5

Quaternions

We saw in the chapter on *Complex Numbers* that it is convenient to use the real algebra of complex numbers $\mathbb{C}$ to represent the rotation group $SO(2)$ of the plane $\mathbb{R}^2$. In this chapter we shall study rotations of the 3-dimensional space $\mathbb{R}^3$. The composition of spatial rotations is no longer commutative, and we need a non-commutative multiplication to represent the rotation group $SO(3)$. This can be done within the real algebra of 3×3-matrices $\mathrm{Mat}(3, \mathbb{R})$, or by the real algebra of **quaternions**, $\mathbb{H}$, invented by Hamilton.

The complex plane $\mathbb{C}$ is a real linear space $\mathbb{R}^2$, and multiplication by a complex number $c = a + ib$, that is, the map $\mathbb{C} \to \mathbb{C}$, $z \to cz$, may be regarded as a real linear map with matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ operating on $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$. The complex plane is also a real quadratic space $\mathbb{R}^{2,0}$, in short $\mathbb{R}^2$, with a quadratic form

$$\mathbb{C} \to \mathbb{R}, \; z = x + iy \to z\bar{z} = x^2 + y^2,$$

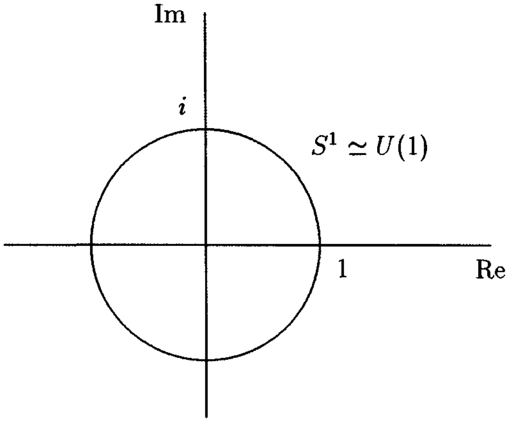
and norm $|z| = \sqrt{z\bar{z}}$. Multiplication of complex numbers preserves the norm, that is, $|cz| = |c||z|$ for all c, $z \in \mathbb{C}$, and so multiplication by c is a rotation of $\mathbb{R}^2$ if, and only if, $|c| = 1$. Conversely, any rotation of $\mathbb{R}^2$ can be represented by a unit complex number c, $|c| = 1$, in $\mathbb{C}$. The unit complex numbers form a group

$$U(1) = \{c \in \mathbb{C} \mid c\bar{c} = 1\},$$

called the *unitary group*, which is isomorphic to the rotation group $SO(2) = \{U \in \mathrm{Mat}(2, \mathbb{R}) \mid U^\top U = I, \; \det U = 1\}$, that is, $U(1) \simeq SO(2)$. The unitary group $U(1)$ can be visualized as the *unit circle*

$$S^1 = \{x + iy \in \mathbb{C} \mid x^2 + y^2 = 1\}$$

of the complex plane $\mathbb{C}$.



Similarly, the algebra of quaternions $\mathbb{H}$ may be used to represent rotations of the 3-dimensional space $\mathbb{R}^3$. It will turn out that quaternions are also convenient to represent the rotations of the 4-dimensional space $\mathbb{R}^4$.

Quaternions as hypercomplex numbers

Quaternions are generalized complex numbers of the form $q = w + ix + jy + kz$ where w, x, y, z are real numbers and the generalized imaginary units i, j, k satisfy the following multiplication rules:

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k = -ji, \quad jk = i = -kj, \quad ki = j = -ik.$$

Note that the multiplication is by definition **non-commutative**. One can show that quaternion multiplication is *associative*. The above multiplication rules can be condensed into the following form:

$$i^2 = j^2 = k^2 = ijk = -1$$

where in the last identity we have omitted parentheses and thereby tacitly assumed associativity.

The generalized imaginary units will be denoted either by i, j, k or by $\mathbf{i}, \mathbf{j}, \mathbf{k}$. They have two different roles: they act as generators of

rotations, that is, they are bivectors, or
translations, that is, they are vectors.

This distinction is not clear-cut since bivectors are dual to vectors in $\mathbb{R}^3$.

