Final Report - Research Internship

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1. Introduction

In the novel paper of Purcell [10], it is shown that any micro-organism trying to swim using a reciprocal movement like the one of a scallop, i.e. swimming by opening and closing a kind of valve, cannot move. This observation, also known as the *scallop theorem*[1] entails the problem of finding the simplest swimming mechanism at these scales; that is the capacity to advance using a periodic change of shape - a swimming *stroke* - in the absence of external forces. A variety of such mechanisms has already been proposed and analyzed, see e.g. [2], [9], [10].

The principal mathematical challenge of this problem stems from the low value of the Reynolds number $\mathrm{Re} = \rho u L/\mu$ which gives an estimate of the relative importance of inertial to viscous forces for an object of characteristic length scale L moving at speed u through a Newtonian fluid of density ρ and dynamic viscosity μ. In the low Reynolds number regime, the inertial forces become irrelevant and consequently, micro-swimmers can only utilize the viscous resistance of the surrounding fluid to move. In mathematical terms, the micro-swimmers are governed by the Stokes equations, which are symmetric under time reversal. In the case of the scallop, this means that whatever forward motion is caused by closing its valves, it will exactly be compensated by the movement produced by reopening them, regardless of the speed of these two processes.

In [2], a whole class of micro-swimmers, able to overcome this problem, is presented. The said paper rigorously proves the controllability of the swimming mechanisms and puts forward a numerical method to address the problem of optimal swimming. However, their explicit dynamics as well as the structure of optimal swimming strokes remain largely unknown. In this paper, we will analyze further the swimmer SPr4 from [2] and shed a light on the latter aspects. The analysis will take place very much in the spirit of the treatment of the swimmer SPr3 in [3], which originally also has been presented in [2] as well. In fact, the swimmer SPr4 is a natural generalization of the swimmer SPr3 capable of moving in the entire $3d$ space instead of just a plane. Although the principal techniques used in this paper are in close analogy to the ones in [3], the more complex geometry of both the position and the shape space cause the analysis to be more involved.

Aim of this paper is to *analytically* address the optimal control problem for SPr4 in the range of *small* strokes.

The rest of the paper is organized as follows: in section 2, we give both a geometric and a kinematic description of parking 4-sphere swimmer (SPr4). Next, we introduce the control system treated in this paper. In section 3, we study the geometric structure of the control system taking advantage of the symmetries it has to satisfy due to the underlying Stokes equations. In section 4, we unravel the properties of the control system in the range of small strokes. Eventually, section 5 addresses the characterization of energy minimizing strokes.

[1] For a proof as well as an elementary introduction to the topic we refer to the encyclopedia article [4].

2. Modeling of the swimmer as a control problem

We restrict ourselves to considering the swimmer SPr4 proposed in [2]. Let (S_1, S_2, S_3, S_4) be a regular reference tetrahedron centered at $c \in \mathbb{R}^3$ such that $\operatorname{dist}(c, S_i) = 1$ for all $i \in \mathbb{N}_4$. Then the swimmer consists of four balls $(B_i)_{i\in\mathbb{N}_4}$ of $\mathbb{R}^3$ centered at $b_i \in \mathbb{R}^3$, all of radius $a > 0$, such that the ball B_i can move along the ray starting at c and passing through S_i. This reflects the situation where the balls are linked together by think jacks that are able to elongate. However, the viscous resistance of these jacks is neglected and therefore the fluid is assumed to permeate the entire open set $\mathbb{R}^3 \setminus \bigcup_{i=1}^4 \overline{B}_i$. The balls do not rotate around their arms which implies that the shape of the swimmer is completely determined by the four lengths $\xi_1, \xi_2, \xi_3, \xi_4$ of its arms, measured from the c to the center of each ball b_i. However, there are no restrictions for the rotation of the swimmer around the center c, i.e. for fixed arm lengths, the swimmer is considered to be a rigid body in a Stokesian fluid. Hence, the geometrical configuration of the swimmer can be described by two sets of variables:

(i) The vector of *shape variables* $\xi := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{M} := (\sqrt{\frac{3}{2}}a, +\infty)^4 \subseteq \mathbb{R}_+^4$, from which one obtains the relative distances $(b_{ij})_{i,j\in\mathbb{N}_4}$ between the balls, where the lower bound in the open intervals is chosen such that the balls cannot overlap.

(ii) The vector of *position variables* $p = (c, R) \in \mathcal{P} := \mathbb{R}^3 \times \mathrm{SO}(3)$, which encode the global position and orientation in space of the swimmer.

To be more precise, we consider the reference tetrahedron convexly spanned by the four unit vectors $z_1 := (2\sqrt{2}/3, 0, -1/3)$, $z_2 := (-\sqrt{2}/3, -\sqrt{2/3}, -1/3)$, $z_3 := (-\sqrt{2}/3, \sqrt{2/3}, -1/3)$ and $z_4 := (0, 0, 1)$. Position and orientation in $\mathbb{R}^3$ are then described by the coordinates of the center $c \in \mathbb{R}^3$ and the rotation $R \in SO(3)$ of the swimmer with respect to the reference orientation induced by the reference tetrahedron, i.e. if the arms are aligned with the reference tetrahedron, then this corresponds to the identity matrix $I \in \mathrm{SO}(3)$. Thus, we set $b_i := c + \xi_i R z_i$ for the center of the ball B_i.

The swimmer is completely described by the parameters $(\xi, p) \in \mathcal{M} \times \mathcal{P}$. Indeed, if we denote by B_a the ball in $\mathbb{R}^3$ of radius a centered at the origin, then for any $r \in \partial B_a$, the position of the current point on the i-th sphere of the swimmer in the state (ξ, p) is given, for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$, by the function

$$r_i(\xi, p, r) := c + R(\xi_i z_i + r). \tag{2.1}$$

Note that the functions $(r_i)_{i\in\mathbb{N}_4}$ are analytic in $\mathcal{M} \times \mathcal{P}$ and thus we can use them to calculate the instantaneous velocity on the i-th sphere B_i, which for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$ and every $i \in \mathbb{N}_4$ is given by

$$u_i(\xi, p, r) = \dot{c} + \omega \times (\xi z_i + r) + R z_i \dot{\xi}_i, \tag{2.2}$$

where ω is the axial vector associated with the skew matrix $\dot{R}R$.

In [2] it is shown that the system SPr4 , i.e. both the shape ξ and the position p, is controllable only using the rate of change $\dot{\xi}$ of the shape. To do so, we have to understand how p changes when we vary $\dot{\xi}$. To that end, the assumptions of *self-propulsion* and *negligible inertia of the swimmer* (which is equivalent to assuming a very low Reynolds number) are made. They imply that the total viscous force and torque exerted by the surrounding fluid on the swimmer must vanish. More precisely, the system can be written as

$$\dot{p} = F(R, \xi)\dot{\xi} := \begin{pmatrix} F_c(R, \xi) \\ \hline F_\theta(R, \xi) \end{pmatrix} \dot{\xi}, \tag{2.3}$$

where $\dot{c} = F_c(R, \xi)\dot{\xi}$ and $\dot{R} = F_\theta(R, \xi)\dot{\xi}$.

In preparation for what follows, let us note that we have $F(R, \xi) \in \mathcal{L}(\mathbb{R}^4, T_p\mathcal{P})$ for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$, where $\mathcal{L}(V, W)$ denotes the linear maps between two vector spaces

V and W. We quickly recall the fact that at any point $R \in \mathrm{SO}(3)$, see e.g. [6] for details, we have

$$T_R\,\mathrm{SO}(3) = R^*\,\mathrm{Skew}_3(\mathbb{R}) = \{RM \mid M \in \mathrm{Skew}_3(\mathbb{R})\}, \tag{2.4}$$

where $\mathrm{Skew}_n(\mathbb{R})$ denotes the set of skew-symmetric real matrices of size $n \times n$. Hence, we have in particular that for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$

$$F_c(R,\xi) \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3) \text{ and } F_\theta(R,\xi) \in \mathcal{L}(\mathbb{R}^4, R^*\,\mathrm{Skew}_3(\mathbb{R})) \tag{2.5}$$

and therefore we can express both $F_c(R,\xi)$ and $F_\theta(R,\xi)$ as real matrices of size 3×4 once we have chosen a basis for the corresponding tangent spaces. Indeed, one verifies quickly that $\mathrm{Skew}_3(\mathbb{R})$ is a three-dimensional vector space over $\mathbb{R}$.

In analogy to [3], it is important to note here that the control system F is independent of c due to the translational invariance of the Stokes equations. However, the translational invariance is not the only symmetry property that SPr4 satisfies. The goal of the following section is to examine the structure of the control system F in consequence of the symmetries it must satisfy being driven by the Stokes equations.

3. Symmetries

For any initial condition $p_0 = (c_0, R_0) \in \mathcal{P}$ and any control curve $\xi : I \subseteq \mathbb{R} \to \mathcal{M}$, with I a neighborhood of zero, we denote $\gamma(c_0, R_0, \xi) : I \to \mathcal{P}$ the solution associated to the dynamical system

$$\dot{p} = F(R,\xi)\dot{\xi}, \quad p(0) := p_0, \tag{3.1}$$

as well as $\gamma_c(c_0, R_0, \xi)$ and $\gamma_\theta(c_0, R_0, \xi)$ its projections on $\mathbb{R}^3$ and $\mathrm{SO}(3)$, respectively, such that for any $t \in I$

$$\dot{\gamma}(c_0, R_0, \xi)(t) = F(\gamma_\theta(c_0, R_0, \xi)(t), \xi(t))\dot{\xi}(t), \tag{3.2}$$

and similarly for the projections $\gamma_c(c_0, R_0, \xi)$ and $\gamma_\theta(c_0, R_0, \xi)$.

