22

Chevalley's Construction and Characteristic 2

Consider an n-dimensional linear space V over a field $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$, and the symmetric bilinear form

$$<\mathbf{x}, \mathbf{y}> = \frac{1}{2}[Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})]$$

associated with the quadratic form Q. Later in this chapter we will discuss the case $\operatorname{char}\mathbb{F} = 2$. As before, we denote the exterior algebra of V by $\bigwedge V$ and the Clifford algebra of Q, with $\mathbf{x}^2 = Q(\mathbf{x})$, by $\mathcal{C}\ell(Q)$ or $\mathcal{C}\ell_{p,q}$ when $V = \mathbb{R}^{p,q}$ and

$$Q(\mathbf{x}) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2, \qquad n = p + q.$$

We shall construct a natural linear isomorphism $\bigwedge V \to \mathcal{C}\ell(Q)$, review how Riesz goes backwards and derives $\bigwedge V$ from $\mathcal{C}\ell(Q)$ and compare Riesz's method to an alternative construction due to Chevalley but known to some theoretical physicists in the disguise of the Kähler-Atiyah isomorphism.

22.1 Construction of the linear isomorphism $\bigwedge V \to \mathcal{C}\ell(Q)$

Here we start from the exterior algebra $\bigwedge V$. Recall that a k-vector $\mathbf{a} \in \bigwedge^k V$ has the grade involute $\hat{\mathbf{a}} = (-1)^k \mathbf{a}$ and the reverse $\tilde{\mathbf{a}} = (-1)^{\frac{1}{2}k(k-1)}\mathbf{a}$. The symmetric bilinear form associated with Q on V can be extended to simple k-vectors in $\bigwedge^k V$ by way of

$$<\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k, \mathbf{y}_1 \wedge \mathbf{y}_2 \wedge \ldots \wedge \mathbf{y}_k> = \det<\mathbf{x}_i, \mathbf{y}_j>,$$

where

$$\det<\mathbf{x}_i, \mathbf{y}_j> = \begin{vmatrix} <\mathbf{x}_1, \mathbf{y}_1> & <\mathbf{x}_1, \mathbf{y}_2> & \cdots & <\mathbf{x}_1, \mathbf{y}_k> \\ <\mathbf{x}_2, \mathbf{y}_1> & <\mathbf{x}_2, \mathbf{y}_2> & \cdots & <\mathbf{x}_2, \mathbf{y}_k> \\ \vdots & \vdots & \ddots & \vdots \\ <\mathbf{x}_k, \mathbf{y}_1> & <\mathbf{x}_k, \mathbf{y}_2> & \cdots & <\mathbf{x}_k, \mathbf{y}_k> \end{vmatrix},$$

and further by linearity to all of $\bigwedge^k V$ and by orthogonality to all of $\bigwedge V$.

Example. Let $Q(x_1\mathrm{e}_1 + x_2\mathrm{e}_2) = ax_1^2 + bx_2^2$. Then $<\mathbf{x},\mathbf{y}> = ax_1y_1 + bx_2y_2$ and $\mathbf{x}\wedge\mathbf{y} = (x_1y_2 - x_2y_1)\mathrm{e}_1\wedge\mathrm{e}_2$. The identity $(ax_1^2+bx_2^2)(ay_1^2+by_2^2) = (ax_1y_1 + bx_2y_2)^2 + ab(x_1y_2 - x_2y_1)^2$ can be written as $Q(\mathbf{x})Q(\mathbf{y}) = <\mathbf{x},\mathbf{y}>^2 + Q(\mathbf{x}\wedge\mathbf{y})$, where $Q(\mathbf{x}\wedge\mathbf{y}) = ab(x_1y_2 - x_2y_1)^2$. ∎

In the case of a non-degenerate Q on V we can introduce the dual of the exterior product called the left contraction $u \rfloor v$ of $v \in \bigwedge V$ by $u \in \bigwedge V$ through the requirement

$$<u \rfloor v, w> = <v, \tilde{u} \wedge w> \quad \text{for all} \quad w \in \bigwedge V.$$

Examples. 1. Let $\mathbf{x},\mathbf{y} \in V$, $w \in \mathbb{F}$. Then $<\mathbf{x} \rfloor \mathbf{y}, w> = <\mathbf{y}, \mathbf{x}\wedge w> = <\mathbf{y}, \mathbf{x}w> = <\mathbf{y},\mathbf{x}>w$ and since $<\mathbf{x} \rfloor \mathbf{y}, w> = <\mathbf{x} \rfloor \mathbf{y}, 1>w$ we have the rule $\mathbf{x} \rfloor \mathbf{y} = <\mathbf{x},\mathbf{y}>$.

2. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{w} \in V$. Then $<\mathbf{x} \rfloor (\mathbf{y}\wedge\mathbf{z}), \mathbf{w}> = <\mathbf{y}\wedge\mathbf{z}, \mathbf{x}\wedge\mathbf{w}>$

$$= \begin{vmatrix} <\mathbf{y},\mathbf{x}> & <\mathbf{y},\mathbf{w}> \\ <\mathbf{z},\mathbf{x}> & <\mathbf{z},\mathbf{w}> \end{vmatrix}$$

$= (\mathbf{x} \rfloor \mathbf{y})<\mathbf{z},\mathbf{w}> - (\mathbf{x} \rfloor \mathbf{z})<\mathbf{y},\mathbf{w}> = <(\mathbf{x} \rfloor \mathbf{y})\mathbf{z} - (\mathbf{x} \rfloor \mathbf{z})\mathbf{y}, \mathbf{w}>$ and so we have the rule $\mathbf{x} \rfloor (\mathbf{y}\wedge\mathbf{z}) = (\mathbf{x} \rfloor \mathbf{y})\mathbf{z} - (\mathbf{x} \rfloor \mathbf{z})\mathbf{y}$.

