$; its elements are graded subgroups (of a graded algebra A)

$$\frac{G}{H}$$

where G is the subgroup determinded by the anti-involution β, $G = \{s \in A \mid \beta(s)s = 1\}$, and H is its even subgroup, $H = B \cap G$ (B is the even part of A).

Table 8. Scalar Product $s\tilde{\psi}\varphi$ in $\mathcal{C}\ell_{p,q}$ and $BWP(\mathbb{R})$.

	$p-q$	0	1	2	3	4	5	6	7
$q=0$	$p+q$								
$O(2\nu)$	0	$O(\nu,\nu)$		$O(\nu,\nu)$		$SO^*(2\nu)$		$SO^*(2\nu)$	
${}^2O(2\nu)$	1		${}^2O(\nu,\nu)$		$O(2\nu,\mathbb{C})$		${}^2SO^*(2\nu)$		$O(2\nu,\mathbb{C})$
$O(2\nu)$	2	$O(\nu,\nu)$		$O(\nu,\nu)$		$SO^*(2\nu)$		$SO^*(2\nu)$	
$U(2\nu)$	3		$GL(2\nu,\mathbb{R})$		$U(\nu,\nu)$		$GL(\nu,\mathbb{H})$		$U(\nu,\nu)$
$Sp(2\nu)$	4	$Sp(2\nu,\mathbb{R})$		$Sp(2\nu,\mathbb{R})$		$Sp(\nu,\nu)$		$Sp(\nu,\nu)$	
${}^2Sp(2\nu)$	5		${}^2Sp(2\nu,\mathbb{R})$		$Sp(2\nu,\mathbb{C})$		${}^2Sp(\nu,\nu)$		$Sp(2\nu,\mathbb{C})$
$Sp(2\nu)$	6	$Sp(2\nu,\mathbb{R})$		$Sp(2\nu,\mathbb{R})$		$Sp(\nu,\nu)$		$Sp(\nu,\nu)$	
$U(2\nu)$	7		$GL(2\nu,\mathbb{R})$		$U(\nu,\nu)$		$GL(\nu,\mathbb{H})$		$U(\nu,\nu)$

Table 9. Scalar Product $s\bar{\psi}\varphi$ in $\mathcal{C}\ell_{p,q}$ and $BWP(\mathbb{R})$.

	$p-q$	0	1	2	3	4	5	6	7
$p=0$	$p+q$								
$O(2\nu)$	0	$O(\nu,\nu)$		$O(\nu,\nu)$		$SO^*(2\nu)$		$SO^*(2\nu)$	
$U(2\nu)$	1		$GL(2\nu,\mathbb{R})$		$U(\nu,\nu)$		$GL(\nu,\mathbb{H})$		$U(\nu,\nu)$
$Sp(2\nu)$	2	$Sp(2\nu,\mathbb{R})$		$Sp(2\nu,\mathbb{R})$		$Sp(\nu,\nu)$		$Sp(\nu,\nu)$	
${}^2Sp(2\nu)$	3		${}^2Sp(2\nu,\mathbb{R})$		$Sp(2\nu,\mathbb{C})$		${}^2Sp(\nu,\nu)$		$Sp(2\nu,\mathbb{C})$
$Sp(2\nu)$	4	$Sp(2\nu,\mathbb{R})$		$Sp(2\nu,\mathbb{R})$		$Sp(\nu,\nu)$		$Sp(\nu,\nu)$	
$U(2\nu)$	5		$GL(2\nu,\mathbb{R})$		$U(\nu,\nu)$		$GL(\nu,\mathbb{H})$		$U(\nu,\nu)$
$O(2\nu)$	6	$O(\nu,\nu)$		$O(\nu,\nu)$		$SO^*(2\nu)$		$SO^*(2\nu)$	
${}^2O(2\nu)$	7		${}^2O(\nu,\nu)$		$O(2\nu,\mathbb{C})$		${}^2SO^*(2\nu)$		$O(2\nu,\mathbb{C})$

The Brauer-Wall-Porteous group $BWP(\mathbb{R})$ is a commutative group of 32 elements,

$$BWP(\mathbb{R}) \simeq \{(x,y) \in \mathbb{Z}_8 \times \mathbb{Z}_8 \mid x, y \in 2\mathbb{Z}\}.$$

We see that the elements of $BWP(\mathbb{R})$ are (graded) automorphism groups of scalar products of spinors for $\mathcal{C}\ell_{p,q}$,

$$\frac{\{s \in \mathcal{C}\ell_{p,q} \mid s\beta(s) = 1\}}{\{s \in \mathcal{C}\ell^+_{p,q} \mid s\beta(s) = 1\}}.$$

The even subgroup $\{s \in \mathcal{C}\ell^+_{p,q} \mid s\beta(s) = 1\}$ is isomorphic to $\{s \in \mathcal{C}\ell_{p,q-1} \mid s\beta(s) = 1\}$, obtained by taking a step to the North-East. Tensor products of real graded central simple algebras with an anti-involution correspond to movements of a bishop on the chessboard.

Recall that the Brauer group $Br(\mathbb{C})$ of $\mathbb{C}$ is a one-element group $\{\mathbb{C}\}$. The Brauer-Wall group $BW(\mathbb{C})$ of $\mathbb{C}$ is a group of two elements

$$\left\{\frac{\mathrm{Mat}(2,\mathbb{C})}{^2\mathbb{C}}, \frac{^2\mathbb{C}}{\mathbb{C}}\right\}.$$

Thus, complex Clifford algebras have a periodicity of 2. The Brauer-Wall-Porteous group $BWP(\mathbb{C})$ of $\mathbb{C}$ is a cyclic group of eight elements; in other words complex Clifford algebras with an anti-involution have a periodicity of 8, see Table 10.

