Chapter 7

General Formulation of Resistance and Mobility Relations

7.1 Introduction

In our effort to understand the behavior of suspensions, we must take into account the effect of particle-particle interactions, for such interactions exist in all but very dilute suspensions. In this part of the book, we consider several methods for calculating *hydrodynamic interactions* between particles. Our task will be guided in part by the knowledge gained in Part II of the flow that induces or is induced by the motion of a particle.

What do we mean by hydrodynamic interaction? When two particles suspended in a viscous fluid approach each other, the motion of each particle is influenced by the other, even in the absence of interparticle interactions, such as van der Waals and electrostatic forces. The velocity field generated by the motion of one particle is transmitted through the fluid medium and influences the motion of, as well as the hydrodynamic force, torque, and stresslet on the other particle. Thus when two particles move towards each other, for example, as a result of an attractive colloidal force, the hydrodynamic interaction between the two retards the motion; using just the single particle mobility in the attractive force law overestimates the rate of aggregation.

For the sake of organization, we divide the discussion on interactions according to particle-particle and particle-wall interactions (see Figure 7.1). For each, the method of attack depends on the separation between the surfaces. For widely separated particles (the distance between closest points on the surfaces is much greater than particle size), a general asymptotic method known as the method of reflection is available. The solution can be expressed analytically as a series in terms of (the small parameter) particle size over separation.

Surfaces near contact present a far more challenging problem, both from an analytical and computational viewpoint. For rigid surfaces in relative motion, the flow in the gap region dominates and *lubrication theory* provides the leading

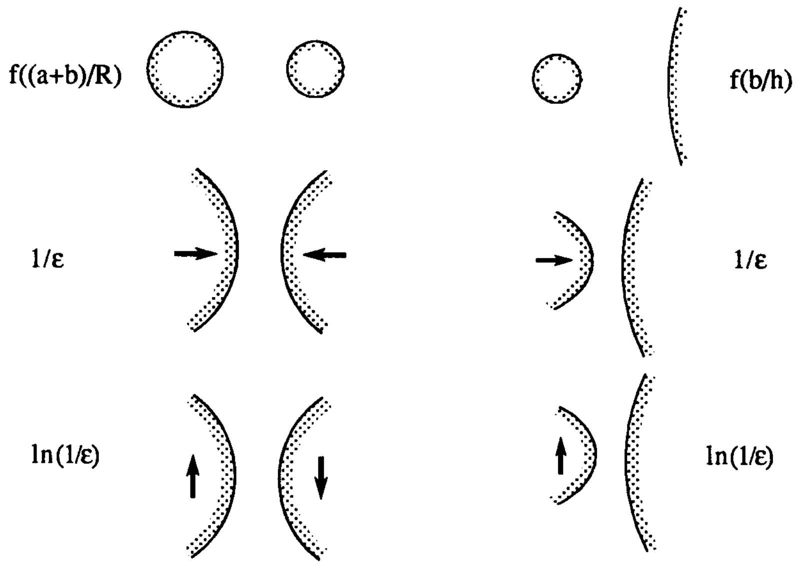

Figure 7.1: The range of hydrodynamic interaction geometries.

terms in an asymptotic expansion. Here, the small parameter ϵ is the gap distance divided by the characteristic size of the particle (usually proportional to the local radius of curvature). For squeezing motions of two rigid surfaces, this leading term is $O(\epsilon^{-1})$ and truly dominates over higher order corrections. On the other hand, for shearing motions (curved surfaces sliding past each other) the leading term is only $O(\ln \epsilon^{-1})$, and thus dominates the solution only in the mathematical sense. Usually, the next term is $O(1)$, and even if we take ϵ based on the ratio of molecular to macroscopic dimensions, $\ln \epsilon^{-1}$ is not a very large number. Unfortunately, the next term cannot be obtained by an asymptotic analysis of the region near the gap; all regions of the particle surface contribute to the $O(1)$ term, and thus a numerical solution is unavoidable.

Relative motion of two viscous drops near contact presents a new wrinkle. Consider the simplest problem of two spherical drops (high surface tension). Squeezing flows now produce a weaker singularity in the leading term of $O(\epsilon^{-1})$ since the surfaces are mobile. For the bubble, the singularity is an even weaker $O(\ln \epsilon^{-1})$. Shearing motions of one drop past another is not singular at all. The drops simply slip past each other with a finite mobility. Since all surface regions contribute to this $O(1)$ result, a matched asymptotic expansion approach is not practical.

