Chapter 9

Particles Near Contact

9.1 Overview

Knowledge of the interactions between particles in close proximity is essential in understanding the suspension phenomena. For a certain class of problems, lubrication theory provides the leading order and singular terms in the resistance functions. If ℓ denotes the characteristic length of the particles, then the gap, defined as the separation between the closest points on the two particle surfaces, will be written as $\epsilon\ell$, with $\epsilon \ll 1$, and ϵ will be the key small paramater throughout this chapter.

For two rigid particles in relative motion, the dominant contributions to the force and torque may be determined by the methods of lubrication theory. In particular, we shall show *via* the lubrication equations that these leading order terms in the force and torque are singular as ϵ^{-1} and $\ln \epsilon^{-1}$ for small ϵ. Two representative problems (next two sections) involving the relative motion of spheres illustrate the essential ideas of lubrication theory. A sphere moving towards a second stationary sphere generates a "squeezing" flow, and we will see that the hydrodynamic resistance scales as ϵ^{-1}. On the other hand, a sphere moving past a second stationary sphere, in a direction transverse to the sphere-sphere axis, generates a "shearing" flow, and we will see that the hydrodynamic force and torque scales as $\ln \epsilon^{-1}$.

In Sections 9.4 and 9.5, we consider relative motion of two viscous drops near contact. We shall see that squeezing flows (Section 9.4) may involve a weaker singularity in the leading term, *viz.* $O(\epsilon^{-1/2})$, while for a bubble the singularity is an even weaker $O(\ln \epsilon^{-1})$. Shearing motions of one drop past another are not singular at all — the drops simply slip past each other with a finite mobility. Since all surface regions contribute to this $O(1)$ result, a matched asymptotic expansions approach is not practical. The numerical procedure for this problem is presented in Chapter 13, but the results are discussed in this chapter to provide a measure of continuity.

As mentioned at the beginning of Part III, lubrication methods do not apply at all for rigid surfaces moving in tandem as a single rigid body, because all regions near the bodies contribute to the $O(1)$ result for the resistance functions.

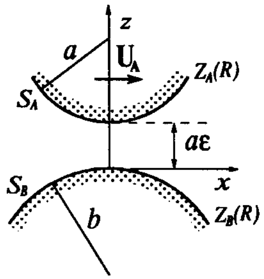

Figure 9.1: Flow produced by shearing motion of a sphere past another.

The numerical computations for these problems are also presented in Chapter 13, but these results will be used here to generate the higher order corrections to the terms obtained from lubrication theory.

9.2 Shearing Motions of Rigid Surfaces

Consider a moving sphere (denoted by S_A) of radius a and a stationary sphere (S_B) of radius $b = \beta a$. As shown in Figure 9.1, we choose a Cartesian coordinate system in which sphere S_A moves in the positive x-direction with velocity U and in which the sphere-sphere axis is coincident with the z-axis. The stationary sphere lies in the lower half space with the xy-plane as a tangent plane at the origin. The gap or clearance between the two spheres is denoted by $a\epsilon$.

The symmetries in the flow field near the origin suggest[1] the use of cylindrical coordinates (r, θ, z), with the corresponding velocity components (v_r, v_θ, v_z) and p written as

$$\begin{aligned} v_r &= U_A U(r,z)\cos\theta \\ v_\theta &= U_A V(r,z)\sin\theta \\ v_z &= U_A W(r,z)\cos\theta \\ p &= \frac{\mu U_A}{a} P(r,z)\cos\theta \, . \end{aligned}$$

The governing equations then become:

$$\frac{1}{a}\frac{\partial P}{\partial r} = L_0^2 U - \frac{2}{r^2}(U+V) \tag{9.1}$$

$$-\frac{1}{a}\frac{P}{r} = L_0^2 V - \frac{2}{r^2}(U+V) \tag{9.2}$$

$$\frac{1}{a}\frac{\partial P}{\partial z} = L_1^2 W \tag{9.3}$$

[1]See Chapter 18 for a more general discussion of the Fourier components.

$$\frac{\partial U}{\partial r} + \frac{U+V}{r} + \frac{\partial W}{\partial z} = 0 \ . \tag{9.4}$$

The operator L_m^2 is defined by

$$L_m^2 = \frac{\partial^2}{r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{z^2} \ .$$

Near the origin, the surfaces of the spheres are given by

$$z_A = a(1+\epsilon) - (a^2 - r^2)^{1/2} \sim a\epsilon + \frac{r^2}{2a}$$

and

$$z_B = -\beta a + (\beta^2 a^2 - r^2)^{1/2} \sim -\frac{1}{2\beta a} r^2$$

for $r \ll a$. These results suggest a new, dimensionless "stretched" r coordinate,

$$R = \epsilon^{-1/2}\frac{r}{a} \ ,$$

while for the z-direction we scale with the gap $a\epsilon$ to obtain

$$Z = \epsilon^{-1}\frac{z}{a} \ .$$

In these new coordinates, the sphere surfaces are given by

$$Z_A = 1 + \frac{1}{2}R^2 + \frac{1}{8}\epsilon R^4 + O(\epsilon^2) \tag{9.5}$$

$$Z_B = -\frac{1}{2\beta}R^2 - \frac{1}{(2\beta)^3}\epsilon R^4 + O(\epsilon^2) \ . \tag{9.6}$$

Now the boundary conditions imply that both U and V are O(1), so from the equation of continuity, Equation 9.4, we obtain the scaling for W as $\epsilon^{1/2}$. The scaling for the pressure then follows immediately from the governing equations as $\epsilon^{3/2}$. Our first (and fortunately correct) inclination is to assume the simplest form, a regular expansion in ϵ of the form

$$P(R,Z) = \epsilon^{-3/2}P_0(R,Z) + \epsilon^{-1/2}P_1(R,Z) + \ldots \tag{9.7}$$

$$U(R,Z) = U_0(R,Z) + \epsilon U_1(R,Z) + \ldots \tag{9.8}$$

$$V(R,Z) = V_0(R,Z) + \epsilon V_1(R,Z) + \ldots \tag{9.9}$$

$$W(R,Z) = \epsilon^{1/2}W_0(R,Z) + \epsilon^{3/2}W_1(R,Z) + \ldots \ . \tag{9.10}$$

Substitution of the asymptotic expansions into the governing equations yields a hierarchy of equations, of which the lowest order is

$$\frac{\partial P_0}{\partial R} = \frac{\partial^2 U_0}{\partial Z^2} \ , \tag{9.11}$$

$$-\frac{P_0}{R} = \frac{\partial^2 V_0}{\partial Z^2} \ , \tag{9.12}$$

$$\frac{\partial P_0}{\partial Z} = 0 \ , \tag{9.13}$$

and

$$\frac{\partial U_0}{\partial R} + \frac{U_0 + V_0}{R} + \frac{\partial W_0}{\partial Z} = 0 . \tag{9.14}$$

