Chapter 6

Transient Stokes Flows

6.1 Time Scales

Throughout Part II, up to this point, we have restricted our attention to steady flow. Strictly speaking, we do not have a steady problem since the moving particle naturally changes the problem geomery. This is especially true for the multiparticle geometries of Part III. However, our analysis made the implicit assumption that these unsteady problems could be tackled as a sequence of steady-state problems consisting of a series of "snap shots" of the flow. Each frame yields a solution for the particle velocity, the geometry is updated by moving the particle by an incremental amount over the small time step Δt, and then the whole process is repeated. (We are focusing, for the moment, on deterministic problems, such as trajectory analysis.) In this chapter we test the validity of this approximate procedure more closely by analyzing the leading corrections due to time-dependent flow.

In Chapter 1 we showed that the importance of the Eulerian acceleration term, $\rho\partial \boldsymbol{v}/\partial t$, depended on the time scale of the phenomena of interest. When this time scale is of order U/ℓ, the acceleration term is $O(\mathrm{Re})$ smaller than the viscous terms. On the other hand, when the phenomena of interest occur over the much faster time scale, ℓ^2/ν, that it takes vorticity to diffuse over a length ℓ, the acceleration term is of the same order as the viscous terms. The time-dependent flows encountered in this chapter will illustrate this, as well as other interesting concepts of transient low Reynolds number flow.

The formal derivation *via* the dimensional analysis proceeds as follows. We pick velocity, length, and time scales V, ℓ, and τ so that the Navier–Stokes equations may be rendered into a dimensionless form:

$$\mathrm{Re}\,\mathrm{Sl}\frac{\partial \boldsymbol{v}}{\partial t} + \mathrm{Re}\,\boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla p + \nabla^2\boldsymbol{v}\ .$$

Now if the Reynolds number and the Strouhal number satisfy

$$\mathrm{Re} = \frac{U\ell}{\nu} \to 0$$

$$\mathrm{Sl} = \frac{\ell}{U\tau} \to \infty\ ,$$

but with $\mathrm{Re\,Sl} = \ell^2/\nu\tau = O(1)$, then all terms except the nonlinear term will be $O(1)$. The nonlinear term will be $O(\mathrm{Re})$ smaller than the rest. If we retain only those $O(1)$ dominant terms, we obtain, in dimensional form, the time-dependent Stokes equations:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p + \mu\nabla^2\boldsymbol{v}\ , \qquad \nabla\cdot\boldsymbol{v} = 0\ .$$

When $\boldsymbol{v} = \widehat{\boldsymbol{v}}\exp(-i\omega t)$ and $p = \widehat{p}\exp(-i\omega t)$ are inserted in these equations, we obtain the Fourier transformed equations,

$$-\nabla\widehat{p} + \mu\nabla^2\widehat{\boldsymbol{v}} - \mu\alpha^2\widehat{\boldsymbol{v}} = \mathbf{0}\ , \qquad \nabla\cdot\widehat{\boldsymbol{v}} = 0\ , \tag{6.1}$$

where $\alpha^2 = -i\omega/\nu$. Here $|\alpha|^{-1}$ has dimensions of length, in fact, the distance over which the vorticity diffuses through this fluid (of kinematic viscosity ν) during a time scale $\tau \sim \omega^{-1}$.

Suppose that the phenomena of interest has a characteristic velocity U, lengths are scaled with $|\alpha|^{-1}$, and pressure with $\mu|\alpha|U$. The governing equations are then rendered into the following dimensionless form:

$$-\nabla\widehat{p} + \nabla^2\widehat{\boldsymbol{v}} - \widehat{\boldsymbol{v}} = \mathbf{0}\ , \qquad \nabla\cdot\widehat{\boldsymbol{v}} = 0\ . \tag{6.2}$$

The boundary conditions (at particle surfaces) will introduce in a natural way the characteristic length scale, $\ell \to a$, where a is a characteristic dimension of the particle. The dimensionless group $\lambda = |\alpha|a$ is a ratio of the length scales, $|\alpha|^{-1}$ and a, and is the key parameter of this chapter. Small λ corresponds to slow temporal variations, and for steady flows λ is identically zero.

If we render the governing equations dimensionless using a for the characteristic length scale, then the scaled equations become

$$-\nabla\widehat{p} + \nabla^2\widehat{\boldsymbol{v}} - \lambda^2\widehat{\boldsymbol{v}} = \mathbf{0}\ , \qquad \nabla\cdot\widehat{\boldsymbol{v}} = 0\ , \tag{6.3}$$

and since λ appears explicitly, this form is useful in studying the limits for small λ. In fact, we will find uses for the dimensional and both nondimensional forms, hence the scaling will be declared explicitly unless the situation is clear from the context.

Our task in this chapter is to determine the leading $O(\lambda)$ transient corrections to the steady solutions of the preceding chapters. These calculations are most readily accomplished in the transformed variables, but their meaning in the time domain will be examined as well. In the next section, we derive the fundamental solution, which forms the basis for much of the subsequent discussion. Borrowing a concept from steady flow, we solve transient flows for the sphere by the singularity method. Among these are Stokes' classical solution for the oscillating sphere (translation motions) and Lamb's solution for a sphere in oscillatory rotation. The $O(\lambda)$ and $O(\lambda^2)$ terms introduce the concept of Basset forces and added mass.

In Section 6.3, we show that the transient Stokes equations also possesses a reciprocal theorem, and from this we derive the integral representation and

Faxén laws. The latter furnishes a good *entré* to the literature on particle tracers of Laser-Doppler Velocimetry (LDV). In the last section, we introduce a general expression for the $O(\lambda)$ correction for a particle of arbitrary shape. The correction is proportional to the square of the steady Stokes resistance tensor, and *thus can be determined from an analysis of steady flows only.*

6.2 The Fundamental Solution

We seek a solution to the fundamental problem,

$$-\nabla\hat{p} + \nabla^2\hat{\boldsymbol{v}} - \hat{\boldsymbol{v}} = -\boldsymbol{F}\delta(\boldsymbol{x}) , \qquad \nabla\cdot\hat{\boldsymbol{v}} = 0 . \tag{6.4}$$

We write this fundamental solution as $\hat{\boldsymbol{v}} = \boldsymbol{F}\cdot\mathcal{G}(\boldsymbol{x})/8\pi$, where $\mathcal{G}$ is (the Fourier transform of) the transient Oseen tensor and $\hat{p} = \boldsymbol{F}\cdot\mathcal{P}(\boldsymbol{x})/8\pi$. The transient Oseen tensor is deduced using the same line of reasoning as in the steady problem, the main difference being that the fundamental solution is expressed in terms of a scalar function that satisfies the modified Helmholtz equation,

$$\nabla^2\psi - \psi = 0 ,$$

instead of the Laplace equation. The final (nondimensional) result is[1]

$$\begin{aligned} \mathcal{G}(\boldsymbol{x}) &= (\nabla\nabla - \boldsymbol{\delta}\nabla^2)\frac{2}{r}(1 - e^{-r}) \\ &= \frac{4}{r^3}[1 - (1+r)e^{-r}]\frac{\boldsymbol{xx}}{r^2} \\ &\quad + \frac{2}{r^3}[(1 + r + r^2)e^{-r} - 1](\boldsymbol{\delta} - \frac{\boldsymbol{xx}}{r^2}) , \end{aligned} \tag{6.5}$$

$$\mathcal{P}(\boldsymbol{x}) = -2\nabla\frac{1}{r} = 2\frac{\boldsymbol{x}}{r^3} . \tag{6.6}$$

For small r we obtain the expansion,

$$\mathcal{G}(\boldsymbol{x}) = \mathcal{G}_0(\boldsymbol{x}) - \frac{4}{3}\boldsymbol{\delta} + \frac{r}{4}(3\boldsymbol{\delta} - \frac{\boldsymbol{xx}}{r^2}) + O(r^2) ,$$

where $\mathcal{G}_0(\boldsymbol{x})$ is the Oseen tensor for steady flow. The dimensional forms are given by

$$\begin{aligned} \mathcal{G}(\boldsymbol{x};\alpha) &= \frac{4\alpha}{(\alpha r)^3}[1 - (1+\alpha r)e^{-\alpha r}]\frac{\boldsymbol{xx}}{r^2} \\ &\quad + \frac{2\alpha}{(\alpha r)^3}[(1 + \alpha r + (\alpha r)^2)e^{-\alpha r} - 1](\boldsymbol{\delta} - \frac{\boldsymbol{xx}}{r^2}) , \\ \mathcal{P}(\boldsymbol{x};\alpha) &= 2\mu\frac{\boldsymbol{x}}{r^3} , \end{aligned}$$

and are obtained from the nondimensional forms by reintroducing the proper scales. The notation $(;\alpha)$ reminds us that this is the dimensional form. In the small αr expansion, the leading term, $\mathcal{G}_0$, does not depend on α as required.

[1] We have used a notational convention to simplify the expressions. Lengths have been "scaled" with α^{-1} instead of $|\alpha|^{-1}$. The distinction is important since α is a complex number.

6.2.1 The Oscillating Sphere

Stokes' celebrated work concerning a sphere oscillating with velocity $\hat{\boldsymbol{U}}e^{-i\omega t}$ provides an illustrative example of time-dependent Stokes flow. Here it is natural to set $\tau = \omega^{-1}$, where ω is the frequency of oscillation.

