1

Vectors and Linear Spaces

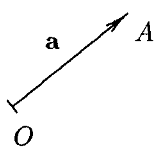
Vectors provide a mathematical formulation for the notion of direction, thus making direction a part of our mathematical language for describing the physical world. This leads to useful applications in physics and engineering, notably in connection with forces, velocities of motion, and electrical fields. Vectors help us to visualize physical quantities by providing a geometrical interpretation. They also simplify computations by bringing algebra to bear on geometry.

1.1 Scalars and vectors

In geometry and physics and their engineering applications we use two kinds of quantities, scalars and vectors. A **scalar** is a quantity that is determined by its magnitude, measured in units on a suitable scale. [1] For instance, mass, temperature and voltage are scalars.

A **vector** is a quantity that is determined by its direction as well as its magnitude; thus it is a *directed quantity* or a *directed line-segment*. For instance, force, velocity and magnetic intensity are vectors.

We denote vectors by boldface letters $\mathbf{a}, \mathbf{b}, \mathbf{r}$, etc. [or indicate them by arrows, $\vec{a}, \vec{b}, \vec{r}$, etc., especially in dimension 3]. A vector can be depicted by an arrow, a line-segment with a distinguished end point. The two end points are called the initial point (tail) and the terminal point (tip):



1. length (of the line-segment OA)
2. direction
   - attitude (of the line OA)
   - orientation (from O to A)

The length of a vector $\mathbf{a}$ is denoted by $|\mathbf{a}|$. Two vectors are equal if and only

1 In this chapter scalars are real numbers (elements of $\mathbb{R}$).

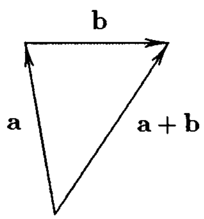
if they have the same length and the same direction. Thus,

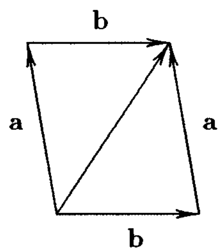
$$\mathbf{a} = \mathbf{b} \iff |\mathbf{a}| = |\mathbf{b}| \quad \text{and} \quad \mathbf{a} \uparrow\uparrow \mathbf{b}.$$

Two vectors have the same direction, if they are parallel as lines (the same attitude) and similarly aimed (the same orientation). The *zero vector* has length zero, and its direction is unspecified. A *unit vector* $\mathbf{u}$ has length one, $|\mathbf{u}| = 1$. A vector $\mathbf{a}$ and its *opposite* $-\mathbf{a}$ are of equal length and parallel, but have opposite orientations.

1.2 Vector addition and subtraction

Given two vectors $\mathbf{a}$ and $\mathbf{b}$, translate the initial point of $\mathbf{b}$ to the terminal point of $\mathbf{a}$ (without rotating $\mathbf{b}$). Then the sum $\mathbf{a}+\mathbf{b}$ is a vector drawn from the initial point of $\mathbf{a}$ to the terminal point of $\mathbf{b}$. Vector addition can be visualized by the triangle formed by vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}+\mathbf{b}$.

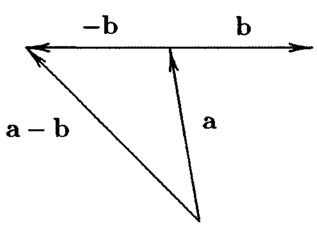


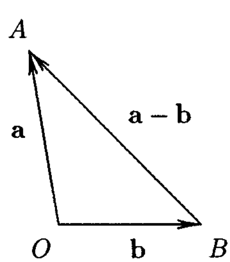


Vector addition

Vector addition is commutative, $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$, as can be seen by inspection of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as sides. It is also associative, $(\mathbf{a}+\mathbf{b})+\mathbf{c} = \mathbf{a}+(\mathbf{b}+\mathbf{c})$, and such that two opposite vectors cancel each other, $\mathbf{a}+(-\mathbf{a}) = 0$.

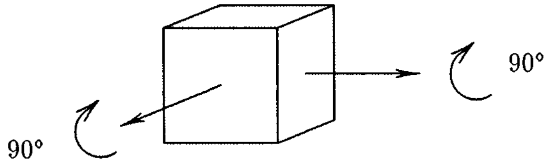
Instead of $\mathbf{a}+(-\mathbf{b})$ we simply write the difference as $\mathbf{a}-\mathbf{b}$. Note the order in $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$ when $\mathbf{a} = \overrightarrow{OA}$ and $\mathbf{b} = \overrightarrow{OB}$.





Vector subtraction

Remark. To qualify as vectors, quantities must have more than just direction and magnitude – they must also satisfy certain rules of combination. For instance, a rotation can be characterized by a direction $\mathbf{a}$, the axis of rotation, and a magnitude $\alpha = |\mathbf{a}|$, the angle of rotation, but rotations are not vectors because their composition fails to satisfy the commutative rule of vector addition, $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$. The lack of commutativity of the composition of rotations can be verified by turning a box around two of its horizontal axes by $90°$:



The terminal attitude of the box depends on the order of operations. The axis of the composite rotation is not even horizontal, so that neither $\mathbf{a}+\mathbf{b}$ nor $\mathbf{b}+\mathbf{a}$ can represent the composite rotation. We conclude that rotation angles are not vectors – they are a different kind of directed quantities. ▮

1.3 Multiplication by numbers (scalars)

Instead of $\mathbf{a}+\mathbf{a}$ we write $2\mathbf{a}$, etc., and agree that $(-1)\mathbf{a} = -\mathbf{a}$, the opposite of $\mathbf{a}$. This suggests the following definition for multiplication of vectors $\mathbf{a}$ by real numbers $\lambda \in \mathbb{R}$: the vector $\lambda\mathbf{a}$ has length $|\lambda\mathbf{a}| = |\lambda||\mathbf{a}|$ and direction given by (for $\mathbf{a} \neq 0$)

$$\begin{aligned} \lambda\mathbf{a} &\uparrow\uparrow \mathbf{a} \quad \text{if} \quad \lambda > 0, \\ \lambda\mathbf{a} &\uparrow\downarrow \mathbf{a} \quad \text{if} \quad \lambda < 0. \end{aligned}$$