5.1 Pure part and cross product

A quaternion $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is a sum of a scalar and a vector, called the *real part*, $\mathrm{Re}(q) = w \in \mathbb{R}$, and the *pure part*, $\mathrm{Pu}(q) = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \in \mathbb{R}^3$. The quaternions form a 4-dimensional real linear space $\mathbb{H}$ which contains the real axis $\mathbb{R}$ and a 3-dimensional real linear space $\mathbb{R}^3$ so that $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$. We denote the pure part also by a boldface letter so that $q = q_0 + \mathbf{q}$ where $q_0 \in \mathbb{R}$ and $\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{R}^3$. The real linear space $\mathbb{R} \oplus \mathbb{R}^3$ with the quaternion product is an associative algebra over $\mathbb{R}$ called the *quaternion algebra* $\mathbb{H}$. The product of two quaternions $a = a_0 + \mathbf{a}$ and $b = b_0 + \mathbf{b}$ can be written as

$$ab = a_0 b_0 - \mathbf{a} \cdot \mathbf{b} + a_0 \mathbf{b} + \mathbf{a} b_0 + \mathbf{a} \times \mathbf{b}.$$

A quaternion $q = q_0 + \mathbf{q}$ is *pure* if its real part vanishes, $q_0 = 0$, so that $q = \mathbf{q} \in \mathbb{R}^3$. A product of two pure quaternions $\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$ and $\mathbf{b} = \mathbf{i}b_1 + \mathbf{j}b_2 + \mathbf{k}b_3$ is a sum of a real number and a pure quaternion:

$$\mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

where we recognize the scalar product $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ and the cross product $\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_2 b_3 - a_3 b_2) + \mathbf{j}(a_3 b_1 - a_1 b_3) + \mathbf{k}(a_1 b_2 - a_2 b_1)$.

The vector space $\mathbb{R}^3$ with the cross product $\mathbf{a} \times \mathbf{b}$ is a real algebra, that is, it is a real linear space with a bilinear map

$$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \ (\mathbf{a}, \mathbf{b}) \to \mathbf{a} \times \mathbf{b}.$$

The cross product satisfies two rules

$$\begin{aligned} &\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}, \\ &\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0, \end{aligned}$$

the latter being called the Jacobi identity; this makes $\mathbb{R}^3$ with the cross product a *Lie algebra*. In particular, the cross product is not associative, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

We can reobtain the cross product of two pure quaternions $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ as the pure part of their quaternion product: $\mathbf{a} \times \mathbf{b} = \mathrm{Pu}(\mathbf{ab})$.

5.2 Quaternion conjugate, norm and inverse

The conjugate $\tilde{q}$ of a quaternion $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ is obtained by changing the sign of the pure part:

$$\tilde{q} = w - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z.$$

We shall also refer to $\tilde{q}$ as the *quaternion conjugate* of q. The conjugation is an anti-automorphism of $\mathbb{H}$; $\overline{ab} = \bar{b}\bar{a}$ for $a, b \in \mathbb{H}$.

A quaternion q multiplied by its conjugate $\bar{q}$ results in a real number $q\bar{q} = w^2 + x^2 + y^2 + z^2$ called the square norm of $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$. The *norm* $|q|$ of q is given by $|q| = \sqrt{q\bar{q}}$ so that

$$|w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

The norm of a product of two quaternions a and b is the product of their norms – as an equation, $|ab| = |a||b|$ for $a, b \in \mathbb{H}$ – which turns $\mathbb{H}$ into a normed algebra.

The *inverse* q^{-1} of a non-zero quaternion q is obtained by $q^{-1} = \bar{q}/|q|^2$ or more explicitly by

$$\frac{1}{w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z} = \frac{w - \mathbf{i}x - \mathbf{j}y - \mathbf{k}z}{w^2 + x^2 + y^2 + z^2}.$$

In particular, $ab = 0$ implies $a = 0$ or $b = 0$, which means that the quaternion algebra is a division algebra (or that the ring of quaternions is a division ring).

5.3 The center of $\mathbb{H}$

The set of those elements in $\mathbb{H}$ which commute with every element of $\mathbb{H}$ forms the *center* of $\mathbb{H}$,

$$\mathrm{Cen}(\mathbb{H}) = \{w \in \mathbb{H} \mid wq = qw \text{ for all } q \in \mathbb{H}\}.$$

The center is of course closed under multiplication. The center of the division ring $\mathbb{H}$ is isomorphic to the field of real numbers $\mathbb{R}$. In contrast to the case of the complex field $\mathbb{C}$, the real axis in $\mathbb{H}$ is the unique subfield which is the center of $\mathbb{H}$.