3.1 Rotational invariance

Rotational invariance of the Stokes equations expresses the fact that the solution of the dynamical system (3.1) is invariant under rotations, i.e. that for any rotation $R \in \mathrm{SO}(3)$ we have for the spatial part of the solution

$$\gamma_c(c_0, RR_0, \xi)(t) = R\gamma_c(c_0, R_0, \xi)(t) + (I - R)c_0 \tag{3.3}$$

and for the angular part of the solution

$$\gamma_\theta(c_0, RR_0, \xi)(t) = R\gamma_\theta(c_0, R_0, \xi)(t) \tag{3.4}$$

at any point in time $t \in I$. Eventually, we can rigorously state the following symmetry property of the control system (3.1) with respect to rotations:

Condition 1 (Rotational invariance)**.** *If $\gamma(c_0, R_0, \xi)$ is a solution of the control system (3.1) then so is $\gamma(c_0, RR_0, \xi)$ and (3.3) and (3.4) hold.*

Remark. To follow the reasoning of [3], the symmetry relations satisfied by SPr4 are stated as hypotheses on the solution γ. In so doing, the results work for any control system of the form (2.3) and satisfying the hypotheses we state, independently of these hypotheses being guaranteed by the invariance of the Stokes equations under a certain group of transformations.

We then have

Proposition 2. *Let $\xi_0 := \xi(0) \in \mathcal{M}$ denote the initial state of the control parameters and by $T_\xi \mathcal{M}$ the tangent space of $\mathcal{M}$ at ξ. If the control system (3.1) is invariant under rotations and for every $\xi \in \mathcal{M}$ it holds that $T_\xi \mathcal{M} \simeq \mathbb{R}^4$, then*

$$F_c(R, \xi) = RF_c(\xi) \text{ and } F_\theta(R, \xi) = RF_\theta(R, \xi), \tag{3.5}$$

for every $(R, \xi) \in \mathrm{SO}(3) \times \mathcal{M}$, where $F_c(\xi) := F_c(I, \xi)$ and $F_\theta(\xi) := F_\theta(I, \xi)$.

Proof. On one hand, we have by definition of the dynamical system (3.1) that

$$\dot{\gamma}_c(c_0, R, \xi) = F_c(\gamma_\theta(c_0, R, \xi), \xi)\dot{\xi}. \tag{3.6}$$

On the other hand, using equation (3.3) and once more the definition of the dynamical system (2.3), we obtain

$$\dot{\gamma}_c(c_0, R, \xi) = R\dot{\gamma}_c(c_0, I, \xi) = RF_c(\gamma_\theta(c_0, I, \xi), \xi)\dot{\xi}. \tag{3.7}$$

Therefore, $F_c(\gamma_\theta(c_0, R, \xi), \xi)\dot{\xi} = RF_c(\gamma_\theta(c_0, I, \xi), \xi)\dot{\xi}$ for every $R \in \mathrm{SO}(3)$. Since $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$, evaluation of the preceding expression at $t = 0$ yields $F_c(R, \xi_0) = RF_c(I, \xi_0)$, as desired. The proof for F_θ is completely analogous and thus is omitted. □

3.2 Permutation of two arms

In this section, we investigate the effect of a swap of two arms on the generic solution of the dynamical system (3.1). To that end, let $P_{ij} \in M_{4\times 4}(\mathbb{R})$ denote the permutation matrix that interchanges the i-th and j-th index of a vector, which corresponds to the swap of the arms $||i$ and $||j$, denoted by $(||i \leftrightsquigarrow ||j)$, if applied to the shape space $\mathcal{M}$. In addition, let S_{ij} denote the reflection of $\mathbb{R}^3$ sending arm $||i$ onto arm $||j$ in the reference orientation I. Geometrical inspection of the reference tetrahedron shows that S_{ij} is always a reflection at a plane containing the remaining arms $||k$ and $||l$.

Before we formulate the symmetry conditions for the interchanging of two arms, we recall some results about how rotations behave under reflections. So far, we have only regarded the orientation of SPR4 as a rotation matrix in SO(3). However, by Euler's rotation theorem to every such rotation matrix $R \in \mathrm{SO}(3)$ there exists a corresponding rotation vector $\omega \in \mathbb{R}^3$ which is collinear to the unique axis of rotation defined by R, i.e. ω is an eigenvector associated to the eigenvalue 1 of R. It's length is given by the angle of rotation around this axis. The rotation vector ω is then directly related to the rotation matrix R via the map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, where $\mathfrak{so}(3) = T_I\,\mathrm{SO}(3) = \mathrm{Skew}_3(\mathbb{R})$ denotes the Lie algebra over SO(3), which we will illustrate in the following paragraphs.

It is clear that $\dim \mathrm{Skew}_3(\mathbb{R}) = 3$. In particular, if $R_1(\theta), R_2(\theta)$ and $R_3(\theta)$ denote the simple rotations around the $\hat{e}_1$-, $\hat{e}_2$- and $\hat{e}_3$-axis, where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the canonical basis vectors of $\mathbb{R}^3$, then the matrices

$$L_1 = \frac{\mathrm{d}}{\mathrm{d}\theta} R_1(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.8}$$

$$L_2 = \frac{d}{d\theta} R_2(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

$$L_3 = \frac{d}{d\theta} R_3(\theta)_{|\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

form a basis of $\mathfrak{so}(3)$ consisting of the infinitesimal rotations around the corresponding axes. If we now write $\mathbf{L} := (L_1, L_2, L_3)^T$ and allow the slight abuse of notation

$$\omega \cdot \mathbf{L} = \omega_1 L_1 + \omega_2 L_2 + \omega_3 L_3, \tag{3.11}$$

we find by direct computation that $\exp(\omega \cdot \mathbf{L}) = R$. This relationship allows us to formulate the behavior of the orientation of SPr4 under reflection and thus under permutation of two arms as we shall see later. Indeed, we have

Lemma 3. *For any orientation of a rigid body characterized by* $R \in \mathrm{SO}(3)$*, the orientation of its mirror image under a reflection* S *is given by*

$$\tilde{R} = SRS. \tag{3.12}$$

Proof. Let us first consider the simple case where S is the reflection of the $\hat{e}_1$-axis. Let ω and $\tilde{\omega}$ be the rotation vectors corresponding to R and $\tilde{R}$, respectively. They are related by $\tilde{\omega} = -S\omega$, where the gain of the minus sign stems from the fact that rotation vectors are in fact pseudovectors. In other words, we not only reflect the axis of rotation but we also reverse the sense of rotation around the axis. It follows then from direct computation that

$$\tilde{\omega} \cdot \mathbf{L} = (-S\omega) \cdot \mathbf{L} = S(\omega \cdot \mathbf{L})S \tag{3.13}$$

and thus we have

$$\tilde{R} = \exp(\tilde{\omega} \cdot \mathbf{L}) = \exp(S(\omega \cdot \mathbf{L})S) = SRS, \tag{3.14}$$

as $S^{-1} = S$.

If now S' is an arbitrary reflection, we always find a rotation $Q \in \mathrm{SO}(3)$ such that $S' = QSQ^T$. Moreover, for any rotation $R' \in \mathrm{SO}(3)$ we find another $R \in \mathrm{SO}(3)$ such that $R' = QRQ^T$. In particular, we have

$$\tilde{R}' = Q\tilde{R}Q^T = QSRSQ^T = S'R'S'^T, \tag{3.15}$$

as desired. □

With this lemma at hand, we can now finally state the following

Condition 4 (Swap $(||i \leftrightsquigarrow ||j)$)**.** *Let the initial position be* $p_0 := (c_0, I)$*. If* $\gamma(c_0, I, P_{ij}\xi)$ *is a solution of the control system (2.3), then so is* $\gamma(S_{ij}c_0, I, \xi)$ *and the following relations hold*

$$\gamma_c(c_0, I, P_{ij}\xi) = S_{ij}\gamma_c(S_{ij}c_0, I, \xi) \tag{3.16}$$

and

$$\gamma_\theta(c_0, I, P_{ij}\xi) = S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{3.17}$$

To avoid chaos in our notation, we treat the the spatial and angular parts now separately. For the spatial part, we find

Proposition 5. *If the control system (3.1) is invariant under the swap* $(||i \leftrightsquigarrow ||j)$ *and* $T_\xi\mathcal{M} \simeq \mathbb{R}^4$ *for all* $\xi \in \mathcal{M}$*, then for all* $\xi \in \mathcal{M}$

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij}. \tag{3.18}$$

Proof. Let $\gamma_c(c_0, R_0, P_{ij}\xi)$ be the spatial part of any solution of the control problem (2.3). The hypothesis of rotational invariance, i.e. (3.3), implies that

$$\gamma_c(c_0, R_0, P_{ij}\xi) = R_0\gamma_c(c_0, I, P_{ij}\xi) + (I - R_0)c_0. \tag{3.19}$$

From condition 3.2, we then get

$$\gamma_c(c_0, R_0, P_{ij}\xi) = R_0S_{ij}\gamma_c(S_{ij}c_0, I, \xi) + (I - R_0)c_0. \tag{3.20}$$

As both $\gamma_c(c_0, R_0, P_{ij}\xi)$ and $\gamma_c(S_{ij}c_0, I, \xi)$ are spatial parts of solutions of the control system (3.1), we have on one hand using Proposition 3.1

$$\dot{\gamma_c}(c_0, R_0, P_{ij}\xi) = \gamma_\theta(c_0, R_0, P_{ij}\xi)F_c(P_{ij}\xi)P_{ij}\dot{\xi}, \tag{3.21}$$

and on the other hand using (3.20) and once more (3.1) together with Proposition 3.1

$$\dot{\gamma_c}(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\dot{\gamma_c}(S_{ij}c_0, I, \xi) = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)F_c(\xi)\dot{\xi}. \tag{3.22}$$

Equating (3.21) and (3.22) at $t = 0$ yields $F_c(P_{ij}\xi_0) = S_{ij}F_c(\xi_0)P_{ij}$, since by hypothesis $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$. As ξ_0 was arbitrary, we conclude. □

For the angular part, we first have to choose a basis and fix some notation. Naturally, we choose the canonical basis $\mathcal{E} := (e_1, e_2, e_3, e_4)$ for $\mathbb{R}^4$. For $\mathfrak{so}(3)$, we choose the basis $\mathcal{L} := (L_1, L_2, L_3)$, where the matrices L_i are defined in (3.8) - (3.10). Then we denote the the matrix representing an arbitrary linear map $T : V \to W$ between two vector spaces V and W with respect to two bases $\mathcal{B}$ and $\mathcal{B}'$ by $[T]^{\mathcal{B}'}_{\mathcal{B}}$. Let S be an arbitrary reflection at a plane in $\mathbb{R}^3$ and define the adjoint isomorphism $T_S : \mathfrak{so}(3) \to \mathfrak{so}(3)$ by $M \mapsto SMS$. Then the calculations leading to (3.13) show that in fact we have $[T_S]^{\mathcal{L}}_{\mathcal{L}} = -S$. This leads to the following