Not surprisingly, these equations are identical to those encountered in lubrication theory [63]. The boundary conditions on the surfaces of the upper and lower sphere are

$$U_0(R, Z_A) = -V_0(R, Z_A) = 1, \qquad W_0(R, Z_A) = 0,$$
$$U_0(R, Z_B) = V_0(R, Z_B) = W_0(R, Z_B) = 0.$$

At the next order, we obtain the following equations for U_1, V_1, W_1, and P_1:

$$\frac{\partial P_1}{\partial R} = \frac{\partial^2 U_1}{\partial Z^2} + \frac{\partial^2 U_0}{\partial R^2} + \frac{1}{R}\frac{\partial U_0}{\partial R} - \frac{2(U_0 + V_0)}{R^2} , \tag{9.15}$$

$$-\frac{P_1}{R} = \frac{\partial^2 V_1}{\partial Z^2} + \frac{\partial^2 V_0}{\partial R^2} + \frac{1}{R}\frac{\partial V_0}{\partial R} - \frac{2(U_0 + V_0)}{R^2} , \tag{9.16}$$

$$\frac{\partial P_1}{\partial Z} = \frac{\partial^2 W_0}{\partial Z^2} , \tag{9.17}$$

and

$$\frac{\partial U_1}{\partial R} + \frac{U_1 + V_1}{R} + \frac{\partial W_1}{\partial Z} = 0 , \tag{9.18}$$

with boundary conditions

$$\begin{aligned} U_1(R, Z_A) &= -\frac{1}{8}R^4\frac{\partial U_0}{\partial Z} , \\ V_1(R, Z_A) &= -\frac{1}{8}R^4\frac{\partial V_0}{\partial Z} , \\ W_1(R, Z_A) &= -\frac{1}{8}R^4\frac{\partial W_0}{\partial Z} , \end{aligned}$$

and

$$\begin{aligned} U_1(R, Z_B) &= \frac{1}{(2\beta)^3}R^4\frac{\partial U_0}{\partial Z} , \\ V_1(R, Z_B) &= \frac{1}{(2\beta)^3}R^4\frac{\partial V_0}{\partial Z} , \\ W_1(R, Z_B) &= \frac{1}{(2\beta)^3}R^4\frac{\partial W_0}{\partial Z} , \end{aligned}$$

accounting for the $O(\epsilon)$ discrepancy in the placement of the sphere surface in the lower order equations.

We now turn our attention to the solution of these equations and the associated results for the force and torque on the spheres. From Equation 9.13, we see that P_0 is a function only of R whence we may integrate Equations 9.11 and 9.12 to obtain the following parabolic profiles for U_0 and V_0:

$$U_0 = -\frac{1}{2}P_0'(Z - Z_B)(Z_A - Z) + \frac{Z - Z_B}{H} \tag{9.19}$$

$$V_0 = \frac{1}{2}\frac{P_0}{R}(Z - Z_B)(Z_A - Z) - \frac{Z - Z_B}{H} . \tag{9.20}$$

Here

$$H(R) = Z_A - Z_B = 1 + \frac{1+\beta}{2\beta} R^2$$

is the gap between the upper and lower surfaces at R, neglecting terms of $O(\epsilon)$.

We solve Equation 9.14 for $\partial W_0/\partial Z$ and integrate from $Z = Z_B$ to $Z = Z_A$. Since $v_z = 0$ on the sphere surfaces, we obtain the result

$$\begin{aligned} W_0(R, Z_A) - W_0(R, Z_B) &= 0 \\ &= -\frac{1}{R}\int_0^H \frac{\partial}{\partial R}(RU_0)d\zeta - \frac{1}{R}\int_0^H V_0\, d\zeta \\ &= -\frac{1}{R}\left[\frac{\partial}{\partial R}\int_0^H (RU_0)d\zeta - R\frac{dZ_A}{dR}\right] - \frac{1}{R}\int_0^H V_0\, d\zeta \ , \end{aligned}$$

with $\zeta = Z - Z_B$. Since Z_A depends on R, we applied the Leibniz rule to change the order of integration and differentation.

We use $dZ_A/dR = R$, $dH/dR = R(1+\beta)/\beta$ and the following results for the velocity integrals,

$$\int_0^H U_0\, d\zeta = -\frac{H^3}{12}P_0' + \frac{H}{2} \qquad \int_0^H V_0\, d\zeta = \frac{H^3}{12}\frac{P_0}{R} - \frac{H}{2} \ ,$$

to obtain the *Reynolds equation*,

$$R^2 P_0'' + \left[R + 3(1+\frac{1}{\beta})\frac{R^3}{H}\right] P_0' - P_0 = -6(1-\frac{1}{\beta})\frac{R^3}{H^3} \ . \tag{9.21}$$

The exact expressions for the contribution to the force and torque from an infinitesimal surface element dS on S_A are

$$\begin{aligned} dF_x &= -p(\boldsymbol{n}\cdot\boldsymbol{\delta}_x) + [(\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_\theta\tau_{\theta r})(\boldsymbol{\delta}_r\cdot\boldsymbol{n}) + (\boldsymbol{\delta}_r\tau_{rz} + \boldsymbol{\delta}_\theta\tau_{\theta z})(\boldsymbol{\delta}_z\cdot\boldsymbol{n})]\cdot\boldsymbol{\delta}_x dS \\ dT_y &= a(n_z dF_x - n_x dF_z) \\ &= n_z\left[(\boldsymbol{\delta}_r\tau_{rr} + \boldsymbol{\delta}_\theta\tau_{\theta r})(\boldsymbol{\delta}_r\cdot\boldsymbol{n}) + (\boldsymbol{\delta}_r\tau_{rz} + \boldsymbol{\delta}_\theta\tau_{\theta z})(\boldsymbol{\delta}_z\cdot\boldsymbol{n})\right]\cdot\boldsymbol{\delta}_x dS \\ &\quad + n_x\left[\tau_{zr}(\boldsymbol{\delta}_r\cdot\boldsymbol{n}) + \tau_{zz}(\boldsymbol{\delta}_z\cdot\boldsymbol{n})\right] dS \ . \end{aligned}$$

We employ spherical polar coordinates (ρ, χ, θ), noting the following identities:

$$\boldsymbol{\delta}_r\cdot\boldsymbol{n} = \sin\chi \ , \qquad \boldsymbol{\delta}_z\cdot\boldsymbol{n} = \cos\chi \ ,$$

$$\boldsymbol{\delta}_r\cdot\boldsymbol{\delta}_x = \cos\theta \ , \qquad \boldsymbol{\delta}_\theta\cdot\boldsymbol{\delta}_x = -\sin\theta \ , \qquad \boldsymbol{n}\cdot\boldsymbol{\delta}_x = \sin\chi\cos\theta \ .$$

We also use the stress tensor expressions in cylindrical coordinates and the velocity representation to obtain the following dimensionless expression for dF_x and dT_y:

$$\begin{aligned} dF_x &= \sin\chi\left[-P\cos^2\theta + 2\frac{\partial U}{\partial r}\cos^2\theta - \frac{\partial V}{\partial r}\sin^2\theta\right] \\ &+ \cos\chi\left[\frac{\partial U}{\partial z}\cos^2\theta - \frac{\partial V}{\partial z}\sin^2\theta + \frac{\partial W}{\partial r}\cos^2\theta\right] \end{aligned}$$

$$
\begin{aligned}
dT_y &= \sin\chi\cos\chi\left[+4\frac{\partial U}{\partial r}\cos^2\theta - \frac{\partial V}{\partial r}\sin^2\theta\right] \\
&+ \cos^2\chi\left[\frac{\partial U}{\partial z}\cos^2\theta - \frac{\partial V}{\partial z}\sin^2\theta + \frac{\partial W}{\partial r}\cos^2\theta\right] \\
&- \sin^2\chi\left[\frac{\partial U}{\partial z}\cos^2\theta + \frac{\partial W}{\partial r}\cos^2\theta\right] .
\end{aligned}
$$

The boundary conditions $U = -V = 1$ and $W = 0$ and the equation of continuity were used to simplify these expressions.

