Chapter 15

Odqvist's Approach for a Single Particle Surface

The integral representation in terms of boundary velocity and tractions was already known to Lorentz. Odqvist realized that the terms in the integral representation could be separately inspected and used to create Stokes solutions. He called these terms the single and the double layer, and established the properties of these for single closed boundary surfaces. Much of the remainder of this and the next chapter is just an extension of his ideas to fluid domains with multiple boundaries and inspection of numerical methods, although the fairly new idea of completing the range [57] plays a significant role in the recent developments.

The formulations presented in this section can be called secondary variable formulations, since instead of the physical variables of primary interest we employ nonphysical density distributions of secondary importance by themselves.

15.1 Smoothness of the Boundary Surfaces

So far we have restricted the boundary surfaces to be Lyapunov-smooth. There are two reasons for the necessity of this restriction. Firstly we need the Fredholm–Riesz–Schauder theory of compact operators, and this is applicable to weakly singular kernels. Without the smoothness restriction the kernels that come up would not be weakly singular any more. Secondly we know that a Stokeslet is nonphysical in the sense that it gives infinite dissipation. In the single layer term Stokeslets are distributed over the surface, and for this to result in a nice Stokes flow with finite (physical) dissipation it is necessary to require some smoothness of both the surface and the distribution density. Thus this second reason is concerned with the relation to Stokes solutions in the three-dimensional domain, instead of just with the solution of boundary integral equations.

The discussion in Ladyzhenskaya shows that for the single layer term it suffices to require Lyapunov-smoothness of the boundary surfaces and continuity of single layer density for the resulting tractions to be well defined and smooth. In this connection Odqvist defined surfaces of class Ah for similar purposes, meaning that the surface can piecewise be represented by one Cartesian coordinate as a function of the other two, so that this function has Hölder-continuous first derivatives.

For the double layer somewhat stronger restrictions are necessary, because the singularity distributed over the boundary is stronger (of higher order). While Ladyzhenskaya does not consider this case, Odqvist showed the following sufficient conditions. The surface must be of class Bh, meaning that Hölder-continuity is required of also the second derivatives in the Cartesian piecewise representation. Furthermore, the double layer density must have Hölder-continuous first derivatives. With these conditions satisfied, the tractions due to the double layer will again exist as the surface is approached, having a continuous limit on the surface. We shall not discuss these mathematically elaborate deductions in detail here.

In order to avoid complications we shall in the central discussion of this chapter always assume that the surfaces are both Lyapunov-smooth and of class Bh. However, the theory can be developed for just Lyapunov-smooth surfaces [37] (see also Section 16.2), but then the mathematical complications may hide the essential physical ideas from the reader to some extent.

Odqvist has shown further that the eigenfunctions of the double layer operator will be not only continuous (as implied by the standard theory of weakly singular IEs), but also their first derivatives will be Hölder-continuous. This follows by noting that an eigenfunction is given by the integral operator acting on this function, and the integral operator has a "smoothing effect." The same conclusion holds for other solutions of the second kind equation than eigenfunctions, provided that the data used are as smooth as we want the solution to be. As a result, we can use the eigenfunctions (or other solution densities with the provision above) to generate nice smooth Stokes flows, such that the tractions are well defined and continuous. The significance of this is that now energy dissipation arguments can be utilized, expressing the dissipation as the rate of work done on the boundaries of the fluid domain (the energy relation), and also the uniqueness theorem for Stokes solutions with velocity boundary conditions is valid.

15.2 Single and Double Layer Potentials, and Some of Their Properties

Consider the terms on the RHSs of the velocity representation 14.15 and the pressure representation 14.16. They will be called the *single layer potentials* and the *double layer potentials*, respectively. The reason for this terminology is, that the single layer potentials are just superpositions of the hydrodynamic potentials of a point force, whereas the double layer potentials are those caused by a layer of sources (or sinks) and doublets of point forces.

Let us look at the double layer a bit more precisely. In 14.15 singularities of the type $\hat{\boldsymbol{n}} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{u}$ are distributed over the surface; in the following the vectors

$\hat{n}$ and u are considered constant. Due to the symmetry of Σ the same result is obtained from $\hat{n}u{:}\Sigma$. Using the definition of Σ in terms of the fundamental solution, this is the scaled sum of

$$\hat{n}u{:}(-\delta\mathcal{P}) = -(\hat{n}\cdot u)\mathcal{P}\ , \tag{15.1}$$

$$\mu\hat{n}u{:}(\nabla\mathcal{G}) = \mu(u\cdot\nabla)(\hat{n}\cdot\mathcal{G})\ , \tag{15.2}$$

and

$$\mu\hat{n}u{:}({}^t\nabla\mathcal{G}) = \mu(\hat{n}\cdot\nabla)(u\cdot\mathcal{G})\ . \tag{15.3}$$

