The optimal control curve

So now we know that ξ is a minimizer of (P), then it must be of the form

$$(1) \quad \xi(t) = \sum_{i \in \mathbb{N}_2} \left[\cos(\sigma_i(\mu)t)\, a_i + \sin(\sigma_i(\mu)t)\, a_i' \right]$$

$$= \sum_{i \in \mathbb{N}_2} \left[\cos(\sigma_i(\mu)t)\, U\tilde{a}_i + \sin(\sigma_i(\mu)t)\, U\tilde{a}_i' \right]$$

- with $\sigma_2(\mu) \,/\, \sigma_1(\mu) \in \mathbb{N}$ and we have shown that it is possible to choose $\mu \in \mathbb{R}^6$ such that this relation holds. Furthermore, we have seen that the $\tilde{a}_i$ and $\tilde{a}_i'$ are pairwise G-orthogonal.

- For the energy, we now have

$$(2) \quad \mathcal{G}(\xi) = \sigma_1(\mu)^2 \left[\Delta_g \tilde{a}_1 \cdot \tilde{a}_1 + \Delta_g \tilde{a}_1' \cdot \tilde{a}_1' \right]$$
$$+ \sigma_2(\mu)^2 \left[\Delta_g \tilde{a}_2 \cdot \tilde{a}_2 + \Delta_g \tilde{a}_2' \cdot \tilde{a}_2' \right]$$

$$= \frac{\sigma_1(\mu)}{2\pi} \left[|\tilde{u}_{\sigma_1}|^2 + |\tilde{v}_{\sigma_1}|^2 \right]$$

$$+ \frac{\sigma_2(\mu)}{2\pi} \left[|\tilde{u}_{\sigma_2}|^2 + |\tilde{v}_{\sigma_2}|^2 \right],$$

with $\tilde{u}_{\sigma_i} := \sqrt{2\pi \Delta_g \sigma_i}\, \tilde{a}_i$, $\tilde{v}_{\sigma_i} := \sqrt{2\pi \Delta_g \sigma_i}\, \tilde{a}_i'^{\,2}$ for $i \in \mathbb{N}_2$. In particular, we then have

(3) $$\sqrt{\det \Delta_g}\,\left(\Delta_h \tilde{\Delta}_g\right)^{-1} \delta_p = \tilde{v}_{\sigma_1} \wedge \tilde{u}_{\sigma_1} + \tilde{v}_{\sigma_2} \wedge \tilde{u}_{\sigma_2}.$$

The latter relation is independent of the index σ_i, so we can choose $\sigma_1 = 2 \geq \sigma_2 = 1$ to minimize the energy in (2). Moreover, note that due to the G-orthogonality of the $\tilde{a}_i$ and $\tilde{a}_i'$, we have

$$\tilde{u}_{\sigma_1} \perp \tilde{v}_{\sigma_1} \quad \text{and} \quad \tilde{u}_{\sigma_2} \perp \tilde{v}_{\sigma_2},$$

so among all simple bivectors realizing $\tilde{v}_{\sigma_1} \wedge \tilde{u}_{\sigma_1}$ and $\tilde{v}_{\sigma_2} \wedge \tilde{u}_{\sigma_2}$, the pairs $(\tilde{u}_{\sigma_i}, \tilde{v}_{\sigma_i})$, $i \in \mathbb{N}_2$ are the ones that minimize the quantities

$$|\tilde{u}_{\sigma_i}|^2 + |\tilde{v}_{\sigma_i}|^2, \quad i \in \mathbb{N}_2.$$

Hence, we cannot further minimize the energy in (2) other than permuting 1 and 2 if

$$|\tilde{u}_{\sigma_2}|^2 + |\tilde{v}_{\sigma_2}|^2 < |\tilde{u}_{\sigma_1}|^2 + |\tilde{v}_{\sigma_1}|^2.$$

RECONSTRUCTION

Finally, if we are given a non-simple net displacement δp, we set again

(5) $$\omega := \sqrt{\det \Delta_g}\,\left(\Delta_n \tilde{\Delta}_g\right)^{-1} \delta p .$$

Then, we write ω as

- $$\omega = v_1 \wedge u_1 + v_2 \wedge u_2,$$

with $u_i \perp v_i$, $i \in \mathbb{N}$ and

(6) $$|v_2|^2 + |u_2|^2 \leq |v_1|^2 + |u_1|^2,$$

which is always possible. Setting

- $$a_1 := \frac{1}{\sqrt{2\pi}} U \Delta_g^{-1/2} u_1$$

$$a_1' := \frac{1}{\sqrt{2\pi}} U \Delta_g^{-1/2} v_1$$

$$a_2 := \frac{1}{\sqrt{4\pi}} U \Delta_g^{-1/2} u_2$$

$$a_2' := \frac{1}{\sqrt{4\pi}} U \Delta_g^{-1/2} v_2,$$

yields the optimial control curve

$$\xi(t) := \cos(t)\, a_1 + \sin(t)\, a_1' + \cos(2t)\, a_2 + \sin(2t)\, a_2' .$$