13

Symmetry and gauge theory

The fundamental forces of nature can all be described in terms of *gauge theories*. Not long after the advent of quantum theory, physicists realised that electromagnetic interactions arise from demanding invariance of quantum wave equations under local changes of phase. This idea was later extended by Yang and Mills, who showed how to construct theories based on more complicated, non-commutative Lie groups. This is the basis for the *standard model* of the electroweak and strong interactions. Around this time physicists also turned their attention to gravitation, and discovered that general relativity could also be formulated as a gauge theory. But this time there was a price to pay. The existence of spinor fields means that the simple geometric structure of general relativity has to be modified by the inclusion of a torsion field, leading to an Einstein–Cartan theory. For clarity, we use the term *general relativity* to refer to the theory defined by Einstein, with zero torsion and the connection given by the Christoffel symbol. The extended theory, with torsion present, is referred to as Einstein–Cartan theory.

While gauge theory is the dominant method in particle physics, it is less popular as a means of analysing gravitational interactions. This is, in part, due to the perception that the gauge theory equations are more complicated than their geometric counterparts. In this and the following chapter we argue that this apparent complexity is a reflection of the inappropriate mathematical techniques typically employed when analysing the gauge theory equations. The spacetime algebra provides the appropriate setting for a gauge formulation of gravity and, applied carefully, this approach is often *easier* to compute with than the metric formulation. We demonstrate that, in the absence of torsion and highly esoteric topology, the gauge and metric approaches produce the same physical predictions.

We begin with a discussion of symmetry in the Maxwell and Dirac theories. Our starting point is the field Lagrangian, which we analyse using Noether's theorem. In particular, we use this to extract the canonical energy-momentum tensor, which is conserved in the absence of external fields. We then turn to the wider subject of gauge theories, before deriving the properties of the gauge fields for gravitation. This chapter concludes with a derivation of the gravitational field equations, and a discussion of the observable quantities in the theory. For the source matter, observables are contained in the functional energy-momentum tensor, which is closely related to the canonical tensor. Applications of the field equations are contained in chapter 14. Throughout the present chapter various results and notation from chapter 11 are assumed without comment.

13.1 Conservation laws in field theory

In section 12.4 we derived the Euler–Lagrange equations for field theory, and demonstrated how to apply these to the cases of elasticity and relativistic fluid dynamics. In this section we concentrate on conservation theorems for Lagrangian field theory. As all of the applications that will concern us are to relativistic field theory, we assume from the outset that the we are describing field theory in a (flat) spacetime. Given a Lagrangian density $\mathcal{L}(\psi_i, \partial_\mu \psi_i)$, where ψ_i, $i = 1, \ldots, n$ are a set of multivector fields, the Euler–Lagrange equations governing the evolution of the system are

$$\frac{\partial \mathcal{L}}{\partial \psi_i} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}\right) = 0, \tag{13.1}$$

where $x^\mu = \gamma^\mu \cdot x$ are a set of fixed orthonormal coordinates. For the applications of interest here the final equations can always be assembled into a frame-free form. Curvilinear coordinates can then be introduced to analyse these equations, if desired.

To obtain a version of Noether's theorem appropriate for field theory we follow the derivation of section 12.1.1. For simplicity we assume that only one field is present. The results are easily extended to the case of more fields by summing over all of the fields present. Suppose that $\psi'(x)$ is a new field obtained from $\psi(x)$ by a scalar-parameterised transformation of the form

$$\psi'(x) = f(\psi(x), \alpha), \tag{13.2}$$

with $\alpha = 0$ corresponding to the identity. We again define

$$\delta\psi = \left.\frac{\partial \psi'}{\partial \alpha}\right|_{\alpha=0}. \tag{13.3}$$

With $\mathcal{L}'$ denoting the original Lagrangian evaluated on the transformed fields we find that

$$\left.\frac{\partial \mathcal{L}'}{\partial \alpha}\right|_{\alpha=0} = (\delta\psi)*\frac{\partial \mathcal{L}}{\partial \psi} + \partial_\mu(\delta\psi)*\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}$$
$$= \frac{\partial}{\partial x^\mu}\left((\delta\psi)*\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\right). \tag{13.4}$$

This equation relates the change in the Lagrangian to the divergence of the current J, where

$$J = \gamma_\mu\,(\delta\psi)*\frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}. \tag{13.5}$$

If the transformation is a symmetry of the system then $\mathcal{L}'$ is independent of α. In this case we immediately establish that the conjugate current is conserved, that is,

$$\nabla\cdot J = 0. \tag{13.6}$$

Symmetries of a field Lagrangian therefore give rise to conserved currents. These in turn define Lorentz-invariant constants via

$$Q = \int d^3x\, J^0, \tag{13.7}$$

where $J^0 = J\cdot\gamma^0$ is the density measured in the γ_0 frame. The fact that this is constant follows from

$$\frac{dQ}{dt} = \int d^3x\,\frac{\partial J^0}{\partial t} = \int d^3x\,\boldsymbol{\nabla}\cdot\boldsymbol{J} = 0, \tag{13.8}$$

where we assume that the current $\boldsymbol{J}$ falls off sufficiently fast at infinity. The value of Q is constant, and independent of the spatial hypersurface used to define the integral.

If the transformation involves a change in the spacetime dependence, Noether's theorem does apply, but we have to be careful in defining the transformation law for $\mathcal{L}$. Suppose that we define

$$\psi'(x) = \psi(x'), \tag{13.9}$$

where

$$x' = f(x). \tag{13.10}$$

The differential is defined in the usual way as

$$\mathsf{f}(a) = a\cdot\nabla f(x). \tag{13.11}$$

The transformed action is

$$S = \int d^4x\,\mathcal{L}(\psi(x'))$$
$$= \int d^4x'\,\det\,(\mathsf{f})^{-1}\mathcal{L}(\psi(x')) \tag{13.12}$$

from which we see that the correct definition of the transformed Lagrangian is

$$\mathcal{L}'(\psi'(x)) = \det\,(\mathsf{f})^{-1}\mathcal{L}(\psi(x')). \tag{13.13}$$

This transformation law demonstrates that $\mathcal{L}$ is indeed a Lagrangian *density*.

13.1.1 Spacetime symmetries

One of the most important spacetime symmetries is translational invariance. All fundamental theories are assumed to give rise to the same physical predictions, independent of the position of the fields in (flat) spacetime. That is, the background space is assumed to be homogeneous. A more careful discussion of this principle, and its relation to gravitation, is contained in section 13.4.1. In terms of the Lagrangian, this principle is encoded in the statement that all x dependence enters $\mathcal{L}$ through the fields. In this case we can apply Noether's theorem to extract a conserved quantity, though we could proceed equally simply by differentiating $\mathcal{L}$ directly to obtain

$$\begin{aligned} a\cdot\nabla\mathcal{L} &= (a\cdot\nabla\psi)*\frac{\partial\mathcal{L}}{\partial\psi} + \big(a\cdot\nabla(\partial_\mu\psi)\big)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} \\ &= \frac{\partial}{\partial x^\mu}\left((a\cdot\nabla\psi)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right), \end{aligned} \tag{13.14}$$

where the field equations have been assumed. We can therefore define the conserved current conjugate to translations by

$$\mathsf{T}(a) = \gamma_\mu(a\cdot\nabla\psi)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - a\mathcal{L}. \tag{13.15}$$

This defines a linear function of a, called the *canonical energy-momentum tensor*. This is a conserved tensor if the system is invariant under translations, so

$$\nabla\cdot\mathsf{T}(a) = 0, \quad \forall\,\text{constant } a. \tag{13.16}$$

The canonical energy-momentum tensor need not be symmetric, and its adjoint is found to be

$$\bar{\mathsf{T}}(a) = \partial_b\langle\mathsf{T}(b)a\rangle = a\cdot\gamma_\mu\,\dot{\nabla}\Big\langle\dot{\psi}*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Big\rangle - a\mathcal{L}. \tag{13.17}$$

The conservation equation for the adjoint tensor is

$$\dot{\bar{\mathsf{T}}}(\dot{\nabla}) = 0. \tag{13.18}$$

If more than one field is present, the energy-momentum tensor is the sum of the individual contributions from each field.

One can similarly define a conserved tensor conjugate to rotations. This time the assumption is that spacetime is isotropic, so does not contain any preferred directions except those defined by the fields themselves. The derivation is slightly more complicated now, as the fields transform in different ways depending on their spins. For all cases we have

$$x' = \tilde{R}xR, \qquad R = \mathrm{e}^{\alpha B/2}, \tag{13.19}$$

and in general we can write

$$\delta\psi = -B\cdot(x\wedge\nabla)\psi + \delta_B\psi, \tag{13.20}$$

where ψ is a general field, and the precise form of $\delta_B\psi$ depends on the spin. The transformation $x' = \tilde{R}xR$ has unit Jacobian, so Noether's theorem gives

$$-B\cdot(x\wedge\nabla)\mathcal{L} = \frac{\partial}{\partial x^\mu}\left(\big(-B\cdot(x\wedge\nabla)\psi + \delta_B\psi\big)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right). \tag{13.21}$$

We can therefore read off the canonical angular momentum tensor $\mathsf{J}(B)$, where

$$\begin{aligned}\mathsf{J}(B) &= \gamma_\mu\big(-B\cdot(x\wedge\nabla)\psi + \delta_B\psi\big)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} + B\cdot x\,\mathcal{L} \\ &= \mathsf{T}(x\cdot B) + (\delta_B\psi)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}.\end{aligned} \tag{13.22}$$

This is a vector-valued linear function of the bivector B, which is conserved for all constant B.

The adjoint function $\bar{\mathsf{J}}(a)$ is often easier to work with. This evaluates to

$$\bar{\mathsf{J}}(a) = \partial_B\langle\mathsf{J}(B)a\rangle = \bar{\mathsf{T}}(a)\wedge x + \mathsf{S}(a), \tag{13.23}$$

which is a bivector-valued linear function of the vector a. The form of $\bar{\mathsf{J}}(a)$ generalises the point-particle definition of angular momentum to the field theory setting. The term $\mathsf{S}(a)$ is the canonical spin tensor,

$$\mathsf{S}(a) = a\cdot\gamma_\mu\,\partial_B\left\langle(\delta_B\psi)\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right\rangle. \tag{13.24}$$

The conservation equation for $\bar{\mathsf{J}}$ states that

$$\dot{\bar{\mathsf{J}}}(\dot{\nabla}) = 0 = \dot{\bar{\mathsf{T}}}(\dot{\nabla})\wedge x + \bar{\mathsf{T}}(\dot{\nabla})\wedge\dot{x} + \dot{\mathsf{S}}(\dot{\nabla}). \tag{13.25}$$

Since the energy-momentum tensor is also conserved, conservation of angular momentum reduces to the equation

$$\bar{\mathsf{T}}(\partial_a)\wedge a + \dot{\mathsf{S}}(\dot{\nabla}) = 0. \tag{13.26}$$

So, in any homogeneous, isotropic, relativistic field theory, the antisymmetric part of the canonical energy-momentum tensor is a total divergence.

13.2 Electromagnetism

As a first application of the preceding results we consider electromagnetism. The dynamical variable in electromagnetism is the vector potential A, and the electromagnetic Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}F\cdot F - A\cdot J, \tag{13.27}$$

where $F = \nabla\wedge A$, and A couples to an external current J. An electromagnetic gauge transformation is defined by

$$A \mapsto A + \nabla\phi(x), \tag{13.28}$$

where $\phi(x)$ is a scalar field. Gauge invariance of the Lagrangian is ensured by requiring that the current J is conserved. The field equation is

$$-J - \frac{\partial}{\partial x^\mu}\left(\frac{1}{2}\frac{\partial}{\partial(\partial_\mu A)}\langle FF\rangle\right) = -J - \frac{\partial}{\partial x^\mu}(\nabla\wedge A)\cdot\gamma^\mu = 0, \tag{13.29}$$

which simplifies to the familiar equation

$$\nabla\cdot F = J. \tag{13.30}$$

The remaining Maxwell equation, $\nabla\wedge F = 0$, follows from the definition of F in terms of A.

13.2.1 The electromagnetic energy-momentum tensor

To calculate the free-field energy-momentum tensor, we set $J = 0$ and work with the Lagrangian density

$$\mathcal{L}_0 = \tfrac{1}{2}\langle F^2\rangle. \tag{13.31}$$

Equation (13.15) yields the energy-momentum tensor

$$\mathsf{T}(a) = (a\cdot\nabla A)\cdot F - \tfrac{1}{2}a\langle F^2\rangle. \tag{13.32}$$

This expression is somewhat unsatisfactory as it stands, as it is not gauge-invariant. In order to find a gauge-invariant form of the energy-momentum tensor we write

$$a\cdot\nabla A = a\cdot F + \dot{\nabla}(\dot{A}\cdot a). \tag{13.33}$$

If we now employ the field equations we can write

$$\mathsf{T}(a) = F\cdot(F\cdot a) - \tfrac{1}{2}a\, F\cdot F + \nabla\cdot(A\cdot a\, F). \tag{13.34}$$

The first two terms are gauge-invariant, and the final term is a total divergence. In most classical applications the total divergence can be ignored, as its integral over any finite volume results in a boundary term which can be set to zero. In quantum field theory the issue of how to handle gauge invariance is more complicated. Typically, manifest gauge invariance is lost at the level of the quantum field equations, and only recovered in the physical predictions of the theory. With the boundary term removed, the remaining terms recover the familiar classical free-field electromagnetic energy-momentum tensor,

$$\begin{aligned}\mathsf{T}_{em}(a) &= F\cdot(F\cdot a) - \tfrac{1}{2}a\,F\cdot F \\ &= \tfrac{1}{2}Fa\tilde{F},\end{aligned} \tag{13.35}$$

as found in section 7.2.3. This tensor is gauge-invariant, traceless and symmetric. It is also equal to the functional energy-momentum tensor, defined in section 13.5.4.

13.2.2 Angular momentum in electromagnetism

The canonical angular momentum is found by considering the symmetry transformation

$$A'(x) = RA(x')\tilde{R}, \tag{13.36}$$

with R and x' as defined in equation (13.19). The transformation law for x implies that

$$\nabla_{x'} = \tilde{R}\nabla R, \tag{13.37}$$

so that the new field satisfies

$$\nabla\wedge A' = R\,\nabla_{x'}\wedge A(x')\,\tilde{R} = RF(x')\tilde{R}. \tag{13.38}$$

It follows that the transformed free-field Lagrangian only depends on α through the transformed position dependence, as required for isotropy. We also find that

$$\delta A = B\cdot A - (B\cdot x)\cdot\nabla A, \tag{13.39}$$

so equation (13.22) gives

$$\mathsf{J}(B) = \big(B\cdot A - (B\cdot x)\cdot\nabla A\big)\cdot F + \tfrac{1}{2}B\cdot x\langle F^2\rangle. \tag{13.40}$$

As with the canonical energy-momentum tensor, the angular momentum tensor is not manifestly gauge-invariant. This time we write

$$\begin{aligned}(B\cdot x)\cdot\nabla A &= (B\cdot x)\cdot(\nabla\wedge A) + \dot{\nabla}(B\cdot x)\cdot\dot{A} \\ &= (B\cdot x)\cdot F + \nabla\big((B\cdot x)\cdot A\big) + B\cdot A,\end{aligned} \tag{13.41}$$

so that

$$\mathsf{J}(B) = -\big((B\cdot x)\cdot F\big)\cdot F + \tfrac{1}{2}B\cdot x\langle F^2\rangle - \nabla\cdot\big((B\cdot x)\cdot A\,F\big). \tag{13.42}$$

The final term is again a total divergence which can be ignored. We therefore define

$$\mathsf{J}_{em}(B) = -\big((B\cdot x)\cdot F\big)\cdot F + \tfrac{1}{2}B\cdot x\langle F^2\rangle = \mathsf{T}_{em}(x\cdot B), \tag{13.43}$$

which is now manifestly gauge-invariant. The adjoint is simply

$$\bar{\mathsf{J}}_{em}(a) = \mathsf{T}_{em}(a)\wedge x. \tag{13.44}$$

Conservation of angular momentum implies that

$$\nabla\cdot\mathsf{T}_{em}(x\cdot B) = \partial_a\cdot\mathsf{T}_{em}(a\cdot B) = (\bar{\mathsf{T}}_{em}(\partial_a)\wedge a)\cdot B = 0. \tag{13.45}$$

This holds because $\mathsf{T}_{em}(a)$ is symmetric.

The redefinition of the energy-momentum and angular momentum tensors for electromagnetism removes the spin term and absorbs it directly into $\mathsf{T}_{em}(a)\wedge x$. This guarantees that the fields are gauge-invariant, but suppresses the spin-1 nature of the electromagnetic field. For gravitational interactions the canonical energy-momentum and spin tensors are not as important as their functional equivalents. In the case of electromagnetism, the latter are guaranteed to be (electromagnetic) gauge-invariant, and the spin contribution does turn out to vanish.

