16

Matrix Representations and Periodicity of 8

The Clifford algebra $\mathcal{C}\ell(Q)$ of a quadratic form Q on a linear space V over a field $\mathbb{F}$ contains an isometric copy of the vector space V. In this chapter we will temporarily forget this special feature of the Clifford algebra $\mathcal{C}\ell(Q)$. Then the Clifford algebra of a non-degenerate quadratic form is nothing but a matrix algebra or a direct sum of two matrix algebras. We have already identified the following Clifford algebras:

$$\begin{aligned}
&\mathcal{C}\ell_2 \simeq \mathrm{Mat}(2,\mathbb{R}), \quad \mathcal{C}\ell_{0,2} \simeq \mathbb{H}, \\
&\mathcal{C}\ell_3 \simeq \mathrm{Mat}(3,\mathbb{C}), \quad \mathcal{C}\ell_{0,3} \simeq \mathbb{H}\oplus\mathbb{H}, \\
&\mathcal{C}\ell_4 \simeq \mathrm{Mat}(2,\mathbb{H}), \quad \mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R}), \quad \mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(2,\mathbb{H}).
\end{aligned}$$

We will find a general pattern for matrix images of Clifford algebras $\mathcal{C}\ell_{p,q}$ of non-degenerate quadratic spaces $\mathbb{R}^{p,q}$. We will see that $\mathcal{C}\ell_{p,q}$ are isomorphic to real matrix algebras with entries in $\mathbb{R},\mathbb{C},\mathbb{H}$ or in ${}^2\mathbb{R}=\mathbb{R}\oplus\mathbb{R}$, ${}^2\mathbb{H}=\mathbb{H}\oplus\mathbb{H}$, that is, their matrix images are

$$\begin{aligned}
&\mathrm{Mat}(d,\mathbb{R}),\ \mathrm{Mat}(d,\mathbb{C}),\ \mathrm{Mat}(d,\mathbb{H}) \quad \text{or} \\
&{}^2\mathrm{Mat}(d,\mathbb{R}) = \mathrm{Mat}(d,{}^2\mathbb{R}), \quad {}^2\mathrm{Mat}(d,\mathbb{H}) = \mathrm{Mat}(d,{}^2\mathbb{H}).
\end{aligned}$$

Review of Matrix Images of $\mathcal{C}\ell_{p,q}$, $p+q<5$

The quadratic space $\mathbb{R}^{p,q}$ is an n-dimensional real vector space $\mathbb{R}^n$, $n=p+q$, with a non-degenerate symmetric *scalar product*

$$\mathbf{x}\cdot\mathbf{y} = x_1y_1+\ldots+x_py_p-x_{p+1}y_{p+1}-\ldots-x_{p+q}y_{p+q}.$$

The scalar product induces the *quadratic form*

$$\mathbf{x}\cdot\mathbf{x} = x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2.$$

A real associative algebra with unity 1 is the *Clifford algebra* $\mathcal{C}\ell_{p,q}$ on $\mathbb{R}^{p,q}$

if it contains $\mathbb{R}^{p,q}$ and $\mathbb{R} = \mathbb{R} \cdot 1 \not\subset \mathbb{R}^{p,q}$ as subspaces so that $\mathbb{R}^{p,q}$ generates $\mathcal{C}\ell_{p,q}$ as a real algebra and

$$\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^{p,q}$. Furthermore, we require that $\mathcal{C}\ell_{p,q}$ is not generated by any proper subspace of $\mathbb{R}^{p,q}$.

The identity $\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x}$ has a polarized form $\mathbf{xy} + \mathbf{yx} = 2\mathbf{x} \cdot \mathbf{y}$. In an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of $\mathbb{R}^{p,q}$ this means

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2g_{ij}$$

where $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ or $g_{ii} = 1$, $i \leq p$, $g_{ii} = -1$, $i > p$, and $g_{ij} = 0$, $i \neq j$. The above identity is a condensed form of the relations

$$\mathbf{e}_i^2 = 1,\ 1 \leq i \leq p, \quad \mathbf{e}_i^2 = -1,\ p < i \leq n, \quad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i,\ i < j.$$

The requirement that no proper subspace of $\mathbb{R}^{p,q}$ generates $\mathcal{C}\ell_{p,q}$ results in the constraint $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n \neq \pm 1$, needed only in the case $p - q = 1 \bmod 4$.

The Clifford algebra $\mathcal{C}\ell_{p,q}$, $p + q = n$, is of dimension 2^n. If the constraint $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n \neq \pm 1$ is omitted, then the resulting algebra could be of dimension 2^n or 2^{n-1}, the lower value being possible only if $p - q = 1 \bmod 4$. In the lower-dimensional case we have $\mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n = \pm 1$, the algebra itself being isomorphic to the two-sided ideal $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots n})\mathcal{C}\ell_{p,q}$. For instance, the negative definite quadratic space $\mathbb{R}^{0,3}$ has an 8-dimensional Clifford algebra $\mathcal{C}\ell_{0,3} \simeq \mathbb{H} \oplus \mathbb{H}$, which is a direct sum of two ideals $\frac{1}{2}(1 \pm \mathbf{e}_{123})\mathcal{C}\ell_{0,3}$, both isomorphic to the 4-dimensional quaternion algebra $\mathbb{H}$.