We seek a solution of the unsteady Stokes equation, with the boundary conditions,

$$\begin{aligned} \hat{\boldsymbol{v}} &= \hat{\boldsymbol{U}} && \text{on the sphere surface, } i.e., r = a, \\ \hat{\boldsymbol{v}} &\to \boldsymbol{0} && \text{for } r \to \infty \\ \hat{p} &\to p_0 && \text{for } r \to \infty \,. \end{aligned}$$

Following the strategy that proved so successful in the steady problem, we conjecture that a singularity solution of the form

$$\hat{\boldsymbol{v}} = 6\pi\mu a\hat{\boldsymbol{U}} \cdot (B_0 + B_2 a^2 \nabla^2)\alpha \frac{\mathcal{G}(\boldsymbol{x})}{8\pi\mu}$$

exists, where B_0 and B_2 are dimensionless functions of $\lambda = \alpha a$. For steady flows, recall that $B_0 = 1$ and $B_2 = 1/6$. Now at $r = a$, we have

$$\begin{aligned} \hat{\boldsymbol{v}} &= \frac{3}{4}\hat{\boldsymbol{U}} \cdot \left\{ \frac{\boldsymbol{xx}}{r^2}\left[\frac{6B_0}{\lambda^2} - \left(\frac{B_0}{\lambda^2} + B_2\right)e^{-\lambda}[6 + 6\lambda + 2\lambda^2]\right] \right. \\ &\left. +\boldsymbol{\delta}\left[-\frac{2B_0}{\lambda^2} + 2\left(\frac{B_0}{\lambda^2} + B_2\right)e^{-\lambda}[1 + \lambda + \lambda^2]\right]\right\} , \end{aligned}$$

so it follows from these boundary conditions that

$$\begin{aligned} 6B_0 - [6 + 6\lambda + 2\lambda^2]e^{-\lambda}(B_0 + \lambda^2 B_2) &= 0 \\ -B_0 + [1 + \lambda + \lambda^2]e^{-\lambda}(B_0 + \lambda^2 B_2) &= \frac{2}{3}\lambda^2 , \end{aligned}$$

or

$$B_0 = 1 + \lambda + \frac{1}{3}\lambda^2 , \qquad B_2 = \lambda^{-2}(e^{\lambda} - B_0) .$$

As in the steady Stokes problem, the force $\hat{\boldsymbol{F}}e^{-i\omega t}$ exerted by the fluid on the sphere is readily extracted from the singularity solution using the properties of the Dirac delta function:

$$\begin{aligned} \hat{\boldsymbol{F}} &= \oint_{S_p} \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}\, dS = \int_{V_p} \nabla \cdot \hat{\boldsymbol{\sigma}}\, dV \\ &= \int_{V_p} (-6\pi\mu a B_0 \hat{\boldsymbol{U}}\delta(\boldsymbol{x}) + \mu\alpha^2\hat{\boldsymbol{U}})\, dV \\ &= -6\pi\mu a[B_0 - \frac{2}{9}(\alpha a)^2]\hat{\boldsymbol{U}} \\ &= -6\pi\mu a[1 + \lambda + \frac{1}{9}\lambda^2]\hat{\boldsymbol{U}} . \end{aligned}$$

Thus the force on an oscillating sphere consists of three parts: a term in phase with the particle motion, a term proportional to $\omega^{1/2}$ and out of phase by $\pi/4$, and a term out of phase by $\pi/2$ or in phase with the acceleration. Note also that the Stokes law for steady translation is recovered in the limit $\lambda \rightarrow 0$.

Later on, in the derivation of the Faxén law for unsteady Stokes flow, we will need the expression for the surface traction from this problem. We may obtain this by first determining the rate-of-strain $\hat{e}$ and pressure $\hat{p}$ from the velocity field. The final expression for the surface tracton is

$$\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}|_{r=a} = -\frac{3\mu}{2a}\hat{\boldsymbol{U}} \cdot \left[(1+\lambda)\boldsymbol{\delta} + \frac{\lambda^2}{3}\boldsymbol{nn}\right] . \tag{6.7}$$

We recover the same expression for $\hat{\boldsymbol{F}}$ as above by integrating this result for the surface tractions. It is interesting to note that for unsteady Stokes flow, the traction is no longer directed along $\hat{\boldsymbol{U}}$; in fact, the $O(\lambda^2)$ term is directed normal to the sphere surface (this is precisely the inviscid pressure).

Basset [4] extended Stokes' solution to arbitrary motion, $\boldsymbol{U}(t)$, and his solution, given below, may be obtained by Fourier inversion[2] of the preceding result (see Landau and Lifshitz [54] for the derivation):

$$\boldsymbol{F}(t) = -6\pi\mu a \boldsymbol{U}(t) - 6\mu a^2\sqrt{\pi/\nu}\int_{-\infty}^{t} \dot{\boldsymbol{U}}(t')\frac{dt'}{\sqrt{t-t'}} - \frac{2}{3}\pi\rho a^3\frac{d\boldsymbol{U}}{dt} .$$

The origins of the first and third terms are readily identified. The first term is the pseudo-steady Stokes drag, and it originates from the O(1), in-phase term in $\hat{\boldsymbol{F}}$. The third term, a force contribution proportional to the particle acceleration, originates from the $\lambda^2 = -i\omega a^2/\nu$ term in $\hat{\boldsymbol{F}}$. This term is also known as the *added mass* term, because the extra force expended to overcome the inertia of the neighboring fluid is equivalent to an apparent increase in the mass of the sphere equal to one half of the mass of the displaced fluid. The second term, a convolution integral involving the sphere's history of motion, is known as the *Basset force* or *Basset memory integral*, and comes from the $O(\lambda)$ term in $\hat{\boldsymbol{F}}$. Note that we may write $\mu a^2\nu^{-1/2}$ as $(\mu a)^{1/2}(\rho a^3)^{1/2}$ so that the intermediate term is in some sense a geometric mean of the other two. The convolution integral is in fact the definition of the "half-derivative."

6.2.2 Sphere Released from Rest

For the case where an external force is applied with $F^e(t) = 0$ for $t < 0$ and arbitrary for $t > 0$ (this includes release from rest as a special case), Basset's formula may be inverted [3] so that in nondimensional form the sphere velocity is given by

$$\boldsymbol{U}(t) = \boldsymbol{F}^e(0)g(t) + \int_0^t \boldsymbol{G}(t')g(t-t')dt' , \tag{6.8}$$

where

$$\boldsymbol{G}(t) = \boldsymbol{F}^e + \frac{d\boldsymbol{F}^e}{dt} - \sqrt{\frac{\beta}{\pi}}\int_0^t \dot{\boldsymbol{F}}^e(t')\frac{dt'}{\sqrt{t-t'}} - \sqrt{\frac{\beta}{\pi t}}\boldsymbol{F}^e(0) ,$$

[2] Basset's work predates the widespread application of integral transforms, hence his solution is not by the Fourier integral.

$$
\begin{aligned}
g(t) &= \frac{1}{\sqrt{\beta(\beta-4)}}[\exp(m_+t)-\exp(m_-t)] \, , \\
\beta &= \frac{9}{2}\Big[\frac{\rho_p}{\rho}+\frac{1}{2}\Big]^{-1}, \qquad m_\pm = \frac{\beta}{2}-1\pm\frac{1}{2}\sqrt{\beta(\beta-4)} \, .
\end{aligned}
$$

Here, $\boldsymbol{F}^e$, $\boldsymbol{U}$, and t have been scaled with a characteristic force F_0, velocity $F_0/6\pi\mu a$, and time $a^2/\nu\beta$, respectively. We now give an outline of the derivation.

Since we have only $t \geq 0$, the derivation of the preceding result is most natural in terms of the Laplace transform. We start with the force balance,

$$
\boldsymbol{F}^e(t) - 6\pi\mu a\boldsymbol{U}(t) - 6\mu a^2\sqrt{\pi/\nu}\int_0^t \dot{\boldsymbol{U}}(t')\frac{dt'}{\sqrt{t-t'}} - \frac{2}{3}\pi\rho a^3\frac{d\boldsymbol{U}}{dt} = \frac{4}{3}\pi\rho_p a^3\frac{d\boldsymbol{U}}{dt} \, .
$$

The transformed equation is

$$
\widehat{\boldsymbol{F}}^e(s) = 6\pi\mu a\widehat{\boldsymbol{U}}(s)\left[1+\frac{as^{1/2}}{\sqrt{\nu}}+\frac{a^2s}{\nu\beta}\right] \, .
$$

If we now scale this equation as stated earlier and solve for $\widehat{\boldsymbol{U}}$, the result is

$$
\widehat{\boldsymbol{U}}(s) = \frac{\widehat{\boldsymbol{F}}^e(s)}{1+\sqrt{\beta}s^{1/2}+s} \, .
$$

At this point, the inversion may be performed by partial fraction expansions, leading to an expression in the time domain in terms of the error function of complex argument. The solution given here is obtained by clearing the denominator of $s^{1/2}$; we rewrite $\widehat{\boldsymbol{U}}$ as

$$
\widehat{\boldsymbol{U}}(s) = \frac{1}{(1+s)^2-\beta s} \times \widehat{\boldsymbol{F}}^e(s)(1+s-\sqrt{\beta}s^{1/2}) = \widehat{g}(s)\widehat{\boldsymbol{h}}(s) \, .
$$

From the convolution theorem, we have

$$
\boldsymbol{U}(t) = \int_0^t g(t-t')\boldsymbol{h}(t')dt' \, .
$$

The first factor, $\widehat{g}(s)$, is inverted by partial fractions by noting that

$$
(1+s)^2 - \beta s = (s-m_+)(s-m_-) \, .
$$

Thus we have

$$
\widehat{g}(s) = \frac{1}{\sqrt{\beta(\beta-4)}}\left[\frac{1}{s-m_+}-\frac{1}{s-m_-}\right] \, , \qquad g(t) = \frac{e^{m_+t}-e^{m_-t}}{\sqrt{\beta(\beta-4)}} \, .
$$

The second factor, $\widehat{\boldsymbol{h}}(s)$, requires another application of the convolution theorem to handle the $s^{1/2}\widehat{\boldsymbol{F}}^e(s)$ term. We define $\widehat{\boldsymbol{h}}(s) = \widehat{\boldsymbol{G}}(s) + \boldsymbol{F}^e(0)$, so that

$$
\begin{aligned}
\widehat{\boldsymbol{G}}(s) &= \widehat{\boldsymbol{F}}^e(s)(1+s-\sqrt{\beta}s^{1/2}) - \boldsymbol{F}^e(0) \\
&= \widehat{\boldsymbol{F}}^e(s) + s\widehat{\boldsymbol{F}}^e(s) - \boldsymbol{F}^e(0) - \frac{\sqrt{\beta}}{s^{1/2}}(s\widehat{\boldsymbol{F}}^e(s)-\boldsymbol{F}^e(0)) - \frac{\sqrt{\beta}}{s^{1/2}}\boldsymbol{F}^e(0)
\end{aligned}
$$

and $\boldsymbol{G}(t)$ is as claimed. In the final step, we insert $\boldsymbol{h}(t) = \boldsymbol{G}(t) + \boldsymbol{F}^e(0)\delta(t)$ into the convolution integral to obtain Equation 6.8.