Numbers multiplying vectors are called *scalars*. Multiplication by scalars, or *scalar multiplication*, satisfies distributivity, $\lambda(\mathbf{a}+\mathbf{b}) = \lambda\mathbf{a}+\lambda\mathbf{b}$, $(\lambda+\mu)\mathbf{a} = \lambda\mathbf{a}+\mu\mathbf{a}$, associativity, $(\lambda\mu)\mathbf{a} = \lambda(\mu\mathbf{a})$, and the unit property, $1\mathbf{a} = \mathbf{a}$, for all real numbers λ, μ and vectors $\mathbf{a}, \mathbf{b}$.

1.4 Bases and coordinates

In the plane any two non-parallel vectors $\mathbf{e}_1, \mathbf{e}_2$ form a *basis* so that an arbitrary vector in the plane can be uniquely expressed as a linear combination $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$. The numbers a_1, a_2 are called *coordinates* or *components* of the vector $\mathbf{a}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

When a basis has been chosen, vectors can be expressed in terms of the coordinates alone, for instance,

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1), \quad \mathbf{a} = (a_1, a_2).$$

If we single out a distinguished point, the origin O, we can use vectors to label the points A by $\mathbf{a} = \overrightarrow{OA}$. In the *coordinate system* fixed by O and $\{\mathbf{e}_1, \mathbf{e}_2\}$ we can denote points and vectors in a similar manner,

$$\text{point} \quad A = (a_1, a_2), \qquad \text{vector} \quad \mathbf{a} = (a_1, a_2),$$

since all the vectors have a common initial point O.

In coordinate form vector addition and multiplication by scalars are just coordinate-wise operations:

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2),$$
$$\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2).$$

Conversely, we may start from the set $\mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$, and equip it with component-wise addition and multiplication by scalars. This construction introduces a real *linear structure* on the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ making it a 2-dimensional real *linear space* $\mathbb{R}^2$. The real linear structure allows us to view the set $\mathbb{R}^2$ intuitively as a plane, the *vector plane* $\mathbb{R}^2$. The two unit points on the axes give the *standard basis*

$$\mathbf{e}_1 = (1,0), \quad \mathbf{e}_2 = (0,1)$$

of the 2-dimensional linear space $\mathbb{R}^2$.

In our ordinary space a basis is formed by three non-zero vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ which are not in the same plane. An arbitrary vector $\mathbf{a}$ can be uniquely represented as a linear combination of the basis vectors:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3.$$

The numbers a_1, a_2, a_3 are coordinates [2] in the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Conversely, coordinate-wise addition and scalar multiplication make the set

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x,y,z) \mid x, y, z \in \mathbb{R}\}$$

a 3-dimensional real *linear space* or *vector space* $\mathbb{R}^3$. In a coordinate system fixed by the origin O and a standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ a point $P = (x,y,z)$ and its *position vector*

$$\overrightarrow{OP} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

have the same coordinates. [3]

2 Some authors speak about components of vectors and coordinates of points.

3 Since a vector beginning at the origin is completely determined by its endpoints, we will sometimes refer to the *point* $\mathbf{r}$ rather than to the *endpoint of the vector* $\mathbf{r}$.

1.5 Linear spaces and linear functions

Above we introduced vectors by visualizing them without specifying the grounds of our study. In an axiomatic approach, one starts with a set whose elements satisfy certain characteristic rules. Vectors then become elements of a mathematical object called a linear space or a vector space V. In a linear space vectors can be added to each other but not multiplied by each other. Instead, vectors are multiplied by numbers, in this context called scalars. [4]

Formally, we begin with a set V and the field of real numbers $\mathbb{R}$. We associate with each pair of elements $\mathbf{a}, \mathbf{b} \in V$ a unique element in V, called the *sum* and denoted by $\mathbf{a}+\mathbf{b}$, and to each $\mathbf{a} \in V$ and each real number $\lambda \in \mathbb{R}$ we associate a unique element in V, called the *scalar multiple* and denoted by $\lambda\mathbf{a}$. The set V is called a **linear space** V over $\mathbb{R}$ if the usual rules of addition are satisfied for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in V$

$$\begin{array}{ll} \mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} & \text{commutativity} \\ (\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) & \text{associativity} \\ \mathbf{a}+0=\mathbf{a} & \text{zero-vector } 0 \\ \mathbf{a}+(-\mathbf{a})=0 & \text{opposite vector } -\mathbf{a} \end{array}$$

and if the scalar multiplication satisfies

$$\begin{array}{ll} \left.\begin{array}{l} \lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b} \\ (\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a} \end{array}\right\} & \text{distributivity} \\ (\lambda\mu)\mathbf{a}=\lambda(\mu\mathbf{a}) & \text{associativity} \\ 1\mathbf{a}=\mathbf{a} & \text{unit property} \end{array}$$

for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in V$. The elements of V are called *vectors*, and the linear space V is also called a vector space. The above axioms of a linear space set up a real *linear structure* on V.

