Final Report - Research Internship

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1 Introduction

2 Modelling of the swimmer as a control problem

We restrict ourselves to considering the swimmer $\mathrm{SPR4}$ proposed in [1]. Let (S_1, S_2, S_3, S_4) be a regular reference tetrahedron centered at $c \in \mathbb{R}^3$ such that $\operatorname{dist}(c, S_i) = 1$ for all $i \in \mathbb{N}_4$. Then the swimmer consists of four balls $(B_i)_{i \in \mathbb{N}_4}$ of $\mathbb{R}^3$ centered at $b_i \in \mathbb{R}^3$, all of radius $a > 0$, such that the ball B_i can move along the ray starting at c and passing through S_i. This reflects the situation where the balls are linked together by think jacks that are able to elongate. However, the viscous resistance of these jacks is neglected and therefore the fluid is assumed to permeate the entire open set $\mathbb{R}^3 \setminus \bigcup_{i=1}^4 \overline{B_i}$. The balls do not rotate around their arms which implies that the shape of the swimmer is completely determined by the four lengths $\xi_1, \xi_2, \xi_3, \xi_4$ of its arms, measured from the c to the center of each ball b_i. However, there are no restrictions for the rotation of the swimmer around the center c, i.e. for fixed arm lengths, the swimmer is considered to be a rigid body in a Stokesian fluid. Hence, the geometrical configuration of the swimmer can be described by two sets of variables:

(i) The vector of *shape variables* $\xi := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{M} := (\sqrt{\frac{3}{2}}a, +\infty)^4 \subseteq \mathbb{R}_+^4$, where the lower bound in the open intervals is chosen such that the balls cannot overlap.

(ii) The vector of *position variables* $p = (c, R) \in \mathcal{P} := \mathbb{R}^3 \times \mathrm{SO}(3)$.

To be more precise, we consider the reference tetrahedron convexly spanned by the four unit vectors $z_1 := (2\sqrt{2}/3, 0, -1/3)$, $z_2 := (-\sqrt{2}/3, -\sqrt{2/3}, -1/3)$, $z_3 := (-\sqrt{2}/3, \sqrt{2/3}, -1/3)$ and $z_4 := (0, 0, 1)$. Position and orientation in $\mathbb{R}^3$ are then described by the coordinates of the center $c \in \mathbb{R}^3$ and the rotation $R \in SO(3)$ of the swimmer with respect to the reference orientation induced by the reference tetrahedron. Thus, we set $b_i := c + \xi_i R z_i$ for the center of the ball B_i.

The swimmer is completely described by the parameters $(\xi, p) \in \mathcal{M} \times \mathcal{P}$. Indeed, if we denote by B_a the ball in $\mathbb{R}^3$ of radius a centered at the origin, then for any $r \in \partial B_a$, the position of the current point on the i-th sphere of the swimmer in the state (ξ, p) is given, for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$, by the function

$$r_i(\xi, p, r) := c + R(\xi_i z_i + r). \tag{1}$$

Note that the functions $(r_i)_{i \in \mathbb{N}_4}$ are analytic in $\mathcal{M} \times \mathcal{P}$ and thus we can use them to calculate the instantaneous velocity on the i-th sphere B_i, which for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$ and every $i \in \mathbb{N}_4$ is given by

$$u_i(\xi, p, r) = \dot{c} + \omega \times (\xi z_i + r) + R z_i \dot{\xi}_i, \tag{2}$$

where ω is the axial vector associated with the skew matrix $\dot{R}R$.

In [1] it is shown that the system $\mathrm{SPR4}$, i.e. both the shape ξ and the position p, is controllable only using the rate of change $\dot{\xi}$ of the shape. To do so, we have to understand how p changes when we vary $\dot{\xi}$. To that and, the assumptions of *self-propulsion* and negligible inertia of the swimmer are made. They imply that the total viscous force and torque exerted by the surrounding fluid on the swimmer must vanish. More precisely, the system can be written as

$$\dot{p} = F(R, \xi)\dot{\xi} := \left(\frac{F_c(R, \xi)}{F_\theta(R, \xi)} \right) \dot{\xi}, \tag{3}$$

where $\dot{c} = F_c(R, \xi)\dot{\xi}$ and $\dot{R} = F_\theta(R, \xi)\dot{\xi}$.

