3

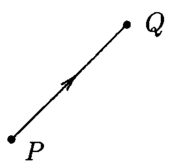
Bivectors and the Exterior Algebra

There are other kinds of directed quantities besides vectors, most notably bivectors. For instance, a moment of a force, angular velocity of a rotating body, and magnetic induction can be described with bivectors. In three dimensions bivectors are dual to vectors, and their use can be circumvented. Scalars, vectors, bivectors and the volume element span the exterior algebra $\bigwedge\mathbb{R}^3$, which provides a multivector structure for the Clifford algebra $\mathcal{C}\ell_3$ of the Euclidean space $\mathbb{R}^3$.

3.1 Bivectors as directed plane segments

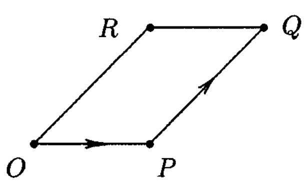
In three dimensions bivectors are oriented plane segments, which have a direction and a magnitude, the area of the plane segment. Two bivectors have the same direction if they are on parallel planes (the same attitude) and are similarly oriented (the same sense of rotation).

Vector (directed line segment)



1. magnitude (length of PQ)
2. direction
   - attitude (line PQ)
   - orientation (toward the point Q)

Bivector (directed plane segment)



1. magnitude (area of $OPQR$)
2. direction
   - attitude (plane OPQ)
   - orientation (sense of rotation)

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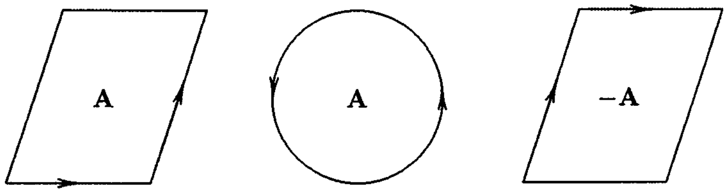
Bivectors are denoted by boldface capital letters $\mathbf{A}, \mathbf{B}$, etc. [1] The area or norm of a bivector $\mathbf{A}$ is denoted by $|\mathbf{A}|$. Two bivectors $\mathbf{A}$ and $\mathbf{B}$ in parallel planes have the same attitude, and we write $\mathbf{A} \parallel \mathbf{B}$. Parallel bivectors $\mathbf{A}$ and $\mathbf{B}$ can be regarded as directed angles turning either the same way, $\mathbf{A} \uparrow\uparrow \mathbf{B}$, or the opposite way, $\mathbf{A} \uparrow\downarrow \mathbf{B}$. If two plane segments have the same area and the same direction (= parallel planes with the same sense of rotation), then the bivectors are equal:

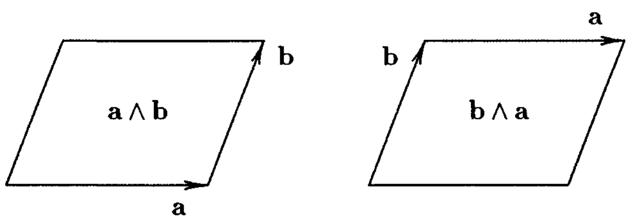
$$\mathbf{A} = \mathbf{B} \quad \Longleftrightarrow \quad |\mathbf{A}| = |\mathbf{B}| \quad \text{and} \quad \mathbf{A} \uparrow\uparrow \mathbf{B}$$



A bivector $\mathbf{A}$ and its *opposite* $-\mathbf{A}$ are of equal area and parallel, but have opposite orientations. A *unit bivector* $\mathbf{A}$ has area one, $|\mathbf{A}| = 1$.

The shape of the area is irrelevant.

Representing a bivector as an oriented parallelogram suggests that a bivector can be thought of as a geometrical product of vectors along its sides. With this in mind we introduce the *exterior product* $\mathbf{a} \wedge \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ as the bivector obtained by sweeping $\mathbf{b}$ along $\mathbf{a}$.



The bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$ have the same area and the same attitude but opposite senses of rotations. This can be simply expressed by writing

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.$$

3.2 Addition of bivectors

The geometric interpretation of bivector addition is most easily seen when the bivectors are expressed in terms of the exterior product with a common

1 In handwriting, bivectors can be distinguished by an angle on top of the letter, *A*, *B* .

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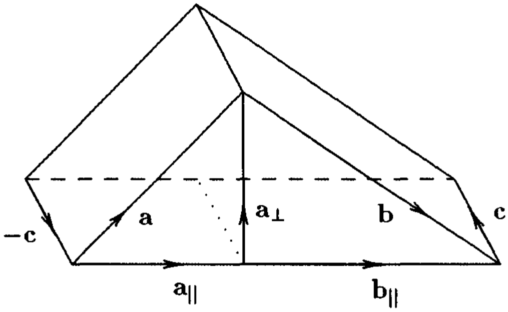
vector factor. In three dimensions this is always possible because any two planes will either be parallel or intersect along a common line. [2] Thus let $\mathbf{A} = \mathbf{a} \wedge \mathbf{c}$ and $\mathbf{B} = \mathbf{b} \wedge \mathbf{c}$; then the bivector $\mathbf{A} + \mathbf{B}$ is defined so that $\mathbf{A} + \mathbf{B} = \mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} + \mathbf{b}) \wedge \mathbf{c}$. The geometric significance of this can be depicted as follows:



By decomposing the vectors $\mathbf{a}$ and $\mathbf{b}$ into components parallel and perpendicular to $\mathbf{a} + \mathbf{b}$, [3] so that

$$\mathbf{a} = \mathbf{a}_{||} + \mathbf{a}_{\perp} \quad \text{and} \quad \mathbf{b} = \mathbf{b}_{||} + \mathbf{b}_{\perp}$$

where $\mathbf{b}_{\perp} = -\mathbf{a}_{\perp}$, we are able to reduce the general addition of bivectors in three dimensions to the addition of coplanar bivectors. This is evident in the equality

$$\mathbf{a} \wedge \mathbf{c} + \mathbf{b} \wedge \mathbf{c} = (\mathbf{a} + \mathbf{b}) \wedge \mathbf{c} = (\mathbf{a}_{||} + \mathbf{b}_{||}) \wedge \mathbf{c} = \mathbf{a}_{||} \wedge \mathbf{c} + \mathbf{b}_{||} \wedge \mathbf{c}.$$

3.3 Basis of the linear space of bivectors

Bivectors can be added and multiplied by scalars. This way the set of bivectors becomes a linear space, denoted by $\bigwedge^2 \mathbb{R}^3$. A basis for the linear space $\bigwedge^2 \mathbb{R}^3$ can be constructed by means of a basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of the linear space $\mathbb{R}^3$. The oriented plane segments of the coordinate planes, obtained by taking the exterior products

$$\mathbf{e}_1 \wedge \mathbf{e}_2, \ \mathbf{e}_1 \wedge \mathbf{e}_3, \ \mathbf{e}_2 \wedge \mathbf{e}_3,$$

2 The two bivectors are first translated in the affine space $\mathbb{R}^3$ so that they induce opposite orientations to their common edge, that is, the terminal side of $\mathbf{A} = \mathbf{a} \wedge \mathbf{c}$ is opposite to the initial side of $\mathbf{B} = (-\mathbf{c}) \wedge \mathbf{b}$.