3. Let $\mathbf{x} \in V$, $\mathbf{x}_i \in V$ and $w = \mathbf{w}_1 \wedge \mathbf{w}_2 \wedge \ldots \wedge \mathbf{w}_{k-1} \in \bigwedge^{k-1} V$. Then $<\mathbf{x} \rfloor (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k), w> = <\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k, \mathbf{x} \wedge \mathbf{w}_1 \wedge \mathbf{w}_2 \wedge \ldots \wedge \mathbf{w}_{k-1}>$

$$= \begin{vmatrix} <\mathbf{x}_1,\mathbf{x}> & <\mathbf{x}_1,\mathbf{w}_1> & <\mathbf{x}_1,\mathbf{w}_2> & \cdots & <\mathbf{x}_1,\mathbf{w}_{k-1}> \\ <\mathbf{x}_2,\mathbf{x}> & <\mathbf{x}_2,\mathbf{w}_1> & <\mathbf{x}_2,\mathbf{w}_2> & \cdots & <\mathbf{x}_2,\mathbf{w}_{k-1}> \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ <\mathbf{x}_k,\mathbf{x}> & <\mathbf{x}_k,\mathbf{w}_1> & <\mathbf{x}_k,\mathbf{w}_2> & \cdots & <\mathbf{x}_k,\mathbf{w}_{k-1}> \end{vmatrix}$$

$$= \sum_{i=1}^{k}(-1)^{i-1}<\mathbf{x},\mathbf{x}_i><\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_{i-1} \wedge \mathbf{x}_{i+1} \wedge \ldots \wedge \mathbf{x}_k, w>$$

and so

$$\mathbf{x} \rfloor (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k) = \sum_{i=1}^{k}(-1)^{i-1}<\mathbf{x},\mathbf{x}_i>\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_{i-1} \wedge \mathbf{x}_{i+1} \wedge \ldots \wedge \mathbf{x}_k.$$

4. $\mathbf{x} \rfloor (u \wedge v) = \mathbf{x} \rfloor ((\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \ldots \wedge \mathbf{u}_i) \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_j))$

$$= \sum_{k=1}^{i}(\mathbf{x} \rfloor \mathbf{u}_k)(-1)^{k-1}\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \ldots \wedge \mathbf{u}_{k-1} \wedge \mathbf{u}_{k+1} \wedge \ldots \wedge \mathbf{u}_i \wedge v$$

$$+(-1)^i \sum_{k=1}^{j} (\mathbf{x} \lrcorner \mathbf{v}_k)(-1)^{k-1} u \wedge \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \ldots \wedge \mathbf{v}_{k-1} \wedge \mathbf{v}_{k+1} \wedge \ldots \wedge \mathbf{v}_j$$

$$= (\mathbf{x} \lrcorner u) \wedge v + (-1)^i u \wedge (\mathbf{x} \lrcorner v).$$

5. $<(u \wedge v) \lrcorner w, z> = <w, (u \wedge v)^{\sim} \wedge z> = <w, \tilde{v} \wedge \tilde{u} \wedge z> = <v \lrcorner w, \tilde{u} \wedge z>$ $= <u \lrcorner (v \lrcorner w), z>$ and so $(u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w)$. ▮

In the case of a non-degenerate Q we have verified the following properties of the contraction:

(*a*) $\quad \mathbf{x} \lrcorner \mathbf{y} = <\mathbf{x}, \mathbf{y}> \quad$ for $\quad \mathbf{x}, \mathbf{y} \in V$

(*b*) $\quad \mathbf{x} \lrcorner (u \wedge v) = (\mathbf{x} \lrcorner u) \wedge v + \hat{u} \wedge (\mathbf{x} \lrcorner v)$

(*c*) $\quad (u \wedge v) \lrcorner w = u \lrcorner (v \lrcorner w) \quad$ for $\quad u, v, w \in \bigwedge V$

(see Helmstetter 1982). These properties also determine the contraction uniquely for an arbitrary, not necessarily non-degenerate Q. The identity (*c*) introduces a scalar multiplication on $\bigwedge V$ making it a left module over $\bigwedge V$. The identity (*b*) means that contraction by $\mathbf{x} \in V$ operates like a derivation. Evidently, $\mathbf{x} \lrcorner \mathbf{a} \in \bigwedge^{k-1} V$ for $\mathbf{a} \in \bigwedge^k V$.

Introduce the *Clifford product* of $\mathbf{x} \in V$ and $u \in \bigwedge V$ by

$$\mathbf{x}u = \mathbf{x} \wedge u + \mathbf{x} \lrcorner u$$

and extend this product by linearity and associativity to all of $\bigwedge V$, which then becomes, as an associative algebra, isomorphic to $\mathcal{C}\ell(Q)$. For instance, the product of a simple bivector $\mathbf{x} \wedge \mathbf{y} \in \bigwedge^2 V$ and an arbitrary element $u \in \bigwedge V$ is given by

$$(\mathbf{x} \wedge \mathbf{y})u = \mathbf{x} \wedge \mathbf{y} \wedge u + \mathbf{x} \wedge (\mathbf{y} \lrcorner u) - \mathbf{y} \wedge (\mathbf{x} \lrcorner u) + \mathbf{x} \lrcorner (\mathbf{y} \lrcorner u)$$

where we have first expanded $(\mathbf{x} \wedge \mathbf{y})u = (\mathbf{x}\mathbf{y} - \mathbf{x} \lrcorner \mathbf{y})u$, then used

$$\mathbf{x}(\mathbf{y}u) = \mathbf{x} \wedge \mathbf{y} \wedge u + \mathbf{x} \wedge (\mathbf{y} \lrcorner u) + \mathbf{x} \lrcorner (\mathbf{y} \wedge u) + \mathbf{x} \lrcorner (\mathbf{y} \lrcorner u)$$

and the derivation rule $\mathbf{x} \lrcorner (\mathbf{y} \wedge u) = (\mathbf{x} \lrcorner \mathbf{y}) \wedge u - \mathbf{y} \wedge (\mathbf{x} \lrcorner u)$.