5.4 Rotations in three dimensions

Take a pure quaternion or a vector

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \in \mathbb{R}^3, \quad \text{where} \quad \mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3,$$

of length $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. For a non-zero quaternion $a \in \mathbb{H}$, the expression $a\mathbf{r}a^{-1}$ is again a pure quaternion with the same length, that is,

$$a\mathbf{r}a^{-1} \in \mathbb{R}^3 \quad \text{and} \quad |a\mathbf{r}a^{-1}| = |\mathbf{r}|.$$

In other words, the mapping

$$\mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{r} \to a\mathbf{r}a^{-1}$$

is a rotation of the quadratic space of pure quaternions $\mathbb{R}^3$. Each rotation in $SO(3) = \{U \in \mathrm{Mat}(3, \mathbb{R}) \mid U^\top U = I, \ \det U = 1\}$ can be so represented, and there are two unit quaternions a and $-a$ representing the same rotation, $a\mathbf{r}a^{-1} = (-a)\mathbf{r}(-a)^{-1}$. In other words, the sphere of unit quaternions,

$$S^3 = \{q \in \mathbb{H} \mid |q| = 1\},$$

is a two-fold covering group of $SO(3)$, that is, $SO(3) \simeq S^3/\{\pm 1\}$.

A rotation has three parameters in dimension 3. In other words, $SO(3)$ and S^3 are 3-dimensional manifolds. The three parameters are the angle of rotation and the two direction cosines of the axis of rotation.

To find the axis of this rotation we take a unit quaternion a, $|a| = 1$, and write it in the form $a = e^{\mathbf{a}/2}$ where $\mathbf{a} \in \mathbb{R}^3$. Note that

$$e^{\mathbf{a}/2} = \cos\frac{\alpha}{2} + \frac{\mathbf{a}}{\alpha}\sin\frac{\alpha}{2}$$

where $\alpha = |\mathbf{a}|$. The rotation $\mathbf{r} \to a\mathbf{r}a^{-1}$ turns $\mathbf{r}$ about the axis $\mathbf{a}$ by the angle α. The sense of the rotation is counter-clockwise when regarded from the arrow-head of $\mathbf{a}$.

The composite of two consecutive rotations, first around $\mathbf{a}$ by the angle $\alpha = |\mathbf{a}|$ and then around $\mathbf{b}$ by the angle $\beta = |\mathbf{b}|$, is again a rotation around some axis, say $\mathbf{c}$. The axis of the composite rotation can be found by inspection of the real and pure parts of the formula $e^{\mathbf{c}/2} = e^{\mathbf{b}/2}e^{\mathbf{a}/2}$. Divide both sides by their real parts and substitute

$$\mathbf{c}' = \frac{\mathbf{c}}{\gamma}\tan\frac{\gamma}{2}, \quad \text{where} \quad \gamma = |\mathbf{c}|,$$

to obtain the *Rodrigues formula*

$$\mathbf{c}' = \frac{\mathbf{a}' + \mathbf{b}' - \mathbf{a}' \times \mathbf{b}'}{1 - \mathbf{a}' \cdot \mathbf{b}'}.$$

5.5 Rotations in four dimensions

The mapping $\mathbb{H} \to \mathbb{H}$, $q \to aqb^{-1}$, where $a, b \in \mathbb{H}$ are unit quaternions $|a| = |b| = 1$, is a rotation of the 4-dimensional space $\mathbb{R}^4 = \mathbb{H}$. In other words, the real linear mapping

$$\mathbb{H} \to \mathbb{H}, \quad q \to aqb^{-1}, \quad \text{where} \quad a, b \in \mathbb{H} \quad \text{and} \quad |a| = |b| = 1,$$

is a rotation of $\mathbb{R}^4$. Each rotation in $SO(4)$ can be so represented, and there are two elements (a, b) and $(-a, -b)$ in $S^3 \times S^3$ representing the same rotation, that is, $aqb^{-1} = (-a)q(-b)^{-1}$. In other words, the group $S^3 \times S^3$ is a two-fold covering group of $SO(4)$, that is,

$$SO(4) \simeq \frac{S^3 \times S^3}{\{(1,1), (-1,-1)\}}.$$

A rotation in dimension 4 can be represented by a pair of unit quaternions, and so it has six parameters, in other words, $\dim SO(4) = \dim(S^3 \times S^3) = 6$. A rotation has two completely orthogonal invariant planes; both the invariant planes can turn arbitrarily; this takes two parameters. Fixing a plane in $\mathbb{R}^4$ takes the remaining four parameters: three parameters for a unit vector in S^3, plus two parameters for another orthogonal unit vector in S^2, minus one parameter for rotating the pairs of such vectors in the plane.