13.2.3 Conformal invariance of free-field electromagnetism

In addition to invariance under Poincaré transformations, free-field electromagnetism is invariant under the full conformal group of spacetime. Conformal geometry is discussed in detail in chapter 10. Here we are interested in the field theory manifestation of conformal invariance. We start by considering an arbitrary displacement, $x' = f(x)$. Gauge invariance tells us that A must transform in the same manner as ∇ (it is a 1-form), so we define

$$A'(x) = \bar{\mathsf{f}}(A(x')). \tag{13.46}$$

The electromagnetic field strength therefore transforms to

$$\nabla\wedge A'(x) = \bar{\mathsf{f}}\big(\bar{\mathsf{f}}^{-1}(\nabla)\wedge A(x')\big) = \bar{\mathsf{f}}\big(F(x')\big), \tag{13.47}$$

where we have made use of the results

$$\dot{\nabla}\wedge\dot{\bar{\mathsf{f}}}(a) = 0 \tag{13.48}$$

and

$$\nabla_{x'} = \bar{\mathsf{f}}^{-1}(\nabla). \tag{13.49}$$

These formulae are derived in section 6.5.6. The transformed Lagrangian density is now

$$\mathcal{L}' = \tfrac{1}{2}\det\,(\mathsf{f})^{-1}\,\langle\bar{\mathsf{f}}\big(F(x')\big)\bar{\mathsf{f}}\big(F(x')\big)\rangle. \tag{13.50}$$

We therefore define a symmetry of the action integral if f satisfies

$$\bar{\mathsf{f}}(A)\cdot\bar{\mathsf{f}}(B) = \det\,(\mathsf{f})\,A\cdot B \tag{13.51}$$

for any pair of bivectors A and B. This is clearly satisfied by any orthogonal transformation, but it is also satisfied by dilations. The Lagrangian for the free electromagnetic field is therefore symmetric under any displacement whose derivative is a local orthogonal transformation coupled with a dilation. This defines the conformal group.

As a simple example, consider the dilation $x' = \exp(\alpha)x$. For this transformation Noether's theorem gives

$$x\cdot\nabla\mathcal{L} = -4\mathcal{L} + \nabla\cdot\left(\gamma_\mu\,(A + x\cdot\nabla\, A)*\frac{\partial\mathcal{L}}{\partial(\partial_\mu A)}\right), \tag{13.52}$$

from which we extract the conserved current

$$J = \mathsf{T}(x) + A\cdot F = \mathsf{T}_{em}(x) + \nabla\cdot(A\cdot x\, F). \tag{13.53}$$

The final term is the divergence of a bivector so is automatically conserved. Dilation invariance therefore tells us that

$$\nabla\cdot\mathsf{T}_{em}(x) = 0, \tag{13.54}$$

which holds because T_{em} is conserved and traceless. The latter property is typical of scale-invariant theories.

Similarly, a special conformal transformation maps the position vector x to x', where

$$x' = f(x) = (x^{-1} + \alpha a)^{-1} = x(1 + \alpha a x)^{-1}. \tag{13.55}$$

The derivative transformation is

$$\mathsf{f}(b) = b\cdot\nabla f(x) = (1 + \alpha x a)^{-1} b (1 + \alpha a x)^{-1}, \tag{13.56}$$

which is a local rotation and dilation. The determinant is

$$\det\,(\mathsf{f}) = (1 + 2\alpha a\cdot x + \alpha^2 a^2 x^2)^{-4}. \tag{13.57}$$

We also find that

$$\left.\frac{\partial x'}{\partial\alpha}\right|_{\alpha=0} = -xax \tag{13.58}$$

and

$$\left.\frac{\partial}{\partial\alpha}\det\,(\mathsf{f})^{-1}\right|_{\alpha=0} = 8x\cdot a. \tag{13.59}$$

Noether's theorem for special conformal transformations can then be shown to produce the conserved tensor $\mathsf{T}_{em}(xax)$. Conservation again follows from the properties of T_{em}.

13.3 Dirac theory

The free-field Dirac Lagrangian is

$$\mathcal{L} = \langle \nabla\psi I\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{13.60}$$

where ψ is a spinor field. Variation with respect to ψ produces the Euler–Lagrange equation

$$(\nabla\psi I\gamma_3)^{\sim} - 2m\tilde{\psi} + \frac{\partial}{\partial x^\mu}(I\gamma_3\tilde{\psi}\gamma^\mu) = 0, \tag{13.61}$$

which reverses to recover the Dirac equation in the form

$$\nabla\psi I\gamma_3 = m\psi. \tag{13.62}$$

This derivation departs from that given in many textbooks, as we do not consider ψ and $\tilde{\psi}$ as independent variables. Instead we view $\mathcal{L}$ as a real scalar function of a single field ψ. An immediate consequence of the field equations is that $\mathcal{L} = 0$ when the Dirac equation is satisfied. This behaviour is typical of first-order systems.

13.3.1 Spacetime transformations

The canonical energy-momentum tensor for the Dirac field is easily found,

$$\begin{aligned}\mathsf{T}_D(a) &= \gamma_\mu\langle a\cdot\nabla\psi I\gamma_3\tilde{\psi}\gamma^\mu\rangle - a\mathcal{L} \\ &= \langle a\cdot\nabla\psi I\gamma_3\tilde{\psi}\rangle_1.\end{aligned} \tag{13.63}$$

This energy-momentum tensor is not symmetric. Its adjoint is

$$\bar{\mathsf{T}}_D(a) = \dot{\nabla}\langle\dot{\psi}I\gamma_3\tilde{\psi}a\rangle, \tag{13.64}$$

and the antisymmetric term is governed by the bivector

$$\partial_a\wedge\mathsf{T}_D(a) = \dot{\nabla}\wedge\langle\dot{\psi}I\gamma_3\tilde{\psi}\rangle_1. \tag{13.65}$$

This bivector can be written as

$$\begin{aligned}\dot{\nabla}\wedge\langle\dot{\psi}I\gamma_3\tilde{\psi}\rangle_1 &= \big\langle\langle\nabla\psi I\gamma_3\tilde{\psi} - \dot{\nabla}\langle\dot{\psi}I\gamma_3\tilde{\psi}\rangle_3\big\rangle_2 \\ &= -\tfrac{1}{2}\nabla\cdot(\psi I\gamma_3\tilde{\psi}).\end{aligned} \tag{13.66}$$

So, as stated in section 13.1.1, the antisymmetric component of the energy-momentum tensor is a total divergence. In this case we can write

$$\partial_a\wedge\mathsf{T}_D(a) = -\tfrac{1}{2}\nabla\cdot S, \tag{13.67}$$

where S is the spin trivector

$$S = \psi I\gamma_3\tilde{\psi}. \tag{13.68}$$

Rotational invariance follows from the transformation

$$\psi'(x) = R\psi(x') \tag{13.69}$$

with R and x' as defined in equation (13.19). The wavefunction ψ is subject to the single-sided transformation law appropriate for spinors. One can easily show that the rotors cancel out of the transformed Lagrangian, and the conjugate angular momentum is

$$\mathsf{J}(B) = \langle(-B\cdot x)\cdot\nabla\psi I\gamma_3\tilde{\psi}\rangle_1 + \tfrac{1}{2}B\cdot(\psi I\gamma_3\tilde{\psi}). \tag{13.70}$$

The adjoint gives

$$\bar{\mathsf{J}}(a) = \bar{\mathsf{T}}(a)\wedge x + \tfrac{1}{2}a\cdot S, \tag{13.71}$$

which neatly exposes the spin contribution to the angular momentum. Comparison with equation (12.56) confirms that the point-particle models discussed in section 12.2.1 do correctly capture the properties of the field angular momentum.

The mass term in the free-field Dirac Lagrangian is the sole term breaking conformal invariance. Spacetime spinors have a conformal weight of 3/2, so dilations are defined by

$$\psi'(x) = \mathrm{e}^{3\alpha/2}\psi(\mathrm{e}^{\alpha}x). \tag{13.72}$$

For this transformation, Noether's theorem gives rise to the canonical vector $\mathsf{T}_D(x)$, which satisfies the partial conservation law

$$\nabla\cdot\mathsf{T}_D(x) = \langle m\psi\tilde{\psi}\rangle. \tag{13.73}$$

Special conformal transformations are also interesting to consider. With the transformation as defined in equation (13.55), we write the derivative transformation as

$$a\cdot\nabla x' = \mathsf{f}(a) = \frac{1}{\rho}Ra\tilde{R}, \tag{13.74}$$

where

$$\rho = 1 + 2\alpha a\cdot x + \alpha^2 a^2 x^2, \qquad R = \frac{1+\alpha a x}{\rho^{1/2}}. \tag{13.75}$$

We define the transformed spinor by

$$\psi'(x) = \frac{1}{\rho^{3/2}}\tilde{R}\psi(x') = (1+\alpha a x)^{-2}(1+\alpha x a)^{-1}\psi(x'). \tag{13.76}$$

This transformation of ψ defines a symmetry of the action because of the remarkable result that

$$\nabla\big((1+\alpha a x)^{-2}(1+\alpha x a)^{-1}\big) = 0. \tag{13.77}$$

It follows that

$$\nabla\psi'(x) = \frac{1}{\rho^{5/2}}\tilde{R}\nabla_{x'}\psi(x'), \tag{13.78}$$

which is precisely the transformation required in the Dirac action. More generally, a special conformal transformation can be applied to any spacetime monogenic to obtain a new monogenic function. Equation (13.77) is an example of the general result that

$$\nabla \left(\frac{1+\alpha x a}{(1+2\alpha x\cdot a+\alpha^2 a^2 x^2)^{n/2}} \right) = 0, \tag{13.79}$$

which holds in an n-dimensional space of arbitrary signature.

The conserved tensor conjugate to special conformal transformations, T_c, is found from Noether's theorem to be

$$\mathsf{T}_c(a) = \mathsf{T}_D(xax) + (a\wedge x)\cdot S. \tag{13.80}$$

The partial conservation law for this is

$$\nabla\cdot \mathsf{T}_c(a) = 2m\, a\cdot x \langle \psi\tilde{\psi}\rangle. \tag{13.81}$$

For both dilations and special conformal transformations we recover a genuine conservation law if the mass m is set to zero. This is the basis for an important technique in quantum field theory. In high-energy experiments it is often a reasonable approximation to treat the particles as massless. One can then take advantage of the conformal symmetry to compute a range of consequences for the outcome of experiment. Typically, these predictions will be valid up to order m/E, where E is the energy.

13.3.2 Internal symmetries and phase invariance

As well as spacetime symmetries there are a number of internal symmetries of the Dirac action we can consider. The first of these is the duality transformation

$$\psi' = \psi e^{I\alpha}. \tag{13.82}$$

Equation (13.4) produces the relation

$$\nabla\cdot(\psi\gamma_3\tilde{\psi}) = 2\langle mI\psi\tilde{\psi}\rangle. \tag{13.83}$$

So the spin vector defines a conserved current in the massless limit. This is the partially-conserved axial current, which is important in scattering calculations.

Further transformations to consider are internal rotations of the form

$$\psi' = \psi e^{\alpha B}, \tag{13.84}$$

where B is a bivector. In this case equation (13.4) reduces to

$$\nabla\cdot\big(\psi\, B\cdot(I\gamma_3)\,\tilde{\psi}\big) = 0, \tag{13.85}$$

where we have applied the Dirac equation. This yields conserved currents for any component of B which commutes with γ_3. This space is spanned by $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$

and $I\boldsymbol{\sigma}_3$. Of these, only $I\boldsymbol{\sigma}_3$ has the additional property of leaving invariant the observable current $\psi\gamma_0\tilde{\psi}$. This is the case of a phase transformation, and the conjugate conserved quantity is precisely the current J, so

$$\nabla\cdot J = 0, \qquad J = \psi\gamma_0\tilde{\psi}. \tag{13.86}$$

This is an example of the general result in quantum theory that phase invariance ensures that probability density is conserved, and wavefunction evolution is unitary.

The phase transformation law

$$\psi \mapsto \psi' = \psi e^{\phi I\boldsymbol{\sigma}_3} \tag{13.87}$$

is a *global* symmetry of the Lagrangian, because ϕ is a constant. If ψ satisfies the Dirac equation, then so to does ψ'. We arrive at a *gauge theory* if we convert this global symmetry to a local one. There are a number of reasons for believing that this is a sensible way to construct interactions in field theory. One motivation is from the structure of the physical statements that can be extracted from Dirac theory. Quantum theory makes predictions about the values of *observables*, which are formed from inner products between spinors, $\langle\psi|\phi\rangle$. These inner products are invariant under local changes of phase. Similarly, quantum theory can make statements about the equality of two spinor expressions, for example

$$\psi = \psi_1 + \psi_2. \tag{13.88}$$

This might decompose ψ into two orthogonal eigenstates of some operator. Again, if all spinors pick up the same locally-varying phase factor then the physical predictions are unchanged. In addition, a global change of phase corresponds to simultaneously changing the phase of the wavefunction everywhere in the universe. While this can be conceived of mathematically, it does not make a great deal of physical sense. The ultimate motivation, however, comes from the fact that gauge theories are spectacularly successful. All of the known fundamental forces can be described by the procedure of turning a global symmetry into a local symmetry.

13.3.3 Covariant derivatives and minimal coupling

Now that we are clear on the motivation, we must find how to modify the Dirac equation in order that phase changes become a local symmetry. This is the prototype gauge theory. We start by writing

$$\psi' = \psi R, \tag{13.89}$$

where R is a position-dependent rotor. We will later set $R = \exp(I\boldsymbol{\sigma}_3\phi(x))$. This slightly more general formulation eases the transition to the more complicated cases of electroweak and gravitational interactions. The equation for ψ' now includes the term

$$\nabla\psi' = \gamma^\mu\big(\partial_\mu\psi R + \psi\partial_\mu R\big). \tag{13.90}$$

We need to modify the ∇ operator to be able to cancel out the term in the derivative of R. We therefore define a new, *covariant* derivative operator D, where

$$D\psi = \gamma^\mu D_\mu\psi. \tag{13.91}$$

The directional covariant derivatives D_μ contain an extra term going as

$$D_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}\psi\Omega_\mu, \tag{13.92}$$

where Ω_μ is a multivector field whose nature and transformation properties we have to determine. (The factor of 1/2 is inserted for later convenience.) The index indicates that Ω_μ is a linear function. We can therefore write

$$\Omega_\mu = \Omega(\gamma_\mu) = \Omega(\gamma_\mu; x), \tag{13.93}$$

which defines the linear function $\Omega(a) = \Omega(a;x)$. The x dependence records the fact that the field will in general be a function of position. This label is usually suppressed. In later applications we will make strong use of the index-free form $\Omega(a)$.

The behaviour we require is that under a local rotation, D should transform in such a way that ψR is still a solution of the modified equation. So, with D transforming to D', we require that

$$D'\,(\psi R) = (D\psi)R \tag{13.94}$$

for any R. We expect that D' should have the same functional form as D, so we also have

$$D'\psi = \gamma^\mu\big(\partial_\mu\psi + \tfrac{1}{2}\psi\Omega'_\mu\big). \tag{13.95}$$

Equation (13.94) therefore gives

$$\begin{aligned} D'\,(\psi R) &= \gamma^\mu\big(\partial_\mu\psi R + \psi\partial_\mu R + \tfrac{1}{2}\psi R\Omega'_\mu\big) \\ &= \gamma^\mu\big(\partial_\mu\psi + \tfrac{1}{2}\psi\Omega_\mu\big)R. \end{aligned} \tag{13.96}$$

From this we can read off that

$$\partial_\mu R + \tfrac{1}{2}R\Omega'_\mu = \tfrac{1}{2}\Omega_\mu R, \tag{13.97}$$

which establishes the transformation law

$$\Omega'_\mu = \tilde{R}\Omega_\mu R - 2\tilde{R}\partial_\mu R. \tag{13.98}$$

Now R is a rotor, so $2\tilde{R}\partial_\mu R$ is a member of the Lie algebra of the rotor group. It follows that this term is a pure bivector, so Ω_μ must also contain a bivector term if it is to cancel a term in $2\tilde{R}\partial_\mu R$. We assume that this is the only term present the Ω_μ field. This is the minimal assumption, and is referred to as defining *minimal coupling*.