Lubrication methods do not apply at all for rigid surfaces moving in tandem as a single rigid body. In fact, the velocity gradient is small in the gap region. In the frame moving with the particles, the gap region is stagnant. Here again, all regions near the bodies contribute to the $O(1)$ result for the resistance and mobility functions and a numerical computation is necessary. Although the flow in the gap is not singular, the gap region provides a challenge to the computational methods. These computations are of additional interest because these $O(1)$ results are closely related to the $O(1)$ corrections to the results of lubrication theory.

In the next section, we discuss the general framework for resistance and mobility problems in the multiparticle setting. As in Part II, we encounter resistance and mobility matrices and rules for transforming one to the other.

In Chapter 8, we discuss the method of reflections for interactions between widely separated particles. We emphasize the importance of direct solutions of mobility problems. For two spheres, we discuss how the method of refletions can be greatly streamlined by the use of addition theorems for spherical harmonics. In essence, the higher order terms in the expansion (which become more important at smaller separations) can be generated on a computer.

In Chapter 9, we discuss the asymptotic methods for two almost-touching particles. As alluded to earlier, we draw a distinction between lubrication flows, such as those produced by shearing or squeezing motions, and non-lubrication flows, for example, the flow produced by two almost-touching particles moving in tandem as a rigid body.

In Chapter 10, we consider interactions between a small "satellite" particle in the vicinity of a much larger particle. These can be described quite efficiently by solutions obtained by image methods. The characteristic feature is the specific relation between three length scales. The separation between the particles is much larger than the dimension of the smaller particle, but is comparable to the dimension of the larger particle. Thus the larger particle is in the "far field" of the smaller particle, so the disturbance field produced by the satellite may be described quite accurately by the first few terms of the multipole expansion. In other words, the smaller particle may be represented by a small collection of Stokes singularities. On the other hand, the smaller particle is not in the far field of the larger particle, and the details of the geometry of the larger particle will be quite important. For each Stokes singularity used in the representation of the smaller particle, we find the exact image field, to satisfy the boundary condition on the surface of the larger particle.

The interaction between two spheres is one of the more frequently used result from microhydrodynamics. Many colloidal systems contain particles that are nearly spherical, as a consequence of the domination of Brownian forces in the aggregation phase of the synthesis. The relatively simple geometry permits the application of a wealth of analytical and numerical techniques. In fact, numerical methods designed for more general problems are quite often tested against the wealth of information available for the two-sphere problem. For these reasons, we gather these results, as well as the numerical results of the later chapters, into a concise summary in Chapter 11.

In Chapter 12, we discuss particle-wall interactions, a degenerate but important case of interactions between two particles. Our discussion also includes a formal derivation of the image system for a planar wall, the so-called Lorentz reflection theorem. We conclude with a brief extension of these ideas to interactions between a drop and a fluid-fluid interface (*Lee reflection lemma*).

In Chapter 13, we present a powerful numerical method for the interaction between a small number of particles, the ***boundary-multipole collocation method.*** The velocity is expanded using a basis set of solutions to the Stokes equations. The coefficients of the expansion are determined by setting the boundary conditions at the collocation points. The key to the method is the availability of a good basis set, so in practice this restricts the method to interactions between particles where each particle alone has a shape that fits a separable coordinate system. An exposition on the convergence of this numerical method is provided.

7.2 Resistance and Mobility Relations

In Chapter 5 we used the linearity of the Stokes equations to affect a decomposition of the resistance and mobility relations for a single particle into smaller subproblems (force on a translating particle, torque on a rotating particle, *etc.*). Here, we apply the same approach to multiparticle hydrodynamic interactions. In the following discussion, it should be clear that the general construction for an N-particle system is, with only minor changes in the notation, essentially the same as the framework for the pair interaction problem. Accordingly, we limit our exposition to pair interactions.

As before, we are primarily interested in the force $\boldsymbol{F}_\alpha$, the torque $\boldsymbol{T}_\alpha$, and the stresslet $\boldsymbol{S}_\alpha$ *exerted by the fluid on the particles.* The particles (labeled by $\alpha = 1,2$) are in rigid-body motion, $\boldsymbol{U}_\alpha + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_\alpha)$, where $\boldsymbol{x}_\alpha$ is a reference point inside particle α, in the ambient field $\boldsymbol{v}^\infty = \boldsymbol{U}^\infty + \boldsymbol{\Omega}^\infty \times \boldsymbol{x} + \boldsymbol{E}^\infty \cdot \boldsymbol{x}$.

