12

Flags, Poles and Dipoles

The classification of spinors is commonly based on representation theory, irreducible representations of the Lorentz group $SO_+(1,3)$. Thus, one customarily speaks about Dirac, Majorana and Weyl spinors. In this chapter spinors are classified in a new way by their bilinear covariants, multivectors of observables. The new classification is geometric, since it is based on multivectors, and physical, since it is based on observables. The classification reveals new spinors, called *flag-dipole spinors*, which reside between the Weyl, Majorana and Dirac spinors.

Dirac spinors describe the electron, and for them $\Omega \neq 0$. Weyl and Majorana spinors describe the neutrino. Weyl spinors are eigenspinors of the helicity projection operators $\frac{1}{2}(1 \pm \gamma_{0123})$, and their bilinear covariants satisfy $\Omega = 0$, $\mathbf{S} = 0$, $\mathbf{K} \neq 0$. Majorana spinors are eigenspinors of the charge conjugation operator, [1] with eigenvalues ± 1, and their bilinear covariants satisfy $\Omega = 0$, $\mathbf{S} \neq 0$, $\mathbf{K} = 0$. [Weyl and Majorana spinors are usually introduced by properties of matrices, see Benn & Tucker 1987 and Crumeyrolle 1990.]

The flag-dipole spinors satisfy $\Omega = 0$ [and cannot be Dirac spinors] and $\mathbf{S} \neq 0$, $\mathbf{K} \neq 0$ [and so they are neither Weyl nor Majorana spinors]. Unlike Weyl and Majorana spinors, the flag-dipole spinors do not form a real linear subspace, because they are characterized by a quadratic constraint. Therefore the superposition principle is violated, and the flag-dipole spinors cannot describe fermions. It has been conjectured (G. Trayling, Windsor) that the flag-dipole spinors are related to the quark confinement.

1 The charge conjugation operator $\mathcal{C}$ is conventionally defined by $\mathcal{C}(\psi) = -i\gamma_2\psi^*$ for $\psi \in \mathbb{C}^4$.

12.1 Classification of spinors by their bilinear covariants

In the following we shall present a classification of spinors ψ based on properties of their bilinear covariants Ω_1, $\mathbf{J}$, $\mathbf{S}$, $\mathbf{K}$, Ω_2, collected as

$$Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}.$$

In other words, we classify the boomerangs $Z = 4\psi\tilde{\psi}^*$.

Recall that $Z = P + iQ$, $P = \Omega_1 + \mathbf{J} + \Omega_2\gamma_{0123}$, $Q = \mathbf{S} + \mathbf{K}\gamma_{0123}$.

Dirac spinors of the electron:

1. $\underline{\Omega_1 \neq 0,\ \Omega_2 \neq 0}$: Using $P^2 = 2\Omega_1 P = -Q^2$ we find the relationship $P = \pm(-\frac{1}{2}Q^2)/\sqrt{\langle -\frac{1}{2}Q^2\rangle_0}$ where the sign is given by $J^0 > 0$ (and coincides with the sign of Ω_1). $P = kQ$, where $k = -(\Omega_2 + \Omega_1\gamma_{0123})^{-1}\mathbf{K}$, $i\psi = k\psi$.

2. $\underline{\Omega_1 \neq 0,\ \Omega_2 = 0}$: P is a multiple of $\frac{1}{2\Omega_1}(\Omega_1 + \mathbf{J})$ which looks like a proper *energy projection operator* and which commutes with the *spin projection operator* $\frac{1}{2}(1 - i\mathbf{K}\gamma_{0123}/\Omega_1)$. $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} = (\Omega_1 + \mathbf{J})(1 - i\gamma_{0123}\mathbf{K}/\Omega_1)$, $\mathbf{S} = \gamma_{0123}\mathbf{J}\mathbf{K}/\Omega_1$. $P = \gamma_{0123}\frac{1}{\Omega_1}\mathbf{K}Q$, $k = \gamma_{0123}\mathbf{K}/\Omega_1$. In this class the Yvon-Takabayasi angle β gets only two values, 0 and π; and the charge superselection rule applies.

3. $\underline{\Omega_1 = 0,\ \Omega_2 \neq 0}$: Using $P^2 = 2\Omega_1 P$ we find that P is nilpotent: $P^2 = 0$. $Z = \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}$, $\mathbf{S} = -\mathbf{J}\mathbf{K}/\Omega_2$. $P = -\frac{1}{\Omega_2}\mathbf{K}Q = \pm\mathbf{K}Q/\sqrt{-\mathbf{K}^2}$ (opposite to the sign of Ω_2), $k = -\mathbf{K}/\Omega_2$.

Singular spinors with a light-like pole/current:

4. $\underline{\Omega_1 = \Omega_2 = 0,\ \mathbf{K} \neq 0,\ \mathbf{S} \neq 0}$: *Flag-dipole spinors*. $Z = \mathbf{J} + i\mathbf{J}\mathrm{s} - ih\gamma_{0123}\mathbf{J}$, $\mathbf{J}^2 = 0$, s is a space-like vector, $\mathrm{s}^2 < 0$, orthogonal to $\mathbf{J}$, $\mathbf{J}\cdot\mathrm{s} = 0$, $\mathbf{S} = \mathbf{J}\mathrm{s}$ $(= \mathbf{J}\wedge\mathrm{s})$, $\mathbf{K} = h\mathbf{J}$, $h^2 = 1 + \mathrm{s}^2 < 1$ (h real, $h \neq 0$). $P = \mathbf{J}$, $Q = \mathbf{J}(\mathrm{s} + h\gamma_{0123})$, $(1 + i\mathrm{s} + ih\gamma_{0123})Z = 0$. Note that $\frac{1}{2}(1 - i\mathrm{s} - ih\gamma_{0123})\psi = \psi$ and $(1 + i\mathrm{s} + ih\gamma_{0123})\psi = 0$. $\tilde{Z}^* = Z$ and $Z^2 = 0$ imply $Z = \mathbf{J}(1 + i\mathrm{s} + ih\gamma_{0123})$ etc. Let $\psi = \frac{1}{4N}Z\eta$, then $Z = 4\psi\tilde{\psi}^*$ implies $(\mathrm{s} + h\gamma_{0123})^2 = -1$. $P = (\mathrm{s} + h\gamma_{0123})Q$, $i\psi = (\mathrm{s} + h\gamma_{0123})\psi$.

