Existence of integer eigenvalues

By the direct calculation, we find that

$$(1) \quad \sigma_{1,2}(\mu) = \frac{2\sqrt{3}}{g_c\sqrt{g_\theta}} \sqrt{A \pm \sqrt{A^2 - K}},$$

where

$$A := \alpha^2 g_c |P_\mu|^2 + \delta^2 g_\theta |Q_\mu|^2 > 0$$

$$K := 4\alpha^2\delta^2 g_c g_\theta |P_\mu \cdot Q_\mu|^2 > 0$$

Clearly, we have $\sigma_1(\mu) \geq \sigma_2(\mu)$. So let us suppose that

$$\sigma_2(\mu) \overset{1)}{=} k \in \mathbb{N} \quad \text{and} \quad \sigma_1(\mu) \overset{2)}{=} \ell k, \ \ell \in \mathbb{N}.$$

From 1) we get

$$(2) \quad \frac{2\sqrt{3}}{g_c\sqrt{g_\theta}} \sqrt{A - \sqrt{A^2 - K}} = k,$$

which implies that

$$(3) \quad A = \frac{g_c^4 g_\theta^2 k^4 + 144K}{24 g_c^2 g_\theta k^2}.$$

On the other hand, 2) implies that

$$(4) \quad \ell\sqrt{A + \sqrt{A^2 - K}} = \sqrt{A - \sqrt{A^2 - K}},$$

which has the positive solution

(5) $$A = \frac{\sqrt{k}\,(1+\ell^2)}{2\ell}.$$

This yields

(6) $$\alpha^2 g_c |P_\mu|^2 + \delta^2 g_\theta |Q_\mu|^2 = \frac{1+\ell^2}{2\ell}\sqrt{g_c g_\theta}\,\alpha\delta\,|P_\mu \cdot Q_\mu|$$

Combining (3) and (4) gives us the two following solutions for K:

$$K = \frac{g_c^4 g_\theta^2 k^4}{144\ell^2} \quad \text{or} \quad K = \frac{g_c^4 g_\theta^2 k^4 \ell^2}{144}$$

which subsequently two admissible values for $|P_\mu \cdot Q_\mu|$, more precisely

(7) $$|P_\mu \cdot Q_\mu| = \frac{g_c^{3/2}\sqrt{g_\theta}\,k^2}{24\alpha\delta\ell} \quad \text{or} \quad |P_\mu \cdot Q_\mu| = \frac{g_c^{3/2}\sqrt{g_\theta}\,k^2\ell}{24\alpha\delta}.$$

Together with (6) this motivates the definition of the quadratic form $\mu \in \mathbb{R}^6 \longmapsto \Gamma\mu\cdot\mu$ with the positive definite matrix Γ given by

$$\Gamma := \frac{48}{g_c^2 g_\theta}\begin{pmatrix} \alpha^2 g_c I_3 & 0 \\ 0 & \delta^2 g_\theta I_3 \end{pmatrix}$$

In fact, now we can express the condition on $\underline{\mu}$ for 1) and 2) to hold by

$$\Gamma \underline{\mu} \cdot \underline{\mu} = \frac{1+e^2}{e} \frac{k^2}{e} \quad \text{or} \quad \frac{1+e^2}{e} k^2 \ell,$$

i.e. $\underline{\mu}$ has to lie in one of two ellipsoids depending only on k, ℓ and the geometry of the problem, which is accounted for by Γ.