In preparation for what follows, let us note that we have $F(R, \xi) \in \mathcal{L}(\mathbb{R}^4, T_p\mathcal{P})$ for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$, where $\mathcal{L}(V, W)$ denotes the linear maps between two vector spaces V and W. We quickly recall the fact that at any point $R \in \mathrm{SO}(3)$ we have

$$T_R \mathrm{SO}(3) = R^* \mathrm{Skew}_3(\mathbb{R}) = \{RM \mid M \in \mathrm{Skew}_3(\mathbb{R})\}, \tag{4}$$

where $\mathrm{Skew}_n(\mathbb{R})$ denotes the set of skew-symmetric real matrices of size $n \times n$. Hence, we have in particular that for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$

$$F_c(R, \xi) \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3) \text{ and } F_\theta(R, \xi) \in \mathcal{L}(\mathbb{R}^4, R^* \mathrm{Skew}_3(\mathbb{R})) \tag{5}$$

and therefore we can express both $F_c(R, \xi)$ and $F_\theta(R, \xi)$ as real matrices of size 3×4 once we have chosen a basis for the corresponding tangent spaces.

In analogy to [2], it is important to note here that the control system F is independent of c due to the translational invariance of the Stokes equations. However, the translational invariance is not the only symmetry that SPR4 satisfies. The goal of the following section is to examine the structure of the control system F in consequence of the symmetries it must satisfy being driven by the Stokes equations.

3 Symmetries

For any initial condition $p_0 = (c_0, R_0) \in \mathcal{P}$ and any control curve $\xi : I \subseteq \mathbb{R} \to \mathcal{M}$, with I a neighborhood of zero, we denote $\gamma(p_0, \xi) : I \to \mathcal{P}$ the solution associated to the dynamical system

$$\dot{p} = F(R, \xi)\dot{\xi}, \quad p(0) := p_0, \tag{6}$$

as well as $\gamma_c(c_0, R_0, \xi)$ and $\gamma_\theta(c_0, R_0, \xi)$ its projections on $\mathbb{R}^3$ and $\mathrm{SO}(3)$, respectively, such that for any $t \in I$

$$\dot{\gamma}(c_0, R_0, \xi)(t) = F(\gamma_\theta(c_0, R_0, \xi)(t), \xi(t))\dot{\xi}(t), \tag{7}$$

and similarly for the projections $\gamma_c(c_0, R_0, \xi)$ and $\gamma_\theta(c_0, R_0, \xi)$.

3.1 Rotational invariance

Rotational invariance of the Stokes equations expresses the fact that the solution of the dynamical system (6) is invariant under rotations, i.e. that for any rotation $R \in \mathrm{SO}(3)$ we have for the spatial part of the solution

$$\gamma_c(c_0, RR_0, \xi)(t) = R\gamma_c(c_0, R_0, \xi)(t) + (I - R)c_0 \tag{8}$$

and for the angular part of the solution

$$\gamma_\theta(c_0, RR_0, \xi)(t) = R\gamma_\theta(c_0, R_0, \xi)(t) \tag{9}$$

at any point in time $t \in I$. Eventually, we can rigorously state the following symmetry property of the control system (6) with respect to rotations:

Condition 1 (Rotational invariance)**.** *If $\gamma(c_0, R_0, \xi)$ is a solution of the control system (6) then so is $\gamma(c_0, RR_0, \xi)$ and (8) and (9) hold.*

Remark. To follow the reasoning of [2], the symmetry relations satisfied by SPR4 are stated as hypotheses on the solution γ. In so doing, the results work for any control system of the form (3) and satisfying the hypotheses we state, independently of these hypotheses being guaranteed by the invariance of the Stokes equations under a certain group of transformations.

We then have

Proposition 1. *Let $\xi_0 := \xi(0) \in \mathcal{M}$ denote the initial state of the control parameters and by $T_\xi\mathcal{M}$ the tangent space of $\mathcal{M}$ at ξ. If the control system (6) is invariant under rotations and for every $\xi \in \mathcal{M}$ it holds that $T_\xi\mathcal{M} \simeq \mathbb{R}^4$, then*

$$F_c(R, \xi) = RF_c(\xi) \text{ and } F_\theta(R, \xi) = RF_\theta(R, \xi), \tag{10}$$

for every $(R, \xi) \in \mathrm{SO}(3) \times \mathcal{M}$, where $F_c(\xi) := F_c(I, \xi)$ and $F_\theta(\xi) := F_\theta(I, \xi)$.