16.1 The Euclidean spaces $\mathbb{R}^n$

In the positive definite case, $p = n$, $q = 0$, of the Euclidean space we abbreviate $\mathbb{R}^{n,0}$ to $\mathbb{R}^n$ and its Clifford algebra $\mathcal{C}\ell_{n,0}$ to $\mathcal{C}\ell_n$. In the Euclidean case we can speak of the *length* $|\mathbf{x}|$ of a vector $\mathbf{x} \in \mathbb{R}^n$ given by $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$. [1]

The Euclidean plane $\mathbb{R}^2$. Consider the Euclidean plane $\mathbb{R}^2$. The Clifford algebra $\mathcal{C}\ell_2$ of $\mathbb{R}^2$ is generated by an orthonormal basis $\mathbf{e}_1, \mathbf{e}_2$ of $\mathbb{R}^2$. We have the multiplication rules

$$\begin{array}{lcl} \mathbf{e}_1^2 = 1,\ \mathbf{e}_2^2 = 1 & \text{corresponding to} & |\mathbf{e}_1| = 1,\ |\mathbf{e}_2| = 1 \\ \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 & & \mathbf{e}_1 \perp \mathbf{e}_2. \end{array}$$

Using $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$ and associativity we find $(\mathbf{e}_1\mathbf{e}_2)^2 = -\mathbf{e}_1^2\mathbf{e}_2^2$ which implies $(\mathbf{e}_1\mathbf{e}_2)^2 = -1$. This indicates that $\mathbf{e}_1\mathbf{e}_2$ is neither a scalar nor a vector, but a

1 In the negative definite case we can also speak of the length $|\mathbf{x}|$ of $\mathbf{x} \in \mathbb{R}^{0,n}$ given by $|\mathbf{x}|^2 = -\mathbf{x} \cdot \mathbf{x}$.

new kind of unit, called a *bivector*. The Clifford algebra $\mathcal{C}\ell_2$ is 4-dimensional with a basis consisting of

1	a scalar
$\mathbf{e}_1, \mathbf{e}_2$	vectors
$\mathbf{e}_1\mathbf{e}_2$	a bivector.

Write for short $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$. The Clifford algebra $\mathcal{C}\ell_2$ has the following multiplication table:

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

The Clifford algebra $\mathcal{C}\ell_2$ of the Euclidean plane $\mathbb{R}^2$ is isomorphic, as an associative algebra, to the matrix algebra of real 2×2-matrices $\mathrm{Mat}(2,\mathbb{R})$. This is seen by the correspondences

$\mathcal{C}\ell_2$	$\mathrm{Mat}(2,\mathbb{R})$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{e}_1, \mathbf{e}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{e}_{12}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

It should be emphasized that the Clifford algebra $\mathcal{C}\ell_2$ has more structure than the matrix algebra $\mathrm{Mat}(2,\mathbb{R})$. The Clifford algebra $\mathcal{C}\ell_2$ is the matrix algebra $\mathrm{Mat}(2,\mathbb{R})$ with a specific subspace singled out (and a quadratic form on that subspace making it isometric to the Euclidean plane $\mathbb{R}^2$). ∎

The 3-dimensional Euclidean space $\mathbb{R}^3$. Consider the 3-dimensional Euclidean space $\mathbb{R}^3$. The Clifford algebra $\mathcal{C}\ell_3$ is generated by an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$. This time there are three linearly independent bivectors $\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}$, each being a square root of -1. In addition, there is the volume element $\mathbf{e}_{123} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ which squares to -1 and commutes with all the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and thereby also with all the elements of the algebra $\mathcal{C}\ell_3$.

The Clifford algebra $\mathcal{C}\ell_3$ is 8-dimensional over $\mathbb{R}$ and has a basis consisting of

1	a scalar
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$	vectors
$\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{23}$	bivectors
$\mathbf{e}_{123}$	a volume element.

The Clifford algebra $\mathcal{C}\ell_3$ is isomorphic, as a real associative algebra, to the matrix algebra $\mathrm{Mat}(2,\mathbb{C})$ of 2×2-matrices with entries in $\mathbb{C}$. The isomorphism $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$ of real associative algebras is fixed by the correspondences

$$\mathbf{e}_1 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_2 \simeq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{e}_3 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices above are known as *Pauli spin matrices*. The multiplication of the unit vectors, $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_{123}$, results in the correspondence

$$\mathbf{e}_{123} \simeq \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

As noted above, the volume element $\mathbf{e}_{123}$, such that $\mathbf{e}_{123}^2 = -1$, commutes with all the elements of the algebra $\mathcal{C}\ell_3$; that is, it belongs to the center of $\mathcal{C}\ell_3$. This enables us to view $\mathcal{C}\ell_3$ as a complex algebra isomorphic, as an associative algebra, to the matrix algebra of complex 2×2-matrices $\mathrm{Mat}(2,\mathbb{C})$. ∎

The 4-dimensional Euclidean space $\mathbb{R}^4$. The Clifford algebra $\mathcal{C}\ell_4$ of the Euclidean space $\mathbb{R}^4$ is isomorphic, as an associative algebra, to the real algebra $\mathrm{Mat}(2,\mathbb{H})$ of 2×2-matrices with entries in the division ring of quaternions $\mathbb{H}$. Using an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ of $\mathbb{R}^4$ we can find the correspondences

$$\mathbf{e}_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}$$
$$\mathbf{e}_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Clifford algebra $\mathcal{C}\ell_4$ is of dimension 16 and has a basis consisting of

1	a scalar
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$	vectors
$\mathbf{e}_{12}, \mathbf{e}_{13}, \mathbf{e}_{14}, \mathbf{e}_{23}, \mathbf{e}_{24}, \mathbf{e}_{34}$	bivectors
$\mathbf{e}_{123}, \mathbf{e}_{124}, \mathbf{e}_{134}, \mathbf{e}_{234}$	3-vectors
$\mathbf{e}_{1234}$	a 4-volume element.