Table 10. $\mathcal{C}\ell(\mathbb{C}^n)$ and $BWP(\mathbb{C})$.

n	$s\tilde{\psi}\varphi$	n	$s\bar{\psi}\varphi$
0	$O(2\nu,\mathbb{C})$	0	$O(2\nu,\mathbb{C})$
1	$^2O(2\nu,\mathbb{C})$	1	$GL(2\nu,\mathbb{C})$
2	$O(2\nu,\mathbb{C})$	2	$Sp(2\nu,\mathbb{C})$
3	$GL(2\nu,\mathbb{C})$	3	$^2Sp(2\nu,\mathbb{C})$
4	$Sp(2\nu,\mathbb{C})$	4	$Sp(2\nu,\mathbb{C})$
5	$^2Sp(2\nu,\mathbb{C})$	5	$GL(2\nu,\mathbb{C})$
6	$Sp(2\nu,\mathbb{C})$	6	$O(2\nu,\mathbb{C})$
7	$GL(2\nu,\mathbb{C})$	7	$^2O(2\nu,\mathbb{C})$

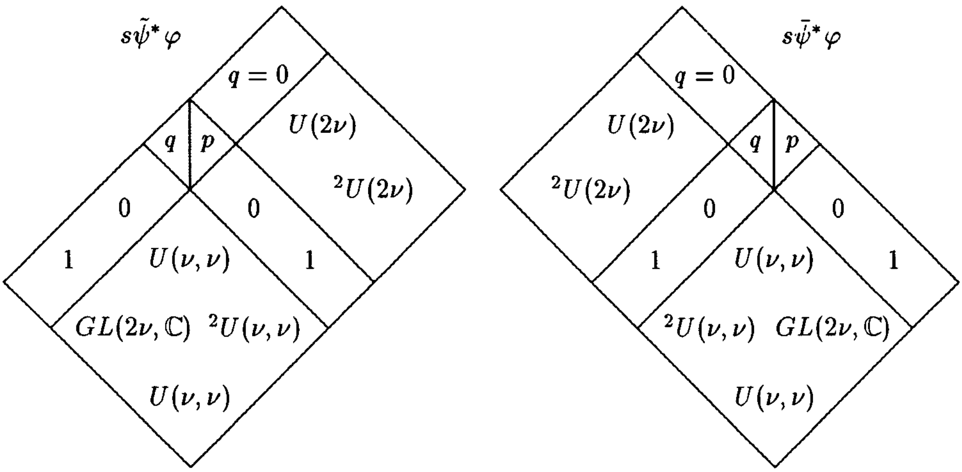
The Brauer-Wall-Porteous group $BWP(^2\mathbb{R})$ of the double ring $^2\mathbb{R}$ with swap is also a cyclic group of eight elements, see Table 11.

Table 11. $^2\mathbb{R} \otimes \mathcal{C}\ell_{p,q}$ and $BWP(^2\mathbb{R})$.

$p-q$	0	1	2	3	4	5	6	7
	$GL(2\nu,\mathbb{R})$	$^2GL(2\nu,\mathbb{R})$	$GL(2\nu,\mathbb{R})$	$GL(2\nu,\mathbb{C})$	$GL(\nu,\mathbb{H})$	$^2GL(\nu,\mathbb{H})$	$GL(\nu,\mathbb{H})$	$GL(2\nu,\mathbb{C})$

Tensoring $\mathcal{C}\ell_{p,q}$ by $\mathbb{C}^*$, the complex field with complex conjugation, results in a Brauer-Wall-Porteous group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, see Table 12.

Table 12. $\mathbb{C}^* \otimes \mathcal{C}\ell_{p,q}$ and $BWP(\mathbb{C}^*)$.



As our last extension we tensor $\mathcal{C}\ell(\mathbb{C}^n)$ by ${}^2\mathbb{C}$ (Table 13).

Table 13. ${}^2\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}^n$ and $BWP({}^2\mathbb{C})$.

n	$s\tilde{\psi}\varphi$ and	$\bar{\psi}\varphi$
0	$GL(2\nu,\mathbb{C})$	
1		${}^2GL(2\nu,\mathbb{C})$

In total, we have the following Brauer-Wall-Porteous groups (of $\mathbb{R}$ and $\mathbb{C}$ and their extensions with an irreducible involution).

$$\begin{array}{ll}
\mathbb{R}^{p,q} & BWP(\mathbb{R}) \simeq (\mathbb{Z}_8 \times \mathbb{Z}_8)/\mathbb{Z}_2 \\
{}^2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}^{p,q} & BWP({}^2\mathbb{R}) \simeq \mathbb{Z}_8 \\
\mathbb{C}^* \otimes \mathbb{R}^{p,q} & BWP(\mathbb{C}^*) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \\
\mathbb{C}^n & BWP(\mathbb{C}) \simeq \mathbb{Z}_8 \\
{}^2\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}^n & BWP({}^2\mathbb{C}) \simeq \mathbb{Z}_2.
\end{array}$$

It is convenient to be able to characterize the automorphism groups of scalar products on spinor spaces S directly by making use of real dimensions of the subspaces $P_\pm = \{\psi \in S \mid \beta(\psi) = \pm\psi\}$, see Table 14.

Table 14. Scalar products on S.

$\dim P_+$	0	1	2	3
$\dim P_-$				
0	$GL(\nu,\mathbb{R})$	$O(\nu,\nu)$	$O(2\nu,\mathbb{C})$	
1	$Sp(2\nu,\mathbb{R})$	$U(\nu,\nu)$		$SO^*(4\nu)$
2	$Sp(2\nu,\mathbb{C})$			
3		$Sp(2\nu,2\nu)$		

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