7

The Cross Product

The cross product is useful in many physical applications. It measures the angular velocity $\vec{\omega} = \vec{r} \times \vec{v}$ about O of a body moving at velocity $\vec{v}$ at the position P, $\vec{r} = \overrightarrow{OP}$. It is used to describe the torque $\vec{r} \times \vec{F}$ about O of a force $\vec{F}$ acting at $\vec{r}$. It also gives the force $\vec{F} = q\vec{v} \times \vec{B}$ acting on a charge q moving at velocity $\vec{v}$ in a magnetic field $\vec{B}$.

The usefulness of the cross product in three dimensions suggests the following questions: Is there a higher-dimensional analog of the cross product of two vectors in $\mathbb{R}^3$? If an analog exists, is it unique?

The first question is usually responded to by giving an answer to a modified question by explaining that there is a higher-dimensional analog of the cross product of $n-1$ vectors in $\mathbb{R}^n$. However, such a reply not only does not answer the original question, but also gives an incomplete answer to the modified question. In this chapter we will give a complete answer to the above questions and their modifications.

7.1 Scalar product in $\mathbb{R}^3$

The linear space $\mathbb{R}^3$ can be given extra structure by introducing the *scalar product* or *dot product*

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

for vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3$ in $\mathbb{R}^3$. The scalar product is scalar valued, $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$, and satisfies

$$\left.\begin{aligned} &(\mathbf{a}+\mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \\ &(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) \end{aligned}\right\} \quad \text{linear in the first factor}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad \text{symmetric}$$

$$\mathbf{a} \cdot \mathbf{a} > 0 \quad \text{for} \quad \mathbf{a} \neq 0 \qquad \text{positive definite.}$$

Linearity with respect to the first argument together with symmetry implies that the scalar product is linear with respect to both arguments, that is, it is *bilinear*. The symmetric bilinear scalar valued product gives rise to the quadratic form

$$\mathbb{R}^3 \to \mathbb{R}, \quad \mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \to \mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2,$$

which makes the linear space $\mathbb{R}^3$ into a *quadratic space* $\mathbb{R}^3$. The quadratic form is positive definite, that is, $\mathbf{a}\cdot\mathbf{a} = 0$ implies $\mathbf{a} = 0$, which allows us to introduce the *length* [1] $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}}$ of a vector $\mathbf{a} \in \mathbb{R}^3$. The real linear space $\mathbb{R}^3$ with a positive definite quadratic form on itself is called a *Euclidean space* $\mathbb{R}^3$. The length and the scalar product satisfy

$$\begin{array}{ll} |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}| & \text{triangle inequality} \\ |\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}| & \text{Cauchy-Schwarz inequality} \end{array}$$

where the latter inequality gives rise to the concept of angle. The angle φ between two directions $\mathbf{a}$ and $\mathbf{b}$ is obtained from

$$\cos\varphi = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Thus, we can write the scalar product in the form

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\varphi,$$

a formula which is usually taken as a definition of the scalar product, although this requires prior introduction of the concepts of length and angle.

7.2 Cross product in $\mathbb{R}^3$

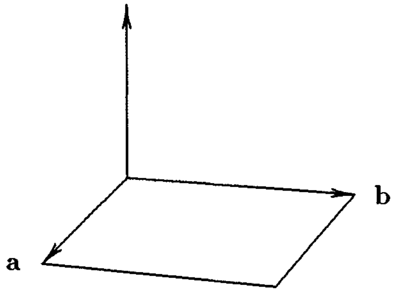
In the Euclidean space $\mathbb{R}^3$ it is convenient to introduce a vector valued product, the *cross product* $\mathbf{a}\times\mathbf{b} \in \mathbb{R}^3$ of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, with the following properties:

$$\begin{array}{ll} (\mathbf{a}\times\mathbf{b}) \perp \mathbf{a},\ (\mathbf{a}\times\mathbf{b}) \perp \mathbf{b} & \text{orthogonality} \\ |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi & \text{length equals area} \\ \mathbf{a}, \mathbf{b}, \mathbf{a}\times\mathbf{b} & \text{right-hand system.} \end{array}$$

In other words, the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$ and $\mathbf{b}$, its length is equal to the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as edges, and the vectors

1 The function $\mathbb{R}^3 \to \mathbb{R}$, $\mathbf{a} \to |\mathbf{a}|$ is a *norm* satisfying $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$, $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$, $|\mathbf{a}| = 0 \Rightarrow \mathbf{a} = 0$. Since this norm can be obtained from a scalar product, it satisfies the parallelogram law $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$.

