7

Classical electrodynamics

Geometric algebra offers a number of new techniques for studying problems in electromagnetism and electrodynamics. These are described in this chapter. We will not attempt a thorough development of electrodynamics, which is a vast subject with numerous specialist areas. Instead we concentrate on a number of selected applications which highlight the advantages that geometric algebra can bring. There are two particularly significant new features that geometric algebra adds to traditional formulations of electrodynamics. The first is that, through employing the spacetime algebra, all equations can be studied in the appropriate spacetime setting. This is much more transparent than the more traditional approach based on a $3+1$ formulation involving retarded times. The spacetime algebra simplifies the study of how electromagnetic fields appear to different observers, and is particularly powerful for handling accelerated charges and radiation. These results build on the applications of spacetime algebra described in section 5.5.3.

The second major advantage of the geometric algebra treatment is a new, compact formulation of Maxwell's equations. The spacetime vector derivative and the geometric product enable us to unite all four of Maxwell's equations into a single equation. This is one of the most impressive results in geometric algebra. And, as we showed in chapter 6, this is more than merely a cosmetic exercise. The vector derivative is invertible directly, without having to pass via intermediate, second-order equations. This has many implications for scattering and propagator theory. Huygen's principle is encoded directly, and the first-order theory is preferable for numerical computation of diffraction effects. In addition, the first-order formulation of electromagnetism means that plane waves are easily handled, as are their polarisation states.

7.1 Maxwell's equations

Before writing down the Maxwell equations, we remind ourselves of the notation introduced in chapter 5. We denote an orthonormal spacetime frame by $\{\gamma_\mu\}$, with coordinates $x_\mu = \gamma_\mu \cdot x$. The spacetime vector derivative is

$$\nabla = \gamma^\mu \partial_\mu, \qquad \partial_\mu = \frac{\partial}{\partial x^\mu}. \tag{7.1}$$

The spacetime split of the vector derivative is

$$\nabla\gamma_0 = (\gamma^0\partial_t + \gamma^i\partial_i)\gamma_0 = \partial_t - \boldsymbol{\sigma}_i\partial_i = \partial_t - \boldsymbol{\nabla}, \tag{7.2}$$

where the $\boldsymbol{\sigma}_i = \gamma_i\gamma_0$ denote a right-handed orthonormal frame for the relative space defined by the timelike vector γ_0. The three-dimensional vector derivative operator is

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}_i \frac{\partial}{\partial x^i} = \boldsymbol{\sigma}_i\partial_i, \tag{7.3}$$

and all relative vectors are written in bold.

The four Maxwell equations, in SI units, are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho, & \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0, \\ -\boldsymbol{\nabla}\times\boldsymbol{E} &= \frac{\partial}{\partial t}\boldsymbol{B}, & \boldsymbol{\nabla}\times\boldsymbol{H} &= \frac{\partial}{\partial t}\boldsymbol{D} + \boldsymbol{J}, \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} \boldsymbol{D} &= \epsilon_0\boldsymbol{E} + \boldsymbol{P}, \\ \boldsymbol{H} &= \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}, \end{aligned} \tag{7.5}$$

and the $\times$ symbol denotes the vector cross product. The cross product is ubiquitous in electromagnetic theory, and it will be encountered at various points in this chapter. To avoid any confusion, the commutator product (denoted by $\times$) will not be employed in this chapter.

The first step in simplifying the Maxwell equations is to assume that we are working in a vacuum region outside isolated sources and currents. We can then remove the polarisation and magnetisation fields $\boldsymbol{P}$ and $\boldsymbol{M}$. We also replace the cross product with the exterior product, and revert to natural units ($c = \epsilon_0 = \mu_0 = 1$), so that the equations now read

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{E} &= \rho, & \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0, \\ \boldsymbol{\nabla}\wedge\boldsymbol{E} &= -\partial_t(I\boldsymbol{B}), & \boldsymbol{\nabla}\wedge\boldsymbol{B} &= I(\boldsymbol{J} + \partial_t\boldsymbol{E}). \end{aligned} \tag{7.6}$$

We naturally assemble equations for the separate divergence and curl parts of the vector derivative. We know that there are many advantages in uniting these into a single equation involving the vector derivative. First we take the two equations for $\boldsymbol{E}$ and combine them into the single equation

$$\boldsymbol{\nabla}\boldsymbol{E} = \rho - \partial_t(I\boldsymbol{B}). \tag{7.7}$$

A similar manipulation combines the $\boldsymbol{B}$-field equations into

$$\boldsymbol{\nabla}(I\boldsymbol{B}) = -\boldsymbol{J} - \partial_t\boldsymbol{E}, \tag{7.8}$$

where we have multiplied through by I. This equation is a combination of (spatial) bivector and pseudoscalar terms, whereas equation (7.7) contains only scalar and vector parts. It follows that we can combine all of these equations into the single multivector equation

$$\boldsymbol{\nabla}(\boldsymbol{E} + I\boldsymbol{B}) + \partial_t(\boldsymbol{E} + I\boldsymbol{B}) = \rho - \boldsymbol{J}. \tag{7.9}$$

This is already a significant compactification of the original equations. We have not lost any information in writing this, since each of the separate Maxwell equations can be recovered by picking out terms of a given grade.

In section 5.5.3 we introduced the Faraday bivector F. This represents the *electromagnetic field strength* and is defined by

$$F = \boldsymbol{E} + I\boldsymbol{B}. \tag{7.10}$$

The combination of relative vectors and bivectors tells us that this quantity is a spacetime bivector. Many authors have noticed that the Maxwell equations can be simplified if expressed in terms of the complex quantity $\boldsymbol{E} + i\boldsymbol{B}$. The reason is that the spacetime pseudoscalar has negative square, so can be represented by the unit imaginary for certain applications. It is important, however, to work with I in the full spacetime setting, as I anticommutes with spacetime vectors.

In terms of the field strength the Maxwell equations reduce to

$$\boldsymbol{\nabla}F + \partial_t F = \rho - \boldsymbol{J}. \tag{7.11}$$

We now wish to convert this to manifestly Lorentz covariant form. We introduce the spacetime current J, which has

$$\rho = J \cdot \gamma_0, \qquad \boldsymbol{J} = J \wedge \gamma_0. \tag{7.12}$$

It follows that

$$\rho - \boldsymbol{J} = \gamma_0 \cdot J + \gamma_0 \wedge J = \gamma_0 J. \tag{7.13}$$

But we know that $\partial_t + \boldsymbol{\nabla} = \gamma_0 \nabla$. We can therefore pre-multiply equation (7.11) by γ_0 to assemble the covariant equation

$$\nabla F = J. \tag{7.14}$$

This unites all four Maxwell equations into a single spacetime equation based on the *geometric* product with the vector derivative. An immediate consequence is seen if we multiply through by ∇, giving

$$\nabla^2 F = \nabla J = \nabla\cdot J + \nabla\wedge J. \tag{7.15}$$

Since ∇^2 is a scalar operator, the left-hand side can only contain bivector terms. It follows that the current J must satisfy the conservation equation

$$\nabla\cdot J = \frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{J} = 0. \tag{7.16}$$

This equation tells us that the total charge generating the fields must be conserved.

The equation $\nabla F = J$ separates into a pair of spacetime equations for the vector and trivector parts,

$$\nabla\cdot F = J, \qquad \nabla\wedge F = 0. \tag{7.17}$$

In tensor language, these correspond to the pair of spacetime equations

$$\partial_\mu F^{\mu\nu} = J^\nu, \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0. \tag{7.18}$$

These two tensor equations are as compact a formulation of the Maxwell equations as tensor algebra can achieve, and the same is true of differential forms. Only geometric algebra enables us to combine the Maxwell equations (7.17) into the single equation $\nabla F = J$.

7.1.1 The vector potential

The fact that $\nabla\wedge F = 0$ tells us that we can introduce a vector field A such that

$$F = \nabla\wedge A. \tag{7.19}$$

The equation $\nabla\wedge F = \nabla\wedge\nabla\wedge A = 0$ then follows automatically. The field A is known as the *vector potential*. We shall see in later chapters that the vector potential is key to the quantum theory of how matter interacts with radiation. The vector potential is also the basis for the Lagrangian treatment of electromagnetism, described in chapter 12.

The remaining source equation tells us that the vector potential satisfies

$$\nabla\cdot(\nabla\wedge A) = \nabla^2 A - \nabla(\nabla\cdot A) = J. \tag{7.20}$$

There is some residual freedom in A beyond the restriction of equation (7.19). We can always add the gradient of a scalar field to A, since

$$\nabla\wedge(A + \nabla\lambda) = \nabla\wedge A + \nabla\wedge(\nabla\lambda) = F. \tag{7.21}$$

For historical reasons, this ability to alter A is referred to as a *gauge* freedom.

Before we can solve the equations for A, we must therefore specify a gauge. A natural way to absorb this freedom is to impose the *Lorentz condition*

$$\nabla \cdot A = 0. \tag{7.22}$$

This does not totally specify A, as the gradient of a solution of the wave equation can still be added, but this remaining freedom can be removed by imposing appropriate boundary conditions. The Lorentz gauge condition implies that $F = \nabla A$. We then recover a wave equation for the components of A, since

$$\nabla F = \nabla^2 A = J. \tag{7.23}$$

One route to solving the Maxwell equations is to solve the associated wave equation $\nabla^2 A = J$, with appropriate boundary conditions applied, and then compute F at the end. In this chapter we explore alternative, more direct routes.

The fact that a gauge freedom exists in the formulation in terms of A suggests that some *conjugate* quantity should be conserved. This is the origin of the current conservation law derived in equation (7.16). Conservation of charge is therefore intimately related to gauge invariance. A more detailed understanding of this will be provided by the Lagrangian framework.

7.1.2 The electromagnetic field strength

In uniting the Maxwell equations we introduced the *electromagnetic field strength* $F = \boldsymbol{E} + I\boldsymbol{B}$. This is a covariant spacetime bivector. Its components in the $\{\gamma^\mu\}$ frame give rise to the tensor

$$F^{\mu\nu} = \gamma^\nu \cdot (\gamma^\mu \cdot F) = (\gamma^\nu \wedge \gamma^\mu) \cdot F. \tag{7.24}$$

These are the components of a rank-2 antisymmetric tensor which, written out as a matrix, has entries

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{7.25}$$

This matrix form of the field strength is often presented in textbooks on relativistic electrodynamics. It has a number of disadvantages. Amongst these are that Lorentz transformations cannot be handled elegantly and the natural complex structure is hidden.

Writing $F = \boldsymbol{E} + I\boldsymbol{B}$ decomposes F into the sum of a relative vector $\boldsymbol{E}$ and a relative bivector $I\boldsymbol{B}$. The separate $\boldsymbol{E}$ and $I\boldsymbol{B}$ fields are recovered from

$$\begin{aligned} \boldsymbol{E} &= \tfrac{1}{2}(F - \gamma_0 F \gamma_0), \\ I\boldsymbol{B} &= \tfrac{1}{2}(F + \gamma_0 F \gamma_0). \end{aligned} \tag{7.26}$$

This shows clearly how the split into $\boldsymbol{E}$ and $I\boldsymbol{B}$ fields depends on the observer velocity (γ_0 here). Observers in relative motion see different fields. For example, suppose that a second observer has velocity $v = R\gamma_0\tilde{R}$ and constructs the rest frame basis vectors

$$\gamma'_\mu = R\gamma_\mu\tilde{R}. \tag{7.27}$$

This observer measures components of an electric field to be

$$E'_i = (\gamma'_i\gamma'_0)\cdot F = (R\boldsymbol{\sigma}_i\tilde{R})\cdot F = \boldsymbol{\sigma}_i\cdot(\tilde{R}FR). \tag{7.28}$$

The effect of a Lorentz transformation can therefore be seen by taking F to $\tilde{R}FR$. The fact that bivectors are subject to the same rotor transformation law as vectors is extremely useful for computations.

Suppose now that two observers measure the F-field at a point. One has 4-velocity γ_0, and the other is moving at relative velocity $\boldsymbol{v}$ in the γ_0 frame. This observer has 4-velocity

$$v = R\gamma_0\tilde{R}, \qquad R = \exp(\alpha\hat{\boldsymbol{v}}/2), \tag{7.29}$$

where $\boldsymbol{v} = \tanh(\alpha)\hat{\boldsymbol{v}}$. The second observer measures the $\{\gamma_\mu\}$ components of $\tilde{R}FR$. To find these we decompose F into terms parallel and perpendicular to $\boldsymbol{v}$,

$$F = F_\parallel + F_\perp, \tag{7.30}$$

where

$$\boldsymbol{v}F_\parallel = F_\parallel\boldsymbol{v}, \qquad \boldsymbol{v}F_\perp = -F_\perp\boldsymbol{v}. \tag{7.31}$$

We quickly see that the parallel components are unchanged, but the perpendicular components transform to

$$\tilde{R}F_\perp R = \exp(-\alpha\hat{\boldsymbol{v}})F_\perp = \gamma(1-\boldsymbol{v})F_\perp, \tag{7.32}$$

where γ is the Lorentz factor $(1-\boldsymbol{v}^2)^{-1/2}$. This result is sufficient to immediately establish the transformation law

$$\begin{aligned} \boldsymbol{E}'_\perp &= \gamma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})_\perp, \\ \boldsymbol{B}'_\perp &= \gamma(\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E})_\perp. \end{aligned} \tag{7.33}$$

Here the primed vectors are formed from $\boldsymbol{E}' = E'_i\boldsymbol{\sigma}_i$, for example. These have the components of F in the new frame, but combined with the original basis vectors.