A subset U of a linear space V is called a linear *subspace* of V if it is closed under the operations of a linear space:

$$\begin{array}{lll} \mathbf{a}+\mathbf{b} \in U & \text{for} & \mathbf{a}, \mathbf{b} \in U, \\ \lambda\mathbf{a} \in U & \text{for} & \lambda \in \mathbb{R},\ \mathbf{a} \in U. \end{array}$$

For instance, $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

A function $L : U \to V$ between two linear spaces U and V is said to be *linear* if for any $\mathbf{a}, \mathbf{b} \in U$ and $\lambda \in \mathbb{R}$,

$$L(\mathbf{a}+\mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b}) \quad \text{and}$$
$$L(\lambda\mathbf{a}) = \lambda L(\mathbf{a}).$$

4 Vectors are not scalars, and scalars are not vectors. Vectors belong to a linear space V, and scalars belong to a field $\mathbb{F}$. In this chapter $\mathbb{F} = \mathbb{R}$.

Linear functions preserve the linear structure. A linear function $V \to V$ is called a linear transformation or an *endomorphism*. An invertible linear function $U \to V$ is a *linear isomorphism*, denoted by $U \simeq V$. [5]

The set of linear functions $U \to V$ is itself a linear space. A composition of linear functions is also a linear function. The set of linear transformations $V \to V$ is a ring denoted by $\mathrm{End}(V)$. Since the endomorphism ring $\mathrm{End}(V)$ is also a linear space over $\mathbb{R}$, it is an associative algebra over $\mathbb{R}$, denoted by $\mathrm{End}_{\mathbb{R}}(V)$. [6]

1.6 Linear independence; dimension

A vector $\mathbf{b} \in V$ is said to be a *linear combination* of vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ if it can be written as a sum of multiples of the vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$, that is,

$$\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k \quad \text{where} \quad \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}.$$

A set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is said to be *linearly independent* if none of the vectors can be written as a linear combination of the other vectors. In other words, a set of vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is linearly independent if $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$ is the only set of real numbers satisfying

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k = 0.$$

In a linear combination

$$\mathbf{b} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \cdots + \lambda_k \mathbf{a}_k$$

of linearly independent vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$ the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ are unique; we call them the *coordinates* of $\mathbf{b}$.

Linear combinations of $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \subset V$ form a subspace of V; we say that this subspace is *spanned* by $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$. A linearly independent set $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\} \subset V$ which spans V is said to be a *basis* of V. All the bases for V have the same number of elements called the *dimension* of V.

Quadratic Structures

Concepts such as *distance* or *angle* are *not* inherent in the concept of a linear structure alone. For instance, it is meaningless to say that two lines in the linear space $\mathbb{R}^2$ meet each other at right angles, or that there is a basis of

5 Finite-dimensional real linear spaces are isomorphic if they are of the same dimension.

6 A ring R is a set with the usual addition and an associative multiplication $R \times R \to R$ which is distributive with respect to the addition. An algebra A is a linear space with a bilinear product $A \times A \to A$.

equally long vectors $\mathbf{e}_1, \mathbf{e}_2$ in $\mathbb{R}^2$. The linear structure allows comparison of lengths of parallel vectors, but it does not enable comparison of lengths of non-parallel vectors. For this, an extra structure is needed, namely the metric or quadratic structure.

The quadratic structure on a linear space $\mathbb{R}^n$ brings along an algebra which makes it possible to calculate with geometric objects. In the rest of this chapter we shall study such a geometric algebra associated with the Euclidean plane $\mathbb{R}^2$.

1.7 Scalar product

We will associate with two vectors a real number, the *scalar product* $\mathbf{a} \cdot \mathbf{b} \in \mathbb{R}$ of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$. This scalar valued product of $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ is defined as

in coordinates	$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$
geometrically	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\varphi$

where φ $[0 \le \varphi \le 180^\circ]$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The geometrical construction depends on the prior introduction of lengths and angles. Instead, the coordinate approach can be used to define the length

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}},$$

which equals $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$, and the angle given by

$$\cos\varphi = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be *orthogonal*, if $\mathbf{a} \cdot \mathbf{b} = 0$. A vector of length one, $|\mathbf{a}| = 1$, is called a *unit vector*. For instance, the standard basis vectors $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ are orthogonal unit vectors, and so form an *orthonormal basis* for $\mathbb{R}^2$.

The scalar product can be characterized by its properties:

$$\left.\begin{aligned} (\mathbf{a}+\mathbf{b}) \cdot \mathbf{c} &= \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \\ (\lambda\mathbf{a}) \cdot \mathbf{b} &= \lambda(\mathbf{a} \cdot \mathbf{b}) \end{aligned}\right\} \quad \text{linear in the first factor}$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad \text{symmetric}$$

$$\mathbf{a} \cdot \mathbf{a} > 0 \quad \text{for} \quad \mathbf{a} \neq 0 \qquad \text{positive definite.}$$

Symmetry and linearity with respect to the first factor together imply bilinearity, that is, linearity with respect to both factors. The real linear space $\mathbb{R}^2$ endowed with a bilinear, symmetric and positive definite product is called a *Euclidean plane* $\mathbb{R}^2$.

All Euclidean planes are isometric [7] to the one with the metric/norm

$$\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \to |\mathbf{r}| = \sqrt{x^2 + y^2}.$$

In the rest of this chapter we assume this metric structure on our vector plane $\mathbb{R}^2$.

Remark. The quadratic form $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 \to |\mathbf{r}|^2 = x^2 + y^2$ enables us to compare lengths of non-parallel line-segments. The linear structure by itself allows only comparison of parallel line-segments. ▮

1.8 The Clifford product of vectors; the bivector

It would be useful to have a multiplication of vectors satisfying the same axioms as the multiplication of real numbers – distributivity, associativity and commutativity – and require that the norm is preserved in multiplication, $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$. Since this is impossible in dimensions $n \geq 3$, we will settle for distributivity and associativity, but drop commutativity. However, we will attach a geometrical meaning to the lack of commutativity.