$\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are oriented according to the right hand rule.



The above definition results in the following multiplication rules:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2 \times \mathbf{e}_1,$$
$$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 = -\mathbf{e}_3 \times \mathbf{e}_2,$$
$$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 = -\mathbf{e}_1 \times \mathbf{e}_3.$$

It is convenient to write the cross product in the form

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

The cross product is uniquely determined by

$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0, \ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$	orthogonality
$\lvert\mathbf{a} \times \mathbf{b}\rvert^2 = \lvert\mathbf{a}\rvert^2 \lvert\mathbf{b}\rvert^2 - (\mathbf{a} \cdot \mathbf{b})^2$	Pythagorean theorem

together with the right hand rule. The Pythagorean theorem can be written using the *Gram determinant* as

$$|\mathbf{a} \times \mathbf{b}|^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} \end{vmatrix}$$

which in coordinate form means *Lagrange's identity*

$$\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2$$
$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2.$$

The cross product satisfies the following rules for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$:

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$	antisymmetry
$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$	interchange rule.

The antisymmetry of the cross product has a geometric meaning: the lack of symmetry measures how much the two directions diverge. The cross product is not associative, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, which results in an inconvenience in computation, because parentheses cannot be omitted. [2]

The cross product is dual to the exterior product of two vectors:

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b})\mathbf{e}_{123}.$$

Taking the exterior product of $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\mathbf{a} \cdot \mathbf{b}) - |\mathbf{a}|^2\mathbf{b}$ and $\mathbf{b}$ one finds that

$$\mathbf{a} \cdot \mathbf{b} = \frac{(\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) \wedge \mathbf{b}}{\mathbf{a} \wedge \mathbf{b}} \quad \text{for} \quad \mathbf{a} \nparallel \mathbf{b},$$

that is, the scalar product can be recaptured from the cross product [you can also replace $\wedge$ by $\times$ in the above formula].

7.3 Cross product of $n-1$ vectors in $\mathbb{R}^n$

We can associate to three given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^4$ a fourth vector

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

which is orthogonal to the factors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and whose length is equal to the volume of the parallelepiped with $\mathbf{a}, \mathbf{b}, \mathbf{c}$ as edges, that is,

$$|\mathbf{a} \times \mathbf{b} \times \mathbf{c}|^2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}.$$

The cross product $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ of three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^4$ is completely antisymmetric and obeys the interchange rule slightly modified:

$$(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot \mathbf{d} = -\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c} \times \mathbf{d})$$

where $\mathbf{d} \in \mathbb{R}^4$. The oriented volume of the 4-dimensional parallelepiped with $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ as edges is the scalar

$$\det(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = (\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \cdot \mathbf{d}$$

multiplied by (the unit oriented volume) $\mathbf{e}_{1234}$.

2 The cross product is antisymmetric, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, and satisfies the Jacobi identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$, which makes the linear space $\mathbb{R}^3$, with cross product on $\mathbb{R}^3$, a non-associative algebra, called a *Lie algebra*. The Jacobi identity can be verified using $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

The cross product of three vectors in $\mathbb{R}^4$ is dual to the exterior product:

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = -(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})\mathbf{e}_{1234}$$

where the latter product is computed in the Clifford algebra $\mathcal{C}\ell_4$.

In a similar manner we can introduce in n dimensions a cross product of $n-1$ factors. The result is a vector orthogonal to the factors, and the length of the vector is equal to the hypervolume of the parallelepiped formed by the factors.