Exercises 1,2,. . . ,10

Remark. Some authors use, instead of the left and right contractions, a more symmetric dot product in $\mathcal{C}\ell(Q)$, defined by the Clifford product for homogeneous elements as (char $\neq 2$)

$$\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{a}\mathbf{b} \rangle_{|i-j|}, \quad \mathbf{a} \in \bigwedge^i V, \ \mathbf{b} \in \bigwedge^j V,$$

and extended by linearity to all of $C\ell(Q)$. The relation between the dot product and the contractions,

$$u \cdot v = u \,\lrcorner\, v + u \,\llcorner\, v - <u, \tilde{v}>,$$

shows that the dot product cannot be expected to have properties, which can be easily proved (using the more fundamental contractions). Some authors try to make the dot product look like derivation, and define exceptionally $\lambda \cdot u = 0$, $u \cdot \lambda = 0$ for $\lambda \in \mathbb{F}$. This only makes things worse, because for this dot product the relation to the contractions is still more complicated:

$$u \cdot v = u \,\lrcorner\, v + u \,\llcorner\, v - <u, \tilde{v}> - \langle u \rangle_0 v - u \langle v \rangle_0 + \langle u \rangle_0 \langle v \rangle_0. \qquad \blacksquare$$

22.2 Chevalley's identification of $C\ell(Q) \subset \mathrm{End}(\bigwedge V)$

Chevalley 1954 pp. 38-42 tried to include the characteristic 2 by embedding the Clifford algebra $C\ell(Q)$ into the endomorphism algebra $\mathrm{End}(\bigwedge V)$ of the exterior algebra $\bigwedge V$. He introduced a linear operator $L'_{\mathbf{x}} = \varphi_{\mathbf{x}} \in \mathrm{End}(\bigwedge V)$ such that

$$\varphi_{\mathbf{x}}(u) = \mathbf{x} \wedge u + \mathbf{x} \,\lrcorner\, u \quad \text{for} \quad \mathbf{x} \in V,\ u \in \bigwedge V.$$

From the derivation rule $\mathbf{x} \,\lrcorner\, (\mathbf{x} \wedge u) = (\mathbf{x} \,\lrcorner\, \mathbf{x}) \wedge u - \mathbf{x} \wedge (\mathbf{x} \,\lrcorner\, u)$ and $\mathbf{x} \wedge \mathbf{x} \wedge u = 0$, $\mathbf{x} \,\lrcorner\, (\mathbf{x} \,\lrcorner\, u) = 0$ we can conclude the identity $(\varphi_{\mathbf{x}})^2 = Q(\mathbf{x})$. Chevalley's inclusion map $V \to \mathrm{End}(\bigwedge V)$, $\mathbf{x} \to \varphi_{\mathbf{x}}$ is then a Clifford map and can be extended to an algebra homomorphism $\psi : C\ell(Q) \to \mathrm{End}(\bigwedge V)$, whose image evaluated at $1 \in \bigwedge V$ yields the map $\phi : \mathrm{End}(\bigwedge V) \to \bigwedge V$. The composite linear map $\theta = \phi \circ \psi$ is the right inverse of the natural map $\bigwedge V \to C\ell(Q)$ and

$$\bigwedge V \to C\ell(Q) \xrightarrow{\psi} \mathrm{End}(\bigwedge V) \xrightarrow{\phi} \bigwedge V$$

is the identity mapping on $\bigwedge V$. The faithful representation ψ sends $C\ell(Q)$ onto an isomorphic subalgebra of $\mathrm{End}(\bigwedge V)$.

Chevalley's identification also works well with a contraction defined by an arbitrary – *not necessarily symmetric* – bilinear form B such that $B(\mathbf{x}, \mathbf{x}) = Q(\mathbf{x})$ and

(a) $\quad \mathbf{x} \,\lrcorner\, \mathbf{y} = B(\mathbf{x}, \mathbf{y}) \quad$ for $\quad \mathbf{x}, \mathbf{y} \in V$

(b) $\quad \mathbf{x} \,\lrcorner\, (u \wedge v) = (\mathbf{x} \,\lrcorner\, u) \wedge v + \hat{u} \wedge (\mathbf{x} \,\lrcorner\, v)$

(c) $\quad (u \wedge v) \,\lrcorner\, w = u \,\lrcorner\, (v \,\lrcorner\, w) \quad$ for $\quad u, v, w \in \bigwedge V$

(see Helmstetter 1982). As before, $\mathbf{x} \lrcorner \mathbf{a} \in \bigwedge^{k-1} V$ for $\mathbf{a} \in \bigwedge^k V$ and

$$\mathbf{x} \lrcorner (\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_k)$$
$$= \sum_{i=1}^{k} (-1)^{i-1} B(\mathbf{x}, \mathbf{x}_i)\, \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \ldots \wedge \mathbf{x}_{i-1} \wedge \mathbf{x}_{i+1} \wedge \ldots \wedge \mathbf{x}_k,$$

and the faithful representation ψ sends the Clifford algebra $\mathcal{C}\ell(Q)$ onto an isomorphic subalgebra of $\mathrm{End}(\bigwedge V)$, which, however, as a subspace depends on B.

Remark. Chevalley introduced his identification $\mathcal{C}\ell(Q) \subset \mathrm{End}(\bigwedge V)$ in order to be able to include the exceptional case of characteristic 2. In characteristic $\neq 2$ the theory of quadratic forms is the same as the theory of symmetric bilinear forms and Chevalley's identification gives the Clifford algebra of the *symmetric* bilinear form $\langle \mathbf{x}, \mathbf{y} \rangle = \frac{1}{2}(B(\mathbf{x}, \mathbf{y}) + B(\mathbf{y}, \mathbf{x}))$ satisfying $\mathbf{xy} + \mathbf{yx} = 2\langle \mathbf{x}, \mathbf{y} \rangle$. ∎

For arbitrary Q but $\mathrm{char}\,\mathbb{F} \neq 2$ there is the natural choice of the unique symmetric bilinear form B such that $B(\mathbf{x}, \mathbf{x}) = Q(\mathbf{x})$ giving rise to the canonical/privileged linear isomorphism $\mathcal{C}\ell(Q) \to \bigwedge V$. The case $\mathrm{char}\,\mathbb{F} = 2$ is quite different. In general, there are no symmetric bilinear forms such that $B(\mathbf{x}, \mathbf{x}) = Q(\mathbf{x})$ and when there is such a symmetric bilinear form, it is not unique since any alternating [1] bilinear form is also symmetric and could be added to the symmetric bilinear form without changing Q. Hence the contraction is not unique if $\mathrm{char}\,\mathbb{F} \neq 2$, and there is an ambiguity in $\varphi_{\mathbf{x}}$.

In characteristic 2 the theory of quadratic forms is not the same as the theory of symmetric bilinear forms.