3 A depiction of addition of bivectors does not require a metric, or perpendicular components. It is sufficient that one component of both $\mathbf{a}$ and $\mathbf{b}$ is parallel to $\mathbf{a} + \mathbf{b}$, so that the two components sum up to $\mathbf{a} + \mathbf{b}$, while the other component can be any non-parallel component.

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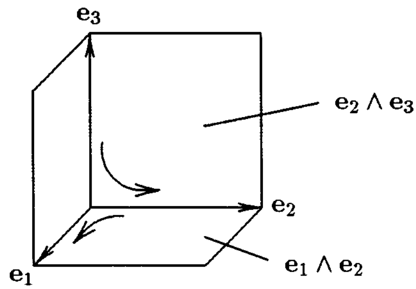
form a basis for the linear space of bivectors $\bigwedge^2 \mathbb{R}^3$.



An arbitrary bivector is a linear combination of the basis elements,

$$\mathbf{B} = B_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + B_{13}\mathbf{e}_1 \wedge \mathbf{e}_3 + B_{23}\mathbf{e}_2 \wedge \mathbf{e}_3,$$

and such linear combinations form the space of bivectors $\bigwedge^2 \mathbb{R}^3$. [4] The construction of bivectors calls only for a linear structure, and no metric is needed.

The scalar product on a Euclidean space $\mathbb{R}^3$ extends to a symmetric bilinear product on the space of bivectors $\bigwedge^2 \mathbb{R}^3$,

$$<\mathbf{x}_1 \wedge \mathbf{x}_2, \mathbf{y}_1 \wedge \mathbf{y}_2> = \begin{vmatrix} \mathbf{x}_1 \cdot \mathbf{y}_1 & \mathbf{x}_1 \cdot \mathbf{y}_2 \\ \mathbf{x}_2 \cdot \mathbf{y}_1 & \mathbf{x}_2 \cdot \mathbf{y}_2 \end{vmatrix}.$$

In particular, $<\mathbf{a} \wedge \mathbf{b}, \mathbf{a} \wedge \mathbf{b}> = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$. The norm or area of $\mathbf{B} = B_{12}\mathbf{e}_1 \wedge \mathbf{e}_2 + B_{13}\mathbf{e}_1 \wedge \mathbf{e}_3 + B_{23}\mathbf{e}_2 \wedge \mathbf{e}_3$ is seen to be

$$|\mathbf{B}| = \sqrt{<\mathbf{B}, \mathbf{B}>} = \sqrt{B_{12}^2 + B_{13}^2 + B_{23}^2}.$$

3.4 The oriented volume element

The exterior product $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ of three vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ and $\mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ represents the oriented volume of the parallelepiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3.$$

It is an element of the 1-dimensional linear space of 3-vectors $\bigwedge^3 \mathbb{R}^3$ with basis $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$. The exterior product is associative,

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}),$$

4 In three dimensions all bivectors are simple, that is, they are exterior products of two vectors, $\mathbf{B} = \mathbf{x} \wedge \mathbf{y}$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. This is no longer true in four dimensions; for instance $\mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_3 \wedge \mathbf{e}_4$ is not simple.

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and antisymmetric,

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}&=\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{a}=\mathbf{c}\wedge\mathbf{a}\wedge\mathbf{b}\\&=-\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a}=-\mathbf{a}\wedge\mathbf{c}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}\wedge\mathbf{c}\end{aligned}$$

for $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^3$.

The exterior product of the orthogonal unit vectors $\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\in\mathbb{R}^3$ is the unit oriented volume element $\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3\in\bigwedge^3\mathbb{R}^3$. The norm or volume $|\mathbf{V}|$ of a 3-vector [5]

$$\mathbf{V}=V\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$$

is $|\mathbf{V}|=|V|$, that is, $|V\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3|=V$ for $V\geq 0$ and $|V\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3|=-V$ for $V<0$.

More formally, the scalar product on $\mathbb{R}^3$ extends to a symmetric bilinear product on $\bigwedge^3\mathbb{R}^3$ by

$$<\mathbf{x}_1\wedge\mathbf{x}_2\wedge\mathbf{x}_3,\mathbf{y}_1\wedge\mathbf{y}_2\wedge\mathbf{y}_3>=\begin{vmatrix}\mathbf{x}_1\cdot\mathbf{y}_1&\mathbf{x}_1\cdot\mathbf{y}_2&\mathbf{x}_1\cdot\mathbf{y}_3\\\mathbf{x}_2\cdot\mathbf{y}_1&\mathbf{x}_2\cdot\mathbf{y}_2&\mathbf{x}_2\cdot\mathbf{y}_3\\\mathbf{x}_3\cdot\mathbf{y}_1&\mathbf{x}_3\cdot\mathbf{y}_2&\mathbf{x}_3\cdot\mathbf{y}_3\end{vmatrix}$$

giving the norm as $|\mathbf{V}|=\sqrt{<\mathbf{V},\mathbf{V}>}$.