The important point in this derivation is that we have used the form of the term $-2\tilde{R}\partial_\mu R$ to say what type of object Ω_μ is. We are *not* asserting that Ω_μ is equal to $-2\tilde{R}\partial_\mu R$. On the contrary, as will become apparent later, if Ω_μ was given by the gradient of a rotor in this manner it would give rise to a vanishing field strength and therefore be of no physical interest. This step, of taking a term arising from a derivative (like $-2\tilde{R}\partial_\mu R$ here), and generalizing it to a field *not* in general derivable from a derivative, is the essence of the gauging process. The Ω_μ term in the covariant derivative is called a *connection*. In general, connections take their values in the Lie algebra of the associated symmetry group. Many of the symmetry groups we consider are rotor groups, so for these the connections are bivector fields.

13.3.4 The minimally coupled Dirac equation

Returning to electromagnetism, we are concerned with the restricted class of rotations that take place entirely in the $\gamma_2\gamma_1$ plane. In this case, writing $R = \exp(I\boldsymbol{\sigma}_3\phi)$, we have

$$-2\tilde{R}\partial_\mu R = -2\mathrm{e}^{-I\boldsymbol{\sigma}_3\phi}\partial_\mu\phi\mathrm{e}^{I\boldsymbol{\sigma}_3\phi}I\boldsymbol{\sigma}_3 = -2\gamma_\mu\cdot(\nabla\phi)I\boldsymbol{\sigma}_3. \tag{13.99}$$

In generalizing to Ω_μ, we see that this must take the form

$$\Omega_\mu = -\lambda\gamma_\mu\cdot A\, I\boldsymbol{\sigma}_3 \tag{13.100}$$

or, in frame-free notation,

$$\Omega(a) = -\lambda a\cdot A\, I\boldsymbol{\sigma}_3. \tag{13.101}$$

Here A is a spacetime vector field, and λ is some coupling constant. We now reassemble our full, covariant Dirac equation to obtain

$$D\psi I\gamma_3 = \gamma^\mu\big(\partial_\mu\psi - \tfrac{1}{2}\lambda\psi\,\gamma_\mu\cdot A\, I\boldsymbol{\sigma}_3\big)I\gamma_3 = m\psi. \tag{13.102}$$

This simplifies to give

$$\nabla\psi I\gamma_3 - \tfrac{1}{2}\lambda A\psi\gamma_0 = m\psi, \tag{13.103}$$

and we see that the contraction between the γ^μ frame and the connection in equation (13.102) assembles to give a vector multiplying ψ from the left. It is clear that for an electron we require $\lambda = 2e$, so the *minimally coupled* Dirac equation is

$$\nabla\psi I\boldsymbol{\sigma}_3 - eA\psi = m\psi\gamma_0, \tag{13.104}$$

as studied in section 8.3. A local phase transformation of ψ now induces the transformation

$$eA \mapsto eA - \nabla\phi, \tag{13.105}$$

which we recognise as an electromagnetic change of gauge. By adding an interaction term solely in A we are making the simplest possible modification to the original equation, which is the essence of minimal coupling. We could, for example, add further terms in F, or F^2 multiplying ψ, and the equation would still be gauge-invariant. It appears, however, that this possibility is not required for describing the fundamental forces. Why this should be so is unknown.

13.3.5 The gauge field strength

Now that we have introduced the gauge fields the next step is to construct the observable (gauge-invariant) quantities associated with them. For electromagnetism we know that these are the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, which form part of the *field strength tensor*. This is found in general by commuting covariant derivatives. We form

$$\begin{aligned}[D_\mu, D_\nu]\psi &= D_\mu\big(\partial_\nu\psi + \tfrac{1}{2}\psi\Omega_\nu\big) - D_\nu\big(\partial_\mu\psi + \tfrac{1}{2}\psi\Omega_\mu\big)\\ &= \tfrac{1}{2}\psi\big(\partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - \Omega_\mu\times\Omega_\nu\big).\end{aligned} \tag{13.106}$$

Despite the fact that we formed commutators of derivatives on ψ, all of the derivatives of ψ have cancelled, and we are left with a single object

$$\mathsf{F}_{\mu\nu} = \mathsf{F}(\gamma_\mu\wedge\gamma_\nu) = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu - \Omega_\mu\times\Omega_\nu. \tag{13.107}$$

This is a bivector-valued linear function of the bivector argument $\gamma_\mu\wedge\gamma_\nu$. The construction of this object guarantees that under a change of gauge

$$\mathsf{F}_{\mu\nu} \mapsto \mathsf{F}'_{\mu\nu} = \tilde{R}\mathsf{F}_{\mu\nu}R. \tag{13.108}$$

This transformation tells us that the field strength transforms *covariantly* under changes of gauge.

Specialising to the case of electromagnetism, where $\Omega_\mu = -2e\gamma_\mu\cdot A\, I\boldsymbol{\sigma}_3$, we find that the term multiplying ψ contains

$$\begin{aligned}(-2e)^{-1}\mathsf{F}_{\mu\nu} &= \partial_\mu(\gamma_\nu\cdot A\, I\boldsymbol{\sigma}_3) - \partial_\nu(\gamma_\mu\cdot A\, I\boldsymbol{\sigma}_3) - \gamma_\mu\cdot A\,\gamma_\nu\cdot A\, I\boldsymbol{\sigma}_3\times I\boldsymbol{\sigma}_3\\ &= (\gamma_\nu\wedge\gamma_\mu)\cdot(\nabla\wedge A)I\boldsymbol{\sigma}_3\\ &= (\gamma_\nu\wedge\gamma_\mu)\cdot F\, I\boldsymbol{\sigma}_3.\end{aligned} \tag{13.109}$$

This is a function that maps the bivector $\gamma_\nu\wedge\gamma_\mu$ linearly onto a pure phase term. For most applications of electromagnetism it is sensible to lose the mapping nature of the field strength and instead work directly with the bivector F. For more complicated gauge fields this is not appropriate. In forming the commutator of covariant derivatives we have extracted the correct field strength, $F = \nabla\wedge A$, which encodes the physically measurable content of the electromagnetic field. The electromagnetic field strength is *invariant* under a change of gauge, as opposed to covariant. This is because the underlying gauge group, U(1), is a commutative group, so the rotors cancel out in equation (13.108). The picture is less simple for non-commutative Lie groups.

13.3.6 Electroweak symmetry

A full treatment of electroweak gauge theory requires the apparatus of quantum field theory, which is beyond the scope of this book. Here we give a simplified treatment, concentrating entirely on the fermionic sector for an electron and a neutrino. The left-handed particles in this sector are assembled into a doublet

$$L_e = \begin{pmatrix} |\nu_e\rangle \\ |e_l\rangle \end{pmatrix} \tag{13.110}$$

and the right-handed particles consist of a singlet state $|e_r\rangle$. The kets denote Dirac spinors, projected into their left-handed or right-handed states. The left-hand doublet is acted on by SU(2) matrices, which transform the upper and lower components into linear superpositions of $|\nu_e\rangle$ and $|e_l\rangle$. To construct an equivalent group action in spacetime algebra, we introduce the spinor ψ_l, where

$$\begin{pmatrix} |\nu_e\rangle \\ |e_l\rangle \end{pmatrix} \leftrightarrow \psi_l = \psi_e \tfrac{1}{2}(1-\boldsymbol{\sigma}_3) - \psi_\nu I\boldsymbol{\sigma}_2 \tfrac{1}{2}(1+\boldsymbol{\sigma}_3). \tag{13.111}$$

Here ψ_e and ψ_ν are the spacetime algebra equivalents of the $|e_l\rangle$ and $|\nu_e\rangle$ spinors, as defined by the map of equation (8.69). This map ensures that the action of the generators of the SU(2) group become

$$\hat{\sigma}_k L_e \leftrightarrow \psi_l \boldsymbol{\sigma}_k, \tag{13.112}$$

and hence

$$iL_e \leftrightarrow -\psi_l I. \tag{13.113}$$

So all transformations are now carried out on the right-hand side of ψ_l, and are of the class discussed in section 13.3.2.

The kinetic term in the Lagrangian for the left-handed doublet is usually written as

$$\bar{L}_e i\not{D} L_e = \langle \bar{\nu}_e | i\not{D} | \nu_e \rangle + \langle \bar{e}_l | i\not{D} | e_l \rangle, \tag{13.114}$$

which has the multivector equivalent

$$\mathcal{L}_l = \langle \nabla\psi_\nu \tfrac{1}{2}(1-\boldsymbol{\sigma}_3) I\gamma_3 \tilde{\psi}_\nu + \nabla\psi_e \tfrac{1}{2}(1-\boldsymbol{\sigma}_3) I\gamma_3 \tilde{\psi}_e \rangle. \tag{13.115}$$

Now

$$\begin{aligned}\tfrac{1}{2}(1-\boldsymbol{\sigma}_3)I\gamma_3\tilde{\psi}_e &= -\tfrac{1}{2}(1-\boldsymbol{\sigma}_3)I\gamma_0\tilde{\psi}_l,\\ \tfrac{1}{2}(1-\boldsymbol{\sigma}_3)I\gamma_3\tilde{\psi}_\nu &= I\boldsymbol{\sigma}_2\tfrac{1}{2}(1+\boldsymbol{\sigma}_3)I\gamma_0\tilde{\psi}_l,\end{aligned} \tag{13.116}$$

so

$$\begin{aligned}\mathcal{L}_l &= -\langle\nabla\big(\psi_e\tfrac{1}{2}(1-\boldsymbol{\sigma}_3)-\psi_\nu I\boldsymbol{\sigma}_2\tfrac{1}{2}(1+\boldsymbol{\sigma}_3)\big)I\gamma_0\tilde{\psi}_l\rangle\\ &= -\langle\nabla\psi_l I\gamma_0\tilde{\psi}_l\rangle.\end{aligned} \tag{13.117}$$

The left-handed fermionic sector of the electroweak Lagrangian is similar to the Dirac Lagrangian, but with γ_3 replaced by γ_0. The internal symmetry group is therefore defined by transformations of the form

$$\psi \mapsto \psi \mathrm{e}^{M}, \tag{13.118}$$

where M is any even multivector that satisfies

$$\exp(M)\gamma_0\exp(\tilde{M}) = \gamma_0. \tag{13.119}$$

This picks out the set of bivectors that commute with γ_0, and the pseudoscalar. The former define an SU(2) group, and the latter is a U(1) phase term. The Lagrangian therefore has the expected SU(2)×U(1) symmetry of electroweak theory, encoded in a very natural way in the spacetime algebra.

The right-handed sector of the electroweak theory involves a singlet state

$$\psi_r = \psi_e\tfrac{1}{2}(1+\boldsymbol{\sigma}_3). \tag{13.120}$$

The kinetic term for this is

$$\langle\nabla\psi_r I\gamma_3\tilde{\psi}_r\rangle = -\langle\nabla\psi_e\tfrac{1}{2}(\gamma_0+\gamma_3)I\tilde{\psi}_e\rangle. \tag{13.121}$$

Mass terms are introduced via interaction with the Higgs field, which can be modelled straightforwardly as an interaction between left-handed and right-handed particles. A global SU(2) transformation is described by

$$\psi_l \mapsto \psi_l R, \tag{13.122}$$

where R is a rotor satisfying $R\gamma_0\tilde{R} = \gamma_0$. This is converted to a local symmetry following the procedure of section 13.3.2, which tells us that the connection consists of bivectors which commute with γ_0. The U(1) connection is a multiple of the pseudoscalar. The field strength is defined similarly, and one can proceed to model spontaneous symmetry breaking using this scheme. At some point, however, it is necessary to adopt a quantum field theory perspective, and replace the wavefunctions described here by operators acting on the quantum vacuum.

13.4 Gauge principles for gravitation

We have so far described electromagnetism and electroweak forces in terms of gauge theories. We now turn our attention to gravity. Our aim is to model gravitational interactions in terms of gauge fields defined in the spacetime algebra. This initially appears to be a radical departure from general relativity, but in fact the two approaches converge in a manner that sheds light on the physical structure of the theory. Spacetime algebra is the geometric algebra of *flat* spacetime, and the introduction of fields cannot alter this basic property. What then are we to make of the standard arguments that spacetime is curved? The answer is that all of these arguments involve light paths, or measuring rods, or similar devices, and all of these processes are also modelled by fields. Since all physical quantities correspond to fields, the *absolute* position and orientation of particles or fields in our background spacetime is not measurable. It drops out of all physical calculations. The only predictions that can be extracted are relative relations between fields. Ensuring that this property is true locally means there is no conflict with any of the principles by which one is traditionally led to general relativity, and naturally guides us in the direction of a gauge theory.

To illustrate these considerations, consider possible relations between quantum fields. Suppose that $\psi_1(x)$ and $\psi_2(x)$ are spinor fields. A physical statement could be a simple relation of equality:

$$\psi_1(x) = \psi_2(x). \tag{13.123}$$

But all this statement says is that at a point where one field has a particular value, then the second field has the same value. This statement is completely independent of where we choose to place the fields in the spacetime algebra. And, more importantly, it is totally independent of where we choose to locate other values of the fields. We could equally well introduce two new fields

$$\psi_1'(x) = \psi_1(x'), \quad \psi_2'(x) = \psi_2(x'), \tag{13.124}$$

where x' is an arbitrary function of position x. The statement $\psi_1'(x) = \psi_2'(x)$ contains precisely the same physical content as the original equation.

The same picture emerges if both fields are acted on by a spacetime rotor, giving rise to new fields

$$\psi_1' = R\psi_1, \quad \psi_2' = R\psi_2. \tag{13.125}$$

Again, the statement $\psi_1' = \psi_2'$ has the same physical content as the original equation. Similar considerations apply to the observables formed from ψ, such as the vector $J = \psi\gamma_0\tilde{\psi}$. Replacing ψ by ψ' produces the new vector $J' = RJ\tilde{R}$. Invariance of the equations under this transformation ensures that the absolute direction of vectors in the spacetime algebra is not measurable, only the relative orientation of two physical vectors is measurable. We now have a clear mathematical statement of the invariance properties we want to establish. The next task is to study the form of the gauge fields needed to enforce this invariance.

13.4.1 Displacements

We write $x' = f(x)$ for an arbitrary (differentiable) map between spacetime position vectors. The transformation we are interested in is where the field $\psi(x)$ is transformed to the new field

$$\psi'(x) = \psi(x'). \tag{13.126}$$

The map $f(x)$ should not be thought of as a map between manifolds, or as moving points around. The function $f(x)$ is just a rule for relating one position vector to another within a single vector space. It is the fields that are transformed in this space. We need a good name for this operation of moving fields around. One possibility is *translation*, but this suggests a rigid map where all fields are translated by the same amount. Mathematicians favour the term *diffeomorphism*, but this usually refers to a map between distinct manifolds. We prefer to use the term *displacement*, which does suggest the concept of moving a field around from one point to another in an arbitrary manner.

The next step is to consider the behaviour of the derivative of ψ. With the displacement denoted by $x' = f(x)$, and the derivative defined by

$$\mathsf{f}(a) = a\cdot\nabla f(x), \tag{13.127}$$

we know that the vector derivative satisfies

$$\nabla_x = \bar{\mathsf{f}}(\nabla_{x'}). \tag{13.128}$$

So, for example, if $\psi(x)$ is a spinor, and $\psi'(x) = \psi(x')$, we have

$$\nabla\psi'(x) = \bar{\mathsf{f}}(\nabla_{x'})\psi(x'). \tag{13.129}$$

To formulate a version of the Dirac action that is invariant under arbitrary displacements, we must introduce a gauge field that removes the effect of the $\bar{\mathsf{f}}$ function. This field will then assemble with the vector derivative to form an object which, under displacements, simply reevaluates to the derivative with respect to the new position vector. We construct such an object by replacing ∇ with a new derivative $\bar{\mathsf{h}}(\nabla)$, where

$$\bar{\mathsf{h}}(a) = \bar{\mathsf{h}}(a;x) \tag{13.130}$$

is a position-dependent linear function of a. We again suppress this position dependence where clarity permits.

Under displacements the gauge field $\bar{\mathsf{h}}$ must transform such that

$$\bar{\mathsf{h}}'(\nabla_{x'}) = \bar{\mathsf{h}}(\nabla_x) = \bar{\mathsf{h}}\bar{\mathsf{f}}(\nabla_{x'}). \tag{13.131}$$

Explicitly, the transformation law for $\bar{\mathsf{h}}$ under displacements must be

$$\bar{\mathsf{h}}'(a;x) = \bar{\mathsf{h}}\big(\bar{\mathsf{f}}^{-1}(a);x'\big), \tag{13.132}$$

or, suppressing the position dependence,

$$\bar{\mathsf{h}}'(a) = \bar{\mathsf{h}}\bar{\mathsf{f}}^{-1}(a). \tag{13.133}$$

This must hold for any arbitrary vector a. This transformation law is different to that encountered in the gauge theories discussed previously, as the gauge field acts directly on ∇. The $\bar{\mathsf{h}}$ field is therefore not a connection in the conventional Yang–Mills sense. It is clear, however, that the $\bar{\mathsf{h}}$ field embodies the idea of ensuring that a symmetry is local, so can sensibly be called a gauge field. Since $\bar{\mathsf{h}}(a)$ is an arbitrary, position-dependent linear function of a, it has $4 \times 4 = 16$ degrees of freedom.