Take two orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ in the vector plane $\mathbb{R}^2$. The length of the vector $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2$ is $|\mathbf{r}| = \sqrt{x^2 + y^2}$. If the vector $\mathbf{r}$ is multiplied by itself, $\mathbf{rr} = \mathbf{r}^2$, [8] a natural choice is to require that the product equals the square of the length of $\mathbf{r}$,

$$\mathbf{r}^2 = |\mathbf{r}|^2.$$

In coordinate form, we introduce a product for vectors in such a way that

$$(x\mathbf{e}_1 + y\mathbf{e}_2)^2 = x^2 + y^2.$$

Use the distributive rule without assuming commutativity to obtain

$$x^2\mathbf{e}_1^2 + y^2\mathbf{e}_2^2 + xy(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) = x^2 + y^2.$$

This is satisfied if the orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ obey the multiplication rules

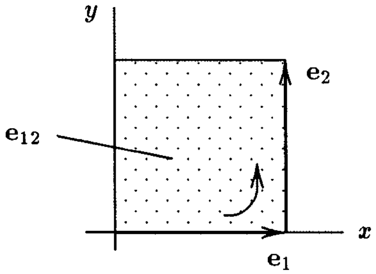
$$\boxed{\begin{array}{l} \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1 \\ \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 \end{array}} \quad \text{which correspond to} \quad \boxed{\begin{array}{l} |\mathbf{e}_1| = |\mathbf{e}_2| = 1 \\ \mathbf{e}_1 \perp \mathbf{e}_2 \end{array}}$$

Use associativity to calculate the square $(\mathbf{e}_1\mathbf{e}_2)^2 = -\mathbf{e}_1^2\mathbf{e}_2^2 = -1$. Since the square of the product $\mathbf{e}_1\mathbf{e}_2$ is negative, it follows that $\mathbf{e}_1\mathbf{e}_2$ is neither a scalar

7 An isometry of quadratic forms is a linear function $f : V \to V'$ such that $Q'(f(\mathbf{a})) = Q(\mathbf{a})$ for all $\mathbf{a} \in V$.

8 The scalar product $\mathbf{a} \cdot \mathbf{b}$ is not the same as the Clifford product $\mathbf{ab}$. Instead, the two products are related by $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba})$.

nor a vector. The product is a new kind of unit, called a **bivector**, representing the oriented plane area of the square with sides $\mathbf{e}_1$ and $\mathbf{e}_2$. Write for short $\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_2$.



We define the *Clifford product* of two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ to be $\mathbf{ab} = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\mathbf{e}_{12}$, a sum of a scalar and a bivector.

1.9 The Clifford algebra $C\ell_2$

The four elements

1	scalar
$\mathbf{e}_1$, $\mathbf{e}_2$	vectors
$\mathbf{e}_{12}$	bivector

form a basis for the **Clifford algebra** $C\ell_2$ [9] of the vector plane $\mathbb{R}^2$, that is, an arbitrary element

$$u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12} \quad \text{in} \quad C\ell_2$$

is a linear combination of a scalar u_0, a vector $u_1\mathbf{e}_1 + u_2\mathbf{e}_2$ and a bivector $u_{12}\mathbf{e}_{12}$. [10]

Example. Compute $\mathbf{e}_1\mathbf{e}_{12} = \mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_2$, $\mathbf{e}_{12}\mathbf{e}_1 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1^2\mathbf{e}_2 = -\mathbf{e}_2$, $\mathbf{e}_2\mathbf{e}_{12} = \mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2^2 = -\mathbf{e}_1$ and $\mathbf{e}_{12}\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2^2 = \mathbf{e}_1$. Note in particular that $\mathbf{e}_{12}$ anticommutes with both $\mathbf{e}_1$ and $\mathbf{e}_2$. ∎

The Clifford algebra $C\ell_2$ is a 4-dimensional real linear space with basis elements

9 These algebras were invented by William Kingdon Clifford (1845-1879). The first announcement of the result was issued in a talk in 1876, which was published posthumously in 1882. The first publication of the invention came out in another paper in 1878.

10 The Clifford algebra $C\ell_n$ of $\mathbb{R}^n$ contains 0-vectors (or scalars), 1-vectors (or just vectors), 2-vectors, ..., n-vectors. The aggregates of k-vectors give the linear space $C\ell_n$ a multivector structure $C\ell_n = \mathbb{R} \oplus \mathbb{R}^n \oplus \bigwedge^2 \mathbb{R}^n \oplus \ldots \oplus \bigwedge^n \mathbb{R}^n$.

1, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_{12}$ which have the multiplication table

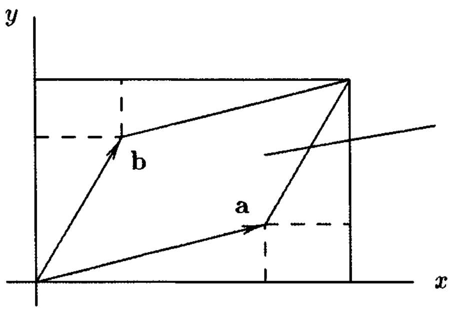
	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

1.10 Exterior product = bivector part of the Clifford product

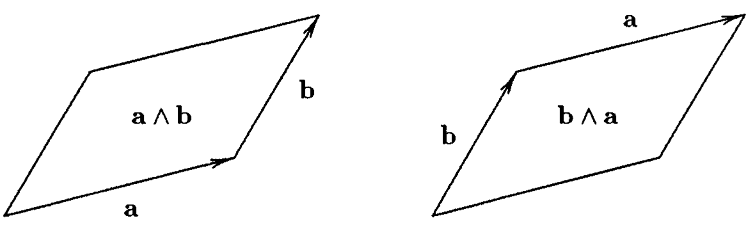
Extracting the scalar and bivector parts of the Clifford product we have as products of two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2, \qquad \text{the scalar product ‘}\mathbf{a}\text{ dot }\mathbf{b}\text{’},$$
$$\mathbf{a} \wedge \mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_{12}, \qquad \text{the exterior product ‘}\mathbf{a}\text{ wedge }\mathbf{b}\text{’}.$$

The bivector $\mathbf{a} \wedge \mathbf{b}$ represents the oriented plane segment of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. The area of this parallelogram is $|a_1b_2 - a_2b_1|$, and we will take the *magnitude* of the bivector $\mathbf{a} \wedge \mathbf{b}$ to be this area $|\mathbf{a} \wedge \mathbf{b}| = |a_1b_2 - a_2b_1|$.