3.5 The cross product

Let $\mathbf{a}=a_1\mathbf{e}_1+a_2\mathbf{e}_2+a_3\mathbf{e}_3$ and $\mathbf{b}=b_1\mathbf{e}_1+b_2\mathbf{e}_2+b_3\mathbf{e}_3$. The bivector

$$\mathbf{a}\wedge\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{e}_2\wedge\mathbf{e}_3+(a_3b_1-a_1b_3)\mathbf{e}_3\wedge\mathbf{e}_1+(a_1b_2-a_2b_1)\mathbf{e}_1\wedge\mathbf{e}_2$$

can be expressed as a 'determinant'

$$\mathbf{a}\wedge\mathbf{b}=\begin{vmatrix}\mathbf{e}_2\wedge\mathbf{e}_3&\mathbf{e}_3\wedge\mathbf{e}_1&\mathbf{e}_1\wedge\mathbf{e}_2\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}.$$

It is customary to introduce a vector with the same coordinates. Thus, we define the *cross product*

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2)\mathbf{e}_1+(a_3b_1-a_1b_3)\mathbf{e}_2+(a_1b_2-a_2b_1)\mathbf{e}_3$$

of $\mathbf{a}$ and $\mathbf{b}$. The cross product can also be represented by a 'determinant'

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{e}_1&\mathbf{e}_2&\mathbf{e}_3\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}.$$

5 V is a real number, positive or negative, while $\mathbf{V}$ is a 3-vector. The usual volume is $|V|$.

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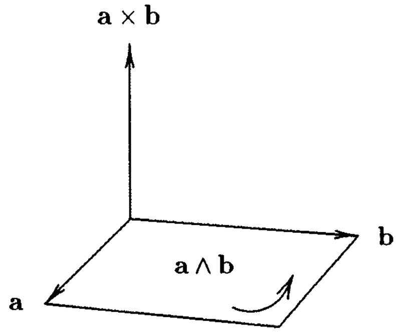

The direction of $\mathbf{a}\times\mathbf{b}$ is perpendicular to the plane of $\mathbf{a}\wedge\mathbf{b}$ and the length/norm of $\mathbf{a}\times\mathbf{b}$ equals the area/norm of $\mathbf{a}\wedge\mathbf{b}$,

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$$

where φ, $0 \le \varphi \le 180°$, is the angle between $\mathbf{a}$ and $\mathbf{b}$.

In spite of the resemblance between the determinant expressions for the exterior product $\mathbf{a}\wedge\mathbf{b}$ and the cross product $\mathbf{a}\times\mathbf{b}$ there is a difference: the exterior product does not require a metric while the cross product requires or induces a metric. The metric gets involved in positioning the vector $\mathbf{a}\times\mathbf{b}$ perpendicular to the bivector $\mathbf{a}\wedge\mathbf{b}$.

3.6 The Hodge dual

Since the vector space $\mathbb{R}^3$ and the bivector space $\bigwedge^2\mathbb{R}^3$ are both of dimension 3, they are linearly isomorphic. We can use the metric on the vector space $\mathbb{R}^3$ to set up a standard isomorphism between the two linear spaces, the Hodge dual sending a vector $\mathbf{a}\in\mathbb{R}^3$ to a bivector $\star\mathbf{a}\in\bigwedge^2\mathbb{R}^3$, defined by

$$\mathbf{b}\wedge\star\mathbf{a} = (\mathbf{b}\cdot\mathbf{a})\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3 \quad \text{for all} \quad \mathbf{b}\in\mathbb{R}^3.$$

The Hodge dual depends not only on the metric but also on the choice of orientation – it is customary to use a right-handed and orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

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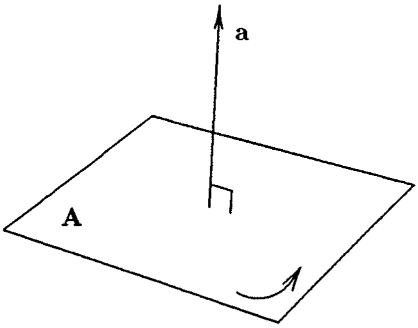

Vector $\mathbf{a}$ and its dual bivector $\mathbf{A} = \mathbf{a}\mathbf{e}_{123}$

Thus, we have assigned to each vector

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathbb{R}^3$$

a bivector

$$\mathbf{A} = \star\mathbf{a} = a_1\mathbf{e}_2 \wedge \mathbf{e}_3 + a_2\mathbf{e}_3 \wedge \mathbf{e}_1 + a_3\mathbf{e}_1 \wedge \mathbf{e}_2 \in \bigwedge^2 \mathbb{R}^3.$$

Using the induced metric on the bivector space $\bigwedge^2 \mathbb{R}^3$ we can extend the Hodge dual to a mapping sending a bivector $\mathbf{A} \in \bigwedge^2 \mathbb{R}^3$ to a vector $\star\mathbf{A} \in \mathbb{R}^3$, defined by

$$\mathbf{B} \wedge \star\mathbf{A} = <\mathbf{B}, \mathbf{A}>\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \quad \text{for all} \quad \mathbf{B} \in \bigwedge^2 \mathbb{R}^3.$$

Using duality, the relation between the cross product and the exterior product can be written as [6]

$$\mathbf{a} \wedge \mathbf{b} = \star(\mathbf{a} \times \mathbf{b}),$$
$$\mathbf{a} \times \mathbf{b} = \star(\mathbf{a} \wedge \mathbf{b}).$$

6 In terms of the Clifford algebra $\mathcal{C}\ell_3$ the relation between the exterior product and the cross product can be written as

$$\mathbf{a} \wedge \mathbf{b} = (\mathbf{a} \times \mathbf{b})\mathbf{e}_{123},$$
$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b})\mathbf{e}_{123}.$$

The metric gets involved in multiplying by $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. Using the Clifford algebra $\mathcal{C}\ell_3$ the Hodge dual can be computed as $\star u = \tilde{u}\mathbf{e}_{123}$. This gives rise to the *Clifford dual* defined as $u\mathbf{e}_{123}$ for $u \in \mathcal{C}\ell_3$. Later we will see that in actual computations the Clifford dual is more convenient than the Hodge dual (although in three dimensions the Hodge dual happens to be symmetric/involutory).