7.2.1 The Resistance Matrix

For the resistance problem, the specified quantities are the particle velocities and the ambient field. The unknown moments of the stress distribution over each particle surface may be expressed, as was done earlier for the single particle, as [5, 45]

$$\begin{pmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \\ \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{pmatrix} = \mu \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \tilde{\boldsymbol{B}}_{11} & \tilde{\boldsymbol{B}}_{12} & \tilde{\boldsymbol{G}}_{11} & \tilde{\boldsymbol{G}}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \tilde{\boldsymbol{B}}_{21} & \tilde{\boldsymbol{B}}_{22} & \tilde{\boldsymbol{G}}_{21} & \tilde{\boldsymbol{G}}_{22} \\ \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \boldsymbol{C}_{11} & \boldsymbol{C}_{12} & \tilde{\boldsymbol{H}}_{11} & \tilde{\boldsymbol{H}}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \boldsymbol{C}_{21} & \boldsymbol{C}_{22} & \tilde{\boldsymbol{H}}_{21} & \tilde{\boldsymbol{H}}_{22} \\ \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{H}_{11} & \boldsymbol{H}_{12} & \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} & \boldsymbol{H}_{21} & \boldsymbol{H}_{22} & \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{v}^\infty(\boldsymbol{x}_1) - \boldsymbol{U}_1 \\ \boldsymbol{v}^\infty(\boldsymbol{x}_2) - \boldsymbol{U}_2 \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\omega}_1 \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\omega}_2 \\ \boldsymbol{E}^\infty \\ \boldsymbol{E}^\infty \end{pmatrix} .$$

As before, the resistance matrix contains second-rank tensors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$; third-rank tensors $\boldsymbol{G}$ and $\boldsymbol{H}$; and a fourth-rank tensors $\boldsymbol{M}$. The subscripts on a typical tensor element in the resistance matrix, as in $\boldsymbol{P}_{\alpha\beta}$, denote that the relation is between the appropriate stress moment on particle α and the appropriate velocity quantity for particle β. As in Chapter 5, tildes are employed to highlight symmetry relations between the tensors.

The proofs for the symmetry relations for multiparticle systems follow from the Lorentz reciprocal theorem in a manner that is completely analogous to the single-particle proofs (see Exercise 7.1). The end results, with the components of $\boldsymbol{P}_{\alpha\beta}$ written as $P_{ij}^{(\alpha\beta)}$, are

$$\begin{aligned}
A_{ij}^{(\alpha\beta)} &= A_{ji}^{(\beta\alpha)} && (7.1)\\
C_{ij}^{(\alpha\beta)} &= C_{ji}^{(\beta\alpha)} && (7.2)\\
M_{ijkl}^{(\alpha\beta)} &= M_{klij}^{(\beta\alpha)} && (7.3)\\
B_{ij}^{(\alpha\beta)} &= \tilde{B}_{ji}^{(\beta\alpha)} && (7.4)\\
G_{ijk}^{(\alpha\beta)} &= \tilde{G}_{kij}^{(\beta\alpha)} && (7.5)\\
H_{ijk}^{(\alpha\beta)} &= \widetilde{H}_{kij}^{(\beta\alpha)}\ . && (7.6)
\end{aligned}$$

7.2.2 The Mobility Matrix

The mobility problem, in which the particle motion and stresslets are the unknowns that are to be related to the given quantities, $\boldsymbol{F}_\alpha$, $\boldsymbol{T}_\alpha$, and the ambient field arises frequently in many physical problems. We can invoke linearity again to write

