19

Möbius Transformations and Vahlen Matrices

Classical complex analysis can be generalized from the complex plane to higher dimensions in three different ways: function theory of several complex variables (commutative), higher-dimensional one-variable hypercomplex analysis (anti-commutative), and conformal transformations (geometric). In this chapter we study the third possibility: conformal transformations in n dimensions, $n \geq 3$.

A function f sending a region in $\mathbb{R}^2 = \mathbb{C}$ into $\mathbb{C}$ is conformal at z, if it is complex analytic and has a non-zero derivative, $f'(z) \neq 0$ (we consider only sense-preserving conformal mappings). The only conformal transformations of the whole plane $\mathbb{C}$ are affine linear transformations: compositions of rotations, dilations and translations. The *Möbius mapping*

$$f(z) = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0,$$

is affine linear when $c = 0$; otherwise it is conformal at each $z \in \mathbb{C}$ except when $z = -\frac{d}{c}$. The Möbius mapping f sends $\mathbb{C} \setminus \{-\frac{d}{c}\}$ onto $\mathbb{C} \setminus \{\frac{a}{c}\}$. If we agree that $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$, then f becomes a (one-to-one) transformation of $\mathbb{C} \cup \{\infty\}$, the complex plane compactified by the point at infinity. [1] These transformations are called *Möbius transformations* of $\mathbb{C} \cup \{\infty\}$. Möbius transformations are compositions of rotations, translations, dilations and transversions. [2] Möbius transformations send circles (and affine lines) to circles (or affine lines). The derivative of a Möbius transformation is a composition of a rotation and a dilation.

By definition, a conformal mapping preserves angles between intersecting curves. Formally, let D be a region in a Euclidean space $\mathbb{R}^n$. A continuously

1 Möbius mappings $f(z) = \frac{az+b}{cz+d}$ are defined almost everywhere in $\mathbb{C}$. The set of Möbius mappings can be used to compactify $\mathbb{C}$, the compactification being $\mathbb{C} \cup \{\infty\}$.

2 A transversion is a composition of an inversion in the unit circle, a translation and another inversion. Thus, transversions are conjugate (by the inversion) to translations.

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differentiable function $g : D \to \mathbb{R}^n$ is *conformal* in D if there is a continuous function $\lambda : D \to \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ such that

$$<f'(\mathbf{x})\mathbf{a}, f'(\mathbf{x})\mathbf{b}> = <\mathbf{a}\lambda(\mathbf{x}), \mathbf{b}\lambda(\mathbf{x})>$$

for all $\mathbf{x} \in D$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. In higher-dimensional Euclidean spaces $\mathbb{R}^n$, $n \geq 3$, the only conformal mappings [sending a region in $\mathbb{R}^n$ into $\mathbb{R}^n$] are restrictions of Möbius transformations of $\mathbb{R}^n \cup \{\infty\}$. [3] The case $n = 3$ was proved by Liouville 1850. The analogous statement for indefinite quadratic spaces is also true by a theorem of Haantjes 1937.

19.1 Quaternion representation of conformal transformations of $\mathbb{R}^4$

Conformal transformations of $\mathbb{R}^4$ can be represented by quaternion computation:

$$\mathbb{R}^4 = \mathbb{H} \to \mathbb{H}, \ q \to (aq + b)(cq + d)^{-1}, \quad a, b, c, d \in \mathbb{H}.$$

In order to exclude constant functions we require the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to be invertible, that is, $|a|^2|d|^2 + |b|^2|c|^2 - 2\,\mathrm{Re}(a\,\bar{c}\,d\,\bar{b}) \neq 0$. This matrix representation renders composition of non-linear conformal transformations into multiplication of matrices.