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3.7 The exterior algebra and the Clifford algebra

The exterior algebra $\bigwedge\mathbb{R}^3$ of the linear space $\mathbb{R}^3$ is a direct sum of the

subspaces of		with basis
scalars	$\mathbb{R}$	1
vectors	$\mathbb{R}^3$	$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$
bivectors	$\bigwedge^2\mathbb{R}^3$	$\mathbf{e}_1\wedge\mathbf{e}_2,\ \mathbf{e}_1\wedge\mathbf{e}_3,\ \mathbf{e}_2\wedge\mathbf{e}_3$
volume elements	$\bigwedge^3\mathbb{R}^3$	$\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3$

We also write $\mathbb{R}=\bigwedge^0\mathbb{R}^3$ and $\mathbb{R}^3=\bigwedge^1\mathbb{R}^3$. Thus, $\bigwedge\mathbb{R}^3$ is a direct sum of its subspaces of homogeneous degrees $0,1,2,3$:

$$\bigwedge\mathbb{R}^3=\mathbb{R}\oplus\mathbb{R}^3\oplus\bigwedge^2\mathbb{R}^3\oplus\bigwedge^3\mathbb{R}^3.$$

The dimensions of $\mathbb{R}$, $\mathbb{R}^3$, $\bigwedge^2\mathbb{R}^3$, $\bigwedge^3\mathbb{R}^3$ and $\bigwedge\mathbb{R}^3$ are $1,3,3,1$ and $2^3=8$, respectively.

The exterior algebra $\bigwedge\mathbb{R}^3$ is an associative algebra with unity 1 satisfying

$$\boxed{\begin{array}{ll}\mathbf{e}_i\wedge\mathbf{e}_j=-\mathbf{e}_j\wedge\mathbf{e}_i & \text{for}\quad i\neq j\\ \mathbf{e}_i\wedge\mathbf{e}_i=0 & \end{array}}$$

for a basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ of the linear space $\mathbb{R}^3$. The exterior product of two homogeneous elements satisfies

$$\mathbf{a}\wedge\mathbf{b}\in\bigwedge^{i+j}\mathbb{R}^3\quad\text{for}\quad\mathbf{a}\in\bigwedge^{i}\mathbb{R}^3,\ \mathbf{b}\in\bigwedge^{j}\mathbb{R}^3.$$

The product of two elements u and v in the Clifford algebra $\mathcal{C}\ell_3$ of the Euclidean space $\mathbb{R}^3$ is denoted by juxtaposition, uv, to distinguish it from the exterior product $u\wedge v$. An orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ of the Euclidean space $\mathbb{R}^3\subset\mathcal{C}\ell_3$ satisfies [7]

$$\boxed{\begin{array}{ll}\mathbf{e}_i\mathbf{e}_j=-\mathbf{e}_j\mathbf{e}_i & \text{for}\quad i\neq j\\ \mathbf{e}_i\mathbf{e}_i=1 & \end{array}}$$

7 These rules were invented by W.K. Clifford in 1882. In an earlier paper Clifford 1878 had considered an associative algebra of dimension 8 with the rules $\mathbf{e}_i\mathbf{e}_i=-1$ for $i=1,2,3$.

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and generates a basis of $\mathcal{C}\ell_3$, corresponding to a basis of $\bigwedge \mathbb{R}^3$,

$\mathcal{C}\ell_3$	$\bigwedge \mathbb{R}^3$
1	1
$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$	$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$
$\mathbf{e}_1\mathbf{e}_2,\ \mathbf{e}_1\mathbf{e}_3,\ \mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_1 \wedge \mathbf{e}_2,\ \mathbf{e}_1 \wedge \mathbf{e}_3,\ \mathbf{e}_2 \wedge \mathbf{e}_3$
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$

The above correspondences induce an identification of the linear spaces $\mathcal{C}\ell_3$ and $\bigwedge \mathbb{R}^3$, and we shall write

$$\mathcal{C}\ell_3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3.$$

This decomposition introduces a *multivector structure* into the Clifford algebra $\mathcal{C}\ell_3$. The multivector structure is unique, that is, an arbitrary element $u \in \mathcal{C}\ell_3$ can be uniquely decomposed into a sum of k-vectors, the k-vector parts $\langle u \rangle_k$ of u,

$$u = \langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 + \langle u \rangle_3 \quad \text{where} \quad \langle u \rangle_k \in \bigwedge^k \mathbb{R}^3.$$

3.8 The Clifford product of two vectors

A new kind of product called the *Clifford product* of vectors $\mathbf{a}$ and $\mathbf{b}$ is obtained by adding the scalar $\mathbf{a} \cdot \mathbf{b}$ and the bivector $\mathbf{a} \wedge \mathbf{b}$:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}.$$

The commutative rule $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ together with the anticommutative rule $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$ implies a relation between $\mathbf{ab}$ and $\mathbf{ba}$. Thus,

$$\mathbf{ba} = \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \wedge \mathbf{b}.$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a} \parallel \mathbf{b}$, when their product is commutative, $\mathbf{ab} = \mathbf{ba}$, and perpendicular, $\mathbf{a} \perp \mathbf{b}$, when their product is anticommutative, $\mathbf{ab} = -\mathbf{ba}$.

Note that if $\mathbf{a}$ is decomposed into components parallel, $\mathbf{a}_{\parallel}$, and perpendicular, $\mathbf{a}_{\perp}$, to $\mathbf{b}$, then $\mathbf{ab} = \mathbf{a}_{\parallel}\mathbf{b} + \mathbf{a}_{\perp}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$.

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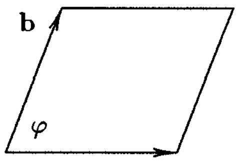
$$\mathbf{a}\cdot\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}+\mathbf{ba})$$
$$\mathbf{a}\wedge\mathbf{b} = \tfrac{1}{2}(\mathbf{ab}-\mathbf{ba})$$
$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\varphi$$
$$|\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$$



Compute the product $\mathbf{abba}$ to get $\mathbf{a}^2\mathbf{b}^2 = (\mathbf{a}\cdot\mathbf{b})^2 - (\mathbf{a}\wedge\mathbf{b})^2$ and use $(\mathbf{a}\wedge\mathbf{b})^2 = -|\mathbf{a}\wedge\mathbf{b}|^2$ to obtain the identity

$$\mathbf{a}^2\mathbf{b}^2 = (\mathbf{a}\cdot\mathbf{b})^2 + |\mathbf{a}\wedge\mathbf{b}|^2.$$

3.9 Even and odd parts

The Clifford algebra is, like the exterior algebra, a direct sum of two of its subspaces,

$$\begin{array}{ll} \text{the even part} & \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3, \\ \text{the odd part} & \mathbb{R}^3\oplus\bigwedge^3\mathbb{R}^3. \end{array}$$

For both algebras the even part is also a subalgebra. The even subalgebra $(\bigwedge\mathbb{R}^3)^+ = \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3$ of $\bigwedge\mathbb{R}^3$ is commutative, but the even subalgebra $\mathcal{C}\ell_3^+ = \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3$ of $\mathcal{C}\ell_3$ is not commutative; instead it is isomorphic to the quaternion algebra: $\mathbb{H}\simeq\mathcal{C}\ell_3^+$. The odd parts are denoted by $\mathcal{C}\ell_3^-$ and $(\bigwedge\mathbb{R}^3)^-$.