An arbitrary element $u = \langle u\rangle_0 + \langle u\rangle_1 + \langle u\rangle_2 + \langle u\rangle_3 + \langle u\rangle_4$ in $\mathcal{C}\ell_4$ is a sum of a scalar $\langle u\rangle_0$, a vector $\langle u\rangle_1$, a bivector $\langle u\rangle_2$, a 3-vector $\langle u\rangle_3$ and a volume element $\langle u\rangle_4$.

Split complex numbers $\mathbb{R}\oplus\mathbb{R}$. The Clifford algebra $\mathcal{C}\ell_1$ of the Euclidean line $\mathbb{R}^1=\mathbb{R}$ is spanned by $1, \mathbf{e}_1$ where $\mathbf{e}_1^2=1$. Its multiplication table is

	1	$\mathbf{e}_1$
1	1	$\mathbf{e}_1$
$\mathbf{e}_1$	$\mathbf{e}_1$	1

The Clifford algebra $\mathcal{C}\ell_1$ is isomorphic, as an associative algebra, to the double-field $\mathbb{R}\oplus\mathbb{R}$ of *split complex numbers*. The product of two elements (α_1,α_2) and (β_1,β_2) in $\mathbb{R}\oplus\mathbb{R}$ is defined component-wise:

$$(\alpha_1,\alpha_2)(\beta_1,\beta_2)=(\alpha_1\beta_1,\alpha_2\beta_2).$$

The isomorphism $\mathcal{C}\ell_1\simeq\mathbb{R}\oplus\mathbb{R}$ can be seen by the correspondences

$\mathcal{C}\ell_1$	$\mathbb{R}\oplus\mathbb{R}$
1	$(1,1)$
$\mathbf{e}_1$	$(1,-1)$

The Clifford algebra $\mathcal{C}\ell_1\simeq\mathbb{R}\oplus\mathbb{R}$ is a direct sum of two ideals spanned by the idempotents $\frac{1}{2}(1+\mathbf{e}_1)\simeq(1,0)$ and $\frac{1}{2}(1-\mathbf{e}_1)\simeq(0,1)$. ∎

The 5-dimensional Euclidean space $\mathbb{R}^5$. The Clifford algebra $\mathcal{C}\ell_5$ of $\mathbb{R}^5$ is isomorphic to ${}^2\mathrm{Mat}(2,\mathbb{H})=\mathrm{Mat}(2,{}^2\mathbb{H})$, as can be seen by the correspondences

$$\mathbf{e}_1=\begin{pmatrix}(0,0) & (\text{-}i,i)\\ (i,\text{-}i) & (0,0)\end{pmatrix},\ \mathbf{e}_2=\begin{pmatrix}(0,0) & (\text{-}j,j)\\ (j,\text{-}j) & (0,0)\end{pmatrix},\ \mathbf{e}_3=\begin{pmatrix}(0,0) & (\text{-}k,k)\\ (k,\text{-}k) & (0,0)\end{pmatrix}$$
$$\mathbf{e}_4=\begin{pmatrix}(1,\text{-}1) & (0,0)\\ (0,0) & (\text{-}1,1)\end{pmatrix},\ \mathbf{e}_5=\begin{pmatrix}(0,0) & (1,\text{-}1)\\ (1,\text{-}1) & (0,0)\end{pmatrix}.$$

The Clifford algebra $\mathcal{C}\ell_5$ has two central idempotents

$$\frac{1}{2}(1+\mathbf{e}_{12345})=\begin{pmatrix}(1,0) & (0,0)\\ (0,0) & (1,0)\end{pmatrix},\ \frac{1}{2}(1-\mathbf{e}_{12345})=\begin{pmatrix}(0,1) & (0,0)\\ (0,0) & (0,1)\end{pmatrix}$$

which both project out of $\mathcal{C}\ell_5$ an isomorphic copy of $\mathrm{Mat}(2,\mathbb{H})$, that is, $\frac{1}{2}(1\pm\mathbf{e}_{12345})\mathcal{C}\ell_5\simeq\mathrm{Mat}(2,\mathbb{H})$. An isomorphic copy of $\frac{1}{2}(1\pm\mathbf{e}_{12345})\mathcal{C}\ell_5$ is constructed within another subspace of $\mathcal{C}\ell_5$ in the following counter-example.