19.2 Möbius transformations of $\mathbb{R}^n$

Möbius transformations might be sense-preserving with $\det f'(\mathbf{x}) > 0$ or sense-reversing with $\det f'(\mathbf{x}) < 0$. The Möbius transformations form a group, the full Möbius group, which has two components, the identity component being the sense-preserving Möbius group. The full Möbius group of $\mathbb{R}^n$ is generated by translations, reflections and the inversion

$$\mathbf{x} \to \mathbf{x}^{-1} = \frac{\mathbf{x}}{\mathbf{x}^2},$$

or equivalently, by reflections in affine hyperplanes and inversions in spheres (not necessarily centered at the origin). The sense-preserving Möbius group is generated by the following four types of transformations:

3 We shall often refer to Möbius transformations of the Euclidean space $\mathbb{R}^n$ whereby we tacitly mean transformations of the compactification $\mathbb{R}^n \cup \{\infty\}$.

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rotations	$\mathbf{x}a\mathbf{x}a^{-1}$	$a \in \mathbf{Spin}(n)$
translations	$\mathbf{x}+\mathbf{b}$	$\mathbf{b} \in \mathbb{R}^n$
dilations	$\mathbf{x}\delta$	$\delta > 0$
transversions	$\dfrac{\mathbf{x}+\mathbf{x}^2\mathbf{c}}{1+2\mathbf{x}\cdot\mathbf{c}+\mathbf{x}^2\mathbf{c}^2}$	$\mathbf{c} \in \mathbb{R}^n$.

Rewriting the transversion into the form $(\mathbf{x}^{-1}+\mathbf{c})^{-1}$, one sees that it is a composition of the inversion, a translation and the inversion. Using the multiplicative notation of the Clifford algebra $\mathcal{C}\ell_n$, the transversion can further be written in the form

$$\mathbf{x} \to \mathbf{x}(\mathbf{c}\mathbf{x}+1)^{-1}.$$

This might suggest the following: Let a, b, c, d be in $\mathcal{C}\ell_n$. If $(a\mathbf{x}+\mathbf{x})(c\mathbf{x}+d)^{-1}$ is in $\mathbb{R}^n$ for almost all $\mathbf{x} \in \mathbb{R}^n$ and if the range of

$$g(\mathbf{x}) = (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1}$$

is dense in $\mathbb{R}^n$, then g is a Möbius transformation of $\mathbb{R}^n$. Although this is true, the group so obtained is too large to be a practical covering group of the full Möbius group. [4] Therefore, we introduce:

Definition (Maass 1949, Ahlfors 1984). [5] The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{C}\ell_n)$ fulfilling the conditions

(i) $a, b, c, d \in \mathbf{\Gamma}_n \cup \{0\}$
(ii) $a\tilde{b},\ \tilde{b}d,\ d\tilde{c},\ \tilde{c}a \in \mathbb{R}^n$
(iii) $a\tilde{d} - b\tilde{c} \in \mathbb{R} \setminus \{0\}$

is called a *Vahlen matrix* of the Möbius transformation g of $\mathbb{R}^n$ given by $g(\mathbf{x}) = (a\mathbf{x}+b)(c\mathbf{x}+d)^{-1}$. ∎

By condition (i) the diagonal entries of a Vahlen matrix are either even or odd. Conditions (i) and (ii) imply that if the diagonal entries are even then the off-diagonal entries must be odd, and if the diagonal entries are odd then

4 This group is the Vahlen group multiplied by the group generated by invertible matrices of the form

$$\begin{pmatrix} \alpha+\beta\mathbf{e}_{12\ldots n} & 0 \\ 0 & \alpha-\beta\hat{\mathbf{e}}_{12\ldots n} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}.$$

5 Vahlen 1902 originally wrote the second condition in the form

$$\text{(ii)} \quad \bar{a}b,\ b\bar{d},\ \bar{d}c,\ c\bar{a} \in \mathbb{R}^n,$$

which gives an equivalent characterization of the Vahlen group.

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the off-diagonal entries must be even. Condition (iii) tells us that the pseudo-determinant $a\tilde{d} - b\tilde{c}$ is real and non-zero, in particular, that the Vahlen matrix is invertible.

The Vahlen matrices form a group under matrix multiplication, the Vahlen group. The Vahlen group has a normalized subgroup where condition (iii) is replaced by

(iii′) $a\tilde{d} - b\tilde{c} = \pm 1$.