7.4 Cross product of two vectors in $\mathbb{R}^7$

Is there a cross product in n dimensions with just two factors? If we require the cross product to be orthogonal to the factors and have length equal to the area of the parallelogram, then the answer is no, unless $n = 3$ or $n = 7$.

The cross product of two vectors in $\mathbb{R}^7$ can be defined in terms of an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_7$ by antisymmetry, $\mathbf{e}_i \times \mathbf{e}_j = -\mathbf{e}_j \times \mathbf{e}_i$, and

$$\begin{array}{lll} \mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_4, & \mathbf{e}_2 \times \mathbf{e}_4 = \mathbf{e}_1, & \mathbf{e}_4 \times \mathbf{e}_1 = \mathbf{e}_2, \\ \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_5, & \mathbf{e}_3 \times \mathbf{e}_5 = \mathbf{e}_2, & \mathbf{e}_5 \times \mathbf{e}_2 = \mathbf{e}_3, \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \mathbf{e}_7 \times \mathbf{e}_1 = \mathbf{e}_3, & \mathbf{e}_1 \times \mathbf{e}_3 = \mathbf{e}_7, & \mathbf{e}_3 \times \mathbf{e}_7 = \mathbf{e}_1. \end{array}$$

The above table can be condensed into the form

$$\mathbf{e}_i \times \mathbf{e}_{i+1} = \mathbf{e}_{i+3}$$

where the indices are permuted cyclically and translated modulo 7.

This cross product of vectors in $\mathbb{R}^7$ satisfies the usual properties, that is,

$$\begin{array}{ll} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0, \ (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0 & \text{orthogonality} \\ |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 & \text{Pythagorean theorem} \end{array}$$

where the second rule can also be written as $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \sphericalangle(\mathbf{a}, \mathbf{b})$. Unlike the 3-dimensional cross product, the 7-dimensional cross product does not satisfy the Jacobi identity, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} + (\mathbf{b} \times \mathbf{c}) \times \mathbf{a} + (\mathbf{c} \times \mathbf{a}) \times \mathbf{b} \neq 0$, and so it does not form a Lie algebra. However, the 7-dimensional cross product satisfies the Malcev identity, a generalization of Jacobi, see Ebbinghaus et al. 1991 p. 279.

In $\mathbb{R}^3$ the direction of $\mathbf{a} \times \mathbf{b}$ is unique, up to two alternatives for the orientation, but in $\mathbb{R}^7$ the direction of $\mathbf{a} \times \mathbf{b}$ depends on a 3-vector defining the cross product; to wit,

$$\mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b}) \,\lrcorner\, \mathbf{v} \quad [\neq -(\mathbf{a} \wedge \mathbf{b})\mathbf{v}]$$

depends on

$$\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^7.$$

In the 3-dimensional space $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ implies that $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are in the same plane, but for the cross product $\mathbf{a} \times \mathbf{b}$ in $\mathbb{R}^7$ there are also other planes than the linear span of $\mathbf{a}$ and $\mathbf{b}$ giving the same direction as $\mathbf{a} \times \mathbf{b}$.

The 3-dimensional cross product is invariant under all rotations of $SO(3)$, while the 7-dimensional cross product is not invariant under all of $SO(7)$, but only under the exceptional Lie group G_2, a subgroup of $SO(7)$. When we let $\mathbf{a}$ and $\mathbf{b}$ run through all of $\mathbb{R}^7$, the image set of the simple bivectors $\mathbf{a} \wedge \mathbf{b}$ is a manifold of dimension $2 \cdot 7 - 3 = 11 > 7$ in $\bigwedge^2 \mathbb{R}^7$, $\dim(\bigwedge^2 \mathbb{R}^7) = \frac{1}{2}7(7-1) = 21$, while the image set of $\mathbf{a} \times \mathbf{b}$ is just $\mathbb{R}^7$. So the mapping

$$\mathbf{a} \wedge \mathbf{b} \to \mathbf{a} \times \mathbf{b} = -(\mathbf{a} \wedge \mathbf{b}) \,\lrcorner\, \mathbf{v}$$

is not a one-to-one correspondence, but only a method of associating a vector to a bivector.