Example. Let $\dim_{\mathbb{F}} V = 2$, $B(\mathbf{x}, \mathbf{y}) = a x_1 y_1 + b x_1 y_2 + c x_2 y_1 + d x_2 y_2$ and $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$. The contraction $\mathbf{x} \lrcorner \mathbf{y} = B(\mathbf{x}, \mathbf{y})$ gives the Clifford product $\mathbf{x}v = \mathbf{x} \wedge v + \mathbf{x} \lrcorner v$ of $\mathbf{x} \in V$, $v \in \bigwedge V$. We will determine the matrix of $v \to uv$, $u = u_0 + u_1 \mathbf{e}_1 + u_2 \mathbf{e}_2 + u_{12} \mathbf{e}_1 \wedge \mathbf{e}_2$ with respect to the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_2\}$ for $\bigwedge V$. The matrix of $\mathbf{e}_1$ is obtained by the following computation:

$$\begin{aligned}
&\mathbf{e}_1 1 = \mathbf{e}_1 \wedge 1 = \mathbf{e}_1 && (\text{first column} = 0100)\\
&\mathbf{e}_1 \mathbf{e}_1 = \mathbf{e}_1 \lrcorner \mathbf{e}_1 = a && (\text{second column} = a000)\\
&\mathbf{e}_1 \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 + \mathbf{e}_1 \lrcorner \mathbf{e}_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 + b && (\text{third column} = b001)\\
&\mathbf{e}_1 (\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_1 \lrcorner (\mathbf{e}_1 \wedge \mathbf{e}_2) = (\mathbf{e}_1 \lrcorner \mathbf{e}_1) \wedge \mathbf{e}_2 - (\mathbf{e}_1 \lrcorner \mathbf{e}_2) \wedge \mathbf{e}_1 = a\mathbf{e}_2 - b\mathbf{e}_1.
\end{aligned}$$

1 Recall that antisymmetric means $B(\mathbf{x}, \mathbf{y}) = -B(\mathbf{y}, \mathbf{x})$ and alternating $B(\mathbf{x}, \mathbf{x}) = 0$; alternating is always antisymmetric, though in characteristic 2 antisymmetric is not necessarily alternating.

The matrix of $\mathbf{e}_1 \wedge \mathbf{e}_2$ is obtained by

$$
\begin{aligned}
&(\mathbf{e}_1 \wedge \mathbf{e}_2)\,1 = \mathbf{e}_1 \wedge \mathbf{e}_2 \\
&(\mathbf{e}_1 \wedge \mathbf{e}_2)\mathbf{e}_1 = (\mathbf{e}_1 \wedge \mathbf{e}_2) \llcorner \mathbf{e}_1 = \mathbf{e}_1 \wedge (\mathbf{e}_2 \llcorner \mathbf{e}_1) - \mathbf{e}_2 \wedge (\mathbf{e}_1 \llcorner \mathbf{e}_1) = c\mathbf{e}_1 - a\mathbf{e}_2 \\
&(\mathbf{e}_1 \wedge \mathbf{e}_2)\mathbf{e}_2 = (\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1 \lrcorner \mathbf{e}_2)\mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2^2 - (\mathbf{e}_1 \lrcorner \mathbf{e}_2)\mathbf{e}_2 = d\mathbf{e}_1 - b\mathbf{e}_2 \\
&(\mathbf{e}_1 \wedge \mathbf{e}_2)(\mathbf{e}_1 \wedge \mathbf{e}_2) = (\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1 \lrcorner \mathbf{e}_2)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&\qquad = \mathbf{e}_1(\mathbf{e}_2 \wedge (\mathbf{e}_1 \wedge \mathbf{e}_2) + \mathbf{e}_2 \lrcorner (\mathbf{e}_1 \wedge \mathbf{e}_2)) - (\mathbf{e}_1 \lrcorner \mathbf{e}_2)(\mathbf{e}_1 \wedge \mathbf{e}_2) \\
&\qquad = \mathbf{e}_1(c\mathbf{e}_2 - d\mathbf{e}_1) - b(\mathbf{e}_1 \wedge \mathbf{e}_2) = -ad + bc + (-b + c)(\mathbf{e}_1 \wedge \mathbf{e}_2).
\end{aligned}
$$

So we have the following matrix representations:

$$
\mathbf{e}_1 = \begin{pmatrix} 0 & a & b & 0 \\ 1 & 0 & 0 & -b \\ 0 & 0 & 0 & a \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad
\mathbf{e}_2 = \begin{pmatrix} 0 & c & d & 0 \\ 0 & 0 & 0 & -d \\ 1 & 0 & 0 & c \\ 0 & -1 & 0 & 0 \end{pmatrix},
$$

$$
\mathbf{e}_1 \wedge \mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 0 & -ad + bc \\ 0 & c & d & 0 \\ 0 & -a & -b & 0 \\ 1 & 0 & 0 & -b + c \end{pmatrix},
$$

or in general

$$
u = \begin{pmatrix} u_0 & au_1 + cu_2 & bu_1 + du_2 & -(ad - bc)u_{12} \\ u_1 & u_0 + cu_{12} & du_{12} & -(bu_1 + du_2) \\ u_2 & -au_{12} & u_0 - bu_{12} & au_1 + cu_2 \\ u_{12} & -u_2 & u_1 & u_0 + (-b + c)u_{12} \end{pmatrix}.
$$

Evidently, the commutation relations $\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1 = b + c$ and $\mathbf{e}_1^2 = a$, $\mathbf{e}_2^2 = d$ are satisfied, and we have the following multiplication table:

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1 \wedge \mathbf{e}_2$
$\mathbf{e}_1$	a	$\mathbf{e}_1 \wedge \mathbf{e}_2 + b$	$-b\mathbf{e}_1 + a\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_1 \wedge \mathbf{e}_2 + c$	d	$-d\mathbf{e}_1 + c\mathbf{e}_2$
$\mathbf{e}_1 \wedge \mathbf{e}_2$	$c\mathbf{e}_1 - a\mathbf{e}_2$	$d\mathbf{e}_1 - b\mathbf{e}_2$	$-ad + bc + (-b + c)\mathbf{e}_1 \wedge \mathbf{e}_2$