Counter-example. Consider the subspace of scalars, vectors and bivectors $\mathbb{R}\oplus\mathbb{R}^5\oplus\bigwedge^2\mathbb{R}^5$ of dimension $1+5+\frac{1}{2}5(5-1)=\frac{1}{2}2^5$. Introduce in this subspace a new product $u\circ v$ defined by (one of the following)

$$u\circ v=\langle uv(1\pm\mathbf{e}_{12345})\rangle_{0,1,2}$$

where $\langle w\rangle_{0,1,2} = \langle w\rangle_0 + \langle w\rangle_1 + \langle w\rangle_2$. This new product is associative and satisfies

$$\mathbf{x} \circ \mathbf{x} = |\mathbf{x}|^2 \quad \text{for} \quad \mathbf{x} \in \mathbb{R}^5.$$

However, $\mathbf{e}_1 \circ \mathbf{e}_2 \circ \mathbf{e}_3 \circ \mathbf{e}_4 \circ \mathbf{e}_5 = \pm 1$. As a sample, this new product satisfies

$$\mathbf{e}_1 \circ \mathbf{e}_2 = \mathbf{e}_{12},\ \mathbf{e}_1 \circ \mathbf{e}_{12} = \mathbf{e}_2,\ \mathbf{e}_{12} \circ \mathbf{e}_{12} = -1,\ \mathbf{e}_{12} \circ \mathbf{e}_{23} = \mathbf{e}_{13}$$
$$\mathbf{e}_1 \circ \mathbf{e}_{23} = \mp \mathbf{e}_{45},\ \mathbf{e}_{12} \circ \mathbf{e}_{34} = \pm \mathbf{e}_5.$$

The multiplication table of this new product is given by the following matrices

$$\mathbf{e}_1 = \pm\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{e}_2 = \pm\begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \mathbf{e}_3 = \pm\begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix},$$
$$\mathbf{e}_4 = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{e}_5 = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This serves as a counter-example to a belief that the Clifford algebra would be uniquely generated by its subspaces $\mathbb{R}$ and $\mathbb{R}^n$. ∎

The 3-dimensional anti-Euclidean space $\mathbb{R}^{0,3}$

The anti-Euclidean space $\mathbb{R}^{0,3}$ has a negative definite quadratic form sending a vector $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ to the scalar

$$\mathbf{x} \cdot \mathbf{x} = -(x_1^2 + x_2^2 + x_3^2).$$

An orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^{0,3}$ obeys the multiplication rules

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1 \quad \text{and}$$
$$\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1,\ \mathbf{e}_1\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_1,\ \mathbf{e}_2\mathbf{e}_3 = -\mathbf{e}_3\mathbf{e}_2.$$

These relations are satisfied by the unit quaternions

$$i = \mathbf{e}_1, \quad j = \mathbf{e}_2, \quad k = \mathbf{e}_3$$

in $\mathbb{H}$. The rule $ijk = -1$, or $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1$, means that the real algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^{0,3}$ is generated by a proper subspace $\mathbb{R}^{0,2}$ of $\mathbb{R}^{0,3}$. In other words, each quaternion can be expressed in the form $x = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_1\mathbf{e}_2$ where $\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2$. This matter is expressed by saying that $\mathbb{H}$ is an *algebra of the quadratic form*

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \to -(x_1^2 + x_2^2 + x_3^2)$$

although it is not the Clifford algebra $\mathcal{C}\ell_{0,3}$. The 8-dimensional Clifford algebra $\mathcal{C}\ell_{0,3}$ is isomorphic, as an associative algebra, to the direct sum $\mathbb{H} \oplus \mathbb{H}$. This can be seen by the correspondences

$C\ell_{0,3}$	$\mathbb{H} \oplus \mathbb{H}$
1	$(1,1)$
$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$	$(i,-i),\ (j,-j),\ (k,-k)$
$\mathbf{e}_{23},\ \mathbf{e}_{31},\ \mathbf{e}_{12}$	$(i,i),\ (j,j),\ (k,k)$
$\mathbf{e}_{123}$	$(-1,1)$

The Clifford algebra $C\ell_{0,3}$ of $\mathbb{R}^{0,3}$ is the **universal object** in the category of *algebras of the quadratic form*

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \to -(x_1^2 + x_2^2 + x_3^2)$$

or for short in the *category of quadratic algebras.* [2] If there are other objects in this category, they are quotients of the universal object with respect to a two-sided ideal. This gives us two other algebras of dimension 4; in one of them we have the relation $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = 1$ and in the other $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1$. These two algebras of dimension 4 are linearly isomorphic to $\mathbb{R} \oplus \mathbb{R}^{0,3}$. In the category of quadratic algebras these two algebras of dimension 4 are *not isomorphic* with each other, which means that the relations $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = 1$ and $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = -1$ prevent the identity mapping on $\mathbb{R}^{0,3}$ being extended to an isomorphism in this category. However, in the category of all associative algebras these two algebras of dimension 4 are isomorphic with each other (and with the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^{0,3}$). The isomorphism can be seen by the mappings

$$\mathbf{e}_1 \to \mathbf{e}_1,\ \mathbf{e}_2 \to \mathbf{e}_2 \quad \text{and} \quad \mathbf{e}_3 \to -\mathbf{e}_3.$$

∎

16.2 Indefinite metrics $\mathbb{R}^{p,q}$

The hyperbolic plane $\mathbb{R}^{1,1}$. The hyperbolic plane is the linear space $\mathbb{R}^2$ endowed with a quadratic form

$$(u,v) \to uv$$

which by change of variables $u = x_1 + x_2$, $v = x_1 - x_2$ is seen to be

$$(x_1, x_2) \to x_1^2 - x_2^2.$$

Thus the hyperbolic plane is indefinite, neutral and has the Lorentz signature $\mathbb{R}^{1,1}$. The Clifford algebra $C\ell_{1,1}$ of $\mathbb{R}^{1,1}$ is isomorphic, as an associative algebra,