The first of these represents a source or a sink depending on the sign of the dot product on its RHS, and the two remaining terms are clearly doublets formed by opposing point forces. The torques due to these doublets are given by $\mu u\times\hat{n}$ and $\mu\hat{n}\times u$, which cancel each other. Thus *the double layer singularities can give no total force or torque.*

To examine the effects of arbitrary *density distributions* given on S for these potentials, substitute $\sigma\cdot\hat{n}\leftarrow -2\psi$ and $u\leftarrow 2\varphi$ to get the following contributions to the velocity and pressure fields:

$$V(y) = \frac{1}{4\pi\mu}\oint_S \frac{1}{r_{\xi y}}(\delta + r^0_{\xi y}r^0_{\xi y})\cdot\psi(\xi)\,dS(\xi) \tag{15.4}$$

$$\begin{aligned}\Omega(y) &= \frac{1}{2\pi}\oint_S \frac{r_{\xi y}}{r^3_{\xi y}}\cdot\psi(\xi)\,dS(\xi)\\ &= -\frac{1}{2\pi}\nabla_y\cdot\oint\frac{\psi(\xi)}{r_{\xi y}}\,dS(\xi)\end{aligned} \tag{15.5}$$

for the single layer density ψ, and correspondingly,

$$W(y) = \frac{3}{2\pi}\oint_S \hat{n}(\xi)\cdot\frac{r_{\xi y}r_{\xi y}r_{\xi y}}{r^5_{\xi y}}\cdot\varphi(\xi)\,dS(\xi) \tag{15.6}$$

$$\begin{aligned}\Pi(y) &= -\frac{\mu}{\pi}\oint_S \nabla_y\left\{\frac{r_{\xi y}}{r^3_{\xi y}}\cdot\hat{n}(\xi)\right\}\cdot\varphi(\xi)\,dS(\xi)\\ &= -\frac{\mu}{\pi}\nabla_y\cdot\oint\left\{\hat{n}(\xi)\frac{r_{\xi y}}{r^3_{\xi y}}\right\}\cdot\varphi(\xi)\,dS(\xi)\end{aligned} \tag{15.7}$$

for the double layer density φ. Here the base pressures are so chosen that as infinity is approached the pressure decays towards zero. The alternative forms of $\Pi(y)$ are due to the symmetry of $\nabla_y\mathcal{P}$, mentioned when the integral representation for the pressure was introduced:

$$\hat{n}(\xi)\cdot\nabla_y P(y-\xi) = \nabla_y\mathcal{P}(y-\xi)\cdot\hat{n}(\xi)\ . \tag{15.8}$$

The kernels giving the velocity and pressure fields of a double layer potential will be denoted by

$$K(y,\xi) = -2\hat{n}(\xi)\cdot\Sigma(y,\xi) = \frac{3}{2\pi}\hat{n}(\xi)\cdot\frac{r_{\xi y}r_{\xi y}r_{\xi y}}{r^5_{\xi y}} \tag{15.9}$$

and

$$\boldsymbol{k}(\boldsymbol{y},\boldsymbol{\xi}) = -\frac{1}{2\pi}\hat{\boldsymbol{n}}(\boldsymbol{\xi}) \cdot \nabla_y \mathcal{P}(\boldsymbol{y}-\boldsymbol{\xi}) = -\frac{\mu}{\pi}\hat{\boldsymbol{n}}(\boldsymbol{\xi}) \cdot \nabla_y \frac{\boldsymbol{r}_{\xi y}}{r_{\xi y}^3} \ . \tag{15.10}$$

Thus the density distribution is premultiplied with $\boldsymbol{K}$ or $\boldsymbol{k}$ and integrated over the surface, so that the *first* argument of each kernel will remain, to get the corresponding velocity and pressure fields.

The natural inner product was defined for square integrable vector surface fields on S as the over S integrated pointwise dot product of these fields. With respect to this inner product, the adjoint kernel to any given kernel $\boldsymbol{K}$ is $\boldsymbol{K}^*$ defined by

$$\boldsymbol{K}^*(\boldsymbol{\eta},\boldsymbol{\xi}) = \boldsymbol{K}^t(\boldsymbol{\xi},\boldsymbol{\eta}) \ , \tag{15.11}$$

so that transformation with $\boldsymbol{K}$ inside an inner product may be flipped to the other side of the inner product by changing the transformation to $\boldsymbol{K}^*$. This corresponds to the transpose of a matrix and its relation to the ordinary dot product (a real inner product) of finite vectors. For complex inner products one also needs to complex conjugate the transpose, to create the adjoint. The "operator notation" $\boldsymbol{W} = \mathcal{K}\boldsymbol{\varphi}$, where $\mathcal{K}$ is the linear operator corresponding to kernel $\boldsymbol{K}$, shall be used as a shorthand for Equation 15.6; similarly $\mathcal{K}^*$ corresponds to $\boldsymbol{K}^*$. Since these operators, corresponding to weakly singular kernels, are compact on L_2, square integrable densities are mapped to square integrable surface fields. The inner product is denoted by $\langle \bullet, \bullet \rangle$, so that the flipping rule looks like