We can now systematically replace every occurrence of ∇ with $\bar{\mathsf{h}}(\nabla)$, and all our equations will be invariant under arbitrary displacements. In particular, the Dirac Lagrangian density is now modified to read

$$\mathcal{L} = \det\,(\mathsf{h})^{-1}\left\langle \bar{\mathsf{h}}(\nabla)\psi I\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\right\rangle. \tag{13.134}$$

This now transforms covariantly under arbitrary displacements of the fields. Similarly, we can consider the proper time or distance along a trajectory $x(\lambda)$. In the absence of gravitational fields this is

$$S = \int d\lambda\,\left|\frac{\partial x}{\partial\lambda}\cdot\frac{\partial x}{\partial\lambda}\right|^{1/2}. \tag{13.135}$$

Under a displacement the path transforms to $f(x(\lambda))$, so the tangent vector transforms to

$$\partial_\lambda f\big(x(\lambda)\big) = \mathsf{f}(\partial_\lambda x). \tag{13.136}$$

We can therefore construct a gauge-invariant interval by setting

$$S = \int d\lambda\,\left|\mathsf{h}^{-1}(x')\cdot\mathsf{h}^{-1}(x')\right|^{1/2}, \tag{13.137}$$

where

$$x' = \frac{\partial x(\lambda)}{\partial\lambda}. \tag{13.138}$$

This distance is now invariant under displacements, so is a physically-observable quantity.

We now see that tangent vectors pick up a factor of h^{-1} and cotangent vectors a factor of $\bar{\mathsf{h}}$. Spinors are not acted on by the h function. Next we establish contact with more familiar constructions of general relativity. Suppose that x^μ denote an arbitrary coordinate system, with frame vectors denoted by

$$e_\mu = \frac{\partial x}{\partial x^\mu}, \qquad e^\mu = \nabla x^\mu. \tag{13.139}$$

In terms of this coordinate system, equation (13.137) involves the term

$$\mathsf{h}^{-1}(x')\cdot\mathsf{h}^{-1}(x') = \frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\mu}{\partial\lambda}\,\mathsf{h}^{-1}(e_\mu)\cdot\mathsf{h}^{-1}(e_\nu). \tag{13.140}$$

If we define the vectors

$$g_\mu = \mathsf{h}^{-1}(e_\mu), \qquad g^\mu = \bar{\mathsf{h}}(e^\mu). \tag{13.141}$$

then we can write the preceding term as

$$\mathsf{h}^{-1}(x')\cdot\mathsf{h}^{-1}(x') = \frac{\partial x^\mu}{\partial\lambda}\frac{\partial x^\mu}{\partial\lambda}\,g_\mu\cdot g_\nu. \tag{13.142}$$

Equation (13.137) is therefore equivalent to the line interval in general relativity if we set the metric equal to

$$g_{\mu\nu} = g_\mu\cdot g_\nu = \mathsf{h}^{-1}(e_\mu)\cdot\mathsf{h}^{-1}(e_\nu). \tag{13.143}$$

The gauge field h is therefore a form of square root of the metric, which allows us to replace the metric inner product with the inner product in the spacetime algebra. In this sense, h is closely related to the concept of a spacetime orthonormal tetrad or *vierbein*. A vierbein is obtained from the h field by defining

$$\begin{aligned} e_\mu{}^i &= g_\mu\cdot\gamma^i, \\ e^\mu{}_i &= g^\mu\cdot\gamma_i, \end{aligned} \tag{13.144}$$

where both i and μ run from 0 to 4. The advantage of working directly with the h field is that it frees us from any coordinate frame. Coordinate frames are best introduced at a later date, when the geometry of a given problem usually dictates the appropriate coordinate system.

Now that we have recovered the metric, the obvious question is what has happened to the original flat space? It has not gone away, as all fields take their values over this space. In fact, there are now three distinct spaces of objects we can discuss. We refer to these as the tangent, cotangent and covariant spaces. Tangent vectors are of the form e_μ. Inner products between these are not gauge-invariant, and hence not physically meaningful. Similarly, cotangent vectors are of the form of e^μ, and the inner product of cotangent vectors is also an unphysical quantity. The inner product between tangent and cotangent vectors does produce a gauge-invariant quantity, so can correspond to a physical observable. Tangent and cotangent vectors can be interchanged via the metric, which maps one space into the other. In frame-free form, we can write

$$a^* = \bar{\mathsf{h}}^{-1}\mathsf{h}^{-1}(a) = \mathsf{g}(a). \tag{13.145}$$

The tangent and cotangent spaces, and the metric map between them, are the traditional elements of general relativity. Our third space, of covariant objects,

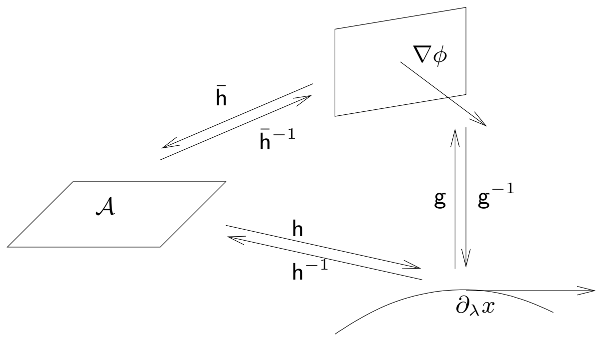

Figure 13.1 *Gauge fields for gravitation.* There are three vector spaces involved, consisting of tangent vectors $\partial_\lambda x$, cotangent vectors $\nabla\phi$ and covariant fields $\mathcal{A}$. The h field maps between these. The metric tensor maps between tangent and cotangent vectors, so is given by $\mathsf{g} = \bar{\mathsf{h}}^{-1}\mathsf{h}^{-1}$. Gauge-invariant quantities are formed from the scalar product of a tangent and cotangent vector, or from a pair of covariant vectors.

is unique to the gauge theory formulation. This space consists of objects whose transformation law under displacements is

$$\phi'(x) = \phi(x'). \tag{13.146}$$

This defines what it means to transform covariantly under displacements. These include velocity vectors of the form $\mathsf{h}^{-1}(\partial_\lambda x)$, gradients of the form $\bar{\mathsf{h}}(\nabla)\phi$, and spinor fields. Inner products between covariant vectors produce covariant scalars, which can be physically observable.

The various fields and spaces involved are depicted in figure 13.1. The advantage of the gauge theory viewpoint, coupled with the application of spacetime algebra, is that we can now take full advantage of the space of covariant objects when analysing the gravitational field equations. This turns out to have many advantages, both conceptually and computationally. The possibilities afforded by this space have been overlooked in most treatments of gauge theory gravity. One immediate question posed by figure 13.1 is whether the insistence on the existence of a map from a curved spacetime onto a flat one has any topological consequences. The answer is yes, though the restrictions are not as severe as one might expect. Many apparently topological constructions, such as cosmic strings and closed universe models, are easily handled in the gauge theory framework. Others, such as wormholes connecting multiple universes, do not fit so easily because they require a modification of the initial assumption that the background space is topologically flat. Models incorporating these effects can be constructed, though their motivation is less clear from the gauge theory perspective, as aspects of the theory have to be put in by hand initially.

13.4.2 Rotations

Now that we have discovered the metric tensor within the gauge approach we could immediately write down the familiar equations of general relativity. But we seek a theory formulated entirely in terms of covariant vectors, and this requires the existence of a second gauge field. As well as invariance under displacements, we require that our wave equation be invariant under the transformation

$$\psi \mapsto \psi' = R\psi, \tag{13.147}$$

where R is an arbitrary, position-dependent spacetime rotor. We are now back in the territory of section 13.3.3, with the difference that the rotor multiplies ψ from the left, instead of the right. To convert ∂_μ into a covariant derivative, we add a bivector connection Ω_μ and define

$$D_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}\Omega_\mu\psi. \tag{13.148}$$

The connection Ω_μ is a position-dependent bivector, subject to the transformation law

$$\Omega_\mu \mapsto \Omega'(a) = R\Omega_\mu\tilde{R} - 2\partial_\mu R\tilde{R}. \tag{13.149}$$

Since R is an arbitrary rotor there is no constraint on the blades that Ω_μ can contain, so Ω_μ has $6 \times 4 = 24$ degrees of freedom.

With the rotation gauge field included, the fully covariant Dirac action now reads, with the electromagnetic term included,

$$S = \int d^4x \det\,(\mathsf{h})^{-1} \left\langle \bar{\mathsf{h}}(\gamma^\mu)(\partial_\mu\psi + \tfrac{1}{2}\Omega_\mu\psi)I\gamma_3\tilde{\psi} - e\bar{\mathsf{h}}(A)\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi} \right\rangle. \tag{13.150}$$

The value of this action should be unchanged under local displacements and rotations. To establish this we need to complete the set of transformation properties for the gravitational gauge fields. First, we need to define how Ω_μ transforms under displacements. For this it is easier to use the notation $\Omega(a;x)$ for the linear argument and position dependence of the connection. Since $\Omega(a)$ picks up a term in $a\cdot\nabla R\tilde{R}$ under local rotations, we see that the appropriate transformation law under displacements is

$$\Omega'(a;x) = \Omega(\mathsf{f}(a);x'). \tag{13.151}$$

The connection in the action of equation (13.150) is contracted to form the object

$$\bar{\mathsf{h}}(\gamma^\mu)\Omega_\mu = \bar{\mathsf{h}}(\partial_a)\Omega(a). \tag{13.152}$$

So under a displacement this transforms to

$$\bar{\mathsf{h}}'(\partial_a)\Omega'(a) = \bar{\mathsf{h}}(\bar{\mathsf{f}}^{-1}(\partial_a);x')\,\Omega(\mathsf{f}(a);x') = \bar{\mathsf{h}}(\partial_a;x')\Omega(a;x'), \tag{13.153}$$

which is precisely the behaviour we require.

Similarly, we can establish the behaviour of the h field under rotations from the kinetic term in the covariant Dirac action. Under a local rotation this transforms to

$$\left\langle \bar{\mathsf{h}}'(\gamma^\mu)(\partial_\mu\psi' + \tfrac{1}{2}\Omega'_\mu\psi')I\gamma_3\tilde{\psi}'\right\rangle = \left\langle \tilde{R}\bar{\mathsf{h}}'(\gamma^\mu)R(\partial_\mu\psi + \tfrac{1}{2}\Omega_\mu\psi)I\gamma_3\tilde{\psi}\right\rangle. \tag{13.154}$$

So under rotations we must have

$$\bar{\mathsf{h}}(a) \mapsto \bar{\mathsf{h}}'(a) = R\bar{\mathsf{h}}(a)\tilde{R}. \tag{13.155}$$

The same transformation law is obeyed by vectors of the form $\mathsf{h}^{-1}(a)$, where a is a tangent vector. This guarantees that inner products between tangent and cotangent vectors are gauge-invariant, as required. The action of equation (13.150) now contains all of the local symmetries we require. The coupling of the electromagnetic vector potential A follows from the fact that A generalises the gradient of a scalar, so is a cotangent vector. This is acted on by $\bar{\mathsf{h}}$ to establish a covariant vector.

13.4.3 The Dirac equation in a gravitational background

We have so far established invariance at the level of the Dirac action, which led us to the action of equation (13.150). We now vary this action with respect to ψ, treating all other fields as external, to obtain the full, minimally-coupled Dirac equation. After reversing, variation with respect to ψ produces the equation

$$\begin{aligned}\bar{\mathsf{h}}(\nabla)\psi I\gamma_3 + \tfrac{1}{2}\bar{\mathsf{h}}(\gamma^\mu)\Omega_\mu\psi I\gamma_3 + \tfrac{1}{2}\Omega_\mu\bar{\mathsf{h}}(\gamma^\mu)\psi I\gamma_3 \\ - 2e\bar{\mathsf{h}}(A)\psi\gamma_0 - 2m\psi = -\frac{\partial}{\partial x^\mu}\big(\det\,(\mathsf{h})^{-1}\bar{\mathsf{h}}(\gamma^\mu)\psi I\gamma_3\big)\det\,(\mathsf{h}).\end{aligned} \tag{13.156}$$

This simplifies to

$$\bar{\mathsf{h}}(\gamma^\mu)(\partial_\mu\psi + \tfrac{1}{2}\Omega_\mu\psi)I\gamma_3 - e\bar{\mathsf{h}}(A)\psi\gamma_0 = m\psi + \tfrac{1}{2}t\psi I\gamma_3, \tag{13.157}$$

where the vector t is defined by

$$t = \det\,(\mathsf{h})\partial_\mu\big(\det\,(\mathsf{h})^{-1}\bar{\mathsf{h}}(\gamma^\mu)\big) + \Omega_\mu\cdot\bar{\mathsf{h}}(\gamma^\mu). \tag{13.158}$$

Here we encounter an initial surprise. The minimally-coupled Dirac action only produces the expected Dirac equation if the vector t is zero. We will establish the circumstances when this holds once we have discovered the full gravitational field equations. With t assumed to equal zero, we obtain the expected equation, which we write as

$$D\psi I\gamma_3 - e\mathcal{A}\psi\gamma_0 = m\psi. \tag{13.159}$$

Here we introduce the notation

$$D\psi = \bar{\mathsf{h}}(\gamma^\mu) D_\mu \psi = \bar{\mathsf{h}}(\gamma^\mu)(\partial_\mu \psi + \tfrac{1}{2}\Omega_\mu \psi) \tag{13.160}$$

and

$$\bar{\mathsf{h}}(A) = \mathcal{A}. \tag{13.161}$$

In this latter definition we begin to introduce the useful notation of writing fully covariant multivectors in calligraphic font.

13.4.4 Covariant derivatives for observables

Having established the form of the gravitational covariant derivative for a spinor, it is a simple matter to establish the form of the derivatives of the observables formed from a spinor. In general, these observables have the form

$$\mathcal{M} = \psi \Gamma \tilde{\psi}, \tag{13.162}$$

where Γ is a constant multivector formed from combinations of γ_0, γ_3 and $I\boldsymbol{\sigma}_3$. The observable $\mathcal{M}$ inherits its transformation properties from the spinor ψ, so under displacements $\mathcal{M}$ transforms as

$$\mathcal{M}(x) \mapsto \mathcal{M}'(x) = \mathcal{M}(x') \tag{13.163}$$

and under rotations $\mathcal{M}$ transforms as

$$\mathcal{M} \mapsto \mathcal{M}' = R\mathcal{M}\tilde{R}. \tag{13.164}$$

Multivectors with these transformation properties are said to be (fully) *covariant*. Scalars formed from inner products of these quantities account for the physical observables in the theory.

If we now form the partial derivative of $\mathcal{M}$ we obtain

$$\partial_\mu \mathcal{M} = (\partial_\mu \psi)\Gamma\tilde{\psi} + \psi\Gamma(\partial_\mu\psi)^\sim. \tag{13.165}$$

There is no need to restrict to orthonormal coordinates, so we can take ∂_μ as the derivative with respect to an arbitrary coordinate system, with coordinate frame $\{e_\mu\}$. We immediately see how to construct a covariant derivative for $\mathcal{M}$. We simply replace spinor directional derivatives with their covariant versions and form

$$\begin{aligned}(D_\mu\psi)\Gamma\tilde{\psi} + \psi\Gamma(D_\mu\psi)^\sim &= \partial_\mu(\psi\Gamma\tilde{\psi}) + \tfrac{1}{2}\Omega_\mu\psi\Gamma\tilde{\psi} - \tfrac{1}{2}\psi\Gamma\tilde{\psi}\Omega_\mu \\ &= \partial_\mu(\psi\Gamma\tilde{\psi}) + \Omega_\mu \times (\psi\Gamma\tilde{\psi}),\end{aligned} \tag{13.166}$$

where

$$\Omega_\mu = \Omega(e_\mu). \tag{13.167}$$

We therefore define the covariant derivative $\mathcal{D}_\mu$ by

$$\mathcal{D}_\mu \mathcal{M} = \partial_\mu \mathcal{M} + \Omega_\mu \times \mathcal{M}. \tag{13.168}$$

This is the form appropriate for acting on covariant multivectors, including observables formed from spinors. The commutator with the bivector Ω_μ has two important properties. The first is that it is grade-preserving, so the full $\mathcal{D}_\mu$ operator preserves grade. The second is that

$$\Omega_\mu \times (AB) = (\Omega_\mu \times A)B + A(\Omega_\mu \times B), \tag{13.169}$$

which holds for any multivectors A and B. This ensures that $\mathcal{D}_a$ is a *derivation.* That is, it satisfies Leibniz's rule

$$\mathcal{D}_\mu(AB) = (\mathcal{D}_\mu A)B + A(\mathcal{D}_\mu B). \tag{13.170}$$

These properties of preserving grade and satisfying Leibniz's rule are necessary for $\mathcal{D}_\mu$ to be a suitable generalisation of a directional derivative.