2 The term quadratic algebra is customarily used for something else: in a quadratic algebra x^2 is linearly dependent on x and 1.

to the matrix algebra $\mathrm{Mat}(2,\mathbb{R})$ by the correspondences

$\mathcal{C}\ell_{1,1}$	$\mathrm{Mat}(2,\mathbb{R})$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbf{e}_1,\ \mathbf{e}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$\mathbf{e}_{12}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Note that the Clifford algebras $\mathcal{C}\ell_{1,1}$ and $\mathcal{C}\ell_2 \simeq \mathrm{Mat}(2,\mathbb{R})$ are isomorphic *as associative algebras* but non-isomorphic *as quadratic algebras.* ∎

The Minkowski space-time $\mathbb{R}^{3,1}$. The elements of an orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4\}$ of $\mathbb{R}^{3,1}$ anticommute, $\mathbf{e}_\mu\mathbf{e}_\nu = -\mathbf{e}_\nu\mathbf{e}_\mu$, and have unit squares, $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$, $\mathbf{e}_4^2 = -1$. The basis vectors are often given the following representation by complex 4×4-matrices:

$$\mathbf{e}_k = \mathbf{e}^k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \text{ for } k=1,2,3 \text{ and } \mathbf{e}_4 = -\mathbf{e}^4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where we recognize the 2×2 Pauli spin matrices $\sigma_1,\sigma_2,\sigma_3$. It is possible to represent $\mathcal{C}\ell_{3,1}$ by real matrices as follows:

$$\mathbf{e}_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix},$$

$$\mathbf{e}_4 = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}.$$

This implies $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$. ∎

The Minkowski time-space $\mathbb{R}^{1,3}$. In the signature $\mathbb{R}^{1,3}$ one usually gives the following representation by complex 4×4-matrices:

$$\gamma_0 = \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and}$$

$$\gamma_k = -\gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \text{for } k = 1,2,3.$$

In addition to the above matrix representation one can represent the Clifford algebra $\mathcal{C}\ell_{1,3}$ by the following 2×2-matrices with quaternion entries:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

Since the Clifford algebra $\mathcal{C}\ell_{1,3}$ and the matrix algebra $\mathrm{Mat}(2, \mathbb{H})$ of 2×2-matrices with entries in $\mathbb{H}$ are both real algebras of dimension 16, the above correspondences establish an isomorphism of associative algebras, that is, $\mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(2, \mathbb{H})$.

A short look at physics: A vector $u = u_0\gamma^0 + u_1\gamma^1 + u_2\gamma^2 + u_3\gamma^3$ with square $u^2 = u_0^2 - u_1^2 - u_2^2 - u_3^2$ can be time-like $u^2 > 0$, null $u^2 = 0$, or space-like $u^2 < 0$. A time-like vector or non-zero null vector can be future oriented $u_0 > 0$ or past oriented $u_0 < 0$. A time-like future oriented unit vector u, $u^2 = 1$, gives the velocity $v < c$ of a real particle by

$$u_0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

▮

Physicists might want to observe that the Clifford algebras $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4, \mathbb{R})$ and $\mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(2, \mathbb{H})$ are *not isomorphic* as associative algebras, even though both of them have the same complexification $\mathrm{Mat}(4, \mathbb{C})$ with the same complex structure but with different real structures (= different real subalgebras). The complexified Clifford algebras $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{3,1}$ have a 4-dimensional irreducible left ideal (8-dimensional real subspace). As a graded left ideal this ideal is also irreducible. The real algebra $\mathcal{C}\ell_{1,3}$ has an 8-dimensional irreducible left ideal, which is also graded. However, the real algebra $\mathcal{C}\ell_{3,1}$ has a 4-dimensional irreducible ideal, which is not graded (that is $\mathcal{C}\ell_{3,1}$ does not have primitive idempotents sitting in $\mathcal{C}\ell_{3,1}^+$), and an 8-dimensional irreducible graded ideal.

The Table of Clifford Algebras

The Clifford algebra $\mathcal{C}\ell_{p,q}$, where $p - q \neq 1 \bmod 4$, is a simple algebra of dimension 2^n, where $n = p + q$, and therefore isomorphic with a full matrix algebra with entries in $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The Clifford algebra $\mathcal{C}\ell_{p,q}$, where $p - q = 1 \bmod 4$, is a semi-simple algebra of dimension 2^n so that the two central idempotents $\frac{1}{2}(1 \pm \mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n)$ project out two copies of a full matrix algebra with entries in $\mathbb{R}$ or $\mathbb{H}$. To put it slightly differently, the Clifford algebra $\mathcal{C}\ell_{p,q}$

has a faithful representation as a matrix algebra with entries in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{H} \oplus \mathbb{H}$. In the rings ${}^2\mathbb{R} = \mathbb{R} \oplus \mathbb{R}$ and ${}^2\mathbb{H} = \mathbb{H} \oplus \mathbb{H}$ the multiplication is defined component-wise:

$$(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2).$$

16.3 Matrix representation $\mathcal{C}\ell_{p+1,q+1} \simeq \mathrm{Mat}(2, \mathcal{C}\ell_{p,q})$

Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbb{R}^{p,q}$, $n = p + q$, generating the Clifford algebra $\mathcal{C}\ell_{p,q}$. The 2×2-matrices

$$\begin{pmatrix} \mathbf{e}_i & 0 \\ 0 & -\mathbf{e}_i \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, n, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

anticommute and generate the Clifford algebra $\mathcal{C}\ell_{p+1,q+1}$. In other words, the Clifford algebra $\mathcal{C}\ell_{p+1,q+1}$ is isomorphic, as an associative algebra, to the algebra of 2×2-matrices with entries in the Clifford algebra $\mathcal{C}\ell_{p,q}$. This can be condensed by writing $\mathcal{C}\ell_{p+1,q+1} \simeq \mathrm{Mat}(2, \mathcal{C}\ell_{p,q})$.