At very short times $t \ll a^2/\nu$, all manifestations of the viscous resistance of the fluid are negligible. For example, a sphere released from rest experiences a constant acceleration and we expect $\boldsymbol{U}(t) \sim \boldsymbol{F}^e(\rho_p + \frac{1}{2}\rho)V_p)^{-1}t$.

The exact solution has the small time behavior given by

$$\boldsymbol{U}(t) \sim \boldsymbol{F}^e t \left[1 - \frac{4}{3}\sqrt{\frac{\beta t}{\pi}}\right] ,$$

or in dimensional form,

$$\boldsymbol{U}(t) \sim \boldsymbol{F}^e \left[(\rho_p + \frac{1}{2}\rho)V_p\right]^{-1} t \left[1 - \frac{4\beta}{3}\sqrt{\frac{\nu t}{\pi a^2}}\right] ,$$

which is consistent with our expectations. The correction term scales with time as $t^{3/2}$ and comes from the Basset integral. In some elementary treatments of transient settling, the Basset effect is neglected. The pseudo-steady Stokes drag introduces a simple damping effect, which leads to a first order differential equation for $\boldsymbol{U}(t)$ and the solution,

$$\boldsymbol{U}(t) = \frac{\boldsymbol{F}^e}{6\pi\mu a}\left[1 - \exp\left(\frac{-6\pi\mu a t}{(\rho_p + \frac{1}{2}\rho)V_p}\right)\right] \qquad \text{(Basset effect neglected).}$$

At small times, this gives the *erroneous* result

$$\boldsymbol{U}(t) \sim \boldsymbol{F}^e \left[(\rho_p + \frac{1}{2}\rho)V_p\right]^{-1} t \left[1 - \frac{\beta\nu t}{2a^2}\right] .$$

On the vorticity diffusion time scale a^2/ν (at low Reynolds numbers, this still corresponds to the brief instant just after the release of the particle), the contributions from the pseudo-steady Stokes drag, the Basset force, and the added mass are comparable and all three effects must be included. Much later, on the time scale of a/U, the pseudo-steady Stokes drag gives the dominant contribution, with the Basset and added mass effects decaying in relative importance as $\mathrm{Re}^{1/2}$ and Re. For a particle flow problem with Re $= 0.01$, the Basset correction is 10% and the cumulative effect can be significant.

6.2.3 Oscillatory Rotation of a Sphere

The velocity field produced by a sphere undergoing oscillatory rotation $\hat{\boldsymbol{\omega}}e^{-i\omega t}\times\boldsymbol{x}$ in a quiescent fluid can be represented by the transient rotlet,

$$\hat{\boldsymbol{v}} = (\hat{\boldsymbol{T}} \cdot \nabla) \cdot \mathcal{G}(\boldsymbol{x})/(8\pi\mu) .$$

On the sphere surface, the rotlet reduces to the desired rotational motion, provided that

$$T_{ij} = \epsilon_{ijk}\hat{\omega}_k \frac{4\pi\mu a^3 e^{\lambda}}{1+\lambda} .$$

The torque on the sphere follow as

$$\begin{aligned}\widehat{T}_i &= \oint_{S_p} \epsilon_{ijk} x_j (\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{n})_k \, dS \\ &= -\epsilon_{ijk} B_0(\lambda) e^{-\lambda} T_{jk} \\ &= -8\pi\mu a^3 \frac{1+\lambda+\lambda^2/3}{1+\lambda} \widehat{\omega}_i \ . \end{aligned} \tag{6.9}$$

The corresponding result in the time domain is

$$\begin{aligned}\boldsymbol{T}(t) &= -8\pi\mu a^3 \boldsymbol{\omega}(t) \\ &- \frac{8}{3}\int_{-\infty}^{t} \dot{\boldsymbol{\omega}}(t') \left[\frac{\mu a^4 \pi^{1/2}}{\sqrt{\nu(t-t')}} - \pi a^5 \rho e^{t-t'} \operatorname{erfc}\sqrt{t-t'} \right] dt' \ . \end{aligned} \tag{6.10}$$

For large λ, $\widehat{\boldsymbol{T}}$ is linear in λ (unlike $\widehat{\boldsymbol{F}}$, which was quadratic) and, correspondingly, in the time domain there is no term in phase with the angular acceleration, $\dot{\boldsymbol{\omega}}(t)$, *i.e.*, no "added moment of inertia."

6.3 Reciprocal Theorem and Applications

The derivation of the reciprocal theorem for unsteady Stokes flow is quite similar to that given in Chapter 2 for steady flow. For our purposes here, we express the theorem as

$$\begin{aligned}&\oint_S \widehat{\boldsymbol{v}}_1 \cdot (\widehat{\boldsymbol{\sigma}}_2 \cdot \boldsymbol{n})\, dS + \int_V \widehat{\boldsymbol{v}}_1 \cdot (\nabla \cdot \widehat{\boldsymbol{\sigma}}_2)\, dV \\ &= \oint_S \widehat{\boldsymbol{v}}_2 \cdot (\widehat{\boldsymbol{\sigma}}_1 \cdot \boldsymbol{n})\, dS + \int_V \widehat{\boldsymbol{v}}_2 \cdot (\nabla \cdot \widehat{\boldsymbol{\sigma}}_1)\, dV \ , \end{aligned} \tag{6.11}$$

where the fields $\widehat{\boldsymbol{v}}_1$, $\widehat{\boldsymbol{\sigma}}_1$, $\widehat{\boldsymbol{v}}_2$, and $\widehat{\boldsymbol{\sigma}}_2$ decay far away from the particle, to the extent that the surface contributions are taken from only the particle surface. Note that we may choose to subtract a volume integral of $\widehat{\boldsymbol{v}}_1 \cdot \widehat{\boldsymbol{v}}_2$ from both sides, to obtain a form closer to the governing equation.

As in steady Stokes flow, the reciprocal theorem may be used to derive the integral representation, Faxén laws, and symmetry relations for the resistance and mobility functions. We shall expand on these ideas in this section.

6.3.1 Integral Representations

We obtain the integral representation for disturbance fields by setting $\widehat{\boldsymbol{v}}_1$ as the fundamental solution and letting $\widehat{\boldsymbol{v}}_2$ be the solution of interest. The procedure is analogous to that used for steady flow and we obtain single and double layer potentials:

$$\begin{aligned}\widehat{\boldsymbol{v}}(\boldsymbol{x}) &= -\frac{1}{8\pi}\oint_{S_p} (\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}) \cdot \mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) \\ &- \oint_{S_p} \widehat{\boldsymbol{v}} \cdot \boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\xi}) \cdot \boldsymbol{n}\, dS(\boldsymbol{\xi}) \ . \end{aligned}$$

For a translating particle, the double layer potential may be replaced by the formula obtained by applying the representation to the uniform field $\widehat{U}$ inside the particle. Here, however, the uniform field carries a pressure field equal to $-\widehat{U}\cdot\boldsymbol{x}$, and so the final result reads

$$\begin{aligned}\widehat{\boldsymbol{v}}(\boldsymbol{x}) &= -\frac{1}{8\pi}\oint_{S_p}(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{n}-\widehat{\boldsymbol{U}}\cdot\boldsymbol{x}\boldsymbol{n})\cdot\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\,dS(\boldsymbol{\xi}) \\ &= -\frac{1}{8\pi}\oint_{S_p}(\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{n})\cdot\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\,dS(\boldsymbol{\xi})+\frac{\widehat{\boldsymbol{U}}\cdot}{8\pi}\int_{V_p}\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})\,dV(\boldsymbol{\xi})\ .\end{aligned}$$

6.3.2 The Faxén Law: Particles of Arbitrary Shape

Faxén laws for unsteady Stokes flow can be derived using the same approach as that employed in the steady problem. Starting from the reciprocal theorem, we show that the Faxén law must be of the same functional form as the associated transient singularity solution. This is illustrated here for the force law and the translational singularity solution.