Proof. On one hand, we have by definition of the dynamical system (6) that

$$\dot{\gamma}_c(c_0, R, \xi) = F_c(\gamma_\theta(c_0, R, \xi), \xi)\dot{\xi}. \tag{11}$$

On the other hand, using equation (8) and once more the definition of the dynamical system (3), we obtain

$$\dot{\gamma}_c(c_0, R\xi) = R\dot{\gamma}_c(c_0, I, \xi) = RF_c(\gamma_\theta(c_0, I, \xi), \xi)\dot{\xi}. \tag{12}$$

Therefore, $F_c(\gamma_\theta(c_0, R, \xi), \xi)\dot{\xi} = RF_c(\gamma_\theta(c_0, I, \xi), \xi)\dot{\xi}$ for every $R \in \mathrm{SO}(3)$. Since $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$, evaluation of the preceding expression at $t = 0$ yields $F_c(R, \xi_0) = RF_c(I, \xi_0)$, as desired. The proof for F_θ is completely analogous. $\qquad\square$

3.2 Permutation of two arms

In this section, we investigate the effect of a swap of two arms on the generic solution of the dynamical system (6). To that end, let $P_{ij} \in M_{4\times4}(\mathbb{R})$ denote the permutation matrix that interchanges the i-th and j-th index, which corresponds to the swap of the arms i and j, denoted by $(||i \leftrightsquigarrow ||j)$, if applied to the shape space $\mathcal{M}$. In addition, let S_{ij} denote the reflection of $\mathbb{R}^3$ sending arm $||i$ onto arm $||j$ in the reference orientation I. Geometrical inspection of the reference tetrahedron shows that S_{ij} is always a reflection at a plane containing the remaining arms $||k$ and $||l$.

Before we formulate the symmetry conditions for the interchanging of two arms, we recall some results about how rotations behave under reflections. So far, we have only regarded the orientation of SPR4 as a rotation matrix in $\mathrm{SO}(3)$. However, by Euler's rotation theorem to every such rotation matrix $R \in \mathrm{SO}(3)$ there exists a corresponding rotation vector $\omega \in \mathbb{R}^3$ which is collinear to the unique axis of rotation defined by R, i.e. ω is an eigenvector associated to the eigenvalue 1 of R. It's length is then given by the angle of rotation around this axis. The rotation vector ω is then directly related to the rotation matrix R via the map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, where $\mathfrak{so}(3) = T_I\,\mathrm{SO}(3) = \mathrm{Skew}_3(\mathbb{R})$ denotes the Lie algebra over $\mathrm{SO}(3)$, which we will illustrate in the following paragraphs.

It is clear that $\dim \mathrm{Skew}_3(\mathbb{R}) = 3$. In particular, if $R_1(\theta), R_2(\theta)$ and $R_3(\theta)$ denote the simple rotations around the $\hat{e}_1$-, $\hat{e}_2$- and $\hat{e}_3$-axis, where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the canonical basis vectors of $\mathbb{R}^3$, then the matrices

$$L_1 = \frac{d}{d\theta}R_1(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{13}$$

$$L_2 = \frac{d}{d\theta}R_2(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{14}$$

$$L_3 = \frac{d}{d\theta}R_3(\theta)_{|\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{15}$$

form a basis of $\mathfrak{so}(3)$ consisting of the infinitesimal rotations around the corresponding axes. If we now write $\mathbf{L} := (L_1, L_2, L_3)^T$ and allow the slight abuse of notation

$$\omega \cdot \mathbf{L} = \omega_1 L_1 + \omega_2 L_2 + \omega_3 L_3, \tag{16}$$

we find by direct computation that $\exp(\omega \cdot \mathbf{L}) = R$. This relationship allows us to formulate the behavior of the orientation of SPR4 under reflection and thus under permutation of two arms as we shall see later. Indeed, we have

Lemma 1. *For any orientation of a rigid body characterized by $R \in \mathrm{SO}(3)$, the orientation of its mirror image under a reflection S is characterized by*

$$\tilde{R} = SRS. \tag{17}$$

Proof. Let us first consider the simple case where S is the reflection of the $\hat{e}_1$-axis. Let ω and $\tilde{\omega}$ be the rotation vectors corresponding to R and $\tilde{R}$, respectively. They are related by

$\tilde{\omega} = -S\omega$, where the gain of the minus sign stems from the fact that rotation vectors are in fact pseudovectors. In other words, we not only reflect the axis of rotation but we also reverse the sense of rotation around the axis. It follows then from direct computation that

$$\tilde{\omega} \cdot \mathbf{L} = (-S\omega) \cdot \mathbf{L} = S(\omega \cdot \mathbf{L})S \tag{18}$$

and thus we have

$$\tilde{R} = \exp(\tilde{\omega} \cdot \mathbf{L}) = \exp(S(\omega \cdot \mathbf{L})S) = SRS, \tag{19}$$

as $S^{-1} = S$.