$$\begin{pmatrix} \boldsymbol{v}^\infty(\boldsymbol{x}_1) - \boldsymbol{U}_1 \\ \boldsymbol{v}^\infty(\boldsymbol{x}_2) - \boldsymbol{U}_2 \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\omega}_1 \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\omega}_2 \\ \mu^{-1}\boldsymbol{S}_1 \\ \mu^{-1}\boldsymbol{S}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \tilde{\boldsymbol{b}}_{11} & \tilde{\boldsymbol{b}}_{12} & \tilde{\boldsymbol{g}}_1 \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \tilde{\boldsymbol{b}}_{21} & \tilde{\boldsymbol{b}}_{22} & \tilde{\boldsymbol{g}}_2 \\ \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{c}_{11} & \boldsymbol{c}_{12} & \tilde{\boldsymbol{h}}_1 \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{c}_{21} & \boldsymbol{c}_{22} & \tilde{\boldsymbol{h}}_2 \\ \boldsymbol{g}_{11} & \boldsymbol{g}_{12} & \boldsymbol{h}_{11} & \boldsymbol{h}_{12} & \boldsymbol{m}_1 \\ \boldsymbol{g}_{21} & \boldsymbol{g}_{22} & \boldsymbol{h}_{21} & \boldsymbol{h}_{22} & \boldsymbol{m}_2 \end{pmatrix} \begin{pmatrix} \mu^{-1}\boldsymbol{F}_1 \\ \mu^{-1}\boldsymbol{F}_2 \\ \mu^{-1}\boldsymbol{T}_1 \\ \mu^{-1}\boldsymbol{T}_2 \\ \boldsymbol{E}^\infty \end{pmatrix} .$$

The mobility matrix contains second-rank tensors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$; third-rank tensors $\boldsymbol{g}$ and $\boldsymbol{h}$; and a fourth-rank tensor $\boldsymbol{m}$, as before. The same convention that was employed for the resistance matrix is used again, so that in $\boldsymbol{p}_{\alpha\beta}$, α and β denote the connection between a velocity on particle α and forces or higher order moments on particle β.

The elements of the mobility matrix also obey symmetry relations (see Exercise 7.1). These are

$$\begin{aligned}
a_{ij}^{(\alpha\beta)} &= a_{ji}^{(\beta\alpha)} && (7.7)\\
c_{ij}^{(\alpha\beta)} &= c_{ji}^{(\beta\alpha)} && (7.8)\\
m_{ijkl}^{(1)} + m_{ijkl}^{(2)} &= m_{klij}^{(1)} + m_{klij}^{(2)} && (7.9)\\
b_{ij}^{(\alpha\beta)} &= \tilde{b}_{ji}^{(\beta\alpha)} && (7.10)\\
g_{ijk}^{(1\alpha)} + g_{ijk}^{(2\alpha)} &= -\tilde{g}_{kij}^{(\alpha)} && (7.11)\\
h_{ijk}^{(1\alpha)} + h_{ijk}^{(2\alpha)} &= -\tilde{h}_{kij}^{(\alpha)}\ , && (7.12)
\end{aligned}$$

with the components of $\boldsymbol{p}_{\alpha\beta}$ written as $p_{ij}^{(\alpha\beta)}$. The minus signs appear once more in the relations for $\boldsymbol{g}$ and $\boldsymbol{h}$.

7.2.3 Relations Between the Resistance and Mobility Tensors

The relation between the resistance and mobility matrices may be established by the same procedure as that used in the single-particle problem. The formal expressions have been arranged in the following set of equations so that the mobility tensors may be obtained from the resistance tensors when complete information on the latter is available.

$$\begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \tilde{\boldsymbol{b}}_{11} & \tilde{\boldsymbol{b}}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \tilde{\boldsymbol{b}}_{21} & \tilde{\boldsymbol{b}}_{22} \\ \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{c}_{11} & \boldsymbol{c}_{12} \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{c}_{21} & \boldsymbol{c}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \tilde{\boldsymbol{B}}_{11} & \tilde{\boldsymbol{B}}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \tilde{\boldsymbol{B}}_{21} & \tilde{\boldsymbol{B}}_{22} \\ \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \boldsymbol{C}_{11} & \boldsymbol{C}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \boldsymbol{C}_{21} & \boldsymbol{C}_{22} \end{pmatrix}^{-1} \tag{7.13}$$

$$\begin{pmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{12} & \boldsymbol{h}_{11} & \boldsymbol{h}_{12} \\ \boldsymbol{g}_{21} & \boldsymbol{g}_{22} & \boldsymbol{h}_{21} & \boldsymbol{h}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} & \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{a}_{11} & \boldsymbol{a}_{12} & \tilde{\boldsymbol{b}}_{11} & \tilde{\boldsymbol{b}}_{12} \\ \boldsymbol{a}_{21} & \boldsymbol{a}_{22} & \tilde{\boldsymbol{b}}_{21} & \tilde{\boldsymbol{b}}_{22} \\ \boldsymbol{b}_{11} & \boldsymbol{b}_{12} & \boldsymbol{c}_{11} & \boldsymbol{c}_{12} \\ \boldsymbol{b}_{21} & \boldsymbol{b}_{22} & \boldsymbol{c}_{21} & \boldsymbol{c}_{22} \end{pmatrix} \tag{7.14}$$