Proposition 6. *If the control system (3.1) is invariant under the swap* $(||i \leftrightsquigarrow ||j)$ *and* $T_\xi\mathcal{M} \simeq \mathbb{R}^4$ *for all* $\xi \in \mathcal{M}$*, then for all* $\xi \in \mathcal{M}$

$$[F_\theta(P_{ij}\xi)]^{\mathcal{L}}_{\mathcal{E}} = -S_{ij}[F_\theta(\xi)]^{\mathcal{L}}_{\mathcal{E}}P_{ij}. \tag{3.23}$$

Proof. Let $\gamma_\theta(c_0, R_0, P_{ij}\xi)$ be the angular part of any solution of the control problem (3.1). By the rotational invariance hypothesis, i.e. (3.4), we have

$$\gamma_\theta(c_0, R_0, P_{ij}\xi) = R_0\gamma(c_0, I, \xi). \tag{3.24}$$

Then Condition 3.2 implies that

$$\gamma_\theta(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{3.25}$$

Since both $\gamma_\theta(c_0, R_0, P_{ij}\xi)$ and $\gamma_\theta(S_{ij}c_0, I, \xi)$ are the angular parts of solutions of the control problem (3.1), we obtain with Proposition 3.1 on one hand

$$\dot{\gamma_\theta}(c_0, R_0, P_{ij}\xi) = \gamma_\theta(c_0, R_0, P_{ij}\xi)F_\theta(P_{ij}\xi)P_{ij}\dot{\xi} \tag{3.26}$$

and on the other hand using (3.25) and once more Proposition 3.1

$$\dot{\gamma_\theta}(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\dot{\gamma_\theta}(S_{ij}c_0, I, \xi)S_{ij} = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)F_\theta(\xi)\dot{\xi}S_{ij}. \tag{3.27}$$

Imposing equality of (3.26) and (3.27) at $t = 0$ yields

$$F_\theta(P_{ij}\xi_0)P_{ij}\dot{\xi}(0) = S_{ij}F_\theta(\xi_0)\dot{\xi}(0)S_{ij}. \tag{3.28}$$

By choice of the canonical basis for $\mathbb{R}^4$ we clearly have $[P_{ij}]^{\mathcal{E}}_{\mathcal{E}} = P_{ij}$. Therefore, by the reasoning concerning the linear map $T_{S_{ij}}$ above, we have

$$[F_\theta(P_{ij}\xi_0)]^{\mathcal{L}}_{\mathcal{E}}P_{ij}[\dot{\xi}(0)]_{\mathcal{E}} = [S_{ij}F_\theta(\xi_0)\dot{\xi}(0)S_{ij}]_{\mathcal{L}} = -S_{ij}[F_\theta(\xi_0)]^{\mathcal{L}}_{\mathcal{E}}[\dot{\xi}(0)]_{\mathcal{E}}. \tag{3.29}$$

Recalling that $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$ as well as the arbitrariness of ξ_0 finish the proof. □

In the following sections, we will always understand $F_\theta(\xi)$ as a matrix of size 3×4 and thus, since no confusion may arise, we will abandon the slightly cumbersome notation and identify $[F_\theta(\xi)]^{\mathcal{L}}_{\mathcal{E}}$ with $F_\theta(\xi)$.

4. The small strokes regime

Let us return to the control equations for SPr4 given by (3.1). The response of the control system is characterized by the two matrix valued functions $F_c, F_\theta : \mathrm{SO}(3) \times \mathbb{R}^4 \to M_{3\times 4}(\mathbb{R})$ which can by Proposition 3.1 can be factorized as:

$$F_c(R,\zeta) = RF_c(\zeta) \text{ and } F_\theta(R,\xi) = RF_\theta(\zeta), \tag{4.1}$$

with $F_c(\zeta) := F_c(I,\zeta)$ and $F_\theta(\zeta) := F_\theta(I,\zeta)$. Hereinafter, we suppose that $\zeta := \xi_0 + \xi$ with $\xi_0 \in \mathcal{M}$ having all its components equal. Furthermore, we set $F_{c,\xi_0}(\xi) := F_c(\xi_0+\xi)$ and analogously $F_{\theta,\xi_0}(\xi) := F_\theta(\xi_0+\xi)$. It has been shown in [2] that F and thus both F_{c,ξ_0} and F_{θ,ξ_0} are analytic functions. Therefore, we can consider their first order expansions in ξ:

$$F_{c,\xi_0}(\xi)\eta = F_{c,0}\eta + \mathcal{H}_{c,0}(\xi\otimes\eta) + \mathcal{O}(|\xi|)\eta \tag{4.2}$$

$$F_{\theta,\xi_0}(\xi)\eta = F_{\theta,0}\eta + \mathcal{H}_{c,0}(\xi\otimes\eta) + \mathcal{O}(|\xi|)\eta, \tag{4.3}$$

where $F_{c,0} := F_c(\xi_0) \in M_{3\times 4}(\mathbb{R})$, $\mathcal{H}_{c,0} \in \mathcal{L}(\mathbb{R}^4 \otimes \mathbb{R}^4, \mathbb{R}^3)$ represents the first order derivative of F_{c,ξ_0} at $\xi = 0$ and for $F_{\theta,\xi}$ the analogous definitions are made. The purpose of this section is to reveal the structure of the different terms in the expansions (4.3) and (4.2) in light of the symmetry properties fulfilled by F_c and F_θ due to Propositions (3.2) and (3.2) , i.e

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij} \qquad \text{and} \qquad F_\theta(P_{ij}\xi) = -S_{ij}F_\theta(\xi)P_{ij} \qquad \forall \xi \in \mathcal{M}. \tag{4.4}$$

The following slightly generalized statement of Lemma 9 in [3] proves useful in our case as well. However, the proof is omitted, it being exactly the same.

Lemma 7. *Let $G : \mathbb{R}^n \to M_{m\times n}(\mathbb{R})$ be an analytic function and $S \in M_{m\times m}(\mathbb{R})$ and $P \in M_{n\times n}(\mathbb{R})$ matrices such that $G(P\xi) = SG(\xi)P$ for every $\xi \in \mathbb{R}^n$. For $\xi_0 \in \mathbb{R}^n$ with all components equal, set $G_{\xi_0}(\xi) := G(\xi_0+\xi)$ and write the first order expansion*

$$G_{\xi_0}(\xi)\eta = G_0\eta + \mathcal{H}_0(\xi\otimes\eta) + \mathcal{O}(|\xi|)\eta. \tag{4.5}$$

Then we have

$$G_0 = SG_0P, \tag{4.6}$$

and

$$\mathcal{H}_0((P\xi)\otimes\eta) = S\mathcal{H}_0(\xi\otimes(P\eta)) \qquad \forall \xi,\eta \in \mathbb{R}^n \tag{4.7}$$

4.1 The zeroth order terms $F_{c,0}$ and $F_{\theta,0}$

By applying Lemma 4 to F_{c,ξ_0} and F_{θ,ξ_0}, we obtain two linear systems of matrix equations in the unknowns $F_{c,0}$ and $F_{\theta,0}$. To solve these two systems, we eventually have to determine at least some of the matrices S_{ij}. Let $S_{kl}(\phi)$ denote the reflection at the plane orthogonal to the $\hat{e}_k - \hat{e}_l$-plane making an angle of ϕ with the $\hat{e}_k$-axis. By geometrical inspection of the reference tetrahedron (S_1, S_2, S_3, S_4), we find that

$$S_{12} = S_{12}\left(\tfrac{2\pi}{3}\right), \quad S_{23} = S_{12}(0), \quad S_{13} = S_{13}\left(\tfrac{\pi-\alpha_{\mathrm{tet}}}{2}\right), \tag{4.8}$$

where $\alpha_{\mathrm{tet}} = \arccos(-1/3)$ denotes the angle between two legs of a regular tetrahedron. Indeed, it happens that these three matrices are enough to determine both terms of order and one finds that $F_{\theta,0} = 0$ and

$$F_{c,0} = \begin{pmatrix} -2\mathfrak{a} & \mathfrak{a} & \mathfrak{a} & 0 \\ 0 & \sqrt{3}\mathfrak{a} & -\sqrt{3}\mathfrak{a} & 0 \\ \frac{1}{\sqrt{2}}\mathfrak{a} & \frac{1}{\sqrt{2}}\mathfrak{a} & \frac{1}{\sqrt{2}}\mathfrak{a} & \frac{-3}{\sqrt{2}}\mathfrak{a} \end{pmatrix} \text{ with } \mathfrak{a} \in \mathbb{R}. \tag{4.9}$$

In the following sections, we will exploit the orthonormal basis of $\mathbb{R}^4$ consisting of the vectors $\tau_1 := \frac{1}{\sqrt{6}}(-2,1,1,0)^T$, $\tau_2 := \frac{1}{\sqrt{2}}(0,1,-1,0)^T$, $\tau_3 := \frac{1}{2\sqrt{3}}(1,1,1,-3)^T$ and $\tau_4 := \frac{1}{2}(1,1,1,1)^T$, in terms of which $F_{c,0}$ can be written as $F_{c,0} = \mathfrak{a}\sqrt{6}[\tau_1|\tau_2|\tau_3]^T$.

Remark. First, we observe that the upper left corner of $F_{c,0}$ corresponding to the arms $||1, ||2$ and $||3$ is exactly the same as for SPr3 in [3]. Furthermore, one notes that physically it is clear that $F_{\theta,0}$ must vanish since by hypothesis ξ_0 has all its components equal and thus the swimmer is in a symmetric shape at $\xi = 0$. Therefore, the balls moving along their axes cannot create any torque. Lastly, one should note that apparently $F_{c,0} \sim (z_1|z_2|z_3|z_4)$ which can be shown directly using the symmetry properties.