Examples. Recall that $\mathcal{C}\ell_1 \simeq {}^2\mathbb{R} = \mathbb{R} \oplus \mathbb{R}$ by setting $\mathbf{e}_1 \simeq (1, -1)$. This implies the isomorphism $\mathcal{C}\ell_{2,1} \simeq {}^2\mathrm{Mat}(2, \mathbb{R})$. Recall that $\mathcal{C}\ell_{0,3} \simeq {}^2\mathbb{H} = \mathbb{H} \oplus \mathbb{H}$, which implies $\mathcal{C}\ell_{1,4} \simeq {}^2\mathrm{Mat}(2, \mathbb{H})$. Recall that $\mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(2, \mathbb{H})$ which implies $\mathcal{C}\ell_{2,4} \simeq \mathrm{Mat}(4, \mathbb{H})$. ∎

Supplement an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^{p,q}$ with two more anticommuting basis vectors $\mathbf{e}_+$ and $\mathbf{e}_-$ such that $\mathbf{e}_+^2 = 1$ and $\mathbf{e}_-^2 = -1$ to form an orthonormal basis of $\mathbb{R}^{p+1,q+1}$. The generators $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \mathbf{e}_+, \mathbf{e}_-$ of $\mathcal{C}\ell_{p+1,q+1}$ correspond to the generators

$$\mathbf{e}_i \simeq \begin{pmatrix} \mathbf{e}_i & 0 \\ 0 & -\mathbf{e}_i \end{pmatrix} \quad \text{for } i = 1, 2, \ldots, n,$$

$$\mathbf{e}_+ \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{e}_- \simeq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathrm{Mat}(2, \mathcal{C}\ell_{p,q})$, so that the element $a \in \mathcal{C}\ell_{p,q}$ is represented by a matrix

$$a \simeq \begin{pmatrix} a & 0 \\ 0 & \hat{a} \end{pmatrix}$$

where the hat means the grade involution $\hat{a} = a_0 - a_1$ with $a_0 = \mathrm{even}(a)$ and $a_1 = \mathrm{odd}(a)$. There is another possibility to embed $\mathcal{C}\ell_{p,q}$ into $\mathrm{Mat}(2, \mathcal{C}\ell_{p,q})$, so that $a \in \mathcal{C}\ell_{p,q}$ is represented by

$$a' = a_0 + a_1 \mathbf{e}_+ \mathbf{e}_- \simeq \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

which is just a multiple of the identity matrix. Since $a' = a_0 + a_1 \mathbf{e}_+\mathbf{e}_-$ commutes with

$$\tfrac{1}{2}(1+\mathbf{e}_+\mathbf{e}_-) \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \tfrac{1}{2}(\mathbf{e}_+ - \mathbf{e}_-) \simeq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\tfrac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-) \simeq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tfrac{1}{2}(1-\mathbf{e}_+\mathbf{e}_-) \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

we have the correspondence $\mathrm{Mat}(2, C\ell_{p,q}) \simeq C\ell_{p+1,q+1}$, given by

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&\simeq a'\tfrac{1}{2}(1+\mathbf{e}_+\mathbf{e}_-) + b'\tfrac{1}{2}(\mathbf{e}_+ - \mathbf{e}_-) + c'\tfrac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-) + d'\tfrac{1}{2}(1-\mathbf{e}_+\mathbf{e}_-) \\
&= a\tfrac{1}{2}(1+\mathbf{e}_+\mathbf{e}_-) + b\tfrac{1}{2}(\mathbf{e}_+ - \mathbf{e}_-) + \hat{c}\tfrac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-) + \hat{d}\tfrac{1}{2}(1-\mathbf{e}_+\mathbf{e}_-) \\
&= \tfrac{1}{2}(1+\mathbf{e}_+\mathbf{e}_-)a + \tfrac{1}{2}(\mathbf{e}_+ - \mathbf{e}_-)\hat{b} + \tfrac{1}{2}(\mathbf{e}_+ + \mathbf{e}_-)c + \tfrac{1}{2}(1-\mathbf{e}_+\mathbf{e}_-)\hat{d}.
\end{aligned}$$

To put all this in another way: The Clifford algebra $C\ell_{p+1,q+1}$ contains an isomorphic copy of $C\ell_{p,q}$ generated by the elements $\mathbf{e}'_i = \mathbf{e}_i\mathbf{e}_+\mathbf{e}_-$, where $i = 1, 2, \ldots, n = p+q$, in such a way that each element of $C\ell_{p,q}$ commutes with every element of a copy of $C\ell_{1,1}$ generated by $\mathbf{e}_+$ and $\mathbf{e}_-$, and further that $C\ell_{p,q}$ and $C\ell_{1,1}$ together generate all of $C\ell_{p+1,q+1}$. These considerations can be condensed by writing

$$\boxed{C\ell_{p,q} \otimes C\ell_{1,1} \simeq C\ell_{p+1,q+1}}$$

where $C\ell_{1,1} \simeq \mathrm{Mat}(2, \mathbb{R})$.