5.6 Matrix representation of quaternion multiplication

The product of two quaternions $q = w+\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$ and $u = u_0+\mathbf{i}u_1+\mathbf{j}u_2+\mathbf{k}u_3$ can be represented by matrix multiplication:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

where $qu = v$. Swapping the multiplication to the right, that is, $uq = v'$, gives a partially transformed matrix:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v'_0 \\ v'_1 \\ v'_2 \\ v'_3 \end{pmatrix}.$$

Let us denote the above matrices respectively by L_q and R_q, that is,

$$L_q(u) = qu \ (= v) \quad \text{and} \quad R_q(u) = uq \ (= v').$$

We find that [1]

$$L_{\mathbf{i}}L_{\mathbf{j}}L_{\mathbf{k}} = -I \quad \text{and} \quad R_{\mathbf{i}}R_{\mathbf{j}}R_{\mathbf{k}} = I.$$

The sets $\{L_q \in \mathrm{Mat}(4,\mathbb{R}) \mid q \in \mathbb{H}\}$ and $\{R_q \in \mathrm{Mat}(4,\mathbb{R}) \mid q \in \mathbb{H}\}$ form two subalgebras of $\mathrm{Mat}(4,\mathbb{R})$, both isomorphic to $\mathbb{H}$. For two arbitrary quaternions $a, b \in \mathbb{H}$ these two matrix representatives commute, that is, $L_aR_b = R_bL_a$. Any real 4×4-matrix is a linear combination of matrices of the form L_aR_b. The above observations together with $(\dim \mathbb{H})^2 = \dim \mathrm{Mat}(4,\mathbb{R})$ imply that

$$\mathrm{Mat}(4,\mathbb{R}) \simeq \mathbb{H} \otimes \mathbb{H},$$

or more informatively $\mathrm{Mat}(4,\mathbb{R}) = \mathbb{H} \otimes \mathbb{H}^*$. [2]

1 Note that $R_{\mathbf{i}}^{\mathsf{T}} R_{\mathbf{j}}^{\mathsf{T}} R_{\mathbf{k}}^{\mathsf{T}} = -I$.

2 For unit quaternions $a, b \in \mathbb{H}$ such that $|a| = |b| = 1$ we may choose $L_a \in Q$ and $R_b \in Q^*$ or alternatively $L_a \in Q^*$ and $R_b \in Q$. For a discussion about the meaning of Q and Q^*, see the chapter on *The Fourth Dimension*.

Take a matrix of the form $U = L_a R_b$ in $\mathrm{Mat}(4, \mathbb{R})$. Then $U^\top U = |a|^2|b|^2 I$, but in general $U + U^\top \neq \alpha I$. Take a matrix of the form $V = L_a + R_b$ in $\mathrm{Mat}(4, \mathbb{R})$. Then $V + V^\top = 2(\mathrm{Re}(a) + \mathrm{Re}(b))I$, but in general $V^\top V \neq \beta I$. Conversely, if $U \in \mathrm{Mat}(4, \mathbb{R})$ is such that $U + U^\top = \alpha I$ and $U^\top U = \beta I$ then the matrix U belongs either to $\mathbb{H}$ or to $\mathbb{H}^*$.

Besides real 4×4-matrices, quaternions can also be represented by complex 2×2-matrices:

$$w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \simeq \begin{pmatrix} w + iz & ix + y \\ ix - y & w - iz \end{pmatrix}.$$

The orthogonal unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are represented by matrices obtained by multiplying each of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ by $i = \sqrt{-1}$:

$$\mathbf{i} \simeq \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \mathbf{j} \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} \simeq \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

5.7 Linear spaces over $\mathbb{H}$

Much of the theory of linear spaces over commutative fields extends to $\mathbb{H}$. Because of the non-commutativity of $\mathbb{H}$ it is, however, necessary to distinguish between two types of linear spaces over $\mathbb{H}$, namely *right* linear spaces and *left* linear spaces.