3.10 The center

The center of an algebra consists of those elements which commute with all the elements of the algebra. The center $\mathrm{Cen}(\mathcal{C}\ell_3) = \mathbb{R}\oplus\bigwedge^3\mathbb{R}^3$ of $\mathcal{C}\ell_3$ is isomorphic to $\mathbb{C}$, and the center of $\bigwedge\mathbb{R}^3$ is $\mathrm{Cen}(\bigwedge\mathbb{R}^3) = \mathbb{R}\oplus\bigwedge^2\mathbb{R}^3\oplus\bigwedge^3\mathbb{R}^3$.

3.11 Gradings and the multivector structure

The exterior products of homogeneous elements satisfy the relations

$$\mathbf{a}\wedge\mathbf{b}\in\bigwedge^{i+j}\mathbb{R}^3 \quad\text{for}\quad \mathbf{a}\in\bigwedge^{i}\mathbb{R}^3 \quad\text{and}\quad \mathbf{b}\in\bigwedge^{j}\mathbb{R}^3.$$

Such a property of an algebra is usually referred to by saying that the algebra is graded over the index group $\mathbb{Z}$. We shall refer to this property of the exterior algebra $\bigwedge\mathbb{R}^3$ as the *dimension grading*, because simple homogeneous elements

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represent subspaces of specified dimension. The homogeneous elements in $\bigwedge \mathbb{R}^3$ satisfy

$$\mathbf{a} \wedge \mathbf{b} = (-1)^{ij} \mathbf{b} \wedge \mathbf{a} \quad \text{for} \quad \mathbf{a} \in \bigwedge^i \mathbb{R}^3, \ \mathbf{b} \in \bigwedge^j \mathbb{R}^3,$$

that is, the exterior algebra $\bigwedge \mathbb{R}^3$ is *graded commutative.* [8]

The Clifford products of even and odd subspaces satisfy the inclusion relations

$$\begin{aligned} \mathcal{C}\ell_3^+ \mathcal{C}\ell_3^+ &\subset \mathcal{C}\ell_3^+, \quad & \mathcal{C}\ell_3^+ \mathcal{C}\ell_3^- &\subset \mathcal{C}\ell_3^-, \\ \mathcal{C}\ell_3^- \mathcal{C}\ell_3^+ &\subset \mathcal{C}\ell_3^-, \quad & \mathcal{C}\ell_3^- \mathcal{C}\ell_3^- &\subset \mathcal{C}\ell_3^+. \end{aligned}$$

These relations can be summarized by saying that the Clifford algebra $\mathcal{C}\ell_3$ has an *even-odd grading*, or that it is graded over the index group $\mathbb{Z}_2 = \{0, 1\}$.

The exterior algebra $\bigwedge \mathbb{R}^3$ is also even-odd graded.

The Clifford algebra $\mathcal{C}\ell_3$ is not graded over $\mathbb{Z}$. However, we can reconstruct the exterior product from the Clifford product in a unique manner. We shall refer to the dimension grading of the associated exterior algebra by saying that the Clifford algebra has a *multivector structure.* Recall that $\mathbb{R}$ and $\mathbb{R}^3$ have, by definition, unique copies in $\mathcal{C}\ell_3$. The exterior product of two vectors equals the antisymmetric part of their Clifford product,

$$\mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx}) \in \bigwedge^2 \mathbb{R}^3 \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^3,$$

whence the space of bivectors $\bigwedge^2 \mathbb{R}^3$ has a unique copy in $\mathcal{C}\ell_3$. The subspace of 3-vectors $\bigwedge^3 \mathbb{R}^3$ can be uniquely reconstructed within $\mathcal{C}\ell_3$ by a completely antisymmetrized Clifford product

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{6}(\mathbf{xyz} + \mathbf{yzx} + \mathbf{zxy} - \mathbf{zyx} - \mathbf{xzy} - \mathbf{yxz}) \in \bigwedge^3 \mathbb{R}^3$$

of three vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$.

Thus, we have established a linear isomorphism sending $\bigwedge \mathbb{R}^3$ to $\mathcal{C}\ell_3$ defined

8 The graded opposite algebra of $\bigwedge \mathbb{R}^3$ is the linear space $\bigwedge \mathbb{R}^3$ with a new product $u \circ v$ defined by

$$(u_0 + u_1) \circ (v_0 + v_1) = v_0 \wedge u_0 + v_0 \wedge u_1 + v_1 \wedge u_0 - v_1 \wedge u_1$$

for $u_0, v_0 \in (\bigwedge \mathbb{R}^3)^+$ and $u_1, v_1 \in (\bigwedge \mathbb{R}^3)^-$. Since $\bigwedge \mathbb{R}^3$ is graded commutative, that is $u \circ v = u \wedge v$, the graded opposite of $\bigwedge \mathbb{R}^3$ is just $\bigwedge \mathbb{R}^3$.

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for k-vectors:

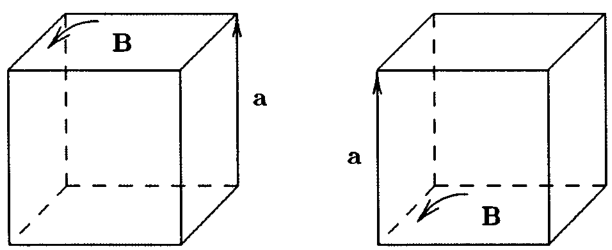
$\bigwedge\mathbb{R}^3$	$\mathcal{C}\ell_3$
α	$= \alpha \in \mathbb{R}$
$\mathbf{x}$	$= \mathbf{x} \in \mathbb{R}^3$
$\mathbf{x} \wedge \mathbf{y}$	$= \frac{1}{2}(\mathbf{xy} - \mathbf{yx}) \in \bigwedge^2 \mathbb{R}^3$
$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}$	$= \frac{1}{6}(\mathbf{xyz} + \mathbf{yzx} + \mathbf{zxy} - \mathbf{zyx} - \mathbf{xzy} - \mathbf{yxz}) \in \bigwedge^3 \mathbb{R}^3$