We can assemble a full, covariant version of the vector derivative by writing

$$\mathcal{D} = \bar{\mathsf{h}}(e^\mu)\mathcal{D}_\mu = g^\mu \mathcal{D}_\mu, \tag{13.171}$$

where $g^\mu = \bar{\mathsf{h}}(e^\mu)$. This acts on covariant multivectors to raise and lower the grade by one. We can also write

$$\mathcal{D}\mathcal{M} = \mathcal{D}\cdot\mathcal{M} + \mathcal{D}\wedge\mathcal{M}, \tag{13.172}$$

where $\mathcal{M}$ is a homogeneous-grade multivector, and

$$\begin{aligned} \mathcal{D}\cdot\mathcal{M} &= g^\mu\cdot(\mathcal{D}_\mu\mathcal{M}), \\ \mathcal{D}\wedge\mathcal{M} &= g^\mu\wedge(\mathcal{D}_\mu\mathcal{M}). \end{aligned} \tag{13.173}$$

It is also sometimes convenient to write the directional covariant derivative as $a\cdot\mathcal{D}$, where

$$a\cdot\mathcal{D}\,\mathcal{M} = a\cdot g^\mu \mathcal{D}_\mu \mathcal{M}. \tag{13.174}$$

We are now beginning to assemble a very powerful, compact notation for the main operators in gauge theory gravitation.

13.5 The gravitational field equations

The price we pay for ensuring that the Dirac action is invariant under local rotations is the introduction of two gauge fields: the vector-valued function $\mathsf{h}(a)$ and the bivector-valued $\Omega(a)$. These in total have 40 degrees of freedom. Our next task is to construct suitable equations for these gauge fields. As with the Dirac equation, our ultimate goal is to formulate the equations in terms of covariant objects, where the physical content of the theory is clearest. The alternative approach is to work entirely in terms of the metric $g_{\mu\nu}$. This is invariant under rotations, so all reference to the rotation gauge is removed. The end result is a set of second-order equations that are notoriously difficult to solve. The gauge theory approach, with its focus on gauge-covariant objects, provides a number of new solution strategies, both for analytical and numerical work.

Our method for constructing covariant field equations is to find a covariant Lagrangian and vary this. The resulting equations are then guaranteed to be covariant. Our first task, then, is to find covariant forms of the field strengths for the gravitational gauge fields. From these we can construct covariant scalar quantities, which can act as a Lagrangian density.

13.5.1 The rotation-gauge field strength

The field strength for the $\Omega(a)$ connection is found in the standard way by considering commutators of covariant derivatives. We define

$$[D_\mu, D_\nu]\psi = \tfrac{1}{2}\mathsf{R}_{\mu\nu}\psi, \tag{13.175}$$

so that

$$\mathsf{R}_{\mu\nu} = \partial_\mu\Omega_\nu - \partial_\nu\Omega_\mu + \Omega_\mu \times \Omega_\nu. \tag{13.176}$$

A frame-free notation is introduced by first writing

$$\mathsf{R}_{\mu\nu} = \mathsf{R}(e_\mu \wedge e_\nu), \tag{13.177}$$

where the $\{e_\mu\}$ vectors are the coordinate frame defined by the x^μ. We can therefore write

$$\mathsf{R}(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b). \tag{13.178}$$

Whenever we adopt this notation we assume that the vector arguments a and b are constant. Since the right-hand side is antisymmetric on a and b, the field strength depends only on the bivector $a\wedge b$. This linear action on bivector blades is extended to general bivectors by defining

$$\mathsf{R}(a\wedge b + c\wedge d) = \mathsf{R}(a\wedge b) + \mathsf{R}(c\wedge d). \tag{13.179}$$

This means that we can write the field strength as

$$\mathsf{R}(B) = \mathsf{R}(B; x), \tag{13.180}$$

which is a position-dependent, linear function of the bivector B. The field strength is a general bivector, as there are no restrictions on the form of $\Omega(a)$. This means that $\mathsf{R}(a\wedge b)$ has 36 degrees of freedom, as opposed to the rather simpler six of electromagnetism.

Unlike the electromagnetic case of equation (13.109), the commutator term $\Omega(a)\times\Omega(b)$ has not cancelled out. This has an important consequence for the field equations — they are no longer *linear*. If we add together two configurations of $\Omega(a)$, the field strength of the resultant $\Omega(a)$ is not the same as that from the superposition of the original field strengths. This makes the gravitational field equations much more difficult to solve than those of electromagnetism.

The definition of $\mathsf{R}(B)$ in terms of commutators makes it easy to establish its transformation properties under rotation gauge transformations. We see that

$$[D'_\mu, D'_\nu,]\psi' = \tfrac{1}{2}\mathsf{R}'(e_\mu\wedge e_\nu)R\psi = R[D_\mu, D_\nu]\psi = \tfrac{1}{2}R\mathsf{R}(e_\mu\wedge e_\nu)\psi, \qquad (13.181)$$

from which we can read off that

$$\mathsf{R}'(B) = R\mathsf{R}(B)\tilde{R}. \qquad (13.182)$$

Unlike electromagnetism, the field strength now transforms under gauge transformations, albeit in a straightforward way.

Under displacements, $\Omega(a)$ transforms as defined in equation (13.153). It follows that the field strength transforms to

$$\begin{aligned}\mathsf{R}'(e_\mu\wedge e_\nu) &= \partial_\mu\Omega'(e_\nu) - \partial_\nu\Omega'(e_\mu) + \Omega'(e_\nu)\times\Omega'(e_\mu)\\ &= \mathsf{f}(e_\mu)\cdot\dot{\nabla}_{x'}\dot{\Omega}\big(\mathsf{f}(e_\nu);x'\big) - \mathsf{f}(e_\nu)\cdot\dot{\nabla}_{x'}\dot{\Omega}\big(\mathsf{f}(e_\mu);x'\big) + \Omega'(e_\mu)\times\Omega'(e_\nu)\\ &\quad + \Omega\big(\partial_\mu\mathsf{f}(e_\nu) - \partial_\nu\mathsf{f}(e_\mu);x'\big)\\ &= \mathsf{R}\big(\mathsf{f}(e_\mu\wedge e_\nu);x'\big) + \Omega\big(\partial_\mu\mathsf{f}(e_\nu) - \partial_\nu\mathsf{f}(e_\mu);x'\big). \end{aligned} \qquad (13.183)$$

But we know that

$$\partial_\mu\mathsf{f}(e_\nu) - \partial_\nu\mathsf{f}(e_\mu) = \partial_\mu\partial_\nu f(x) - \partial_\nu\partial_\mu f(x) = 0, \qquad (13.184)$$

so the field strength has the simple displacement transformation law

$$\mathsf{R}(B) \mapsto \mathsf{R}'(B) = \mathsf{R}\big(\mathsf{f}(B);x'\big). \qquad (13.185)$$

We see that $\mathsf{R}'(B)$ picks up a term in $\mathsf{f}(B)$ under displacements, so is not fully covariant. To form a covariant tensor we insert a term in $\mathsf{h}(a)$ into $\mathsf{R}(B)$ and define the covariant field strength

$$\mathcal{R}(B) = \mathsf{R}\big(\mathsf{h}(B)\big). \qquad (13.186)$$

The factor of $\mathsf{h}(B)$ in this definition alters the transformation properties under rotations. Since $\bar{\mathsf{h}}$ transforms according to equation (13.155), the adjoint transforms as

$$\mathsf{h}(a) \mapsto \mathsf{h}'(a) = \partial_b\langle aR\bar{\mathsf{h}}(b)\tilde{R}\rangle = \mathsf{h}(\tilde{R}aR). \qquad (13.187)$$

The transformation properties of $\mathcal{R}(B)$ are therefore summarised by:

$$\begin{aligned} \text{displacements:}&\quad \mathcal{R}'(B,x) = \mathcal{R}(B,x'),\\ \text{rotations:}&\quad \mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R}. \end{aligned} \qquad (13.188)$$

These are precisely the properties we require, and they define a *covariant tensor*. The rotation law may look complicated, but it is quite natural. For example, suppose that $\mathcal{R}(B)$ simply amounts to the instruction 'dilate all fields by the factor α'. This is a physical statement, so ought to be true in all gauges. The original statement corresponds to

$$\mathcal{R}(B) = \alpha B. \tag{13.189}$$

The transformed field is then

$$\mathcal{R}'(B) = R\mathcal{R}(\tilde{R}BR)\tilde{R} = R(\alpha\tilde{R}BR)\tilde{R} = \alpha B, \tag{13.190}$$

so does contain to the same physical information. The function $\mathcal{R}(B)$ plays the same role in the gauge theory approach as the curvature tensor in general relativity, so we refer to $\mathcal{R}(B)$ as the *Riemann tensor*. We continue to employ the notational device of writing covariant tensors in calligraphic symbols to help keep track of which objects are gauge-invariant.

13.5.2 The displacement-gauge field strength

The displacement gauge field couples to the vector derivative to form the object $\bar{\mathsf{h}}(\nabla)$. This coupling is different to that of the connection for the rotation gauge field, and we cannot use the commutator of covariant derivatives to obtain the field strength. Indeed, the precise definition and meaning of the field strength for the displacement gauge are unclear. Here we motivate a definition that has the desired properties and is physically plausible.

The main property we require of a field strength is that it should vanish if the field is obtained by a pure gauge transformation. If we start with the identity and apply a displacement, the induced h field is given by

$$\bar{\mathsf{h}}(a) = \bar{\mathsf{f}}^{-1}(a). \tag{13.191}$$

One of the properties satisfied by a pure displacement is that

$$\nabla\wedge\bar{\mathsf{f}}(a) = 0. \tag{13.192}$$

So h will define a pure gauge transformation if it satisfies

$$\nabla\wedge\bar{\mathsf{h}}^{-1}(a) = 0, \tag{13.193}$$

where we temporarily ignore the rotation gauge. The left-hand side is our candidate object for the field strength. The task now is to make it covariant.

We know that the vector derivative ∇ picks up a factor of $\bar{\mathsf{h}}$ to convert it to covariant form. Since $\bar{\mathsf{h}}^{-1}$ transforms in the same way as ∇, we can define a displacement-gauge covariant object $H(a)$ as

$$H(a) = -\bar{\mathsf{h}}\big(\nabla\wedge\bar{\mathsf{h}}^{-1}(a)\big) = \bar{\mathsf{h}}(\dot{\nabla})\wedge\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a). \tag{13.194}$$

This is a bivector-valued function of its vector argument. The final step is to convert the derivative to one that is covariant under rotations. This is straightforward since $\bar{\mathsf{h}}$ transforms as a vector under rotations. We therefore define

$$\mathcal{H}(a) = \bar{\mathsf{h}}(\partial_b)\wedge\Big(b\cdot\dot{\nabla}\dot{\bar{\mathsf{h}}}\bar{\mathsf{h}}^{-1}(a) + \Omega(b)\cdot a\Big), \tag{13.195}$$

or, in terms of a coordinate frame,

$$\mathcal{H}(g^\mu) = g^\alpha\wedge(\mathcal{D}_\alpha g_\mu) = \mathcal{D}\wedge g^\mu, \tag{13.196}$$

where we have applied that result that $\nabla\wedge e^\mu = 0$.

The tensor $\mathcal{H}(a)$ is covariant under displacements and rotations, so transforms covariantly as

$$\begin{aligned} \text{displacements:} \quad & \mathcal{H}'(a,x) = \mathcal{H}(a,x'), \\ \text{rotations:} \quad & \mathcal{H}'(a) = R\mathcal{H}(\tilde{R}aR)\tilde{R}. \end{aligned} \tag{13.197}$$

As we will soon see, the object we have defined is in fact the *torsion* tensor, a bivector-valued function of a vector with $6\times 4 = 24$ degrees of freedom. This is the appropriate number for the field strength of the displacement gauge, as a displacement is specified by four degrees of freedom. In the simplest formulation of the field equations, the torsion is equated with the spin of the matter. It is therefore a pure contact term, and usually extremely small. One can justify this on dimensional grounds. The two field strengths we have defined, $\mathcal{H}(a)$ and $\mathcal{R}(B)$, differ in dimensions by a factor of length. This is because $\Omega(a)$ has dimensions of $(\text{length})^{-1}$, whereas $\bar{\mathsf{h}}(a)$ is dimensionless. The only fundamental length scale that could relate these is the Planck length, l_P, which is tiny. The natural scale for $\mathsf{S}(a)$ is therefore l_P times $\mathcal{R}(B)$, making it negligible compared to the Riemann tensor.

13.5.3 The gravitational action

We have now defined two covariant tensors from the gravitational gauge fields — the Riemann and torsion tensors. We next require a scalar term to act as the Lagrangian density for gravitation. There are a number of quadratic scalars we can derive from the gauge fields, but only one scalar is linear in the field strength. This is important, as one can again argue on dimensional grounds that higher order terms should be reduced by factors of the Planck length.

We first define the contractions of the Riemann tensor. The first is the *Ricci tensor*:

$$\mathcal{R}(b) = \partial_a\cdot\mathcal{R}(a\wedge b). \tag{13.198}$$

By construction, this is a tensor. The Ricci tensor can be contracted further to defined the *Ricci scalar*

$$\mathcal{R} = \partial_a\cdot\mathcal{R}(a). \tag{13.199}$$

We use the same symbol to denote the Riemann tensor, Ricci tensor and Ricci scalar, and distinguish between these by their argument. The Ricci scalar is a covariant scalar field, so is invariant under rotations and transforms covariantly under displacements. The Ricci scalar is the first scalar observable we have constructed from the gravitational fields, and is the simplest candidate for the Lagrangian density. We therefore suppose that the overall action integral is of the form

$$S = \int |d^4x| \det(\mathsf{h})^{-1}(\tfrac{1}{2}\mathcal{R} + \Lambda - \kappa\mathcal{L}_m), \tag{13.200}$$

where $\mathcal{L}_m$ describes the matter content and $\kappa = 8\pi G$. We have also included the cosmological constant Λ, though for most applications we set this to zero. The independent dynamical variables are $\bar{\mathsf{h}}(a)$ and $\Omega(a)$, and we assume that $\mathcal{L}_m$ contains no second-order derivatives, so that $\bar{\mathsf{h}}(a)$ and $\Omega(a)$ appear undifferentiated in the matter Lagrangian.

The $\bar{\mathsf{h}}$ field is undifferentiated in the entire action, as we have not included any terms in $\mathcal{H}(a)$. The Euler–Lagrange equation for $\bar{\mathsf{h}}$ is simply

$$\partial_{\bar{\mathsf{h}}(a)}\big(\det(\mathsf{h})^{-1}(\mathcal{R}/2 + \Lambda - \kappa\mathcal{L}_m)\big) = 0. \tag{13.201}$$

Employing the results of section 11.1.2 we find that

$$\partial_{\bar{\mathsf{h}}(a)}\det(\mathsf{h})^{-1} = -\det(\mathsf{h})^{-1}\mathsf{h}^{-1}(a) \tag{13.202}$$

and

$$\begin{aligned}\partial_{\bar{\mathsf{h}}(a)}\mathcal{R} &= \partial_{\bar{\mathsf{h}}(a)}\left\langle \bar{\mathsf{h}}(\partial_c\wedge\partial_b)\mathsf{R}(b\wedge c)\right\rangle \\ &= 2\bar{\mathsf{h}}(\partial_b)\cdot\mathsf{R}(b\wedge a).\end{aligned} \tag{13.203}$$

It follows that

$$\partial_{\bar{\mathsf{h}}(a)}\Big(\mathcal{R}\det(\mathsf{h})^{-1}\Big) = 2\mathcal{G}\Big(\mathsf{h}^{-1}(a)\Big)\det(\mathsf{h})^{-1}, \tag{13.204}$$

where $\mathcal{G}$ is the Einstein tensor,

$$\mathcal{G}(a) = \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}. \tag{13.205}$$

We now define the *functional* matter energy-momentum tensor $\mathcal{T}(a)$ by

$$\det(\mathsf{h})\partial_{\bar{\mathsf{h}}(a)}(\mathcal{L}_m\det(\mathsf{h})^{-1}) = \mathcal{T}\Big(\mathsf{h}^{-1}(a)\Big). \tag{13.206}$$

We therefore arrive at the first of our field equations,

$$\mathcal{G}(a) - \Lambda a = \kappa\mathcal{T}(a). \tag{13.207}$$

This is the gauge theory statement of Einstein's equation. The source term in the Einstein equations in the functional energy-momentum tensor, not the canonical one. The form of this is discussed once we have found the remaining field equations for the rotation gauge field.