5. $\underline{\Omega_1 = \Omega_2 = \mathbf{K} = 0,\ \mathbf{S} \neq 0}$: *Flag-pole spinors* for which $Z = \mathbf{J} + i\mathbf{J}\mathrm{s}$ is a pole $\mathbf{J}$ plus a flag $\mathbf{S} = \mathbf{J}\mathrm{s}$ $(= \mathbf{J}\wedge\mathrm{s})$, $\mathbf{J}\cdot\mathrm{s} = 0$, $\mathrm{s}^2 = -1$. $P = \mathrm{s}Q$, $i\psi = \mathrm{s}\psi$. The flag-pole spinors are eigenspinors of the charge conjugation operator with eigenvalues $\lambda \in U(1)$, thus $\mathcal{C}(\psi) = \lambda\psi$, $|\lambda| = 1$.
— Write $\mathbf{K}_k = \Psi\gamma_k\tilde{\Psi}$, $\mathbf{S}_k = \Psi\gamma_{ij}\tilde{\Psi}$ (ijk cycl.) with $\mathbf{K}_3 = \mathbf{K}$, $\mathbf{S}_3 = \mathbf{S}$. Then $\mathbf{K}_1 = \mathbf{J}$, $\mathbf{K}_2 = \mathbf{K}_3 = 0$ and $\mathbf{S}_1 = 0$, $\mathbf{S}_2 = \mathbf{J}\mathrm{s}_2$ $(= \mathbf{J}\wedge\mathrm{s}_2)$, $\mathbf{S}_3 = \mathbf{J}\mathrm{s}_3$ where $\mathrm{s}_3 = \mathrm{s}$, $\mathrm{s}_2^2 = -1$, $\mathrm{s}_2\cdot\mathrm{s}_3 = 0$.
— Given an arbitrary Dirac spinor ψ with covariants $\mathbf{J}$, $\mathbf{K}$ (and with $\mathbf{K}_1$, $\mathbf{S}_2$) we may construct, as special cases of flag-pole spinors, two *Majorana spinors*

$\psi_\pm = \frac{1}{2}(\psi \pm \psi_C)$, which are seen to be eigenspinors of the charge conjugation $\mathcal{C}(\psi_\pm) = \pm\psi_\pm$, and whose bilinear covariants $\mathbf{J}_\pm$, $\mathbf{S}_\pm$ satisfy $\mathbf{K}\cdot\mathbf{J}_\pm = 0$, $\mathbf{K}\cdot\mathbf{S}_\pm = -\Omega_2\mathbf{J}_\pm$ and $\mathbf{J} = \mathbf{J}_+ + \mathbf{J}_-$, $\mathbf{S} = \mathbf{S}_+ + \mathbf{S}_-$. [Note that $\mathbf{J}_\pm = \frac{1}{2}(\mathbf{J} \pm \mathbf{K}_1)$ and $\mathbf{S}_\pm = \frac{1}{2}(\mathbf{S} \mp \mathbf{S}_2\gamma_{0123})$.] The charge conjugations of $i\psi_\pm$ are $\mathcal{C}(i\psi_+) = -i\psi_+ \neq \pm\psi_-$ and $\mathcal{C}(i\psi_-) = i\psi_- \neq \pm\psi_+$.

6. $\underline{\Omega_1 = \Omega_2 = \mathbf{S} = 0,\ \mathbf{K} \neq 0}$: *Weyl spinors* (of massless neutrinos) are eigenspinors of the chirality operator $\gamma_{0123}\psi_\pm = \pm i\psi_\pm$. $Z = \mathbf{J} \mp i\gamma_{0123}\mathbf{J}$, $\mathbf{J} = \pm\mathbf{K}$, $h = \pm 1$, $\psi_\pm = \frac{1}{2}(1 \mp i\gamma_{0123})\psi_\pm$. Note that $\mathrm{even}(\psi_\pm) = \psi_\pm\frac{1}{2}(1 \mp \gamma_{03})$, $\mathrm{odd}(\psi_\pm) = \psi_\pm\frac{1}{2}(1 \pm \gamma_{03})$. $P = \pm\gamma_{0123}Q$.
— Write $\mathbf{K}_k = \Psi\gamma_k\tilde{\Psi}$, $\mathbf{S}_k = \Psi\gamma_{ij}\tilde{\Psi}$ as before. Then $\mathbf{K}_1 = \mathbf{K}_2 = 0$, $\mathbf{S}_1 = \mathbf{J}\mathbf{s}_1$ $(= \mathbf{J}\wedge\mathbf{s}_1)$, $\mathbf{S}_2 = \mathbf{J}\mathbf{s}_2$ $(= \mathbf{J}\wedge\mathbf{s}_2)$ where $\mathbf{s}_1^2 = \mathbf{s}_2^2 = -1$, $\mathbf{s}_1\cdot\mathbf{s}_2 = 0$.
— Given an arbitrary Dirac spinor ψ with covariants $\mathbf{J}, \mathbf{K}$ we may construct two Weyl spinors $\psi_\pm = \frac{1}{2}(1 \mp i\gamma_{0123})\psi$ with covariants $\mathbf{J}_\pm = \frac{1}{2}(\mathbf{J} \pm \mathbf{K})$, $\mathbf{K}_\pm = \frac{1}{2}(\mathbf{K} \pm \mathbf{J})$. Weyl spinors are *pure*: $\tilde{\psi}_\pm\gamma_{0123}\gamma_\mu\psi_\pm = 0$ [no complex conjugation; for arbitrary Dirac spinors $\tilde{\psi}\psi = 0$, $\tilde{\psi}\gamma_\mu\psi = 0$, $\tilde{\psi}\gamma_{0123}\psi = 0$ though $\tilde{\psi}\gamma_{\mu\nu}\psi \neq 0$, $\tilde{\psi}\gamma_{0123}\gamma_\mu\psi \neq 0$ (and also $\bar{\psi}\psi = 0$, $\bar{\psi}\gamma_{0123}\gamma_\mu\psi = 0$, $\bar{\psi}\gamma_{0123}\psi = 0$, though $\bar{\psi}\gamma_\mu\psi \neq 0$, $\bar{\psi}\gamma_{\mu\nu}\psi \neq 0$)]. $\mathcal{C}(\psi_\pm)$ is of helicity $h = \mp 1$ with covariants $\mathbf{J}_\pm$, $-\mathbf{K}_\pm$.