4.2 The first order terms $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$

Following the approach in [3], we evaluate the tensors $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$ on the basis $(e_i \otimes e_j)_{i,j\in\mathbb{N}_4}$. Setting $A_k := (\mathcal{H}_{c,0}(e_i \otimes e_j)\cdot \hat{e}_k)_{i,j\in\mathbb{N}_4}$ and $B_k := (\mathcal{H}_{\theta,0}(e_i \otimes e_j)\cdot \hat{e}_k)_{i,j\in\mathbb{N}_4}$ for $k \in \mathbb{N}_3$, we can write the vectors $\mathcal{H}_{c,0}(\xi\otimes\eta), \mathcal{H}_{\theta,0}(\xi\otimes\eta) \in \mathbb{R}^3$ for any $\xi, \eta \in \mathbb{R}^3$ as

$$\mathcal{H}_{c,0}(\xi\otimes\eta) = \sum_{k\in\mathbb{N}_3} (A_k\eta\cdot\xi)\hat{e}_k, \tag{4.10}$$

and similarly

$$\mathcal{H}_{\theta,0}(\xi\otimes\eta) = \sum_{k\in\mathbb{N}_3} (B_k\eta\cdot\xi)L_k. \tag{4.11}$$

We could pursue the approach of [3] and directly calculate the matrices A_k and B_k. However, as we shall see later, the dynamics of SPr4 , up to higher order terms in the norm of the control curve ξ, will only be governed by their skew symmetric parts. Thus, we will evade this strenuous task and we determine the skew symmetric matrices $M_k := \frac{1}{2}(A_k - A_k^T)$ and $M_{k+3} := \frac{1}{2}(B_k - B_k^T)$ for $k \in \mathbb{N}_3$, up to two scalar parameters, using a more abstract argument.

To that end, we notice that Lemma 4 together with the fact that $(P_{ij})^2 = I$ yields for all $i, j \in \mathbb{N}_4$ and for all $\xi, \eta \in \mathbb{N}_4$

$$S_{ij}\mathcal{H}_{c,0}(P_{ij}\xi\otimes P_{ij}\eta) = \mathcal{H}_{c,0}(\xi\otimes\eta), \tag{4.12}$$

as well as

$$-S_{ij}\mathcal{H}_{\theta,0}(P_{ij}\xi\otimes P_{ij}\eta) = \mathcal{H}_{\theta,0}(\xi\otimes\eta). \tag{4.13}$$

Next, we define $\mathcal{K}_{c,0}(\xi\otimes\eta) := \frac{1}{2}[\mathcal{H}_{c,0}(\xi\otimes\eta) - \mathcal{H}_{c,0}(\eta\otimes\xi)]$ and similarly $\mathcal{K}_{\theta,0}$ such that

$$M_k = (\mathcal{K}_{c,0}(e_i\otimes e_j)\cdot\hat{e}_k)_{i,j\in\mathbb{N}_4} \text{ and } M_{k+3} = (\mathcal{K}_{\theta,0}(e_i\otimes e_j)\cdot\hat{e}_k)_{i,j\in\mathbb{N}_4},\ \ k\in\mathbb{N}_3. \tag{4.14}$$

In particular, it is clear that $\mathcal{K}_{c,0}$ and $\mathcal{K}_{\theta,0}$ satisfy the same symmetry relations as $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$, respectively, and additionally, we have $\mathcal{K}_{c,0}(e_i\otimes e_j) = -\mathcal{K}_{c,0}(e_j\otimes e_i)$ and $\mathcal{K}_{\theta,0}(e_i\otimes e_j) = -\mathcal{K}_{\theta,0}(e_j\otimes e_i)$.

For the spatial part, we deduce from the symmetry properties above that for all $i, j \in \mathbb{N}_4$

$$\mathcal{K}_{c,0}(e_i\otimes e_j) = S_{ij}\mathcal{K}_{c,0}(e_j\otimes e_i) = -S_{ij}\mathcal{K}_{c,0}(e_i\otimes e_j) \tag{4.15}$$

and therefore $\mathcal{K}_{c,0}(e_i\otimes e_j)$ is an eigenvector associated to the eigenvalue -1 of the reflection S_{ij}. The reflection S_{ij} taking place at the plane passing through the two remaining arms of the reference tetrahedron z_k and z_l, implies that $\mathcal{K}_{c,0}(e_i\otimes e_j) = \alpha_{ij}(z_k\times z_l)$ for some scalar $\alpha_{ij}\in\mathbb{R}$. Additionally, we have $\mathcal{K}_{c,0}(e_i\otimes e_j) = S_{jk}\mathcal{K}_{c,0}(e_i\otimes e_k) = \alpha_{ik}(z_j\times z_l)$ and since S_{jk} is orthogonal, we have $|\alpha_{ij}| = |\alpha_{ik}|$ as the vectors z_i are normalized. Eventually, one quickly verifies that the quantity

$$\mathcal{K}_{c,0}(e_i\otimes e_j)\cdot \mathrm{sgn}(ijkl)(z_k\times z_l), \tag{4.16}$$

where $\mathrm{sgn}(ijkl)$ denotes the parity of the permutation $(ijkl)$ of $\mathbb{N}_4$, stays constant under any permutation of the indices as well as any symmetry condition. Hence, we may conclude that

$$\mathcal{K}_{c,0}(e_i\otimes e_j) = \alpha\,\mathrm{sgn}(ijkl)z_k\times z_l \tag{4.17}$$

for all $i \neq j \in \mathbb{N}_4$ and some scalar $\alpha \in \mathbb{R}$. Clearly, the symmetry conditions imply that $\mathcal{K}_{c,0}(e_i \otimes e_i) = 0$ for all $i \in \mathbb{N}_4$. Thus, we have determined the matrices M_1, M_2, M_3 up to one scalar parameter. By explicitly calculating the cross products $z_i \times z_j$, we find

$$M_1 = \alpha \begin{pmatrix} 0 & 3 & 3 & 2 \\ -3 & 0 & 0 & -1 \\ -3 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{pmatrix}, \qquad M_2 = \sqrt{3}\alpha \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -2 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{4.18}$$

and

$$M_3 = 2\sqrt{2}\alpha \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.19}$$

Similarly, for the angular part, we find that

$$\mathcal{K}_{\theta,0}(e_i \otimes e_j) = -S_{kl}\mathcal{K}_\theta(e_i \otimes e_j) \tag{4.20}$$

and therefore $\mathcal{K}_{\theta,0}(e_i \otimes e_j) = \delta_{ij} e_i \times e_j$. By noticing that this time the quantity $\mathcal{K}_{\theta,0}(e_i \otimes e_j) \cdot (z_i \times z_j)$ stays constant, a similar argument to the one above shows that

$$\mathcal{K}_{\theta,0}(e_i \otimes e_j) = \delta z_i \times z_j, \tag{4.21}$$

for all $i \neq j \in \mathbb{N}_4$. Again, we have $\mathcal{K}_\theta(e_i \otimes e_i) = 0$ for all $i \in \mathbb{N}_4$. A calculation similar to the one above now yields

$$M_4 = \delta \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & -2 & 3 \\ 1 & 2 & 0 & -3 \\ 0 & -3 & 3 & 0 \end{pmatrix}, \qquad M_5 = \sqrt{3}\delta \begin{pmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{pmatrix}, \tag{4.22}$$

and

$$M_6 = 2\sqrt{2}\delta \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.23}$$

Remark. At this point, let us point out the apparent similarity between the matrices M_1, M_2 and M_6 with their corresponding matrices in [3]. In fact, in the upper left corner, i.e. the entries that relate the first three arms to each other, we retrieve the same matrices as in [3] up to rescaling α and δ, which very well reflects the similarity between SPr3 and SPr4 . However, the fact that it is the first three arms that corresponds to the three arms in SPr3 merely stems from our choice of the reference orientation.

4.3 The linearized control equations

Herein, we denote by J the closed interval $[0, 2\pi] \subset \mathbb{R}$ and we define the so-called *strokes space* as $H^1_\sharp(J, \mathbb{R}^4)$, i.e. the Sobolev space of 2π-periodic vector valued functions of $L^2_\sharp(J, \mathbb{R}^4)$ having first order weak derivative in $L^2_\sharp(\mathbb{R}^4)$. For every $f \in L^2_\sharp(J, \mathbb{R}^4)$ we denote by $\langle f \rangle := (2\pi)^{-1} \int_J f(s)\mathrm{d}s$ the average of f on J.

In the previous section, we have seen that the control system governing the evolution of SPr4 under the action of the control parameters $\zeta \in \mathcal{M}$ is given by

$$\begin{cases} \dot{c} &= RF_c(\zeta)\dot{\zeta} \\ \dot{R} &= RF_\theta(\zeta)\dot{\zeta}, \end{cases} \tag{4.24}$$

where $(c, R) \in \mathcal{P} = \mathbb{R}^3 \times \mathrm{SO}(3)$, the systems $F_c, F_\theta : \mathcal{M} \to M_{3\times 4}(\mathbb{R})$ are given by (4.1) and $\dot{\zeta} \in T_\zeta \mathcal{M}$. Furthermore, we have seen previously, c.f. (4.4), (4.10) and (4.11), that if we set $\zeta = \xi_0 + \xi$, the response of the system around $\xi = 0$, up to higher order terms, simplifies to

$$\begin{cases} \dot{c} &= RF_{c,0}\dot{\xi} + R\sum_{k\in\mathbb{N}_3}(A_k\dot{\xi}\cdot\xi)\hat{e_k} \\ \dot{R} &= R\sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi)L_k. \end{cases} \tag{4.25}$$

In particular, if we fix $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ and define $\Gamma := \sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi) : J \to \mathfrak{so}(3)$, then the dynamics of R can be written as an ordinary differential equation on the Lie group SO(3):

$$\begin{cases} \dot{R}(t) = R(t)\Gamma(t) \\ R(0) := R_0. \end{cases} \tag{4.26}$$