The parallelogram can be regarded as a kind of geometrical product of its sides:



The bivectors $\mathbf{a} \wedge \mathbf{b}$ and $\mathbf{b} \wedge \mathbf{a}$ have the same magnitude but opposite senses of rotation. This can be expressed simply by writing

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}.$$

Using the multiplication table of the Clifford algebra $C\ell_2$ we notice that the Clifford product

$$(a_1\mathbf{e}_1 + a_2\mathbf{e}_2)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)\mathbf{e}_{12}$$

of two vectors $\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ and $\mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2$ is a sum of a scalar $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$ and a bivector $\mathbf{a}\wedge\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_{12}$. [11] In an equation,

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}. \qquad (a)$$

The commutative rule $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ together with the anticommutative rule $\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$ implies a relation between $\mathbf{ab}$ and $\mathbf{ba}$. Thus,

$$\mathbf{ba} = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\wedge\mathbf{b}. \qquad (b)$$

Adding and subtracting equations (a) and (b), we find

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) \quad \text{and} \quad \mathbf{a}\wedge\mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}).$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a}\parallel\mathbf{b}$, when they commute, $\mathbf{ab} = \mathbf{ba}$, that is, $\mathbf{a}\wedge\mathbf{b} = 0$ or $a_1b_2 = a_2b_1$, and orthogonal, $\mathbf{a}\perp\mathbf{b}$, when they anticommute, $\mathbf{ab} = -\mathbf{ba}$, that is, $\mathbf{a}\cdot\mathbf{b} = 0$. Thus,

$$\begin{aligned} \mathbf{ab} = \mathbf{ba} \quad &\Longleftrightarrow \quad \mathbf{a}\parallel\mathbf{b} \quad \Longleftrightarrow \quad \mathbf{a}\wedge\mathbf{b} = 0 \quad \Longleftrightarrow \quad \mathbf{ab} = \mathbf{a}\cdot\mathbf{b}, \\ \mathbf{ab} = -\mathbf{ba} \quad &\Longleftrightarrow \quad \mathbf{a}\perp\mathbf{b} \quad \Longleftrightarrow \quad \mathbf{a}\cdot\mathbf{b} = 0 \quad \Longleftrightarrow \quad \mathbf{ab} = \mathbf{a}\wedge\mathbf{b}. \end{aligned}$$

1.11 Components of a vector in given directions

Consider decomposing a vector $\mathbf{r}$ into two components, one parallel to $\mathbf{a}$ and the other parallel to $\mathbf{b}$, where $\mathbf{a}\nparallel\mathbf{b}$. This means determining the coefficients α and β in the decomposition $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$. The coefficient α may be obtained by forming the exterior product $\mathbf{r}\wedge\mathbf{b} = (\alpha\mathbf{a} + \beta\mathbf{b})\wedge\mathbf{b}$ and using $\mathbf{b}\wedge\mathbf{b} = 0$; this results in $\mathbf{r}\wedge\mathbf{b} = \alpha(\mathbf{a}\wedge\mathbf{b})$. Similarly, $\mathbf{a}\wedge\mathbf{r} = \beta(\mathbf{a}\wedge\mathbf{b})$. In the last two equations both sides are multiples of $\mathbf{e}_{12}$ and we may write, symbolically, [12]

$$\alpha = \frac{\mathbf{r}\wedge\mathbf{b}}{\mathbf{a}\wedge\mathbf{b}}, \qquad \beta = \frac{\mathbf{a}\wedge\mathbf{r}}{\mathbf{a}\wedge\mathbf{b}}.$$

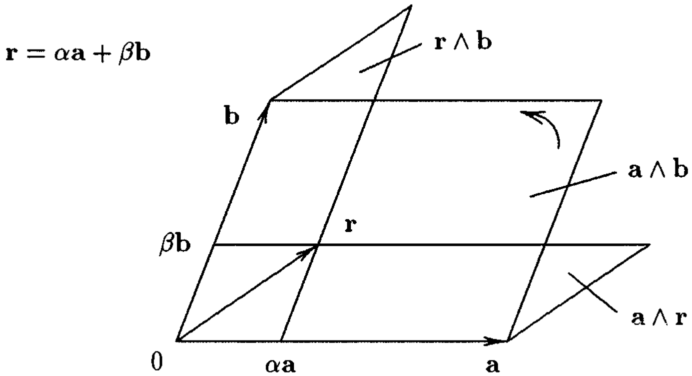
11 The bivector valued exterior product $\mathbf{a}\wedge\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_{12}$, which represents a plane area, should not be confused with the vector valued cross product $\mathbf{a}\times\mathbf{b} = (a_1b_2 - a_2b_1)\mathbf{e}_3$, which represents a line segment.

12 As an element of the exterior algebra $\bigwedge\mathbb{R}^2$ the bivector $\mathbf{a}\wedge\mathbf{b}$ is not invertible. As an element of the Clifford algebra $C\ell_2$ a non-zero bivector $\mathbf{a}\wedge\mathbf{b}$ is invertible, but since the multiplication in $C\ell_2$ is non-commutative, it is more appropriate to write

$$\alpha = (\mathbf{r}\wedge\mathbf{b})(\mathbf{a}\wedge\mathbf{b})^{-1} \quad \text{and} \quad \beta = (\mathbf{a}\wedge\mathbf{r})(\mathbf{a}\wedge\mathbf{b})^{-1}.$$

However, since $\mathbf{r}\wedge\mathbf{b}$, $\mathbf{a}\wedge\mathbf{r}$ and $\mathbf{a}\wedge\mathbf{b}$ commute, our notation is also acceptable.