Further useful information about the F field is contained in its square, which defines a pair of Lorentz-invariant terms. We form

$$F^2 = \langle FF\rangle + \langle FF\rangle_4 = a_0 + Ia_4, \tag{7.34}$$

which is easily seen to be Lorentz-invariant,

$$(\tilde{R}FR)(\tilde{R}FR) = \tilde{R}FFR = a_0 + Ia_4. \tag{7.35}$$

Both the scalar and pseudoscalar terms are independent of the frame in which they are measured. In the γ_0 frame these are

$$\alpha = \langle(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = \boldsymbol{E}^2 - \boldsymbol{B}^2 \tag{7.36}$$

and

$$\beta = -\langle I(\boldsymbol{E}+I\boldsymbol{B})(\boldsymbol{E}+I\boldsymbol{B})\rangle = 2\boldsymbol{E}\cdot\boldsymbol{B}. \tag{7.37}$$

The former yields the Lagrangian density for the electromagnetic field. The latter is seen less often. It is perhaps surprising that $\boldsymbol{E}\cdot\boldsymbol{B}$ is a full Lorentz invariant, rather than just being invariant under rotations.

7.1.3 Dielectric and magnetic media

The Maxwell equations inside a medium, with polarisation and magnetisation fields $\boldsymbol{P}$ and $\boldsymbol{M}$, were given in equation (7.4). These separate into a pair of spacetime equations. We introduce the spacetime bivector field G by

$$G = \boldsymbol{D} + I\boldsymbol{H}. \tag{7.38}$$

Maxwell's equations are now given by the pair of equations

$$\begin{aligned} \nabla\wedge F &= 0, \\ \nabla\cdot G &= J. \end{aligned} \tag{7.39}$$

The first tells us that F has vanishing curl, so can still be obtained from a vector potential, $F = \nabla\wedge A$. The second equation tells us how the $\boldsymbol{D}$ and $\boldsymbol{H}$ fields respond to the presence of free sources. These equations on their own are insufficient to fully describe the behaviour of electromagnetic fields in matter. They must be augmented by constitutive relations which relate F and G. The simplest examples of these are for linear, isotropic, homogeneous materials, in which case the constitutive relations amount to specifying a relative permittivity ϵ_r and permeability μ_r. The fields are then related by

$$\boldsymbol{D} = \epsilon_r \boldsymbol{E}, \qquad \boldsymbol{B} = \mu_r \boldsymbol{H}. \tag{7.40}$$

More complicated models for matter can involve considering responses to different frequencies, and the presence of preferred directions on the material. The subject of suitable constitutive relations is one of heuristic model building. We are, in effect, seeking models which account for the quantum properties of matter in bulk, without facing the full multiparticle quantum equations.

7.2 Integral and conservation theorems

A number of important integral theorems exist in electromagnetism. Indeed, the subject of integral calculus was largely shaped by considering applications to electromagnetism. Here the results are all derived as examples of the fundamental theorem of integral calculus, derived in chapter 6.

7.2.1 Static fields

We start by deriving a number of results for static field configurations. When the fields are static the Maxwell equations reduce to the pair

$$\boldsymbol{\nabla}\boldsymbol{E} = \frac{\rho}{\epsilon_0}, \qquad \boldsymbol{\nabla}\boldsymbol{B} = \mu_0 I\boldsymbol{J}, \tag{7.41}$$

where (for this section) we have reinserted the constants ϵ_0 and μ_0. A current $\boldsymbol{J}$ is static if the charge flows at a constant rate. The fact that $\boldsymbol{\nabla}\wedge\boldsymbol{E} = 0$ implies that around any closed path

$$\oint_{\partial\sigma} \boldsymbol{E}\cdot d\boldsymbol{l} = 0, \tag{7.42}$$

which applies for all static configurations. We can therefore introduce a potential ϕ such that

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi. \tag{7.43}$$

The potential ϕ is the timelike component of the vector potential A, $\phi = \gamma_0\cdot A$. One can formulate many of the main results of electrostatics directly in terms of ϕ. Here we adopt a different approach and work directly with the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

An extremely important integral theorem is a straightforward application of Gauss' law (indeed this *is* Gauss' original law)

$$\oint_{\partial V} \boldsymbol{E}\cdot\boldsymbol{n}\,|dA| = \frac{1}{\epsilon_0}\int_V \rho\,|dX| = \frac{Q}{\epsilon_0}, \tag{7.44}$$

where Q is the enclosed charge. In this formula $\boldsymbol{n}$ is the outward pointing normal, formed from $d\boldsymbol{A} = I\boldsymbol{n}|dA|$, where $d\boldsymbol{A}$ is the directed measure over the surface, and the scalar measure $|dX|$ is simply

$$|dX| = dx\,dy\,dz. \tag{7.45}$$

For the next application, recall from section 6.4.7 the form of the Green's function for the vector derivative in three dimensions,

$$\boldsymbol{G}(\boldsymbol{r};\boldsymbol{r}') = \frac{1}{4\pi}\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}. \tag{7.46}$$

An application of the fundamental theorem tells us that

$$\int_V (\dot{\boldsymbol{G}}\dot{\boldsymbol{\nabla}}\boldsymbol{E} + \boldsymbol{G}\,\boldsymbol{\nabla}\boldsymbol{E})|dX| = -I\oint_{\partial V} \boldsymbol{G}\,d\boldsymbol{A}\,\boldsymbol{E}. \tag{7.47}$$

If we assume that the sources are localised, so that $\boldsymbol{E}$ falls off at large distance, we can take the integral over all space and the right-hand side will vanish. Replacing $\boldsymbol{G}$ by the Green's function above we find that the field from a static charge distribution is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\boldsymbol{r}')(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,|dX'|. \tag{7.48}$$

If ρ is a single δ-function source, $\rho = Q\delta(\boldsymbol{r}'-\boldsymbol{r}_0)$, we immediately recover the Coulomb field

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{Q}{4\pi\epsilon_0}\frac{(\boldsymbol{r}-\boldsymbol{r}_0)}{|\boldsymbol{r}-\boldsymbol{r}_0|^3}. \tag{7.49}$$

Unsurprisingly, this is simply a weighted Green's function.

For the magnetic field $\boldsymbol{B}$, the absence of magnetic monopoles is encoded in the integral equation

$$\oint \boldsymbol{B}\cdot d\boldsymbol{A} = 0. \tag{7.50}$$

This tells us that the integral curves of $\boldsymbol{B}$ always form closed loops. This is true both inside and outside matter, and holds in the time-dependent case as well. Next we apply the integral theorem of equation (7.47) with $\boldsymbol{E}$ replaced by $\boldsymbol{B}$. If we again assume that the fields are produced by localised charges and fall off at large distances, we derive

$$I\boldsymbol{B}(\boldsymbol{r}) = -\frac{\mu_0}{4\pi}\int \frac{(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\boldsymbol{J}(\boldsymbol{r}')\,|dX'|. \tag{7.51}$$

The scalar term in the integrand vanishes as a consequence of the static conservation law $\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0$. The bivector term gives the magnetic field bivector $I\boldsymbol{B}$. Now suppose that the current is carried entirely in an 'ideal' wire. This is taken as an infinitely thin wire carrying a current J,

$$\boldsymbol{J} = J\int d\lambda\,\frac{d\boldsymbol{y}(\lambda)}{d\lambda}\delta\big(\boldsymbol{r}-\boldsymbol{y}(\lambda)\big) = J\int d\boldsymbol{l}\,\delta\big(\boldsymbol{r}-\boldsymbol{y}(\lambda)\big). \tag{7.52}$$

We have little option but to use J for the current as the more standard symbol I is already taken for the pseudoscalar. The result is that the $\boldsymbol{B}$-field is determined by a line integral along the wire. This is the Biot–Savart law, which can be written

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 J}{4\pi}\int \frac{d\boldsymbol{l}'\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{7.53}$$

where $\boldsymbol{r}'$ is the position vector to the line element $d\boldsymbol{l}'$.

A further integral theorem for magnetic fields is found if we consider the integral around a loop enclosing a surface σ. We have

$$\oint_{\partial\sigma} \boldsymbol{B}\cdot d\boldsymbol{l} = \int_{\sigma} (\dot{\boldsymbol{B}}\wedge\dot{\boldsymbol{\nabla}})\cdot d\boldsymbol{A} = \mu_0 \int_{\sigma} \boldsymbol{J}\cdot(-I\, d\boldsymbol{A}). \tag{7.54}$$

Again, we write $d\boldsymbol{A} = I\boldsymbol{n}|dA|$, where $\boldsymbol{n}$ is the unit right-handed normal. That is, if we grip the surface in our right hands in the manner specified by the line integral, our thumbs point in the normal direction. The result is that we integrate $\boldsymbol{J}\cdot\boldsymbol{n}$ over the surface. This returns the total current through the loop, J, recovering Ampère's law,

$$\oint_{\partial\sigma} \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 J. \tag{7.55}$$

This is routinely used for finding the magnetic fields surrounding electrical circuits.

7.2.2 Time-varying fields

If the fields vary in time, some of the preceding formulae remain valid, and others only require simple modifications. The two applications of Gauss' law, equations (7.44) and (7.50), remain unchanged. The two applications of Stokes' theorem acquire an additional term. For the $\boldsymbol{E}$-field we have

$$\oint_{\partial\sigma} \boldsymbol{E}\cdot d\boldsymbol{l} = \frac{d}{dt}\int_{\sigma} (I\boldsymbol{B})\cdot d\boldsymbol{A} = -\frac{d\Phi}{dt}, \tag{7.56}$$

where Φ is the linked magnetic flux. The flux is the integral of $\boldsymbol{B}\cdot\boldsymbol{n}$ over the area enclosed by the loop, with $\boldsymbol{n}$ the unit normal. Magnetic flux is an important concept for understanding inductance in circuits.

For the magnetic field we can derive a similar formula,

$$\oint_{\partial\sigma} \boldsymbol{B}\cdot d\boldsymbol{l} = \mu_0 J + \epsilon_0\mu_0 \frac{d}{dt}\int_{\sigma} \boldsymbol{E}\cdot\boldsymbol{n}\,|dA|. \tag{7.57}$$

This is useful when studying boundary conditions at surfaces of media carrying time-varying currents. The equations involving the Euclidean Green's function are no longer valid when the sources vary with time. In section 7.5 we discuss an alternative Green's function suitable for the important case of electromagnetic radiation.

7.2.3 The energy-momentum tensor

The energy density contained in a vacuum electromagnetic field, measured in the γ_0 frame, is

$$\varepsilon = \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2), \tag{7.58}$$

where we have reverted to natural units. In section 7.1.2 we saw that the quantity $\boldsymbol{E}^2 - \boldsymbol{B}^2$ is Lorentz-invariant. This is not true of the energy density, which should clearly depend on the observer performing the measurement. The total energy in a volume V is found by integrating ε over the volume. If we look at how this varies in time, assuming no sources are present, we find that

$$\begin{aligned} \frac{d}{dt}\int_V |dX|\, \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) &= \int_V |dX|\, \langle -\boldsymbol{E}\boldsymbol{\nabla}(I\boldsymbol{B}) + I\boldsymbol{B}\boldsymbol{\nabla}\boldsymbol{E}\rangle \\ &= \oint_{\partial V} |dA|\, \boldsymbol{n}\cdot\big(\boldsymbol{E}\cdot(I\boldsymbol{B})\big). \end{aligned} \tag{7.59}$$

We therefore establish that the field momentum is described by the Poynting vector

$$\boldsymbol{P} = -\boldsymbol{E}\cdot(I\boldsymbol{B}) = \boldsymbol{E}\times\boldsymbol{B}. \tag{7.60}$$

The energy and momentum should be the components of a spacetime 4-vector P, so we form

$$\begin{aligned} P = (\varepsilon + \boldsymbol{P})\gamma_0 &= \tfrac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\gamma_0 + \tfrac{1}{2}(I\boldsymbol{B}\boldsymbol{E} - \boldsymbol{E}I\boldsymbol{B})\gamma_0 \\ &= \tfrac{1}{2}(\boldsymbol{E} + I\boldsymbol{B})(\boldsymbol{E} - I\boldsymbol{B})\gamma_0 \\ &= \tfrac{1}{2}F(-\gamma_0 F\gamma_0)\gamma_0 = -\tfrac{1}{2}F\gamma_0 F. \end{aligned} \tag{7.61}$$

This quantity is still observer-dependent as it contains a factor of γ_0. We have in fact constructed the *energy-momentum tensor* of the electromagnetic field. We write this as

$$\mathsf{T}(a) = -\tfrac{1}{2}FaF = \tfrac{1}{2}Fa\tilde{F}. \tag{7.62}$$

This is clearly a linear function of a and, since it is equal to its own reverse, the result is automatically a vector. It is instructive to contrast our neat form of the energy-momentum tensor with the tensor formula

$$\mathsf{T}^{\mu}{}_{\nu} = \tfrac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta} + F^{\mu\alpha}F_{\alpha\nu}. \tag{7.63}$$

The geometric algebra form of equation (7.62) does a far better job of capturing the geometric content of the electromagnetic energy-momentum tensor.