The normalized Vahlen group is a four-fold, or rather double two-fold, covering group of the full Möbius group of $\mathbb{R}^n$; the identity Möbius transformation is represented by the following four matrices:

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} \mathbf{e}_{12\ldots n} & 0 \\ 0 & -\hat{\mathbf{e}}_{12\ldots n} \end{pmatrix}.$$

The sense-preserving Möbius group has a non-trivial two-fold covering group formed by normalized Vahlen matrices with even diagonal (and odd off-diagonal) and pseudo-determinant equal to 1. The full Möbius group has a non-trivial two-fold covering group with two components, the non-identity component consisting of normalized Vahlen matrices with odd diagonal (and even off-diagonal) and pseudo-determinant equal to -1.

19.3 Opposite of a Euclidean space

Consider the $(n-1)$-dimensional real quadratic space $\mathbb{R}^{0,n-1}$ having a negative definite quadratic form

$$\mathbf{x} \to \mathbf{x}^2 = -x_1^2 - \ldots - x_{n-1}^2.$$

The sums of scalars and vectors are called *paravectors*. Paravectors span the linear space $\mathbb{R} \oplus \mathbb{R}^{0,n-1}$, which we denote by

$$\$R^n = \mathbb{R} \oplus \mathbb{R}^{0,n-1}.$$

The linear space of paravectors, $\$R^n$, can be made isometric to the Euclidean space $\mathbb{R}^n$ by introducing for $x = x_0 + \mathbf{x} \in \mathbb{R} \oplus \mathbb{R}^{0,n-1}$, where $x_0 \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^{0,n-1}$, a quadratic form

$$x = x_0 + \mathbf{x} \to x\bar{x} = x_0^2 - \mathbf{x}^2 = x_0^2 + x_1^2 + \ldots + x_{n-1}^2.$$

As an extension of the Lipschitz group Porteous 1969 pp. 254-259 introduced the group of products of invertible paravectors, defined equivalently by

$$\$\Gamma_n = \{s \in C\ell_{0,n-1} \mid \forall x \in \$R^n,\ sx\hat{s}^{-1} \in \$R^n\}.$$

For a non-zero paravector $a \in \$R^n$ the mapping $x \to ax\bar{a}^{-1}$ is a rotation of

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$\$R^n$. Thus, we have a group isomorphism $\$\Gamma_n \simeq \mathbf{\Gamma}_n^+$. Note that $\mathbf{\Gamma}_{0,n-1} \subset \$\Gamma_n$ and $\mathbf{\Gamma}_{0,n-1}^{\pm} = \$\Gamma_n^{\pm}$.

Vahlen originally considered the sense-preserving Möbius group of the paravector space $\$R^n$.

Definition (Vahlen 1902).[6] The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{C}\ell_{0,n-1})$ fulfilling the conditions

(i) $a, b, c, d \in \$\Gamma_n \cup \{0\}$
(ii) $\bar{a}b,\ b\bar{d},\ \bar{d}c,\ c\bar{a} \in \$R^n$
(iii) $a\tilde{d} - b\tilde{c} = 1$

is a *Vahlen matrix*, with pseudo-determinant or norm 1, of the sense-preserving Möbius transformation g of $\$R^n$ given by $g(x) = (ax+b)(cx+d)^{-1}$. ∎

These Vahlen matrices with norm 1 form a group, which is a non-trivial two-fold cover of the sense-preserving Möbius group of $\$R^n$.

19.4 Indefinite quadratic spaces

The full Möbius group of $\mathbb{R}^{p,q}$ contains two components (if either p or q is even) or four components (if both p and q are odd).