If now S' is an arbitrary reflection, we always find a rotation $Q \in \mathrm{SO}(3)$ such that $S' = QSQ^T$. Moreover, for any rotation $R' \in \mathrm{SO}(3)$ we find another $R \in \mathrm{SO}(3)$ such that $R' = QRQ^T$. In particular, we have

$$\tilde{R}' = Q\tilde{R}Q^T = QSRSQ^T = S'R'S'^T, \tag{20}$$

as desired. $\qquad\square$

With this lemma at hand, we can now finally state the following

Condition 2 (Swap $(\|i \leftrightsquigarrow \|j)$)**.** *Let the initial position be* $p_0 := (c_0, I)$. *If* $\gamma(c_0, I, P_{ij}\xi)$ *is a solution of the control system (3), then so is* $\gamma(S_{ij}c_0, I, \xi)$ *and the following relations hold*

$$\gamma_c(c_0, I, P_{ij}\xi) = S_{ij}\gamma_c(S_{ij}c_0, I, \xi) \tag{21}$$

and

$$\gamma_\theta(c_0, I, P_{ij}\xi) = S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{22}$$

To avoid chaos in our notation, we treat the the spatial and angular parts now separately. For the spatial part, we find

Proposition 2. *If the control system (6) is invariant under the swap* $(\|i \leftrightsquigarrow \|j)$ *and* $T_\xi\mathcal{M} \simeq \mathbb{R}^4$ *for all* $\xi \in \mathcal{M}$, *then for all* $\xi \in \mathcal{M}$

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij}. \tag{23}$$

Proof. Let $\gamma_c(c_0, R_0, P_{ij}\xi)$ be the spatial part of any solution of the control problem (3). The hypothesis of rotational invariance, i.e. (8), implies that

$$\gamma_c(c_0, R_0, P_{ij}\xi) = R_0\gamma_c(c_0, I, P_{ij}\xi) + (I - R_0)c_0. \tag{24}$$

From condition 2, we then get

$$\gamma_c(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\gamma_c(S_{ij}c_0, I, \xi) + (I - R_0)c_0. \tag{25}$$

As both $\gamma_c(c_0, R_0, P_{ij}\xi)$ and $\gamma_c(S_{ij}c_0, I, \xi)$ are spatial parts of solutions of the control system (6), we have on one hand using Proposition 1

$$\dot{\gamma}_c(c_0, R_0, P_{ij}\xi) = \gamma_\theta(c_0, R_0, P_{ij}\xi)F_c(P_{ij}\xi)P_{ij}\dot{\xi}, \tag{26}$$

and on the other hand using (25) and once more (6) together with Proposition 1

$$\dot{\gamma}_c(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\dot{\gamma}_c(S_{ij}c_0, I, \xi) = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)F_c(\xi)\dot{\xi}. \tag{27}$$

Equating (26) and (27) at $t = 0$ yields $F_c(P_{ij}\xi_0) = S_{ij}F_c(\xi_0)P_{ij}$, since by hypothesis $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$. As ξ_0 was arbitrary, we conclude. $\qquad\square$

For the angular part, we first have to choose a basis and fix some notation. Naturally, we choose the canonical basis $\mathcal{E} := (e_1, e_2, e_3, e_4)$ for $\mathbb{R}^4$. For $\mathfrak{so}(3)$, we choose the basis $\mathcal{L} := (L_1, L_2, L_3)$, where the matrices L_i are defined in (13) - (15). Then we denote the the matrix representing an arbitrary linear map $T : V \to W$ between two vector spaces V and W with respect to two bases $\mathcal{B}$ and $\mathcal{B}'$ by $[T]_{\mathcal{B}}^{\mathcal{B}'}$. Let S be an arbitrary reflection at a plane in $\mathbb{R}^3$ and define the adjoint isomorphism $T_S : \mathfrak{so}(3) \to \mathfrak{so}(3)$ by $M \mapsto SMS$. Then the calculations leading to (18) show that in fact we have $[T_S]_{\mathcal{L}}^{\mathcal{L}} = -S$. This leads to the following