$$\langle \mathcal{K}\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \rangle = \langle \boldsymbol{\varphi}_1, \mathcal{K}^*\boldsymbol{\varphi}_2 \rangle \ , \tag{15.12}$$

which is easy to verify just by interchanging the order of the implied surface integrations. To indicate an argument position vector at which an image (function) given by, say, $\mathcal{K}$ is to be evaluated, notation like $\mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\xi})$ shall be used.

The surface jumps for $\boldsymbol{V}$ and $\boldsymbol{W}$ were discussed when the BIE in the primary variables was derived for smooth (continuous) density distributions. Later on density distributions in L_2 shall be considered with the integral equations for secondary variables, but the smoothness assumption is essential for the following "jump conditions." Explicitly, $\boldsymbol{V}$ is continuous through the surface, whereas the double layer potential has the jump (see Figure 14.3)

$$\left.\begin{aligned} \boldsymbol{W}(\boldsymbol{\eta})_{(i)} &= \ \ \boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) \\ \boldsymbol{W}(\boldsymbol{\eta})_{(e)} &= -\boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) \end{aligned}\right\} \ , \tag{15.13}$$

where the obvious notation

$$\boldsymbol{W}(\boldsymbol{\eta})_{(i)} = \lim_{\epsilon \to 0+} \boldsymbol{W}\left(\boldsymbol{\eta} + \epsilon\hat{\boldsymbol{n}}(\boldsymbol{\eta})\right) \tag{15.14}$$

is used for the limiting value on the interior side (to which the surface normal points), with similar notation for the exterior. The one-dimensional limit on the RHS here is just representative and clearly shows the direction of approach; the limits exists in the ordinary three-dimensional sense as the surface point is approached from either side. The surface tractions for the single layer potential $\boldsymbol{V}$ will also show a similar jump, as is shown shortly.

Consider now the stress field (with a particular choice of base pressure) for the single layer potential 15.4. On interpreting the surface integral as a superposition of fundamental solutions $\mathcal{G}$, and superposing the corresponding stress fields $\boldsymbol{\Sigma}$, we get

$$\boldsymbol{\sigma}(\boldsymbol{y};\boldsymbol{V}) = 2\oint_S \boldsymbol{\Sigma}(\boldsymbol{y},\boldsymbol{\xi})\cdot\boldsymbol{\psi}(\boldsymbol{\xi})\,dS(\boldsymbol{\xi}) \ . \tag{15.15}$$

The kernel here is $\mathrm{O}(r_{\xi y}^{-2})$, and is no more weakly singular on substituting $\boldsymbol{y} \leftarrow \boldsymbol{\eta} \in S$. Multiply with the surface normal at $\boldsymbol{\eta}$:

$$\hat{\boldsymbol{n}}(\boldsymbol{\eta})\cdot\boldsymbol{\sigma}(\boldsymbol{y};\boldsymbol{V}) = 2\oint_S \hat{\boldsymbol{n}}(\boldsymbol{\eta})\cdot\boldsymbol{\Sigma}(\boldsymbol{y};\boldsymbol{\xi})\cdot\boldsymbol{\psi}(\boldsymbol{\xi})\,dS(\boldsymbol{\xi}) \ . \tag{15.16}$$

Now the integrand on the RHS is weakly singular as one substitutes $\boldsymbol{y} \leftarrow \boldsymbol{\eta}$, and the kernel $\boldsymbol{K}^*(\boldsymbol{\eta},\boldsymbol{\xi})$ results. Adding the double layer velocity field $\mathcal{K}\boldsymbol{\psi}(\boldsymbol{y})$ to the RHS of 15.16 gives a function that is continuous as $\boldsymbol{y}$ passes through S at $\boldsymbol{\eta}$. This shows that the tractions of the single layer potential (with suitable choice of base pressures on both sides of S) have a jump similar to the velocity jump of the double layer. Explicitly,

$$\left.\begin{array}{rl} \hat{\boldsymbol{n}}(\boldsymbol{\eta})\cdot\boldsymbol{\sigma}(\boldsymbol{\eta};\boldsymbol{V})_{(i)} = & -\boldsymbol{\psi}(\boldsymbol{\eta}) + \mathcal{K}^*\boldsymbol{\psi}(\boldsymbol{\eta}) \\ \hat{\boldsymbol{n}}(\boldsymbol{\eta})\cdot\boldsymbol{\sigma}(\boldsymbol{\eta};\boldsymbol{V})_{(e)} = & \boldsymbol{\psi}(\boldsymbol{\eta}) + \mathcal{K}^*\boldsymbol{\psi}(\boldsymbol{\eta}) \end{array}\right\} , \tag{15.17}$$

As stated earlier, the results above will be referred to as "jump conditions." Verbally, *on passing from the exterior to the interior, the double layer velocity field will have a jump equal to twice the double layer density, and the single layer tractions (with suitable base pressures) jump by minus twice the single layer density.*

From the physical picture of the double layer, having just sources and pairs of opposing forces, one can infer that there will be no traction jump through it. On any surface element of S there is nothing to generate a total force, and so the tractions $\times$ surface element area must be equal on the two sides. Mathematically the situation is more complicated.