There is another construction of the subspace of 3-vectors $\bigwedge^3 \mathbb{R}^3$, obtained by using the reversion, $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{2}(\mathbf{xyz} - \mathbf{zyx}) \in \bigwedge^3 \mathbb{R}^3$ for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, related to the following recursive construction, via an intermediate step in $\bigwedge^2 \mathbb{R}^3$:

$$\mathbf{x} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{xB} + \mathbf{Bx}) \in \bigwedge^3 \mathbb{R}^3 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3,\ \mathbf{B} \in \bigwedge^2 \mathbb{R}^3.$$

3.12 Products of vectors and bivectors, visualization

A vector $\mathbf{a} \in \mathbb{R}^3$ and a bivector $\mathbf{B} \in \bigwedge \mathbb{R}^3$ can be multiplied to give a 3-vector $\mathbf{a} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{a} \in \bigwedge^3 \mathbb{R}^3$. The exterior product of a vector and a bivector can be depicted as an oriented volume:



$\mathbf{a} \wedge \mathbf{B}$ $\qquad$ $\mathbf{B} \wedge \mathbf{a}$

The orientation is obtained by putting the arrows in succession. The commutativity of the exterior product $\mathbf{a} \wedge \mathbf{B} = \mathbf{B} \wedge \mathbf{a}$ means that the screws of $\mathbf{a} \wedge \mathbf{B}$ and $\mathbf{B} \wedge \mathbf{a}$ can be rotated onto each other (without reflection).

A vector $\mathbf{x} \in \mathbb{R}^3$ and a bivector $\mathbf{B} \in \bigwedge^2 \mathbb{R}^3$ can also be multiplied so that the result is a vector $\mathbf{B} \llcorner \mathbf{x} \in \mathbb{R}^3$. Consider a vector $\mathbf{x}$ tilted by an angle φ out of the plane of a bivector $\mathbf{B}$. Let $\mathbf{a}$ be the orthogonal projection of $\mathbf{x}$ in the plane of $\mathbf{B}$. Then $|\mathbf{a}| = |\mathbf{x}| \cos\varphi$. The right *contraction* of the bivector $\mathbf{B}$ by the vector $\mathbf{x}$ is a vector $\mathbf{y} = \mathbf{B} \llcorner \mathbf{x}$ in the plane of $\mathbf{B}$ such that

(i) $|\mathbf{y}| = |\mathbf{B}||\mathbf{a}|$,

(ii) $\mathbf{y} \perp \mathbf{a} \quad$ and $\quad \mathbf{a} \wedge \mathbf{y} \uparrow\uparrow \mathbf{B}$.

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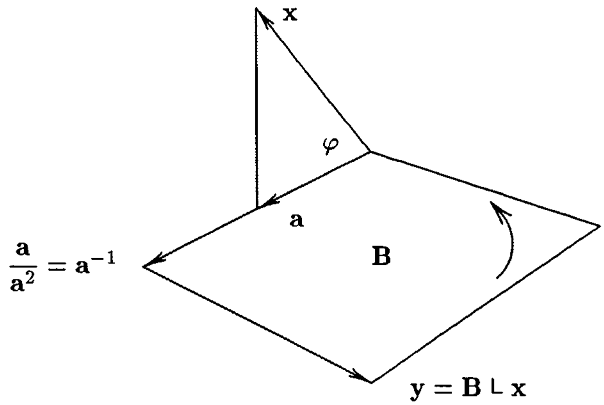
By convention, we agree that

$$\mathbf{x} \rfloor \mathbf{B} = -\mathbf{B} \lfloor \mathbf{x},$$

that is, the left and right contractions have opposite signs.



[The inverse vector $\mathbf{a}^{-1}$ of $\mathbf{a}$ has a geometrical meaning in this figure: it gives the area of the rectangle, $|\mathbf{B}| = |\mathbf{a}^{-1}||\mathbf{y}|$.]

Write $\mathbf{x}_{||} = \mathbf{a}$ and $\mathbf{x}_\perp = \mathbf{x} - \mathbf{x}_{||}$. Then $\mathbf{x} \rfloor \mathbf{B} = \mathbf{x}_{||}\mathbf{B}$ and $\mathbf{x} \wedge \mathbf{B} = \mathbf{x}_\perp \mathbf{B}$ so that

$\mathbf{x}_{\vert\vert} = (\mathbf{x} \rfloor \mathbf{B})\mathbf{B}^{-1}$	parallel component
$\mathbf{x}_\perp = (\mathbf{x} \wedge \mathbf{B})\mathbf{B}^{-1}$	perpendicular component

where $\mathbf{B}^{-1} = \mathbf{B}/\mathbf{B}^2$, $\mathbf{B}^2 = -|\mathbf{B}|^2$.

3.13 Contractions and the derivation

The Clifford product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is a sum of a scalar $\mathbf{a} \cdot \mathbf{b}$ and a bivector $\mathbf{a} \wedge \mathbf{b}$,

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b},$$

so that the terms on the right hand side can be recaptured from the Clifford product:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}), \qquad \mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}).$$

The product of a vector $\mathbf{a}$ and a bivector $\mathbf{B}$ is a sum of a vector and a 3-vector:

$$\mathbf{aB} = \mathbf{a} \rfloor \mathbf{B} + \mathbf{a} \wedge \mathbf{B}$$

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where

$$\mathbf{a} \lrcorner \mathbf{B} = \frac{1}{2}(\mathbf{aB} - \mathbf{Ba}), \qquad \mathbf{a} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{aB} + \mathbf{Ba}).$$

In general, the Clifford product of a vector $\mathbf{x} \in \mathbb{R}^3$ and an arbitrary element $u \in \mathcal{C}\ell_3$ can be decomposed into a sum of the left contraction and the exterior product as follows: [9]

$$\mathbf{x}u = \mathbf{x} \lrcorner u + \mathbf{x} \wedge u$$

where we can write, in the case where u is a k-vector in $\bigwedge^k \mathbb{R}^3$,

$$\mathbf{x} \lrcorner u = \tfrac{1}{2}(\mathbf{x}u - (-1)^k u\mathbf{x}) \in \textstyle\bigwedge^{k-1} \mathbb{R}^3,$$
$$\mathbf{x} \wedge u = \tfrac{1}{2}(\mathbf{x}u + (-1)^k u\mathbf{x}) \in \textstyle\bigwedge^{k+1} \mathbb{R}^3.$$

The exterior product and the left contraction by a homogeneous element, respectively, raise or lower the degree, that is,

$$\mathbf{a} \wedge \mathbf{b} \in \bigwedge^{i+j} \mathbb{R}^3, \qquad \mathbf{a} \lrcorner \mathbf{b} \in \bigwedge^{j-i} \mathbb{R}^3$$

for $\mathbf{a} \in \bigwedge^i \mathbb{R}^3$ and $\mathbf{b} \in \bigwedge^j \mathbb{R}^3$.