A *right* linear space over $\mathbb{H}$ consists of an additive group V and a map

$$V \times \mathbb{H} \to V, \quad (\mathbf{x}, \lambda) \to \mathbf{x}\lambda$$

such that the usual distributivity and unity axioms hold and such that, for all λ, $\mu \in \mathbb{H}$ and $\mathbf{x} \in V$,

$$(\mathbf{x}\lambda)\mu = \mathbf{x}(\lambda\mu).$$

A *left* linear space over $\mathbb{H}$ consists of an additive group V and a map

$$\mathbb{H} \times V \to V, \ (\lambda, \mathbf{x}) \to \lambda\mathbf{x}$$

such that the usual distributivity and unity axioms hold and such that, for all λ, $\mu \in \mathbb{H}$ and $\mathbf{x} \in V$,

$$\lambda(\mu\mathbf{x}) = (\lambda\mu)\mathbf{x}.$$

A mapping $L : V \to U$ between two right linear spaces V and U is a *right linear map* if it respects addition and, for all $\mathbf{x} \in V$, $\lambda \in \mathbb{H}$, $L(\mathbf{x}\lambda) = (L(\mathbf{x}))\lambda$.

Comment. In the matrix form the above definition means that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \lambda \right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1\lambda \\ x_2\lambda \end{pmatrix} = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] \lambda. \qquad \blacksquare$$

Remark. Although there are linear spaces over $\mathbb{H}$, there are no algebras over $\mathbb{H}$, since non-commutativity of $\mathbb{H}$ precludes bilinearity over $\mathbb{H}$: $\lambda(xy) = (\lambda x)y \neq (x\lambda)y = x(\lambda y) \neq x(y\lambda) = (xy)\lambda$. ∎

5.8 Function theory of quaternion variables

The richness of complex analysis suggests that there might be a function theory of quaternion variables. There are several different ways to generalize the theory of complex variables to the theory of quaternion functions of quaternion variables, $f : \mathbb{H} \to \mathbb{H}$. However, many generalizations are uninteresting, the classes of functions are too small or too large. In the following we will first eliminate the uninteresting generalizations.

First, consider quaternion differentiable functions such that

$$f'(q) = \lim_{h\to 0}[f(q+h) - f(q)]h^{-1}, \quad \text{where} \quad q, h \in \mathbb{H},$$

exists. The derivative $f'(q)$ is a real linear function

$$\mathbb{R}^4 \to \mathbb{R}^4 : h \to f'(q)h$$

corresponding to multiplication by a quaternion $a \in \mathbb{H}$ on the left, $f'(q)h = ah$ for $h \in \mathbb{H} = \mathbb{R}^4$. However, since $ah \neq ha$ the only quaternion differentiable functions are the affine right $\mathbb{H}$-linear functions

$$f(q) = aq + b \quad \text{where} \quad a, b \in \mathbb{H}.$$

We conclude that the set of quaternion differentiable functions reduces to a small and uninteresting set.

Second, if we consider power series in a quaternion variable $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, then we get the set of all power series in the four real variables w, x, y, z. For instance, the coordinates are first-order functions

$$\begin{aligned}
w &= \tfrac{1}{4}(q - \mathbf{i}q\mathbf{i} - \mathbf{j}q\mathbf{j} - \mathbf{k}q\mathbf{k}),\\
x &= \tfrac{1}{4}(q - \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k})\mathbf{i}^{-1},\\
y &= \tfrac{1}{4}(q + \mathbf{i}q\mathbf{i} - \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k})\mathbf{j}^{-1},\\
z &= \tfrac{1}{4}(q + \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} - \mathbf{k}q\mathbf{k})\mathbf{k}^{-1},
\end{aligned}$$

and so the set of power series in q, with left and right quaternion coefficients, is the set of all power series in the real variables w, x, y, z. This set is too big to be interesting.

Third, we could consider power series in q with real coefficients, that is, functions of type $f(q) = a_0 + a_1 q + a_2 q^2 + \ldots$ where $a_0, a_1, a_2, \ldots$ are real.

Restrict such a function to the complex subfield $\mathbb{C} \subset \mathbb{H}$, and send $z = x + iy$ to $f(z) = u + iv$, where $u = u(x, y)$ and $v = v(x, y)$. Decompose the quaternion q into real and vector parts, $q = q_0 + \mathbf{q}$, and note that $\mathbf{q}/|\mathbf{q}|$ is a generalized imaginary unit, $(\mathbf{q}/|\mathbf{q}|)^2 = -1$. Then

$$f(q_0 + \mathbf{q}) = u(q_0, |\mathbf{q}|) + \frac{\mathbf{q}}{|\mathbf{q}|} v(q_0, |\mathbf{q}|).$$

So this generalization just rotates the graph of $\mathbb{C} \to \mathbb{C}$, $z \to f(z)$, or rather makes $i = \mathbf{i}$ sweep all of $S^2 = \{\mathbf{r} \in \mathbb{R}^3 \mid |\mathbf{r}| = 1\}$, and thus gives only (a subclass of) axially symmetric functions.