The 3-dimensional cross product is the pure/vector part of the quaternion product of two pure quaternions, that is,

$$\mathbf{a} \times \mathbf{b} = \operatorname{Im}(\mathbf{ab}) \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathbb{H}.$$

In terms of the Clifford algebra $\mathcal{C}\ell_3 \simeq \operatorname{Mat}(2, \mathbb{C})$ of the Euclidean space $\mathbb{R}^3$ the cross product could also be expressed as

$$\mathbf{a} \times \mathbf{b} = -\langle \mathbf{abe}_{123} \rangle_1 \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathcal{C}\ell_3.$$

In terms of the Clifford algebra $\mathcal{C}\ell_{0,3} \simeq \mathbb{H} \times \mathbb{H}$ of the negative definite quadratic space $\mathbb{R}^{0,3}$ the cross product can be expressed not only as

$$\mathbf{a} \times \mathbf{b} = -\langle \mathbf{abe}_{123} \rangle_1 \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{0,3} \subset \mathcal{C}\ell_{0,3}$$

but also as [3]

$$\mathbf{a} \times \mathbf{b} = \langle \mathbf{ab}(1 - \mathbf{e}_{123}) \rangle_1 \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^{0,3} \subset \mathcal{C}\ell_{0,3}.$$

Similarly, the 7-dimensional cross product is the pure/vector part of the octonion product of two pure octonions, that is, $\mathbf{a} \times \mathbf{b} = \langle \mathbf{a} \circ \mathbf{b} \rangle_1$. The octonion algebra $\mathbb{O}$ is a norm-preserving algebra with unity 1, whence its pure/imaginary part is an algebra with cross product, that is, $\mathbf{a} \times \mathbf{b} = \frac{1}{2}(\mathbf{a} \circ \mathbf{b} - \mathbf{b} \circ \mathbf{a})$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^7 \subset \mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7$. The octonion product in turn is given by

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta - \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

3 This expression is also valid for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \subset \mathcal{C}\ell_3$, but the element $1 - \mathbf{e}_{123}$ does not pick up an ideal of $\mathcal{C}\ell_3$. Recall that $\mathcal{C}\ell_3$ is simple, that is, it has no proper two-sided ideals.

for $a = \alpha + \mathbf{a}$ and $b = \beta + \mathbf{b}$ in $\mathbb{R} \oplus \mathbb{R}^7$. If we replace the Euclidean space $\mathbb{R}^7$ by the negative definite quadratic space $\mathbb{R}^{0,7}$, then not only

$$a \circ b = \alpha\beta + \alpha\mathbf{b} + \mathbf{a}\beta + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

for $a, b \in \mathbb{R} \oplus \mathbb{R}^{0,7}$, but also

$$a \circ b = \langle ab(1 - \mathbf{v}) \rangle_{0,1}$$

where $\mathbf{v} = \mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713} \in \bigwedge^3 \mathbb{R}^{0,7}$.

7.5 Cross products of k vectors in $\mathbb{R}^n$

If one reformulates the question about the existence of a cross product of two vectors in $\mathbb{R}^n$, and also allows $n-1$ factors, then one is led to a more general problem on the existence of a cross product of k factors in $\mathbb{R}^n$. If we were looking for a vector valued product of k factors in $\mathbb{R}^n$, then we should first formalize our problem by modifying the Pythagorean theorem, a candidate being the Gram determinant. A natural thing to do is to consider a vector valued product $\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k$ satisfying

$$\begin{array}{ll} (\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k) \cdot \mathbf{a}_i = 0 & \text{orthogonality} \\ |\mathbf{a}_1 \times \mathbf{a}_2 \times \cdots \times \mathbf{a}_k|^2 = \det(\mathbf{a}_i \cdot \mathbf{a}_j) & \text{Gram determinant} \end{array}$$

where the second condition means that the length of $\mathbf{a}_1 \times \mathbf{a}_2 \times \ldots \times \mathbf{a}_k$ equals the volume of the parallelepiped with $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ as edges.