We may integrate the $\cos^2\theta$ and $\sin^2\theta$ factors over $0 \le \theta \le 2\pi$ and convert to the stretched variables to arrive at the following expressions for the force and torque:

$$
\begin{aligned}
\frac{F_x}{\pi\mu a U_A} &= \int_0^\pi \left\{\left[-P + \epsilon^{-1/2}\left(2\frac{\partial U}{\partial R} - \frac{\partial V}{\partial R}\right)\right]\sin\chi\right. \\
&\left. + \left[\epsilon^{-1}\left(\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z}\right) + \epsilon^{-1/2}\frac{\partial W}{\partial R}\right]\cos\chi\right\}\sin\chi\, d\chi \\
\frac{T_y}{\pi\mu a^2 U_A} &= \int_0^\pi \left\{\left[\epsilon^{-1}\left(\frac{\partial U}{\partial Z} - \frac{\partial V}{\partial Z}\right) + \epsilon^{-1/2}\frac{\partial W}{\partial R}\right]\cos^2\chi\right. \\
&-\epsilon^{-1/2}\left(4\frac{\partial U}{\partial R} - \frac{\partial V}{\partial R}\right)\sin\chi\cos\chi \\
&- \left.\left[\epsilon^{-1}\frac{\partial U}{\partial Z} + \epsilon^{-1/2}\frac{\partial W}{\partial R}\right]\sin^2\chi\right\}\sin\chi\, d\chi .
\end{aligned}
$$

The extent of the gap region is denoted by the angle χ_0, with $\pi - \chi_0 \ll 1$; now change variables as

$$
\begin{aligned}
R &= \epsilon^{-1/2}\sin\chi \\
d\chi &= \frac{\epsilon^{1/2} dR}{\sqrt{1-\epsilon R^2}} .
\end{aligned}
$$

The resulting integrals for the force and torque are then expanded in powers of ϵ as

$$
\begin{aligned}
\frac{F_x}{\pi\mu a U_A} &= \int_0^{R_0}\left[-P_0 R + \left(\frac{\partial V_0}{\partial Z} - \frac{\partial U_0}{\partial Z}\right)\right] R dR \\
&+ \epsilon\int_0^{R_0}\left[\frac{P_0}{2}R^3 + P_1 R - \left(2\frac{\partial U_0}{\partial R} - \frac{\partial V_0}{\partial R}\right) R\right. \\
&\left. + \left(\frac{\partial U_1}{\partial Z} - \frac{\partial V_1}{\partial Z}\right) + \frac{\partial W_0}{\partial R}\right] R dR \\
&= -\frac{16\beta(2+\beta+2\beta^2)}{5(1+\beta)^3}\ln R_0 \qquad (9.22) \\
&- \frac{16(16 - 45\beta + 58\beta^2 - 45\beta^3 + 16\beta^4)}{125(1+\beta)^4}\epsilon\ln R_0 + \ldots
\end{aligned}
$$

$$
\begin{aligned}
\frac{T_y}{\pi\mu a^2 U_A} &= \int_0^{R_0}\left(\frac{\partial U_0}{\partial Z}-\frac{\partial V_0}{\partial Z}\right)RdR \\
&\quad+\epsilon\int_0^{R_0}\left[-\frac{3}{2}R^2\frac{\partial U_0}{\partial Z}-\frac{1}{2}R^2\frac{\partial V_0}{\partial Z}+R\left(4\frac{\partial U_0}{\partial R}-\frac{\partial V_0}{\partial R}\right)\right. \\
&\qquad\left.(1-R^2)\frac{\partial W_0}{\partial R}+\frac{\partial U_1}{\partial Z}-\frac{\partial V_1}{\partial Z}\right]RdR \\
&= \frac{8\beta(4+\beta)}{5(1+\beta)^2}\ln R_0 \qquad (9.23)\\
&\quad+\frac{8(32-33\beta+83\beta^2+43\beta^3)}{125(1+\beta)^3}\epsilon\ln R_0+\dots\,.
\end{aligned}
$$

Here, $R_0 = r_0/\epsilon^{1/2}$ corresponds to the limit of the lubrication region. As discussed by ONeill and Stewartson [59], a precise assignment of R_0 is not required; instead, a matching procedure is used. The above "inner" solution is matched in the limit of large R_0 with the corresponding entity from the "outer" solution. The principal result is that the leading term may be obtained by replacing $\ln R_0$ with $-(1/2)\ln\epsilon$, with corrections of $O(1)$ and $O(\epsilon)$ determined from the details of the matching with the outer solution. The lubrication results for the force and torque on sphere A are thus [39]

$$
\begin{aligned}
\frac{F_x}{6\pi\mu a U_A} &= -\frac{4\beta(2+\beta+2\beta^2)}{15(1+\beta)^3}\ln\frac{1}{\epsilon}+A(\beta) \qquad (9.24)\\
&\quad-\frac{4(16-45\beta+58\beta^2-45\beta^3+16\beta^4)}{375(1+\beta)^4}\epsilon\ln\frac{1}{\epsilon}+O(\epsilon)\\
\frac{T_y}{8\pi\mu a^2 U_A} &= \frac{\beta(4+\beta)}{10(1+\beta)^2}\ln\frac{1}{\epsilon}+B(\beta) \qquad (9.25)\\
&\quad+\frac{(32-33\beta+83\beta^2+43\beta^3)}{250(1+\beta)^3}\epsilon\ln\frac{1}{\epsilon}+O(\epsilon)\,.
\end{aligned}
$$

In principle, the $O(1)$ terms $A(\beta)$ and $B(\beta)$ should be determined by matching with the outer solution. In practice, values for $A(\beta)$ and $B(\beta)$ may be obtained by a curve fit of numerical results for small ϵ.

The companion problem of calculating forces and torques produced by the rotation of sphere A about the y-axis (see Figure 9.2) can be solved with almost identical steps, with U_A replaced by $\omega_A a$. The final result for the force and torque are [39]

$$
\begin{aligned}
\frac{F_x}{8\pi\mu a^2\omega_A} &= \frac{\beta(4+\beta)}{10(1+\beta)^2}\ln\frac{1}{\epsilon}+B(\beta) \qquad (9.26)\\
&\quad+\frac{(32-33\beta+83\beta^2+43\beta^3)}{250(1+\beta)^3}\epsilon\ln\frac{1}{\epsilon}+O(\epsilon)\\
\frac{T_y}{8\pi\mu a^3\omega_A} &= -\frac{2\beta}{5(1+\beta)}\ln\frac{1}{\epsilon}+C(\beta) \qquad (9.27)\\
&\quad-\frac{2(8+6\beta+33\beta^2)}{125(1+\beta)^2}\epsilon\ln\frac{1}{\epsilon}+O(\epsilon)\,.
\end{aligned}
$$

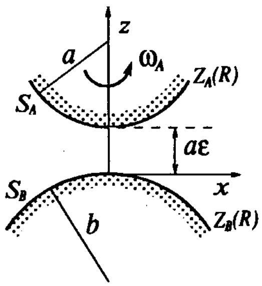

Figure 9.2: Flow produced by a shearing rotation of one sphere near another.

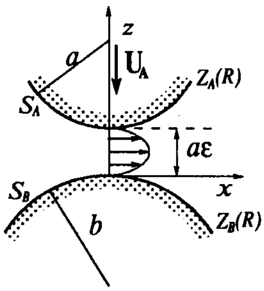

Figure 9.3: Squeezing flow produced by a sphere approaching a fixed sphere.

Note that the same coefficient appears in Equations 9.26 and 9.25 as required by the Lorentz reciprocal theorem.