Proposition 3. *If the control system (6) is invariant under the swap ($\|i \rightsquigarrow \|j$) and $T_\xi \mathcal{M} \simeq \mathbb{R}^4$ for all $\xi \in \mathcal{M}$, then for all $\xi \in \mathcal{M}$*

$$[F_\theta(P_{ij}\xi)]_{\mathcal{E}}^{\mathcal{L}} = -S_{ij}[F_\theta(\xi)]_{\mathcal{E}}^{\mathcal{L}}P_{ij}. \tag{28}$$

Proof. Let $\gamma_\theta(c_0, R_0, P_{ij}\xi)$ be the angular part of any solution of the control problem (6). By the rotational invariance hypothesis, i.e. (9), we have

$$\gamma_\theta(c_0, R_0, P_{ij}\xi) = R_0\gamma(c_0, I, \xi). \tag{29}$$

Then Condition 2 implies that

$$\gamma_\theta(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{30}$$

Since both $\gamma_\theta(c_0, R_0, P_{ij}\xi)$ and $\gamma_\theta(S_{ij}c_0, I, \xi)$ are the angular parts of solutions of the control problem (6), we obtain with Proposition 1 on one hand

$$\dot{\gamma}_\theta(c_0, R_0, P_{ij}\xi) = \gamma_\theta(c_0, R_0, P_{ij}\xi)F_\theta(P_{ij}\xi)P_{ij}\dot{\xi} \tag{31}$$

and on the other hand using (30) and once more Proposition 1

$$\dot{\gamma}_\theta(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\dot{\gamma}_\theta(S_{ij}c_0, I, \xi)S_{ij} = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)F_\theta(\xi)\dot{\xi}S_{ij}. \tag{32}$$

Imposing equality of (31) and (32) at $t = 0$ yields

$$F_\theta(P_{ij}\xi_0)P_{ij}\dot{\xi}(0) = S_{ij}F_\theta(\xi_0)\dot{\xi}(0)S_{ij}. \tag{33}$$

By choice of the canonical basis for $\mathbb{R}^4$ we clearly have $[P_{ij}]_{\mathcal{E}}^{\mathcal{E}} = P_{ij}$. Therefore, by the reasoning concerning the linear map $T_{S_{ij}}$ above, we have

$$[F_\theta(P_{ij}\xi_0)]_{\mathcal{E}}^{\mathcal{L}}P_{ij}[\dot{\xi}(0)]_{\mathcal{E}} = [S_{ij}F_\theta(\xi_0)\dot{\xi}(0)S_{ij}]_{\mathcal{L}} = -S_{ij}[F_\theta(\xi_0)]_{\mathcal{E}}^{\mathcal{L}}[\dot{\xi}(0)]_{\mathcal{E}}. \tag{34}$$

Recalling that $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$ as well as the arbitrariness of ξ_0 finishes the proof. $\square$

In the following sections, we will always understand $F_\theta(\xi)$ as a matrix of size 3×4 and thus, since no confusion may arise, we will abandon the slightly cumbersome notation and identify $[F_\theta(\xi)]_{\mathcal{E}}^{\mathcal{L}}$ with $F_\theta(\xi)$.

4   The control problem in the regime of small strokes

Let us return to the control equations for SPR4 given by (6). The response of the control system is characterized by the two matrix valued functions $F_c, F_\theta : \text{SO}(3) \times \mathbb{R}^4 \to M_{3\times 4}(\mathbb{R})$ which can by Proposition 1 can be factorized as:

$$F_c(R, \zeta) = RF_c(\zeta) \quad \text{and} \quad F_\theta(R, \xi) = RF_\theta(\zeta), \tag{35}$$

with $F_c(\zeta) := F_c(I, \zeta)$ and $F_\theta(\zeta) := F_\theta(I, \zeta)$. Hereinafter, we suppose that $\zeta := \xi_0 + \xi$ with $\xi_0 \in \mathcal{M}$ having all its components equal. Furthermore, we set $F_{c,\xi_0}(\xi) := F_c(\xi_0 + \xi)$ and analogously $F_{\theta,\xi_0}(\xi) := F_\theta(\xi_0 + \xi)$. It has been shown in [1] that F and thus both F_{c,ξ_0} and F_{θ,ξ_0} are analytic functions. Therefore, we can consider their first order expansions in ξ:

$$F_{c,\xi_0}(\xi)\eta = F_{c,0}\eta + \mathcal{H}_{c,0}(\xi \otimes \eta) + \mathcal{O}(|\xi|)\eta \tag{36}$$
$$F_{\theta,\xi_0}(\xi)\eta = F_{\theta,0}\eta + \mathcal{H}_{c,0}(\xi \otimes \eta) + \mathcal{O}(|\xi|)\eta, \tag{37}$$