The Euler–Lagrange field equation from $\Omega(a)$ is, after multiplying through by det (h),

$$\frac{\partial \mathcal{R}}{\partial \Omega(a)} - \det\,(\mathsf{h}) \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{R}}{\partial(\partial_\mu \Omega(a))} \det\,(\mathsf{h})^{-1} \right) = 2\kappa \frac{\partial \mathcal{L}_m}{\partial \Omega(a)}, \tag{13.208}$$

where we have employed the assumption that $\Omega(a)$ does not contain any coupling to matter through its derivatives, and have temporarily reverted to an orthonormal coordinate system. The right-hand side defines the matter *spin tensor*

$$\mathsf{S}(a) = \frac{\partial \mathcal{L}_m}{\partial \Omega(a)}. \tag{13.209}$$

This has the covariant form

$$\mathcal{S}(a) = \mathsf{S}\big(\bar{\mathsf{h}}^{-1}(a)\big), \tag{13.210}$$

which is a covariant tensor. For the left-hand side we use the results

$$\partial_{\Omega(a)} \left\langle \bar{\mathsf{h}}(\partial_d \wedge \partial_c) \Omega(c) \times \Omega(d) \right\rangle = 2\Omega(b) \times \bar{\mathsf{h}}(\partial_b \wedge a)$$

and

$$\frac{\partial}{\partial(\partial_\mu \Omega(a))} \left\langle \bar{\mathsf{h}}(\partial_d \wedge \partial_c) \big(c \cdot \nabla \Omega(d) - d \cdot \nabla \Omega(c)\big) \right\rangle = 2\bar{\mathsf{h}}(a \wedge \gamma^\mu). \tag{13.211}$$

Combining these results, equation (13.208) becomes

$$\begin{aligned} \bar{\mathsf{h}}(\dot{\nabla}) \wedge \dot{\bar{\mathsf{h}}}(a) + \det\,(\mathsf{h}) \partial_\mu \Big(\bar{\mathsf{h}}(\gamma^\mu) \det\,(\mathsf{h})^{-1} \Big) \wedge \bar{\mathsf{h}}(a) \\ + \Omega(b) \times \bar{\mathsf{h}}(\partial_b \wedge a) = \kappa \mathsf{S}(a). \end{aligned} \tag{13.212}$$

Recalling the definitions of $\mathcal{H}(a)$ and t, from equations (13.195) and (13.158) respectively, the second field equation has the covariant form

$$\mathcal{H}(a) + t \wedge a = \kappa \mathcal{S}(a). \tag{13.213}$$

So, as stated, $\mathcal{H}$ is governed by the matter spin density.

The second field equation (13.213) simplifies further once we form the contraction of the torsion tensor $\mathcal{H}(a)$. This is

$$\partial_a \cdot \mathcal{H}(a) = \mathcal{D}_\mu \bar{\mathsf{h}}(\gamma^\mu) - \bar{\mathsf{h}}(\dot{\nabla})\, \mathsf{h}^{-1}(\gamma_\mu) \cdot \dot{\bar{\mathsf{h}}}(\gamma^\mu). \tag{13.214}$$

But we can now use

$$\begin{aligned} \mathsf{h}^{-1}(\gamma_\mu) \cdot \big(\partial_\nu \bar{\mathsf{h}}(\gamma^\mu)\big) &= \left\langle \big(\partial_\nu \bar{\mathsf{h}}(\gamma_0)\big) \wedge \bar{\mathsf{h}}(\gamma_1 \wedge \gamma_2 \wedge \gamma_3) I^{-1} \det\,(\mathsf{h})^{-1} \right\rangle + \cdots \\ &= \det\,(\mathsf{h})^{-1} \partial_\nu \det\,(\mathsf{h}), \end{aligned} \tag{13.215}$$

to write

$$\partial_a \cdot \mathcal{H}(a) = \det\,(\mathsf{h}) \mathcal{D}_\mu \big(\det\,(\mathsf{h})^{-1} \bar{\mathsf{h}}(\gamma^\mu) \big) = t. \tag{13.216}$$

So the vector t which appeared in the Dirac equation is the contraction of the torsion tensor. On contracting equation (13.213) we find that

$$-2t = \kappa \partial_a \cdot \mathcal{S}(a), \tag{13.217}$$

which directly relates t to the matter spin density. The second field equation can now be written as

$$\mathcal{H}(a) = \kappa \mathcal{S}(a) + \tfrac{1}{2}\kappa\big(\partial_b \cdot \mathcal{S}(b)\big)\wedge a. \tag{13.218}$$

This equation directly relates the torsion to the matter spin density.

13.5.4 The matter content

To illustrate the structure of the source terms we return to the covariant Maxwell and Dirac Lagrangian densities. First consider free-field electromagnetism. Under displacements, the vector potential A transforms as a cotangent vector (1-form):

$$A(x) \mapsto A'(x) = \bar{\mathsf{f}}\big(A(x')\big), \tag{13.219}$$

and the field strength F transforms as a 2-form:

$$F \mapsto F'(x) = \nabla\wedge A'(x) = \bar{\mathsf{f}}\big(F(x')\big). \tag{13.220}$$

The covariant field strength is therefore defined by

$$\mathcal{F} = \bar{\mathsf{h}}(F) = \bar{\mathsf{h}}(\nabla\wedge A), \tag{13.221}$$

and the covariant Lagrangian density for the electromagnetic field is

$$\mathcal{L}_{em} = \tfrac{1}{2}\mathcal{F}\cdot\mathcal{F}. \tag{13.222}$$

The functional energy-momentum tensor is defined by

$$\begin{aligned} \mathcal{T}_{em}\big(\mathsf{h}^{-1}(a)\big) &= \det\,(\mathsf{h})\partial_{\bar{\mathsf{h}}(a)}\Big(\tfrac{1}{2}\mathcal{F}\cdot\mathcal{F}\det\,(\mathsf{h})^{-1}\Big) \\ &= \bar{\mathsf{h}}(a\cdot F)\cdot\mathcal{F} - \mathsf{h}^{-1}(a). \end{aligned} \tag{13.223}$$

So we obtain

$$\mathcal{T}_{em}(a) = (a\cdot\mathcal{F})\cdot\mathcal{F} - a = -\tfrac{1}{2}\mathcal{F}a\mathcal{F}. \tag{13.224}$$

This is precisely the form we would expect for the covariant generalisation of the electromagnetic field strength. Unlike the canonical definition, there is no issue about the tensor being electromagnetic gauge-invariant, and the tensor is automatically symmetric. Furthermore, there is no coupling to $\Omega(a)$, so the electromagnetic spin density is zero. We will discover in section 13.6 that, if the spin tensor is zero, the functional energy-momentum tensor must also be symmetric.

As an example of a field with non-vanishing spin density we next consider the

Dirac theory. With the electromagnetic coupling included, the covariant action is defined by equation (13.150). The functional energy-momentum tensor is simply

$$\mathcal{T}_D(a) = \left\langle a\cdot g^{\mu} D_{\mu}\psi I\gamma_3\tilde{\psi}\right\rangle_1 - ea\cdot\mathcal{A}\,\psi\gamma_0\tilde{\psi}. \tag{13.225}$$

This is manifestly a covariant tensor, though it is not necessarily symmetric. The spin density is

$$\mathsf{S}_D(a) = \tfrac{1}{2}\bar{\mathsf{h}}(a)\cdot\left(\psi I\gamma_3\tilde{\psi}\right) \tag{13.226}$$

or, covariantly,

$$\mathcal{S}_D(a) = \tfrac{1}{2}a\cdot\left(\psi I\gamma_3\tilde{\psi}\right) = \tfrac{1}{2}a\cdot S, \tag{13.227}$$

where S is the spin trivector. In the limit where gravitational interactions are turned off, the functional definitions agree with the canonical energy-momentum and angular momentum tensors.

The form of the Dirac spin has an important consequence. If we form the contraction we find that

$$2\partial_a\cdot\mathcal{S}(a) = \partial_a\cdot(a\cdot S) = 0, \tag{13.228}$$

so the torsion vector t vanishes. This is reassuring, as it implies that the minimally-coupled Dirac action produces the minimally-coupled Dirac equation on variation. Equation (13.228) is satisfied by scalar, Dirac and Yang–Mills fields. An exception is provided by a vector field that is often introduced to ensure local dilation invariance. There are good reasons for introducing such a field, though any interactions it might generate are likely to be on the scale of quantum gravity and are not discussed here.

As a further example of a source field for gravitation, we consider the case of an ideal fluid. This is the simplest form of matter energy-momentum tensor one can consider, and generates an important class of models. The action for an ideal fluid was introduced in section 12.4.2, and the only modification required to convert to a covariant action is multiplication of the energy density by $\det(\mathsf{h})^{-1}$:

$$S = \int d^4x\,\left(-\det(\mathsf{h})^{-1}\varepsilon + J\cdot(\nabla\lambda) - \mu J\cdot\nabla\eta\right). \tag{13.229}$$

The Lagrange multiplier terms are both unaffected by the presence of a gravitational field. The covariant current density is

$$\mathcal{J} = \det(\mathsf{h})\mathsf{h}^{-1}(J) = \rho v, \tag{13.230}$$

where $v^2 = 1$ (see section 13.5.6). The energy density ε therefore depends on the h field through its dependence on ρ. We find that

$$\partial_{\bar{\mathsf{h}}(a)}\rho^2 = 2\rho^2\left(\mathsf{h}^{-1}(a) - \mathsf{h}^{-1}(a)\cdot v\,v\right), \tag{13.231}$$

so the functional stress-energy tensor is

$$\mathcal{T}(a) = -\rho(a - a\cdot v\, v)\frac{\partial \varepsilon}{\partial \rho} + a\varepsilon. \tag{13.232}$$

Recalling the definition of the pressure from equation (12.158), we are left with

$$\begin{aligned}\mathcal{T}(a) &= -(a - a\cdot v\, v)(\varepsilon + P) + a\varepsilon \\ &= (\varepsilon + P)a\cdot v\, v - Pa.\end{aligned} \tag{13.233}$$

This is precisely the form we expect, with v now a covariant vector satisfying the constraint $v^2 = 1$. The actual form of v is gauge-dependent, a fact we can exploit to our advantage in applications by choosing a gauge where v has a simple form.

13.5.5 The torsion-free equations and general relativity

For many applications the matter spin density is negligible. It is a quantum effect, and the macroscopic spin of an object is usually extremely small as all of the individual constituents cancel out. In the case where the spin can be ignored the second field equation becomes

$$\mathcal{H}(a) = 0. \tag{13.234}$$

If we replace a by a general cotangent vector A, this equation can be written

$$\mathcal{D}\wedge\bar{\mathsf{h}}(A) = \bar{\mathsf{h}}(\nabla\wedge A), \tag{13.235}$$

which is extremely useful in practice. This equation says that antisymmetrised partial and covariant derivatives produce the same result. We will now establish that the spinless gauge field equations are (locally) equivalent to those of general relativity. Many of the relevant equations for Riemannian geometry were derived in section 6.5.5.

To begin, we define the connection by

$$\mathcal{D}_\mu g_\nu = \Gamma^\alpha_{\mu\nu} g_\alpha, \tag{13.236}$$

so that

$$\Gamma^\lambda_{\mu\nu} = g^\lambda\cdot(\mathcal{D}_\mu g_\nu). \tag{13.237}$$

It follows that the directional covariant derivative of a vector $\mathcal{A} = A^\mu g_\mu$ has components

$$\begin{aligned}\mathcal{D}_\mu \mathcal{A} &= \mathcal{D}_\mu(A^\alpha g_\alpha) \\ &= (\partial_\mu A^\alpha) g_\alpha + A^\alpha \Gamma^\beta_{\mu\alpha} g_\beta \\ &= (\partial_\mu A^\alpha + \Gamma^\alpha_{\mu\beta} A^\beta) g_\alpha,\end{aligned} \tag{13.238}$$

which recovers the general relativistic expression.

If we recall from equation (13.143) that the metric is given by $g_{\mu\nu} = g_\mu \cdot g_\nu$, we can now write

$$\partial_\mu g_{\nu\lambda} = (\mathcal{D}_\mu g_\nu)\cdot g_\lambda + g_\nu \cdot (\mathcal{D}_\mu g_\lambda), \tag{13.239}$$

so that

$$\partial_\mu g_{\nu\lambda} = \Gamma^\alpha_{\mu\nu} g_{\alpha\lambda} + \Gamma^\alpha_{\mu\lambda} g_{\alpha\nu}. \tag{13.240}$$

This is the metric compatibility condition for the connection. The second important condition on the connection, for pure general relativity, is antisymmetry. This follows from the torsion-free condition, since

$$\begin{aligned} 0 = (g_\mu \wedge g_\nu)\cdot(\mathcal{D}\wedge g^\alpha) &= g_\mu(\mathcal{D}_\nu g^\alpha) - g_\nu(\mathcal{D}_\mu g^\alpha) \\ &= g^\alpha \cdot (\mathcal{D}_\mu g_\nu - \mathcal{D}_\nu g_\mu). \end{aligned} \tag{13.241}$$

We can therefore read off that

$$\mathcal{D}_\mu g_\nu - \mathcal{D}_\nu g_\mu = 0. \tag{13.242}$$

It follows that, in the absence of torsion,

$$\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = 0. \tag{13.243}$$

This equation and equation (13.240) together define the Christoffel connection. The equations can be inverted to recover the connection in terms of derivatives of the metric. Rather than reproduce the standard derivation at this point, we will instead demonstrate how to invert equation (13.234) to find $\Omega(a)$ in terms of the h field.

Returning to the definition of the $H(a)$ and $\mathcal{H}(a)$ tensors of equations (13.194) and (13.195), the absence of torsion tells us that

$$-H(a) = \bar{\mathsf{h}}(\partial_b)\wedge\big(\Omega(b)\cdot a\big). \tag{13.244}$$

At this point it is useful to introduce the displacement-gauge-covariant connection

$$\omega(a) = \Omega\big(\mathsf{h}(a)\big). \tag{13.245}$$

Under displacements this transforms covariantly,

$$\omega'(a;x) = \omega(a;x'). \tag{13.246}$$

Under rotations the transformation law for $\omega(a)$ is somewhat more complicated than that for $\Omega(a)$, so it is usually preferable to deal with the latter when discussing rotation-gauge transformations. Equation (13.244) now becomes

$$\partial_b \wedge \big(\omega(b)\cdot a\big) = -H(a), \tag{13.247}$$

which gives $\omega(a)$ in terms of h and its derivatives. To solve this we first compute

$$\partial_a \wedge \partial_b \wedge \big(\omega(b)\cdot a\big) = 2\partial_b \wedge \omega(b) = -\partial_b \wedge H(b). \tag{13.248}$$

Now, taking the inner product with a again, we obtain

$$\omega(a) - \partial_b\wedge(a\cdot\omega(b)) = -\tfrac{1}{2}a\cdot\big(\partial_b\wedge H(b)\big). \tag{13.249}$$

We can therefore write

$$\omega(a) = H(a) - \tfrac{1}{2}a\cdot\big(\partial_b\wedge H(b)\big), \tag{13.250}$$

which enables us to compute $\omega(a)$ directly. In the presence of spin an additional term built from the spin tensor is added to the right-hand side. One can now convert the solution for $\omega(a)$ into a set of Christoffel coefficients, if desired. One disadvantage of the latter is that they mix up gauge terms with terms induced by a choice of curvilinear coordinates. From the manifold viewpoint this is sensible, but it is less natural in the gauge theory context.

Next we turn to the form of the Riemann tensor in general relativity. In terms of the connection, this is

$$\begin{aligned} R_{\mu\nu\rho}{}^{\sigma} &= \partial_\mu\Gamma^\sigma_{\nu\rho} - \partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\mu\alpha}\Gamma^\alpha_{\nu\rho} - \Gamma^\sigma_{\nu\alpha}\Gamma^\alpha_{\mu\rho} \\ &= \partial_\mu\big(g^\sigma\cdot(\mathcal{D}_\nu g_\rho)\big) - \partial_\nu\big(g^\sigma\cdot(\mathcal{D}_\mu g_\rho)\big) - (\mathcal{D}_\mu g^\sigma)\cdot(\mathcal{D}_\nu g_\rho) + (\mathcal{D}_\nu g^\sigma)\cdot(\mathcal{D}_\mu g_\rho) \\ &= g^\sigma\cdot(\mathcal{D}_\mu\mathcal{D}_\nu g_\rho - \mathcal{D}_\nu\mathcal{D}_\mu g_\rho), \end{aligned} \tag{13.251}$$

from which we can read off that

$$R_{\mu\nu\rho}{}^{\sigma} = \mathcal{R}(g_\mu\wedge g_\nu)\cdot(g_\rho\wedge g^\sigma). \tag{13.252}$$

This converts directly between the gauge theory and tensor formulations of gravity. One can also check that the contractions defined earlier are all equivalent to their general relativistic counterparts, so the gauge theory equation (13.207), in the torsion-free case, has the same content as the Einstein equations. The main differences between the two theories are topological in nature, and one can argue that such considerations are beyond the scope of the (local) theory of general relativity anyway.