To simplify equations (4.25) further, we are interested in the solution of (4.26) in the regime of a small stroke $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ or equivalently in the regime of a small vector field Γ. Intuitively, the solution R of (4.26) should not deviate too much from the initial value R_0 if the vector field Γ driving the differential equation is small. The solution of (4.26) and its relation to a small vector field Γ as well as the notion of smallness for the vector field Γ in the first place, are mathematically formalized by the concept of *chronological calculus*. For details on the topic, we refer to [1] but essentially it works as follows: First, one identifies any smooth manifold M with the space $C^\infty(M)$, on which one defines a certain metric topology, the Whitney topology. Then, the solution of a differential equation $\dot{q}(t) = q(t)V(t)$ for V a non-autonomous vecor field on M is given by

$$q(t) = q(0)\overrightarrow{\exp}\int_J^t V(s)\mathrm{d}s, \tag{4.27}$$

where $\overrightarrow{\exp}$ is a special operator called the *right chronological exponential*, which is defined as a limit of an iterated integral. For the series expansion

$$S_m(t) := I + \sum_{n=1}^{m-1}\int_{\Delta_n(t)}\cdots\int V(s_n)\circ\cdots\circ V(s_1)\mathrm{d}s_n\cdots\mathrm{d}s_1, \tag{4.28}$$

with $\Delta_n(t) = \{(s_1, \ldots, s_n) \in \mathbb{R}^n | 0 \le s_n \le \cdots \le s_1 \le t\}$, we have

$$q(t) = S_m(t) + \mathcal{O}(t), \qquad t \downarrow 0. \tag{4.29}$$

In particular, for an equation of the form $\dot{q}(t) = q(t)\varepsilon V(t)$, it is shown in [1], that

$$q(t) = S^\varepsilon_m(t) + \mathcal{O}(\varepsilon^m), \qquad \varepsilon \downarrow 0, \tag{4.30}$$

where S^ε_m denotes the series expansion (4.28)for the vector field εV.

With the estimate (4.30) at hand, let $\hat{\xi} \in H^1_\sharp(J, \mathbb{R}^4)$ a normalized stroke, i.e. $||\hat{\xi}||_{H^1_\sharp} = 1$, and $\varepsilon > 0$. Set $\xi := \varepsilon\hat{\xi}$ as well as $\Gamma_\varepsilon := \sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi)L_k$ such that $\Gamma_1 = \sum_{k\in\mathbb{N}_3}(B_k\dot{\hat{\xi}}\cdot\hat{\xi})L_k$ and $\Gamma_\varepsilon = \varepsilon^2\Gamma_1$. Writing S^ϵ_m for the expansion (4.28) of the vector field Γ_ε, we find by (4.30)

$$R(t) = R_0\left(I + \int_0^t \Gamma_\varepsilon(\tau)\mathrm{d}\tau\right) + \mathcal{O}(\epsilon^4), \qquad \varepsilon \downarrow 0. \tag{4.31}$$

Hence, choosing $R_0 = I$, we have in particular the following approximations for any $t \in J$

$$\begin{cases} \dot{c} &= \left(I + \int_0^t \Gamma_\varepsilon(\tau)\mathrm{d}\tau\right)\left(F_{c,0}\dot{\xi} + \sum_{k\in\mathbb{N}_3}(A_k\dot{\xi}\cdot\xi)\hat{e}_k\right) + \mathcal{O}(\varepsilon^4) \\ \dot{R} &= \left(I + \int_0^t \Gamma_\varepsilon(\tau)\mathrm{d}\tau\right)\sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi)L_k + \mathcal{O}(\varepsilon^4), \end{cases} \tag{4.32}$$

for $\varepsilon \downarrow 0$. By integrating the previous two relations over J, we find an estimate of the net displacement undergone by the center c of SPR4 as well as its orientation R after a small stroke. Moreover, with equations (4.32) we can express the net displacements δc and δR as maps $H^1_\sharp(J, \mathbb{R}^4) \to \mathbb{R}^4$ given by $\xi \mapsto 2\pi\langle\dot{c}(\xi)\rangle$ and $\xi \mapsto 2\pi\langle\dot{R}(\xi)\rangle$, respectively. Consequently, let us prove that

Proposition 8. *For any $\xi \in H^1_\sharp(J,\mathbb{R}^4), \xi : J \to \mathbb{R}^4$, in a neighborhood of $0 \in H^1_\sharp(J,\mathbb{R}^4)$, the following estimates hold*

$$\begin{aligned} \delta c(\xi) &= 2\pi \sum_{k\in\mathbb{N}_3} \langle A_k\dot{\xi}\cdot\xi\rangle \hat{e}_k + \mathcal{O}(||\xi||^3_{H^1_\sharp}), \\ \delta R(\xi) &= 2\pi \sum_{k\in\mathbb{N}_3} \langle B_k\dot{\xi}\cdot\xi\rangle L_k + \mathcal{O}(||\xi||^4_{H^1_\sharp}). \end{aligned} \tag{4.33}$$

Proof. First, let us note that the term $\langle F_{c,0}\dot{\xi}\rangle$ vanishes due to the periodicity of the stroke ξ. Next, we observe that it suffices to prove that the scalar quantities of the form

$$\left\langle \left(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau\right) A_l\dot{\hat{\xi}}\cdot\hat{\xi}\right\rangle, \qquad k,l\in\mathbb{N}_3, \tag{4.34}$$

as well as $\langle (\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau)\dot{\hat{\xi}}_i\rangle$, $i\in\mathbb{N}_4$ are bounded, where we again set $\xi = \varepsilon\hat{\xi}$. We focus on the terms of the latter form, since the others can be treated in the same manner. We have

$$\begin{aligned} \left|\int_J \left(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau\right)\dot{\hat{\xi}}_i(t)\mathrm{d}t\right| &= \left|\int_0^{2\pi} B_k\dot{\hat{\xi}}(t)\cdot\hat{\xi}(t)\int_t^{2\pi}\dot{\hat{\xi}}_i(s)\mathrm{d}s\mathrm{d}t\right| \\ &\le ||B_k||_{op}\int_J |\dot{\hat{\xi}}(t)|\cdot|\hat{\xi}(t)-\hat{\xi}(0)|^2\mathrm{d}t \end{aligned} \tag{4.35}$$

The Sobolev-Morrey embedding $H^1_\sharp(J,\mathbb{R}^4)\subseteq L^\infty_\sharp(J,\mathbb{R}^4)$ guarantees the existence of a $c_S>0$ such that $||\xi||_\infty \le c_S||\xi||_{H^1_\sharp}$ for every $\xi\in H^1_\sharp(J,\mathbb{R}^4)$. Hence, we have

$$\begin{aligned} \left|\left\langle\left(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau\right)\dot{\hat{\xi}}_i\right\rangle\right| &\le ||B_k||_{op}||\hat{\xi}||^2_\infty ||||_{H^1_\sharp} \\ &\le c_S||B_k||_{op}||\hat{\xi}||^3_{H^1_\sharp} = c_S||B_k||_{op}, \end{aligned} \tag{4.36}$$

which is clearly bounded. This finishes the proof. □

To end this section, we note that on the one hand, we have $\langle A_k\dot{\xi}\cdot\xi\rangle = \langle M_k\dot{\xi}\cdot\xi\rangle$ and $\langle B_k\dot{\xi}\cdot\xi\rangle = \langle M_{k+3}\dot{\xi}\cdot\xi\rangle$ for all $k\in\mathbb{N}_3$. Indeed, if A is a symmetric matrix, we have by integration by parts that $\langle A\xi\cdot\dot{\xi}\rangle = \langle A\dot{\xi}\cdot\xi\rangle = -\langle A\xi\cdot\dot{\xi}\rangle$ and thus only the skew-symmetric parts of the matrices A_k and B_k contribute to the net displacement. Furthermore, similarly to [3], we can represent the terms $M_k\dot{\xi}\cdot\xi$ in terms of certain operations of the orthonormal basis $\{\tau_i\}_{i\in\mathbb{N}_4}$ of $\mathbb{R}^4$. In fact, we find by straightforward calculation using that

$$M_k\dot{\xi}\cdot\xi = -2\sqrt{6}\,\alpha\det(\xi|\dot{\xi}|\tau_{k+1}|\tau_{k+2}), \qquad k\in\mathbb{N}_3 \tag{4.37}$$

$$M_{3+k}\dot{\xi}\cdot\xi = -2\sqrt{6}\,\delta\det(\xi|\dot{\xi}|\tau_k|\tau_4), \qquad k\in\mathbb{N}_3, \tag{4.38}$$

where $\det(\xi|\dot{\xi}|\tau_j|\tau_k)$ denotes the determinant of the matrix $(\xi|\dot{\xi}|\tau_j|\tau_k)$ and where the index l is reduced mod 3 to simplify the notation. Ultimately, using that $\mathbb{R}^3\times\mathfrak{so}(3)\simeq\mathbb{R}^6$, we can write the net displacement in position and orientation simultaneously as

$$\frac{\delta p}{2\pi} = -2\sqrt{6}\alpha\sum_{k\in\mathbb{N}_3}\det(\xi|\dot{\xi}|\tau_{k+1}|\tau_{k+2})f_k - 2\sqrt{6}\delta\sum_{k\in\mathbb{N}_3}\det(\xi|\dot{\xi}|\tau_k|\tau_4)f_{k+3}, \tag{4.39}$$

where $\{f_i\}_{i\in\mathbb{N}_6}$ denotes the canonical Basis of $\mathbb{R}^6$ and the index k is once more reduced mod 3. This representation will prove particularly useful in the following section.