The energy-momentum tensor $\mathsf{T}(a)$ returns the flux of 4-momentum across the hypersurface perpendicular to a. This is the relativistic extension of the stress tensor, and it is as fundamental to field theory as momentum is to the mechanics of point particles. All relativistic fields, classical or quantum, have an associated energy-momentum tensor that contains information about the distribution of energy in the fields, and acts as a source of gravitation. The electromagnetic energy-momentum tensor demonstrates a number of properties that turn out to be quite general. The first is that the energy-momentum tensor is (usually) symmetric. For example, we have

$$a\cdot\mathsf{T}(b) = -\tfrac{1}{2}\langle aFbF\rangle = -\tfrac{1}{2}\langle FaFb\rangle = \mathsf{T}(a)\cdot b. \tag{7.64}$$

The reason for qualifying the above statement is that quantum spin gives rise to an antisymmetric contribution to the (matter) energy-momentum tensor. This will be discussed in more details when we look at Dirac theory.

A second property of the electromagnetic energy-momentum tensor is that the energy density $v \cdot \mathsf{T}(v)$ is positive for any timelike vector v. This is clear from the definition of ε in equation (7.58). The expression for ε is appropriate the γ_0 frame, but the sign of ε cannot be altered by transforming to a different frame. The reason is that

$$\langle vFvF\rangle = \langle R\gamma_0\tilde{R}FR\gamma_0\tilde{R}F\rangle = \langle \gamma_0 F' \gamma_0 F'\rangle, \tag{7.65}$$

where $F' = \tilde{R}FR$. Transforming to a different velocity is equivalent to back-transforming the fields in the γ_0 frame, so keeps the energy density positive. Matter which does not satisfy the inequality $v \cdot \mathsf{T}(v) \geq 0$ is said to be '*exotic*', and has curious properties when acting as a source of gravitational fields.

The third main property of energy-momentum tensors is that, in the absence of external sources, they give rise to a set of conserved vectors. This is because we have

$$\nabla \cdot \mathsf{T}(a) = 0 \quad \forall \text{ constant } a. \tag{7.66}$$

Equivalently, we can use the symmetry of $\mathsf{T}(a)$ to write

$$\dot{\mathsf{T}}(\dot{\nabla}) \cdot a = 0, \quad \forall\, a, \tag{7.67}$$

which implies that

$$\dot{\mathsf{T}}(\dot{\nabla}) = 0. \tag{7.68}$$

For the case of electromagnetism, this result is straightforward to prove:

$$\dot{\mathsf{T}}(\dot{\nabla}) = -\tfrac{1}{2}[\dot{F}\dot{\nabla}F + F\nabla F] = 0, \tag{7.69}$$

which follows since $\nabla F = \dot{F}\dot{\nabla} = 0$ in the absence of sources.

Conservation of the energy-momentum tensor implies that the total flux of energy-momentum over a closed hypersurface is zero:

$$\int_{\partial V} |dA|\, \mathsf{T}(n) = 0, \tag{7.70}$$

where ∂V is a closed 3-surface with directed measure $dA = nI\,|dA|$. That the flux vanishes is a simple application of the fundamental theorem of integral calculus (in flat spacetime),

$$\int_{\partial V} \mathsf{T}(n\,|dA|) = \int_{\partial V} \mathsf{T}(dAI^{-1}) = \int_V \dot{\mathsf{T}}(\dot{\nabla})\, dX\, I^{-1} = 0. \tag{7.71}$$

Given that $\mathsf{T}(\gamma_0)$ is the energy-momentum density in the γ_0 frame, the total 4-momentum is

$$P_{tot} = \int |dX|\, \mathsf{T}(\gamma_0). \tag{7.72}$$

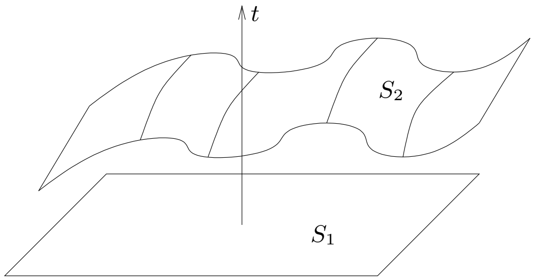

Figure 7.1 *Hypersurface integration.* The integral over a hypersurface of a (spacetime) conserved current is independent of the chosen hypersurface. The two surfaces S_1 and S_2 can be joined at spatial infinity (provided the fields vanish there). The difference is therefore the integral over a closed 3-surface, which vanishes by the divergence theorem.

The conservation equation (7.68) guarantees that, in the absence of charges, the total energy-momentum is conserved. We see that

$$\frac{d}{dt}P_{tot} = \int |dX|\, \partial_t \mathsf{T}(\gamma_0) = \int |dX|\, \dot{\mathsf{T}}(\dot{\boldsymbol{\nabla}}\gamma_0), \tag{7.73}$$

where we have used the fact that $\nabla = \gamma_0 \partial_t - \boldsymbol{\nabla}\gamma_0$. The final integral here is a total derivative and so gives rise to a boundary term, which vanishes provided the fields fall off sufficiently fast at large distances. Similarly, we can also see that P_{tot} is independent of the chosen timelike axis. It is a covariant (non-local) property of the field configuration. The proof comes from considering the integral over two distinct spacelike hypersurfaces (figure 7.1). If the integrals are joined at infinity (which introduces zero contribution) we form a closed integral of $\mathsf{T}(n)$. This vanishes from the conservation equation, so the total energy-momentum is independent of the choice of hypersurface.

In the presence of additional sources the electromagnetic energy-momentum tensor is no longer conserved. The total energy-momentum tensor, including both the matter and electromagnetic content will be conserved, however. This is a general feature of field theory in a flat spacetime, though the picture is altered somewhat if gravitational fields are present. The extent to which the separate tensors for each field are not conserved contains useful information about the flow of energy-momentum. For example, suppose that an external current is present, so that

$$\dot{\mathsf{T}}(\dot{\nabla}) = -\tfrac{1}{2}(-JF + FJ) = J\cdot F. \tag{7.74}$$

An expression of the form $J\cdot F$ was derived in the Lorentz force law, discussed in section 5.5.3. In the γ_0 frame, $J\cdot F$ decomposes into

$$J\cdot F = \langle (\rho + \boldsymbol{J})\gamma_0(\boldsymbol{E} + I\boldsymbol{B})\rangle_1 = -\big(\boldsymbol{J}\cdot\boldsymbol{E} + \rho\boldsymbol{E} + \boldsymbol{J}\times\boldsymbol{B}\big)\gamma_0. \tag{7.75}$$

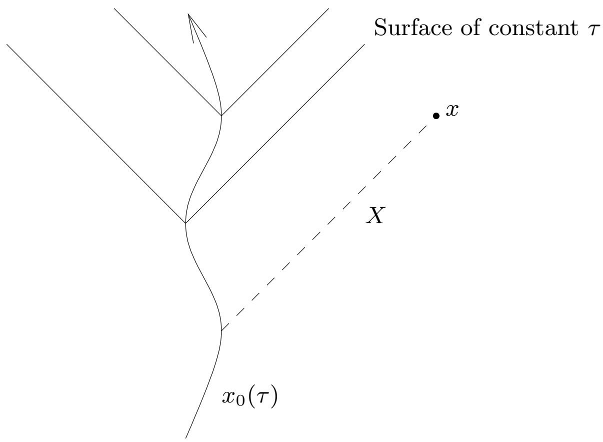

Figure 7.2 *Field from a moving point charge.* The charge follows the trajectory $x_0(\tau)$, and $X = x - x_0(\tau)$ is the retarded null vector connecting the point x to the worldline. The time τ can be viewed as a scalar field with each value of τ extended out over the forward null cone.

The timelike component, $\boldsymbol{J}\cdot\boldsymbol{E}$, is the work done — the rate of change of energy density. The relative vector term is the rate of change of field momentum, and so is closely related to the force on a point particle.

7.3 The electromagnetic field of a point charge

We now derive a formula for the electromagnetic fields generated by a radiating charge. This is one of the most important results in classical electromagnetic theory. Suppose that a charge q moves along a worldline $x_0(\tau)$, where τ is the proper time along the worldline (see figure 7.2). An observer at spacetime position x receives an electromagnetic influence from the point where the charge's worldline intersects the observer's past light-cone. The vector

$$X = x - x_0(\tau) \tag{7.76}$$

is the separation vector down the light-cone, joining the observer to this intersection point. Since this vector must be null, we can view the equation

$$X^2 = 0 \tag{7.77}$$

as defining a map from spacetime position x to a value of the particle's proper time τ. That is, for every spacetime position x there is a unique value of the (retarded) proper time along the charge's worldline for which the vector connecting x to the worldline is null. In this sense, we can write $\tau = \tau(x)$, and treat τ as a scalar field.

The Liénard–Wiechert potential for the retarded field from a point charge moving with an arbitrary velocity $v = \dot{x}_0$ is

$$A = \frac{q}{4\pi} \frac{v}{|X \cdot v|}. \tag{7.78}$$

This solution is obtained from the wave equation $\nabla^2 A = J$ using the appropriate retarded Green's function

$$G_{ret}(\boldsymbol{r}, t) = \frac{1}{4\pi|\boldsymbol{r}|}\delta(|\boldsymbol{r}| - t). \tag{7.79}$$

A similar solution exists if the advanced Green's function is used. The question of which is the correct one to use is determined experimentally by the fact that no convincing detection of an advanced (acausal) field has ever been reported. A deeper understanding of these issues is provided by the quantum treatment of radiation.

If the charge is at rest in the γ_0 frame, we have

$$x_0(\tau) = \tau\gamma_0 = (t - r)\gamma_0, \tag{7.80}$$

where r is the relative 3-space distance from the observer to the charge. The null vector X is therefore

$$X = r(\gamma_0 + e_r). \tag{7.81}$$

For this simple case the 4-potential A is a pure $1/r$ electrostatic field:

$$A = \frac{q}{4\pi} \frac{\gamma_0}{|X \cdot \gamma_0|} = \frac{q}{4\pi r}\gamma_0. \tag{7.82}$$

The same result is obtained if the advanced Green's function is used. The difference between the advanced and retarded solutions is only seen when the charge radiates. We know that radiation is not handled satisfactorily in the classical theory because it predicts that atoms are not stable and should radiate. Issues concerning the correct Green's function cannot be fully resolved without a quantum treatment.

7.3.1 The field strength

The aim now is to differentiate the potential of equation (7.78) to find the field strength. First, we differentiate the equation $X^2 = 0$ to obtain

$$\begin{aligned} 0 = \gamma^\mu(\partial_\mu X)\cdot X &= \dot{\nabla}\, \dot{x}\cdot X - \nabla\tau\,(\partial_\tau x_0)\cdot X \\ &= X - \nabla\tau\,(v\cdot X). \end{aligned} \tag{7.83}$$

It follows that

$$\nabla\tau = \frac{X}{X\cdot v}. \tag{7.84}$$

The gradient of τ points along X, which is the direction of constant τ. This is a peculiarity of null surfaces that was first encountered in chapter 6. In finding an expression for $\nabla\tau$ we have demonstrated how the particle proper time can be treated as a spacetime scalar field. Fields of this type are known as *adjunct* fields — they carry information, but do not exist in any physical sense.

To differentiate A we need an expression for $\nabla(X\cdot v)$. We find that

$$\begin{aligned}\nabla(X\cdot v) &= \dot{\nabla}(\dot{X})\cdot v + \nabla\tau\, X\cdot(\partial_\tau v)\\ &= v - \nabla\tau + \nabla\tau\, X\cdot\dot{v},\end{aligned} \tag{7.85}$$

where $\dot{v} = \partial_\tau v$. Provided X is defined in terms of the retarded time, $X\cdot v$ will always be positive and there is no need for the modulus in the denominator of equation (7.78). We are now in a position to evaluate ∇A. We find that

$$\begin{aligned}\nabla A &= \frac{q}{4\pi}\left(\frac{\nabla v}{X\cdot v} - \frac{1}{(X\cdot v)^2}\nabla(X\cdot v)v\right)\\ &= \frac{q}{4\pi}\left(\frac{X\dot{v}}{(X\cdot v)^2} - \frac{1}{(X\cdot v)^2} - \frac{(X\, X\cdot\dot{v} - X)v}{(X\cdot v)^3}\right)\\ &= \frac{q}{4\pi}\left(\frac{X\wedge\dot{v}}{(X\cdot v)^2} + \frac{X\wedge v - X\cdot\dot{v}\, X\wedge v}{(X\cdot v)^3}\right).\end{aligned} \tag{7.86}$$

The result is a pure bivector, so $\nabla\cdot A = 0$ and the A field of equation (7.78) is in the Lorentz gauge. This is to be expected, since the solution is obtained from the wave equation $\nabla^2 A = J$.