13.5.6 Currents and Killing vectors

The gauge theory we have constructed is founded on an action principle in a flat spacetime. It follows that Noether's theorem still holds, and that symmetries of the action result in a conserved vector current J. Every such vector has a corresponding covariant equivalent. To find this we first write

$$\nabla\cdot J = I\nabla\wedge(IJ) = 0, \tag{13.253}$$

so, assuming no torsion is present, we have

$$\bar{\mathsf{h}}\big((\nabla\wedge(IJ)\big) = \mathcal{D}\wedge\bar{\mathsf{h}}(IJ) = 0. \tag{13.254}$$

We can therefore write

$$I\mathcal{J} = \bar{\mathsf{h}}(IJ) = I\mathsf{h}^{-1}(J)\det(\mathsf{h}), \tag{13.255}$$

which defines the covariant current $\mathcal{J}$ in terms of J. The covariant vector $\mathcal{J}$ then satisfies

$$\mathcal{D}\cdot\mathcal{J} = 0. \tag{13.256}$$

There is a vector $\mathcal{J}$ conjugate to each continuous symmetry of the action. If we attempt to find conserved vectors conjugate to translations and rotations, however, we do not discover any new information. In both cases the conjugate tensor turns out to be zero once the field equations are employed. This is due to the manner of the coupling of the h field. Variation with respect to h can be viewed as defining the total energy-momentum tensor, and this is zero because there is no derivative term for the h field in the action. It is traditional, of course, to single out (minus) the gravitational contribution to the total energy-momentum tensor (the Einstein tensor), and then equate this to the matter energy-momentum tensor.

A covariantly-conserved vector $\mathcal{J}$ gives rise to a conserved scalar because it can always be converted back to a non-covariant vector J satisfying $\nabla\cdot J = 0$. The same is not true of covariant conservation of a tensor, such as $\mathcal{G}(a)$. Tensors only give rise to useful conserved quantities in the presence of additional symmetries of the Lagrangian. This is the case when the h field is independent of the derivative along a global vector field. In this case one can construct a coordinate system such that the metric $g_{\mu\nu}$ is independent of one of the coordinates. If we call this x^0, we have

$$\frac{\partial}{\partial x^0} g_{\mu\nu} = g_\mu\cdot(g_0\cdot\mathcal{D}g_\nu) + g_\nu\cdot(g_0\cdot\mathcal{D}g_\mu) = 0. \tag{13.257}$$

But, for a coordinate frame in the absence of torsion, equation (13.242) holds and we have

$$g_\mu\cdot(g_\nu\cdot\mathcal{D}\mathcal{K}) + g_\nu\cdot(g_\mu\cdot\mathcal{D}\mathcal{K}) = 0, \tag{13.258}$$

where $\mathcal{K} = g_0$ is the covariant Killing vector. In coordinate-free form we can write

$$a\cdot(b\cdot\mathcal{D}\mathcal{K}) + b\cdot(a\cdot\mathcal{D}\mathcal{K}) = 0 \tag{13.259}$$

for any two vector fields a and b. This can be used as an alternative definition for a Killing vector. Contracting with $\partial_a\cdot\partial_b$ immediately tells us that $\mathcal{K}$ is divergenceless.

13.5.7 Point particle motion

General relativity typically models observers as point particles following geodesic paths, as defined by the geodesic equation. But the gauge approach has dealt solely with the properties of classical and quantum fields. To complete the proof of the equivalence of the gauge approach and general relativity, we must recover the geodesic equation from the minimally-coupled Dirac equation. In coordinate form, the geodesic equation is

$$\dot{v}^\mu + v^\alpha v^\beta \Gamma^\mu_{\alpha\beta} = 0, \tag{13.260}$$

where $v^\mu = \dot{x}^\mu$ and the overdots denote the derivative with respect to proper time. This is defined such that

$$g_{\mu\nu} v^\mu v^\nu = 1. \tag{13.261}$$

To convert to covariant form we introduce the vector

$$v = v_\mu g^\mu = \mathsf{h}^{-1}(\dot{x}), \qquad v^2 = 1. \tag{13.262}$$

This is a covariant vector, though for aesthetic reasons we do not write this in a calligraphic font. The derivative with respect to proper time is

$$\partial_\tau = \dot{x}^\mu \partial_\mu = v \cdot \bar{\mathsf{h}}(\nabla). \tag{13.263}$$

The geodesic equation (13.260) can be now be written

$$\partial_\tau v - v^\mu \partial_\tau g_\mu + v^\alpha v^\beta (\mathcal{D}_\alpha g_\beta) = \dot{v} + \omega(v) \cdot v = 0. \tag{13.264}$$

The gauge theory form of the the geodesic equation is therefore

$$v \cdot \mathcal{D} v = \dot{v} + \omega(v) \cdot v = 0. \tag{13.265}$$

This equation is also recovered by finding the paths that minimise the proper time interval

$$S = \int d\lambda \, |\mathsf{h}^{-1}(x') \cdot \mathsf{h}^{-1}(x')|^{1/2}. \tag{13.266}$$

Geodesics are classified into timelike, lightlike or spacelike according to the value of v^2, which can be $+1$, 0 or -1 respectively. Point particles with mass follow timelike geodesics.

The process by which classical paths are recovered from Dirac theory is discussed in section 12.2.1. The essential term in the action is the kinetic one, which we manipulate in the same way to write

$$\det\, (\mathsf{h})^{-1} \langle D\psi I \gamma_3 \tilde{\psi} \rangle = \det\, (\mathsf{h})^{-1} \langle \mathcal{J} D\psi I \boldsymbol{\sigma}_3 \psi^{-1} \rangle, \tag{13.267}$$

where $\mathcal{J} = \psi \gamma_0 \tilde{\psi}$. Equation (13.255) relates the covariant current $\mathcal{J}$ to the divergenceless current J. The classical limit is formed by concentrating the density onto a single streamline of J and ignoring terms in the action perpendicular to the flow. The action therefore contains the term

$$\det\, (\mathsf{h})^{-1} \left\langle \mathcal{J} \cdot g^\mu \, D_\mu \psi I \boldsymbol{\sigma}_3 \psi^{-1} \right\rangle = \left\langle (\psi' + \tfrac{1}{2} \Omega(x') \psi) I \boldsymbol{\sigma}_3 \psi^{-1} \right\rangle. \tag{13.268}$$

Separating out the rotor dependence, as before, and converting to proper time derivatives, the equations of motion are

$$v\cdot\mathcal{D}S + 2p\wedge v = 0 \tag{13.269}$$

and

$$v\cdot\mathcal{D}\,p = 0. \tag{13.270}$$

Here $v = \mathsf{h}^{-1}(x') = R\gamma_0\tilde{R}$ and $S = RI\boldsymbol{\sigma}_3\tilde{R}$. Classical point-particle motion is recovered by setting the spin to zero, so that p and v are aligned, and fixing $p\cdot v = m$. In this case we recover precisely the geodesic equation.

This derivation is unusual, but it is important for two reasons. The geodesic equation tells us that point particles follow the same paths regardless of their mass and so implies the equivalence of gravitational and inertial mass. This is the weak equivalence principle, a fundamental ingredient in general relativity. From the gauge theory perspective, the weak equivalence principle is derived from the classical limit of the Dirac equation. The only principle invoked in constructing the covariant Dirac equation was minimal coupling, so at one level this has the consequence of enforcing the weak equivalence principle. One can also argue that minimal coupling is the essence of the full equivalence principle, which tells us how physics should appear locally to a freely-falling observer. The second important feature of this derivation is that it points out the limitations of the weak equivalence principle. Both the wave nature of matter and the existence of quantum spin ensure that the geodesic equation is an approximation, and there are many quantum effects in gravitational backgrounds (such as black hole absorption) where the particle mass is important.

If a Killing vector is present, equation (13.259) tells us that

$$v\cdot(v\cdot\mathcal{D}\mathcal{K}) = 0. \tag{13.271}$$

So, for a particle satisfying the geodesic equation, we find that

$$\partial_\tau(v\cdot\mathcal{K}) = v\cdot\mathcal{D}(v\cdot\mathcal{K}) = \mathcal{K}\cdot(v\cdot\mathcal{D}v) + v\cdot(v\cdot\mathcal{D}\mathcal{K}) = 0. \tag{13.272}$$

It follows that the quantity $v\cdot\mathcal{K}$ is conserved along the worldline of a freely-falling particle. For stationary matter configurations, this can be used to define the conserved energy of the particle.

13.5.8 Electromagnetism in a gravitational background

The electromagnetic vector potential A ensures that the Dirac equation is covariant under local phase transformations. In equation (13.222) we found that the covariant action integral for the electromagnetic field in a gravitational background is given by

$$S = \int |d^4x| (\det\,\mathsf{h})^{-1}\,\tfrac{1}{2}\mathcal{F}\cdot\mathcal{F}, \tag{13.273}$$

where

$$\mathcal{F} = \bar{\mathsf{h}}(F). \tag{13.274}$$

The field strength $\mathcal{F}$ is covariant under local translations and rotations, as well as being phase-invariant.

We can include a source term by adding an $\mathcal{A}\cdot\mathcal{J}$ term, where $\mathcal{J}$ is a covariant vector. For example, when coupling to a fermion $\mathcal{J}$ is given by the Dirac current $\psi\gamma_0\tilde{\psi}$. The full action integral is therefore

$$S = \int |d^4x| (\det \mathsf{h})^{-1} \left(\tfrac{1}{2}\mathcal{F}\cdot\mathcal{F} + \mathcal{A}\cdot\mathcal{J}\right). \tag{13.275}$$

To find the field equations for electromagnetism we vary this integral with respect to the underlying dynamical variable A, with $\bar{\mathsf{h}}$ and $\mathcal{J}$ treated as external fields. The result is the equation

$$\nabla\cdot\left(\mathsf{h}\bar{\mathsf{h}}(\nabla\wedge A)\det\,(\mathsf{h})^{-1}\right) = J, \tag{13.276}$$

where

$$J = \det\,(\mathsf{h})^{-1}\mathsf{h}(\mathcal{J}). \tag{13.277}$$

Equation (13.276) combines with the identity $\nabla\wedge F = 0$ to form the full set of Maxwell equations in a gravitational background. Some insight into these equations is provided by performing a spacetime split and writing

$$\begin{aligned} \boldsymbol{E} + cI\boldsymbol{B} &= F, \\ \boldsymbol{D} + I\boldsymbol{H}/c &= \epsilon_0 \mathsf{h}\bar{\mathsf{h}}(F)\det\,(\mathsf{h})^{-1}, \end{aligned} \tag{13.278}$$

where we have temporarily included the factors of c and ϵ_0. In terms of these variables Maxwell's equations can be written in the familiar forms

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{B} &= 0, & \boldsymbol{\nabla}\cdot\boldsymbol{D} &= \rho, \\ \boldsymbol{\nabla}\wedge\boldsymbol{E} + I\frac{\partial \boldsymbol{B}}{\partial t} &= 0, & \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{\nabla}\cdot(I\boldsymbol{H}) &= -\boldsymbol{J}, \end{aligned} \tag{13.279}$$

where $J\gamma_0 = \rho + \boldsymbol{J}$. These forms of the equations illustrate how the $\det\,(\mathsf{h})^{-1}\mathsf{h}\bar{\mathsf{h}}$ is a generalized permittivity/permeability tensor, defining the properties of the space through which the electromagnetic field propagates. For example, the bending of light by the sun can be easily understood in terms of the properties of the dielectric defined by the $\bar{\mathsf{h}}$ field exterior to it.

So far, however, we have failed to achieve a covariant form of the Maxwell equations. We have, furthermore, failed to unite the separate equations into a single equation. To find a covariant equation, we simplify matters by ignoring torsion effects, so that we can write

$$\mathcal{D}\wedge\mathcal{F} = \bar{\mathsf{h}}(\nabla\wedge F) = 0. \tag{13.280}$$

Next, we use a double-duality transformation to write the left-hand side of equation (13.276) as

$$\begin{aligned}\nabla\cdot\big(\mathsf{h}(\mathcal{F})\det\,(\mathsf{h})^{-1}\big) &= I\nabla\wedge(I\mathsf{h}(\mathcal{F})\det\,(\mathsf{h})^{-1}) \\ &= I\nabla\wedge(\bar{\mathsf{h}}^{-1}(I\mathcal{F})) \\ &= I\bar{\mathsf{h}}^{-1}\big(\mathcal{D}\wedge(I\mathcal{F})\big).\end{aligned} \tag{13.281}$$

Equation (13.276) now becomes

$$\mathcal{D}\cdot\mathcal{F} = \mathcal{J}, \tag{13.282}$$

and equations (13.280) and (13.282) combine into the single covariant equation

$$\mathcal{D}\mathcal{F} = \mathcal{J}. \tag{13.283}$$

This achieves our objective. Equation (13.283) is manifestly covariant and generalises the free-field Maxwell equations to a gravitational background in an obvious and natural manner. In the presence of torsion an additional term appears in the covariant expression of the Maxwell equations. But in such circumstances the spin fields generating the torsion are likely to interact strongly with the electromagnetic field and swamp most interesting gravitational effects.

13.6 The structure of the Riemann tensor

The Riemann tensor $\mathcal{R}(B)$ contains a remarkable amount of algebraic structure, much of which is hidden in the tensor calculus approach. Again, we assume that there is no torsion present, so that the second field equation reduces to (13.234). Writing $\mathcal{A} = \bar{\mathsf{h}}(A)$ we have

$$\mathcal{D}\wedge\mathcal{A} = \bar{\mathsf{h}}(\nabla\wedge A), \tag{13.284}$$

so

$$\mathcal{D}\wedge(\mathcal{D}\wedge\mathcal{A}) = \bar{\mathsf{h}}(\nabla\wedge\nabla\wedge A) = 0. \tag{13.285}$$

It follows that

$$\begin{aligned}g^\mu\wedge\big(\mathcal{D}_\mu\big(g^\nu\wedge(\mathcal{D}_\nu\mathcal{A})\big)\big) &= g^\mu\wedge g^\nu\wedge\big(\mathcal{D}_\mu\mathcal{D}_\nu\mathcal{A}\big) \\ &= \tfrac{1}{2}g^\mu\wedge g^\nu\wedge\big([\mathcal{D}_\mu,\mathcal{D}_\nu]\mathcal{A}\big) \\ &= g^\mu\wedge g^\nu\wedge(\mathsf{R}_{\mu\nu}\times\mathcal{A}).\end{aligned} \tag{13.286}$$

So, for any multivector $\mathcal{M}$,

$$\partial_a\wedge\partial_b\wedge(\mathcal{R}(a\wedge b)\times\mathcal{M}) = 0, \tag{13.287}$$

which is a covariant equation.

To analyse equation (13.287) further we set $\mathcal{M}$ equal to the vector c, and protract with ∂_c to form

$$\partial_c \wedge \partial_a \wedge \partial_b \wedge \big(\mathcal{R}(a \wedge b) \times c\big) = -2\partial_a \wedge \partial_b \wedge \mathcal{R}(a \wedge b) = 0. \tag{13.288}$$

Now forming the inner product with c we obtain

$$2\partial_a \wedge \mathcal{R}(a \wedge c) + \partial_a \wedge \partial_b \wedge \big(\mathcal{R}(a \wedge b) \times c\big) = 0, \tag{13.289}$$

so that we are are left with the compact identity

$$\partial_a \wedge \mathcal{R}(a \wedge b) = 0. \tag{13.290}$$

This summarises *all* of the symmetries of $\mathcal{R}(B)$ in the case of zero torsion. Equation (13.290) says that the trivector $\partial_a \wedge \mathcal{R}(a \wedge b)$ vanishes for all values of the vector b, so gives a set of $4 \times 4 = 16$ equations. These reduce the number of independent degrees of freedom in $\mathcal{R}(B)$ from 36 to 20, the expected number for general relativity. Contracting equation (13.290) we obtain

$$\partial_b \cdot \big(\partial_a \wedge \mathcal{R}(a \wedge b)\big) = \partial_a \wedge \mathcal{R}(a) = 0, \tag{13.291}$$

which shows that the Ricci tensor $\mathcal{R}(a)$ is symmetric. The same is therefore true of the Einstein tensor. In the absence of any spin-torsion interactions, the matter energy-momentum tensor must also be symmetric, as is the case for electromagnetism and the relativistic fluid. The covariant Riemann tensor satisfies the further useful identities,

$$\begin{aligned} \partial_c \wedge \big(a \cdot \mathcal{R}(c \wedge b)\big) &= \mathcal{R}(a \wedge b), \\ (B \cdot \partial_a) \cdot \mathcal{R}(a \wedge b) &= -\partial_a \, B \cdot \mathcal{R}(a \wedge b). \end{aligned} \tag{13.292}$$

It follows that

$$\partial_b \wedge \big((B \cdot \partial_a) \cdot \mathcal{R}(a \wedge b)\big) = -2\mathcal{R}(B) = -\partial_b \wedge \partial_a \langle B\mathcal{R}(a \wedge b)\rangle. \tag{13.293}$$

The Riemann tensor is therefore also symmetric,

$$B_1 \cdot \mathcal{R}(B_2) = B_2 \cdot \mathcal{R}(B_1). \tag{13.294}$$

That is, $\mathcal{R}(B) = \bar{\mathcal{R}}(B)$.