where $F_{c,0} := F_c(\xi_0) \in M_{3\times 4}(\mathbb{R})$, $\mathcal{H}_{c,0} \in \mathcal{L}(\mathbb{R}^4 \otimes \mathbb{R}^4, \mathbb{R}^3)$ represents the first order derivative of F_{c,ξ_0} at $\xi = 0$ and for $F_{\theta,\xi}$ the analogous definitions are made. The purpose of this section is to reveal the structure of the different terms in the expansions (37) and (36) in light of the symmetry properties fulfilled by F_c and F_θ due to Propositions (2) and (3) , i.e

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij} \quad \text{and} \quad F_\theta(P_{ij}\xi) = -S_{ij}F_\theta(\xi)P_{ij} \quad \forall \xi \in \mathcal{M}. \tag{38}$$

The following slightly generalized statement of Lemma 9 in [2] proves useful in our case as well. However, the proof is omitted, it being exactly the same.

Lemma 2. *Let $G : \mathbb{R}^n \to M_{m \times n}(\mathbb{R})$ be an analytic function and $S \in M_{m \times m}(\mathbb{R})$ and $P \in M_{n \times n}(\mathbb{R})$ matrices such that $G(P\xi) = SG(\xi)P$ for every $\xi \in \mathbb{R}^n$. For $\xi_0 \in \mathbb{R}^n$ with all components equal, set $G_{\xi_0}(\xi) := G(\xi_0 + \xi)$ and write the first order expansion*

$$G_{\xi_0}(\xi)\eta = G_0\eta + \mathcal{H}_0(\xi \otimes \eta) + \mathcal{O}(|\xi|)\eta. \tag{39}$$

Then we have

$$G_0 = SG_0P, \tag{40}$$

and

$$\mathcal{H}_0((P\xi) \otimes \eta) = S\mathcal{H}_0(\xi \otimes (P\eta)) \qquad\qquad \forall \xi, \eta \in \mathbb{R}^n \tag{41}$$

4.1   The zeroth order terms $F_{c,0}$ and $F_{\theta,0}$

By applying Lemma 2 to F_{c,ξ_0} and F_{θ,ξ_0}, we obtain two linear systems of matrix equations in the unknowns $F_{c,0}$ and $F_{\theta,0}$. To solve these two systems, we eventually have to determine at least some of the matrices S_{ij}. Let $S_{kl}(\phi)$ denote the reflection at the plane orthogonal to the $\hat{e}_k - \hat{e}_l$-plane making an angle of ϕ with the $\hat{e}_k$-axis. By geometrical inspection of the reference tetrahedron (S_1, S_2, S_3, S_4), we find that

$$S_{12} = S_{12}\left(\tfrac{2\pi}{3}\right), \qquad\qquad S_{23} = S_{12}(0), \qquad\qquad S_{13} = S_{13}\left(\tfrac{\pi - \alpha_{\text{tet}}}{2}\right), \tag{42}$$

where $\alpha_{\text{tet}} = \arccos(-1/3)$ denotes the angle between two legs of a regular tetrahedron. Indeed, it happens that these three matrices are enough to determine both terms of order and one finds that $F_{\theta,0} = 0$ and

$$F_{c,0} = \begin{pmatrix} -2\mathfrak{a} & \mathfrak{a} & \mathfrak{a} & 0 \\ 0 & \sqrt{3}\mathfrak{a} & -\sqrt{3}\mathfrak{a} & 0 \\ \tfrac{1}{\sqrt{2}}\mathfrak{a} & \tfrac{1}{\sqrt{2}}\mathfrak{a} & \tfrac{1}{\sqrt{2}}\mathfrak{a} & \tfrac{-3}{\sqrt{2}}\mathfrak{a} \end{pmatrix} \text{ with } \mathfrak{a} \in \mathbb{R}. \tag{43}$$

In the following sections, we will exploit the orthonormal basis of $\mathbb{R}^4$ consisting of the vectors $\tau_1 := \tfrac{1}{\sqrt{6}}(-2, 1, 1, 0)^T$, $\tau_2 := \tfrac{1}{\sqrt{2}}(0, 1, -1, 0)^T$, $\tau_3 := \tfrac{1}{2\sqrt{3}}(1, 1, 1, -3)^T$ and $\tau_4 := \tfrac{1}{2}(1, 1, 1, 1)^T$, in terms of which $F_{c,0}$ can be written as $F_{c,0} = \mathfrak{a}\sqrt{6}[\tau_1|\tau_2|\tau_3]^T$.

