Convexity of the constraint

We want to establish strong duality for the functional $L : A \times B \to \mathbb{R}$ with[1] $A = H^1_\sharp(J, \mathbb{R}^4)$, $B = \mathbb{R}^6$, and

$$L(\eta, \mu) = \int_J \Lambda_{\mathfrak{g}} \dot{\eta}(t) \cdot \eta(t) dt + \mu^T \left[\int_J \dot{\eta}(t) \wedge \eta(t) dt - \omega \right], \tag{1}$$

for some bivector $\omega \in \bigwedge^2 \mathbb{R}^4 \simeq \mathbb{R}^6$. The map $\mu \mapsto L(\eta, \mu)$ is concave for any $\eta \in A$ since it is an affine function. For the map $\eta \mapsto L(\eta, \mu)$ we note that the matrix $\Lambda_{\mathfrak{g}}$ being positive definite implies that the map $\eta \mapsto \int_J (\Lambda_{\mathfrak{g}} \dot{\eta}(t) \cdot \dot{\eta}(t)) dt$ is convex. Next, we observe that for $\alpha \in [0, 1]$, we have for an arbitrary $\mu \in B$ that

$$\begin{aligned}
&(\alpha\dot{\xi} + (1-\alpha)\dot{\eta}) \wedge (\alpha\xi + (1-\alpha)\eta) \\
&= \alpha^2(\dot{\xi} \wedge \xi) + (1-\alpha)^2(\dot{\eta} \wedge \eta) + \alpha(1-\alpha)[\dot{\xi} \wedge \eta + \xi \wedge \dot{\eta}] \qquad (2)\\
&= \alpha^2(\dot{\xi} \wedge \xi) + (1-\alpha)^2(\dot{\eta} \wedge \eta) + \alpha(1-\alpha)\tfrac{\mathrm{d}}{\mathrm{d}t}[\xi \wedge \eta]. \qquad (3)
\end{aligned}$$

By integration over J of this expression, the last terms vanishes due to the periodicity of ξ and η. Thus, we have

$$\begin{aligned}
L(\alpha\xi + (1-\alpha)\eta, \mu) \leq \alpha \int_J \Lambda_{\mathfrak{g}} \dot{\xi}(t) \cdot \dot{\xi}(t) dt + (1-\alpha) \int_J \Lambda_{\mathfrak{g}} \dot{\eta}(t) \cdot \dot{\eta}(t) dt \\
+\alpha^2 \mu^T \int_J \dot{\xi}(t) \wedge \xi(t) dt + (1-\alpha)^2 \mu^T \int_J \dot{\eta}(t) \wedge \eta(t) dt - \mu^T \omega.
\end{aligned} \tag{4}$$

Now, as $\alpha \in [0, 1]$, we have $\alpha^2 \leq \alpha$ and $(1-\alpha)^2 \leq (1-\alpha)$. Therefore, we have

$$L(\alpha\xi + (1-\alpha)\eta, \mu) \leq \alpha L(\xi, \mu) + (1-\alpha) L(\eta, \mu), \tag{5}$$

i.e. the functional $\eta \mapsto L(\eta, \mu)$ is convex for any $\mu \in B$.

[1] Here $J = [0, 2\pi]$ and $H^1_\sharp(J, \mathbb{R}^4)$ denotes all the 2π-periodic functions $J \to \mathbb{R}^4$ with square-integrable weak derivative.