13.6.1 The Weyl tensor

The structure of the Riemann tensor is more clearly seen by separating out the matter content, as contained in the Ricci tensor. Since the contraction of $\mathcal{R}(a \wedge b)$ results in the Ricci tensor $\mathcal{R}(a)$, we expect that $\mathcal{R}(a \wedge b)$ will contain a term in $\mathcal{R}(a) \wedge b$. This must be matched with a term in $a \wedge \mathcal{R}(b)$, since it is only the sum of these that is a function of $a \wedge b$. Contracting this sum we obtain

$$\begin{aligned} \partial_a \cdot \big(\mathcal{R}(a) \wedge b + a \wedge \mathcal{R}(b)\big) &= b\mathcal{R} - \mathcal{R}(b) + 4\mathcal{R}(b) - \mathcal{R}(b) \\ &= 2\mathcal{R}(b) + b\mathcal{R}, \end{aligned} \tag{13.295}$$

and it follows that

$$\partial_a \cdot \Big(\tfrac{1}{2}\big(\mathcal{R}(a)\wedge b + a\wedge\mathcal{R}(b)\big) - \tfrac{1}{6} a\wedge b\mathcal{R} \Big) = \mathcal{R}(b). \tag{13.296}$$

We can therefore write

$$\mathcal{R}(a\wedge b) = \mathcal{W}(a\wedge b) + \tfrac{1}{2}\big(\mathcal{R}(a)\wedge b + a\wedge\mathcal{R}(b)\big) - \tfrac{1}{6} a\wedge b\mathcal{R}, \tag{13.297}$$

where $\mathcal{W}(B)$ is the *Weyl tensor*.

From its definition the Weyl tensor must satisfy

$$\partial_a \cdot \mathcal{W}(a\wedge b) = 0. \tag{13.298}$$

As the Ricci tensor is symmetric, we also have

$$\partial_a \wedge \big(\tfrac{1}{2}\left(\mathcal{R}(a)\wedge b + a\wedge\mathcal{R}(b)\right) - \tfrac{1}{6} a\wedge b\mathcal{R} \big) = 0, \tag{13.299}$$

so the Weyl tensor also satisfies

$$\partial_a \wedge \mathcal{W}(a) = 0. \tag{13.300}$$

Equations (13.298) and (13.300) combine into the single equation

$$\partial_a \mathcal{W}(a\wedge b) = 0. \tag{13.301}$$

This compact equation is unique to the geometric algebra formulation, as it involves the geometric product. To study the consequences of equation (13.301) it is useful to introduce the $\{\gamma_\mu\}$ frame and write the four equations for b equalling each of the γ_μ vectors as

$$\begin{aligned}
\boldsymbol{\sigma}_1\mathcal{W}(\boldsymbol{\sigma}_1) + \boldsymbol{\sigma}_2\mathcal{W}(\boldsymbol{\sigma}_2) + \boldsymbol{\sigma}_3\mathcal{W}(\boldsymbol{\sigma}_3) &= 0,\\
\boldsymbol{\sigma}_1\mathcal{W}(\boldsymbol{\sigma}_1) - I\boldsymbol{\sigma}_2\mathcal{W}(I\boldsymbol{\sigma}_2) - I\boldsymbol{\sigma}_3\mathcal{W}(I\boldsymbol{\sigma}_3) &= 0,\\
-I\boldsymbol{\sigma}_1\mathcal{W}(I\boldsymbol{\sigma}_1) + \boldsymbol{\sigma}_2\mathcal{W}(\boldsymbol{\sigma}_2) - I\boldsymbol{\sigma}_3\mathcal{W}(I\boldsymbol{\sigma}_3) &= 0,\\
-I\boldsymbol{\sigma}_1\mathcal{W}(I\boldsymbol{\sigma}_1) - I\boldsymbol{\sigma}_2\mathcal{W}(I\boldsymbol{\sigma}_2) + \boldsymbol{\sigma}_3\mathcal{W}(\boldsymbol{\sigma}_3) &= 0.
\end{aligned} \tag{13.302}$$

Summing the final three equations, and employing the first, produces

$$I\boldsymbol{\sigma}_k\mathcal{W}(I\boldsymbol{\sigma}_k) = 0. \tag{13.303}$$

Substituting this into each of the final three equations produces

$$\mathcal{W}(I\boldsymbol{\sigma}_k) = I\mathcal{W}(\boldsymbol{\sigma}_k), \tag{13.304}$$

and it follows that the Weyl tensor satisfies

$$\mathcal{W}(IB) = I\mathcal{W}(B). \tag{13.305}$$

This says that the Weyl tensor is *self-dual*. In the two-spinor formalism of Penrose and Rindler the duality of the Weyl tensor is expressed in terms of a complex formulation. The spacetime algebra shows that this complex structure arises geometrically through the properties of the pseudoscalar.

Given the self-duality of the Weyl tensor, the remaining content of equation (13.302) is summarised by

$$\boldsymbol{\sigma}_k \mathcal{W}(\boldsymbol{\sigma}_k) = 0. \tag{13.306}$$

This equation says that, viewed as a three-dimensional complex linear function, $\mathcal{W}(B)$ is symmetric and traceless. This gives $\mathcal{W}(B)$ five complex, or ten real degrees of freedom. The gauge-invariant information is held in the complex eigenvalues of $\mathcal{W}(B)$, since these are invariant under rotations. As these must sum to zero, only two are independent. This leaves a set of four real intrinsic scalar quantities.

Overall, $\mathcal{R}(B)$ has 20 degrees of freedom, six of which are contained in the freedom to perform arbitrary local rotations. Of the remaining 14 physical degrees of freedom, four are contained in the two complex eigenvalues of $\mathcal{W}(B)$, and a further four in the real eigenvalues of the matter stress-energy tensor. The six remaining physical degrees of freedom determine the rotation between the frame that diagonalises $\mathcal{G}(a)$ and the frame that diagonalises $\mathcal{W}(B)$. This identification of the physical degrees of freedom contained in $\mathcal{R}(B)$ is physically very revealing and extremely useful in guiding solution strategies.

13.6.2 The Bianchi identities

Further information about the Riemann tensor is contained in the Bianchi identities. These follow immediately from the Jacobi identity in the form

$$[\mathcal{D}_\alpha, [\mathcal{D}_\beta, \mathcal{D}_\gamma]]\mathcal{A} + \text{cyclic permutations} = 0. \tag{13.307}$$

It follows that

$$\mathcal{D}_\alpha \mathsf{R}_{\beta\gamma} + \text{cyclic permutations} = 0, \tag{13.308}$$

which we need to express as a fully covariant relation. We start by forming the adjoint relation,

$$\partial_a \wedge \partial_b \wedge \partial_c \langle \big(a\cdot\nabla R(b\wedge c) + \Omega(a)\times R(b\wedge c)\big)B\rangle = 0, \tag{13.309}$$

which simplifies to

$$\nabla\wedge\bar{\mathsf{R}}(B) - \partial_a\wedge\bar{\mathsf{R}}\big(\Omega(a)\times B\big) = 0, \tag{13.310}$$

where B is a constant bivector. To make further progress we again assume that the torsion vanishes. The Riemann tensor is then symmetric, so

$$\bar{\mathsf{R}}(B) = \bar{\mathsf{h}}^{-1}\mathsf{R}\mathsf{h}(B) = \bar{\mathsf{h}}^{-1}\mathcal{R}(B). \tag{13.311}$$

We can therefore write

$$\nabla\wedge\big(\bar{\mathsf{h}}^{-1}\mathcal{R}(B)\big) - \partial_a\wedge\bar{\mathsf{h}}^{-1}\mathcal{R}\big(\Omega(a)\times B\big) = 0. \tag{13.312}$$

Now acting on this equation with $\bar{\mathsf{h}}$ and using equation (13.235), we establish the covariant result

$$\mathcal{D}\wedge\mathcal{R}(B) - \partial_a\wedge\mathcal{R}(\omega(a)\times B) = 0. \tag{13.313}$$

This result takes a more natural form when B becomes an arbitrary function of position, and we write the Bianchi identity as

$$\partial_a\wedge\big(a\cdot\mathcal{D}\mathcal{R}(B) - \mathcal{R}(a\cdot\mathcal{D}B)\big) = 0. \tag{13.314}$$

We can extend the overdot notation of section 11.1 in the natural manner to write equation (13.314) as

$$\dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(B) = 0. \tag{13.315}$$

This is a highly compact, elegant expression of the Bianchi identity, though it is often easier to use the more explicit form of equation (13.314).

The contracted Bianchi identity is obtained from

$$\begin{aligned}(\partial_a\wedge\partial_b)\cdot\big(\dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(a\wedge b)\big) &= \partial_a\cdot\big(\dot{\mathcal{R}}(a\wedge\dot{\mathcal{D}}) + \dot{\mathcal{D}}\dot{\mathcal{R}}(a)\big)\\ &= 2\dot{\mathcal{R}}(\dot{\mathcal{D}}) - \mathcal{D}\mathcal{R},\end{aligned} \tag{13.316}$$

from which we find

$$\dot{\mathcal{G}}(\dot{\mathcal{D}}) = 0. \tag{13.317}$$

The adjoint form of this equation is sometimes more useful:

$$\dot{\mathcal{D}}\cdot\dot{\mathcal{G}}(a) = \mathcal{D}\cdot\mathcal{G}(a) - \partial_b\cdot\mathcal{G}(b\cdot\mathcal{D}\,a) = 0. \tag{13.318}$$

This is the covariant expression of conservation of the Einstein tensor. It follows that the total matter energy-momentum tensor must satisfy the same relation. With the gravitational interaction turned off, the free-field (or flat-space) energy-momentum tensor must be symmetric and divergence-free. This is the case for the functional electromagnetic and fluid energy-momentum tensors. This is not true of the Dirac theory, where the presence of spin alters many of the preceding results and distorts much of the elegant structure of pure general relativity.

The covariant conservation equation (13.318) does not give rise to conserved vector currents, and hence conserved scalars, unless a further symmetry is present in the gravitational fields. In this case one can construct a Killing vector $\mathcal{K}$ satisfying equation (13.259). This is sufficient to prove that

$$\mathcal{G}(\partial_a)\cdot(a\mathcal{D}\mathcal{K}) = 0, \tag{13.319}$$

which holds because $\mathcal{G}(a)$ is a symmetric tensor. It follows that

$$\mathcal{D}\cdot\big(\mathcal{G}(\mathcal{K})\big) = \dot{\mathcal{D}}\cdot\dot{\mathcal{G}}(\mathcal{K}) - \partial_a\cdot\mathcal{G}(a\cdot\mathcal{D}\mathcal{K}) = 0, \tag{13.320}$$

which yields a covariantly conserved vector. This can be converted to a spacetime current and hence to a conserved scalar quantity.

13.7 Notes

Lagrangian field theory is discussed in many textbooks, particularly those that go on to treat quantum field theory. The texts by Itzykson & Zuber (1980) and Bjorken & Drell (1964) are again recommended, as are the book by Cheng & Li (1984) and the set of lecture notes by Coleman (1985). The history of gauge theories in the twentieth century is described in the set of collected papers edited by Taylor (2001). The use of the multivector derivative in analysing field Lagrangians was introduced in the paper by Lasenby, Doran & Gull (1993a), and further refinements are contained in the thesis by Doran (1994).

The discovery that gravity could be treated as a gauge theory was made initially by Utiyama (1956) and Kibble (1961). An attempt at a quantum treatment along the lines suggested by Kibble was made by Feynman and is contained in the *Feynman Lectures on Gravitation* (Feynman, Morningo & Wagner, 1995). The application of spacetime algebra in the context of classical general relativity was promoted by Hestenes in the book *Space-Time Algebra* (1966) and the paper 'Curvature calculations with spacetime algebra' (1986). Many other authors have followed this route and a considerable literature now exists on applications of Clifford algebra in general relativity. Rather than attempt to list all of these, and run the risk of offending anyone we miss out, we recommend searching the main pre-print archives on the keyword 'Clifford'.

The particular combination of the gauge treatment of gravity and the spacetime algebra developed here was first presented in full in the paper 'Gravity, gauge theories and geometric algebra', by Lasenby, Doran & Gull (1998). This contains an extensive list of references and we refer the reader there for further material. The form of the field equations in the presence of torsion is discussed in Doran et al.(1998). Readers of these papers, and the preceding chapter, will notice that the notation and conventions for this subject have not yet settled down. We believe that this chapter represents an advance over previous work, but doubtless there is still room for improvement. While it has not been employed in this chapter, we do recommend the underbar/overbar notation for linear functions in hand-written work. This helps keep track of the form of various objects, and avoids the problem of using different fonts to distinguish objects. Unfortunately, this notation tends to look too cluttered when typeset, which is why underbars are not employed in this book.

13.8 Exercises

13.1 The physical energy-momentum tensor for free-field electromagnetism is defined by

$$\mathsf{T}_{em}(a) = -\tfrac{1}{2}FaF.$$

Prove that each of $\mathsf{T}_{em}(x)$, $\mathsf{T}_{em}(a)$, $\mathsf{T}_{em}(xax)$ and $\mathsf{T}_{em}(B\cdot x)$ is conserved. How many independent conserved constants can one construct from these? How does this relate to the dimension of the spacetime conformal group?

13.2 Prove that, in a space of dimension n,

$$\nabla\left(\frac{1+xa}{(1+2x\cdot a+a^2x^2)^{n/2}}\right)=0,$$

where a is an arbitrary vector.

13.3 The field ψ satisfies the minimally-coupled Dirac equation. Prove that

$$\begin{aligned}\nabla\cdot(\psi\gamma_1\tilde{\psi}) &= 2eA\cdot(\psi\gamma_2\tilde{\psi}),\\ \nabla\cdot(\psi\gamma_2\tilde{\psi}) &= -2eA\cdot(\psi\gamma_1\tilde{\psi}).\end{aligned}$$

Can you derive these relations from a transformation applied to the Dirac Lagrangian?

13.4 The coupled Maxwell–Dirac Lagrangian is defined by

$$\mathcal{L}=\langle\nabla\psi I\gamma_3\tilde{\psi}-eA\psi\gamma_0\tilde{\psi}-m\psi\tilde{\psi}\rangle.$$

Find the canonical energy-momentum tensor. Prove that $\mathcal{L}$ is unchanged in form by the transformations

$$\psi(x)\mapsto R\psi(x'),\qquad A(x)\mapsto RA(x')\tilde{R},$$

where $x'=\tilde{R}xR$ and R is a constant rotor. Find the conserved tensor conjugate to this transformation.

13.5 The gravitational field strength is defined in terms of the bivector connection Ω_μ by

$$\mathsf{R}_{\mu\nu}=\partial_\mu\Omega_\nu-\partial_\nu\Omega_\mu+\Omega_\mu\times\Omega_\nu.$$

Verify that this vanishes if

$$\Omega_\mu=-2\partial_\mu R\tilde{R},$$

where R is a spacetime rotor.

13.6 Prove that, for non-vanishing spin, the $\omega(a)$ field is given by

$$\omega(a)=H(a)-\frac{1}{2}a\cdot(\partial_b\wedge H(b))+\kappa\mathcal{S}(a)-\frac{3}{2}\kappa a\cdot\big(\partial_b\wedge\mathcal{S}(b)\big).$$

13.7 Prove that, in the case of zero torsion, timelike paths which minimise the proper time

$$S=\int d\lambda\,\big(\mathsf{h}^{-1}(x')\cdot\mathsf{h}^{-1}(x')\big)^{1/2}$$

satisfy the geodesic equation $v\cdot\mathcal{D}\,v=0$, where $v=\mathsf{h}^{-1}(\dot{x})$ and $v^2=1$.