5. Energy minimizing strokes

In the spirit of [2] and [3], we follow the notion of swimming efficiency suggested by Lighthill [7] and we adopt the following notion of optimality: energy minimizing strokes are the ones that minimize the kinematic energy dissipated while trying to reach a given net displacement $\delta p \in \mathbb{R}^3 \times \mathfrak{so}(3) \simeq \mathbb{R}^6$. Mathematically speaking, the total energy dissipation due to a stroke $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ can be evaluated through an adequate quadratic energy functional, c.f. [2],

$$\mathcal{G}_S(\xi) := \int_J \mathfrak{g}(\xi(t))\dot{\xi}(t) \cdot \dot{\xi}(t)\mathrm{d}t, \tag{5.1}$$

where the energy density $\mathfrak{g} \in C^1(\mathbb{R}^4)$ is a function with values in the space of symmetric and positive definite matrices $M_{4\times 4}(\mathbb{R})$. In other words, $\mathfrak{g}$ defines a continuous Riemannian metric on $\mathcal{M}$. In the small strokes regime, we can approximate the energy density by $\mathfrak{g}(\xi) = \mathfrak{g}(0)+o(1)$, where $\mathfrak{g}(0) \in M_{4\times 4}(\mathbb{R})$ is symmetric and positive definite. More precisely,

$$\mathcal{G}(\xi) := \int_J Q_{\mathfrak{g}}(\dot{\xi}(t))\mathrm{d}t, \tag{5.2}$$

with $Q_{\mathfrak{g}}(\eta) := \mathfrak{g}(0)\eta \cdot \eta$. For the same symmetry reasons as discussed in section 3, we necessarily have for all $\eta \in \mathbb{R}^4$

$$Q_{\mathfrak{g}}(P_{ij}\eta) = Q_{\mathfrak{g}}(\eta),\ i, j \in \mathbb{N}_4, \tag{5.3}$$

where P_{ij} denotes the permutation matrix swapping the i-th and j-th entries. By direct computation, on finds that the symmetric positive matrix G representing the quadratic form $Q_{\mathfrak{g}}$ is of the form

$$G = \begin{pmatrix} \kappa & h & h & h \\ h & \kappa & h & h \\ h & h & \kappa & h \\ h & h & h & \kappa \end{pmatrix}, \tag{5.4}$$

for two parameters h and $\kappa > \max(h, -3h)$. In particular, we observe that $G\tau_k = (\kappa - h)\tau_k$ for $k \in \mathbb{N}_3$ and $G\tau_4 = (\kappa + 3h)\tau_4$. In the following, we denote by $\mathfrak{g}_1 := \mathfrak{g}_2 := \mathfrak{g}_3 := \kappa - h$ and $\mathfrak{g}_4 := \kappa + 3h$ the eigenvalues of G. Furthermore, the eigenvalues $(\mathfrak{g}_i)_{i\in\mathbb{N}_4}$ allow us to diagonalize G as

$$G = U\Lambda_{\mathfrak{g}}U^T, \quad U := [\tau_1|\tau_2|\tau_3|\tau_4], \quad \Lambda_{\mathfrak{g}} := \mathrm{diag}(\mathfrak{g}_i). \tag{5.5}$$

The goal of this section is the minimization of $\mathcal{G}$ in $H^1_\sharp(J, \mathbb{R}^4)$ subject to a prescribed net displacement $\delta p \in \mathbb{R}^6$, i.e. subject to the constraint (c.f. (4.39))

$$\begin{aligned} \delta p = \mathfrak{h}_c \sum_{k\in\mathbb{N}_3} \left(\int_J \det(\xi(t)|\dot{\xi}(t)|\tau_{k+1}|\tau_{k+2})\mathrm{d}t \right) f_k \\ + \mathfrak{h}_\theta \sum_{k\in\mathbb{N}_3} \left(\int_J \det(\xi(t)|\dot{\xi}(t)|\tau_k|\tau_4)\mathrm{d}t \right) f_{k+3}, \end{aligned} \tag{5.6}$$

with $\mathfrak{h}_c = -2\sqrt{6}\alpha$ and $\mathfrak{h}_\theta = -2\sqrt{6}\delta$.

5.1 G-orthogonalisation

We begin by rewriting the energy functional (5.2) and the constraint (5.6) in terms of the orthonormal basis of eigenvectors $(\tau_i)_{i\in\mathbb{N}_4}$ of the matrix G. The change of variables $\eta(t) := U^T\xi(t) \in H^1_\sharp(J, \mathbb{R}^4)$, allows us to write

$$\mathcal{G}_U(\eta) = \int_J \Lambda_{\mathfrak{g}}\dot{\eta}(t) \cdot \dot{\eta}(t)\mathrm{d}t, \tag{5.7}$$

with $\mathcal{G}_U(\eta) := \mathcal{G}(\xi) = \mathcal{G}(U\eta)$. For the constraint, we note that

$$\det(\xi|\dot{\xi}|\tau_i|\tau_j) = \det U \det(\eta|\dot{\eta}|e_i|e_j) = \det(\dot{\eta}|\eta|e_i|e_j), \tag{5.8}$$

since $\det U = -1$. Eventually, we can express the determinants more elegantly in terms of exterior products. To that end, one notes that the scalar product on $\mathbb{R}^n$ extends to the 2nd power $\bigwedge^2 \mathbb{R}^n$, cf. [8], by

$$(x_1 \wedge x_2, y_1 \wedge y_2) = \det \begin{pmatrix} x_1 \cdot y_1 & x_1 \cdot y_2 \\ x_2 \cdot y_1 & x_2 \cdot y_2 \end{pmatrix}. \tag{5.9}$$

In our case, this yields $\det(\dot\eta|\eta|e_k|e_4) = (\dot\eta \wedge \eta, e_{k+1} \wedge e_{k+2})$ and $\det(\dot\eta|\eta|e_{k+1}|e_{k+2}) = (\dot\eta \wedge \eta, e_k \wedge e_4)$, for $k \in \mathbb{N}_3$ taken mod 3. Since $\dim \bigwedge^2 \mathbb{R}^4 = 6$, the isomorphism sending the basis $\{f_i\}_{i\in\mathbb{N}_6}$ of $\mathbb{R}^6$ onto the basis

$$(e_{14}, e_{24}, e_{34}, e_{23}, e_{31}, e_{12}) \tag{5.10}$$

of $\bigwedge^2 \mathbb{R}^4$, where we write $e_{ij} := e_i \wedge e_j$, allows us to rewrite (5.6) as

$$\Lambda_{\mathfrak{h}}^{-1} \delta p = \int_J \dot\eta(t) \wedge \eta(t) \mathrm{d}1t, \tag{5.11}$$

with $\Lambda_{\mathfrak{h}} := \operatorname{diag}(\mathfrak{h}_c, \mathfrak{h}_c, \mathfrak{h}_c, \mathfrak{h}_\theta, \mathfrak{h}_\theta, \mathfrak{h}_\theta)$.

5.2 Fourier transformation of the minimization problem

We denote by $\ell^2(\mathbb{R}^4)$ the space of sequences $\mathbf{u} := (u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^4$ such that the norm

$$||\mathbf{u}||_{\ell^2(\mathbb{R}^4)} := \sqrt{\sum_{n\in\mathbb{N}} |u_n|^2} \tag{5.12}$$

is finite. Consequently, we denote by $\dot\ell^2(\mathbb{R}^4)$ the Hilbert space of sequences $\mathbf{u} = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{R}^4)$ such that $(nu_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{R}^4)$. As the elements in $H^1_\sharp(J, \mathbb{R}^4)$ are 2π-periodic, we can express η in terms of its Fourier series as

$$\eta(t) := \frac{1}{2} a_0 + \sum_{n\in\mathbb{N}} \cos(nt) a_n + \sin(nt) b_n, \tag{5.13}$$

with $(a_n, b_n) \in \dot\ell^2(\mathbb{R}^4) \times \dot\ell^2(\mathbb{R}^4)$. Substitution of the Fourier series of $\dot\eta$ into the energy functional (5.7) yields due to Parseval's equality

$$\mathcal{G}_U(\eta) := \int_J \Lambda_{\mathfrak{g}} \dot\eta(t) \cdot \dot\eta dt = \pi \sum_{n\in\mathbb{N}} n^2 (\Lambda_{\mathfrak{g}} a_n \cdot a_n + \Lambda_{\mathfrak{g}} b_n \cdot b_n) \tag{5.14}$$

$$= \frac{1}{2} ||\mathbf{u}||^2_{\ell^2(\mathbb{R}^4)} + \frac{1}{2} ||\mathbf{v}||^2_{\ell^2(\mathbb{R}^4)}, \tag{5.15}$$

where we have set

$$\mathbf{u} := (u_n)_{n\in\mathbb{N}} := \sqrt{2\pi\Lambda_{\mathfrak{g}}} (na_n)_{n\in\mathbb{N}} \text{ and } \mathbf{v} := (v_n)_{n\in\mathbb{N}} := \sqrt{2\pi\Lambda_{\mathfrak{g}}} (nb_n)_{n\in\mathbb{N}}. \tag{5.16}$$

Clearly, we have $(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$. As a result of the $L^2_\sharp(J, \mathbb{R}^4)$-orthogonality of the Fourier trigonometric system, we can express the constraint (5.11) in terms of Fourier coefficients as $2\pi \sum_{n\in\mathbb{N}} \frac{1}{n} (nb_n) \wedge (na_n) = \Lambda_{\mathfrak{h}}^{-1} \delta p$. Returning to our old notation for a moment, we note that for any $n \in \mathbb{N}$, we have

$$2\pi n^2 \det(b_n|a_n|e_i|e_j) = \frac{\sqrt{\mathfrak{g}_i \mathfrak{g}_j}}{\sqrt{\det \Lambda_{\mathfrak{g}}}} \det(v_n|u_n|e_i|e_j), \tag{5.17}$$

and hence by setting $\tilde\Lambda_{\mathfrak{g}} := \operatorname{diag}(\mathfrak{g}_c, \mathfrak{g}_c, \mathfrak{g}_c, \sqrt{\mathfrak{g}_c \mathfrak{g}_\theta}, \sqrt{\mathfrak{g}_c \mathfrak{g}_\theta}, \sqrt{\mathfrak{g}_c \mathfrak{g}_\theta})$, with $\mathfrak{g}_1 := \mathfrak{g}_2 := \mathfrak{g}_3 := \mathfrak{g}_c$ and $\mathfrak{g}_4 := \mathfrak{g}_\theta$, we eventually find

$$\sqrt{\det \Lambda_{\mathfrak{g}}} (\Lambda_{\mathfrak{h}} \tilde\Lambda_{\mathfrak{g}})^{-1} \delta p = \sum_{n\in\mathbb{N}} \frac{v_n \wedge u_n}{n} \tag{5.18}$$

We thus have proved the following

Proposition 9. *The $H^1_\sharp(J,\mathbb{R}^4)$ minimization of the functional $\mathcal{G}_U$ given by (5.7) under the constraint (5.11) is equivalent to the minimization of the functional*

$$\mathcal{F}(\mathbf{u},\mathbf{v}) := \frac{1}{2}||\mathbf{u}||^2_{\ell^2(\mathbb{R}^4)} + \frac{1}{2}||\mathbf{v}||^2_{\ell^2(\mathbb{R}^4)}, \tag{5.19}$$

defined in the product Hilbert space $\ell^2(\mathbb{R}^4)\times\ell^2(\mathbb{R}^4)$ and subject to the constraint

$$\sum_{n\in\mathbb{N}} \frac{1}{n} v_n \wedge u_n = \omega \text{ with } \omega := \sqrt{\det\Lambda_{\mathfrak{g}}}(\Lambda_{\mathfrak{h}}\tilde{\Lambda}_{\mathfrak{g}})^{-1}\delta p, \tag{5.20}$$

where $\delta p \in \mathbb{R}^6$ is a prescribed net displacement of position and orientation.