Fourth, we could consider functions which are conformal almost everywhere in $\mathbb{R}^4$. This leads to Möbius transformations of $\mathbb{R}^4$, or its one-point compactification $\mathbb{R}^4 \cup \{\infty\}$. The Möbius transformations are compositions of the four mappings sending q to

aqb^{-1}	$a, b \in S^3$	rotations
$q + b$	$b \in \mathbb{H}$	translations
$q\lambda$	$\lambda > 0$	dilations
$(q^{-1} + c)^{-1}$	$c \in \mathbb{H}$	transversions.

A nice thing about quaternions is that all Möbius transformations of $\mathbb{R}^4$ can be written in the form $(aq + b)(cq + d)^{-1}$, where $a, b, c, d \in \mathbb{H}$.

Fifth, we could focus our attention on a generalization of the Cauchy-Riemann equations,

$$\frac{\partial f}{\partial w} + \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = 0 \quad \text{where} \quad f : \mathbb{H} \to \mathbb{H}.$$

Using the differential operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

the above equation can be put into the form

$$\frac{\partial f_0}{\partial w} + \frac{\partial \mathbf{f}}{\partial w} + \nabla f_0 - \nabla \cdot \mathbf{f} + \nabla \times \mathbf{f} = 0$$

where $f = f_0 + \mathbf{f}$ with $f_0 : \mathbb{H} \to \mathbb{R}$ and $\mathbf{f} : \mathbb{H} \to \mathbb{R}^3$. This decomposes into scalar and vector parts

$$\frac{\partial f_0}{\partial w} - \nabla \cdot \mathbf{f} = 0 \quad \text{and} \quad \frac{\partial \mathbf{f}}{\partial w} + \nabla f_0 + \nabla \times \mathbf{f} = 0.$$

There are three linearly independent first-order solutions to these equations

$$q_x = x - \mathbf{i}w, \; q_y = y - \mathbf{j}w, \; q_z = z - \mathbf{k}w.$$

Higher-order homogeneous solutions are linear combinations of symmetrized products of q_x, q_y, q_z. For instance, the symmetrized product of degrees $2, 1, 0$ with respect to q_x, q_y, q_z is seen to be

$$q_x^2 q_y + q_x q_y q_x + q_y q_x^2 = 3(x^2 - w^2)y - 6wxy\mathbf{i} + (w^3 - 3wx^2)\mathbf{j}.$$

This already shows that the last alternative results in an interesting class of new functions, to some extent analogous to the class of holomorphic functions of a complex variable.

Historical survey

Hamilton invented his quaternions in 1843 when he tried to introduce a product for vectors in $\mathbb{R}^3$ similar to the product of complex numbers in $\mathbb{C}$. The present-day formalism of vector algebra was extracted out of the quaternion product of two vectors, $\mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$, by Gibbs in 1901.

Hamilton tried to find an algebraic system which would do for the space $\mathbb{R}^3$ the same thing as complex numbers do for the plane $\mathbb{R}^2$. In particular, Hamilton wanted to find a multiplication rule for triplets $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ so that $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$, that is, a multiplicative product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. However, no such bilinear products exist (at least not over the rationals), since $3 \times 21 = 63 \neq n_1^2 + n_2^2 + n_3^2$ for any integers n_1, n_2, n_3 though $3 = 1^2 + 1^2 + 1^2$ and $21 = 1^2 + 2^2 + 4^2$ (no integer of the form $4^a(8b+7)$, with $a \geq 0$, $b \geq 0$, is a sum of three squares, a result of Legendre in 1830).