In addition to the above six classes *there are no other classes* based on distinctions between bilinear covariants. This can be seen by the following reasoning. First, we always have $\mathbf{J} \neq 0$, because $J^0 > 0$. Secondly, $\Omega \neq 0$ implies $\mathbf{S} \neq 0$ and $\mathbf{K} \neq 0$. Thirdly, $\Omega = 0$ implies $Z = \mathbf{J}(1 + i(\mathbf{s} + h\gamma_{0123}))$ where $(\mathbf{s} + h\gamma_{0123})^2 = -1$ so that we have a non-vanishing $\mathbf{J}(\mathbf{s} + h\gamma_{0123}) = \mathbf{S} + \mathbf{K}\gamma_{0123}$.

Comments:

For classes 1, 2 the element $\frac{1}{4\Omega_1}Z$ is a primitive idempotent in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$.

Classes 1, 2, 3 are *Dirac spinors for the electron*. A spinor operator Ψ has a unique (up to a sign) *polar decomposition* $\Psi = \sqrt{\Omega}u$, $u \in \mathbf{Spin}_+(1,3)$. In particular, writing $\mathbf{K}_k = \Psi\gamma_k\tilde{\Psi}$ we have an orthogonal basis $\{\mathbf{J}, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3\}$ $(\mathbf{K}_3 = \mathbf{K})$ of $\mathbb{R}^{1,3}$.

Class 4 consists of *flag-dipole spinors* with a flag $\mathbf{S}$ on a dipole of two poles $\mathbf{J}$ and $\mathbf{K}$. Class 5 consists of *flag-pole spinors* with a flag $\mathbf{S}$ on a pole $\mathbf{J}$. Class 6 consists of *dipole spinors* with two poles $\mathbf{J}$ and $\mathbf{K}$.

In classes 4, 5, 6 the vectors $\mathbf{J}$, $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$ no longer form a basis but collapse into a null-line $\mathbf{J}$ (also $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$ intersect along $\mathbf{J}$). The even elements $\mathbf{J}(\gamma_0 - \mathbf{s}\gamma_{12} - h\gamma_3)$ and Ψ differ only up to a complex factor $x - y\gamma_{12}$ (on the right).

In addition to the electron (classes 1, 2, 3) the massless neutrino (class 6) has also been discussed by Hestenes [1967 p. 808 (8.13) and 1986 p. 343] who quite correctly observed that $\mathbf{J}_\pm = \frac{1}{2}\Psi(\gamma_0 \pm \gamma_3)\tilde{\Psi}$; note also that $\tilde{\Psi}_\pm \gamma_\mu \Psi_\pm = (\gamma_\mu \cdot \mathbf{J}_\pm)(\gamma_0 \mp \gamma_3)$. Hestenes has not discussed classes 4 and 5. Holland in *Found. Phys.* 1986, pp. 708-709, does not discuss classes 3, 4, 5, 6 with a nilpotent Z, $Z^2 = 0$, but focuses on a nilpotent ψ, $\psi^2 = 0$.

Majorana spinors $\Psi \in \mathcal{C}\ell^+_{1,3}\,\frac{1}{2}(1 \mp \gamma_{01})$ are not stable under the $U(1)$-gauge transformation $\Psi \to \Psi e^{\alpha\gamma_{12}} \notin \mathcal{C}\ell^+_{1,3}\,\frac{1}{2}(1 \mp \gamma_{01})$.

Given a Weyl spinor ψ with bilinear covariants $\mathbf{J}, \mathbf{K}$ we can associate to it two Majorana spinors $\psi_\pm = \frac{1}{2}(\psi \pm \psi_C)$ with *Penrose flags* $Z_\pm = \frac{1}{2}(\mathbf{J} \mp i\mathbf{S}_2\gamma_{0123})$. [2]

The number of parameters in the sets of bilinear covariants (or spinors without $U(1)$-gauge) is seen to be

class no.	1	2	3	4	5	6
parameters	7	6	6	5	4	3

If the $U(1)$-gauge is taken into consideration, then the number of parameters will be raised by one unit in all classes except in class 5 of Majorana spinors [Weyl spinors with $U(1)$-gauge and Majorana spinors both have four parameters and can be mapped bijectively onto each other – which enables Penrose flags also to be attached to Weyl spinors].