9.3 Squeezing Motions of Rigid Surfaces

Here we follow closely the development in [37]. Consider a moving sphere (again denoted by S_A) of radius a and a stationary sphere (S_B) of radius $b = \beta a$. As shown in Figure 9.3, we choose a Cartesian coordinate system in which sphere S_A moves in the negative z-direction, with velocity U_A, and in which the sphere-sphere axis is coincident with the z-axis. The stationary sphere lies in the lower half space with the xy-plane as a tangent plane at the origin. The clearance between the two spheres is denoted by $a\epsilon$. In this case, the motion of one sphere toward the other results in a squeezing motion of the interstitial fluid and produces a stronger singularity in the hydrodynamic force.

Since this flow is axisymmetric, we find it convenient to use the Stokes streamfunction ψ, with

$$\begin{aligned} v_r &= U_A \frac{1}{r}\frac{\partial \psi}{\partial z} \\ v_z &= -U_A \frac{1}{r}\frac{\partial \psi}{\partial r} \\ v_\theta &= 0 \,. \end{aligned}$$

The governing equations then become

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}\right)^2 \psi = 0 \,, \tag{9.28}$$

with the boundary conditions

$$\psi = \frac{1}{2}r^2 \,, \qquad \frac{\partial \psi}{\partial z} = 0 \qquad \text{on } z = z_A,$$

$$\psi = \frac{\partial \psi}{\partial z} = 0 \qquad \text{on } z = z_B,$$

and with z_A and z_B as in the previous subsection on shearing motions. We convert again to the "stretched" coordinates,

$$R = \epsilon^{-1/2}\frac{r}{a} \qquad Z = \epsilon^{-1}\frac{z}{a} \,.$$

In these new coordinates, the sphere surfaces are given by

$$Z_A = 1 + \frac{1}{2}R^2 + \frac{1}{8}\epsilon R^4 + O(\epsilon^2) \tag{9.29}$$

$$Z_B = -\frac{1}{2\beta}R^2 - \frac{1}{(2\beta)^3}\epsilon R^4 + O(\epsilon^2) \,, \tag{9.30}$$

and the governing equation becomes

$$\left[\frac{\partial^2}{\partial Z^2} + \epsilon\left(\frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R}\right)\right]^2 \psi = 0 \,. \tag{9.31}$$

Now the boundary conditions imply that v_z/U_A is O(1), so the streamfunction must scale as ϵ. We insert a regular expansion,

$$\psi(R,Z) = a^2\epsilon(\psi_0(R,Z) + \epsilon\psi_1(R,Z) + \psi_2(R,Z) + \ldots) \,, \tag{9.32}$$

into the governing equation to obtain a hierarchy of equations,

$$\begin{aligned} \frac{\partial^4 \psi_0}{\partial Z^4} &= 0 \\ \frac{\partial^4 \psi_1}{\partial Z^4} &= -2\Upsilon \frac{\partial^2 \psi_0}{\partial Z^2} \\ \frac{\partial^4 \psi_2}{\partial Z^4} &= -2\Upsilon \frac{\partial^2 \psi_1}{\partial Z^2} - \Upsilon^2 \psi_0 \,, \end{aligned}$$

where

$$\Upsilon = \frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} .$$

The boundary conditions are obtained by taking the Taylor expansions at $Z = Z_A$ and $Z = Z_B$. For ψ_0, they are

$$\psi_0 = \frac{1}{2}R^2 \qquad \frac{\partial \psi_0}{\partial Z} = 0 , \qquad \text{on } Z = Z_A$$

$$\psi_0 = 0 \qquad \frac{\partial \psi_0}{\partial Z} = 0 , \qquad \text{on } Z = Z_B .$$

For ψ_1, they are

$$\psi_1 = -\frac{1}{8}R^4 \frac{\partial \psi_0}{\partial Z} = 0 , \qquad \frac{\partial \psi_1}{\partial Z} = -\frac{1}{8}R^4 \frac{\partial^2 \psi_0}{\partial Z^2} , \qquad \text{on } Z = Z_A$$

$$\psi_1 = \frac{1}{8\beta^3}R^4 \frac{\partial \psi_0}{\partial Z} = 0 , \qquad \frac{\partial \psi_1}{\partial Z} = \frac{1}{8\beta^3}R^4 \frac{\partial^2 \psi_0}{\partial Z^2} , \qquad \text{on } Z = Z_B .$$

For ψ_2, they are

$$\begin{aligned}
\psi_2 &= -\frac{1}{8}R^4 \frac{\partial \psi_1}{\partial Z} - \frac{1}{16}R^6 \frac{\partial \psi_0}{\partial Z} - \frac{1}{128}R^8 \frac{\partial^2 \psi_0}{\partial Z^2} \\
&= \frac{1}{128}R^8 \frac{\partial^2 \psi_0}{\partial Z^2} \qquad \text{on } Z = Z_A \\
\frac{\partial \psi_2}{\partial Z} &= -\frac{1}{8}R^4 \frac{\partial^2 \psi_1}{\partial Z^2} - \frac{1}{16}R^6 \frac{\partial^2 \psi_0}{\partial Z^2} - \frac{1}{128}R^8 \frac{\partial^3 \psi_0}{\partial Z^3} \qquad \text{on } Z = Z_A \\
\psi_2 &= \frac{1}{128\beta^6}R^8 \frac{\partial^2 \psi_0}{\partial Z^2} \qquad \text{on } Z = Z_B \\
\frac{\partial \psi_2}{\partial Z} &= \frac{1}{8\beta^3}R^4 \frac{\partial^2 \psi_1}{\partial Z^2} + \frac{1}{16\beta^5}R^6 \frac{\partial^2 \psi_0}{\partial Z^2} - \frac{1}{128\beta^6}R^8 \frac{\partial^3 \psi_0}{\partial Z^3} \qquad \text{on } Z = Z_B .
\end{aligned}$$

We now turn our attention to the solution of these equations and the associated result for the force on the spheres. The exact expression for the contribution to the force is

$$\frac{F_z}{\pi \mu a U_A} = \int_0^\pi r^3 \frac{\partial}{\partial n} \frac{E^2 \psi}{r^2} \, ds ,$$

where ds is an element of arc length (in radians) along the meridian. On the moving sphere,

$$\begin{aligned}
ds &= \epsilon^{1/2}(1 + \frac{\epsilon}{2}R^2 + \frac{3\epsilon^2}{8}R^4)dR + O(\epsilon^{5/2}) \\
\frac{\partial}{\partial n} &= -\epsilon^{-1}\frac{\partial}{\partial Z} + \frac{R^2}{2}\frac{\partial}{\partial Z} + R\frac{\partial}{\partial R} + \epsilon\frac{R^4}{8}\frac{\partial}{\partial Z} + O(\epsilon^2) .
\end{aligned}$$

On the stationary sphere,

$$\begin{aligned}
ds &= \epsilon^{1/2}(1 + \frac{\epsilon}{2\beta^2}R^2 + \frac{3\epsilon^2}{8\beta^4}R^4)dR + O(\epsilon^{5/2}) \\
\frac{\partial}{\partial n} &= \epsilon^{-1}\frac{\partial}{\partial Z} - \frac{R^2}{2\beta^2}\frac{\partial}{\partial Z} - \frac{R}{\beta}\frac{\partial}{\partial R} - \epsilon\frac{R^4}{8\beta^4}\frac{\partial}{\partial Z} + O(\epsilon^2) .
\end{aligned}$$