We can gain some insight into the expression for F by writing

$$X\cdot v\, X\wedge\dot{v} - X\cdot\dot{v}\, X\wedge v = -X\big(X\cdot(\dot{v}\wedge v)\big) = \tfrac{1}{2}X\dot{v}\wedge vX, \tag{7.87}$$

which uses the fact that $X^2 = 0$. Writing $\Omega_v = \dot{v}\wedge v$ for the acceleration bivector of the particle, we arrive at the compact formula

$$F = \frac{q}{4\pi}\frac{X\wedge v + \frac{1}{2}X\Omega_v X}{(X\cdot v)^3}. \tag{7.88}$$

One can proceed to show that, away from the worldline, F satisfies the free-field equation $\nabla F = 0$. The details are left as an exercise. The solution (7.88) displays a clean split into a velocity term proportional to $1/(\text{distance})^2$ and a long-range radiation term proportional to $1/(\text{distance})$. The term representing the distance is simply $X\cdot v$. This is just the distance between the events x and $x_0(\tau)$ as measured in the rest frame of the charge at its retarded position. The first term in equation (7.88) is the Coulomb field in the rest frame of the charge. The second, radiation, term:

$$F_{rad} = \frac{q}{4\pi}\frac{\frac{1}{2}X\Omega_v X}{(X\cdot v)^3}, \tag{7.89}$$

is proportional to the rest frame acceleration projected down the null vector X.

The fact that this term falls of as 1/(distance) implies that the energy-momentum tensor contains a term which falls of as the inverse square of distance. This gives a non-vanishing surface integral at infinity in equation (7.73) and describes how energy is carried away from the source.

7.3.2 Constant velocity

A charge with constant velocity v has the trajectory

$$x_0(\tau) = v\tau, \tag{7.90}$$

where we have chosen an origin so that the particle passes through this point at $\tau = 0$. The intersection of $x_0(\tau)$ with the past light-cone through x is determined by

$$(x - v\tau)^2 = 0 \quad \Rightarrow \tau = v\cdot x - \big((v\cdot x)^2 - x^2\big)^{1/2}. \tag{7.91}$$

We have chosen the earlier root to ensure that the intersection lies on the past light-cone. We now form $X\cdot v$ to find

$$X\cdot v = (x - v\tau)\cdot v = \big((v\cdot x)^2 - x^2\big)^{1/2}. \tag{7.92}$$

We can write this as $|x\wedge v|$ since

$$|x\wedge v|^2 = x\cdot\big(v\cdot(x\wedge v)\big) = (x\cdot v)^2 - x^2. \tag{7.93}$$

The acceleration bivector vanishes since v is constant, and $X\wedge v = x\wedge v$. It follows that the Faraday bivector is simply

$$F = \frac{q}{4\pi}\frac{x\wedge v}{|x\wedge v|^3}. \tag{7.94}$$

This is the Coulomb field solution with the velocity γ_0 replaced by v. This solution could be obtained by transforming the Coulomb field via

$$F \mapsto F' = RF(\tilde{R}xR)\tilde{R}, \tag{7.95}$$

where $v = R\gamma_0\tilde{R}$. Covariance of the field equations ensures that this process generates a new solution.

We next decompose F into electric and magnetic fields in the γ_0 frame. This requires the spacetime split

$$x\wedge v = \langle x\gamma_0\gamma_0 v\rangle_2 = \gamma\langle(t + \boldsymbol{r})(1 - \boldsymbol{v})\rangle_2 = \gamma(\boldsymbol{r} - \boldsymbol{v}t) - \gamma\boldsymbol{r}\wedge\boldsymbol{v}, \tag{7.96}$$

where $\boldsymbol{v}$ is the relative velocity and γ is the Lorentz factor. We now have

$$\boldsymbol{E} = \frac{q\gamma}{4\pi d^3}(\boldsymbol{r} - \boldsymbol{v}t), \qquad \boldsymbol{B} = \frac{q\gamma}{4\pi d^3}I\boldsymbol{r}\wedge\boldsymbol{v}. \tag{7.97}$$

Here, the effective distance d can be written

$$d^2 = \gamma^2(|\boldsymbol{v}|t - \boldsymbol{v}\cdot\boldsymbol{r}/|\boldsymbol{v}|)^2 + \boldsymbol{r}^2 - (\boldsymbol{r}\cdot\boldsymbol{v})^2/\boldsymbol{v}^2. \tag{7.98}$$

The electric field points towards the actual position of the charge at time t, and not its retarded position at time τ. The same is true of the advanced field, hence the retarded and advanced solutions are equal for charges with constant velocity.

7.3.3 Linear acceleration

Suppose that an accelerating charged particle follows the trajectory

$$x_0(\tau) = a\big(\sinh(g\tau)\gamma_0 + \cosh(g\tau)\gamma_3\big), \tag{7.99}$$

where $a = g^{-1}$ (see figure 7.3). The velocity is given by

$$v(\tau) = \cosh(g\tau)\gamma_0 + \sinh(g\tau)\gamma_3 = \mathrm{e}^{g\tau\boldsymbol{\sigma}_3}\gamma_0 \tag{7.100}$$

and the acceleration bivector is simply

$$\dot{v}v = g\boldsymbol{\sigma}_3. \tag{7.101}$$

The charge has constant (relativistic) acceleration in the γ_3 direction. We again seek the retarded solution of $X^2 = 0$. This is more conveniently expressed in a cylindrical polar coordinate system, with

$$\boldsymbol{r} = \rho(\cos(\phi)\,\boldsymbol{\sigma}_1 + \sin(\phi)\,\boldsymbol{\sigma}_2) + z\boldsymbol{\sigma}_3, \tag{7.102}$$

so that $r^2 = \rho^2 + z^2$. We then find the following equivalent expressions for the retarded proper time:

$$\begin{aligned} e^{g\tau} &= \frac{1}{2a(z-t)}\Big(a^2 + r^2 - t^2 - \big((a^2+r^2-t^2)^2 - 4a^2(z^2-t^2)\big)^{1/2}\Big), \\ e^{-g\tau} &= \frac{1}{2a(z+t)}\Big(a^2 + r^2 - t^2 + \big((a^2+r^2-t^2)^2 - 4a^2(z^2-t^2)\big)^{1/2}\Big). \end{aligned} \tag{7.103}$$

These equations have a solution provided $z + t > 0$. As the trajectory assumes that the charge has been accelerating for ever, a *horizon* is formed beyond which no effects of the charge are felt (figure 7.3). Constant eternal acceleration of this type is unphysical and in practice we only consider the acceleration taking place for a short period.

We can now calculate the radiation from the charge. First we need the effective distance

$$X\cdot v = \frac{\big((a^2+r^2-t^2)^2 - 4a^2(z^2-t^2)\big)^{1/2}}{2a}. \tag{7.104}$$

This vanishes on the path of the particle ($\rho = 0$ and $z^2 - t^2 = a^2$), as required.

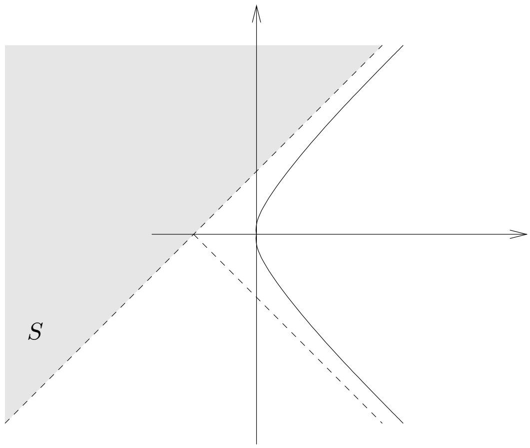

Figure 7.3 *Constant acceleration.* The spacetime trajectory of a particle with constant acceleration is a hyperbola. The asymptotes are null vectors and define future and past horizons. Any signal sent from within the shaded region S will never be received by the particle.

The remaining factor in F is

$$
\begin{aligned}
X \wedge v + \tfrac{1}{2} X \dot{v} v X &= x \wedge v - a\boldsymbol{\sigma}_3 + \frac{1}{2a}(x - x_0)\boldsymbol{\sigma}_3(x - x_0) \\
&= \frac{1}{2a} x \boldsymbol{\sigma}_3 x - \frac{a}{2}\boldsymbol{\sigma}_3 \\
&= \frac{1}{2a}(z^2 - \rho^2 - t^2 - a^2)\boldsymbol{\sigma}_3 + \frac{z\rho}{a}\boldsymbol{\sigma}_\rho + \frac{t\rho}{a} I \boldsymbol{\sigma}_\phi, \qquad (7.105)
\end{aligned}
$$

where $\boldsymbol{\sigma}_\rho$ and $\boldsymbol{\sigma}_\phi$ are the unit spatial axial and azimuthal vectors respectively. An instructive way to display the information contained in the expression for F is to plot the field lines of $\boldsymbol{E}$ at a fixed time. We assume that the charge starts accelerating at $t = t_1$, and stops again at $t = t_2$. There are then discontinuities in the electric field line directions on the two appropriate light-spheres. In figure 7.4 the acceleration takes place for a short period of time, so that a pulse of radiation is sent outwards. In figure 7.5 the charge began accelerating from rest at $t = -10a$. The pattern is well developed, and shows clearly the refocusing of the field lines onto the 'image charge'. The image position corresponds to the place the charge would have reached had it not started accelerating. Of course, the image charge is not actually present, and the field lines diverge after they cross the light-sphere corresponding to the start of the acceleration.

For many applications we are only interested in the fields a long way from the source. In this region the fields can usually be approximated by simple dipole or higher order multipole fields. Suppose that the charge accelerates for a short

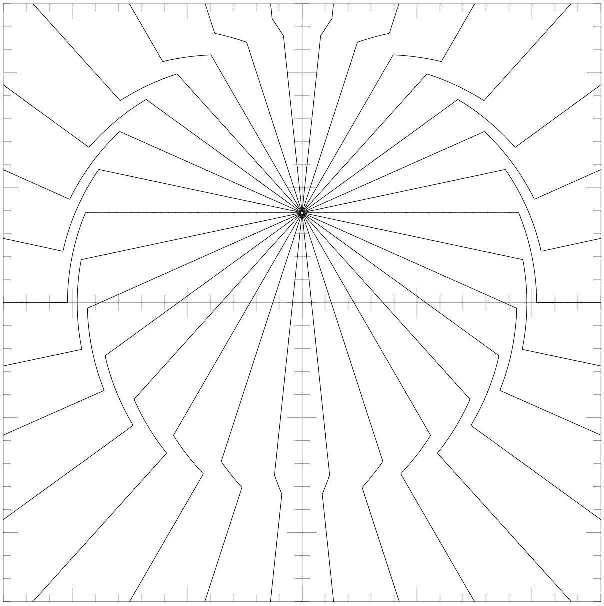

Figure 7.4 *Field lines from an accelerated charge I.* The charge accelerated for $-0.2a < t < 0.2a$, leaving an outgoing pulse of transverse radiation field. The field lines were computed at $t = 5a$.

period and emits a pulse of radiation. In the limit $r \gg a$ the pulse will arrive at some time which, to a good approximation, is centred around the time that minimises $X\cdot v$. This time is given by

$$t_0 = \sqrt{r^2 - a^2}. \tag{7.106}$$

At $t = t_0$ the proper distance $X\cdot v$ evaluates to ρ, the distance from the z axis. The point on the axis ρ away from the observer is where the charge would appear to be if it were not accelerating. For the large distance approximation to be valid we therefore also require that ρ is large, so that the proper distance from the source is large. (For small ρ and $z > a$ a different procedure can be used.) We can now obtain an approximate formula for the radiation field at a fixed location $\boldsymbol{r}$, with $r, \rho \gg a$, around $t = t_0$. For this we define

$$\delta_t = t - t_0 \tag{7.107}$$

so that the proper distance is approximated by

$$X\cdot v \approx \left(\rho^2 + r^2\delta_t^2/a^2\right)^{1/2}. \tag{7.108}$$

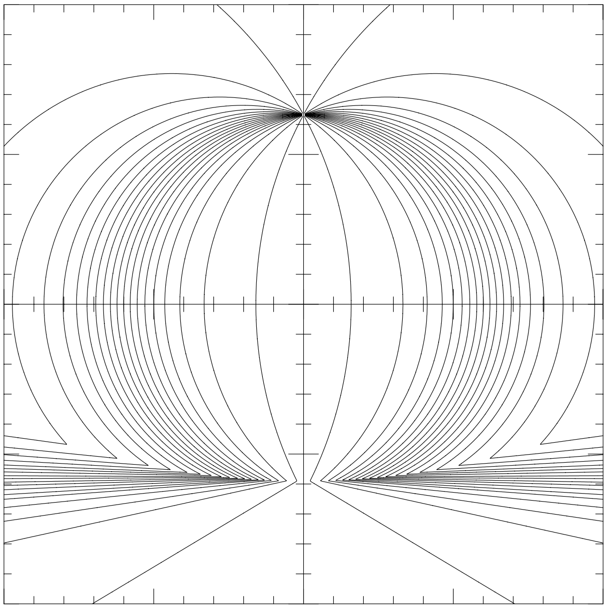

Figure 7.5 *Field lines from an accelerated charge II.* The charge began its acceleration at $t_1 = -10a$ and has thereafter accelerated uniformly. The field lines are plotted at $t = 3a$.