The coefficients α and β could be obtained visually by comparing the oriented areas (instead of lengths) in the following figure:



Exercise 5

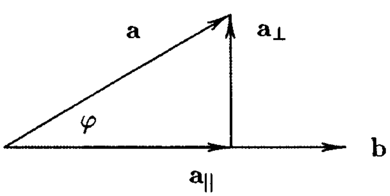
1.12 Perpendicular projections and reflections

Let us calculate the component of $\mathbf{a}$ in the direction of $\mathbf{b}$ when the two vectors diverge by an angle φ, $0 < \varphi < 180°$. The parallel component $\mathbf{a}_{\|}$ is a scalar multiple of the unit vector $\mathbf{b}/|\mathbf{b}|$:

$$\mathbf{a}_{\|} = |\mathbf{a}| \cos\varphi \frac{\mathbf{b}}{|\mathbf{b}|} = |\mathbf{a}||\mathbf{b}| \cos\varphi \frac{\mathbf{b}}{|\mathbf{b}|^2}.$$

In other words, the parallel component $\mathbf{a}_{\|}$ is the scalar product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\varphi$ multiplied by the vector $\mathbf{b}^{-1} = \mathbf{b}/|\mathbf{b}|^2$, called the inverse [13] of the vector $\mathbf{b}$. Thus,

$$\mathbf{a}_{\|} = (\mathbf{a} \cdot \mathbf{b}) \frac{\mathbf{b}}{|\mathbf{b}|^2} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1}.$$



The last formula tells us that the length of $\mathbf{b}$ is irrelevant when projecting into the direction of $\mathbf{b}$.

The perpendicular component $\mathbf{a}_\perp$ is given by the difference

$$\begin{aligned}\mathbf{a}_\perp &= \mathbf{a} - \mathbf{a}_{\|} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1} \\ &= (\mathbf{ab} - \mathbf{a} \cdot \mathbf{b})\mathbf{b}^{-1} = (\mathbf{a} \wedge \mathbf{b})\mathbf{b}^{-1}.\end{aligned}$$

13 The inverse $\mathbf{b}^{-1}$ of a non-zero vector $\mathbf{b} \in \mathbb{R}^2 \subset \mathcal{C}\ell_2$ satisfies $\mathbf{b}^{-1}\mathbf{b} = \mathbf{b}\mathbf{b}^{-1} = 1$ in the Clifford algebra $\mathcal{C}\ell_2$. A vector and its inverse are parallel vectors.

Note that the bivector $\mathbf{e}_{12}$ anticommutes with all the vectors in the $\mathbf{e}_1\mathbf{e}_2$-plane, therefore

$$(\mathbf{a}\wedge\mathbf{b})\mathbf{b}^{-1} = -\mathbf{b}^{-1}(\mathbf{a}\wedge\mathbf{b}) = \mathbf{b}^{-1}(\mathbf{b}\wedge\mathbf{a}) = -(\mathbf{b}\wedge\mathbf{a})\mathbf{b}^{-1}.$$

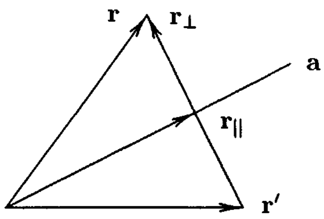
The area of the parallelogram with sides $\mathbf{a},\mathbf{b}$ is seen to be

$$|\mathbf{a}_\perp\mathbf{b}| = |\mathbf{a}\wedge\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\varphi$$

where $0 < \varphi < 180°$.

The reflection of $\mathbf{r}$ across the line $\mathbf{a}$ is obtained by sending $\mathbf{r} = \mathbf{r}_{||} + \mathbf{r}_\perp$ to $\mathbf{r}' = \mathbf{r}_{||} - \mathbf{r}_\perp$, where $\mathbf{r}_{||} = (\mathbf{r}\cdot\mathbf{a})\mathbf{a}^{-1}$. The mirror image $\mathbf{r}'$ of $\mathbf{r}$ with respect to $\mathbf{a}$ is then

$$\begin{aligned}\mathbf{r}' &= (\mathbf{r}\cdot\mathbf{a})\mathbf{a}^{-1} - (\mathbf{r}\wedge\mathbf{a})\mathbf{a}^{-1}\\ &= (\mathbf{r}\cdot\mathbf{a} - \mathbf{r}\wedge\mathbf{a})\mathbf{a}^{-1}\\ &= (\mathbf{a}\cdot\mathbf{r} + \mathbf{a}\wedge\mathbf{r})\mathbf{a}^{-1}\\ &= \mathbf{a}\mathbf{r}\mathbf{a}^{-1}\end{aligned}$$



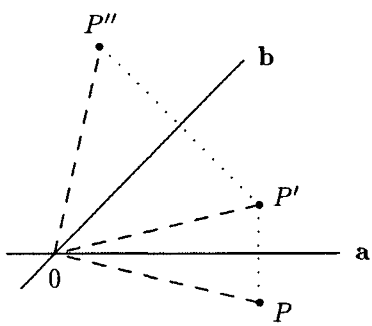
and further

$$\begin{aligned}\mathbf{r}' &= (2\mathbf{a}\cdot\mathbf{r} - \mathbf{r}\mathbf{a})\mathbf{a}^{-1}\\ &= 2\frac{\mathbf{a}\cdot\mathbf{r}}{\mathbf{a}^2}\mathbf{a} - \mathbf{r}.\end{aligned}$$

The formula $\mathbf{r}' = \mathbf{a}\mathbf{r}\mathbf{a}^{-1}$ can be obtained directly using only commutation properties of the Clifford product: decompose $\mathbf{r} = \mathbf{r}_{||} + \mathbf{r}_\perp$, where $\mathbf{a}\mathbf{r}_{||}\mathbf{a}^{-1} = \mathbf{r}_{||}\mathbf{a}\mathbf{a}^{-1} = \mathbf{r}_{||}$, while $\mathbf{a}\mathbf{r}_\perp\mathbf{a}^{-1} = -\mathbf{r}_\perp\mathbf{a}\mathbf{a}^{-1} = -\mathbf{r}_\perp$.