We set $\widehat{\boldsymbol{v}}_1$ to be the solution for a particle oscillating with velocity $\widehat{\boldsymbol{U}}_1$ in a quiescent fluid. We note that $\nabla\cdot\widehat{\boldsymbol{\sigma}}_1-\widehat{\boldsymbol{v}}_1=0$ in V. For $\widehat{\boldsymbol{v}}_2$, we take the velocity field generated by a point force $\widehat{\boldsymbol{F}}\cdot\mathcal{G}(\boldsymbol{x}-\boldsymbol{y})/8\pi$, where $\boldsymbol{y}$ lies outside the particle, and the particle stationary. Then $\widehat{\boldsymbol{v}}_2=0$ on S, and in V, $\nabla\cdot\widehat{\boldsymbol{\sigma}}_2=-\widehat{\boldsymbol{F}}\delta(\boldsymbol{x}-\boldsymbol{y})$. When these boundary conditions and identities are inserted into Equation 6.11, the result is

$$\widehat{\boldsymbol{U}}_1\cdot\widehat{\boldsymbol{F}}_2-\widehat{\boldsymbol{v}}_1(\boldsymbol{y})\cdot\widehat{\boldsymbol{F}}=0\ , \tag{6.12}$$

where $\widehat{\boldsymbol{F}}_2$ is the force on the particle generated by the surface traction $\widehat{\boldsymbol{\sigma}}_2\cdot\boldsymbol{n}$. Now, due to linearity of the Stokes equation, we may factor $\widehat{\boldsymbol{U}}_1$ from $\widehat{\boldsymbol{v}}_1$. Furthermore, suppose that $\widehat{\boldsymbol{v}}_1$ is written as a singularity solution, then

$$\widehat{\boldsymbol{v}}_1(\boldsymbol{x})=\widehat{\boldsymbol{U}}_1\cdot\mathcal{F}\{\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi})/8\pi\}\ , \tag{6.13}$$

where $\mathcal{F}$ is a linear functional and $\boldsymbol{\xi}$ represents the region over which the singularities are distributed. Then Equation 6.12 becomes

$$(\widehat{\boldsymbol{F}}_2)_i=\mathcal{F}\{\widehat{F}_j\mathcal{G}_{ji}(\boldsymbol{y}-\boldsymbol{\xi})/8\pi\}\ , \tag{6.14}$$

but since $\widehat{F}_j\mathcal{G}_{ji}(\boldsymbol{y}-\boldsymbol{\xi})/8\pi=F_j\mathcal{G}_{ij}(\boldsymbol{\xi}-\boldsymbol{y})/8\pi$ is the ambient field evaluated over the image region, we have shown that

$$\widehat{\boldsymbol{F}}_2=\mathcal{F}\{\widehat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})\}\ . \tag{6.15}$$

But all ambient fields $\widehat{\boldsymbol{v}}^\infty$ that satisfy the unsteady Stokes equation can be constructed from an appropriate set of images, so we have derived the general result as well.

The Faxén law for a moving particle is obtained by adding the contributions for the particle moving through a quiescent fluid to the results for the stationary particle.

6.3.3 The Faxén Law: Force on a Rigid Sphere

We shall derive two different but equivalent forms of the Faxén law. The first form is obtained by starting with the integral representation for the oscillating sphere (the surface tractions was given earlier in this chapter):

$$
\begin{aligned}
\hat{\boldsymbol{v}}(\boldsymbol{x}) &= 6\pi\mu a\hat{\boldsymbol{U}}\cdot\left[\frac{1+\lambda}{4\pi a^2}\oint_{S_p}\frac{\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi};\alpha)}{8\pi\mu}\,dS(\boldsymbol{\xi})\right.\\
&\quad\left.+\frac{\lambda^2}{12\pi a^2}\oint_{S_p}\boldsymbol{nn}\cdot\frac{\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi};\alpha)}{8\pi\mu}\,dS(\boldsymbol{\xi})\right]\\
&\quad+\frac{\lambda^2}{8\pi a^2}\int_{V_p}\hat{\boldsymbol{U}}\cdot\mathcal{G}(\boldsymbol{x}-\boldsymbol{\xi};\alpha)\,dV(\boldsymbol{\xi})\ .
\end{aligned}
$$

The Faxén relation for the stationary sphere follows as

$$
\begin{aligned}
\hat{\boldsymbol{F}} &= 6\pi\mu a\left[\frac{1+\lambda}{4\pi a^2}\oint_{S_p}\hat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})dS(\boldsymbol{\xi})+\frac{\lambda^2}{12\pi a^2}\oint_{S_p}\boldsymbol{nn}\cdot\hat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})dS(\boldsymbol{\xi})\right.\\
&\quad\left.+\frac{\lambda^2}{6\pi a^3}\int_{V_p}\hat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})dV(\boldsymbol{\xi})\right]\ .
\end{aligned}
$$

We wish to collect all terms of $O(\lambda^2)$ as a volume integral. This may be accomplished along the lines

$$
\begin{aligned}
\oint_{S_p} n_i n_j \hat{v}_j^\infty(\boldsymbol{\xi})\,dS(\boldsymbol{\xi}) &= a^{-1}\oint_{S_p} x_i n_j \hat{v}_j^\infty(\boldsymbol{\xi})\,dS(\boldsymbol{\xi})\\
&= a^{-1}\int_{V_p}\frac{\partial}{\partial x_j}(x_i\hat{v}_j^\infty(\boldsymbol{\xi})\,dS(\boldsymbol{\xi})\\
&= a^{-1}\int_{V_p}[\delta_{ij}\hat{v}_j^\infty + x_i(\nabla\cdot\hat{\boldsymbol{v}}^\infty)]\,dS(\boldsymbol{\xi})\\
&= a^{-1}\int_{V_p}\hat{v}_i^\infty(\boldsymbol{\xi})\,dS(\boldsymbol{\xi})\ .
\end{aligned}
$$

Thus we obtain the Faxén law, derived by Mazur and Bedeaux [59], for the force on an oscillating sphere in an arbitrary Stokes ambient field:

$$
\hat{\boldsymbol{F}} = 6\pi\mu a\left[\frac{1+\lambda}{4\pi a^2}\oint_{S_p}\hat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})\,dS(\boldsymbol{\xi})+\frac{\lambda^2}{4\pi a^3}\int_{V_p}\hat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})\,dV(\boldsymbol{\xi})-\hat{\boldsymbol{U}}\left(1+\lambda+\frac{\lambda^2}{9}\right)\right]
$$

where for the general case we have simply added the additional contribution for a sphere moving through a quiescent fluid. In the time domain, these results become

$$
\begin{aligned}
\boldsymbol{F}(t) &= 6\pi\mu a[\boldsymbol{v}_S^\infty(t)-\boldsymbol{U}(t)]+6\mu a^2\sqrt{\pi/\nu}\int_{-\infty}^{t}(\dot{\boldsymbol{v}}_S^\infty(t')-\dot{\boldsymbol{U}}(t'))\frac{dt'}{\sqrt{t-t'}}\\
&\quad+\frac{2}{3}\pi\rho a^3\left(\frac{\partial}{\partial t}\boldsymbol{v}_V^\infty-\frac{d\boldsymbol{U}}{dt}\right)+\frac{4}{3}\pi\rho a^3\frac{\partial}{\partial t}\boldsymbol{v}_V^\infty\ ,
\end{aligned}
\tag{6.16}
$$

a form also given by Mazur and Bedeaux, where the ambient velocity appears in the form of averages over the sphere surface and volume,

$$\begin{aligned} \boldsymbol{v}_S^\infty(t) &= \frac{1}{4\pi a^2}\oint_{S_p} \widehat{\boldsymbol{v}}^\infty(\boldsymbol{\xi},t)\,dS(\boldsymbol{\xi})\ , \\ \boldsymbol{v}_V^\infty(t) &= \frac{3}{4\pi a^3}\int_{V_p} \widehat{\boldsymbol{v}}^\infty(\boldsymbol{\xi},t)\,dV(\boldsymbol{\xi})\ . \end{aligned}$$

We summarize the result for unsteady, spatially inhomogeous flows: The drag depends on the *relative* velocity and acceleration between the sphere and ambient fluid, plus a "buoyancy" contribution equal to the force required to accelerate the mass of fluid displaced by the sphere. The Faxén correction for curvature in the ambient velocity field is implicit in the averaged fields $\boldsymbol{v}_S^\infty$ and $\boldsymbol{v}_V^\infty$. While our development is aimed at transient effects in Stokes flow, Equation 6.16 also plays a prominent role in the analysis of tracer particle motions in Laser-Doppler Velocimetry (LDV) at finite Reynolds numbers (see, for example, [58] and references therein). The key idea is that a tracer particle essentially moves with the fluid element, and thus in the frame of reference moving with the particle the flow is at low Reynolds number (based on particle size and relative velocity). The force calculations involve Equation 6.16, with appropriate corrections for the fictitious forces in this noninertial reference frame.

A second form of the Faxén law follows from the singularity solution as

$$\widehat{\boldsymbol{F}} = 6\pi\mu a[B_0(\lambda) + B_2(\lambda)a^2\nabla^2]\widehat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})|_{x=0} - 6\pi\mu a\widehat{\boldsymbol{U}}\left(1+\lambda+\frac{\lambda^2}{9}\right)\ .$$

This form can also be derived directly from the first, by noting that with $(\nabla^2)^n\widehat{\boldsymbol{v}}^\infty = \alpha^{2n-2}\nabla^2\widehat{\boldsymbol{v}}^\infty$,

$$\begin{aligned} \frac{1}{4\pi a^2}\oint_{S_p} \widehat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})\,dS(\boldsymbol{\xi}) &= \widehat{\boldsymbol{v}}^\infty(0) + a^2\sum_{n=1}^{\infty}\frac{\lambda^{2n-2}\nabla^2\widehat{\boldsymbol{v}}^\infty(0)}{(2n+1)!} \\ \frac{3}{4\pi a^3}\int_{V_p} \widehat{\boldsymbol{v}}^\infty(\boldsymbol{\xi})\,dV(\boldsymbol{\xi}) &= \widehat{\boldsymbol{v}}^\infty(0) + a^2\sum_{n=1}^{\infty}\frac{3\lambda^{2n-2}\nabla^2\widehat{\boldsymbol{v}}^\infty(0)}{(2n+3)(2n+1)!}\ . \end{aligned}$$

The two infinite series together yield precisely the B_2 term in the second form of the Faxén law. This second form is useful at small λ, since all field variables are evaluated at a single point.