The Weyl and Majorana spinors can be written with spinor operators in the form

$$\Psi\frac{1}{2}(1 + \gamma_0\mathbf{u}) \qquad [\Psi \in \mathcal{C}\ell^+_{1,3}]$$

where $\mathbf{u} = \pm\gamma_3$ for Weyl spinors and $\mathbf{u} = \pm\gamma_1$ for Majorana spinors. The flag-pole spinors can be written in a similar form with $\mathbf{u} = \gamma_1\cos\phi + \gamma_2\sin\phi$. It is easy to see that all elements of the form $\Psi\frac{1}{2}(1 + \gamma_0\mathbf{u})$, $\Psi \in \mathcal{C}\ell^+_{1,3}$ are flag-dipole spinors, when $\mathbf{u}$ is a spatial unit vector, $\mathbf{u}\cdot\gamma_0 = 0$, $\mathbf{u}^2 = -1$, which is not on the γ_3-axis or in the $\gamma_1\gamma_2$-plane. About the converse the following has been presented:

Conjecture (C. Doran, 1995): All the flag-dipole spinors can be written in the form $\Psi\frac{1}{2}(1 + \gamma_0\mathbf{u})$, where $\Psi \in \mathcal{C}\ell^+_{1,3}$, $\mathbf{u} \in \mathbb{R}^3$, $\mathbf{u}^2 = -1$. ∎

When $\mathbf{u}$ varies in the unit sphere S^2 in $\mathbb{R}^3$ (orthogonal to γ_0), the flag-dipole spinor sweeps around the 'paraboloid' $\Psi\tilde{\Psi} = 0$. If the conjecture is true, it would be nice to know the relation between $\mathbf{s}, h$ and $\mathbf{u}$. [Clearly, $h = \mathbf{u}\cdot\gamma_3$.]

2 Our flag-pole $Z = \mathbf{J} + i\mathbf{S}$ is invariant under rotations $\Psi \to \Psi e^{\alpha\gamma_{12}}$, whereas the Penrose flags $Z_\pm = \frac{1}{2}(\mathbf{J} \mp i\mathbf{S}_2\gamma_{0123})$ make a 720° turn under a rotation of 360°.

12.2 Projection operators in $\mathrm{End}(\mathcal{C}\ell_{1,3})$

Write as before $P = \Omega_1 + \mathbf{J} + \Omega_2\gamma_{0123}$, $Q = \mathbf{S} + \mathbf{K}\gamma_{0123}$, $Z = P + iQ = P\Sigma = \Sigma P$, $\Sigma = 1 - i\gamma_{0123}\mathbf{J}\mathbf{K}^{-1}$. Then

$$\frac{1}{4\Omega_1}Z\psi = \psi, \qquad \frac{1}{2\Omega_1}P\psi = \psi, \qquad \text{when } \Omega_1 \neq 0$$
$$\frac{1}{2}\Sigma\psi = \psi, \qquad \text{when } \Omega \neq 0.$$

Define for $u \in \mathcal{C}\ell_{1,3}$ (or $u \in \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$)

$$P_\pm(u) = \frac{1}{2\Omega_1}(\Omega_1 \pm \mathbf{J} \pm \Omega_2\gamma_{0123})u, \qquad \Omega_1 \neq 0$$
$$\Sigma_\pm(u) = \frac{1}{2}(u \pm \gamma_{0123}\mathbf{J}\mathbf{K}^{-1}u\gamma_{12}), \qquad \Sigma_\pm \in \mathrm{End}(\mathcal{C}\ell_{1,3}).$$

Then

$$\begin{array}{llll} P_+(\psi) = \psi & P_+(\Phi) = \Phi & P_+(\phi) = \phi & \\ P_-(\psi) = 0 & P_-(\Phi) = 0 & P_-(\phi) = 0 & \\ \Sigma_+(\psi) = \psi & \Sigma_+(\Phi) = \Phi & \Sigma_+(\phi) = \phi & \Sigma_+(\Psi) = \Psi \\ \Sigma_-(\psi) = 0 & \Sigma_-(\Phi) = 0 & \Sigma_-(\phi) = 0 & \Sigma_-(\Psi) = 0. \end{array}$$

In general, for $u \in \mathcal{C}\ell_{1,3}$, $P_\pm^2(u) = P_\pm(u)$, $\Sigma_\pm^2(u) = \Sigma_\pm(u)$ and $P_\pm(\Sigma_\pm(u)) = \Sigma_\pm(P_\pm(u))$, that is, $P_\pm$ and $\Sigma_\pm$ are commuting projection operators. For an arbitrary η in $\mathcal{C}\ell_{1,3}\,\frac{1}{2}(1+\gamma_0)$ [or in $\mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03})$] the spinor $P_+(\Sigma_+(\eta))$ is parallel to Φ [or to ϕ], that is, the bilinear covariants of $P_+(\Sigma_+(\eta))$ are proportional to P, Q. However, for an arbitrary $u \in \mathcal{C}\ell_{1,3}$, $P_+(\Sigma_+(u)) \notin \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1+\gamma_0)$ [or $P_+(\Sigma_+(u)) \notin \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03})$].

Define

$$\Sigma_\pm^I(u) = \frac{1}{2}(u \mp \gamma_{0123}\mathbf{J}\mathbf{K}^{-1}u\gamma_{0123})$$

where I stands for ideal spinor. Then for an arbitrary $u \in \mathcal{C}\ell_{1,3}$ we have $\Sigma_+(\Sigma_+^I(u)) \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03})$, and $P_+(\Sigma_+(\Sigma_+^I(u)))$ is an ideal spinor parallel to ϕ (with bilinear covariants proportional to P, Q). Furthermore, $\Sigma_+^I(\phi) = \phi$, $\Sigma_-^I(\phi) = 0$, and $\Sigma_\pm^I$ are projection operators commuting with $P_\pm$, $\Sigma_\pm$.