The solution to this problem is that there are vector valued cross products in

3	dimensions with	2	factors
7	dimensions with	2	factors
n	dimensions with	$n-1$	factors
8	dimensions with	3	factors

and no others – except if one allows degenerate solutions, when there would also be in all even dimensions n, $n \in 2\mathbb{Z}$, a vector product with only one factor (and in one dimension an identically vanishing cross product with two factors).

The cross product of three vectors in $\mathbb{R}^8$ can be expressed as

$$\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}) \,\lrcorner\, (\mathbf{w} - \mathbf{v}\mathbf{e}_8) = \langle (\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})(1 - \mathbf{e}_{12\ldots8})\mathbf{w} \rangle_1$$

where

$$\begin{aligned} \mathbf{w} &= -(\mathbf{e}_{124} + \mathbf{e}_{235} + \mathbf{e}_{346} + \mathbf{e}_{457} + \mathbf{e}_{561} + \mathbf{e}_{672} + \mathbf{e}_{713})\mathbf{e}_{12\ldots7} \\ &= \mathbf{e}_{1236} - \mathbf{e}_{1257} - \mathbf{e}_{1345} + \mathbf{e}_{1467} + \mathbf{e}_{2347} - \mathbf{e}_{2456} - \mathbf{e}_{3567} \end{aligned}$$

and $\mathbf{w} \in \bigwedge^4 \mathbb{R}^7 \subset \bigwedge^4 \mathbb{R}^8$.

The trivial cross product with one factor in an even number of dimensions rotates all vectors by 90°. Thus, let n be even and let $\mathbf{a}$ be the only factor of a trivial cross product with value $\mathbf{b}$, $|\mathbf{b}| = |\mathbf{a}|$, $\mathbf{b} \cdot \mathbf{a} = 0$. This can be accomplished by

$$\mathbf{b} = \mathbf{a} \lrcorner (\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_3\mathbf{e}_4 + \ldots + \mathbf{e}_{n-1}\mathbf{e}_n).$$

Exercises

1. Show that the cross product $\mathbf{a} \times \mathbf{r}$ can be represented by a matrix multiplication $A\mathbf{r} = \mathbf{a} \times \mathbf{r}$, where
$$A\mathbf{r} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$
2. Express the rotation matrix e^A in terms of I, A and A^2. Hint: use the Cayley-Hamilton theorem, $A^3 + |\mathbf{a}|^2 A = 0$.
3. Express the rotated vector $e^A\mathbf{r}$ as a linear combination of $\mathbf{r}$, $\mathbf{a} \times \mathbf{r}$ and $(\mathbf{a} \cdot \mathbf{r})\mathbf{a}$. Hint: $A^2\mathbf{r} = (\mathbf{a} \cdot \mathbf{r})\mathbf{a} - \mathbf{a}^2\mathbf{r}$.
4. Compute the square of $\mathbf{w} = -\mathbf{v}\mathbf{e}_{12\ldots 7} \in \bigwedge^4 \mathbb{R}^7$.
5. Show that $\frac{1}{8}(1 + \mathbf{w})$ is an idempotent of $\mathcal{C}\ell_7 \simeq \mathrm{Mat}(8, \mathbb{C})$.

Solutions

2. $e^A = I + \dfrac{A}{\alpha}\sin\alpha + \dfrac{A^2}{\alpha^2}(1 - \cos\alpha)$, where $\alpha = |\mathbf{a}|$.
3. $e^A\mathbf{r} = \cos\alpha\,\mathbf{r} + \dfrac{\sin\alpha}{\alpha}\mathbf{a} \times \mathbf{r} + \dfrac{1 - \cos\alpha}{\alpha^2}(\mathbf{a} \cdot \mathbf{r})\mathbf{a}$.
4. $\mathbf{w}^2 = 7 + 6\mathbf{w}$.

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