The remaining terms in F become

$$X \wedge v + \tfrac{1}{2} X \dot{v} v X \approx \frac{r\rho}{a} \left(\boldsymbol{\sigma}_\theta + I \boldsymbol{\sigma}_\phi \right), \tag{7.109}$$

where $\boldsymbol{\sigma}_\theta$ and $\boldsymbol{\sigma}_\phi$ are unit spherical-polar basis vectors. The final formula is

$$F \approx \frac{q}{4\pi} \frac{r\rho}{a} \left(\rho^2 + \frac{r^2 \delta_t^2}{a^2} \right)^{-3/2} \left(\boldsymbol{\sigma}_\theta + I \boldsymbol{\sigma}_\phi \right), \tag{7.110}$$

which describes a pure, outgoing radiation field a large distance from a linearly accelerating source. The magnitude of the acceleration is controlled by $g = a^{-1}$.

7.3.4 Circular orbits and synchrotron radiation

As a further application, consider a charge moving in a circular orbit. The worldline is defined by

$$x_0 = \tau \cosh(\alpha)\, \gamma_0 + a\big(\cos(\omega\tau)\gamma_1 + \sin(\omega\tau)\gamma_2\big), \tag{7.111}$$

where $a = \omega^{-1} \sinh(\alpha)$. The particle velocity is

$$v = \cosh(\alpha)\, \gamma_0 + \sinh(\alpha)\big(-\sin(\omega\tau)\gamma_1 + \cos(\omega\tau)\gamma_2\big) = R\gamma_0\tilde{R}, \tag{7.112}$$

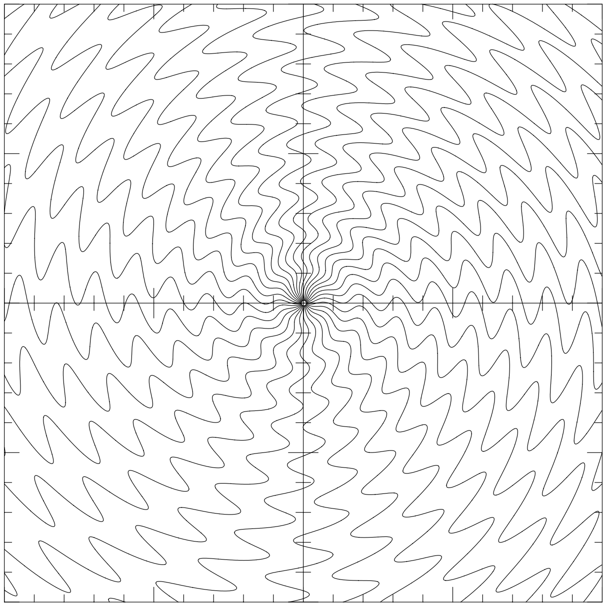

Figure 7.6 *Field lines from a rotating charge I.* The charge has $\alpha = 0.1$, which gives rise to a smooth, wavy pattern.

where the rotor R is given by

$$R = \mathrm{e}^{-\omega\tau I\boldsymbol{\sigma}_3/2}\mathrm{e}^{\alpha\boldsymbol{\sigma}_2/2}. \tag{7.113}$$

We must first locate the retarded null vector X. The equation $X^2 = 0$ reduces to

$$t = \tau\cosh(\alpha) + \big(r^2 + a^2 - 2a\rho\cos(\omega\tau - \phi)\big)^{1/2}, \tag{7.114}$$

which is an implicit equation for $\tau(x)$. No simple analytic solution exists, but a numerical solution is easy to achieve. This is aided by the observation that, for fixed $\boldsymbol{r}$, the mapping between t and τ is monotonic and τ is bounded by the conditions

$$t - \big(r^2 + 2a\rho + a^2\big)^{1/2} < \tau\cosh(\alpha) < t - \big(r^2 - 2a\rho + a^2\big)^{1/2}. \tag{7.115}$$

Once we have a satisfactory procedure for locating τ on the retarded light-cone, we can straightforwardly employ the formula for F in numerical simulations. The first term required is the effective distance $X\cdot v$, which is given by

$$X\cdot v = \cosh(\alpha)\big(r^2 + a^2 - 2a\rho\cos(\omega\tau - \phi)\big)^{1/2} + \rho\sinh(\alpha)\,\sin(\omega\tau - \phi). \tag{7.116}$$

The remaining term to compute, $X\wedge v + X\dot{v}vX/2$, is more complicated, as can be seen from the behaviour shown in figures 7.6, 7.7 and 7.8. They show the

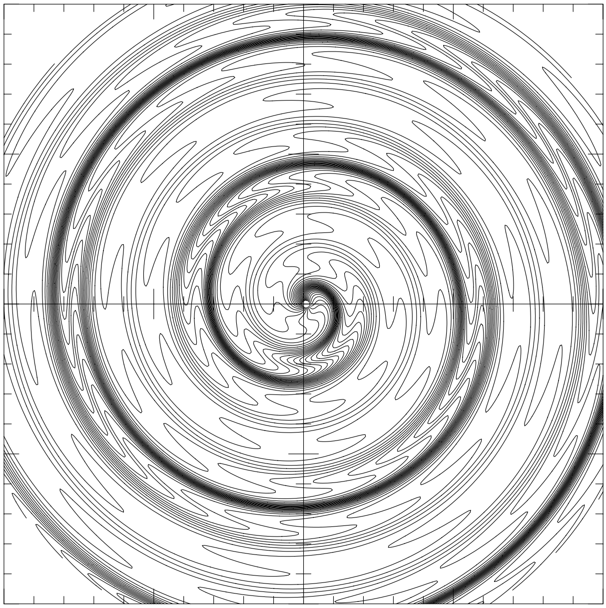

Figure 7.7 *Field lines from a rotating charge II.* The charge has an intermediate velocity, with $\alpha = 0.4$. Bunching of the field lines is clearly visible.

field lines in the equatorial plane of a rotating charge with $\omega = 1$. For 'low' speeds we get the gentle, wavy pattern of field lines shown in figure 7.6. The case displayed in figure 7.7 is for an intermediate velocity ($\alpha = 0.4$), and displays many interesting features. By $\alpha = 1$ (figure 7.8) the field lines have concentrated into synchrotron pulses, a pattern which continues thereafter.

Synchrotron radiation is important in many areas of physics, from particle physics through to radioastronomy. Synchrotron radiation from a radiogalaxy, for example, has $a \approx 10^8$ m and $r \approx 10^{25}$ m. A power-series expansion in a/r is therefore quite safe! Typical values of $\cosh(\alpha)$ are 10^4 for electrons producing radio emission. In the limit $r \gg a$, the relation between t and τ simplifies to

$$t - r \approx \tau \cosh(\alpha) - a \sin(\theta) \cos(\omega\tau - \phi). \tag{7.117}$$

The effective distance reduces to

$$X \cdot v \approx r \cosh(\alpha)\big(1 + \tanh(\alpha) \sin(\theta) \sin(\omega\tau - \phi)\big), \tag{7.118}$$

and the null vector X given by the simple expression

$$X \approx r(\gamma_0 + e_r). \tag{7.119}$$

In the expression for F of equation (7.88) we can ignore the $X \wedge v$ (Coulomb)

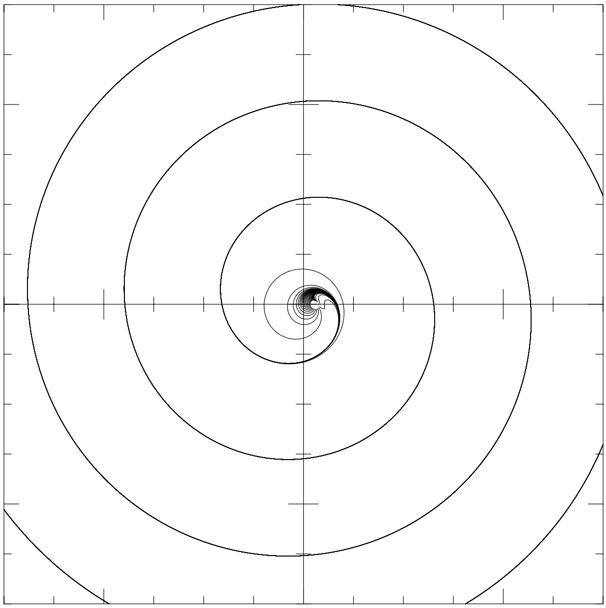

Figure 7.8 *Field lines from a rotating charge III.* The charge is moving at a highly relativistic velocity, with $\alpha = 1$. The field lines are concentrated into a series of synchrotron pulses.

term, which is negligible compared with the long-range radiation term. For the radiation term we need the acceleration bivector

$$\dot{v}v = -\omega \sinh(\alpha)\cosh(\alpha)\big(\cos(\omega\tau)\boldsymbol{\sigma}_1 + \sin(\omega\tau)\boldsymbol{\sigma}_2\big) + \omega \sinh^2(\alpha) I\boldsymbol{\sigma}_3. \quad (7.120)$$

The radiation term is governed by $X\Omega_v X/2$, which simplifies to

$$\begin{aligned}\tfrac{1}{2}X\dot{v}vX \approx\, &\omega r^2 \cosh(\alpha)\sinh(\alpha)(\cos(\theta)\cos(\omega\tau - \phi)\boldsymbol{\sigma}_\theta(1-\boldsymbol{\sigma}_r)\\ &+ \omega r^2 \sinh(\alpha)\big(\cosh(\alpha)\sin(\omega\tau-\phi) + \sinh(\alpha)\sin(\theta)\big)\boldsymbol{\sigma}_\phi(1-\boldsymbol{\sigma}_r). \quad (7.121)\end{aligned}$$

These formulae are sufficient to initiate studying synchrotron radiation. They contain a wealth of physical information, but a detailed study is beyond the scope of this book.

7.4 Electromagnetic waves

For many problems in electromagnetic theory it is standard practice to adopt a complex representation of the electromagnetic field, with the implicit assumption that only the real part represents the physical field. This is particularly convenient when discussing electromagnetic waves and diffraction, as studied in this and the following section. We have seen, however, that the field strength

F is equipped with a natural complex structure through the pseudoscalar I. We should therefore not be surprised to find that, in certain cases, the formal imaginary i plays the role of the pseudoscalar. This is indeed the case for circularly-polarised light. But one cannot always identify i with I, as is clear when handling plane-polarised light. The formal complexification retains its usefulness in such applications and we accordingly adopt it here. It is important to remember that this is a formal exercise, and that real parts must be taken before forming bilinear objects such as the energy-momentum tensor. The study of electromagnetic waves is an old and well-developed subject. Unfortunately, it suffers from the lack of a single, universal set of conventions. As far as possible, we have followed the conventions of Jackson (1999).

We seek vacuum solutions to the Maxwell equations which are purely oscillatory. We therefore start by writing

$$F = \mathrm{Re}\big(F_0 \mathrm{e}^{-ik\cdot x}\big). \tag{7.122}$$

The vacuum equation $\nabla F = 0$ then reduces to the algebraic equation

$$kF_0 = 0. \tag{7.123}$$

Pre-multiplying by k we immediately see that $k^2 = 0$, as expected of the wavevector. The constant bivector F_0 must contain a factor of k, as nothing else totally annihilates k. We therefore must have

$$F_0 = k\wedge n = kn, \tag{7.124}$$

where n is some vector satisfying $k\cdot n = 0$. We can always add a further multiple of k to n, since

$$k(n + \lambda k) = kn + \lambda k^2 = k\wedge n. \tag{7.125}$$

This freedom in n can be employed to ensure that n is perpendicular to the velocity vector of some chosen observer.

As an example, consider a wave travelling in the γ_3 direction with frequency ω as measured in the γ_0 frame. This implies that $\gamma_0\cdot k = \omega$, so the wavevector is given by

$$k = \omega(\gamma_0 + \gamma_3), \tag{7.126}$$

and the phase term is

$$-ik\cdot x = -i\omega(t - z). \tag{7.127}$$

The vector n can be chosen to just contain γ_1 and γ_2 components, so we can write

$$\begin{aligned} F &= -(\gamma_0 + \gamma_3)(\alpha_1\gamma_1 + \alpha_2\gamma_2)\cos(k\cdot x) \\ &= (1 + \boldsymbol{\sigma}_3)(\alpha_1\boldsymbol{\sigma}_1 + \alpha_2\boldsymbol{\sigma}_2)\cos(k\cdot x). \end{aligned} \tag{7.128}$$

This solution represents plane-polarised light, as both the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields lie in fixed planes, 90° apart, and only their magnitudes oscillate in time.