The extent of the gap region where lubrication effects dominate is denoted by R_0 as before, and the integral for the force expands in powers of ϵ as

$$\frac{F_x}{\pi\mu a U_A} = \epsilon^{-1} I_0 + I_1 + \epsilon I_2 ,$$

with

$$\begin{aligned}
I_0 &= \int_0^{R_0} \left[R\frac{\partial^3\psi_0}{\partial Z^3} \right]_{Z=Z_A} dR \\
I_1 &= \int_0^{R_0} \left[R\frac{\partial^3\psi_1}{\partial Z^3} - \frac{\partial^2\psi_0}{\partial Z\partial R} + 2\frac{\partial^2\psi_0}{\partial Z^2} + R\frac{\partial^3\psi_0}{\partial Z\partial R^2} - R^2\frac{\partial^3\psi_0}{\partial Z^2\partial R} \right]_{Z=Z_A} dR \\
I_2 &= \int_0^{R_0} \left[R\frac{\partial^3\psi_2}{\partial Z^3} - \frac{\partial^2\psi_1}{\partial Z\partial R} + R\frac{\partial^2\psi_1}{\partial Z^2} + 2R\frac{\partial^3\psi_1}{\partial Z\partial R^2} - R^2\frac{\partial^3\psi_1}{\partial Z^2\partial R} \right. \\
&\quad \left. -3\frac{\partial\psi_0}{\partial R} + 3\frac{\partial^2\psi_0}{\partial R^2} + R^3\frac{\partial^2\psi_0}{\partial Z^2} - R^2\frac{\partial^3\psi_0}{\partial R^3} - \frac{R^2}{4}\frac{\partial^3\psi_0}{\partial Z^2\partial R} \right]_{Z=Z_A} dR .
\end{aligned}$$

These expressions indicate the specific information required from the asymptotic expansion of the streamfunction.

From the equation for ψ_0, we get

$$\psi_0 = A_0(R)Z^3 + B_0(R)Z^2 + C_0(R)Z + D_0(R) ,$$

with

$$A_0 = R^2/H^3 , \qquad B_0 = \frac{3}{2}R^2(Z_A + Z_B)/H^3 ,$$

$$C_0 = -3R^2 Z_A Z_B/H^3 , \qquad D_0 = \frac{1}{2}R^2(3Z_A - Z_B)Z_B^2/H^3 ,$$

determined from the boundary conditions.

The solution for ψ_1 is

$$\psi_1 = A_1(R)Z^3 + B_1(R)Z^2 + C_1(R)Z + D_1(R) - \frac{Z^5}{10}\Upsilon A_0 - \frac{Z^4}{6}\Upsilon B_0 ,$$

with

$$\begin{aligned}
A_1 &= (3Z_A^2 + 4Z_A Z_B + 3Z_B^2)\frac{1}{10}\Upsilon A_0 + (Z_A + Z_B)\frac{1}{3}\Upsilon B_0 \\
&\quad + \frac{3(1+\beta^3)}{8\beta^3}\frac{R^6}{H^4} \\
B_1 &= -(Z_A + Z_B)(Z_A^2 + 3Z_A Z_B + Z_B^2)\frac{1}{5}\Upsilon A_0 - (Z_A^2 + 4Z_A Z_B + Z_B^2)\frac{1}{6}\Upsilon B_0 \\
&\quad - \frac{3}{8}\frac{R^6}{H^4}[Z_A + 2Z_B + \frac{1}{\beta^3}(Z_B + 2Z_A)] \\
C_1 &= Z_A Z_B(4Z_A^2 + 7Z_A Z_B + 4Z_B^2)\frac{1}{10}\Upsilon A_0 + Z_A Z_B(Z_A + Z_B)\frac{1}{3}\Upsilon B_0 \\
&\quad + \frac{3}{8}\frac{R^6}{H^4}[Z_B(2Z_A + Z_B) + \frac{1}{\beta^3}Z_A(2Z_B + Z_A)]
\end{aligned}$$

$$
\begin{aligned}
D_1 &= -Z_A^2 Z_B^2 (Z_A + Z_B)\frac{1}{5}\Upsilon A_0 - Z_A^2 Z_B^2 \frac{1}{6}\Upsilon B_0 \\
&\quad - \frac{3}{8}\frac{R^6}{H^4} Z_A Z_B (Z_B + \frac{1}{\beta^3} Z_A) .
\end{aligned}
$$

The solution for ψ_2 is

$$
\begin{aligned}
\psi_2 &= A_2(R)Z^3 + B_2(R)Z^2 + C_2(R)Z + D_2(R) \\
&\quad + \frac{Z^7}{280}\Upsilon^2 A_0 + \frac{Z^6}{120}\Upsilon^2 B_0 - \frac{Z^5}{10}(\Upsilon A_1 + \frac{1}{12}\Upsilon^2 C_0 \\
&\quad - \frac{Z^4}{6}(\Upsilon B_1 + \frac{1}{4}\Upsilon^2 D_0 .
\end{aligned}
$$

From the expression for the force, we see that only A_2 is needed and this turns out to be

$$
\begin{aligned}
A_2 &= -(5Z_A^4 + 8Z_A^3 Z_B + 9Z_A^2 Z_B^2 + 8Z_A Z_B^3 + 5Z_B^4)\frac{1}{280}\Upsilon^2 A_0 \\
&\quad - (2Z_A^3 + 3Z_A^2 Z_B + 3Z_A Z_B^2 + 2Z_B^3)\frac{1}{60}\Upsilon^2 B_0 \\
&\quad + \frac{1}{10}(3Z_A^2 + 4Z_A Z_B + 3Z_B^2)(\Upsilon A_1 + \frac{1}{12}\Upsilon^2 C_0) \\
&\quad + \frac{1}{3}(Z_A + Z_B)(\Upsilon B_1 + \frac{1}{4}\Upsilon^2 D_0) \\
&\quad + \frac{3}{32}\frac{(1+\beta^6)}{\beta^6}\frac{R^{10}}{H^5} + \frac{3}{16}\frac{(1+\beta^5)}{\beta^5}\frac{R^8}{H^4} \\
&\quad \frac{1}{4}\frac{R^4}{H^2}[(Z_A^3 - \frac{1}{\beta^3}Z_B^3)\Upsilon A_0 + (Z_A^2 - \frac{1}{\beta^3}Z_B^2)\Upsilon B_0 \\
&\quad - 3A_1(Z_A - \frac{1}{\beta^3}Z_B) - B_1(1 - \frac{1}{\beta^3})] .
\end{aligned}
$$

The integrand for I_0 is

$$
6A_0 = -\frac{6R^3}{H^3} ,
$$

and the asymptotic behavior of those for I_1 and I_2 are

$$
-\frac{2(1 + 7\beta + \beta^2)}{5(1+\beta)^3 R}
$$

and

$$
\frac{24 + 61\beta - 206\beta^2 - 499\beta^3 - 256}{525(1+\beta)^3}R - \frac{2(1 + 18\beta - 29\beta^2 + 18\beta^3 + \beta^4)}{21(1+\beta)^4 R} .
$$

As discussed in O'Neill and Stewartson [59], the $O(R^{-1})$ terms will produce the $\ln\epsilon$ type singularity, while the $O(R)$ terms will be canceled during the matching process with the outer solution. Thus the lubrication result for the

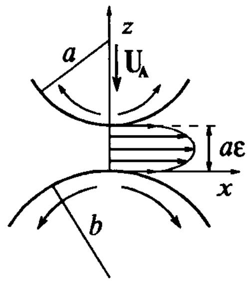

Figure 9.4: Squeezing flow produced by a drop approaching another drop.

force on sphere A is

$$\frac{F_x}{6\pi\mu a U_A} = -\frac{\beta^2}{(1+\beta)^2}\epsilon^{-1} - \frac{(1+7\beta+\beta^2)}{5(1+\beta)^3}\ln\frac{1}{\epsilon} + K(\beta) \qquad (9.33)$$
$$- \frac{(1+18\beta-29\beta^2+18\beta^3+\beta^4)}{21(1+\beta)^4}\epsilon\ln\frac{1}{\epsilon} + O(\epsilon) \ .$$

In principle, the $K(\beta)$ and $L(\beta)$ should be determined by matching with the outer solution. In practice, these values may be obtained by a curve fit of numerical results for small ϵ. It should also be pointed out that a direct asymptotic analysis of the exact solution in bispherical coordinates has been done for two equal spheres ($\beta = 1$) [23, 26] and for the sphere and plane ($\beta = \infty$) [8], and that these results are consistent with the form proposed in Equation 9.33.