The composition of two reflections, first across $\mathbf{a}$ and then across $\mathbf{b}$, is given by

$$\mathbf{r} \to \mathbf{r}' = \mathbf{a}\mathbf{r}\mathbf{a}^{-1} \to \mathbf{r}'' = \mathbf{b}\mathbf{r}'\mathbf{b}^{-1} = \mathbf{b}(\mathbf{a}\mathbf{r}\mathbf{a}^{-1})\mathbf{b}^{-1} = (\mathbf{b}\mathbf{a})\mathbf{r}(\mathbf{b}\mathbf{a})^{-1}.$$



The composite of these two reflections is a rotation by twice the angle between $\mathbf{a}$ and $\mathbf{b}$. As a consequence, if a triangle ABC with angles α,β,γ is turned about its vertices A, B, C by the angles $2\alpha, 2\beta, 2\gamma$ in the same direction, the result is an identity rotation.

Exercises 6,7

1.13 Matrix representation of $\mathcal{C}\ell_2$

In this chapter we have introduced the Clifford algebra $\mathcal{C}\ell_2$ of the Euclidean plane $\mathbb{R}^2$. The Clifford algebra $\mathcal{C}\ell_2$ is a 4-dimensional algebra over the reals $\mathbb{R}$. It is isomorphic, as an associative algebra, to the matrix algebra of real 2×2-matrices $\mathrm{Mat}(2, \mathbb{R})$, as can be seen by the correspondences

$$1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\mathbf{e}_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e}_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{e}_{12} \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

However, in the Clifford algebra $\mathcal{C}\ell_2$ there is more structure than in the matrix algebra $\mathrm{Mat}(2, \mathbb{R})$. In the Clifford algebra $\mathcal{C}\ell_2$ we have singled out by definition a privileged subspace, namely the subspace of vectors or 1-vectors $\mathbb{R}^2 \subset \mathcal{C}\ell_2$. No similar privileged subspace is incorporated in the definition of the matrix algebra $\mathrm{Mat}(2, \mathbb{R})$. [14]

For arbitrary elements the above correspondences mean that

$$u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12} \simeq \begin{pmatrix} u_0 + u_1 & u_2 + u_{12} \\ u_2 - u_{12} & u_0 - u_1 \end{pmatrix}$$

and

$$\frac{1}{2}[(a+d) + (a-d)\mathbf{e}_1 + (b+c)\mathbf{e}_2 + (b-c)\mathbf{e}_{12}] \simeq \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In this representation the transpose of a matrix,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\top} = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

corresponds to the *reverse*

$$\tilde{u} = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 - u_{12}\mathbf{e}_{12}$$

14 For instance, we might choose $\mathbf{u}_1 = \sqrt{2}\mathbf{e}_1 + \mathbf{e}_{12}$, $\mathbf{u}_2 = \mathbf{e}_2$. This also results in the commutation relations $\mathbf{u}_1^2 = 1$, $\mathbf{u}_2^2 = 1$, $\mathbf{u}_1\mathbf{u}_2 + \mathbf{u}_2\mathbf{u}_1 = 0$, which define a different representation of $\mathcal{C}\ell_2$ as $\mathrm{Mat}(2, \mathbb{R})$.

of $u = u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12}$ in $C\ell_2$. The complementary (or adjoint) matrix

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \left[= (ad-bc)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \quad \text{for} \quad ad-bc \neq 0 \right]$$

corresponds to the *Clifford-conjugate* [15]

$$\bar{u} = u_0 - u_1\mathbf{e}_1 - u_2\mathbf{e}_2 - u_{12}\mathbf{e}_{12}.$$

The reversion and Clifford-conjugation are anti-involutions, that is, involutory anti-automorphisms,

$$\tilde{\tilde{u}} = u, \quad \widetilde{uv} = \tilde{v}\tilde{u},$$
$$\bar{\bar{u}} = u, \quad \overline{uv} = \bar{v}\bar{u}.$$

We still need the *grade involute*

$$\hat{u} = u_0 - u_1\mathbf{e}_1 - u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12}$$

for which $\hat{u} = \tilde{u}^{-} = \bar{u}^{\sim}$.