The left contraction can be obtained from the exterior product and the Clifford product as follows:

$$u \lrcorner v = [u \wedge (v\mathbf{e}_{123})]\mathbf{e}_{123}^{-1}.$$

This means that the left contraction is dual to the exterior product. The *left contraction* can be directly defined by its characteristic properties

1) $\mathbf{x} \lrcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y}$,
2) $\mathbf{x} \lrcorner (u \wedge v) = (\mathbf{x} \lrcorner u) \wedge v + \hat{u} \wedge (\mathbf{x} \lrcorner v)$,
3) $(u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w)$,

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $u, v, w \in \bigwedge \mathbb{R}^3$. Recalling that $\hat{u} = (-1)^k u$ for $u \in \bigwedge^k \mathbb{R}^3$, the second rule can also be written as

$$\mathbf{x} \lrcorner (u \wedge v) = (\mathbf{x} \lrcorner u) \wedge v + (-1)^k u \wedge (\mathbf{x} \lrcorner v),$$

when $u \in \bigwedge^k \mathbb{R}^3$. The second rule means that the left contraction by a vector is a *derivation* of the exterior algebra $\bigwedge \mathbb{R}^3$. It happens that the left contraction by a vector is also a derivation of the Clifford algebra, that is,

$$\mathbf{x} \lrcorner (uv) = (\mathbf{x} \lrcorner u)v + \hat{u}(\mathbf{x} \lrcorner v) \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^3, \ u, v \in \mathcal{C}\ell_3.$$

9 A scalar product on $\mathbb{R}^3 \subset \bigwedge \mathbb{R}^3$ induces a contraction on $\bigwedge \mathbb{R}^3$ which can be used to introduce a new product $\mathbf{x}u = \mathbf{x} \lrcorner u + \mathbf{x} \wedge u$ for $\mathbf{x} \in \mathbb{R}^3$ and $u \in \bigwedge \mathbb{R}^3$, which extends by linearity and associativity to all of $\bigwedge \mathbb{R}^3$. The linear space $\bigwedge \mathbb{R}^3$ provided with this new product is the Clifford algebra $\mathcal{C}\ell_3$.

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3.14 The Clifford algebra versus the exterior algebra

Both the Clifford algebra $\mathcal{C}\ell_3$ and the exterior algebra $\bigwedge \mathbb{R}^3$ contain a copy of $\mathbb{R}^3$, which enables application of calculations to the geometry of $\mathbb{R}^3$. The feature distinguishing $\mathcal{C}\ell_3$ from $\bigwedge \mathbb{R}^3$ is that the Clifford multiplication of vectors preserves the norm, $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, whereas $|\mathbf{a} \wedge \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. The equality $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$ enables more calculations to be carried out in $\mathbb{R}^3$, most notably rotations become represented as operations within one algebra, the Clifford algebra $\mathcal{C}\ell_3$.

Historical survey

The exterior algebra $\bigwedge \mathbb{R}^3$ of the linear space $\mathbb{R}^3$ was constructed by Grassmann in 1844. Grassmann's exterior algebra $\bigwedge \mathbb{R}^3$ has a basis

$$\begin{array}{l} 1 \\ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \\ \mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_3 \\ \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \end{array}$$

satisfying the multiplication rules

$$\begin{aligned} &\mathbf{e}_i \wedge \mathbf{e}_j = -\mathbf{e}_j \wedge \mathbf{e}_i \quad \text{for} \quad i \neq j, \\ &\mathbf{e}_i \wedge \mathbf{e}_i = 0. \end{aligned}$$

Clifford introduced a new product into the exterior algebra; he kept the first rule

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \quad \text{for} \quad i \neq j,$$

that is $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$, but replaced the second rule by

$$\begin{aligned} &\mathbf{e}_i\mathbf{e}_i = 1 \quad &&\text{in 1882,} \qquad \text{and} \\ &\mathbf{e}_i\mathbf{e}_i = -1 \quad &&\text{in 1878.} \end{aligned}$$

These two algebras generated are Clifford's geometric algebras

$$\mathcal{C}\ell_3 = \mathcal{C}\ell_{3,0} \simeq \mathrm{Mat}(2, \mathbb{C}) \quad \text{and} \quad \mathcal{C}\ell_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$$

of the positive definite and negative definite quadratic spaces $\mathbb{R}^3 = \mathbb{R}^{3,0}$ and $\mathbb{R}^{0,3}$, respectively.

Exercises

1. Find the area of the triangle with vertices $(1, -4, -6)$, $(5, -4, -2)$ and $(0, 0, 0)$.

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2. Find the volume of the parallelepiped with edges $\mathbf{a} = 2\mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$, $\mathbf{b} = \mathbf{e}_1 + 2\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{c} = 3\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$.
3. Compute the square of the volume element $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ (square with respect to the Clifford product).
4. Show that $\mathbf{e}_{123}$ commutes with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
5. Find the inverse of the bivector $\mathbf{B} = 3\mathbf{e}_{12} + \mathbf{e}_{23}$ (inverse with respect to the Clifford product).
6. Let $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 7\mathbf{e}_3$ and $\mathbf{B} = 4\mathbf{e}_{12} + 5\mathbf{e}_{13} - \mathbf{e}_{23}$. Compute $\mathbf{a} \wedge \mathbf{B}$ and $\mathbf{a} \lrcorner \mathbf{B}$.
7. Let $\mathbf{a} = 3\mathbf{e}_1 + 4\mathbf{e}_2 + 7\mathbf{e}_3$ and $\mathbf{B} = 7\mathbf{e}_{12} + \mathbf{e}_{13}$. Compute the perpendicular and parallel components of $\mathbf{a}$ in the plane of $\mathbf{B}$.
8. Show that the Clifford product of a bivector $\mathbf{B} \in \bigwedge^2 \mathbb{R}^3$ and an arbitrary element $u \in \mathcal{C}\ell_3$ can be decomposed as
$$\mathbf{B}u = \mathbf{B} \lrcorner u + \frac{1}{2}(\mathbf{B}u - u\mathbf{B}) + \mathbf{B} \wedge u.$$
9. Reconstruct the dot product $\mathbf{a} \cdot \mathbf{b}$ with the help of the cross product $\mathbf{a} \times \mathbf{b}$ and the exterior product $\mathbf{a} \wedge \mathbf{b}$. Hint: $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - \mathbf{a}^2\mathbf{b}$.