$$\begin{pmatrix} \boldsymbol{m}_1 \\ \boldsymbol{m}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{11} + \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} + \boldsymbol{M}_{22} \end{pmatrix} + \begin{pmatrix} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} & \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{G}_{21} & \boldsymbol{G}_{22} & \boldsymbol{H}_{21} & \boldsymbol{H}_{22} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{g}}_1 \\ \tilde{\boldsymbol{g}}_2 \\ \tilde{\boldsymbol{h}}_1 \\ \tilde{\boldsymbol{h}}_2 \end{pmatrix} . \tag{7.15}$$

7.2.4 Axisymmetric Geometries

We have already discussed the special forms taken by the resistance and mobility tensors when the particle shape is axisymmetric. An analogous development is possible for multiparticle resistance and mobility tensors in axisymmetric geometries. An important example of such axisymmetry is encountered in the geometry of two spheres.

The forms taken by the tensors are identical to those encountered earlier in the single-particle problem. These are reproduced in Table 7.1. Labels α and β are employed to keep track of the various particle-particle interactions. Our task then is to find the relation between the *scalar* functions that describe the resistance and mobility properties. Note that since we will be dealing with matrices with scalar elements, the inversion operations that appear in the following discussion are the usual matrix inverse operations.

It is intuitively obvious, and also readily shown by algebraic arguments, that the mobility functions x depend only on the X resistance functions, that the y functions depend only on the Y functions, and that the z functions depend only on the Z functions. For example, translational motions along the particle axis induce forces that are also directed *only* along the axis. Conversely, forces directed along the axis produce translational motions *only* in the axial direction. These two problems are inverses of each other, and thus the functions $x^a_{\alpha\beta}$ are related only to $X^A_{\alpha\beta}$.

For axisymmetric geometries, problems involving translation along the axis and rotation about the axis are decoupled as far as the forces and torques are concerned (this is why the scalar functions of the form X^B do not appear in the table). Thus the reduced version of Equation 7.13 for the scalar function is block diagonal, so that we obtain

$$\begin{pmatrix} x^a_{11} & x^a_{12} \\ x^a_{12} & x^a_{22} \end{pmatrix} = \begin{pmatrix} X^A_{11} & X^A_{12} \\ X^A_{12} & X^A_{22} \end{pmatrix}^{-1} \tag{7.16}$$

and

$$\begin{pmatrix} x^c_{11} & x^c_{12} \\ x^c_{12} & x^c_{22} \end{pmatrix} = \begin{pmatrix} X^C_{11} & X^C_{12} \\ X^C_{12} & X^C_{22} \end{pmatrix}^{-1} . \tag{7.17}$$

Note that the symmetry relations have been used. The x^g functions may now be obtained from the following:

$$\begin{pmatrix} x^g_{11} & x^g_{12} \\ x^g_{21} & x^g_{22} \end{pmatrix} = \begin{pmatrix} X^G_{11} & X^G_{12} \\ X^G_{21} & X^G_{22} \end{pmatrix} \begin{pmatrix} x^a_{11} & x^a_{12} \\ x^a_{12} & x^a_{22} \end{pmatrix} . \tag{7.18}$$

The relations for x^m are given below along with the expressions for y^m and z^m.

Relations between the y and Y functions involve simultaneous inversion of the full complement of the functions associated with the A, B, and C tensors, because calculations of the force-translation and torque-rotation relations are coupled for transverse motions. Explicitly, we have

$$\begin{pmatrix} y^a_{11} & y^a_{12} & y^b_{11} & y^b_{21} \\ y^a_{12} & y^a_{22} & y^b_{12} & y^b_{22} \\ y^b_{11} & y^b_{12} & y^c_{11} & y^c_{12} \\ y^b_{21} & y^b_{22} & y^c_{12} & y^c_{22} \end{pmatrix} = \begin{pmatrix} Y^A_{11} & Y^A_{12} & Y^B_{11} & Y^B_{21} \\ Y^A_{12} & Y^A_{22} & Y^B_{12} & Y^B_{22} \\ Y^B_{11} & Y^B_{12} & Y^C_{11} & Y^C_{12} \\ Y^B_{21} & Y^B_{22} & Y^C_{12} & Y^C_{22} \end{pmatrix}^{-1} \tag{7.19}$$