6.3.4 The Faxén Law: Viscous Drop

The Faxén law for the force on a viscous drop must also be of the same functional form as the singularity solution for the oscillating drop. The proof is similar to that used for the rigid particle. We start with the reciprocal theorem and insert the velocity field produced by an oscillating drop for $\widehat{\boldsymbol{v}}_1$, while $\widehat{\boldsymbol{v}}_2$ is taken to be the field of a *stationary* drop near a point force at $\boldsymbol{y}$. The proof proceeds along identical lines to the one used for the rigid particle, up to the point where the boundary conditions are inserted into Equation 6.11. Now we must have

$$\oint_{S_p} \widehat{\boldsymbol{v}}_1\cdot(\widehat{\boldsymbol{\sigma}}_2\cdot\boldsymbol{n})\,dS + \widehat{\boldsymbol{v}}_1(\boldsymbol{y})\cdot\widehat{\boldsymbol{F}} = \oint_{S_p} \widehat{\boldsymbol{v}}_2\cdot(\widehat{\boldsymbol{\sigma}}_1\cdot\boldsymbol{n})\,dS\ . \qquad (6.17)$$

At this point, it proves convenient to extract $\hat{U}$ from $\hat{v}_1$ so that the preceding equation becomes

$$\hat{U} \cdot \hat{F}_2 + \hat{v}_1(y) \cdot \hat{F} = -\oint_{S_p} (\hat{v}_1 - \hat{U}) \cdot (\hat{\sigma}_2 \cdot n)\, dS + \oint_{S_p} \hat{v}_2 \cdot (\hat{\sigma}_1 \cdot n)\, dS \,. \qquad (6.18)$$

The boundary conditions at the surface of the drop require that $n \cdot \hat{v}_2 = 0$ and $n \cdot (\hat{v}_1 - \hat{U}) = 0$, so only the *tangential* components of the surface traction are retained in the dot products. But an application of the reciprocal theorem to the inner fields associated with $\hat{v}_1 - \hat{U}$ and $\hat{v}_2$ yields the relation,

$$\oint_{S_p} (\hat{v}_1^{(i)} - \hat{U}) \cdot (\hat{\sigma}_2^{(i)} \cdot n)\, dS = \oint_{S_p} \hat{v}_2^{(i)} \cdot (\hat{\sigma}_1^{(i)} \cdot n)\, dS \,, \qquad (6.19)$$

where again it is understood that only the tangential component of the surface traction is retained. But in both problem 1 and 2, the tangential component of the traction is continuous across the interface. The preceding equation then implies that the two surface integrals in Equation 6.18 cancel.

Again, we suppose that $\hat{v}_1$, the field produced by the oscillating drop, is available as a singularity solution, so that

$$\hat{v}_1(x) = \hat{U}_1 \cdot \mathcal{F}\{\mathcal{G}(x - \xi)/8\pi\} \,, \qquad (6.20)$$

where $\mathcal{F}$ is a linear functional and ξ represents the region over which the singularities are distributed. We now may use the same arguments as used before for the rigid particle to arrive at the conclusion that the Faxén law is of the same functional form as the singularity solution.

6.3.5 The Oscillating Spherical Drop

We consider a viscous drop oscillating with velocity $\hat{U}e^{-i\omega t}$ in a quiescent fluid. We assume that surface tension forces dominate over viscous forces so that the drop retains its spherical shape. Once again, for convenience, we shall adopt the notational convention of scaling lengths with α^{-1}, and the associated scales for the pressure, *etc.*

The following solutions of the transient Stokes equations are bounded in the interior of a sphere of radius a:

$$\hat{v} = \hat{U} \,, \qquad \hat{v} = (\nabla\hat{U} \cdot \nabla - U\nabla^2)\frac{\sinh r}{r} \,;$$

in fact, the linear combination

$$\hat{v} = \frac{1}{\lambda^2}\hat{U} + \frac{3}{2}(\nabla\hat{U} \cdot \nabla - U\nabla^2)\frac{\sinh r}{r}$$

reduces to the Stokeson of Chapter 3 in the limit of small λ.[3] Therefore, for the velocity field inside the drop, we assume the form

$$\hat{v} = D_0\hat{U} + D_2(\nabla\hat{U} \cdot \nabla - U\nabla^2)\frac{\sinh r}{r} \,, \qquad (6.21)$$

[3]More explicitly, in terms of the dimensional variables, we rewrite αr as $\lambda r/a$ and take the limit of small λ while keeping r/a fixed.

while for outside the drop, we assume the familiar form,

$$\hat{\boldsymbol{v}} = \frac{3\lambda}{4}\hat{\boldsymbol{U}} \cdot (C_0 + C_2\lambda^2\nabla^2)\mathcal{G}(\boldsymbol{x}) \ , \tag{6.22}$$

which already satisfies the boundary condition for large r. At $r = a$, the boundary conditions on the radial and tangential velocities and the tangential component of the surface traction $\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{n}$ are

1. $\boldsymbol{n} \cdot \hat{\boldsymbol{v}}^{(o)} = \boldsymbol{n} \cdot \hat{\boldsymbol{U}}$
2. $\boldsymbol{n} \cdot \hat{\boldsymbol{v}}^{(i)} = \boldsymbol{n} \cdot \hat{\boldsymbol{U}}$
3. $\hat{\boldsymbol{v}}^{(o)} - \boldsymbol{nn} \cdot \hat{\boldsymbol{v}}^{(o)} = \hat{\boldsymbol{v}}^{(i)} - \boldsymbol{nn} \cdot \hat{\boldsymbol{v}}^{(i)}$
4. $(\hat{\boldsymbol{e}}^{(o)} \cdot \boldsymbol{n}) \cdot (\boldsymbol{\delta} - \boldsymbol{nn}) = \kappa(\hat{\boldsymbol{e}}^{(i)} \cdot \boldsymbol{n}) \cdot (\boldsymbol{\delta} - \boldsymbol{nn})$,

where $\kappa = \mu^{(i)}/\mu^{(o)}$, the ratio of the drop and solvent viscosities.[4]

These conditions yield, respectively, the four equations,

$$\begin{aligned} W - (1+\lambda)X &= \frac{\lambda^2}{3} \\ Y + (2\tanh\lambda - 2\lambda)Z &= 1 \\ -W + (1+\lambda+\lambda^2)X - \frac{2}{3}\lambda^2 Y + \frac{2}{3}\lambda^2(\lambda^2\tanh\lambda - \lambda + \tanh\lambda)Z &= 0 \\ 9W - \frac{3}{2}(6+6\lambda+3\lambda^2+\lambda^3) - \kappa\lambda^2\left[(6+3\lambda^2)\tanh\lambda - 6\lambda - \lambda^3\right]Z &= 0 \ , \end{aligned}$$

where $W = C_0$, $X = e^{-\lambda}(C_0 + \lambda^2 C_2)$, $Y = D_0$, and $Z = D_2(\cosh\lambda)/\lambda^3$. The solution is

$$\begin{aligned} C_0 &= B_0(\lambda) - \frac{(1+\lambda)^2 f(\lambda)}{D(\lambda,\kappa)} \\ C_2 &= B_2(\lambda) - \frac{e^\lambda - (1+\lambda)}{\lambda^2}\frac{(1+\lambda)f(\lambda)}{D(\lambda,\kappa)} \\ D_0 &= 1 - \frac{(1+\lambda)(3\tanh\lambda - 3\lambda)}{D(\lambda,\kappa)} \\ D_2 &= \frac{3\lambda^3\mathrm{sech}\lambda(1+\lambda)}{2D(\lambda,\kappa)} \ , \end{aligned}$$

where

$$f(\lambda) = \lambda^2\tanh\lambda - 3\lambda + 3\tanh\lambda$$

and

$$D(\lambda,\kappa) = \kappa[\lambda^3 - \lambda^2\tanh\lambda - 2f(\lambda)] + (\lambda+3)f(\lambda) \ .$$

[4]In much of the literature, the viscosity ratio is denoted by λ. Unfortunately, λ is also the accepted notation for the frequency parameter αa. To avoid confusion, throughout this chapter we will use κ for the viscosity ratio.

In the limit of large κ (a very viscous drop) with λ fixed, we recover the solution for the rigid sphere, with C_0 and C_2 becoming B_0 and B_2. On the other hand, with κ fixed, we obtain at low frequencies (small λ) the solution

$$\begin{aligned} C_0 &= \frac{2+3\kappa}{3(1+\kappa)}\left[1+\frac{2+3\kappa}{3(1+\kappa)}\lambda\right]+O(\lambda^2) \\ C_2 &= \frac{\kappa}{6(1+\kappa)}\left[1+\frac{2+3\kappa}{3(1+\kappa)}\lambda\right]+O(\lambda^2)\,. \end{aligned}$$

The $O(1)$ terms give the Hadamard–Rybczynski solution for steady translation. Note also that for C_0, the coefficient of the $O(\lambda)$ term, is the square of the $O(1)$ coefficient. We will show at the end of this chapter that this is the general form taken by the low-frequency correction.