In characteristic $\neq 2$ we find

$$
\frac{1}{2}(\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1) = \mathbf{e}_1 \wedge \mathbf{e}_2 + \frac{1}{2}(b - c)
$$

and more generally for $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2$

$$
\frac{1}{2}(\mathbf{xy} - \mathbf{yx}) = (x_1y_2 - x_2y_1)\,\mathbf{e}_1 \wedge \mathbf{e}_2 + \frac{1}{2}(b - c)(x_1y_2 - x_2y_1)
$$

$$= \mathbf{x} \wedge \mathbf{y} + A(\mathbf{x}, \mathbf{y})$$

with an alternating scalar valued form $A(\mathbf{x}, \mathbf{y}) = \frac{1}{2}(B(\mathbf{x}, \mathbf{y}) - B(\mathbf{y}, \mathbf{x}))$. For non-zero $A(\mathbf{x}, \mathbf{y})$ the quotient $\mathbf{x} \wedge \mathbf{y} / A(\mathbf{x}, \mathbf{y})$ is independent of $\mathbf{x}, \mathbf{y} \in V$. Note that the matrix of $\frac{1}{2}(\mathbf{xy} - \mathbf{yx})$ is traceless. The symmetric bilinear form associated with $Q(\mathbf{x})$ is

$$\mathbf{x} \cdot \mathbf{y} = \frac{1}{2}(B(\mathbf{x}, \mathbf{y}) + B(\mathbf{y}, \mathbf{x})) = a x_1 y_1 + \frac{1}{2}(b + c)(x_1 y_2 + x_2 y_1) + d x_2 y_2$$

and we have $\mathbf{xy} + \mathbf{yx} = 2\mathbf{x} \cdot \mathbf{y}$ for $\mathbf{x}, \mathbf{y} \in V \subset \mathcal{C}\ell(Q)$. Orthogonal vectors $\mathbf{x} \perp \mathbf{y}$ anticommute, $\mathbf{xy} = -\mathbf{yx}$ and $(\mathbf{xy})^2 = -\mathbf{x}^2\mathbf{y}^2$, even though $\mathbf{xy} = \mathbf{x} \wedge \mathbf{y} + A(\mathbf{x}, \mathbf{y})$. [In this special case $A(\mathbf{x}, \mathbf{y}) = B(\mathbf{x}, \mathbf{y}) \neq 0$ while $\mathbf{x} \cdot \mathbf{y} = 0$ implies $B(\mathbf{y}, \mathbf{x}) = -B(\mathbf{x}, \mathbf{y})$.] ∎

It is convenient to regard $\bigwedge V$ as the subalgebra of $\mathrm{End}(\bigwedge V)$ with the canonical choice of the symmetric $B = 0$. We may also regard $\mathcal{C}\ell(Q)$ as a subalgebra of $\mathrm{End}(\bigwedge V)$ obtained with some B such that $B(\mathbf{x}, \mathbf{x}) = Q(\mathbf{x})$ and choose the symmetric B in char $\neq 2$.

The following example shows that for $Q = 0$ and $B = 0$, Chevalley's process results in the original multiplication of the exterior algebra $\bigwedge V$, but that for $Q = 0$ and alternating $B \neq 0$, the process gives an isomorphic but different exterior multiplication on $\bigwedge V$.

Example. Take a special case of the previous example, the Clifford algebra with $Q = 0$ and $B(\mathbf{x}, \mathbf{y}) = b(x_1 y_2 - x_2 y_1)$. Send the matrix of the exterior product (with the symmetric bilinear form $= 0$) to a matrix of the isomorphic Clifford product (determined by the alternating bilinear form $= B$):

$$\begin{pmatrix} u_0 & 0 & 0 & 0 \\ u_1 & u_0 & 0 & 0 \\ u_2 & 0 & u_0 & 0 \\ u_{12} & -u_2 & u_1 & u_0 \end{pmatrix} \xrightarrow{\beta} \begin{pmatrix} u_0 & -bu_2 & bu_1 & -b^2 u_{12} \\ u_1 & u_0 - bu_{12} & 0 & -bu_1 \\ u_2 & 0 & u_0 - bu_{12} & -bu_2 \\ u_{12} & -u_2 & u_1 & u_0 - 2bu_{12} \end{pmatrix}.$$

In this case

$$\beta(\mathbf{x})\beta(\mathbf{y}) = \beta(\mathbf{x} \wedge \mathbf{y} + B(\mathbf{x}, \mathbf{y})).$$

In particular, $\beta(\mathbf{e}_1)\beta(\mathbf{e}_2) = \beta(\mathbf{e}_1 \wedge \mathbf{e}_2 + b)$, $\beta(\mathbf{e}_2)\beta(\mathbf{e}_1) = -\beta(\mathbf{e}_1 \wedge \mathbf{e}_2 + b)$ and $\beta(\mathbf{e}_1 \wedge \mathbf{e}_2 + b)\beta(\mathbf{e}_i) = 0$, $\beta(\mathbf{e}_i)\beta(\mathbf{e}_1 \wedge \mathbf{e}_2 + b) = 0$. We have already met this situation in the chapter on the *Definitions of the Clifford Algebra* in the section on the *Uniqueness and the definition by generators and relations* except that here the exterior algebra and the Clifford algebra (determined by the alternating B) are regarded as different subspaces of $\mathrm{End}(\bigwedge V)$ [although they are isomorphic subalgebras of $\mathrm{End}(\bigwedge V)$]. ∎

The above example shows that those who do not accept the existence of k-vectors in a Clifford algebra $C\ell(Q)$ over $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$, should also exclude fixed subspaces $\bigwedge^k V \subset \bigwedge V$.

In general, consider two copies of $C\ell(Q)$ in $\mathrm{End}(\bigwedge V)$ so that $Q(\mathbf{x})$ equals $B_1(\mathbf{x},\mathbf{x}) = B_2(\mathbf{x},\mathbf{x})$, which determine $\beta_1(\mathbf{x})\beta_1(\mathbf{y}) = \beta_1(\mathbf{x}\wedge\mathbf{y} + B_1(\mathbf{x},\mathbf{y}))$ and $\beta_2(\mathbf{x})\beta_2(\mathbf{y}) = \beta_2(\mathbf{x}\wedge\mathbf{y} + B_2(\mathbf{x},\mathbf{y}))$. A transition between the two copies is given by an alternating bilinear form $B(\mathbf{x},\mathbf{y}) = B_1(\mathbf{x},\mathbf{y}) - B_2(\mathbf{x},\mathbf{y})$ and $\beta(\mathbf{x})\beta(\mathbf{y}) = \beta(\mathbf{x}\mathbf{y} + B(\mathbf{x},\mathbf{y}))$.