$$\begin{pmatrix} y^g_{11} & y^g_{12} \\ y^g_{21} & y^g_{22} \end{pmatrix} = \begin{pmatrix} Y^G_{11} & Y^G_{12} & -Y^H_{11} & -Y^H_{12} \\ Y^G_{21} & Y^G_{22} & -Y^H_{21} & -Y^H_{22} \end{pmatrix} \begin{pmatrix} y^a_{11} & y^a_{12} \\ y^a_{12} & y^a_{22} \\ y^b_{11} & y^b_{12} \\ y^b_{21} & y^b_{22} \end{pmatrix} \tag{7.20}$$

$$\begin{pmatrix} -y^h_{11} & -y^h_{12} \\ -y^h_{21} & -y^h_{22} \end{pmatrix} = \begin{pmatrix} Y^G_{11} & Y^G_{12} & -Y^H_{11} & -Y^H_{12} \\ Y^G_{21} & Y^G_{22} & -Y^H_{21} & -Y^H_{22} \end{pmatrix} \begin{pmatrix} y^b_{11} & y^b_{21} \\ y^b_{12} & y^a_{22} \\ y^c_{11} & y^c_{12} \\ y^c_{21} & y^c_{22} \end{pmatrix} \tag{7.21}$$

$$
\begin{aligned}
A_{ij}^{(\alpha\beta)} &= X_{\alpha\beta}^{A} d_i d_j + Y_{\alpha\beta}^{A} (\delta_{ij} - d_i d_j) \\
B_{ij}^{(\alpha\beta)} &= Y_{\alpha\beta}^{B} \epsilon_{ijk} d_k \\
C_{ij}^{(\alpha\beta)} &= X_{\alpha\beta}^{C} d_i d_j + Y_{\alpha\beta}^{C} (\delta_{ij} - d_i d_j) \\
G_{ijk}^{(\alpha\beta)} &= X_{\alpha\beta}^{G} (d_i d_j - \frac{1}{3}\delta_{ij}) d_k + Y_{\alpha\beta}^{G} (d_i \delta_{jk} + d_j \delta_{ik} - 2 d_i d_j d_k) \\
H_{ijk}^{(\alpha\beta)} &= Y_{\alpha\beta}^{H} (\epsilon_{ikl} d_l d_j + \epsilon_{jkl} d_l d_i) \\
M_{ijkl}^{(\alpha\beta)} &= X_{\alpha\beta}^{M} d_{ijkl}^{(0)} + Y_{\alpha\beta}^{M} d_{ijkl}^{(1)} + Z_{\alpha\beta}^{M} d_{ijkl}^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
a_{ij}^{(\alpha\beta)} &= x_{\alpha\beta}^{a} d_i d_j + y_{\alpha\beta}^{a} (\delta_{ij} - d_i d_j) \\
b_{ij}^{(\alpha\beta)} &= y_{\alpha\beta}^{b} \epsilon_{ijk} d_k \\
c_{ij}^{(\alpha\beta)} &= x_{\alpha\beta}^{c} d_i d_j + y_{\alpha\beta}^{c} (\delta_{ij} - d_i d_j) \\
g_{ijk}^{(\alpha\beta)} &= x_{\alpha\beta}^{g} (d_i d_j - \frac{1}{3}\delta_{ij}) d_k + y_{\alpha\beta}^{g} (d_i \delta_{jk} + d_j \delta_{ik} - 2 d_i d_j d_k) \\
h_{ijk}^{(\alpha\beta)} &= y_{\alpha\beta}^{h} (\epsilon_{ikl} d_l d_j + \epsilon_{jkl} d_l d_i) \\
m_{ijkl}^{(\alpha)} &= x_{\alpha\beta}^{m} d_{ijkl}^{(0)} + y_{\alpha\beta}^{m} d_{ijkl}^{(1)} + z_{\alpha\beta}^{m} d_{ijkl}^{(2)}
\end{aligned}
$$

where

$$
\begin{aligned}
d^{(0)} &= \frac{3}{2}(d_i d_j - \frac{1}{3}\delta_{ij})(d_k d_l - \frac{1}{3}\delta_{kl}) \\
d^{(1)} &= \frac{1}{2}(d_i \delta_{jl} d_k + d_j \delta_{il} d_k + d_i \delta_{jk} d_l + d_j \delta_{ik} d_l - 4 d_i d_j d_k d_l) \\
d^{(2)} &= \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il} - \delta_{ij}\delta_{kl} + d_i d_j \delta_{kl} + \delta_{ij} d_k d_l \\
&\quad - d_i \delta_{jl} d_k - d_j \delta_{il} d_k - d_i \delta_{jk} d_l - d_j \delta_{ik} d_l + d_i d_j d_k d_l)
\end{aligned}
$$