Remark. First, we observe that the upper left corner of $F_{c,0}$ corresponding to the arms $||1, ||2$ and $||3$ is exactly the same as for SPR3 in [2]. Furthermore, one notes that physically it is quite clear that $F_{\theta,0}$ must vanish since by hypothesis ξ_0 has all its components equal and thus the swimmer is in a symmetric shape at $\xi = 0$. Therefore, the balls moving along their axes cannot create any torque.

4.2   The first order terms $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$

Following the approach in [2], we evaluate the tensors $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$ on the basis $(e_i \otimes e_j)_{i,j \in \mathbb{N}_4}$. Setting $A_k := (\mathcal{H}_{c,0}(e_i \otimes e_j) \cdot \hat{e})_{i,j \in \mathbb{N}_4}$ and $B_k := (\mathcal{H}_{\theta,0}(e_i \otimes e_j) \cdot \hat{e})_{i,j \in \mathbb{N}_4}$ for $k \in \mathbb{N}_3$, we can write the vectors $\mathcal{H}_{c,0}(\xi \otimes \eta), \mathcal{H}_{\theta,0}(\xi \otimes \eta) \in \mathbb{R}^3$ for any $\xi, \eta \in \mathbb{R}^3$ as

$$\mathcal{H}_{c,0}(\xi \otimes \eta) = \sum_{k \in \mathbb{N}_3} (A_k \eta \cdot \xi)\hat{e}_k, \tag{44}$$

and similarly

$$\mathcal{H}_{\theta,0}(\xi \otimes \eta) = \sum_{j \in \mathbb{N}_3} (B_k \eta \cdot \xi)L_k. \tag{45}$$

We could follow pursue the approach of [2] and directly calculate the matrices A_k and B_k. However, as we shall see later, the dynamics of SPR4 , up to higher order terms in the norm of the control curve ξ, will only be governed by their skew symmetric parts. Thus, we will evade this strenuous task and we will determine the skew symmetric matrices $M_k := \tfrac{1}{2}(A_k - A_k^T)$ and $M_{k+3} := \tfrac{1}{2}(B_k - B_k^T)$ for $k \in \mathbb{N}_3$, up to two scalar parameters, using a more abstract argument.

To that end, we notice that Lemma 2 together with the fact that $(P_{ij})^2 = I$ yields for all $i, j \in \mathbb{N}_4$ and for all $\xi, \eta \in \mathbb{N}_4$

$$S_{ij}\mathcal{H}_{c,0}(P_{ij}\xi \otimes P_{ij}\eta) = \mathcal{H}_{c,0}(\xi \otimes \eta), \tag{46}$$

as well as

$$-S_{ij}\mathcal{H}_{\theta,0}(P_{ij}\xi \otimes P_{ij}\eta) = \mathcal{H}_{\theta,0}(\xi \otimes \eta). \tag{47}$$

Next, we define $\mathcal{K}_{c,0}(\xi \otimes \eta) := \frac{1}{2}[\mathcal{H}_{c,0}(\xi \otimes \eta) - \mathcal{H}_{c,0}(\eta \otimes \xi)]$ and similarly $\mathcal{K}_{\theta,0}$ such that

$$M_k = (\mathcal{K}_{c,0}(e_i \otimes e_j) \cdot \hat{e}_k)_{i,j \in \mathbb{N}_4} \text{ and } M_{k+3} = (\mathcal{K}_{\theta,0}(e_i \otimes e_j) \cdot \hat{e}_k)_{i,j \in \mathbb{N}_4} \tag{48}$$

for $k \in \mathbb{N}_3$. In particular, it is clear that $\mathcal{K}_{c,0}$ and $\mathcal{K}_{\theta,0}$ satisfy the same symmetry relations as $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$, respectively, and additionally, we have $\mathcal{K}_{c,0}(e_i \otimes e_j) = -\mathcal{K}_{c,0}(e_j \otimes e_i)$ and $\mathcal{K}_{\theta,0}(e_i \otimes e_j) = -\mathcal{K}_{\theta,0}(e_j \otimes e_i)$.