An arbitrary phase can be added to the cosine term, so the most general solution for a wave travelling in the $+z$ direction is

$$F = (1+\boldsymbol{\sigma}_3)\big((\alpha_1\boldsymbol{\sigma}_1 + \alpha_2\boldsymbol{\sigma}_2)\cos(k\cdot x) + (\beta_1\boldsymbol{\sigma}_1 + \beta_2\boldsymbol{\sigma}_2)\sin(k\cdot x)\big), \tag{7.129}$$

where the constants α_i and β_i, are all real. This general solution can describe all possible states of polarisation. A convenient representation is to introduce the complex coefficients

$$c_1 = \alpha_1 + i\beta_1, \quad c_2 = \alpha_2 + i\beta_2. \tag{7.130}$$

These form the components of the complex *Jones vector* (c_1, c_2). In terms of these components we can write

$$F = \mathrm{Re}\big((1+\boldsymbol{\sigma}_3)(c_1\boldsymbol{\sigma}_1 + c_2\boldsymbol{\sigma}_2)\mathrm{e}^{-ik\cdot x}\big), \tag{7.131}$$

and it is a straightforward matter to read off the separate $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

The multivector $(1+\boldsymbol{\sigma}_3)$ has a number of interesting properties. It absorbs factors of $\boldsymbol{\sigma}_3$, as can be seen from

$$\boldsymbol{\sigma}_3(1+\boldsymbol{\sigma}_3) = 1+\boldsymbol{\sigma}_3. \tag{7.132}$$

In addition, $(1+\boldsymbol{\sigma}_3)$ squares to give a multiple of itself,

$$(1+\boldsymbol{\sigma}_3)^2 = 1 + 2\boldsymbol{\sigma}_3 + \boldsymbol{\sigma}_3^2 = 2(1+\boldsymbol{\sigma}_3). \tag{7.133}$$

This property implies that $(1+\boldsymbol{\sigma}_3)$ does not have an inverse, so in a multivector expression it acts as a projection operator. The combination $(1+\boldsymbol{\sigma}_3)/2$ has the particular property of squaring to give itself back again. Multivectors with this property are said to be *idempotent* and are important in the general classification of Clifford algebras and their spinor representations. In spacetime applications idempotents invariably originate from a null vector, in the manner that $(1+\boldsymbol{\sigma}_3)$ originates from a spacetime split of $\gamma_0 + \gamma_3$.

7.4.1 Circularly-polarised light

Many problems are more naturally studied using a basis of circularly-polarised states, as opposed to plane-polarised ones. These arise when c_1 and c_2 are $\pi/2$ out of phase. One form is given by $\alpha_1 = -\beta_2 = E_0$ and $\alpha_2 = \beta_1 = 0$, where E_0 denotes the magnitude of the electric field. For this solution we can write

$$\begin{aligned} F &= E_0(1+\boldsymbol{\sigma}_3)\big(\boldsymbol{\sigma}_1\cos(k\cdot x) - \boldsymbol{\sigma}_2\sin(k\cdot x)\big) \\ &= E_0(1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1\mathrm{e}^{-I\boldsymbol{\sigma}_3\omega(t-z)}. \end{aligned} \tag{7.134}$$

In a plane of constant z (a wavefront) the $\boldsymbol{E}$ field rotates in a clockwise (negative) sense, when viewed looking back towards the source (figure 7.9). In the optics

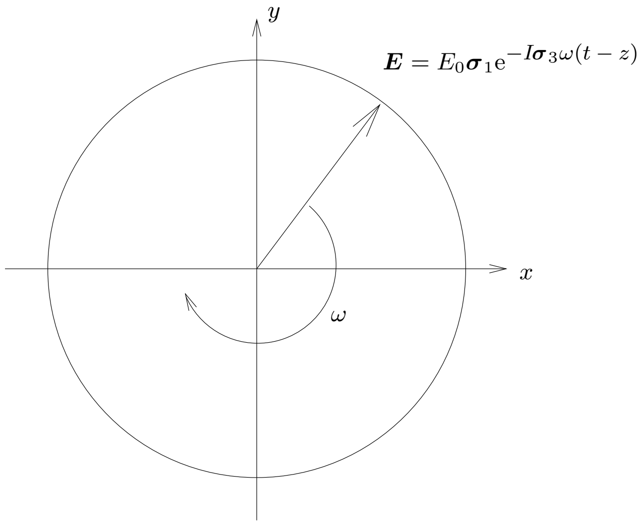

Figure 7.9 *Right-circularly-polarised light.* In the $z=0$ plane the $\boldsymbol{E}$ vector rotates clockwise, when viewed from above. The wave vector points out of the page. In space, at constant time, the $\boldsymbol{E}$ field sweeps out a right-handed helix.

literature this is known as *right-circularly-polarised* light. The reason for this is that, at constant time, the $\boldsymbol{E}$ field sweeps out a helix in space which defines a right-handed screw. If you grip the helix in your right hand, your thumb points in the direction in which the helix advances if tracked along in the sense defined by your grip. This definition of handedness for a helix is independent of which way round you chose to grip it.

Left-circularly-polarised light has the $\boldsymbol{E}$ field rotating with the opposite sense. The general form of this solution is

$$F = (1+\boldsymbol{\sigma}_3)(\alpha_1\boldsymbol{\sigma}_1 + \alpha_2\boldsymbol{\sigma}_2)\mathrm{e}^{I\boldsymbol{\sigma}_3 k\cdot x}. \tag{7.135}$$

Particle physicists prefer an alternative labelling scheme for circularly-polarised light. The scheme is based, in part, on the quantum definition of angular momentum. In the quantum theory, the total angular momentum consists of a spatial part and a spin component. Photons, the quanta of electromagnetic radiation, have spin-1. The spin vector for these can either point in the direction of propagation, or against it, depending on the orientation of rotation of the $\boldsymbol{E}$ field. It turns out that for *right*-circularly-polarised light the spin vector points against the direction of propagation, which is referred to as a state of *negative helicity*. Conversely, *left*-circularly-polarised light has *positive* helicity.

Equation (7.132) enables us to convert phase rotations with the bivector $I\boldsymbol{\sigma}_3$

into duality rotations governed by the pseudoscalar I. This relies on the relation

$$\begin{aligned}(1+\boldsymbol{\sigma}_3)\mathrm{e}^{I\boldsymbol{\sigma}_3\phi} &= (1+\boldsymbol{\sigma}_3)\big(\cos(\phi)+I\boldsymbol{\sigma}_3\sin(\phi)\big)\\ &= (1+\boldsymbol{\sigma}_3)\big(\cos(\phi)+I\sin(\phi)\big) = (1+\boldsymbol{\sigma}_3)\mathrm{e}^{I\phi}.\end{aligned} \tag{7.136}$$

The general solution for right-circularly-polarised light can now be written

$$\begin{aligned}F &= (1+\boldsymbol{\sigma}_3)\mathrm{e}^{I\boldsymbol{\sigma}_3 k\cdot x}(\alpha_1\boldsymbol{\sigma}_1+\alpha_2\boldsymbol{\sigma}_2)\\ &= (1+\boldsymbol{\sigma}_3)(\alpha_1\boldsymbol{\sigma}_1+\alpha_2\boldsymbol{\sigma}_2)\mathrm{e}^{Ik\cdot x}.\end{aligned} \tag{7.137}$$

In this case the complex structure is now entirely geometric, generated by the pseudoscalar. This means that there is no longer any need to take the real part of the solution, as the bivector is already entirely real. A similar trick can be applied to write the constant terms as

$$(1+\boldsymbol{\sigma}_3)(\alpha_1\boldsymbol{\sigma}_1+\alpha_2\boldsymbol{\sigma}_2) = (1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1(\alpha_1 - I\alpha_2), \tag{7.138}$$

so that the coefficient also becomes 'complex' on the pseudoscalar. The general form for right-hand circularly-polarised light solution can now be written

$$F = (1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1\alpha_R\mathrm{e}^{Ik\cdot x}, \tag{7.139}$$

where α_R is a scalar + pseudoscalar combination. Left-hand circularly-polarised light is described by reversing the sign of the exponent to $-Ik\cdot x$. General polarisation states can be built up as linear combinations of these circularly polarised modes, so we can write

$$F = (1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1\big(\alpha_R\mathrm{e}^{Ik\cdot x}+\alpha_L\mathrm{e}^{-Ik\cdot x}\big). \tag{7.140}$$

Here both the coefficients α_L and α_R are scalar + pseudoscalar combinations. The complexification is now based on the pseudoscalar, and we can use α_R and α_L as alternative, geometrically meaningful, complex coefficients for describing general polarisation states. For completeness, the α_L and α_R parameters are related to the earlier plane-polarised coefficients α_i and β_i by

$$\begin{aligned}\alpha_R &= \tfrac{1}{2}(\alpha_1-\beta_2)+\tfrac{1}{2}(\alpha_2+\beta_1)I,\\ \alpha_L &= \tfrac{1}{2}(\alpha_1+\beta_2)+\tfrac{1}{2}(\alpha_2-\beta_1)I.\end{aligned} \tag{7.141}$$

The preceding solutions all assume that the wave vector is entirely in the $\boldsymbol{\sigma}_3$ direction. More generally, we can introduce a right-handed coordinate frame $\{\mathsf{e}_i\}$, with e_3 pointing along the direction of propagation. The solutions then all generalise straightforwardly. In more covariant notation the circularly-polarised modes can also be written

$$F = kn\big(\alpha_R\mathrm{e}^{Ik\cdot x}+\alpha_L\mathrm{e}^{-Ik\cdot x}\big), \tag{7.142}$$

where $k\cdot n = 0$.

7.4.2 Stokes parameters

A useful way of describing the state of polarisation in light emitted from some source is through the Stokes parameters. The general definition of these involves time averages of the fields, which we denote here with an overbar. To start with we assume that the light is coherent, so that all modes are in the same state. We first define the Stokes parameters in terms of the plane-polarised coefficients. The electric field is given by

$$\boldsymbol{E} = \mathrm{Re}\big((c_1\boldsymbol{\sigma}_1 + c_2\boldsymbol{\sigma}_2)\mathrm{e}^{-ik\cdot x}\big) = \mathrm{Re}(\mathcal{E}), \tag{7.143}$$

where $\mathcal{E}$ denotes the complex amplitude. The first Stokes parameter gives the magnitude of the electric field,

$$s_0 = 2\overline{\boldsymbol{E}^2} = \langle \mathcal{E}\mathcal{E}^* \rangle, \tag{7.144}$$

where the star denotes complex conjugation. This evaluates straightforwardly to

$$s_0 = |c_1|^2 + |c_2|^2. \tag{7.145}$$

The remaining three Stokes parameters describe the relative amounts of radiation present in various polarisation states. If we denote the real components of $\boldsymbol{E}$ by E_x and E_y the parameters are defined by

$$\begin{aligned} s_1 &= 2(\overline{E_x^2} - \overline{E_y^2}) = |c_1|^2 - |c_2|^2 \\ s_2 &= 4\overline{E_x E_y} = 2\mathrm{Re}(c_1 c_2^*) \\ s_3 &= 4\overline{E_x(t)E_y(t+\pi/(2\omega))} = -2\mathrm{Im}(c_1 c_2^*). \end{aligned} \tag{7.146}$$

The Stokes parameters can equally well be written in terms of the α_L and α_R coefficients of circularly-polarised modes:

$$\begin{aligned} s_0 &= 2(|\alpha_L|^2 + |\alpha_R|^2), \\ s_1 &= 4\langle \alpha_L \alpha_R \rangle, \\ s_2 &= -4\langle I\alpha_L \alpha_R \rangle, \\ s_3 &= 2(|\alpha_L^2| - |\alpha_R|^2). \end{aligned} \tag{7.147}$$

For coherent light the Stokes parameters are related by

$$s_0^2 = s_1^2 + s_2^2 + s_3^2. \tag{7.148}$$

The s_μ can therefore be viewed algebraically as the components of a null vector, though its direction in space has no physical significance. This representation for 'observables' in terms of a null vector is typical of a two-state quantum system. We can bring this out neatly in the spacetime algebra by introducing the three-dimensional rotor

$$\kappa = \langle \alpha_L \rangle + \langle I\alpha_L \rangle I\boldsymbol{\sigma}_3 - \langle \alpha_R \rangle I\boldsymbol{\sigma}_2 - \langle I\alpha_R \rangle I\boldsymbol{\sigma}_1. \tag{7.149}$$

The (quantum) origin of this object is explained in section 8.1. The rotor κ satisfies

$$\kappa\kappa^\dagger = \tfrac{1}{2}s_0, \qquad \kappa\boldsymbol{\sigma}_3\kappa^\dagger = \tfrac{1}{2}s_i\boldsymbol{\sigma}_i. \tag{7.150}$$

It follows that in spacetime

$$2\kappa(\gamma_0+\gamma_3)\tilde{\kappa} = 2\kappa(1+\boldsymbol{\sigma}_3)\kappa^\dagger\gamma_0 = s_0\gamma_0 + s_i\gamma_i, \tag{7.151}$$

and since we have rotated a null vector we automatically obtain a null vector. The unit spatial vector

$$\hat{\boldsymbol{s}} = \frac{\boldsymbol{s}}{s_0}, \qquad \boldsymbol{s} = s_i\boldsymbol{\sigma}_i \tag{7.152}$$

can be represented by a point on a sphere. For light polarisation states this is called the *Poincaré sphere*. For spin-1/2 systems the equivalent construction is known as the *Bloch sphere*. The construction is also useful for describing partially coherent light. In this case the light can be viewed as originating from a set of discrete (incoherent) sources. The single null vector is replaced by an average over the sources,

$$s = \sum_{k=1}^{n} s_k \tag{7.153}$$

and the unit vector $\hat{\boldsymbol{s}}$ is replaced by

$$\boldsymbol{s} = \frac{s\wedge\gamma_0}{s\cdot\gamma_0} = \sum_{k=1}^{n}\frac{\omega_k}{\omega}\hat{\boldsymbol{s}}_k, \qquad \omega = \sum_{k=1}^{n}\omega_k. \tag{7.154}$$

The resulting polarisation vector $\boldsymbol{s}$ has $\boldsymbol{s}^2 \leq 1$, so now defines a vector inside the Poincaré sphere. The length of this vector directly encodes the relative amounts of coherent and incoherent light present.