5.3 Preliminaries for the reduction to finite dimension

To continue our analysis of SPr4 and to reduce the minimization problem defined by (5.19) and (5.20), we have to discuss the notions *bivector* and *wedge product* in more detail and introduce the *Clifford product* to illustrate the fundamental differences between the two swimmers SPr3 and SPr4 . For an extensive introduction to this topic, we refer to [8] and [5].

For our purposes, the abstract definition of k-vectors as alternating tensors is not very useful. We merely illustrate them in $\mathbb{R}^3$ since the notion then generalizes easily to higher dimensions. In $\mathbb{R}^3$, a bivector is an oriented plane segment; that is, a small piece of surface having a magnitude given by the area of the surface element as well as a direction given by the attitude of the plane the surface element lies in as well as a sense of rotation. Naturally, we represent a bivector ω as a small parallelogram which suggests that we can think of it as some product of the two vectors along its sides. This is realized by the *exterior product*, also called *wedge product*, $u \wedge v$ of two vectors u and v. The product $u\wedge v$ then represents the bivector obtained by sweeping v along u. This operation yields a direct link between $\mathbb{R}^3$ and the vector space $\bigwedge^2\mathbb{R}^3$ of bivectors, a basis of which is given by

$$\{\hat{e}_1\wedge\hat{e}_2, \hat{e}_1\wedge\hat{e}_3, \hat{e}_2\wedge\hat{e}_3\}, \tag{5.21}$$

if $\{\hat{e}_1,\hat{e}_2,\hat{e}_3\}$ is a basis of $\mathbb{R}^3$. In fact, the standard scalar product on $\mathbb{R}^3$ extends to a scalar product on $\bigwedge^2\mathbb{R}^3$ by

$$\langle u_1\wedge u_2, v_1\wedge v_2\rangle = \det\begin{pmatrix} u_1\cdot v_1 & u_1\cdot v_2 \\ u_2\cdot v_1 & u_2\cdot v_2\end{pmatrix}. \tag{5.22}$$

In particular, $\langle u\wedge v, u\wedge v\rangle = |u|^2|v|^2\sin^2\psi$, where ψ is the angle between the vectors u and v. Eventually, the norm of a bivector $\omega = \omega_{12}\hat{e}_1\wedge\hat{e}_2 + \omega_{13}\hat{e}_1\wedge\hat{e}_3 + \omega_{23}\hat{e}_2\wedge\hat{e}_3$ is given by

$$|\omega| = \sqrt{\omega_{12}^2+\omega_{13}^2+\omega_{23}^2}. \tag{5.23}$$

These definitions then extend naturally to all higher dimensions and one can also defined the exterior algebra which connects all possible wedge products, c.f. [8]. In particular, we note that if $\{e_1,e_2,e_3,e_4\}$ again denotes the canonical basis of $\mathbb{R}^4$, a basis of the space $\bigwedge^2\mathbb{R}^4$ is given by

$$\{e_{12},e_{13},e_{14},e_{23},e_{24},e_{34}\}, \tag{5.24}$$

where we write $e_{ij} := e_i\wedge e_j$ to simplify the notation.

One peculiarity of the bivectors in $\mathbb{R}^3$ is that they are isomorphic to the space $\mathbb{R}^3$ itself. This is realized by the so-called *Hodge dual operator* $\star$, c.f. [8] p. 38, which is defined in a way such that for any two vectors $u,v\in\mathbb{R}^3$ one has

$$u\wedge v = \star(u\times v), \tag{5.25}$$

where $\times$ denotes the usual cross products. This entails two things: First, every bivector in $\mathbb{R}^3$ is *simple*; that is, it can be written as the wegde product of two vectors. Second, every bivector in $\mathbb{R}^3$ defines one unique plane in $\mathbb{R}^3$. It is due to this underlying geometrical fact that in [3] one could always reduce the Fourier coefficients of an optimal control curve to one pair of vectors by which one attained the same net displacement at the same energy consumption. Furthermore, this implied that the optimal control curves are situated in a certain plane defined by the vector of the net displacement.

Yet, in $\mathbb{R}^4$, the geometry of its bivectors proves to be more involved. As $\dim \bigwedge^2 \mathbb{R}^4 = 6$, it is already clear that the bivectors in $\mathbb{R}^4$ are not isomorphic to $\mathbb{R}^4$ itself. In particular, in $\mathbb{R}^4$ not all bivectors are simple. To understand why this is the case and to characterize nevertheless the simple bivectors of $\mathbb{R}^4$, we introduce the so-called *Clifford product*. For two vectors $u, v \in \mathbb{R}^n$, the Clifford product is given by

$$uv = u \cdot v + u \wedge v. \tag{5.26}$$

If $\{e_i\}_{i\in\mathbb{N}_n}$ is a basis of $\mathbb{R}^n$, the Clifford algebra is then characterized by

$$\begin{aligned} e_i e_j &= -e_j e_i, \quad i \neq j \\ e_i e_i &= 1. \end{aligned} \tag{5.27}$$

There is also a basis-free definition using quadratic forms which is more general for sure, but we will not make any use of it in this paper. However, we will make use of the identities $|u \wedge v| = |u||v| \sin\psi$ and $(u \wedge v)^2 = -|u \wedge v|^2$. In fact, we have the following two results which characterize the simple bivectors in four dimensions:

Proposition 10. *A bivector $\omega \in \bigwedge^2 \mathbb{R}^4$ is simple if an only if its square ω^2 is real.*

Proof. If ω is simple, there exist two vectors $u, v \in \mathbb{R}^4$ such that $\omega = u \wedge v$. Then the preceding identity immediately yields

$$\omega^2 = (u \wedge v)^2 = -|u \wedge v|^2 \in \mathbb{R}. \tag{5.28}$$

□

By direct computation, then one finds that for a general bivector $\omega \in \bigwedge^2 \mathbb{R}^4$

$$\omega^2 = -|\omega|^2 + 2(\omega_{12}\omega_{34} + \omega_{23}\omega_{14} - \omega_{13}\omega_{24}). \tag{5.29}$$

Hence, we have the following

Corollary 11. *A bivector $\omega = \sum_{i,j} \omega_{ij} e_{ij} \in \bigwedge^2 \mathbb{R}^4$ is simple if and only if*

$$\omega_{12}\omega_{34} + \omega_{23}\omega_{14} + \omega_{31}\omega_{24} = 0. \tag{5.30}$$

We note that the condition (5.30) is in particular satisfied if all coefficients corresponding to a certain index are zero, e.g. $\omega_{i4} = 0$ for $i \in \mathbb{N}_4$. This yields four subspaces D^*_{ijk} of $\bigwedge^2 \mathbb{R}^4$ consisting only of simple bivectors. Since net displacement δp is merely scaled and multiplied by diagonal matrices to get the constraint vector in (5.20), its structure is not changed and thus by inspection of the basis of $\bigwedge^2 \mathbb{R}^4$, we have the following correspondences:

$D^*_{123} \longleftrightarrow$ rotations around all three axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$

$D^*_{124} \longleftrightarrow$ translation in the $\hat{e}_1\hat{e}_2$-plane, rotation around the $\hat{e}_3$-axis

$D^*_{134} \longleftrightarrow$ translation in the $\hat{e}_1\hat{e}_3$-plane, rotation around the $\hat{e}_2$-axis

$D^*_{234} \longleftrightarrow$ translation in the $\hat{e}_2\hat{e}_3$-plane, rotation around the $\hat{e}_1$-axis

In particular, the subspaces D^*_{ijk} are of dimension three and correspond to the subspaces $D_{ijk} := \operatorname{span}\{e_i, e_j, e_k\} \subset \mathbb{R}^4$. In fact, we have $D^*_{ijk} = \bigwedge^2 D_{ijk}$ and furthermore the isomorphism given by the Hodge star operator $\star$, i.e.

$$\omega = \omega_{ij}e_{ij} + \omega_{ik}e_{ik} + \omega_{jk}e_{jk} \mapsto \omega_{ij}e_k + \omega_{ik}e_j + \omega_{jk}e_i. \tag{5.31}$$

This especially allows us to relate cross products in the subspace D_{ijk} to a unique bivector in D^*_{ijk}. More precisely, we have for $u, v \in D_{ijk}$ that $u \wedge v = \star(u \times v)$ just like in (5.25), where the cross product is understood in the subspace D_{ijk}.