Hamilton also tried to find a generalized complex number system in three dimensions. However, no such associative hypercomplex numbers exist in three dimensions. This can be seen by considering generalized imaginary units $\mathbf{i}$ and $\mathbf{j}$ such that $\mathbf{i}^2 = \mathbf{j}^2 = -1$, and such that $1, \mathbf{i}, \mathbf{j}$ span $\mathbb{R}^3$. [3] The product must be of the form $\mathbf{ij} = \alpha + \mathbf{i}\beta + \mathbf{j}\gamma$ for some real α, β, γ. Then

$$\begin{aligned} \mathbf{i}(\mathbf{ij}) &= \mathbf{i}\alpha - \beta + (\mathbf{ij})\gamma = \mathbf{i}\alpha - \beta + (\alpha + \mathbf{i}\beta + \mathbf{j}\gamma)\gamma \\ &= -\beta + \alpha\gamma + \mathbf{i}(\alpha + \beta\gamma) + \mathbf{j}\gamma^2, \end{aligned}$$

whereas by associativity $\mathbf{i}(\mathbf{ij}) = \mathbf{i}^2\mathbf{j} = -\mathbf{j}$ which leads to a contradiction since $\gamma^2 \geq 0$ for all real γ.

Hamilton's great idea was to go to four dimensions and consider elements of the form $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ where the hypercomplex units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy the following **non-commutative** multiplication rules

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \\ &\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \ \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \ \mathbf{ki} = \mathbf{j} = -\mathbf{ik}. \end{aligned}$$

3 Actually, it is not necessary to assume that $\mathbf{j}^2 = -1$. The computation shows that there is no embedding $\mathbb{C} \subset \mathbb{R}^3$, where $\mathbb{R}^3$ is an associative algebra.

Hamilton named his four-component elements **quaternions**. Quaternions form a division ring which we have denoted by $\mathbb{H}$ in honor of Hamilton.

Cayley in 1845 was the first one to publish the quaternionic representation of rotations of $\mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{r} \to a\mathbf{r}a^{-1}$, but he mentioned that the result was known to Hamilton. Cayley, in 1855, also discovered the quaternionic representation of 4-dimensional rotations:

$$\mathbb{R}^4 \to \mathbb{R}^4, \; q \to aqb^{-1},$$

where we have identified $\mathbb{R}^4 = \mathbb{H}$.

The differential operator $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$ is due to Hamilton, although his symbol for nabla was turned $30°$. The first one to study solutions of

$$\frac{\partial f}{\partial w} + \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = 0, \quad \text{where} \quad f : \mathbb{H} \to \mathbb{H},$$

was Fueter 1935.

Comment

The quaternion formalism might seem awkward to a physicist or an engineer, for two reasons: first, the squares of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are negative, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, and second, one invokes a 4-dimensional space which is beyond our ability of visualization.

Exercises

1. Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^3$, $|\mathbf{u}| = 1$. Show that $\mathbb{R}^3 \to \mathbb{R}^3$, $\mathbf{x} \to \mathbf{uxu}$ is a reflection across the plane $\mathbf{u}^\perp$.
2. Determine square roots of the quaternion $q = q_0 + \mathbf{q}$.
3. Hurwitz integral quaternions $q = w + ix + jy + kz$ are $\mathbb{Z}$-linear combinations of i, j, k and $\frac{1}{2}(1 + i + j + k)$, that is, either all w, x, y, z are integers or of the form $n + \frac{1}{2}$. Show that $|q|^2$ is an integer, and that the set

$$\{w + ix + jy + kx \mid w, x, y, z \in \mathbb{Z} \quad \text{or} \quad w, x, y, z \in \mathbb{Z} + \tfrac{1}{2}\}$$

is closed under multiplication.

4. Clearly, $ab = ba$ implies $e^a e^b = e^{a+b}$, but does $e^a e^b = e^{a+b}$ imply $ab = ba$?
5. Denote

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}.$$

Show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} a\bar{d}d - b\bar{d}c & c\bar{b}b - d\bar{b}a \\ b\bar{c}c - a\bar{c}d & d\bar{a}a - c\bar{a}b \end{pmatrix}^{-}$$

for a non-zero $\Delta = |a|^2|d|^2 + |b|^2|c|^2 - 2\,\mathrm{Re}(a\,\bar{c}\,d\,\bar{b})$.

6. Verify that only one of the matrices

$$a = \begin{pmatrix} 1 & i \\ j & k \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 & j \\ i & k \end{pmatrix}$$

is invertible.

7. Does an involutory automorphism of the real algebra $\mathrm{Mat}(2,\mathbb{H})$ necessarily send a diagonal matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{where} \quad a \in \mathbb{H}$$

to a diagonal matrix?