In characteristic $\neq 2$ this means that the symmetric bilinear form such that $<\mathbf{x},\mathbf{x}> = Q(\mathbf{x})$ gives rise to the natural choice $\mathbf{x}\mathbf{y} = \mathbf{x}\wedge\mathbf{y} + <\mathbf{x},\mathbf{y}>$ among the Clifford products $\mathbf{x}\mathbf{y} + B(\mathbf{x},\mathbf{y})$ with an alternating B. In other words, the Clifford product $\mathbf{x}\mathbf{y}$ has a distinguished decomposition into the sum $\mathbf{x}\wedge\mathbf{y} + <\mathbf{x},\mathbf{y}>$ where $<\mathbf{x},\mathbf{y}> = \frac{1}{2}(\mathbf{x}\mathbf{y}+\mathbf{y}\mathbf{x})$ is a scalar and $\mathbf{x}\wedge\mathbf{y} = \frac{1}{2}(\mathbf{x}\mathbf{y}-\mathbf{y}\mathbf{x})$ is a bivector [this decomposition is unique among all the possible decompositions with antisymmetric part $\mathbf{x}\dot\wedge\mathbf{y} = \frac{1}{2}(\mathbf{x}\mathbf{y}-\mathbf{y}\mathbf{x})$ equaling a new kind of bivector $\mathbf{x}\dot\wedge\mathbf{y} = \mathbf{x}\wedge\mathbf{y} + B(\mathbf{x},\mathbf{y})$ where $\mathbf{x}\dot\wedge\mathbf{y} \in \dot{\bigwedge}^2 V$ and $\mathbf{x}\wedge\mathbf{y} \in \bigwedge^2 V$]. [Similarly, a completely antisymmetric product of three vectors equals a new kind of 3-vector $\mathbf{x}\dot\wedge\mathbf{y}\dot\wedge\mathbf{z} = \mathbf{x}\wedge\mathbf{y}\wedge\mathbf{z} + \mathbf{x}B(\mathbf{y},\mathbf{z}) + \mathbf{y}B(\mathbf{z},\mathbf{x}) + \mathbf{z}B(\mathbf{x},\mathbf{y})$.]

Example. Let $\mathbb{F} = \{0,1\}$, $\dim_{\mathbb{F}} V = 2$ and $Q(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = x_1x_2$. There are only two bilinear forms B_i such that $B_i(\mathbf{x},\mathbf{x}) = Q(\mathbf{x})$, namely $B_1(\mathbf{x},\mathbf{y}) = x_1y_2$ and $B_2(\mathbf{x},\mathbf{y}) = x_2y_1$, and neither is symmetric. The difference $A = B_1 - B_2$, $A(\mathbf{x},\mathbf{y}) = x_1y_2 - x_2y_1$ $(= x_1y_2 + x_2y_1)$ is alternating (and thereby symmetric). Therefore, there are two representations of $C\ell(Q)$ in $\mathrm{End}(\bigwedge V)$:

$$\text{for } B_1: \qquad u = \begin{pmatrix} u_0 & 0 & u_1 & 0 \\ u_1 & u_0 & 0 & -u_1 \\ u_2 & 0 & u_0 - u_{12} & 0 \\ u_{12} & -u_2 & u_1 & u_0 - u_{12} \end{pmatrix}$$

$$\text{for } B_2: \qquad u = \begin{pmatrix} u_0 & u_2 & 0 & 0 \\ u_1 & u_0 + u_{12} & 0 & 0 \\ u_2 & 0 & u_0 & u_2 \\ u_{12} & -u_2 & u_1 & u_0 + u_{12} \end{pmatrix}$$

These representations have the following multiplication tables with respect to the basis $\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\wedge\mathbf{e}_2\}$ for $\bigwedge V$:

B_1	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1\wedge\mathbf{e}_2$
$\mathbf{e}_1$	0	$1+\mathbf{e}_1\wedge\mathbf{e}_2$	$-\mathbf{e}_1$
$\mathbf{e}_2$	$-\mathbf{e}_1\wedge\mathbf{e}_2$	0	0
$\mathbf{e}_1\wedge\mathbf{e}_2$	0	$-\mathbf{e}_2$	$-\mathbf{e}_1\wedge\mathbf{e}_2$

B_2	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_1 \wedge \mathbf{e}_2$
$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{e}_2$	0
$\mathbf{e}_2$	$1 - \mathbf{e}_1 \wedge \mathbf{e}_2$	0	$\mathbf{e}_2$
$\mathbf{e}_1 \wedge \mathbf{e}_2$	$\mathbf{e}_1$	0	$\mathbf{e}_1 \wedge \mathbf{e}_2$

In this case there are only two linear isomorphisms $\bigwedge V \to C\ell(Q)$ which are identity mappings when restricted to $\mathbb{F} \oplus V$ and which preserve parity (send even elements to even elements and odd to odd). It is easy to verify that the above multiplication tables actually describe the only representations of $C\ell(Q)$ in $\bigwedge V$. In this case **there are no canonical linear isomorphisms** $\bigwedge V \to C\ell(Q)$, in other words, neither of the above multiplication tables can be preferred over the other. In particular, $\bigwedge^2 V$ cannot be canonically embedded in $C\ell(Q)$, and **there are no bivectors in characteristic 2**. ∎

The need for a simplification of Chevalley's presentation is obvious. For instance, van der Waerden 1966 said that 'the ideas underlying Chevalley's proof (p. 40) are not easy to discern' and gave another proof, equivalent but easier to follow. [Also Crumeyrolle 1990 p. xi claims that 'Chevalley's book proved too abstract for most physicists' and in a *Bull. AMS* review Lam 1989 p. 122 admits that 'Chevalley's book on spinors is ... not the easiest book to read.'] It might be helpful to get acquainted with a simpler and more direct method of relating $\bigwedge V$ and $C\ell(Q)$ due to M. Riesz 1958/1993 pp. 61-67. Riesz introduced a second product in $C\ell(Q)$ making it isomorphic with $\bigwedge V$ without resorting to the usual completely antisymmetric Clifford product of vectors and constructed a privileged linear isomorphism $C\ell(Q) \to \bigwedge V$.