Table 7.1: Resistance and mobility tensors for axisymmetric geometries.

$$\begin{aligned}
x_\alpha^m &= X_{\alpha 1}^M + X_{\alpha 2}^M - \frac{2}{3}X_{\alpha 1}^G(x_{11}^g + x_{21}^g) - \frac{2}{3}X_{\alpha 2}^G(x_{12}^g + x_{22}^g) && (7.22)\\
y_\alpha^m &= Y_{\alpha 1}^M + Y_{\alpha 2}^M - 2Y_{\alpha 1}^G(y_{11}^g + y_{21}^g) - 2Y_{\alpha 2}^G(y_{12}^g + y_{22}^g)\\
&\quad - 2Y_{\alpha 1}^H(y_{11}^h + y_{21}^h) - 2Y_{\alpha 2}^H(y_{12}^h + y_{22}^h) && (7.23)\\
z_\alpha^m &= Z_{\alpha 1}^M + Z_{\alpha 2}^M\,. && (7.24)
\end{aligned}$$

This completes the derivation of the relation between the resistance and mobility functions for axisymmetric gemetries. In a given problem, if we know the complete set of resistance functions, we may then construct any mobility function of interest. Since it is in the resistance formulation that the boundary conditions are given, this solution strategy is quite popular. However, its usefulness is limited to problems involving only a small number of particles. In simulations involving many particles, inversion of the resistance formulation is impractical, and we shall pay particular attention to the question of direct solution of mobility problems.

Exercises

Exercise 7.1 Symmetry Relations for the Resistance and Mobility Tensors.
Use the Lorentz reciprocal theorem to derive the following symmetry relations for the resistance and mobility tensors.

Resistance tensors:

$$\begin{aligned}
A_{ij}^{(\alpha\beta)} &= A_{ji}^{(\beta\alpha)}\\
C_{ij}^{(\alpha\beta)} &= C_{ji}^{(\beta\alpha)}\\
M_{ijkl}^{(\alpha\beta)} &= M_{klij}^{(\beta\alpha)}\\
B_{ij}^{(\alpha\beta)} &= \tilde{B}_{ji}^{(\beta\alpha)}\\
G_{ijk}^{(\alpha\beta)} &= \tilde{G}_{kij}^{(\beta\alpha)}\\
H_{ijk}^{(\alpha\beta)} &= \widetilde{H}_{kij}^{(\beta\alpha)}
\end{aligned}$$

Mobility tensors:

$$\begin{aligned}
a_{ij}^{(\alpha\beta)} &= a_{ji}^{(\beta\alpha)}\\
c_{ij}^{(\alpha\beta)} &= c_{ji}^{(\beta\alpha)}\\
m_{ijkl}^{(1)} + m_{ijkl}^{(2)} &= m_{klij}^{(1)} + m_{klij}^{(2)}\\
b_{ij}^{(\alpha\beta)} &= \tilde{b}_{ji}^{(\beta\alpha)}\\
g_{ijk}^{(1\alpha)} + g_{ijk}^{(2\alpha)} &= -\tilde{g}_{kij}^{(\alpha)}\\
h_{ijk}^{(1\alpha)} + h_{ijk}^{(2\alpha)} &= -\tilde{h}_{kij}^{(\alpha)}
\end{aligned}$$

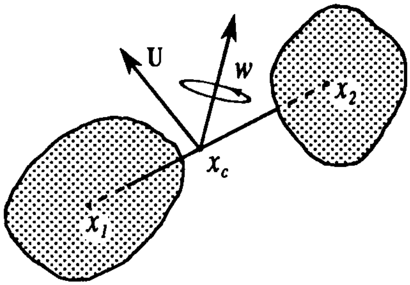

Figure 7.2: The generalized dumbbell.