In the high-frequency limit, the drag coefficient has the asymptotic behavior

$$C_0 - \frac{2\lambda^2}{9} = \frac{\lambda^2}{9} + \frac{\kappa\lambda}{\kappa+1}\,,$$

which shows that the added mass is independent of κ, as expected, since the potential solution outside the sphere applies for all values of κ. Following Lawrence and Weinbaum [55], we may write the force on the drop as

$$\widehat{\boldsymbol{F}} = 6\pi\mu a\widehat{U}\left[\frac{2+3\kappa}{3(1+\kappa)} + B^{\infty}\lambda + \frac{\lambda^2}{9} + (B^0 - B^{\infty})\lambda L(\lambda,\kappa)\right]\,,$$

where

$$B^0 = \frac{(2+3\kappa)^2}{9(1+\kappa)^2} \qquad \text{and} \qquad B^{\infty} = \frac{\kappa}{\kappa+1}$$

are the $O(\lambda)$ (or Basset) coefficients for small and large λ, and $L(\lambda,\kappa)$ is a dimensionless function of λ, which varies from 1 for small λ to 0 for large λ. For the rigid sphere, the L-term is not present because $B^0 = B^{\infty}$.

6.3.6 The Faxén Law: Force on a Spherical Drop

The Faxén law for the drag on a viscous drop in an arbitrary unsteady Stokes flow is

$$\widehat{\boldsymbol{F}} = 6\pi\mu a[C_0(\lambda) + C_2(\lambda)a^2\nabla^2]\widehat{\boldsymbol{v}}^{\infty}(\boldsymbol{\xi})|_{x=0} - 6\pi\mu a\widehat{U}[C_0 - \frac{2}{9}\lambda^2]\,,$$

with C_0 and C_2 from the preceding discussion. The form analogous to the one derived by Mazur and Bedeaux for the rigid sphere follows immediately as

$$\begin{aligned} \widehat{\boldsymbol{F}} &= 6\pi\mu a\left[\frac{\gamma_0(\lambda)}{4\pi a^2}\oint_{S_p}\widehat{\boldsymbol{v}}^{\infty}(\boldsymbol{\xi})dS(\boldsymbol{\xi}) + \frac{3\gamma_2(\lambda)}{4\pi a^3}\int_{V_p}\widehat{\boldsymbol{v}}^{\infty}(\boldsymbol{\xi})dV(\boldsymbol{\xi})\right] \\ &\quad - 6\pi\mu a\widehat{U}\left[C_0 - \frac{2}{9}\lambda^2\right]\,, \end{aligned}$$

with

$$\begin{aligned}\gamma_0(\lambda) &= 1+\lambda-\frac{\lambda^2}{3}\left(1+\frac{\lambda^2\sinh\lambda}{3(\lambda e^{\lambda}-B_0\sinh\lambda)}\right)\frac{(1+\lambda)f}{D(\lambda,\kappa)}\ ,\\ \gamma_2(\lambda) &= \frac{\lambda^2}{3}+\frac{\lambda^2}{3}\left(1+\frac{\lambda^2\sinh\lambda}{3(\lambda e^{\lambda}-B_0\sinh\lambda)}\right)\frac{(1+\lambda)f}{D(\lambda,\kappa)}\ ,\end{aligned}$$

and f and D defined as in solution for the oscillating drop.

6.4 The Low-Frequency Limit

From the preceding discussion, it appears that in the limit of small frequencies, the unsteady Stokes results reduce to the steady Stokes solution plus a correction of $O(\lambda)$, with the $O(\lambda)$ coefficient expressed as a square of the steady result. Here, we investigate this more closely to establish a general description of the low-frequency limit.

A convenient place to start is the integral representation, with the fundamental solution expanded in small λ. The integral equation (dimensionless form) for the traction becomes

$$\hat{\boldsymbol{U}} = -\frac{1}{8\pi}\oint_{S_p}(\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n})\cdot[\mathcal{G}_0(\boldsymbol{x}-\boldsymbol{\xi})-\frac{4}{3}\lambda\boldsymbol{\delta}]\,dS+O(\lambda)^2\ ,$$

or, if we expand the stress as $\hat{\boldsymbol{\sigma}}=\hat{\boldsymbol{\sigma}}_0+\lambda\hat{\boldsymbol{\sigma}}_1+\cdots$,

$$\hat{\boldsymbol{U}}-\frac{\lambda\boldsymbol{F}_0}{6\pi} = -\frac{1}{8\pi}\oint_{S_p}[(\hat{\boldsymbol{\sigma}}_0+\lambda\hat{\boldsymbol{\sigma}}_1)\cdot\boldsymbol{n}]\cdot\mathcal{G}_0(\boldsymbol{x}-\boldsymbol{\xi})\,dS\ .$$

Thus we obtain the integral equation for *steady* Stokes flow, but with an effective uniform velocity[5] $\hat{\boldsymbol{U}}-\lambda\boldsymbol{F}_0/6\pi$.

If the surface traction and force in the steady Stokes problem have the form

$$\boldsymbol{\sigma}\cdot\boldsymbol{n}=-\frac{3}{2}\mathcal{S}\cdot\boldsymbol{U}\ ,\qquad \boldsymbol{F}=-6\pi\boldsymbol{A}\cdot\boldsymbol{U}\ ,$$

then the low-frequency limit must be of the form

$$\begin{aligned}\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n} &= -\frac{3}{2}\mathcal{S}\cdot(\boldsymbol{U}+\lambda\boldsymbol{A}\cdot\boldsymbol{U})\\ \boldsymbol{F} &= -6\pi\boldsymbol{A}\cdot(\boldsymbol{U}+\lambda\boldsymbol{A}\cdot\boldsymbol{U})\ .\end{aligned}$$

The result derived earlier for the sphere is consistent with these general expressions, with $\boldsymbol{A}=\mathcal{S}=\boldsymbol{\delta}$.

The Faxén law also follows directly from the surface tractions for the translating particle. Thus, at low frequencies the Faxén law (dimensional form) for the general particle, correct to $O(\lambda)$, is

$$\hat{\boldsymbol{F}}=6\pi\mu a(\boldsymbol{\delta}+\lambda\boldsymbol{A})\cdot\frac{1}{4\pi a^2}\oint_{S_p}\mathcal{S}\cdot\hat{\boldsymbol{v}}^{\infty}dS\ +\cdots\ .$$

[5]The steady force F_0 here has been scaled with μaU.

We may apply the low-frequency expansion to the resistance expressions for an arbitrary particle undergoing translation and rotation. Suppose that the steady forces and torques are described by the resistance relation

$$\begin{aligned} \boldsymbol{F} &= -6\pi \boldsymbol{A} \cdot \boldsymbol{U} - 6\pi \tilde{\boldsymbol{B}} \cdot \boldsymbol{\omega} \\ \boldsymbol{T} &= -6\pi \boldsymbol{B} \cdot \boldsymbol{U} - 8\pi \boldsymbol{C} \cdot \boldsymbol{\omega} \ . \end{aligned}$$

Once again, the $O(\lambda)$ correction appears *via* an effective uniform velocity given by $-\lambda \boldsymbol{F}_0/(6\pi)$, so the resistance relations follow as

$$\begin{aligned} \hat{\boldsymbol{F}} &= -6\pi \boldsymbol{A} \cdot \left[\hat{\boldsymbol{U}} + \lambda \boldsymbol{A} \cdot \hat{\boldsymbol{U}} + \lambda \tilde{\boldsymbol{B}} \cdot \hat{\boldsymbol{\omega}} \right] - 6\pi \tilde{\boldsymbol{B}} \cdot \hat{\boldsymbol{\omega}} \\ \hat{\boldsymbol{T}} &= -6\pi \boldsymbol{B} \cdot \left[\hat{\boldsymbol{U}} + \lambda \boldsymbol{A} \cdot \hat{\boldsymbol{U}} + \lambda \tilde{\boldsymbol{B}} \cdot \hat{\boldsymbol{\omega}} \right] - 8\pi \boldsymbol{C} \cdot \hat{\boldsymbol{\omega}} \ . \end{aligned}$$

We draw the following observations from the preceding example. The tensors that couple $\hat{\boldsymbol{F}}$ to $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{T}}$ to $\hat{\boldsymbol{U}}$ are

$$-6\pi(\boldsymbol{\delta} + \lambda \boldsymbol{A}) \cdot \tilde{\boldsymbol{B}} \qquad \text{and} \qquad -6\pi \boldsymbol{B} \cdot (\boldsymbol{\delta} + \lambda \boldsymbol{A}) \ ,$$

respectively, so the resistance tensor is symmetric, as required by the reciprocal theorem. Finally, we see that the coupling of $\hat{\boldsymbol{F}}$ to $\hat{\boldsymbol{\omega}}$ and $\hat{\boldsymbol{T}}$ to $\hat{\boldsymbol{U}}$ occurs at $O(\lambda)$ if and only if it also exists for steady Stokes flow.

Exercises

Exercise 6.1 Oscillatory Rotation of a Sphere.
Consider the velocity field produced by a sphere undergoing oscillatory rotation $\hat{\boldsymbol{\omega}} e^{-i\omega t} \times \boldsymbol{x}$ in a quiescent fluid. Starting with the transient rotlet,

$$\hat{\boldsymbol{v}} = (\hat{\boldsymbol{T}} \cdot \nabla) \cdot \mathcal{G}(\boldsymbol{x})/(8\pi\mu) \ .$$

Show that on the sphere surface, the rotlet reduces to the desired rotational motion if

$$T_{ij} = \epsilon_{ijk} \hat{\omega}_k \frac{4\pi\mu a^3 e^{\lambda}}{1+\lambda} \ .$$

Calculate the surface tractions and verify that the torque on the sphere is given by Equation 6.9. Invert this result and show that in the time domain

$$\begin{aligned} \boldsymbol{T}(t) &= -8\pi\mu a^3 \boldsymbol{\omega}(t) \\ &- \frac{8}{3} \int_{-\infty}^{t} \dot{\boldsymbol{\omega}}(t') \left[\frac{\mu a^4 \pi^{1/2}}{\sqrt{\nu(t-t')}} - \pi a^5 \rho e^{t-t'} \operatorname{erfc} \sqrt{t-t'} \right] dt' \ . \end{aligned}$$

Exercise 6.2 The Reciprocal Theorem for Unsteady Flow.
Derive the reciprocal theorem for unsteady flow, Equation 6.11.