22.3 Riesz's introduction of an exterior product in $C\ell(Q)$

In the following we review a construction of M. Riesz 1958/1993 pp. 61-67. Start from the Clifford algebra $C\ell(Q)$ over $\mathbb{F}$, $\operatorname{char}\mathbb{F} \neq 2$. The isometry $\mathbf{x} \to -\mathbf{x}$ of V when extended to an automorphism of $C\ell(Q)$ is called the *grade involution* $u \to \hat{u}$. Define the *exterior product* of $\mathbf{x} \in V$ and $u \in C\ell(Q)$ by

$$\mathbf{x} \wedge u = \frac{1}{2}(\mathbf{x}u + \hat{u}\mathbf{x}), \qquad u \wedge \mathbf{x} = \frac{1}{2}(u\mathbf{x} + \mathbf{x}\hat{u})$$

and extend it by linearity to all of $C\ell(Q)$, which then becomes isomorphic to $\bigwedge V$. The exterior products of two vectors $\mathbf{x} \wedge \mathbf{y} = \frac{1}{2}(\mathbf{xy} - \mathbf{yx})$ are simple *bivectors* and they span $\bigwedge^2 V$. The exterior product of a vector and a bivector, $\mathbf{x} \wedge \mathbf{B} = \frac{1}{2}(\mathbf{xB} + \mathbf{Bx})$, is a 3-vector in $\bigwedge^3 V$. The subspace of k-vectors is constructed recursively by

$$\mathbf{x} \wedge \mathbf{a} = \frac{1}{2}(\mathbf{xa} + (-1)^{k-1}\mathbf{ax}) \in \bigwedge^k V \quad \text{for} \quad \mathbf{a} \in \bigwedge^{k-1} V.$$

We may deduce associativity of the exterior product as follows. First, the definition implies for $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in V$

$$\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z}) = \frac{1}{4}(\mathbf{xyz} - \mathbf{xzy} + \mathbf{yzx} - \mathbf{zyx})$$
$$(\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z} = \frac{1}{4}(\mathbf{xyz} - \mathbf{yxz} + \mathbf{zxy} - \mathbf{zyx}).$$

Then $\mathbf{x} \wedge (\mathbf{y} \wedge \mathbf{z}) = (\mathbf{x} \wedge \mathbf{y}) \wedge \mathbf{z}$ since

$$\mathbf{xyz} - \mathbf{zyx} = \mathbf{xyz} - \mathbf{zyx} + (\mathbf{zy} + \mathbf{yz})\mathbf{x} - \mathbf{x}(\mathbf{yz} + \mathbf{zy}) = \mathbf{yzx} - \mathbf{xzy}.$$

This last result implies

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{6}(\mathbf{xyz} + \mathbf{yzx} + \mathbf{zxy} - \mathbf{zyx} - \mathbf{xzy} - \mathbf{yxz})$$

when $\operatorname{char} \mathbb{F} \neq 2, 3$ (note the resemblance with antisymmetric tensors). [Similarly, we may conclude that $\mathbf{xyz} + \mathbf{zyx} = \mathbf{x}(\mathbf{yz} + \mathbf{zy}) - (\mathbf{xz} + \mathbf{zx})\mathbf{y} + \mathbf{z}(\mathbf{xy} + \mathbf{yx})$ is a vector in V.] Riesz's construction shows that **bivectors exist in all characteristics** $\neq 2$.

Introduce the *contraction* of $u \in C\ell(Q)$ by $\mathbf{x} \in V$ so that

$$\mathbf{x} \,\lrcorner\, u = \frac{1}{2}(\mathbf{x}u - \hat{u}\mathbf{x})$$

and show that this contraction is a derivation of $C\ell(Q)$ since

$$\mathbf{x} \,\lrcorner\, (uv) = \frac{1}{2}(\mathbf{x}uv - \widehat{uv}\mathbf{x}) = \frac{1}{2}(\mathbf{x}uv - \hat{u}\hat{v}\mathbf{x})$$
$$= \frac{1}{2}(\mathbf{x}uv - \hat{u}\mathbf{x}v + \hat{u}\mathbf{x}v - \hat{u}\hat{v}\mathbf{x}) = (\mathbf{x} \,\lrcorner\, u)v + \hat{u}(\mathbf{x} \,\lrcorner\, v).$$

Thus one and the same contraction is indeed a derivation for both the exterior product and the Clifford product. [Kähler 1962 p. 435 (4.4) and p. 456 (10.3) was aware of the equations

$$\mathbf{x} \,\lrcorner\, (u \wedge v) = (\mathbf{x} \,\lrcorner\, u) \wedge v + \hat{u} \wedge (\mathbf{x} \,\lrcorner\, v) \quad \text{and}$$
$$\mathbf{x} \,\lrcorner\, (uv) = (\mathbf{x} \,\lrcorner\, u)v + \hat{u}(\mathbf{x} \,\lrcorner\, v).]$$

Provided with the scalar multiplication $(u \wedge v) \,\lrcorner\, w = u \,\lrcorner\, (v \,\lrcorner\, w)$, the exterior algebra $\bigwedge V$ and the Clifford algebra $C\ell(Q)$ are linearly isomorphic as left $\bigwedge V$-modules.