Define (O stands for spinor operator)

$$P_\pm^O(u) = \frac{1}{2\Omega_1}((\Omega_1 \pm \Omega_2\gamma_{0123})u \pm \mathbf{J}u\gamma_0), \qquad u \in \mathcal{C}\ell_{1,3}^+,$$

which are projection operators commuting with $\Sigma_\pm$.

Exercise. (Inspired by Crawford 1985) Define

$$\Gamma_\pm(u) = \frac{1}{2}(u \mp (\Omega_2 + \Omega_1\gamma_{0123})^{-1}\mathbf{K}u\gamma_{12})$$

and show that $\Gamma_\pm$ are projection operators commuting with $\Sigma_\pm$ [but not with $P_\pm$ unless $\Omega_2 = 0$; recall here the factorization of Crawford]. Show that $P_+(\Gamma_+(\Phi)) = \Phi$, $P_+(\Gamma_+(\phi)) = \phi$. How would you define $\Gamma_\pm$ for a spinor operator Ψ?
[Answer: $\Gamma^O_\pm(u) = \frac{1}{2}(u \mp (\Omega_2 + \Omega_1\gamma_{0123})^{-1}\mathbf{K}u\gamma_{012})$ for $u \in \mathcal{C}\ell^+_{1,3}$.] ∎

Remark. Define $\Gamma_\pm$ for an ideal spinor ϕ (I stands for ideal):

$$\Gamma^I_\pm(u) = \frac{1}{2}(u \pm (\Omega_2 + \Omega_1\gamma_{0123})^{-1}\mathbf{K}u\gamma_{0123}), \qquad u \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03}).$$

In the special case $\Omega_2 = 0$ of type 2 these take the form

$$\Gamma^I_\pm(u) = \frac{1}{2}(u \mp \gamma_{0123}\frac{1}{\Omega_1}\mathbf{K}u\gamma_{0123}), \qquad u \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03}),$$

and commute with $P_\pm$ [this special case was also observed by Hestenes 1986 p. 336 (2.32)]. ∎

12.3 Projection operators for Majorana and Weyl spinors

Treat first the general case (class 4) $\Omega_1 = 0 = \Omega_2$, $\mathbf{K} \neq 0 \neq \mathbf{S}$. Recall that $(1 + is + ih\gamma_{0123})\psi = 0$ or $i\psi = (s + h\gamma_{0123})\psi$. Define

$$\Sigma^G_\pm(u) = \frac{1}{2}(u \pm (s + h\gamma_{0123})u\gamma_{12}).$$

Then $\Sigma^G_+(\Phi) = \Phi$, $\Sigma^G_+(\phi) = \phi$. Majorana and Weyl spinors are now the limiting cases

$$\Sigma^M_\pm(u) = \frac{1}{2}(u \pm su\gamma_{12}), \qquad \Sigma^W_\pm(u) = \frac{1}{2}(u \pm \gamma_{0123}u\gamma_{12}).$$

Exercises 1,2,3,4,5

12.4 Charge conjugate $\psi_C = \mathcal{C}(\psi)$

The charge conjugate spinor $\psi_C = -i\gamma_2\psi^*$ sits in $\mathbb{C}^4$ [3] or in the same minimal left ideal $\mathrm{Mat}(4,\mathbb{C})f$; it satisfies

$$\gamma^\mu(i\partial_\mu + eA_\mu)\psi_C = m\psi_C.$$

Charge conjugation is an anti-linear operation, that is, $\mathcal{C}(i\psi) = -i\mathcal{C}(\psi)$. Other characteristics of charge conjugation are $\psi_C^\dagger\gamma_0\psi_C = -\psi^\dagger\gamma_0\psi$ and $\psi_C^\dagger\gamma_0\gamma_\mu\psi_C = +\psi^\dagger\gamma_0\gamma_\mu\psi$.

In the notation of $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ we have $\psi_C = -i\gamma_2\gamma_{013}\psi^*\gamma_{013}^{-1} = \hat{\psi}^*\gamma_1$, which also sits in the minimal left ideal $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$, while γ_1 swaps the signs of both factors of the primitive idempotent $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_1\gamma_2)$. The bilinear covariants are transformed as follows under the charge conjugation

$$\begin{array}{lll}
\langle\tilde{\psi}_C^*\psi_C\rangle_0 & = -\langle\tilde{\psi}^*\psi\rangle_0 & \text{(as above)} \\
\langle\tilde{\psi}_C^*\gamma_\mu\psi_C\rangle_0 & = +\langle\tilde{\psi}^*\gamma_\mu\psi\rangle_0 & \text{(as above)} \\
\langle\tilde{\psi}_C^* i\gamma_{\mu\nu}\psi_C\rangle_0 & = +\langle\tilde{\psi}^* i\gamma_{\mu\nu}\psi\rangle_0 & \\
\langle\tilde{\psi}_C^* i\gamma_{0123}\gamma_\mu\psi_C\rangle_0 & = -\langle\tilde{\psi}^* i\gamma_{0123}\gamma_\mu\psi\rangle_0 & \\
\langle\tilde{\psi}_C^*\gamma_{0123}\psi_C\rangle_0 & = -\langle\tilde{\psi}^*\gamma_{0123}\psi\rangle_0 &
\end{array}$$

and $4\psi_C\tilde{\psi}_C^* = -\hat{Z}^* = -\hat{P} + i\hat{Q} = -\Omega_1 + \mathbf{J} + i\mathbf{S} + i\gamma_{0123}\mathbf{K} - \gamma_{0123}\Omega_2$, since $4\psi_C\tilde{\psi}_C^* = 4\hat{\psi}^*\gamma_1(\hat{\psi}^*\gamma_1)^{\tilde{}*} = 4\hat{\psi}^*\gamma_1\gamma_1\bar{\psi} = -4(\psi\tilde{\psi}^*)^{\hat{}*}$