8. Suppose A $(\neq \mathbb{R})$ is a simple real associative algebra of dimension ≤ 4 with center $\mathbb{R}$. Show that A is $\mathbb{H}$ or $\mathrm{Mat}(2,\mathbb{R})$.
9. Suppose A $(\neq \mathbb{R})$ is a simple real associative algebra with center $\mathbb{R}$ and an anti-automorphism $x \to \alpha(x)$ such that $x + \alpha(x) \in \mathbb{R}$ and $x\alpha(x) \in \mathbb{R}$. Show that A is $\mathbb{H}$ or $\mathrm{Mat}(2,\mathbb{R})$.
10. Show that all the subgroups of $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ are normal, that is, for a subgroup $H \subset Q_8$ and elements $g \in Q_8$, $h \in H$, $ghg^{-1} \in H$.
11. Take two vectors $\mathbf{a}, \mathbf{b}$ in $\mathbb{R}^3$, such that $|\mathbf{a}| = |\mathbf{b}|$, and $a = e^{\mathbf{a}}$, $b = e^{\mathbf{b}}$ in S^3. Determine the point-wise invariant plane of the simple rotation $q \to aqb^{-1}$ of $\mathbb{R}^4$.

Solutions

2. If $q = 0$, then there is only one square root, 0. If $\mathbf{q} = 0$, $q_0 > 0$, then there are two square roots, $\pm\sqrt{q_0}$. If $\mathbf{q} = 0$, $q_0 < 0$, then there is an infinity of square roots, $\sqrt{-q_0}\,\mathbf{u}$, where $\mathbf{u}$ is a unit pure quaternion $\mathbf{u} \in \mathbb{R}^3 \subset \mathbb{H}$, $|\mathbf{u}| = 1$. If $\mathbf{q} \neq 0$, then there are two square roots,

$$\sqrt{\tfrac{1}{2}(|q| + q_0)} + \frac{\mathbf{q}}{|\mathbf{q}|}\sqrt{\tfrac{1}{2}(|q| - q_0)}$$

and its opposite.

4. Hint: consider the quaternions $a = 3\pi i$ and $b = 4\pi j$, or the matrices

$$a = 3\pi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b = 4\pi\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

6. a is invertible, but b is not.
11. If $a = b^{-1}$, the point-wise invariant plane is $\mathbf{a}^{\perp}$ in $\mathbb{R}^3$. Otherwise the point-wise invariant plane is spanned by $\mathbf{a} + \mathbf{b}$ and $|\mathbf{a}||\mathbf{b}| - \mathbf{ab} = |\mathbf{a}||\mathbf{b}| + \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \times \mathbf{b}$.

Bibliography

S.L. Altmann: *Rotations, Quaternions, and Double Groups*. Oxford University Press, Oxford, 1986.

G.M. Dixon: *Division Algebras: Octonions, Quaternions, Complex Numbers and the Algebraic Design of Physics*. Kluwer, Dordrecht, The Netherlands, 1994.

P. du Val: *Homographies, Quaternions and Rotations*. Oxford University Press, Oxford, 1964.

H.-D. Ebbinghaus et al. (eds.): *Numbers*. Springer, New York, 1991.

R. Fueter: Über die analytische Darstellung der regulären Funktionen einer Quaternionenvariablen. *Comment. Math. Helv.* **8** (1935), 371-378.

K. Gürlebeck, W. Sprössig: *Quaternionic Analysis and Elliptic Boundary Value Problems*. Akademie-Verlag, Berlin, 1989. Birkhäuser, Basel, 1990.

W.R. Hamilton: *Elements of Quaternions*. Longmans Green, London, 1866. Chelsea, New York, 1969.

T.L. Hankins: *Sir William Rowan Hamilton*. Johns Hopkins University Press, Baltimore, MD, 1980.

K. Imaeda: *Quaternionic Formulation of Classical Electrodynamics and the Theory of Functions of a Biquaternion Variable*. Department of Electronic Science, Okayama University of Science, 1983.

I.R. Porteous: *Topological Geometry*. Van Nostrand Reinhold, London, 1969. Cambridge University Press, Cambridge, 1981.

M. Riesz: *Clifford Numbers and Spinors*. The Institute for Fluid Dynamics and Applied Mathematics, Lecture Series No. **38**, University of Maryland, 1958. Reprinted as facsimile (eds.: E.F. Bolinder, P. Lounesto) by Kluwer, Dordrecht, The Netherlands, 1993.

A. Sudbery: Quaternionic Analysis. *Math. Proc. Cambridge Philos. Soc.* **85** (1979), 199-225.