Exercise 6.3 The Integral Representation for Unsteady Flow.
Derive the integral representation (in terms of single layer and double layer potentials) for unsteady Stokes flow. Show that the disturbance field can be expressed using just the single layer potential.

Now consider the vector field defined by $\hat{\boldsymbol{\Omega}} e^{-i\omega t} \times \boldsymbol{x}$ over some region V (with $\boldsymbol{\Omega}$ a constant pseudo-vector). Does this vector field satisfy the time-dependent Stokes equations?

Hint: Consider the following experiment. A closed container completely filled with a viscous fluid is subjected to oscillatory rigid-body rotation. Will the fluid inside undergo oscillatory rigid-body rotation in step with the boundary of the container? What are the implications for the existence of a single layer representation for unsteady particulate Stokes flows?

Exercise 6.4 Viscous Drop at the Low-Frequency Limit.
Show that the $O(\lambda)$ term for the drag on the viscous drop is given by the square of the resistance tensor for the drop in steady flow.

Exercise 6.5 Rigid Sphere in a Linear Field.
Consider a fixed rigid sphere centered in the rate-of-strain field, $\hat{\boldsymbol{v}}^{\infty} = \boldsymbol{E}^{\infty} \cdot \boldsymbol{x}$. Construct the disturbance solution using the transient stresslet and degenerate octupole.

Exercise 6.6 Impulse on a Rigid Sphere.
Consider an impulsive external force, $\boldsymbol{F}^e(t) = \boldsymbol{P}\delta(t)$, and find the formal expression for the motion of the sphere, $\boldsymbol{U}(t)$.

Exercise 6.7 Sphere Released from Rest.
For a sphere released from rest, we have $\boldsymbol{F}^e(t) = 0$, for $t < 0$, and $\boldsymbol{F}^e(t) = \boldsymbol{F}_0$ constant, for $t \geq 0$. Show that the expression for $\boldsymbol{U}(t)$ may be written in the more simple form,

$$\boldsymbol{U}(t) = \boldsymbol{F}_0 g(t) + \boldsymbol{F}_0 \int_0^t g(t-t')\,dt' - \boldsymbol{F}_0 \left[\frac{e^{m_+ t}\mathrm{erf}(\sqrt{m_+ t})}{\sqrt{m_+}(\beta - 4)} - \frac{e^{m_- t}\mathrm{erf}(\sqrt{m_- t})}{\sqrt{m_-}(\beta - 4)} \right] ,$$

$$g(t) = \frac{e^{m_+ t} - e^{m_- t}}{m_+ - m_-} = 2\exp\left\{ \left(\frac{\beta}{2} - 1 \right) t \right\} \frac{\sin\left(\frac{1}{2}\sqrt{\beta(4-\beta)}\,t \right)}{\sqrt{\beta(4-\beta)}} .$$

The force, velocity, and time have been scaled with F_0, $F_0/6\pi\mu a$, and $a^2/\nu\beta$. Use this result to derive asymptotic expansions for small t and for large t. For $\rho_p/\rho = 1 + \epsilon$, $0 < \epsilon \ll 1$, (for example, a particle tracer) what is the nature of the decay to the terminal Stokes velocity?

Although m_+ and m_- can become complex, the above can always be rewritten as a real expression, using integral representations for the error function (see [1]). In any event, the complex form is more convenient for working out the asymptotic expansions.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1972.

[2] A. D. Alawneh and R. P. Kanwal. Singularity methods in mathematical physics. *SIAM Rev.*, 19:437–470, 1977.

[3] L. Arminski and S. Weinbaum. Effect of waveform and duration of impulse on the solution to the Basset–Langevin equation. *Phys. Fluids*, 22:404–411, 1979.

[4] A. B. Basset. *A Treatise on Hydrodynamics*, Volume 2. Deighton, Bell and Co., Cambridge, 1888.

[5] G. K. Batchelor. Slender-body theory for particles of arbitrary cross-section in Stokes flow. *J. Fluid Mech.*, 44:419–440, 1970.

[6] J. M. Bernal and J. de la Torre. Transport properties and hydrodynamic centers of rigid macromolecules with arbitrary shape. *Biopolymers*, 19:751–766, 1980.

[7] R. B. Bird, O. Hassager, R. C. Armstrong, and C. Curtiss. *Dynamics of Polymeric Liquids*, Volume 2, 2nd edition. Wiley, New York, 1987.

[8] F. Booth. The cataphoresis of spherical particles in strong fields. *J. Chem. Phys.*, 18:1361-1364, 1950.

[9] H. Brenner. The slow motion of a sphere through a viscous fluid towards a plane surface. *Chem. Eng. Sci.*, 16:242–251, 1961.

[10] H. Brenner. The Stokes resistance of an arbitrary particle. *Chem. Eng. Sci.*, 18:1-25, 1963.

[11] H. Brenner. The Stokes resistance of a slightly deformed sphere. *Chem. Eng. Sci.*, 19:519–539, 1964.

[12] H. Brenner. The Stokes resistance of an arbitrary particle — V. Symbolic operator representation of intrinsic resistance. *Chem. Eng. Sci.*, 21:97–109, 1966.

[13] H. Brenner. Coupling between the translational and rotational Brownian motions of rigid particles of arbitrary shape. *J. Colloid Interface Sci.*, 23:407–436, 1967.

[14] H. Brenner. Suspension rheology. *Prog. Heat and Mass Transfer*, 5:89–129, 1972.

[15] H. Brenner and M. E. O'Neill. On the Stokes resistance of multiparticle systems in a linear shear field. *Chem. Eng. Sci.*, 27:1421–1439, 1972.

[16] H. Brenner. Rheology of a dilute suspension of axisymmetric Brownian particles. *Int. J. Multiphase Flow*, 1:195–341, 1974.

[17] S. Broersma. Rotational diffusion coefficient of a cylindrical particle. *J. Chem. Phys.*, 32:1626–1631, 1960.

[18] S. Broersma. Viscous force constant for a closed cylinder. *J. Chem. Phys.*, 32:1632–1635, 1960.

[19] J. M. Burgers. On the motion of small particles of elongated form suspended in a viscous liuid. Second report on viscosity and plasticity, Chap. III. *Kon. Ned. Akad. Wet.*, 16:113–184, 1938.

[20] A. T. Chwang and T. Y. Wu. Hydromechanics of low-Reynolds-number flow. Part 2. Singularity method for Stokes flows. *J. Fluid Mech.*, 67:787–815, 1975.

[21] W. D. Collins. A note on the axisymmetric Stokes flow of viscous fluid past a spherical cap. *Mathematika*, 10:72–78, 1963.

[22] R. G. Cox. The deformation of a drop in a general time-dependent fluid flow. *J. Fluid Mech.*, 37:601–623, 1969.

[23] R. G. Cox. The motion of long slender bodies in a viscous fluid. Part 1. General theory. *J. Fluid Mech.*, 44:791–810, 1970.

[24] P. Debye and E. Hückel. Bemerkungen zu einem Satze über die kataphoretische Wanderungsgeschwindigkeit suspendierter Teilchen (Remarks on a theorem concerning the cataphoretic velocity of suspended particles). *Physik. Z.*, 25:49–52, 1924. Also in *Collected Papers*, Interscience, New York, 1954.

[25] S. S. Dukhin and B. V. Derjaguin. *Electrokinetic Phenomena in Surface and Colloid Science*, Vol. 7, E. Matijević, Editor. John Wiley and Sons, New York, 1974.

[26] D. Edwardes. Steady motion of a viscous liquid in which an ellipsoid is constrained to rotate about a principal axis. *Q. J. Math.*, 26:70–78, 1892.

[27] M. Fair and J. L. Anderson. Electrophoresis of nonuniformly charged ellipsoidal particles. *J. Colloid Interface Sci.*, 127:388–400, 1989.

[28] H. Faxén. Der Widerstand gegen die Bewegung einer starren Kugel in einer zähen Flüssigkeit, die zwischen zwei parallelen Ebenen Wänden eingeschlossen ist (The resistance against the movement of a rigid sphere in a viscous fluid enclosed between two parallel planes). *Annalen der Physik*, 4(68):89–119, 1922.

[29] H. Faxén. Der Widerstand gegen die Bewegung einer starren Kugel in einer zähen Flüssigkeit, die zwischen zwei parallelen Ebenen Wänden eingeschlossen ist (The resistance against the movement of a rigid sphere in a viscous fluid enclosed between two parallel planes). *Arkiv fur Matematik, Astronomi och Fysik*, 18(29):1–52, 1924.

[30] F. R Gantmacher. *The Theory of Matrices.* Chelsea, New York, 1960.

[31] H. Giesekus. Elektro-viskose Flüssigkeiten, für die in stationären Schichtströmungen sämtliche Normalspannungskomponenten verschieden groß sind (Electro-viscous fluids with different normal stress components in steady-state laminar flow). *Rheol. Acta*, 2:50–62, 1962.

[32] M. J. Gluckman, R. Pfeffer, and S. Weinbaum. A new technique for treating multiparticle slow viscous flow: axisymmetric flow past spheres and spheroids. *J. Fluid Mech.*, 50:705–740, 1971.

[33] S. Haber, G. Hetsroni, and A. Solan. On the low Reynolds number motion of two droplets. *Intl. J. Multiphase Flow*, 1:57–71, 1973.

[34] R. A. Handelsman and J. B. Keller. Axially symmetric potential flow around a slender body. *J. Fluid Mech.*, 28:131–147, 1967.