Symmetry $C\ell_{p,q} \simeq C\ell_{q+1,p-1}$. Take an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^{p,q}$ where $p \geq 1$ and set

$$\mathbf{e}'_1 = \mathbf{e}_1 \quad \text{and} \quad \mathbf{e}'_i = \mathbf{e}_i\mathbf{e}_1 \quad \text{for} \quad i > 1.$$

The elements $\mathbf{e}'_i$ where $i = 1, 2, \ldots, n$ anticommute with each other so that $\mathbf{e}'^2_1 = \mathbf{e}^2_1$ and $\mathbf{e}'^2_i = -\mathbf{e}^2_i$ for $i > 1$. Therefore, the subset $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ of $C\ell_{p,q}$ is a generating set for $C\ell_{q+1,p-1}$. This proves the isomorphism

$$\boxed{C\ell_{p,q} \simeq C\ell_{q+1,p-1}}$$

when $p \geq 1$.

Examples. Recall that $C\ell_3 \simeq \mathrm{Mat}(2, \mathbb{C})$, which by symmetry implies $C\ell_{1,2} \simeq \mathrm{Mat}(2, \mathbb{C})$. Recall that $C\ell_{3,1} \simeq \mathrm{Mat}(4, \mathbb{R})$, which implies $C\ell_{2,2} \simeq \mathrm{Mat}(4, \mathbb{R})$. From $C\ell_{0,4} \simeq \mathrm{Mat}(2, \mathbb{H})$ we can first deduce $C\ell_{1,5} \simeq \mathrm{Mat}(4, \mathbb{H})$ (by adding a hyperbolic plane) which implies $C\ell_6 \simeq \mathrm{Mat}(4, \mathbb{H})$. ∎

16.4 Periodicity of 8

Table 1, of Clifford algebras, contains or continues with two kinds of periodicities of 8, namely for algebras of the same dimension $\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p-4,q+4}$ where $p \geq 4$, and for algebras of different dimension $\mathcal{C}\ell_{p+8,q} \simeq \mathrm{Mat}(16, \mathcal{C}\ell_{p,q})$. Let us first prove $\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p-4,q+4}$ where $p \geq 4$. Take an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^{p,q}$ and set

$$\begin{aligned} \mathbf{e}'_i &= \mathbf{e}_i \mathbf{h} \qquad \text{for} \quad i = 1,2,3,4, \\ \mathbf{e}'_i &= \mathbf{e}_i \qquad \text{for} \quad i > 4, \end{aligned}$$

where $\mathbf{h} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$. Then the subset $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ of $\mathcal{C}\ell_{p,q}$ is a generating set for $\mathcal{C}\ell_{p-4,q+4}$, which implies the isomorphism

$$\mathcal{C}\ell_{p,q} \simeq \mathcal{C}\ell_{p-4,q+4}$$

where $p \geq 4$. These isomorphisms are due to Cartan 1908 p. 464.

Examples. Recall that $\mathcal{C}\ell_6 \simeq \mathrm{Mat}(4, \mathbb{H})$, which implies $\mathcal{C}\ell_{2,4} \simeq \mathrm{Mat}(4, \mathbb{H})$. From $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2, \mathbb{C})$ deduce first $\mathcal{C}\ell_{4,1} \simeq \mathrm{Mat}(4, \mathbb{C})$, which implies $\mathcal{C}\ell_{0,5} \simeq \mathrm{Mat}(4, \mathbb{C})$. From $\mathcal{C}\ell_{2,2} \simeq \mathrm{Mat}(4, \mathbb{R})$ we first deduce $\mathcal{C}\ell_{3,3} \simeq \mathrm{Mat}(8, \mathbb{R})$; then by $\mathcal{C}\ell_{3,3} \simeq \mathcal{C}\ell_{4,2}$ and $\mathcal{C}\ell_{4,2} \simeq \mathcal{C}\ell_{0,6}$ we find $\mathcal{C}\ell_{0,6} \simeq \mathrm{Mat}(8, \mathbb{R})$. From $\mathcal{C}\ell_{3,3} \simeq \mathrm{Mat}(8, \mathbb{R})$ we find $\mathcal{C}\ell_{4,4} \simeq \mathrm{Mat}(16, \mathbb{R})$ and also $\mathcal{C}\ell_8 \simeq \mathrm{Mat}(16, \mathbb{R})$ and $\mathcal{C}\ell_{0,8} \simeq \mathrm{Mat}(16, \mathbb{R})$. ∎

Next, prove $\mathcal{C}\ell_{p+8,q} \simeq \mathrm{Mat}(16, \mathcal{C}\ell_{p,q})$ by showing that $\mathcal{C}\ell_{p,q+8} \simeq \mathrm{Mat}(16, \mathcal{C}\ell_{p,q})$. Take an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1}, \ldots, \mathbf{e}_{n+8}\}$ of $\mathbb{R}^{p,q+8}$ where $n = p + q$ and set

$$\mathbf{e}'_i = \mathbf{e}_i \mathbf{e}_{n+1} \ldots \mathbf{e}_{n+8} \quad \text{for} \quad i = 1, 2, \ldots, n = p + q.$$