The preceding discussion also makes it a simple matter to compute how the Stokes parameters appear to observers moving at different velocities. Suppose that a second observer with velocity $v = \mathsf{e}_0$ sets up a frame $\{\mathsf{e}_\mu\}$. This is done in such a way that the wave vector still travels in the e_3 direction, which requires that

$$\mathsf{e}_3 = \frac{k - k\cdot v\, v}{k\cdot v}. \tag{7.155}$$

If the old and new frames are related by a rotor, $\mathsf{e}_\mu = R\gamma_\mu\tilde{R}$, then equation (7.155) restricts R to satisfy

$$Rk\tilde{R} = \lambda k. \tag{7.156}$$

Rather than work in the new frame, it is simpler to back-transform the field F and work in the original $\{\gamma_\mu\}$ frame. We define

$$F' = \tilde{R}F(Rx\tilde{R})R = \frac{1}{\lambda}kn'\left(\alpha_R e^{Ik\cdot x/\lambda} + \alpha_L e^{-Ik\cdot x/\lambda}\right), \tag{7.157}$$

where $n' = \tilde{R}nR$ and $k = \omega(\gamma_0+\gamma_3)$. We can again choose n' to be perpendicular to γ_0 by adding an appropriate multiple of k. It follows that the only change to the final vector n can be a rotation in the $I\boldsymbol{\sigma}_3$ plane. Performing a spacetime split on γ_0, and assuming that the original n was $-\gamma_1$, we obtain

$$F' = \frac{1}{\lambda}(1+\boldsymbol{\sigma}_3)\boldsymbol{\sigma}_1 \mathrm{e}^{-\phi I\boldsymbol{\sigma}_3}\left(\alpha_R \mathrm{e}^{Ik\cdot x/\lambda} + \alpha_L \mathrm{e}^{-Ik\cdot x/\lambda}\right), \tag{7.158}$$

where ϕ is the angle of rotation in the $I\boldsymbol{\sigma}_3$ plane. The rotation can again be converted to a phase factor on I, so the overall change is that α_R and α_L are multiplied by $\lambda^{-1}\exp(I\phi)$. The rescaling has no effect on the unit vector on the Poincaré sphere, so the only change is a rotation through 2ϕ in the $I\boldsymbol{\sigma}_3$ plane. This implies that the $\boldsymbol{\sigma}_3$ component of the vector on the Poincaré sphere is constant, which is sensible. This component determines the relative amounts of left and right-circularly-polarised light present, and this ratio is independent of which observer measures it. Similar arguments apply to the case of partially coherent light.

7.5 Scattering and diffraction

We turn now to the related subjects of the scattering and diffraction of electromagnetic waves. This is an enormous subject and our aim here is to provide little more than an introduction, highlighting in particular a unified approach based on the free-space multivector Green's function. This provides a first-order formulation of the scattering problem, which is valuable in numerical computation. We continue to adopt a complex representation for the electromagnetic field, and will concentrate on waves of a single frequency. The time dependence is then expressed via

$$F(x) = F(\boldsymbol{r})\mathrm{e}^{-i\omega t}, \tag{7.159}$$

so that the Maxwell equations reduce to

$$\boldsymbol{\nabla}F - i\omega F = 0. \tag{7.160}$$

This is the first-order equivalent of the vector Helmholtz equation. Throughout this section we work with the full, complex quantities, and suppress all factors of $\exp(i\omega t)$. All quadratic quantities are assumed to be time averaged.

If sources are present the Maxwell equations become

$$(\boldsymbol{\nabla} - i\omega)F = \rho - \boldsymbol{J}. \tag{7.161}$$

Current conservation tells us that the (complex) current satisfies

$$i\omega\rho = \boldsymbol{\nabla}\cdot\boldsymbol{J}. \tag{7.162}$$

Provided that all the sources are localised in some region in space, there can be no electric monopole term present. This follows because

$$Q = \int |dX| \rho = \frac{1}{i\omega} \oint \boldsymbol{J} \cdot \boldsymbol{n} \, |dA|, \tag{7.163}$$

where $\boldsymbol{n}$ is the outward normal. Taking the surface to totally enclose the sources, so that $\boldsymbol{J}$ vanishes over the surface of integration, we see that $Q = 0$.

7.5.1 First-order Green's function

The main result we employ in this section is Green's theorem in three dimensions in the general form

$$\int_V (\dot{G} \dot{\boldsymbol{\nabla}} F + G \, \boldsymbol{\nabla} F) \, |dX| = \oint_{\partial V} G \boldsymbol{n} F \, dA \tag{7.164}$$

where $\boldsymbol{n}$ is the outward-pointing normal vector over the surface ∂V. If F satisfies the vacuum Maxwell equations, we have

$$\oint_{\partial V} G \boldsymbol{n} F \, dA = \int_V (\dot{G} \dot{\boldsymbol{\nabla}} + i\omega G) F \, |dX|. \tag{7.165}$$

We therefore seek a Green's function satisfying

$$\dot{G} \dot{\boldsymbol{\nabla}} + i\omega G = \delta(\boldsymbol{r}). \tag{7.166}$$

It will turn out that G only contains (complex) scalar and vector terms, so (by reversing both sides) this equation is equivalent to

$$(\boldsymbol{\nabla} + i\omega) G = \delta(\boldsymbol{r}). \tag{7.167}$$

The Green's function is easily found from the Green's function for the (scalar) Helmholtz equation,

$$\phi(\boldsymbol{r}) = -\frac{1}{4\pi r} \mathrm{e}^{i\omega r}. \tag{7.168}$$

This is appropriate for *outgoing* radiation. Choosing the outgoing Green's function is equivalent to imposing causality by working with retarded fields. The function ϕ satisfies

$$(\boldsymbol{\nabla}^2 + \omega^2)\phi = \delta(\boldsymbol{r}) = (\boldsymbol{\nabla} + i\omega)(\boldsymbol{\nabla} - i\omega)\phi. \tag{7.169}$$

We therefore see that the required first-order Green's function is

$$\begin{aligned} G(\boldsymbol{r}) &= (\boldsymbol{\nabla} - i\omega)\phi \\ &= \frac{\mathrm{e}^{i\omega r}}{4\pi} \left(\frac{i\omega}{r} (1 - \boldsymbol{\sigma}_r) + \frac{\boldsymbol{r}}{r^3} \right), \end{aligned} \tag{7.170}$$

where $\boldsymbol{\sigma}_r = \boldsymbol{r}/r$ is the unit vector in the direction of $\boldsymbol{r}$. This Green's function is the key to much of scattering theory. With a general argument it satisfies

$$(\boldsymbol{\nabla} + i\omega) G(\boldsymbol{r} - \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{7.171}$$

or, equivalently,

$$(\boldsymbol{\nabla}' - i\omega)G(\boldsymbol{r}-\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{7.172}$$

where $\boldsymbol{\nabla}'$ denotes the vector derivative with respect to $\boldsymbol{r}'$.

7.5.2 Radiation and multipole fields

As a first application, suppose that a localised system of charges in free space, with sinusoidal time dependence, generates outgoing radiation fields. We could find these by generalising our point source solutions of section 7.3, but here we wish to exploit our new Green's function. We can now immediately write down the solution

$$F(\boldsymbol{r}) = -\int_V G(\boldsymbol{r}'-\boldsymbol{r})\big(\rho(\boldsymbol{r}') - \boldsymbol{J}(\boldsymbol{r}')\big)\,|dX'|, \tag{7.173}$$

where the integral is over a volume enclosing all of the sources. Equation (7.172) guarantees that this equation solves the Maxwell equations (7.161), subject to the boundary condition that only outgoing waves are present at large distances. It is worth stressing that the geometric algebra formulation is crucial to the way we have a single integral yielding both the electric and magnetic fields.

Often, one is mainly interested in the radiation fields present at large distances from the source. These are the contributions to F which fall off as $1/r$. To isolate these terms we use the expansion

$$\mathrm{e}^{i\omega|\boldsymbol{r}-\boldsymbol{r}'|} = \mathrm{e}^{i\omega r}\mathrm{e}^{-i\omega\boldsymbol{\sigma}_r\cdot\boldsymbol{r}'} + \mathrm{O}(r^{-1}), \tag{7.174}$$

so that the Green's function satisfies

$$\lim_{r\mapsto\infty} G(\boldsymbol{r}'-\boldsymbol{r}) = \frac{i\omega}{4\pi r}\mathrm{e}^{i\omega r}(1+\boldsymbol{\sigma}_r)\mathrm{e}^{-i\omega\boldsymbol{\sigma}_r\cdot\boldsymbol{r}'}. \tag{7.175}$$

We therefore find that the limiting form of F can be written

$$F(\boldsymbol{r}) = -\frac{i\omega}{4\pi r}\mathrm{e}^{i\omega r}(1+\boldsymbol{\sigma}_r)\int \mathrm{e}^{-i\omega\boldsymbol{\sigma}_r\cdot\boldsymbol{r}'}\big(\rho(\boldsymbol{r}') - \boldsymbol{J}(\boldsymbol{r}')\big)\,|dX'|. \tag{7.176}$$

As expected, the multivector is controlled by the idempotent term $(1+\boldsymbol{\sigma}_r) = (\gamma_0 + e_r)\gamma_0$, appropriate for outgoing radiation.

A multipole expansion of the radiation field is achieved by expanding (7.176) in a series in ωd, where d is the dimension of the source. To leading order, and recalling that no monopole term is present, we find that

$$\begin{aligned}\int \mathrm{e}^{-i\omega\boldsymbol{\sigma}_r\cdot\boldsymbol{r}'}\big(\rho(\boldsymbol{r}') - \boldsymbol{J}(\boldsymbol{r}')\big)\,|dX'| &\approx \int(-\boldsymbol{J} - i\omega\rho\boldsymbol{\sigma}_r\cdot\boldsymbol{r}')|dX'| \\ &= \int(-\boldsymbol{J} + \boldsymbol{\sigma}_r\cdot\boldsymbol{J})\,|dX'|,\end{aligned} \tag{7.177}$$

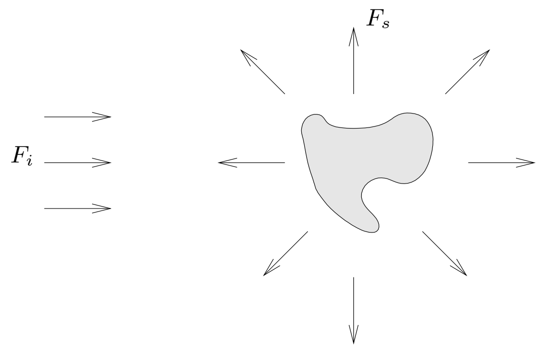

Figure 7.10 *Scattering by a localised object.* The incident field F_i sets up oscillating currents in the object, which generate an outgoing radiation field F_s.

where we have integrated by parts to obtain the final expression. This result is more commonly expressed in terms of the *electric dipole moment* $\boldsymbol{p}$, via

$$\int \boldsymbol{J}\,|dX| = -\int \boldsymbol{r}\,\boldsymbol{\nabla}\cdot\boldsymbol{J}\,|dX| = -i\omega \int \boldsymbol{r}\rho(\boldsymbol{r})\,|dX| = -i\omega\boldsymbol{p}. \tag{7.178}$$

The result is that the F field is given by

$$F(\boldsymbol{r}) = \frac{\omega^2}{4\pi r}\mathrm{e}^{i\omega r}(1+\boldsymbol{\sigma}_r)(\boldsymbol{p}-\boldsymbol{\sigma}_r\cdot\boldsymbol{p}). \tag{7.179}$$

An immediate check is that the scalar term in F vanishes, as it must. The electric and magnetic dipole fields can be read off easily now as

$$\boldsymbol{E} = \frac{\omega^2}{4\pi r}\mathrm{e}^{i\omega r}\boldsymbol{\sigma}_r\,\boldsymbol{\sigma}_r\wedge\boldsymbol{p}, \qquad I\boldsymbol{B} = \frac{\omega^2}{4\pi r}\mathrm{e}^{i\omega r}\boldsymbol{\sigma}_r\wedge\boldsymbol{p}. \tag{7.180}$$

These formulae are quite general for any (classical) radiating object.

7.6 Scattering

The geometry of a basic scattering problem is illustrated in figure 7.10. A (known) field F_i is incident on a localised object. Usually the incident radiation is taken to be a plane wave. This radiation sets up oscillating currents in the scatterer, which in turn generate a scattered field F_s. The total field F is given by

$$F = F_i + F_s, \tag{7.181}$$

and both F_i and F_s satisfy the vacuum Maxwell equations away from the scatterers.