Exercise 7.2 Another Definition of the Resistance Relations.
Consider once again particles (labelled by $\alpha = 1, 2$) in rigid-body motion, $\boldsymbol{U}_\alpha + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_\alpha)$, where $\boldsymbol{x}_\alpha$ is a reference point inside particle α, in the ambient field $\boldsymbol{v}^\infty(\boldsymbol{x}) = \boldsymbol{U}^\infty + \boldsymbol{\Omega}^\infty \times \boldsymbol{x} + \boldsymbol{E}^\infty \cdot \boldsymbol{x}$. In the resistance formulation, we may choose to use the uniform velocity relative to the origin, $\boldsymbol{U}^\infty - \boldsymbol{U}_1$, instead of the relative velocity, $\boldsymbol{v}^\infty(\boldsymbol{x}_\alpha) - \boldsymbol{U}_\alpha$, resulting in

$$\begin{pmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \\ \boldsymbol{S}_1 \\ \boldsymbol{S}_2 \end{pmatrix} = \mu \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \tilde{\mathcal{B}}_{11} & \tilde{\mathcal{B}}_{12} & \tilde{\mathcal{G}}_{11} & \tilde{\mathcal{G}}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \tilde{\mathcal{B}}_{21} & \tilde{\mathcal{B}}_{22} & \tilde{\mathcal{G}}_{21} & \tilde{\mathcal{G}}_{22} \\ \mathcal{B}_{11} & \mathcal{B}_{12} & \mathcal{C}_{11} & \mathcal{C}_{12} & \tilde{\mathcal{H}}_{11} & \tilde{\mathcal{H}}_{12} \\ \mathcal{B}_{21} & \mathcal{B}_{22} & \mathcal{C}_{21} & \mathcal{C}_{22} & \tilde{\mathcal{H}}_{21} & \tilde{\mathcal{H}}_{22} \\ \mathcal{G}_{11} & \mathcal{G}_{12} & \mathcal{H}_{11} & \mathcal{H}_{12} & \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{G}_{21} & \mathcal{G}_{22} & \mathcal{H}_{21} & \mathcal{H}_{22} & \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{U}^\infty - \boldsymbol{U}_1 \\ \boldsymbol{U}^\infty - \boldsymbol{U}_2 \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\omega}_1 \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\omega}_2 \\ \boldsymbol{E}^\infty \\ \boldsymbol{E}^\infty \end{pmatrix} .$$

Find the relations between these tensors and the standard ones of Section 7.2.

Exercise 7.3 The Generalized Dumbbell.
Consider a generalized dumbbell formed by two particles of arbitrary shape connected by a rigid rod with negligible drag coefficient. Consider the rigid-body motion, $\boldsymbol{U} + \boldsymbol{\omega} \times (\boldsymbol{x} - \boldsymbol{x}_c)$, in a quiescent fluid. Here $\boldsymbol{x}_c$ is a point on the rod (see Figure 7.2). Relate the single particle resistance tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ to the two-particle tensors of Section 7.2.

Exercise 7.4 An Important Mobility Tensor.
Consider the two particle interaction problem with $\boldsymbol{F}_2 = -\boldsymbol{F}_1$. Most pair interaction forces are of this form. In phenomena ranging from particle aggregation to Brownian diffusion, the key quanity is the relative velocity, $\boldsymbol{U}_2 - \boldsymbol{U}_1$, and it is common to relate this to the interparticle force using the mobility tensor, $\boldsymbol{\zeta}^{-1}$, as in $\boldsymbol{U}_2 - \boldsymbol{U}_1 = \boldsymbol{\zeta}^{-1} \cdot \boldsymbol{F}_2$. Relate $\boldsymbol{\zeta}^{-1}$ to the two-particle mobility tensors, $\boldsymbol{a}_{\alpha\beta}$. Derive a special form for axisymmetric geometries (*e.g.*, two spheres) in terms of the mobility functions $x^a_{\alpha\beta}$ and $y^a_{\alpha\beta}$.

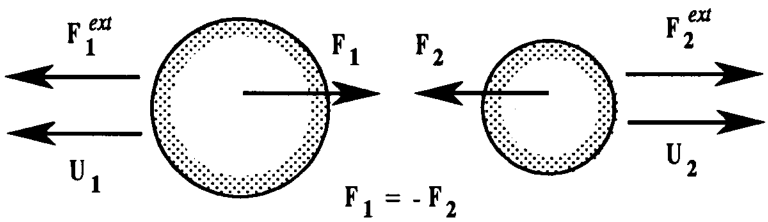

Figure 7.3: The origin of the mobility tensor for pair potentials.