The identity component of the Möbius group of $\mathbb{R}^{p,q}$ is generated by rotations, translations, dilations and transversions which are represented, respectively, as follows:

$$\mathbf{axa}^{-1} \qquad a \in \mathbf{Spin}_+(p,q) \qquad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

$$\mathbf{x} + \mathbf{b} \qquad \mathbf{b} \in \mathbb{R}^{p,q} \qquad \begin{pmatrix} 1 & \mathbf{b} \\ 0 & 1 \end{pmatrix}$$

$$\mathbf{x}\delta \qquad \delta > 0 \qquad \begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & 1/\sqrt{\delta} \end{pmatrix}$$

$$\frac{\mathbf{x} + \mathbf{x}^2\mathbf{c}}{1 + 2\mathbf{x}\cdot\mathbf{c} + \mathbf{x}^2\mathbf{c}^2} \qquad \mathbf{c} \in \mathbb{R}^{p,q} \qquad \begin{pmatrix} 1 & 0 \\ \mathbf{c} & 1 \end{pmatrix}.$$

On the right we have the Vahlen matrices of the respective Möbius transformations.

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6 Maass 1949 and Ahlfors 1984 presented an equivalent characterization of Vahlen matrices where the second condition was replaced by

$$\text{(ii)} \quad a\tilde{b},\ \tilde{b}d,\ d\tilde{c},\ \tilde{c}a \in \$R^n.$$

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Theorem (J. Maks 1989). Consider four Vahlen matrices which represent a rotation, a translation, a dilation and a transversion. A product of these four matrices, in any order, always has an invertible entry in its diagonal (there are $4! = 24$ such products).

Proof. To complete the proof of the fact that a product of a rotation, a translation, a dilation and a transversion, in any order, is such that its Vahlen matrix always has an invertible entry in its diagonal, one can (or rather must) check the claim for all the 24 orderings. For instance, in the product

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & \mathbf{b} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & 1/\sqrt{\delta} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \mathbf{c} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a\sqrt{\delta} + a\mathbf{bc}/\sqrt{\delta} & a\mathbf{b}/\sqrt{\delta} \\ a\mathbf{c}/\sqrt{\delta} & a/\sqrt{\delta} \end{pmatrix}$$

the lower right-hand diagonal element $a/\sqrt{\delta}$ is invertible. We leave the verification of the remaining 23 orderings to the reader. ∎

Counter-example (Maks 1989). In the general case ($p \neq 0$, $q \neq 0$) J. Maks 1989 p. 41 gave an example of a Vahlen matrix where none of the entries is invertible (and all are non-zero).

Consider the Minkowski space-time $\mathbb{R}^{3,1}$ and its Clifford algebra $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$ generated by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ satisfying $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$, $\mathbf{e}_4^2 = -1$. Take a Vahlen matrix

$$M = \frac{1}{2}\begin{pmatrix} 1 + \mathbf{e}_{14} & \mathbf{e}_1 + \mathbf{e}_4 \\ -\mathbf{e}_1 + \mathbf{e}_4 & 1 - \mathbf{e}_{14} \end{pmatrix}.$$

By the theorem of Maks the matrix M cannot be a product of just one rotation, one translation, one dilation and one transversion (in any order). However, the matrix M is in the identity component of the normalized Vahlen group, the four-fold covering group of the Möbius group of the Minkowski space-time. This can be concluded while M has pseudo-determinant equal to 1 and even diagonal. This can also be deduced by factoring M into a product of a transversion, a translation and a transversion as follows:

$$M = \begin{pmatrix} 1 & 0 \\ \frac{1}{2}(-\mathbf{e}_1 + \mathbf{e}_4) & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_4) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2}(-\mathbf{e}_1 + \mathbf{e}_4) & 1 \end{pmatrix}.$$

Topologically, we can see this by connecting M to the identity matrix by the following path (here β grows from 0 to $\pi/4$):

$$M = M_{\pi/4}, \qquad M_\beta = \exp\left\{\beta\begin{pmatrix} 0 & \mathbf{e}_1 + \mathbf{e}_4 \\ -\mathbf{e}_1 + \mathbf{e}_4 & 0 \end{pmatrix}\right\}.$$

Maks' counter-example proves that condition (i) has to be modified in the

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definition of a Vahlen matrix. ∎

Recall that the Lipschitz group $\mathbf{\Gamma}_{p,q}$ consists of products of non-isotropic vectors of $\mathbb{R}^{p,q}$. In the sequel we need the set $\mathbf{\Pi}_{p,q}$ of products of vectors, possibly isotropic, of $\mathbb{R}^{p,q}$. The set $\mathbf{\Pi}_{p,q}$ is the closure of $\mathbf{\Gamma}_{p,q}$. [7]