Then the subset $\{\mathbf{e}'_1, \mathbf{e}'_2, \ldots, \mathbf{e}'_n\}$ of $\mathcal{C}\ell_{p,q+8}$ generates a subalgebra isomorphic to $\mathcal{C}\ell_{p,q}$. The subalgebra generated by $\mathbf{e}_{n+1}, \ldots, \mathbf{e}_{n+8}$ is isomorphic to $\mathcal{C}\ell_{0,8} \simeq \mathrm{Mat}(16, \mathbb{R})$. These two subalgebras commute with each other element-wise and generate all of $\mathcal{C}\ell_{p,q+8}$, which shows that

$$\boxed{\mathcal{C}\ell_{p,q+8} \simeq \mathcal{C}\ell_{p,q} \otimes \mathrm{Mat}(16, \mathbb{R}) \simeq \mathrm{Mat}(16, \mathcal{C}\ell_{p,q})}$$

Similarly, $\mathcal{C}\ell_{p+8,q} \simeq \mathcal{C}\ell_{p,q} \otimes \mathrm{Mat}(16, \mathbb{R}) \simeq \mathrm{Mat}(16, \mathcal{C}\ell_{p,q})$. These isomorphisms are due to Cartan 1908.

Example. Note that $\mathcal{C}\ell_{0,1} \simeq \mathbb{C}$ which implies $\mathcal{C}\ell_{8,1} \simeq \mathrm{Mat}(16, \mathbb{C})$. Recall that $\mathcal{C}\ell_{1,1} \simeq \mathrm{Mat}(2, \mathbb{R})$ and so $\mathcal{C}\ell_{1,9} \simeq \mathrm{Mat}(32, \mathbb{R})$. ∎

Table 1. Clifford Algebras $C\ell_{p,q}$, $p+q<8$.

$p+q$ \ $p-q$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
0								$\mathbb{R}$							
1							$\mathbb{C}$		$^2\mathbb{R}$						
2						$\mathbb{H}$		$\mathbb{R}(2)$		$\mathbb{R}(2)$					
3					$^2\mathbb{H}$		$\mathbb{C}(2)$		$^2\mathbb{R}(2)$		$\mathbb{C}(2)$				
4				$\mathbb{H}(2)$		$\mathbb{H}(2)$		$\mathbb{R}(4)$		$\mathbb{R}(4)$		$\mathbb{H}(2)$			
5			$\mathbb{C}(4)$		$^2\mathbb{H}(2)$		$\mathbb{C}(4)$		$^2\mathbb{R}(4)$		$\mathbb{C}(4)$		$^2\mathbb{H}(2)$		
6		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$		$\mathbb{R}(8)$		$\mathbb{R}(8)$		$\mathbb{H}(4)$		$\mathbb{H}(4)$	
7	$^2\mathbb{R}(8)$		$\mathbb{C}(8)$		$^2\mathbb{H}(4)$		$\mathbb{C}(8)$		$^2\mathbb{R}(8)$		$\mathbb{C}(8)$		$^2\mathbb{H}(4)$		$\mathbb{C}(8)$

$\mathbb{A}(d)$ means the real algebra of $d \times d$-matrices $\mathrm{Mat}(d, \mathbb{A})$ with entries in the ring $\mathbb{A} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $^2\mathbb{R}$, $^2\mathbb{H}$.

16.5 Complex Clifford algebras and their periodicity of 2

Complex quadratic spaces $\mathbb{C}^n$ have quadratic forms

$$z_1^2 + z_2^2 + \ldots + z_n^2.$$

The type of their Clifford algebras $C\ell(\mathbb{C}^n)$ depends only on the parity of n. Denote $\ell = \lfloor n \rfloor$. In even dimensions $C\ell(\mathbb{C}^n) \simeq \mathrm{Mat}(2^\ell, \mathbb{C})$ and in odd dimensions $C\ell(\mathbb{C}^n) \simeq {}^2\mathrm{Mat}(2^\ell, \mathbb{C})$.

Table 2. Complex Clifford Algebras $C\ell(\mathbb{C}^n)$, $n<8$.

n	
0	$\mathbb{C}$
1	$^2\mathbb{C}$
2	$\mathbb{C}(2)$
3	$^2\mathbb{C}(2)$
4	$\mathbb{C}(4)$
5	$^2\mathbb{C}(4)$
6	$\mathbb{C}(8)$
7	$^2\mathbb{C}(8)$

Exercises

1. Show that $\mathcal{C}\ell_{p,q}^{+} \simeq \mathcal{C}\ell_{p,q-1}$ and $\mathcal{C}\ell_{n}^{+} = \mathcal{C}\ell_{n,0}^{+} \simeq \mathcal{C}\ell_{0,n-1}$.
2. Show that all the algebra isomorphisms presented in this chapter are special cases of the following:

$$\mathcal{C}\ell(V_1 \oplus V_2, Q_1 \perp Q_2) \simeq \mathcal{C}\ell(V_1, \lambda Q_1) \otimes \mathcal{C}\ell(V_2, Q_2),$$
$$(\mathbf{x}, \mathbf{y}) \to \mathbf{x} \otimes \omega + 1 \otimes \mathbf{y},$$

where Q_2 is non-degenerate and V_2 is even-dimensional, $\dim V_2 = 2k$, $\omega \in \bigwedge^{2k} V_2$, $\omega^2 = \lambda \in \mathbb{R} \setminus \{0\}$.

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