The charge conjugate of the mother spinor is

$$\Phi_C = \hat{\Phi}\gamma_1 = 4\,\mathrm{Re}(\hat{\psi}^*\gamma_1);$$

it satisfies $\Phi_C \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1+\gamma_0)$, and has the same properties as were listed above for the charge conjugation. The charge conjugate of the ideal spinor is $\phi_C = \hat{\phi}\gamma_1 = \hat{\Phi}\gamma_1\frac{1}{2}(1-\gamma_{03})$. Its bilinear covariants are (as above)

$$\begin{array}{ll}
\phi_C\gamma_0\tilde{\phi}_C = -\hat{P}, & \phi_C\gamma_3\bar{\phi}_C = -\hat{Q}\gamma_{0123} \\
\phi_C\gamma_3\tilde{\phi}_C = -P, & \phi_C\gamma_0\bar{\phi}_C = -\hat{Q}\gamma_{0123}.
\end{array}$$

The charge conjugate of the spinor operator is $\Psi_C = \Psi\gamma_1\gamma_0 = \mathrm{even}(\hat{\Phi}\gamma_1)$, where $\hat{\Phi}\gamma_1 = (\Phi_0 - \Phi_0\gamma_0)\gamma_1$.

Exercise. Show that the operator form of a Majorana spinor $\frac{1}{2}(\psi \pm \psi_C)$ is

3 In this case the complex conjugate is

$$\psi^* = \begin{pmatrix}\psi_1^*\\ \psi_2^*\\ \psi_3^*\\ \psi_4^*\end{pmatrix} \quad \text{for} \quad \psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}.$$

$\Psi\frac{1}{2}(1 \mp \gamma_{01}) \in \mathcal{C}\ell^+_{1,3}\,\frac{1}{2}(1 \mp \gamma_{01})$. ∎

The Wigner time-reversal is $\psi_{\mathcal{T}} = -i\gamma_{13}\psi^* \in \text{Mat}(4,\mathbb{C})f$ or $\psi_{\mathcal{T}} = \gamma_{123}\hat{\psi}^*\gamma_1 \in (\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$ and the parity involution of space is $\psi_{\mathcal{P}} = \gamma_0\psi$. So

$\psi_{\mathcal{T}} = \gamma_{123}\hat{\psi}^*\gamma_1$	$\psi_{\mathcal{P}} = \gamma_0\psi$	$\psi_{\mathcal{C}} = \hat{\psi}^*\gamma_1$	Dirac
$\Phi_{\mathcal{T}} = \gamma_{123}\hat{\Phi}\gamma_1$	$\Phi_{\mathcal{P}} = \gamma_0\Phi$	$\Phi_{\mathcal{C}} = \hat{\Phi}\gamma_1$	mother
$\phi_{\mathcal{T}} = \gamma_{123}\hat{\phi}\gamma_1$	$\phi_{\mathcal{P}} = \gamma_0\phi$	$\phi_{\mathcal{C}} = \hat{\phi}\gamma_1$	ideal
$\Psi_{\mathcal{T}} = \gamma_{123}\Psi\gamma_1$	$\Psi_{\mathcal{P}} = \gamma_0\Psi\gamma_0^{-1}$	$\Psi_{\mathcal{C}} = \Psi\gamma_1\gamma_0$	operator

and $\psi_{\mathcal{TPC}} = \gamma_{0123}\psi$. Note that charge conjugation $\mathcal{C}$ anticommutes with both parity involution $\mathcal{P}$ and time-reversal $\mathcal{T}$.

Exercises 6,7,8,9,10

Appendix: Crumeyrolle's spinoriality transformation

Crumeyrolle introduced a number of *spinoriality groups* to be able to treat the complicated situations with spinors. However, one relevant problem remained unsolved: how can the usual bilinear covariants be obtained from Crumeyrolle's spinors? The bilinear covariants of Crumeyrolle's spinors mix the Dirac current vector $\mathbf{J}$ and the electromagnetic moment bivector $\mathbf{S}$. A solution to this problem can be given by a variation of Crumeyrolle's spinoriality group. In this appendix it is shown how to extract the standard bilinear covariants (see the standard textbooks on quantum mechanics, like Bjorken & Drell 1964) from Crumeyrolle's or Cartan's pure spinors.

Crumeyrolle considered the complexification $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ of the Clifford algebra $\mathcal{C}\ell_{1,3}$ of the Minkowski space $\mathbb{R}^{1,3}$. In the complex linear space $\mathbb{C}\otimes\mathbb{R}^{1,3}$ Crumeyrolle picked up a maximal totally null subspace spanned by the orthogonal null vectors

$$\frac{1}{2}(\gamma_0 - \gamma_3) \quad \text{and} \quad \frac{1}{2}(\gamma_1 - i\gamma_2).$$

Denote the product of these vectors by

$$\mathbf{v} = \frac{1}{2}(\gamma_0 - \gamma_3)\frac{1}{2}(\gamma_1 - i\gamma_2)$$

which is the volume element of the totally null subspace. Crumeyrolle chose as his spinor space the minimal left ideal $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})\mathbf{v}$ of the complex Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$. The difficulty with this choice is that the bilinear covariants of such a spinor are not directly related to those of a column spinor in standard textbooks on quantum mechanics. [4] To overcome this difficulty, first note that the element

$$g = \gamma_{01}\mathbf{v} = \frac{1}{2}(1 - \gamma_{03})\frac{1}{2}(1 + i\gamma_{12})$$

is a primitive idempotent generating Crumeyrolle's spinor space, that is,

$$(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})\mathbf{v} = (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})g.$$

