The Euler-Lagrange Eq.

Now let us establish the Euler-Lagrange equation for (P) more explicitely. Note that

$$\nabla_{\dot{\xi}} \left(G\dot{\xi}\cdot\dot{\xi} + \sum_{i\in\mathbb{N}_6} \mu_i K_i(\xi,\dot{\xi},t) \right)$$

$$= 2G\dot{\xi} + \sum_{i\in\mathbb{N}_6} \mu_i \nabla_{\dot{\xi}} K_i(\xi,\dot{\xi},t)$$

$$= 2G\dot{\xi} - \Omega(\underline{\mu})\xi.$$

Similarly, we find that

$$\nabla_{\xi}\left(G\dot{\xi}\cdot\dot{\xi} + \sum_{i\in\mathbb{N}_6} \mu_i K_i(\xi,\dot{\xi},t)\right)$$

$$= \Omega(\underline{\mu})\dot{\xi}.$$

Hence, after rescaling the μ, the Euler-Lagrange equation reads

$$\text{(EL)} \quad G\ddot{\xi} - \Omega(\underline{\mu})\dot{\xi} = 0.$$

Integrating once and using the fact that ξ is of mean zero, we have

$$(EL') \quad G\dot{\xi} - \Omega(\mu)\xi = 0,$$

and finally by setting $\eta := G^{1/2}\xi$ we find

$$(EL'') \quad \dot{\eta} - \tilde{\Omega}(\mu)\eta = 0,$$

where $\tilde{\Omega}(\mu) := \sum_{i \in \mathbb{N}_6} \mu_i G^{-1/2} M_i G^{-1/2}$.

So the solution of (EL'') is simply given by $\eta(t) = \exp(\tilde{\Omega}(\mu)t)\eta_0$ where $\eta_0 := \eta(0)$.

Note that $\tilde{\Omega}(\mu) \in \mathrm{Skew}_4(\mathbb{R})$. Hence, we find $Q \in \mathcal{O}(4)$ (c.f. previous section) such that $\tilde{\Omega}(\mu) = Q\tilde{\Sigma}(\mu)Q^T$ with

$$\tilde{\Sigma}(\mu) = \begin{pmatrix} 0 & \sigma_1(\mu) & 0 & 0 \\ -\sigma_1(\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2(\mu) \\ 0 & 0 & -\sigma_2(\mu) & 0 \end{pmatrix}$$

Setting $\phi := Q\eta$ yields $\phi(t) = \exp(\tilde{\Sigma}(\mu)t)\phi_0$ with $\phi_0 := Q\eta_0$.

A straightforward computation shows that

$$\phi(t) = \cos(\sigma_1(\mu)t)\begin{pmatrix}\phi_{0,1}\\ \phi_{0,2}\\ 0\\ 0\end{pmatrix} + \sin(\sigma_1(\mu)t)\begin{pmatrix}\phi_{0,2}\\ -\phi_{0,1}\\ 0\\ 0\end{pmatrix}$$

$$+ \cos(\sigma_2(\mu)t)\begin{pmatrix}0\\ 0\\ \phi_{0,3}\\ \phi_{0,4}\end{pmatrix} + \sin(\sigma_2(\mu)t)\begin{pmatrix}0\\ 0\\ -\phi_{0,4}\\ \phi_{0,3}\end{pmatrix}.$$

- Resubstituting the basis transformations, ~~which all were orthogonal~~, we find that a solution ξ of (EL) must be of the form

$$\xi(t) = \cos(\sigma_1(\mu)t)\, a + \sin(\sigma_2(\mu)t)\, a'$$

$$+ \cos(\sigma_2(\mu)t) \quad + \sin(\sigma_2(\mu)t) \,{}'$$

- such that $\mathrm{span}(a, a') \oplus \mathrm{span}(\;,\;') = \mathbb{R}^4$.

Now it remains to show that we can indeed find $\mu \in \mathbb{R}^6$ such that $\sigma_1(\mu), \sigma_2(\mu) \in \mathbb{Z}$ to satisfy the periodicity assumption. Subsequently it suffices to show that we find $\sigma_1(\mu) = 1$ and $\sigma_2(\mu) = 2$ up to permutation and that this is the energy optimal choice.

However, it is important to note that we cannot have $\sigma_1(\mu) = \sigma_2(\mu)$ since this would imply that δ_p is simple, c.f. the relationship between δ_p and the Fourier coefficients.