Exercises 11,12,. . . ,20

Exercises

Show that

1. $\mathbf{x} \wedge (\mathbf{y} \rfloor u) - \mathbf{y} \wedge (\mathbf{x} \rfloor u) = \mathbf{x} \rfloor (\mathbf{y} \wedge u) - \mathbf{y} \rfloor (\mathbf{x} \wedge u)$ for $\mathbf{x}, \mathbf{y} \in V$.
2. $\mathbf{x} \wedge \mathbf{y} \wedge (\mathbf{z} \rfloor u) - \mathbf{x} \wedge \mathbf{z} \wedge (\mathbf{y} \rfloor u) + \mathbf{y} \wedge \mathbf{z} \wedge (\mathbf{x} \rfloor u)$
   $= \mathbf{x} \rfloor (\mathbf{y} \wedge \mathbf{z} \wedge u) - \mathbf{y} \rfloor (\mathbf{x} \wedge \mathbf{z} \wedge u) + \mathbf{z} \rfloor (\mathbf{x} \wedge \mathbf{y} \wedge u)$.
3. $\mathbf{x} \wedge (\mathbf{y} \rfloor (\mathbf{z} \rfloor u)) - \mathbf{y} \wedge (\mathbf{x} \rfloor (\mathbf{z} \rfloor u)) + \mathbf{z} \wedge (\mathbf{x} \rfloor (\mathbf{y} \rfloor u))$
   $= (\mathbf{x} \wedge \mathbf{y}) \rfloor (\mathbf{z} \wedge u) - (\mathbf{x} \wedge \mathbf{z}) \rfloor (\mathbf{y} \wedge u) + (\mathbf{y} \wedge \mathbf{z}) \rfloor (\mathbf{x} \wedge u)$
   $= \mathbf{x} \rfloor (\mathbf{y} \rfloor (\mathbf{z} \wedge u)) - \mathbf{x} \rfloor (\mathbf{z} \rfloor (\mathbf{y} \wedge u)) + \mathbf{y} \rfloor (\mathbf{z} \rfloor (\mathbf{x} \wedge u))$.
4. $(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z})u = \mathbf{x} \wedge \mathbf{y} \wedge (\mathbf{z} \rfloor u) - \mathbf{x} \wedge \mathbf{z} \wedge (\mathbf{y} \rfloor u) + \mathbf{y} \wedge \mathbf{z} \wedge (\mathbf{x} \rfloor u)$
   $+\mathbf{x} \wedge (\mathbf{y} \rfloor (\mathbf{z} \rfloor u)) - \mathbf{y} \wedge (\mathbf{x} \rfloor (\mathbf{z} \rfloor u)) + \mathbf{z} \wedge (\mathbf{x} \rfloor (\mathbf{y} \rfloor u))$
   $+\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} \wedge u + \mathbf{x} \rfloor (\mathbf{y} \rfloor (\mathbf{z} \rfloor u))$.
5. $\mathbf{a} \rfloor \mathbf{b} \in \bigwedge^{j-i} V$ for $\mathbf{a} \in \bigwedge^i V$, $\mathbf{b} \in \bigwedge^j V$ $(\operatorname{char}\mathbb{F} \neq 2)$.

In the next five exercises we have a non-degenerate Q. Define the right contraction $u \llcorner v$ by $<u \llcorner v, w> = <u, w \wedge \tilde{v}>$ for all $w \in \bigwedge V$ (we say that u is contracted by the contractor v). Show that

6. $u \rfloor v = v_0 \llcorner u_0 - v_0 \llcorner u_1 + v_1 \llcorner u_0 + v_1 \llcorner u_1 = v \llcorner u - 2v_0 \llcorner u_1$
   $(v_0 = \operatorname{even}(v),\ u_1 = \operatorname{odd}(u))$.
7. $u \llcorner v = v_0 \rfloor u_0 + v_0 \rfloor u_1 - v_1 \rfloor u_0 + v_1 \rfloor u_1 = v \rfloor u - 2v_1 \rfloor u_0$
   $(v_1 = \operatorname{odd}(v),\ u_0 = \operatorname{even}(u))$.

Show that (when $\operatorname{char}\mathbb{F} \neq 2$)

8. $\mathbf{a} \rfloor \mathbf{b} = (-1)^{i(j-i)} \mathbf{b} \llcorner \mathbf{a}$ for $\mathbf{a} \in \bigwedge^i V$, $\mathbf{b} \in \bigwedge^j V$.
9. $\mathbf{a} \in \bigwedge^k V$, $\mathbf{a} \neq 0$, $\mathbf{x} \in V$, $\mathbf{x} \rfloor \mathbf{a} = 0 \Leftrightarrow \mathbf{x} = \mathbf{a} \rfloor \mathbf{b}$ for some $\mathbf{b} \in \bigwedge^{k+1} V$.
10. $\mathbf{b} \in \bigwedge^k V$ is simple $\Leftrightarrow (\mathbf{a} \rfloor \mathbf{b}) \wedge \mathbf{b} = 0$ for all $\mathbf{a} \in \bigwedge^{k-1} V$.
11. $\mathbf{x}$ and $\mathbf{x} \wedge \mathbf{y}$ anticommute for vectors $\mathbf{x}, \mathbf{y} \in V$.
12. $\mathbf{x}$ and $\mathbf{x} \rfloor \mathbf{B}$ anticommute for a bivector $\mathbf{B} \in \bigwedge^2 V$.
13. $(\mathbf{x} \wedge \mathbf{y})^2 = (\mathbf{x} \rfloor \mathbf{y})^2 - \mathbf{x}^2 \mathbf{y}^2$ (Lagrange's identity).
14. $(\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z}) \rfloor u = (\mathbf{x} \wedge \mathbf{y}) \rfloor (\mathbf{z} \rfloor u) = \mathbf{x} \rfloor (\mathbf{y} \rfloor (\mathbf{z} \rfloor u))$.
15. $(\mathbf{xyz} - \mathbf{zyx})^2 \in \mathbb{F}$, $\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \frac{1}{2}(\mathbf{xyz} - \mathbf{zyx})$.
16. $\mathbf{a} \wedge \mathbf{b} = (-1)^{ij} \mathbf{b} \wedge \mathbf{a}$ for $\mathbf{a} \in \bigwedge^i V$ and $\mathbf{b} \in \bigwedge^j V$.
17. $u \wedge v = v_0 \wedge u_0 + v_0 \wedge u_1 + v_1 \wedge u_0 - v_1 \wedge u_1 = v \wedge u - 2v_1 \wedge u_1$ where
    $u, v \in \bigwedge V$, $u_0 = \operatorname{even}(u)$, $v_0 = \operatorname{even}(v)$, $u_1 = \operatorname{odd}(u)$, $v_1 = \operatorname{odd}(v)$.
18. $\mathbf{B}u = \mathbf{B} \wedge u + \frac{1}{2}(\mathbf{B}u - u\mathbf{B}) + \mathbf{B} \rfloor u$ for $\mathbf{B} \in \bigwedge^2 V$.
    [Hint: $(\mathbf{x} \wedge \mathbf{y}) \wedge u + (\mathbf{x} \wedge \mathbf{y}) \rfloor u = \mathbf{x} \wedge (\mathbf{y} \wedge u) + \mathbf{x} \rfloor (\mathbf{y} \rfloor u)$.]
19. $Q(u) = \langle \tilde{u} u \rangle_0$, $<u, v> = \langle \tilde{u} \rfloor v \rangle_0$ (= the scalar part of $\tilde{u} \rfloor v$).

In the last exercise we have a non-degenerate Q:

20. Q on V extends to a neutral or anisotropic Q on $\mathcal{C}\ell(Q)$.

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