9.4 Squeezing Flow Between Viscous Drops

Two viscous drops approaching each other produce a squeezing flow that is different from that produced by the motion of rigid particles, as shown by Davis *et al.* [12]. Because of the mobility of the fluid-fluid interface, there is a slip velocity superposed on top of the parabolic profile (see Figure 9.4), and the pressure buildup is much less than that of the rigid case. The force singularity then scales as $\epsilon^{-1/2}$ instead of ϵ^{-1}, so that two drops pushed toward each other by a constant attractive force can meet in a finite time.

We write the radial component of the velocity as the sum of a uniform slip velocity, $u_t(r)$, and the familiar parabolic profile, $u_p(r,z)$, of the previous section:

$$v_r(r,z) = u_t(r) + u_p(r,z) \ ,$$

with

$$u_p(r,z) = \frac{1}{2\mu}\frac{\partial p}{\partial r}(z-z_A)(z-z_B) \ .$$

To find the expression for this unknown slip velocity, we examine the tractions at the fluid-fluid interface. To leading order, we may neglect surface curvature in the gap region so that the surface normals point in the z-direction. We denote the relevant component of the surface traction by f_t, and

$$f_t = -\mu \frac{\partial v_r}{\partial z}|_{z=z_A} = \mu \frac{\partial v_r}{\partial z}|_{z=z_B} = -(z_A - z_B)\frac{\partial p}{\partial r} \ . \tag{9.34}$$

The mass balance provides one relation between the velocities and the traction:

$$\pi r^2 U_A = 2\pi r \int_{z_A}^{z_B} v_r \, dz = 2\pi r \left((z_A - z_B) u_t + \frac{(z_A - z_B)^2}{6\mu} f_t \right) \ . \tag{9.35}$$

We obtain a second equation from the integral representation for the flow *inside* the drop:

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{8\pi\lambda\mu} \oint_S (\boldsymbol{\sigma} \cdot \boldsymbol{n}) \cdot \mathcal{G}(\boldsymbol{x} - \boldsymbol{\xi}) \, dS(\boldsymbol{\xi}) + \oint_S \boldsymbol{v}(\boldsymbol{\xi}) \cdot \boldsymbol{\Sigma}(\boldsymbol{x} - \boldsymbol{\xi}) \cdot \boldsymbol{n} \, dS(\boldsymbol{\xi}) \ .$$

Here, $\lambda\mu$ is the viscosity of the drop and $\boldsymbol{n}$ points from the drop interior to the exterior. Now consider the interfacial point $\boldsymbol{\eta}$ in the gap region. We know that in the gap (where the dominant contribution originates) the interface is essentially flat and thus the double layer term vanishes. Recalling the jump condition, we replace the LHS of the velocity representation with $\boldsymbol{v}(\boldsymbol{\eta})/2$ and insert the appropriate expressions for the surface tractions to simplify the integral equation. As in the rigid case, we extend the gap region to infinity with the understanding that this approximation does not affect the leading order term. The final result is

$$\begin{aligned} u_t(r) &= \frac{1}{\lambda\mu} \int_0^\infty K(r,s) f_t(s) \, ds \\ K(r,s) &= \frac{1}{2\pi} \frac{s}{\sqrt{r^2+s^2}} \int_0^\pi \frac{\cos\phi \, d\phi}{\sqrt{1-k^2\cos\phi}} \\ k^2 &= \frac{2rs}{r^2+s^2} \ . \end{aligned} \tag{9.36}$$

The kernel $K(r,s)$ involves an elliptic integral, and at $k = 1$ $(r = s)$ it possesses a logarithmic singularity that we readily attribute to the behavior of the Oseen tensor $\mathcal{G}(\boldsymbol{\eta} - \boldsymbol{\xi})$ for $\boldsymbol{\eta}$ near $\boldsymbol{\xi}$.

A scaling analysis is required at this point to delineate different regimes that differ in the degree of surface mobility. The most important parameter is the ratio u_t/u_p. When this parameter is small, we expect essentially the same behavior as in the rigid case (where this parameter is identically zero). On the other hand, if u_t/u_p is large, then the interface is *mobile* and we may obtain solutions quite different from the rigid case.

At the fluid-fluid interface, we have the traction boundary condition,

$$\mu \frac{\partial u_p}{\partial z} = \lambda\mu \frac{\partial v_r}{\partial z} \ .$$

In the exterior fluid, we scale $\partial u_p/\partial z \sim u_p/(a\epsilon)$. Inside the drop, $v_r \sim u_t$ and *both* the radial and axial lengths scale as $\sqrt{\bar{a}(a\epsilon)}$, where $\bar{a} = ab/(a+b)$ is the "reduced radius." This then implies $\partial u/\partial z \sim u_t/\sqrt{\bar{a}a\epsilon}$. This motivates Davis *et al.*'s [12] definition of the *interfacial mobility* parameter, m:

$$\frac{u_t}{u_p} \sim \lambda^{-1}\sqrt{\frac{\bar{a}}{a\epsilon}} = m \ . \tag{9.37}$$

Drops with nearly immobile interfaces have $m \ll 1$ (in other words, $u_t \ll u_p$) and behave essentially as rigid particles. On the other hand, under conditions where $m \gg 1$ ($u_t \gg u_p$), we obtain results that are quite different from those seen previously for rigid particles.

9.4.1 Nearly Rigid Drops

For nearly rigid drops ($m \ll 1$), the system of equations may be solved by a regular perturbation in m, with the leading term of each expression equal to the corresponding term from the rigid case [12]. The lubrication result for the force may be written, to leading order in ϵ, as

$$\frac{F_z}{6\pi\mu\bar{a}U_A} = \frac{\bar{a}}{a\epsilon}(1 - 1.31m + 1.78m^2 - 2.46m^3 + 3.44m^4 - 4.83m^5 + \cdots) \ .$$

9.4.2 Fully Mobile Interfaces

For fully mobile interfaces ($m \gg 1$) the drops offer no resistance to the outflow, so we simply neglect the parabolic profile, and obtain $u_t(r) = U_A r/(a\epsilon)$. We put this into the integral representation, Equation 9.36, and solve for f_t. This inversion is greatly simplified by the exact formula from Jansons and Lister [35],

$$f_t(r) = 8\lambda\mu\int_0^\infty K(r,s)\left(\frac{u_t}{s^2} - \frac{1}{s}\frac{du_t}{ds} - \frac{d^2u_t}{ds^2}\right) ds \ . \tag{9.38}$$

Now that f_t is known, the pressure distribution $p(r)$ can be obtained by integrating Equation 9.34, with $p(\infty) = 0$. The force follows as

$$F_z = 2\pi\int_0^\infty p(r)r\,dr = 6\pi\mu\bar{a}(2U_A)\left(\frac{\bar{a}}{a\epsilon}\right)\left(\frac{0.876}{m}\right) \approx 33.0\lambda\mu\bar{a}U_A\sqrt{\frac{\bar{a}}{a\epsilon}} \ .$$

Thus for a fully mobile interface, the force behaves as the *inverse square root* of the dimensionless gap width.