Define the right contraction by $u \llcorner v = \mathbf{e}_{123}^{-1}[(\mathbf{e}_{123}u) \wedge v]$ for $u, v \in \mathcal{C}\ell_3$.

10. Show that the following properties – the characteristic properties – of the right contraction hold:
$$\begin{aligned} &1) \quad \mathbf{x} \llcorner \mathbf{y} = \mathbf{x} \cdot \mathbf{y}, \\ &2) \quad (u \wedge v) \llcorner \mathbf{x} = u \wedge (v \llcorner \mathbf{x}) + (u \llcorner \mathbf{x}) \wedge \hat{v}, \\ &3) \quad u \llcorner (v \wedge w) = (u \llcorner v) \llcorner w, \end{aligned}$$
for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ and $u, v, w \in \bigwedge \mathbb{R}^3$.
11. Show that $\mathbf{a} \llcorner \mathbf{b} \in \bigwedge^{i-j} \mathbb{R}^3$ for $\mathbf{a} \in \bigwedge^i \mathbb{R}^3$ and $\mathbf{b} \in \bigwedge^j \mathbb{R}^3$.
12. Show that $(u \lrcorner v) \llcorner w = u \lrcorner (v \llcorner w)$.
13. Show that $u \lrcorner v = \star(\star^{-1}(v) \wedge \tilde{u})$ and $u \llcorner v = \star^{-1}(\tilde{v} \wedge \star(u))$.
14. Show that
$$u\mathbf{x} = u \llcorner \mathbf{x} + u \wedge \mathbf{x}$$
where, for a k-vector $u \in \bigwedge^k \mathbb{R}^3$,
$$\begin{aligned} u \llcorner \mathbf{x} &= \tfrac{1}{2}(u\mathbf{x} - (-1)^k \mathbf{x}u) \in \textstyle\bigwedge^{k-1} \mathbb{R}^3, \\ u \wedge \mathbf{x} &= \tfrac{1}{2}(u\mathbf{x} + (-1)^k \mathbf{x}u) \in \textstyle\bigwedge^{k+1} \mathbb{R}^3. \end{aligned}$$
15. Show that $u \wedge v - v \wedge u \in \bigwedge^2 \mathbb{R}^3$ and $uv - vu \in \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3$.

Let $\mathbf{a} \in \mathbb{R}^3$, $\mathbf{B} \in \bigwedge^2 \mathbb{R}^3$, $u = 1 + \mathbf{a} + \mathbf{B}$.

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16. The exterior inverse of u is $u^{\wedge(-1)} = 1 - \mathbf{a} - \mathbf{B} + \alpha\mathbf{a} \wedge \mathbf{B}$ with some $\alpha \in \mathbb{R}$. Determine α. Hint: use power series or $u \wedge u^{\wedge(-1)} = 1$.
17. The exterior square root of u is $u^{\wedge(1/2)} = 1 + \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{B} + \beta\mathbf{a} \wedge \mathbf{B}$ with some $\beta \in \mathbb{R}$. Determine β. Hint: $u^{\wedge(1/2)} \wedge u^{\wedge(1/2)} = u$.
18. Show that $1 \rfloor u = u$ for all $u \in \bigwedge \mathbb{R}^3$.

Solutions

1. $\mathbf{a} = \mathbf{e}_1 - 4\mathbf{e}_2 - 6\mathbf{e}_3$, $\mathbf{b} = 5\mathbf{e}_1 - 4\mathbf{e}_2 - 2\mathbf{e}_3$, $\mathbf{a} \wedge \mathbf{b} = 16\mathbf{e}_{12} + 28\mathbf{e}_{13} - 16\mathbf{e}_{23}$, $\frac{1}{2}|\mathbf{a} \wedge \mathbf{b}| = \frac{1}{2}\sqrt{16^2 + 28^2 + 16^2} = 18$.
2. $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = -7\mathbf{e}_{123}$, $|\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}| = 7$.
3. $\mathbf{e}_{123}^2 = -1$.
5. $\mathbf{B}^2 = -10$, $|\mathbf{B}| = \sqrt{10}$, $\mathbf{B}^{-1} = -\frac{1}{10}(3\mathbf{e}_{12} + \mathbf{e}_{23})$.
6. $\mathbf{a} \wedge \mathbf{B} = 11\mathbf{e}_{123}$, $\mathbf{a} \rfloor \mathbf{B} = -47\mathbf{e}_1 + 15\mathbf{e}_2 + 7\mathbf{e}_3$.
7. $\mathbf{a}_\perp = -0.9\mathbf{e}_2 + 6.3\mathbf{e}_3$, $\mathbf{a}_{\|} = 3\mathbf{e}_1 + 4.9\mathbf{e}_2 + 0.7\mathbf{e}_3$.
9. Take a wedge product with $\mathbf{b}$ to obtain $(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) \wedge \mathbf{b} = (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \wedge \mathbf{b})$, and
$$\mathbf{a} \cdot \mathbf{b} = \frac{(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}} \quad \text{for} \quad \mathbf{a} \nparallel \mathbf{b}$$
(the division is carried out in the Clifford algebra $C\ell_3$, or it is just a ratio of two parallel bivectors).
16. $\alpha = 2$.
17. $\beta = -\frac{1}{4}$.
18. $1 \rfloor u = (1 \wedge 1) \rfloor u = 1 \rfloor (1 \rfloor u)$ and so the contraction by 1 is a projection with eigenvalues 0 and 1. The only idempotents of $\bigwedge \mathbb{R}^3$ are 0 and 1, and so $1 \rfloor u = 0$ or $1 \rfloor u = u$, identically. The latter must be chosen, since $1 \rfloor (\mathbf{x} \cdot \mathbf{y}) = 1 \rfloor (\mathbf{x} \rfloor \mathbf{y}) = (1 \wedge \mathbf{x}) \rfloor \mathbf{y} = \mathbf{x} \rfloor \mathbf{y} = \mathbf{x} \cdot \mathbf{y} \neq 0$ for some $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

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