[35] J. Happel and H. Brenner. *Low Reynolds Number Hydrodynamics.* Martinus Nijhoff, The Hague, 1983.

[36] L. B. Harris. Simplified calculation of electrophoretic mobility of nonspherical particles when the electric double layer is very extended. *J. Colloid Interface Sci.*, 34:322–325, 1970.

[37] D. C. Henry. The cataphoresis of suspended particles. Part I. The equation of cataphoresis. *Proc. R. Soc.*, A133:106–129, 1931.

[38] G. Hetsroni and S. Haber. Flow in and around a droplet or bubble submerged in an unbounded arbitrary velocity field. *Rheol. Acta*, 9:488–496, 1970.

[39] E. J. Hinch. Note on the symmetries of certain material tensors for a particle in Stokes flow. *J. Fluid Mech.*, 54:423–425, 1972.

[40] E. J. Hinch and L. G. Leal. The effect of Brownian motion on the rheological properties of a suspension of non-spherical particles. *J. Fluid Mech.*, 52:683–712, 1972.

[41] E. J. Hinch and L. G. Leal. Time-dependent shear flows of a suspension of particles with weak Brownian rotations. *J. Fluid Mech.*, 57:753–767, 1973.

[42] E. W. Hobson. *The Theory of Spherical and Ellipsoidal Harmonics.* Chelsea, New York, 1965.

[43] E. Hückel. Die Kataphorese der Kugel (The cataphoresis of the sphere). *Physik. Z.*, 25:204–210, 1924.

[44] R. J. Hunter. *Zeta Potential in Colloid Science.* Academic Press, New York, 1981.

[45] J. D. Jackson. *Classical Electrodynamics*, 2nd edition. Wiley, New York, 1975.

[46] G. B. Jeffery. The motion of ellipsoidal particles immersed in a viscous fluid. *Proc. R. Soc.*, A102:161–179, 1922.

[47] R. E. Johnson. An improved slender-body theory for Stokes flow. *J. Fluid Mech.*, 99:411–431, 1980.

[48] H. J. Keh and J. L. Anderson. Boundary effects on electrophoretic motion of colloidal spheres. *J. Fluid Mech.*, 153:417–439, 1985.

[49] S. Kim and X. Fan. A perturbation solution for rigid dumbbell suspensions in steady shear flow. *J. Rheol.*, 28(2):117–122, 1984.

[50] S. Kim. A note on Faxén laws for nonspherical particles. *Intl. J. Multiphase Flow*, 11(5):713–719, 1985.

[51] S. Kim and R. T. Mifflin. The resistance and mobility functions of two equal spheres in low-Reynolds-number flow. *Phys. Fluids*, 28:2033–2045, 1985.

[52] S. Kim. Singularity solutions for ellipsoids in low-Reynolds-number flows: with applications to the calculation of hydrodynamic interactions in suspensions of ellipsoids. *Intl. J. Multiphase Flow*, 12:469–491, 1986.

[53] H. Lamb. *Hydrodynamics*, 6th edition. Dover, New York, 1932.

[54] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics.* Pergamon Press, New York, 1959.

[55] C. J. Lawrence and S. Weinbaum. The force on an axisymmetric body in linearized, time-dependent motion: a new memory term. *J. Fluid Mech.*, 171:209–218, 1987.

[56] L. G. Leal and E. J. Hinch. The effect of weak Brownian rotations on particles in shear flow. *J. Fluid Mech.*, 46:685–703, 1971.

[57] W. H. Liao and D. Krueger. Multipole expansion calculation of slow viscous flow about spheroids of different sizes. *J. Fluid Mech.*, 96:223–241, 1980.

[58] M. R. Maxey and J. J. Riley. Equation of motion for a small rigid sphere in a nonuniform flow. *Phys. Fluids*, 26:883–889, 1983.

[59] P. Mazur and D. Bedeaux. A generalization of Faxén's theorem to nonsteady motion of a sphere through an incompressible fluid in arbitrary flow. *Physica*, 76:235–246, 1974.

[60] L. M. Milne-Thomson. *Theoretical Hydrodynamics*, 4th edition. Macmillan, New York, 1960.

[61] T. Miloh. The ultimate image singularities for external ellipsoidal harmonics. *SIAM J. Appl. Math.*, 26(2):334–344, 1974.

[62] Jr. Morrison, F. A. Electrophoresis of a particle of arbitrary shape. *J. Colloid Interface Sci.*, 34:210–214, 1970.

[63] A. Oberbeck. Über stationäre Flüssigkeitsbewegungen mit Berücksichtigung der inneren Reibung (On steady-state flow under consideration of inner friction). *J. Reine. Angew. Math.*, 81:62–80, 1876.

[64] R. W. O'Brien and L. R. White. Electrophoretic mobility of a spherical colloidal particle. *J. Chem. Soc., Faraday Trans.*, 74(2):1607–1626, 1978.

[65] L. E. Payne and W. H. Pell. The Stokes flow problem for a class of axially symmetric bodies. *J. Fluid Mech.*, 7:529–549, 1960.

[66] W. H. Pell and L. E. Payne. On Stokes flow about a torus. *Mathematika*, 7:78–92, 1960.

[67] A. Peterlin. Über die Viskosität von verdünnten Lösungen und Suspensionen in Abhängigkeit von der Teilchenform (On the viscosity of dilute solutions and suspensions governed by particle shape). *Z. Phys.*, 111:232–263, 1938.

[68] C. Pozrikidis. The instability of a moving viscous drop. *J. Fluid Mech.*, 210:1–21, 1990.

[69] J. M. Rallison. Note on the Faxén relations for a particle in Stokes flow. *J. Fluid Mech.*, 88:529–533, 1978.

[70] J. M. Rallison and A. Acrivos. A numerical study of the deformation and burst of a viscous drop in an extensional flow. *J. Fluid Mech.*, 89((1)):191–200, 1978.

[71] R. A. Sampson. On Stokes' current function. *Phil. Trans. R. Soc. Lond.*, A182:449–518, 1891.

[72] H. A. Scheraga. Non-Newtonian viscosity of solutions of ellipsoidal particles. *J. Chem. Phys.*, 23:1526–1532, 1955.

[73] R. Schmitz and B. U. Felderhof. Creeping flow about a spherical particle. *Physica*, 113A:90–102, 1982.

[74] L. E. Scriven. Dynamics of a fluid interface. *Chem. Eng. Sci.*, 12:98–108, 1960.

[75] D. J. Shaw. *Introduction to Colloid and Surface Chemistry*, 3rd edition. Butterworths, London, 1980.

[76] M. Smoluchowski. Elektrische Endosmose und Strömungsströme (Electoosmosis and current flow), in *Handbuch der Elektrizität und des Magnetismus Volume 2*, L. Graetz, Editor. Barth, Leipzig, 1921.

[77] W. E. Stewart and J. P. Sϕrensen. Hydrodynamic interaction effects in rigid dumbbell suspensions. II. Computations for steady shear flow. *Trans. Soc. Rheol.*, 16:1–13, 1972.

[78] D. Stigter. Electrophoresis of highly-charged colloidal cylinders in univalent salt solutions. 1. Mobility in transverse field. *J. Phys. Chem.*, 82:1417–1423, 1978.

[79] D. Stigter. Electrophoresis of highly-charged colloidal cylinders in univalent salt solutions. 2. Random orientation in external field and application to polyelectrolytes. *J. Phys. Chem.*, 82:1424–1429, 1978.

[80] M. Stimson and G. B. Jeffery. The motion of two spheres in a viscous fluid. *Proc. R. Soc.*, A111:110–116, 1926.

[81] S. R. Strand. *Rheological and Rheo-optical Properties of Dilute Suspensions of Dipolar Brownian Particles.* Ph.D. Dissertation, University of Wisconsin, Madison, WI, 1989.

[82] S. R. Strand, S. Kim, and S. J. Karrila. Computation of rheological properties of suspensions of rigid rods: stress growth after inception of steady shear flow. *J. Non-Newtonian Fluid Mech.*, 24:311–329, 1987.

[83] G. Strang. *Linear Algebra and Its Applications.* Academic Press, New York, 1976.

[84] G. I. Taylor. Motion of axisymmetric bodies in viscous fluids, in *Problems of Hydrodynamics and Continuum Mechanics*, SIAM, Philadelphia, 1969.

[85] M. Teubner. The motion of charged colloidal particles in electric fields. *J. Chem. Phys.*, 76:5564–5573, 1982.

[86] J. P. K. Tillett. Axial and transverse Stokes flow past slender axisymmetric bodies. *J. Fluid Mech.*, 44:401–417, 1970.

[87] E. O. Tuck. Some methods for flows past blunt slender bodies. *J. Fluid Mech.*, 18:619–635, 1964.

[88] M. van Dyke. *Perturbation Methods in Fluid Mechanics.* Parabolic Press, Stanford, CA, 1975.

[89] E. J. W. Verwey and J. Th. G. Overbeek. *Theory of The Stability of Lyophobic Colloids.* Elsevier, Amsterdam, 1948.

[90] W. A. Wegener. Hydrodynamic resistance and diffusion coefficients of a freely-hinged rod. *Biopolymers*, 19:1899–1908, 1980.

[91] P. H. Wiersema, A. L. Loeb, and J. Th. G. Overbeek. Calculation of the electrophoretic mobility of a spherical colloid particle. *J. Colloid Interface Sci.*, 22:78–99, 1966.

[92] B. J. Yoon and S. Kim. Electrophoresis of spheroidal particles. *J. Colloid and Interface Sci.*, 128:275–288, 1988.

[93] C. F. Zukoski and D. A. Saville. Electrokinetic properties of particles in concentrated suspensions. *J. Colloid Interface Sci.*, 115:422–436, 1987.