Unfortunately, $g = \frac{1}{2}(1 - \gamma_{03})\frac{1}{2}(1 + i\gamma_{12})$ is an even element and does not contain as a factor the 'energy projection' operator $\frac{1}{2}(1 + \gamma_0)$. The physical observables are obtained from column spinors sitting in $\mathbb{C}^4$ which are related to Clifford algebraic spinors sitting in $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$, where

$$f = \frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_{12}).$$

In order to move from the spinor space $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})g$ to the spinor space $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$ we must find a transformation law for spinors $\psi_g \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})g$ and $\psi_f \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$. [5] This transformation law is a slight variation of Crumeyrolle's spinoriality transformation. [6]

Before giving our variation of the spinoriality transformation let us recall that ψ_g is a sum of two Weyl spinors

$$\frac{1}{2}(1 + i\gamma_{0123})\psi_g \in \mathcal{C}\ell_{1,3}^{-} \otimes \mathbb{C}$$
$$\frac{1}{2}(1 - i\gamma_{0123})\psi_g \in \mathcal{C}\ell_{1,3}^{+} \otimes \mathbb{C}$$

so that the components are of homogeneous parity [the correspondence between the even/odd parts and the negative/positive helicities is irrelevant, since it could be swapped by a different choice of g, for instance, by $g = \frac{1}{2}(1+\gamma_{03})\frac{1}{2}(1+i\gamma_{12})$ for which $g = \gamma_{01}\mathbf{v}$ with $\mathbf{v} = \frac{1}{2}(\gamma_0 + \gamma_3)\frac{1}{2}(\gamma_1 - i\gamma_2)$].

Our variation of the spinoriality transformation is carried out by the element

$$z = \frac{1}{\sqrt{2}}(1 + \gamma_3)$$

4 The bilinear covariants of Crumeyrolle's spinors either vanish identically or else, as in Crumeyrolle 1990 p. 229 formula 24, mix the Dirac current $\mathbf{J}$ and the electromagnetic moment $\mathbf{S}$.

5 Recall that, for instance, $J_\mu = 4\langle\tilde{\psi}_f^*\gamma_\mu\psi_f\rangle_0$. In contrast, these bilinear covariants of ψ_g vanish: $4\langle\tilde{\psi}_g^*\gamma_\mu\psi_g\rangle_0 = 0$. However, ψ_g does carry all the information of $\mathbf{J}$: $J_\mu = 4\langle\tilde{\psi}_g^*\gamma_\mu\psi_g\rangle_1 \cdot \gamma_0$.

6 For Crumeyrolle the spinoriality group meant a number of things, with different adjectives added as specification. First, it is the subgroup of those $s \in \mathbf{Spin}(1,3)$ for which $s\mathbf{v} = \pm\mathbf{v}$, see Crumeyrolle 1990 p. 145. Secondly, it is the group of invertible elements z in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ such that the primitive idempotents g and zgz^{-1} determine the same minimal left ideal $(\mathbb{C} \otimes \mathcal{C}\ell_{1,3})g = (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})g'$, see p. 277. Thirdly, it is, if normalized, the intersection of the previous group with $\mathbf{Spin}(1,3)$, see p. 281.

for which $g = zfz^{-1}$ or $f = z^{-1}gz$. The latter rule gives us a relation between Crumeyrolle's nilpotent induced spinors ψ_g and idempotent induced spinors ψ_f (directly related to the standard column spinors like those in Bjorken & Drell 1964),

$$\psi_f = \psi_g z.$$

(Earlier we wrote $\psi = \psi_f$ but here it is necessary to indicate to which minimal left ideal the spinor belongs.)

Now we can compute the spinor operator $\Psi = \mathrm{even}(4\,\mathrm{Re}(\psi_f))$ and the bilinear covariants, for instance, $\mathbf{J} = \Psi\gamma_0\tilde{\Psi}$. For later convenience note that $\Psi = \mathrm{Oper}(\psi_f)$ where

$$\mathrm{Oper}(\psi_f) = \frac{1}{2}(\mathrm{even}(4\,\mathrm{Re}(\psi_f)) + \mathrm{odd}(4\,\mathrm{Re}(\psi_f))\gamma_0).$$

Recall the aggregate of bilinear covariants

$$Z = 4\psi_f\tilde{\psi}_f^*$$

and note that $4\psi_f\tilde{\psi}_f^* = 4\psi_f\gamma_0\tilde{\psi}_f^*$. Our variation of the spinoriality group is the group of those elements z in $\mathcal{C}\ell_{1,3}$ or $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ which preserve the aggregate Z under the transformation $\psi_f \to \psi_f z^{-1}$. Crumeyrolle's spinoriality groups preserve the ideals whereas our spinoriality groups preserve the physical observables. The spinoriality groups are seen to be the following (see Lounesto 1981 p. 733):

	$\mathcal{C}\ell_{1,3}$	$\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$
$Z = 4\psi_f\tilde{\psi}_f^*$	$Sp(2,2)$	$U(2,2)$
$Z = 4\psi_f\gamma_0\tilde{\psi}_f^*$	$Sp(4)$	$U(4)$

where, as an example, the Lie algebra of $Sp(4) \simeq \mathbf{Spin}(5)$ is spanned by the elements

$$\begin{array}{l} \gamma_1,\ \gamma_2,\ \gamma_3 \\ \gamma_{12},\ \gamma_{13},\ \gamma_{23} \\ \gamma_{012},\ \gamma_{013},\ \gamma_{023} \\ \gamma_{0123}. \end{array}$$