Definition (Fillmore & Springer 1990). The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{C}\ell_{p,q})$ fulfilling the conditions

(i) $a, b, c, d \in \mathbf{\Pi}_{p,q}$
(ii) $\bar{a}b,\ b\bar{d},\ \bar{d}c,\ c\bar{a} \in \mathbb{R}^{p,q}$
(iii) $a\tilde{d} - b\tilde{c} \in \mathbb{R} \setminus \{0\}$

is a *Vahlen matrix* of the Möbius transformation g of $\mathbb{R}^{p,q}$ given by $g(\mathbf{x}) = (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}$. ∎

The Vahlen matrices form a group under matrix multiplication, the *Vahlen group*. The normalized Vahlen matrices, with pseudo-determinant satisfying $a\tilde{d} - b\tilde{c} = \pm 1$, form a four-fold, possibly trivial, covering group of the full Möbius group of $\mathbb{R}^{p,q}$. When both p and q are odd, the normalized Vahlen group is a non-trivial four-fold covering group of the full Möbius group of $\mathbb{R}^{p,q}$. When either p or q is even, we may find a non-trivial two-fold covering group of the full Möbius group of $\mathbb{R}^{p,q}$. It consists of the identity component of the normalized Vahlen group, that is, normalized Vahlen matrices with even diagonal and pseudo-determinant equal to 1, and another component not containing the (non-trivial) pre-images of the identity:

$$\pm \begin{pmatrix} \mathbf{e}_{12\ldots n} & 0 \\ 0 & -\hat{\mathbf{e}}_{12\ldots n} \end{pmatrix}.$$

The identity component of the normalized Vahlen group is a two-fold (either p or q is even) or four-fold (both p and q are odd) covering group of the sense-preserving Möbius group.

Conditions (i), (iii) and $\bar{a}b,\ b\bar{d},\ \bar{d}c,\ c\bar{a} \in \mathbb{R}^{p,q}$ imply $a\tilde{b},\ \tilde{b}d,\ d\tilde{c},\ \tilde{c}a \in \mathbb{R}^{p,q}$. In contrast to the Euclidean case, conditions (i), (iii) and $a\tilde{b},\ \tilde{b}d,\ d\tilde{c},\ \tilde{c}a \in \mathbb{R}^{p,q}$ do not imply $\bar{a}b,\ b\bar{d},\ \bar{d}c,\ c\bar{a} \in \mathbb{R}^{p,q}$.

Counter-example (Cnops 1996). Consider the Minkowski space-time $\mathbb{R}^{3,1}$ and its Clifford algebra $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4, \mathbb{R})$ generated by $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ satisfying $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1,\ \mathbf{e}_4^2 = -1$. The Vahlen matrix

$$C = \frac{1}{2} \begin{pmatrix} 1 + \mathbf{e}_{14} & (\mathbf{e}_1 + \mathbf{e}_4)\mathbf{e}_{23} \\ (-\mathbf{e}_1 + \mathbf{e}_4)\mathbf{e}_{23} & 1 - \mathbf{e}_{14} \end{pmatrix}$$

7 The set $\mathbf{\Pi}_{p,q} \subset \mathcal{C}\ell_{p,q} \simeq \bigwedge \mathbb{R}^n$, considered as a subset of the exterior algebra $\bigwedge \mathbb{R}^n$, is independent of p, q for a fixed $n = p + q$.