For the spatial part, we deduce from the symmetry properties above that for all $i \neq j \in \mathbb{N}_4$

$$\mathcal{K}_{c,0}(e_i \otimes e_j) = S_{ij}\mathcal{K}_{c,0}(e_j \otimes e_i) = -S_{ij}\mathcal{K}_{c,0}(e_i \otimes e_j) \tag{49}$$

and therefore $\mathcal{K}_{c,0}(e_i \otimes e_j)$ is an eigenvector associated to the eigenvalue -1 of the reflection S_{ij}. The reflection S_{ij} taking place at the plane passing through the two remaining arms z_k and z_l, implies that $\mathcal{K}_{c,0}(e_i \otimes e_j) = \alpha_{ij}(z_k \times z_l)$ for some scalar $\alpha_{ij} \in \mathbb{R}$. Additionally, we have $\mathcal{K}_{c,0}(e_i \otimes e_j) = S_{jk}\mathcal{K}_{c,0}(e_i \otimes e_k) = \alpha_{ik}(z_j \times z_l)$ and since S_{jk} is orthogonal, we have $|\alpha_{ij}| = |\alpha_{ik}|$ since the vectors z_i are normalized. Eventually, one quickly verifies that the quantity

$$\mathcal{K}_{c,0}(e_i \otimes e_j) \cdot \text{sgn}(ijkl)(z_k \times z_l), \tag{50}$$

where $\text{sgn}(ijkl)$ denotes the parity of the permutation $(ijkl)$ of $\mathbb{N}_4$, stays constant under any permutation of the indices as well as any symmetry condition. Hence, we may conclude that

$$\mathcal{K}_{c,0}(e_i \otimes e_j) = \alpha \,\text{sgn}(ijkl)z_k \times z_l \tag{51}$$

for all $i \neq j \in \mathbb{N}_4$ and some scalar $\alpha \in \mathbb{R}$. Clearly, the symmetry conditions imply that $\mathcal{K}_{c,0}(e_i \otimes e_i) = 0$ for all $i \in \mathbb{N}_4$. Thus, we have determined the matrices M_1, M_2, M_3 up to one scalar parameter. By explicitly calculating the cross products $z_i \times z_j$, we find

$$M_1 = \alpha \begin{pmatrix} 0 & 3 & 3 & 2 \\ -3 & 0 & 0 & -1 \\ -3 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{pmatrix}, \qquad M_2 = \sqrt{3}\alpha \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -2 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{52}$$

and

$$M_3 = 2\sqrt{2}\alpha \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{53}$$

Similarly, for the angular part, we find that

$$\mathcal{K}_{\theta,0}(e_i \otimes e_j) = -S_{kl}\mathcal{K}_\theta(e_i \otimes e_j) \tag{54}$$

and therefore $\mathcal{K}_{\theta,0}(e_i \otimes e_j) = \delta_{ij}e_i \times e_j$. By noticing that this time the quantity $\mathcal{K}_{\theta,0}(e_i \otimes e_j) \cdot (z_i \times z_j)$ stays constant, a similar argument to the one above shows that

$$\mathcal{K}_{\theta,0}(e_i \otimes e_j) = \delta z_i \times z_j, \tag{55}$$

for all $i \neq j \in \mathbb{N}_4$. Again, we have $\mathcal{K}_\theta(e_i \otimes e_i) == 0$ for all $i \in \mathbb{N}_4$. A calculation similar to the one above now yields

$$M_4 = \delta \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & -2 & 3 \\ 1 & 2 & 0 & -3 \\ 0 & -3 & 3 & 0 \end{pmatrix}, \qquad M_5 = \sqrt{3}\delta \begin{pmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{pmatrix}, \tag{56}$$

and

$$M_6 = 2\sqrt{2} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{57}$$

4.3 The linearized control equations

Herein, we denote by J the closed interval $[0,2] \subset \mathbb{R}$ and we define the so-called *strokes space* as $H^1_\sharp(J, \mathbb{R}^4)$, i.e. the Sobolev space of 2π-periodic vector valued functions of $L^2_\sharp(J, \mathbb{R}^4$ having first order weak derivative in $L^2_\sharp(\mathbb{R}^4)$. For every $f \in L^2_\sharp(J, \mathbb{R}^4)$ we denote by $\langle f \rangle :=$ $(2\pi)^{-1} \int_J f(s)ds$ the average of f on J.

In the previous section, we have seen that the control system governing the evolution of SPR4 under the action of the control parameters ζ is given by $\dot{p} = RF(\zeta)\dot{\zeta}$, where $p = (c, R)$ and $F = (F_c, F_\theta) : \mathcal{M} \to M_{6\times4}(\mathbb{R})$ is given by (35) and $\dot{\zeta} \in T_\zeta\mathcal{M}$.

5 Energy optimizing strokes

References

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