9.4.3 Solution in Bispherical Coordinates

The problem of two viscous drops translating along their line of centers can be solved by separating the equation for the streamfunction, $E^4\psi = 0$, in bispherical coordinates, in a manner analogous to the Stimson–Jeffrey solution for rigid spheres. Haber *et al.* [24] consider a very general situation with unequal drop sizes, velocities, and viscosities. For the case of two identical drops moving toward each other with equal speed, their solution reduces to

$$\frac{F_1}{6\pi\mu a U_1} = \frac{2}{3}\sinh\alpha \sum_{n=1}^{\infty} C_n \frac{N_n^{(0)}(\alpha,\lambda) + \lambda N_n^{(1)}(\alpha,\lambda)}{D_n^{(0)}(\alpha,\lambda) + \lambda D_n^{(1)}(\alpha,\lambda)} , \tag{9.39}$$

with

$$\begin{aligned} N_n^{(0)}(\alpha,\lambda) &= 2[(2n+1)\sinh 2\alpha + 2\cosh 2\alpha - 2e^{-(2n+1)\alpha}] \\ N_n^{(1)}(\alpha,\lambda) &= (2n+1)^2\cosh 2\alpha - 2(2n+1)\sinh 2\alpha \\ &\quad - (2n+3)(2n-1) + 4e^{-(2n+1)\alpha} \\ D_n^{(0)}(\alpha,\lambda) &= 4\sinh(n-\tfrac{1}{2})\alpha \sinh(n+\tfrac{3}{2})\alpha \\ D_n^{(1)}(\alpha,\lambda) &= 2\sinh(2n+1)\alpha - (2n+1)\sinh 2\alpha \\ C_n &= \frac{n(n+1)}{(2n-1)(2n+3)} . \end{aligned}$$

Here, F_1 and U_1 denote the force and speed of drop 1, and α is related to the drop-drop separation by $R = 2a\cosh\alpha$. Beshkov *et al.* [3] have considered the same problem, but their result for the force contains an error (effectively, their expression for C_n is $(n+1)/((2n+2)(2n-1))$ instead of the correct result above), so that their asymptotic expressions for the force between two nearly touching drops is also in error — by a factor of exactly $\sqrt{2}$, as inferred correctly by Davis *et al.* after a comparison with their numerical solution. For large λ, this expression above also reduces to the result for rigid spheres.

For small gaps, $\alpha \sim \epsilon^{1/2}$ is also small, and therefore many terms are needed in the bispherical solution. Following Cox and Brenner [8], we break the sum into an "inner sum" $\sum_{n=1}^{N}$ and an "outer sum" $\sum_{n=N+1}^{\infty}$. The breakpoint N is determined by requiring $\alpha N \sim 1$. Then in the inner sum, $\alpha n \ll 1$ for $\alpha \to 0$, and the leading expression in the asymptotic expansion for small α simplifies to

$$\begin{aligned} \frac{F_1^{(inner)}}{6\pi\mu a U_1} &= \frac{2\lambda}{3\alpha}\sum_{n=1}^{N}\frac{8n(n+1)}{(2n-1)^2(2n+3)^2} \\ &= \frac{2\lambda}{3\alpha}\left\{\frac{5}{16}\sum_{n=1}^{N}\left[\frac{1}{2n-1}-\frac{1}{2n+3}\right]+\frac{3}{8}\sum_{n=1}^{N}\left[\frac{1}{(2n-1)^2}-\frac{1}{(2n+3)^2}\right]\right\} \\ &= \frac{2\lambda}{3\alpha}\left\{\frac{5}{16}\left[\frac{4}{3}+O(N^{-1})\right]+\frac{3}{8}\left[-\frac{10}{9}+O(N^{-1})\right]\right\} \\ &= \frac{2\lambda}{3\alpha}\left\{\frac{3\pi^2}{32}+O(N^{-1})\right\} . \end{aligned}$$

As $\alpha \to 0$, the outer sum is bounded, so the leading term comes solely from the inner sum. For two equal spheres, the reduced radius is given by $\bar{a} = a/2$, and the final result for the leading behavior of the force may be written as

$$F_1 = \frac{3}{4}\pi^3\sqrt{2}\lambda\mu\bar{a}U_1\sqrt{\frac{\bar{a}}{a\epsilon}} \approx 32.887\lambda\mu\bar{a}U_1\sqrt{\frac{\bar{a}}{a\epsilon}} ,$$

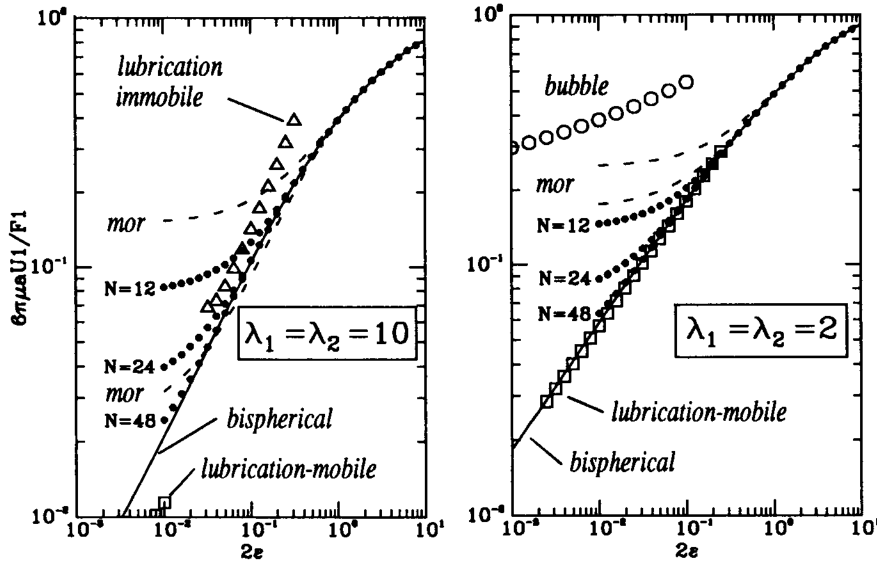

Figure 9.5: Mobility of two equal drops approaching each other: a comparison of different methods.

which is in excellent agreement with the result of Davis *et al.* The mobility functions obtained from lubrication theory, separation in bispherical coordinates, and the image method (Chapter 10) are shown in the figure. We compare the asymptotic solutions with the exact result for $\lambda = 0.1$, 2, and 10. The lubrication and image solutions are in reasonable agreement with the exact solution over their respective domain of validity and overlap quite nicely at moderate drop-drop gaps.

9.4.4 Nearly Touching Bubbles

When $\lambda \ll 1$ (bubble), the flow in the gap no longer gives the dominant contribution to the hydrodynamic force on the drop, and on intuitive grounds we would expect a much weaker singularity than $\epsilon^{-1/2}$ — perhaps even a regular solution, meaning that the bubbles do not offer any significant resistance to coalescense. The truth is actually somewhere in between, since the force is in fact singular, but it is the weak logarithmic singularity.