For those z in $\mathcal{C}\ell_{1,3}$ which preserve $Z = 4\psi_f\gamma_0\tilde{\psi}_f^*$, under the replacement $\psi_f \to \psi_f z^{-1}$, that is $z \in Sp(4)$, we may find that the spinor operator is preserved under the following transformations:

$$\Psi = \frac{1}{2\langle w\rangle_0}(\mathrm{even}(4\,\mathrm{Re}(\psi_f w^{-1})) + \mathrm{odd}(4\,\mathrm{Re}(\psi_f w^{-1}))\gamma_0)$$

where $w = z\tilde{z}$, that is,

$$\Psi = \frac{1}{\langle w\rangle_0}\mathrm{Oper}(\psi_f w^{-1}).$$

To put all this in a nutshell: our variation of

spinoriality transformation preserves bilinear covariants

However, this preservation should be distinguished from our use of the particular element $z = \frac{1}{\sqrt{2}}(1+\gamma_3) \in Sp(4)$ to retrieve the aggregate of bilinear covariants $Z = 4\psi_f\tilde{\psi}_f^*$ by sending ψ_g to $\psi_g z = \psi_f$.

Exercises

1. Recall that $\Psi\gamma_2\gamma_1 = \mathrm{s}\Psi\gamma_0 + h\gamma_{0123}\Psi$. How would you define $\Sigma_\pm^G$ for a spinor operator Ψ?
2. Recall that $\phi\gamma_{0123} = \phi\gamma_2\gamma_1$. How would you define another pair $\Sigma_\pm^G$ for an ideal spinor ϕ?
3. Show that up to a unit complex factor $e^{\gamma_{12}\alpha}$:
   $\Psi \simeq \frac{1}{4N}(\Omega_1 + \mathbf{J}\gamma_0 - \mathbf{S}\gamma_{12} - \mathbf{K}\gamma_3 + \Omega_2\gamma_{0123})$, when $N = \sqrt{\langle Zf\rangle_0} \neq 0$.
4. Show that the operator form of a Weyl spinor is $\Psi\frac{1}{2}(1 \mp \gamma_{03})$.
5. Show that Weyl spinors $\frac{1}{2}(1 \mp i\gamma_{0123})\psi$ correspond to even and odd parts of the ideal spinor $\phi = \phi_0 + \phi_1$.

Write $W = 4\psi\tilde{\psi}_C^*$, [7] note that $\psi\bar{\psi}_C^* = 0$, and show that

6. $W = -(Q_1 + iQ_2)\gamma_{0123}$ where $Q_k = \frac{1}{2}(\Phi\gamma_k\bar{\Phi})\gamma_{0123}$ or $Q_k = \Psi(1+\gamma_0)\gamma_{ij}\tilde{\Psi}$ (ijk cycl.).
7. $W = \mathcal{K} - \mathcal{S}\gamma_{0123}$ where $\mathcal{K} = \mathbf{K}_1 + i\mathbf{K}_2$ and $\mathcal{S} = \mathbf{S}_1 + i\mathbf{S}_2$, where as before $\mathbf{K}_k = \Psi\gamma_k\tilde{\Psi}$, $\mathbf{S}_k = \Psi\gamma_{ij}\tilde{\Psi}$ (ijk cycl.).
8. $W^2 = 0$.
9. $WZ = 0$.
10. $ZW = 4\Omega_1 W$ and so the 3-vector part vanishes:
    $\langle ZW\rangle_3 = -\mathbf{J}\wedge(\gamma_{0123}\mathcal{S}) + i\mathbf{S}\wedge\mathcal{K} + \gamma_{0123}\Omega_2\mathcal{K} + i\mathbf{K}\wedge\mathcal{S} = 0.$
11. Show that $4\psi_g\tilde{\psi}_g^* = 0$.
12. Write $\Psi_f = \mathrm{Oper}(\psi_f)$ and $\Psi_g = \mathrm{Oper}(\psi_g)$. Show that
    $\Psi_f\gamma_0\tilde{\Psi}_f = 2\Psi_g\gamma_0\tilde{\Psi}_g$.
13. Write $\Psi_f = \mathrm{even}(4\,\mathrm{Re}(\psi_f)) = \mathrm{Oper}(\psi_f)$ and
    $\Psi_g = \mathrm{even}(4\,\mathrm{Re}(\psi_g)) \neq \mathrm{Oper}(\psi_g)$. Show that $\Psi_g\tilde{\Psi}_g = 0$, $\Psi_g\gamma_{12}\tilde{\Psi}_g = 0$ and $\Psi_g\gamma_0\tilde{\Psi}_g = \mathbf{J} + \mathbf{K}$ where $\mathbf{J} = \Psi_f\gamma_0\tilde{\Psi}_f$ and $\mathbf{K} = \Psi_f\gamma_3\tilde{\Psi}_f$.

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[7] $\tilde{\psi}_C^* = (\psi_C)^{\tilde{}*} \neq (\tilde{\psi}^*)_C$.

Solutions

1. $\Sigma_{\pm}^{GO}(u) = \frac{1}{2}(u \pm \mathrm{s}u\gamma_{012} \pm h\gamma_{0123}u\gamma_{12})$ for $u \in \mathcal{C}\ell_{1,3}^{+}$.
2. $\Sigma_{\pm}^{GI}(u) = \frac{1}{2}(u \mp (\mathrm{s} + h\gamma_{0123})u\gamma_{0123})$, $u \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1 - \gamma_{03})$.
4. Hint: compute the even part of $4\,\mathrm{Re}(\frac{1}{2}(1 \mp i\gamma_{0123})\psi)$ in the decomposition $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$.

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