5.4 The simple case $\omega \in D^*_{ijk}$

With the remarks from the preceding section, we are able to solve the constrained minimization problem of Proposition 5.2 in a similar manner to [3]. In fact, we retrieve essentially the same result, i.e. that the optimal control curves are ellipses in a certain plane defined by the net displacement. Let us prove

Proposition 12. *If $\omega = D_{ijk}$, then for any $(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$ such that the constraint (5.20) holds, there exist two vectors $u, v \in \mathbb{R}^4$ such that for the sequences $\mathbf{u}_\star := \mathbf{e}_1 u$ and $\mathbf{v}_\star := \mathbf{e}_1 v \in \ell^2(\mathbb{R}^4)$ one has*

$$\mathcal{F}(\mathbf{u}_\star, \mathbf{v}_\star) = \mathcal{F}(\mathbf{u}, \mathbf{v}) \text{ and } v \wedge u = \omega. \tag{5.32}$$

Proof. If $\omega = 0$, then the proof is trivial. Thus, let us denote by $\hat{\omega}$ the unit bivector associated to ω. For a couple $(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$, we then choose $u, v \in \mathbb{R}^4$ such that the following relations hold:

$$|u| = ||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}, \quad |v| = ||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}, \quad \frac{u \wedge v}{|u \wedge v|} = \hat{\omega}. \tag{5.33}$$

The latter is possible since $\hat{\omega}$ is a simple bivector by hypothesis. Furthermore, we have $u \wedge v = ||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}(\sin\psi)\hat{\omega}$, where ψ is the angle between u and v. Therefore, the equality $u \wedge v = \omega$ can be satisfied by choosing the angle $\psi \in (0, \pi)$ such that

$$\sin\psi = \frac{|\omega|}{||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}} \tag{5.34}$$

This is possible under the condition that the right hand side of the previous equation is not greater than one. In fact, we have using the Cauchy-Schwarz inequality

$$|\omega| \leq \sum_{n\in\mathbb{N}} \frac{1}{n}|v_n \wedge u_n| \leq \sum_{n\in\mathbb{N}} |v_n||u_n| \leq ||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}. \tag{5.35}$$

Finally, from (5.33) we obtain

$$\mathcal{F}(\mathbf{u}_\star, \mathbf{v}_\star) = \frac{1}{2}|u| + \frac{1}{2}|v| = \frac{1}{2}||\mathbf{u}||_{\ell^2(\mathbb{R}^4)} + \frac{1}{2}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}, \tag{5.36}$$

which concludes the proof. ∎

We immediately have

Corollary 13. *The minimization problem for $\mathcal{F}$ in $\ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$, under the constraint (5.20), is equivalent to the minimization in $\mathbb{R}^4 \times \mathbb{R}^4$ of the function*

$$f(u, v) := \frac{1}{2}|u|^2 + \frac{1}{2}|v|^2 \tag{5.37}$$

under the constraint

$$v \wedge u = \omega. \tag{5.38}$$

Proof. It suffices to observe that if $\mathcal{V}_\omega$ denotes the subset of $\ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$ satisfying the constraint (5.20) and by V_ω the subset of $(u, v) \in \mathbb{R}^4 \times \mathbb{R}^4$ such that $u \times v = \omega$, then Proposition 5.4 yields

$$\min_{(\mathbf{u},\mathbf{v})\in\mathcal{V}_\omega} \mathcal{F}(\mathbf{u}, \mathbf{v}) = \min_{(u,v)\in V_\omega} \mathcal{F}(\mathbf{e}_1 u, \mathbf{e}_1, v) = \min_{(u,v)\in V_\omega} f(u, v). \tag{5.39}$$

∎

Let us now prove the following

Proposition 14. *Any couple of vectors $(u_\star, v_\star) \in \mathbb{R}^4 \times \mathbb{R}^4$ minimizing the function f given in (5.37) and subject to the constraint (5.38) with $\omega \in D^*_{ijk}$, is characterized by the following conditions:*

$$u_\star, v_\star \in D_{ijk}, \quad |u_\star|^2 = |v_\star|^2, \quad u_\star \cdot v_\star = 0. \tag{5.40}$$

Therefore, for any $\sigma \in D_{ijk}$ such that $\sigma \cdot \hat{\omega} = 0$ and $|\sigma|^2 = |\omega|$, the couple

$$(\sigma, \sigma \times \star\omega) \in D_{ijk} \times D_{ijk} \subset \mathbb{R}^4 \times \mathbb{R}^4 \tag{5.41}$$

is a (global) constrained minimizer for f.

Proof. First, we note that it is clear that $u_\star, v_\star \in D_{ijk}$ by definition of D^*_{ijk}. Next, to find the minimizers of the problem (5.37) - (5.38), we note that the constraint $u \wedge v = \omega$ implies the existence of a $\psi \in (0, \pi)$ such that $|u||v| \sin\psi = |\omega|$. Hence, the constrained minimization for f is equivalent to the unconstrained minimization of the function $\hat{f} : \mathbb{R}^4 \times (0, \pi) \to \mathbb{R}$ defined by

$$(u, \psi) \mapsto \frac{1}{2}|u|^2 + \frac{1}{2}\frac{|\omega|^2}{|u|^2 \sin^2\psi}, \tag{5.42}$$

whose stationary satisfy $\psi_\star = \frac{\pi}{2}$ and $|u_\star|^2 = |\omega|$. This shows the necessity of the conditions stated in (5.40). To show sufficiency of the condition, one observes that for any such points one has $\hat{f}(u_\star, \psi_\star) = |\omega|$. Indeed, for any $(u, \psi) \in D_{ijk,\psi} \in \mathbb{R}^4 \times (0, \pi)$ we have

$$\hat{f}(u, \psi) \geq \frac{1}{2}\frac{|u|^4 + |\omega|^2}{|u|^2} = |\omega| + \frac{1}{2}\frac{(|\omega| - |u|^2)^2}{|u|^2} \geq |\omega| = \hat{f}(u_\star, \psi_\star). \tag{5.43}$$

This proves the first statement. For the second part, we set $\hat{\omega} := \omega/|\omega|$ and we consider a vector $\sigma \in D_{ijk}$ such that

$$\sigma \cdot \hat{\omega} \text{ and } |\sigma|^2 = |\omega|. \tag{5.44}$$

By a trivial computation, on finds that the vectors $v := \sigma \times \star\hat{\omega}$ and $u := \sigma$ satisfy the relations $u \cdot v = 0$, $|u|^2 = |v|^2 = |\omega|$ and $v \wedge u = \omega$. □

Pasting everything worked out above together leads to the final result of this section. We have

Theorem 15. *Let $\delta p \in \mathbb{R}^3 \times \mathfrak{so}(3) \simeq \bigwedge^2 \mathbb{R}^4$ be a prescribed net displacement. Moreover, assume that $\delta p \in D^*_{ijk}$ for some combination of indices $i, j, k \in \mathbb{N}_4$. Then, any minimizer $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ of the energy functional (5.2) subject to the constraint (5.6) is of the form*

$$\xi(t) := (\cos t)a + (\sin t)b, \tag{5.45}$$

i.e. an ellipse of $D_{ijk} \subset \mathbb{R}^4$ centered at the origin and contained in the plane spanned by the vectors a and b. The vectors $a, b \in D_{ijk}$ are obtained as follows:

(i) We compute the vector ω via the relation

$$\omega := \operatorname{diag}\left(\frac{\sqrt{\mathfrak{g}_c\mathfrak{g}_\theta}}{\mathfrak{h}_c}, \frac{\sqrt{\mathfrak{g}_c\mathfrak{g}_\theta}}{\mathfrak{h}_c}, \frac{\sqrt{\mathfrak{g}_c\mathfrak{g}_\theta}}{\mathfrak{h}_c}, \frac{\mathfrak{g}_c}{\mathfrak{g}_\theta}, \frac{\mathfrak{g}_c}{\mathfrak{g}_\theta}, \frac{\mathfrak{g}_c}{\mathfrak{g}_\theta}\right) \delta p \in D_{ijk}. \tag{5.46}$$

Then we consider a vector $u \in D_{ijk}$ in the plane orthogonal to $\star\omega$ in D_{ijk} and such that

$$|u|^2 = |\omega|, \tag{5.47}$$

e.g. $u := \sqrt{|\omega|}\frac{\mu \times \star\omega}{|\mu \times \star\omega|}$, with $\mu \in D_{ijk}$ linearly independent from $\star\omega$. Furthermore, we set $v := u \times \star\omega$.

(ii) We set $\hat{\omega} := \omega/|\omega|$ and we calculate the vectors a and b via the relations

$$a := \frac{U\Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{2\pi}}u, \quad b := \frac{U\Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{2\pi}}v. \tag{5.48}$$

We then have $v \wedge u = \omega$ and the minimum value of $\mathcal{G}$ is equal to $|\omega|$.

In addition, the vectors a and b are $\mathfrak{g}$-orthogonal, i.e. with respect to the inner product defined for every $x, y \in \mathbb{R}^4$ by $(x, y)_{\mathfrak{g}} := 2\pi\Lambda_{\mathfrak{g}} x \cdot y$, and have the same $\mathfrak{g}$-norm $|a|_{\mathfrak{g}}^2 = |b|_{\mathfrak{g}}^2 = |\omega|$.

Proof. From Proposition 5.4, Corollary 5.4 and then Proposition 5.2, we get that any $\sigma \in D_{ijk}$ satisfying the relations

$$u \cdot \star\hat{\omega}, \quad |u|^2 = |\omega|, \quad \omega := \sqrt{\det \Lambda_{\mathfrak{g}}}(\Lambda_{\mathfrak{h}}\tilde{\Lambda}_{\mathfrak{g}})^{-1}\delta p, \tag{5.49}$$

is associated to a (global) constrained minimizer for $\mathcal{G}_U$, via the curve $\eta(t) := (\cos t)\tilde{a} + (\sin t)\tilde{b}$, where the Fourier coefficients $\tilde{a}, \tilde{b} \in \mathbb{R}^4$ are related to ω (c.f. 5.16) by $(\sqrt{2\pi\Lambda_{\mathfrak{g}}})\tilde{a} = u$ and $(\sqrt{2\pi\Lambda_{\mathfrak{g}}})\tilde{b} = (u \times \star\hat{\omega})$. The minimum value of the energy is then $\mathcal{G}_U(\eta) = |\omega|$.

Finally, in the $\mathfrak{g}$-orthogonal reference frame, the inner product is defined by $(x, y)_{\mathfrak{g}} := 2\pi\Lambda_{\mathfrak{g}} x \cdot y$ for $x, y \in \mathbb{R}^4$. Let us denote by $|\cdot|_{\mathfrak{g}}$ the associated norm. Then we have the following relations:

$$|\tilde{a}|_{\mathfrak{g}}^2 = |\tilde{b}|_{\mathfrak{g}}^2 = |\omega| \text{ and } (\tilde{a}, \tilde{b})_{\mathfrak{g}} = 0. \tag{5.50}$$

Applying the orthogonal map U to $\tilde{a}$ and $\tilde{b}$ finishes the proof. □

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