The Lagrangian Setting for the Optimization Prob.

Our optimization problem reads

$$
(P)\quad \begin{cases} \text{Find } \displaystyle\min_{\xi \in \dot{H}^1_\#} \int_J G\dot{\xi}(t)\cdot\dot{\xi}(t)\,dt \\ \text{under the constraints} \\ \qquad \displaystyle\int_J M_i\,\xi(t)\cdot\dot{\xi}(t)\,dt = \delta p_i\,, \quad i \in \mathbb{N}_6, \end{cases}
$$

where $\delta p \in \mathbb{R}^6 \simeq \overset{2}{\Lambda}\mathbb{R}^4 \simeq \mathbb{R}^3 \times so(3)$ is a prescribed net displacement. So we are in the setting of a variational problem with six isoperimetric constraints. We already know that (P) admits a solution. Now we want to characterize it in terms of a differential equation, the Euler-Lagrange equation.

Denote by $K_i : \dot{H}^1_\# \times L^2_\# \times J \longrightarrow \mathbb{R}$ the map $K_i(\xi, \eta, t) := M_i\,\xi(t)\cdot\eta(t)$ for $i \in \mathbb{N}_6$.
Furthermore, denote by $\mathcal{K}_i : \dot{H}^1_\# \longrightarrow \mathbb{R}$ the functional $\mathcal{K}_i(\xi) = \int_J K_i(\xi, \dot{\xi}, t)\,dt$. Finally, we denote by $\delta\mathcal{G}$ and $\delta\mathcal{K}_i$ the first variations of $\mathcal{G}$ and $\mathcal{K}_i$, respectively.

Then a slight adaption of Prop. 2.1.3 in [Kielhöfer] shows that if $\xi \in \dot{H}^1_{\#}$ is a minimizer of (P) and if ξ is not critical for the constraints, i.e. $\delta\mathcal{K}_1(\xi), \dots, \delta\mathcal{K}_6(\xi)$ are linearly independent, then ξ satisfies the Euler-Lagrange equation:

$$\frac{d}{dt}\left(G\dot{\xi}(t)\cdot\dot{\xi}(t) + \sum_{i=1}^{6}\mu_i K_i\right)_q = \left(G\dot{\xi}(t)\cdot\dot{\xi}(t) + \sum_{i=1}^{6}\mu_i K_i\right)_p,$$

for some $\mu \in \mathbb{R}^6$.

Let now δ_p be a net displacement which identifies to a non-simple bivector. Then we have

<u>Prop.</u> Let ξ be a minimizer of (P) with δ_p as above. Then the functionals $\delta\mathcal{K}_1(\xi), \dots, \delta\mathcal{K}_6(\xi)$ are linearly independent.

Proof. Assume that $\lambda_1, \dots, \lambda_6 \in \mathbb{R}$ are such that $\sum_{i=1}^{6}\lambda_i\,\delta\mathcal{K}_i(\xi) \in (\dot{H}^1_{\#})^*$ is the zero functional.

Note that

$$\delta \mathcal{K}_i(\xi)h = \int_J K_{i,\xi}(\xi,\dot{\xi},t)\,h + K_{i,\dot{\xi}}(\xi,\dot{\xi},t)\,\dot{h}\,dt$$

$$= \int_J \xi^T M_i \dot{h} - \dot{\xi}^T M_i h\,dt$$

$$\overset{\text{I.B.P.}}{=} -2\int_J \dot{\xi}^T M_i h\,dt.$$

- Setting $\Omega(\underline{\lambda}) := \sum_{i=1}^{6} \lambda_i M_i$, for $\underline{\lambda} \in \mathbb{R}^6$

yields

$$\sum_{i=1}^{6} \lambda_i \,\delta \mathcal{K}_i(\xi)h = -2\int_J \Omega(\underline{\lambda})\dot{\xi}\cdot h\,dt$$

and thus $\sum_{i=1}^{6} \lambda_i \delta\mathcal{K}_i(\xi) \equiv 0$ is equivalent to $h \longmapsto \int_J \Omega(\underline{\lambda})\dot{\xi}\cdot h \equiv 0$.

- Choosing $h = \Omega(\underline{\lambda})\dot{\xi}^{(*)}$, we get (*) Not necessarily H^1 ....

$$0 = \int_J \Omega(\underline{\lambda})\dot{\xi}\cdot h\,dt = \int_J |h|^2\,dt,$$

which implies that $h \equiv 0$ a.e.

This can only happen in four cases:

i) $\lambda_1 = \dots = \lambda_6 = 0$, then we are done

ii) $\dot{\xi}(t) \in \bigcap_{i\in\mathbb{N}_6} \ker M_i = \{0\}$ $\forall t \Rightarrow \dot{\xi} \equiv 0$, this is excluded since $\gamma_p \neq 0$.

iii) $\dot{\xi} \equiv 0$, which is excluded for the same reason as ii)

iv) Note that $\Omega(\underline{\lambda})$ is skew-symmetric. Therefore its complex eigenvalues are two pairs $\pm\lambda_{\pm} i$ of imaginary eigenvalues. In particular, one finds $S \in \mathcal{O}(4)$ such that $\Omega(\underline{\lambda}) = S\,\Sigma(\underline{\lambda})\,S^T$

Where $$\Sigma(\underline{\lambda}) = \begin{pmatrix} 0 & \lambda_+ & & \\ -\lambda_+ & 0 & & 0 \\ & & 0 & \lambda_- \\ 0 & & -\lambda_- & 0 \end{pmatrix}.$$

~~Th~~ Denoting by P and Q the orthogonal projections on the first and the last three coordinates of $\mathbb{R}^6$, respectively, we find

$$\lambda_{\pm} = 2\sqrt{3}\sqrt{A \pm \sqrt{A^2 - K}},$$

with $$A := \alpha^2 |P\underline{\lambda}|^2 + \delta^2 |Q\underline{\lambda}|^2$$

$$K := 4\alpha^2\delta^2 \, |P\underline{\lambda}\cdot Q\underline{\lambda}|^2.$$

So we note that $h \equiv 0$ can happen for $P\underline{\lambda}\cdot Q\underline{\lambda} = 0$. But then ξ and therefore ξ must lie in a subspace of dimension 2 of $\mathbb{R}^4$. This is excluded by the

Lemma below since δp was assumed to be non-simple. $\square$

<u>Lemma.</u> Let $\xi \in \dot{H}^1_\#$ be a control curve. Suppose that $\xi(t) \in D$ $\forall t \in I$, where $D \subset \mathbb{R}^4$ is a plane through the origin. Then the net displacement produced by ξ is a simple bivector.

Proof. Recall the relation between the produced net displacement δp and the rescaled Fourier coefficients $(u_n)_n$, $(v_n)_n$ of ξ from the simple case. There we had

$$\sqrt{\det \Delta_g}\,\left(\Delta_h \hat{\Delta}_g\right)^{-1} \delta p = \sum_{n\in\mathbb{N}} \frac{v_n \wedge u_n}{n},$$

where on the RHS we have an absolutely convergent series in $\overset{2}{\wedge}\mathbb{R}^4$. However, by assumption $u_n, v_n \in D$ $\forall n \in \mathbb{N}$, thus

$$\sum_{n\in\mathbb{N}} \frac{v_n \wedge u_n}{n} \in \overset{2}{\wedge} D.$$

The plane D being of dimension 2 implies that $\sum_n \frac{u_n \wedge v_n}{n}$ must be a simple bivector in $\overset{2}{\wedge} D$ which we then can naturally embed into $\overset{2}{\wedge} \mathbb{R}^4$. This proves the claim. $\square$.