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satisfies $a, b, c, d \in \mathbf{\Pi}_{3,1}$, $a\tilde{d} - b\tilde{c} = 1$ and $a\tilde{b}$, $\tilde{b}d$, $d\tilde{c}$, $\tilde{c}a = 0 \in \mathbb{R}^{3,1}$, but even then $\bar{a}b$, $b\bar{d}$, $\bar{d}c$, $c\bar{a} \notin \mathbb{R}^{3,1}$. The mapping $g_C(\mathbf{x}) = (a\mathbf{x} + b)(c\mathbf{x} + d)^{-1}$ is conformal. If the matrix C is multiplied on either side by

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + \mathbf{e}_{1234} & 0 \\ 0 & 1 - \mathbf{e}_{1234} \end{pmatrix},$$

then $B = CD = DC$ is such that $g_B(\mathbf{x}) = g_C(\mathbf{x})$ for almost all $\mathbf{x} \in \mathbb{R}^{3,1}$. Furthermore, B does satisfy $\bar{a}b$, $b\bar{d}$, $\bar{d}c$, $c\bar{a} \in \mathbb{R}^{3,1}$.

The matrices satisfying $a, b, c, d \in \mathbf{\Pi}_{3,1}$, $a\tilde{d} - b\tilde{c} = 1$ and $a\tilde{b}$, $\tilde{b}d$, $d\tilde{c}$, $\tilde{c}a \in \mathbb{R}^{3,1}$ do not form a group, but only a set which is not closed under multiplication. This set generates a group which is the Vahlen group with norm 1 multiplied by the group consisting of the matrices

$$\begin{pmatrix} \cos\varphi + \mathbf{e}_{1234}\sin\varphi & 0 \\ 0 & \cos\varphi - \mathbf{e}_{1234}\sin\varphi \end{pmatrix}.$$

All these matrices are pre-images of the identity Möbius transformation. ∎

19.5 Indefinite paravectors

Let $\$\Pi_{q+1,p}$ be the set of products of paravectors in $\$R^{q+1,p} = \mathbb{R} \oplus \mathbb{R}^{p,q}$.

Definition. The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{C}\ell_{p,q})$ fulfilling the conditions

(i) $a, b, c, d \in \$\Pi_{q+1,p}$
(ii) $\bar{a}b$, $b\bar{d}$, $\bar{d}c$, $c\bar{a} \in \$R^{q+1,p}$
(iii) $a\tilde{d} - b\tilde{c} = 1$

is a *Vahlen matrix* with norm 1 of the sense-preserving Möbius transformation g of $\$R^{q+1,p}$ given by $g(x) = (ax + b)(cx + d)^{-1}$. ∎

The Vahlen matrices with norm 1 form a two-fold or four-fold covering group of the sense-preserving Möbius group of $\$R^{q+1,p}$. Conditions (i), (ii), (iii) imply $a\tilde{b}$, $\tilde{b}d$, $d\tilde{c}$, $\tilde{c}a \in \$R^{q+1,p}$ [although (i), (iii) and $a\tilde{b}$, $\tilde{b}d$, $d\tilde{c}$, $\tilde{c}a \in \$R^{q+1,p}$ do not imply $\bar{a}b$, $b\bar{d}$, $\bar{d}c$, $c\bar{a} \in \$R^{q+1,p}$].

19.6 The derivative of a Möbius transformation

The difference of the Möbius transformations of x, y in $\$R^{q+1,p}$ is given by

$$g(x) - g(y) = (cy + d)^{\sim -1}(x - y)(cx + d)^{-1}.$$

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Letting x approach y we may compute the derivative of a Möbius transformation. Denoting $z = cx + d$ and using $N(z) = z\bar{z} \in \mathbb{R}$, we see that in the case $N(z) \neq 0$ the derivative of $x \to g(x)$ is the composition of the rotation

$$x \to \hat{z}xz^{-1}$$

and the dilation

$$x \to \frac{x}{N(z)}.$$

19.7 The Lie algebra of the Vahlen group

If the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}(2, \mathcal{C}\ell_n)$ is in the Lie algebra of the Vahlen group of $\mathbb{R}^n$, then $Ax + B - \mathbf{x}C\mathbf{x} - \mathbf{x}D \in \mathbb{R}^n$ for all $\mathbf{x} \in \mathbb{R}^n$. It follows that $B, C \in \mathbb{R}^n$ and $A, D \in \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2 \oplus \bigwedge^n \mathbb{R}^n$ so that $\langle A \rangle_2 = \langle D \rangle_2$ and $\langle A \rangle_n = \pm \langle D \rangle_n$. Actually, for the Lie algebra of the Vahlen group $\langle A \rangle_n$, $\langle D \rangle_n$ vanish and for the Lie algebra of the normalized Vahlen group $\langle A \rangle_0 = -\langle D \rangle_0$. In fact, matrices in the Lie algebra of the normalized Vahlen group can be characterized by

(i) $A, D \in \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^n$
(ii) $B, C \in \mathbb{R}^n$
(iii) $A + \tilde{D} = 0$.