We set $\lambda = 0$ in Equation 9.39 and consider the behavior of the inner and outer sums. The inner sum simplifies to

$$\frac{F_1^{(inner)}}{6\pi\mu a U_1} = \frac{2}{3}\sum_{n=1}^{N}\left\{\frac{1}{2n-1} + \frac{1}{2n+3} + \frac{3}{4(2n-1)^2} - \frac{3}{4(2n+3)^2}\right\} .$$

The sums may be expanded for large N as

$$\sum_{n=1}^{N} \frac{2}{2n-1} \sim \ln N + \gamma + 2\ln 2 + O(N^{-2}) \ ,$$

where $\gamma = 0.577216\ldots$ is Euler's constant. Therefore, one set of sums simplify as

$$\sum_{n=1}^{N} \left\{ \frac{1}{2n+3} + \frac{1}{2n-1} \right\} \sim \ln N + \gamma + 2\ln 2 - \frac{4}{3} + o(1) \ ,$$

while the other can telescoped to

$$\frac{3}{4} \sum_{n=1}^{N} \left\{ \frac{1}{(2n-1)^2} - \frac{1}{(2n+3)^2} \right\} \sim \frac{5}{6} + O(N^{-2}) \ .$$

We define (see [8]) $X = N\alpha$ so that the inner sum may be written as

$$\frac{F_1^{(inner)}}{6\pi\mu a U_1} \sim \frac{2}{3} \left\{ \ln \alpha^{-1} + \gamma + 2\ln 2 - \frac{1}{2} + \ln X \right\} \ . \tag{9.40}$$

In the outer sum, we fix $\nu = n\alpha$ and let $\alpha \to 0$. The outer sum then simplifies to

$$\frac{F_1^{(outer)}}{6\pi\mu a U_1} \sim \frac{2}{3} \sum_{n=N+1}^{\infty} \Delta\nu \frac{(2\nu + 1 - e^{-2\nu})\Delta\nu}{4\sinh^2 \nu} \ .$$

When $n = N+1$, we have $\nu = (N+1)\alpha \sim X$, and the sum may be approximated using the Euler–MacLaurin formula, so that

$$\frac{F_1^{(outer)}}{6\pi\mu a U_1} \sim \frac{2}{3} \int_X^\infty f(\nu)\, d\nu - \frac{\alpha}{2}[f(\infty) - f(X)] \ ,$$

with

$$f(\nu) = \frac{2\nu + 1 - e^{-2\nu}}{4\sinh^2 \nu} \ .$$

We match with the inner sum by looking at small X. The integral has the asymptotic expansion,

$$\begin{aligned} \int_X^\infty f(\nu)\, d\nu &= \int_{2X}^\infty \frac{t + 1 - e^{-t}}{4(\cosh t - 1)} dt \\ &\sim -\ln X + \frac{1}{2} - \ln 2 + o(1) \ , \end{aligned}$$

so that

$$\frac{F_1^{(outer)}}{6\pi\mu a U_1} = -\ln X + \frac{1}{2} - \ln 2 + o(1) \ . \tag{9.41}$$

Combining Equations 9.40 and 9.41, we obtain the result for the drag on the bubble,

$$\begin{aligned} \frac{F_1}{6\pi\mu a U_1} &= \frac{2}{3}\left[\ln \alpha^{-1} + \gamma + \ln 2\right] + o(1) \\ &= \frac{1}{3}\ln\left(\frac{a}{h}\right) + \frac{2}{3}(\gamma + \ln 2) + o(1) \ . \end{aligned} \tag{9.42}$$

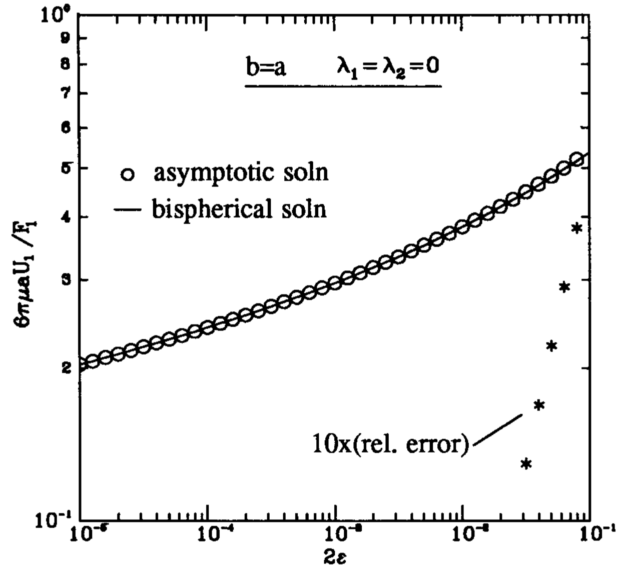

Figure 9.6: Mobility of two bubbles approaching each other: comparison of the asymptotic and exact results.

In the figure, we compare this asymptotic solution with the exact solution obtained by separation in bispherical coordinates. As shown in the figure, the $O(1)$ correction gives a dramatic improvement in accuracy.

9.5 Shearing Flow Between Viscous Drops

The flow between drops in transverse translations is not dominated by the gap region and thus lubrication theory does not apply. The solution must be obtained by numerical methods, such as the boundary collocation method of Chapter 13, but we present the results here to complete the picture of drops in relative motion. In the figure, we show the mobility for two equal drops (at various drop viscosities) for geometries ranging from near contact to large separations (the Hadamard–Rybczynski solution).[2] These results are consistent with those obtained by Zinchenko by bispherical coordinates (see the series of translated articles [72] to [76]).

[2]We acknowledge the assistance of Mr. John Geisz and Dr. Osman Basaran of Oak Ridge National Laboratory in the development of these results.

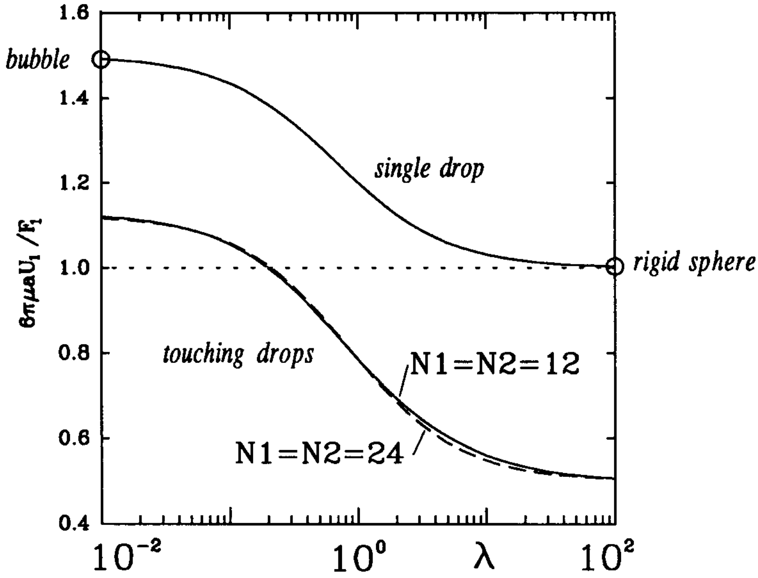

Figure 9.7: Mobility of two drops in transverse translation.

Exercises

Exercise 9.1 Crossed-Cylinders Force Apparatus.
The "crossed-cylinders" force apparatus is used to measure the colloidal forces between surfaces [32, 33, 34]. The colloidal force is deduced from the squeezing motion of the two surfaces — hence the solution for the leading term of $O(\epsilon^{-1})$ is required. Use lubrication theory to calculate this result and compare your answer with that given in the references. Also find the logarithmic singularity for one cylinder sliding over another.

Exercise 9.2 Contact Times for Two Drops.
Consider two equal-sized drops, one heavy and the other neutrally buoyant. If the heavy drop is released above the neutrally buoyant drop, with a small eccentric displacement, estimate the "contact time" between the two surfaces as a function of eccentricity and drop viscosity. Compare the predictions with the experimental data in the article by Yoon Luttrell [70]