The essential difficulty is how to solve for the currents set up by the incident radiation. This is extremely complex and a number of distinct approaches are described in the literature. One straightforward result is for scattering from a small uniform dielectric sphere. For this situation we have

$$\boldsymbol{p} = 4\pi a^3 \frac{\epsilon_r - 1}{\epsilon_r + 2} \boldsymbol{E}_i, \tag{7.182}$$

where a is the radius of the sphere. From equation (7.180) we see that the ratio of incident to scattered radiation is controlled by ω^2. This ratio determines the differential cross section via

$$\frac{d\sigma}{d\Omega} = r^2 \frac{|\boldsymbol{e}^* \cdot \boldsymbol{E}_s|^2}{|\boldsymbol{e}^* \cdot \boldsymbol{E}_i|^2}, \tag{7.183}$$

where the complex vector $\boldsymbol{e}$ determines the polarisation. The cross section clearly depends of the polarisation of the incident wave. Summing over polarisations the differential cross section is

$$\frac{d\sigma}{d\Omega} = \omega^4 a^6 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2} \right)^2 \frac{1 + \cos^2(\theta)}{2}. \tag{7.184}$$

The factor of $\omega^4 = \lambda^{-4}$ is typical of Rayleigh scattering. These results are central to Rayleigh's explanation of blue skies and red sunsets.

Suppose now that we know the fields over a closed surface enclosing a volume V. Provided that F satisfies the vacuum Maxwell equations throughout V we can compute F_s directly from

$$F_s(\boldsymbol{r}') = \oint_{\partial V} G(\boldsymbol{r} - \boldsymbol{r}') \boldsymbol{n} F_s(\boldsymbol{r}) \, |dS|. \tag{7.185}$$

We take the volume V to be bounded by two surfaces, S_1 and S_2, as shown in figure 7.11. The surface S_1 is assumed to lie just outside the scatterers, so that $J = 0$ over S_1. The surface S_2 is assumed to be spherical, and is taken out to infinity. In this limit only the $1/r$ terms in G and F can contribute to the surface integral over S_2. But from equation (7.175) we know that

$$\lim_{r \mapsto \infty} G(\boldsymbol{r} - \boldsymbol{r}') = \frac{i\omega}{4\pi r} \mathrm{e}^{i\omega r} (1 - \boldsymbol{\sigma}_r) \mathrm{e}^{-i\omega \boldsymbol{\sigma}_r \cdot \boldsymbol{r}'}, \tag{7.186}$$

whereas F_s contains a factor of $(1 + \boldsymbol{\sigma}_r)$. It follows that the integrand $G\boldsymbol{n}F_s$ contains the term

$$(1 - \boldsymbol{\sigma}_r) \boldsymbol{\sigma}_r (1 + \boldsymbol{\sigma}_r) = 0. \tag{7.187}$$

This is identically zero, so there is no contribution from the surface at infinity. The result is that the scattered field is given by

$$F_s(\boldsymbol{r}) = \frac{1}{4\pi} \oint_{S_1} \mathrm{e}^{i\omega d} \left(\frac{i\omega}{d} + \frac{i\omega(\boldsymbol{r} - \boldsymbol{r}')}{d^2} - \frac{\boldsymbol{r} - \boldsymbol{r}'}{d^3} \right) \boldsymbol{n}' F_s(\boldsymbol{r}') \, |dS(\boldsymbol{r}')|, \tag{7.188}$$

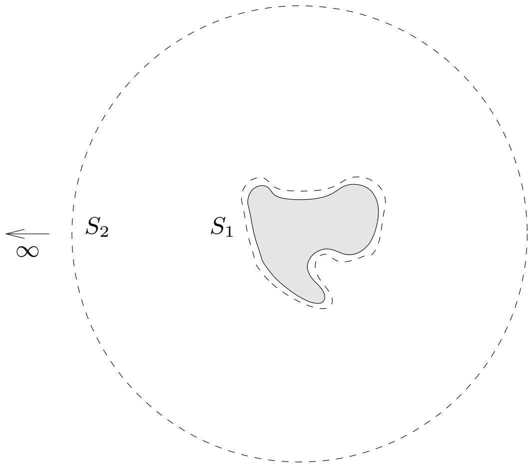

Figure 7.11 *Surfaces for Green's theorem.* The surface S_2 can be taken out to infinity, and S_1 lies just outside the scattering surface.

where

$$d = |\boldsymbol{r} - \boldsymbol{r}'|. \tag{7.189}$$

Since $\boldsymbol{n}$ is the outward pointing normal to the volume, this points *into* the scatterers. This result contains all the necessary polarisation and obliquity factors, often derived at great length in standard optics texts.

A significant advantage of this first-order approach is that it clearly embodies Huygens' principle. The scattered field F_s is propagated into the interior simply by multiplying it by a Green's function. This accords with Huygen's original idea of reradiation of wavelets from any given wavefront. Two significant problems remain, however. The first is how to specify F_s over the surface of integration. This requires detailed modelling of the polarisation currents set up by the incident radiation. A subtlety here is that we do not have complete freedom to specify F over the surface. The equation $\boldsymbol{\nabla} F = i\omega F$ implies that the components of $\boldsymbol{E}$ and $\boldsymbol{B}$ perpendicular to the boundary surface are determined by the derivatives of the components in the surface. This reduces the number of degrees of freedom in the problem from six to four, as is required for electromagnetism.

A further problem is that, even if F_s has been found, the integrals in equation (7.188) cannot be performed analytically. One can approximate to the large r regime and, after various approximations, recover Fraunhofer and Fresnel optics. Alternatively, equation (7.188) can be used as the basis for numerical simulations of scattered fields. Figure 7.12 shows the type of detailed patterns that can emerge. The plot was calculated using the two-dimensional equivalent of equation (7.188). The total energy density is shown, where the scattering

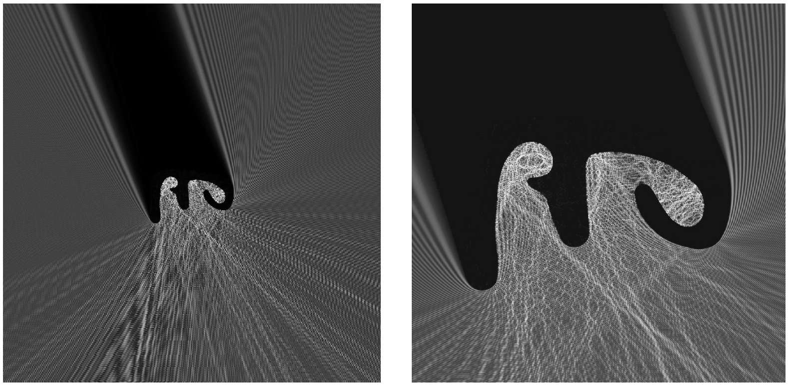

Figure 7.12 *Scattering in two dimensions.* The plots show the intensity of the electric field, with higher intensity coloured lighter. The incident radiation enters from the bottom right of the diagram and scatters off a conductor with complicated surface features. The conductor is closed in the shadow region. Various diffraction effects are clearly visible. The right-hand plot is a close-up near the surface and shows the complicated pattern of hot and cold regions that can develop.

is performed by a series of perfect conductors. A good check that the calculations have been performed correctly is that all the expected shadowing effects are present.

7.7 Notes

There is a vast literature on electromagnetism and electrodynamics. For this chapter we particularly made use of the classic texts by Jackson (1999) and Schwinger et al. (1998), both entitled *Classical Electrodynamics*. The former of these also contains an exhaustive list of further references. Applications of geometric algebra to electromagnetism are discussed in the book *Multivectors and Clifford Algebra in Electrodynamics* by Jancewicz (1989). This is largely an introductory text and stops short of tackling the more advanced applications.

We are grateful to Stephen Gull for producing the figures in section 7.3 and for stimulating much of the work described in this chapter. Further material can be found in the Banff series of lectures by Doran et al (1996a). Readers interested in the action at a distance formalism of Wheeler and Feynman can do no better than return to their original 1949 paper. It is a good exercise to convert their arguments into a more streamlined geometric algebra notation!

7.8 Exercises

7.1 A circular current loop has radius a and lies in the $z = 0$ plane with its centre at the origin. The loop carries a current J. Write down an integral expression for the $\boldsymbol{B}$ field, and show that on the z axis,

$$\boldsymbol{B} = \frac{\mu_0 J a^2}{2(a^2 + z^2)^{3/2}} \boldsymbol{\sigma}_3.$$

7.2 An extension to the Maxwell equations which is regularly discussed is how they are modified in the presence of magnetic monopoles. If ρ_m and $\boldsymbol{J}_m$ denote magnetic charges and currents, the relevant equations are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho_e, & \boldsymbol{\nabla}\cdot\boldsymbol{B} &= \rho_m, \\ -\boldsymbol{\nabla}\times\boldsymbol{E} &= \frac{\partial}{\partial t}\boldsymbol{B} + \boldsymbol{J}_m, & \boldsymbol{\nabla}\times\boldsymbol{H} &= \frac{\partial}{\partial t}\boldsymbol{D} + \boldsymbol{J}_e. \end{aligned}$$

Prove that in free space these can be written

$$\nabla F = J_e + J_m I,$$

where $J_m = (\rho_m + \boldsymbol{J}_m)\gamma_0$. A duality transformation of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields is defined by

$$\boldsymbol{E}' = \boldsymbol{E}\cos(\alpha) + \boldsymbol{B}\sin(\alpha), \qquad \boldsymbol{B}' = \boldsymbol{B}\cos(\alpha) - \boldsymbol{B}\sin(\alpha).$$

Prove that this can be written compactly as $F' = F\mathrm{e}^{-I\alpha}$. Hence find the equivalent transformation law for the source terms such that the equations remain invariant, and prove that the electromagnetic energy-momentum tensor is also invariant under a duality transformation.

7.3 A particle follows the trajectory $x_0(\tau)$, with velocity $v = \dot{x}$ and acceleration $\dot{v}$. If X is the retarded null vector connecting the point x to the worldline, show that the electromagnetic field at x is given by

$$F = \frac{q}{4\pi} \frac{X\wedge v + \frac{1}{2} X \Omega_v X}{(X\cdot v)^3},$$

where $\Omega_v = \dot{v}\wedge v$. Prove directly that F satisfies $\nabla F = 0$ off the particle worldline.

7.4 Prove the following formulae relating the retarded A and F fields for a point charge to the null vector X:

$$A = -\frac{q}{8\pi\epsilon_0}\nabla^2 X, \qquad F = -\frac{q}{8\pi\epsilon_0}\nabla^3 X.$$

These expressions are of interest in the 'action at a distance' formulation of electrodynamics, as discussed by Wheeler and Feynman (1949).

7.5 Confirm that, at large distances for the source, the radiation fields due to both linearly and circularly accelerating charges go as

$$F_{rad} \approx \frac{1}{r}(1+\boldsymbol{\sigma}_r)\boldsymbol{a},$$

where $\boldsymbol{\sigma}_r \cdot \boldsymbol{a} = 0$.

7.6 From the solution for the fields due to a point charge in a circular orbit (section 7.3.4), explain why synchrotron radiation arrives in pulses.

7.7 For the κ defined in equation (7.149), verify that $\kappa\boldsymbol{\sigma}_3\kappa^\dagger = s_i\boldsymbol{\sigma}_i$, where s_i are Stokes parameters.

7.8 A rotor R relates two frames by $\mathsf{e}'_\mu = R\mathsf{e}_\mu\tilde{R}$. In both frames the vector e_3 vector is defined by

$$\mathsf{e}_3 = \mathsf{e}'_3 = \frac{k - k\cdot\mathsf{e}_0\,\mathsf{e}_0}{k\cdot\mathsf{e}_0},$$

where k is a fixed null vector. Prove that for this relation to be valid for both frames we must have

$$Rk\tilde{R} = \lambda k.$$

How many degrees of freedom are left in the rotor R if this equation holds?

7.9 In optical problems we are regularly interested in the effects of a planar aperture on incident plane waves. Suppose that the aperture lies in the $z=0$ plane, and we are interested in the fields in the region $z>0$. By introducing the Green's function

$$G'(\boldsymbol{r};\boldsymbol{r}') = G(\boldsymbol{r}-\boldsymbol{r}') - G(\boldsymbol{r}-\bar{\boldsymbol{r}}'),$$

where $\bar{\boldsymbol{r}} = -\boldsymbol{\sigma}_3\boldsymbol{r}\boldsymbol{\sigma}_3$, prove that the field in the region $z>0$ is given by

$$F_s(\boldsymbol{r}') = \int dx\,dy\,\frac{z'\mathrm{e}^{I\omega d}}{2\pi d^3}(1-i\omega d)F_s(x,y,0), \tag{E7.1}$$

where $d = |\boldsymbol{r}-\boldsymbol{r}'|$. In the Kirchoff approximation we assume that F_s over the aperture can be taken as the incident plane wave. By working in the large r and small angle limit, prove the Fraunhofer result that the transmitted amplitude is controlled by the Fourier transform of the aperture function.

7.10 Repeat the analysis of the previous question for a two-dimensional arrangement. You will need to understand some of the properties of Hankel functions.