The Lie algebra is spanned by the matrices

$$L_{\mu\nu} = \begin{pmatrix} -\frac{1}{2}\mathrm{e}_{\mu\nu} & 0 \\ 0 & -\frac{1}{2}\mathrm{e}_{\mu\nu} \end{pmatrix}, \quad P_\mu = \begin{pmatrix} 0 & \mathrm{e}_\mu \\ 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad K_\mu = \begin{pmatrix} 0 & 0 \\ \mathrm{e}_\mu & 0 \end{pmatrix}.$$

These matrices represent rotations, translations, dilations and transversions.

19.8 Compactification and the isotropic cone at infinity

The set of Möbius mappings on $\mathbb{R}^{p,q}$ can be used to compactify $\mathbb{R}^{p,q}$. The compactification is homeomorphic to

$$\frac{S^p \times S^q}{\mathbb{Z}_2}.$$

In particular, the compactification of a Euclidean space $\mathbb{R}^n$ is the sphere S^n, and the compactification of the hyperbolic plane $\mathbb{R}^{1,1}$ is the torus $S^1 \times S^1$. The conformal compactification adjoins an isotropic cone at infinity to the quadratic space.

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Questions

1. Is an element in the identity component of the conformal group necessarily a product of a rotation, a translation, a dilation and a transversion?
2. The group $SU(2,2)$ is a covering group of the identity component of the conformal group of $\mathbb{R}^{1,3}$. Is it a two-fold or a four-fold covering group?

Answers

1. No, as the counter-example of Maks shows.
2. As the identity component of the normalized Vahlen group it is a four-fold covering group.

Exercises

1. The counter-example M of Maks can be factored into a product of two 'diversions':
$$M = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathbf{e}_1 \\ -\mathbf{e}_1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathbf{e}_4 \\ \mathbf{e}_4 & 1 \end{pmatrix}.$$
Show that a 'diversion' is a product of just one transversion, one dilation, one translation and one rotation.
2. Show that in the case of a Euclidean space $\mathbb{R}^n$ the conditions $a\tilde{b}$, $\tilde{b}d$, $d\tilde{c}$, $\tilde{c}a \in \mathbb{R}^n$ and $\bar{a}b$, $b\bar{d}$, $\bar{d}c$, $c\bar{a} \in \mathbb{R}^n$ are equivalent.
3. Show that the conformal compactification of the Minkowksi space $\mathbb{R}^{1,3}$ is homeomorphic to $U(2)$.

Solutions

1. The first factor is a product of just one transversion, one dilation and one translation as follows:
$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathbf{e}_1 \\ -\mathbf{e}_1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathbf{e}_1 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & \mathbf{e}_1 \\ 0 & 1 \end{pmatrix}.$$
One can insert the identity rotation as the last factor.
2. For $a \in \mathbf{\Gamma}_n$, $\bar{a} = \tilde{a}$. If $\tilde{a}b \in \mathbb{R}^n$, then we have two cases to consider: either a is zero, and so $a\tilde{b}$ is a vector, or a is in the Lipschitz group $\mathbf{\Gamma}_n$, but then $a(\tilde{a}b)\tilde{a}$ is a vector and $a\tilde{a} \in \mathbb{R}\setminus\{0\}$, and so $b\tilde{a}$ is a vector, which implies $\tilde{a}b = (b\tilde{a})^{\sim} \in \mathbb{R}^n$.
3. This follows as a special case from the matrix isomorphism
$$U(n) \simeq \frac{SU(n)\times U(1)}{\mathbb{Z}_n}.$$

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