Exercises

1. Let $a = \mathbf{e}_2 - \mathbf{e}_{12}$, $b = \mathbf{e}_1 + \mathbf{e}_2$, $c = 1 + \mathbf{e}_2$. Compute ab, ac. What did you learn by completing this computation?
2. Let $a = \mathbf{e}_2 + \mathbf{e}_{12}$, $b = \frac{1}{2}(1 + \mathbf{e}_1)$. Compute ab, ba. What did you learn?
3. Let $a = 1 + \mathbf{e}_1$, $b = -1 + \mathbf{e}_1$, $c = \mathbf{e}_1 + \mathbf{e}_2$. Compute ab, ba, ac, ca, bc and cb. What did you learn?
4. Let $a = \frac{1}{2}(1 + \mathbf{e}_1)$, $b = \mathbf{e}_1 + \mathbf{e}_{12}$. Compute a^2, b^2.
5. Let $\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2$, $\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{r} = 5\mathbf{e}_1 - \mathbf{e}_2$. Compute α, β in the decomposition $\mathbf{r} = \alpha\mathbf{a} + \beta\mathbf{b}$.
6. Let $\mathbf{a} = 8\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2$. Compute $\mathbf{a}_{\|}$, $\mathbf{a}_{\perp}$.
7. Let $\mathbf{r} = 4\mathbf{e}_1 - 3\mathbf{e}_2$, $\mathbf{a} = 3\mathbf{e}_1 - \mathbf{e}_2$, $\mathbf{b} = 2\mathbf{e}_1 + \mathbf{e}_2$. Reflect first $\mathbf{r}$ across $\mathbf{a}$ and then the result across $\mathbf{b}$.
8. Show that for any $u \in C\ell_2$, $u\bar{u} = \bar{u}u \in \mathbb{R}$, and that u is invertible, if $u\bar{u} \neq 0$, with inverse
$$u^{-1} = \frac{\bar{u}}{u\bar{u}}.$$
9. Let $u = 1 + \mathbf{e}_1 + \mathbf{e}_{12}$. Compute u^{-1}. Show that $u^{-1} = \hat{u}(u\hat{u})^{-1} \neq (u\hat{u})^{-1}\hat{u}$, $u^{-1} = (\hat{u}u)^{-1}\hat{u} \neq \hat{u}(\hat{u}u)^{-1}$ and $u^{-1} = \tilde{u}(u\tilde{u})^{-1} \neq (u\tilde{u})^{-1}\tilde{u}$, $u^{-1} = (\tilde{u}u)^{-1}\tilde{u} \neq \tilde{u}(\tilde{u}u)^{-1}$.

15 In some countries a vector $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 \in \mathbb{R}^2$ is denoted by $\vec{u}$ in handwriting, but this practice clashes with our notation for the Clifford-conjugate.

10. Consider the four anti-involutions of $\mathrm{Mat}(2,\mathbb{R})$ sending

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Define two anti-automorphisms α, β to be similar, if there is an intertwining automorphism γ such that $\alpha\gamma = \gamma\beta$. Determine which ones of these four anti-involutions are similar or dissimilar to each other. Hint: keep track of what happens to the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with squares I, I, and $-I$.

Remark. In completing the exercises, note that an arbitrary element of $\mathcal{C}\ell_2$ is most easily perceived when written in the order of increasing indices as $u_0 + u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_{12}\mathbf{e}_{12}$. ∎

Solutions

1. $ab = ac = 1 - \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{12}$; one can learn that $ab = ac \not\Rightarrow b = c$.
2. $ab = 0,\ ba = \mathbf{e}_2 + \mathbf{e}_{12}$; one can learn that $ab = 0 \not\Rightarrow ba = 0$ (and also that $ba = a \not\Rightarrow b = 1$).
3. $ab = ba = 0,\ ac = 1 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{12},\ ca = 1 + \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_{12}$, $bc = 1 - \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_{12},\ cb = 1 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_{12}$; one can learn that $ab = ba = 0 \not\Rightarrow ac = 0$ or $ca = 0$.
4. $a^2 = a,\ b^2 = 0$.
5. $\mathbf{r} = 2\mathbf{a} + 3\mathbf{b}$.
6. $\mathbf{a}_{\|} = 6\mathbf{e}_1 + 3\mathbf{e}_2,\ \mathbf{a}_{\perp} = 2\mathbf{e}_1 - 4\mathbf{e}_2$.
7. $\mathbf{r}' = \mathbf{ara}^{-1} = 5\mathbf{e}_1,\ \mathbf{r}'' = \mathbf{br'b}^{-1} = 3\mathbf{e}_1 + 4\mathbf{e}_2$.
8. $u\bar{u} = \bar{u}u = u_0^2 - u_1^2 - u_2^2 + u_{12}^2 \in \mathbb{R}$.
9. $u^{-1} = 1 - \mathbf{e}_1 - \mathbf{e}_{12}$ and $(u\hat{u})^{-1}\hat{u} = \tilde{u}(\tilde{u}u)^{-1} = 1 + 3\mathbf{e}_1 - 4\mathbf{e}_2 - 5\mathbf{e}_{12}$ and $\hat{u}(\hat{u}u)^{-1} = (u\tilde{u})^{-1}\tilde{u} = 1 + 3\mathbf{e}_1 + 4\mathbf{e}_2 - 5\mathbf{e}_{12}$.
10. Only two of the anti-involutions are similar,

$$\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ -b & d \end{pmatrix}, \quad \beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix},$$

as can be seen by choosing the intertwining automorphism

$$\gamma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

for which $\alpha\gamma = \gamma\beta$.

Bibliography

W.K. Clifford: Applications of Grassmann's extensive algebra. *Amer. J. Math.* **1** (1878), 350-358.

W.K. Clifford: On the classification of geometric algebras; pp. 397-401 in R. Tucker (ed.): *Mathematical Papers by William Kingdon Clifford*, Macmillan, London, 1882. (Reprinted by Chelsea, New York, 1968.) Title of talk announced already in *Proc. London Math. Soc.* **7** (1876), p. 135.

M.J. Crowe: *A History of Vector Analysis*. University of Notre Dame Press, 1967. Reprinted by Dover, New York, 1985.

D.C. Lay: *Linear Algebra and its Applications. Instructor's Edition*. Addison-Wesley, Reading, MA, 1994.

M. Riesz: *Clifford Numbers and Spinors*. The Institute for Fluid Dynamics and Applied Mathematics, Lecture Series No. **38**, University of Maryland, 1958. Reprinted as facsimile (eds.: E.F. Bolinder, P. Lounesto) by Kluwer, Dordrecht, The Netherlands, 1993.

G. Strang: *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Cambridge, MA, 1993.