For a Lyapunov surface in general, the kernel that comes up on forming the stresses of the double layer will not be weakly singular, even after dotting with the surface normal to get the tractions. There is no guarantee that a finite limit will exist for the stresses as the surface is approached. Odqvist [53] has shown that for this limit to exist it is sufficient that the double layer density and its first derivatives along the surface are *Hölder continuous* with a similar requirement for the second derivatives of one Cartesian coordinate of points on the surface in terms of the other two coordinates (the coordinate system may be suitably chosen for each surface patch). For this reason the article by Karrila and Kim [38] is not completely rigorous; there tractions due to a double layer were used without requiring more than Lyapunov-smoothness of the surfaces. The results presented there are correct though, and they have been re-established with different arguments in the thesis of Karrila [37]. To keep the theoretical considerations simple, we shall here restrict the boundary surfaces to be both Lyapunov-smooth and of class Bh as defined by Odqvist. Then the null functions shall be sufficiently smooth for the generated Stokes fields to always have well-defined tractions on these surfaces, and the energy relation and jump-conditions hold.

The energy relation 14.14 shall be applied to unbounded domains with the single layer generated velocity field. A single layer on a closed surface generates a Stokes velocity field both inside and outside the surface, and since mass is conserved on the inside and the velocity is continuous through the surface, the total generation of fluid as observed from the outside is zero. (Physically the single layer simply has no sources.) Therefore the energy equation can be applied, neglecting the "surface at infinity," regardless of how the base pressure is chosen. Choosing the base pressure for the single layer so that its pressure field is decaying (as was done on defining Ω in this section) will enable the use of Lorentz reciprocal theorem with any other decaying velocity field, regardless of how the base pressure for this is chosen or whether this has sources within the particle surfaces.

15.3 Results for a Single Closed Surface

Assume now that a (homogeneous) Stokes solution is given by the double layer alone. Then according to 15.13

$$\boldsymbol{u}(\boldsymbol{\eta}) = -\lambda\boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) , \tag{15.18}$$

where λ is -1 or $+1$ according as we are considering the interior or the exterior. Assuming that the velocity surface field $\boldsymbol{u}$ has Hölder-continuous first derivative and admits a solution, the solution $\boldsymbol{\varphi}$ will also be similarly smooth. Since the jump condition 15.13 holds for such densities, the corresponding double layer potential $\boldsymbol{W}$ will be a Stokes solution having $\boldsymbol{u}$ as its pointwise limit as the surface S is approached, and further the corresponding tractions are continuous. Thus for smooth enough data $\boldsymbol{u}$ on S it suffices to solve the BIE 15.18 to get the Stokes solution in the fluid domain corresponding to these velocity boundary conditions, assuming that the boundary S consists of closed Lyapunov surfaces of class Bh. Soon it shall be investigated when, given $\boldsymbol{u} \in L_2$, one can solve for $\boldsymbol{\varphi} \in L_2$. In doing that, the adjoint single layer problem shall be used:

$$\boldsymbol{T}(\boldsymbol{\eta}) = -\lambda^*\boldsymbol{\psi}(\boldsymbol{\eta}) + \mathcal{K}^*\boldsymbol{\psi}(\boldsymbol{\eta}) . \tag{15.19}$$

Comparing this with 15.17 it is easy to see that, with some smoothness restrictions, this corresponds to the interior (exterior) problem with $\lambda = \lambda^*$ set to $+1$ (-1) and $\boldsymbol{T}$ set equal to the surface tractions $\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(;\boldsymbol{V})$. The stresses corresponding to these tractions, given by 15.15, are decaying. So a double layer representation corresponds to velocity boundary conditions, while a single layer representation corresponds to traction boundary conditions, and on switching from one formulation above to the adjoint (keeping the same $\lambda \in \{-1, 1\}$) interior problems are switched to exterior problems and *vice versa*.

In the following, "1" is also used for the identity operator, since the effect of multiplying with one is just the identity mapping. Furthermore, the main interest here is with the double layer equation for the interior and therefore mostly the eigenvalue $\lambda = -1$ is considered; the theorems will be labeled according to the interior.

Theorem 1 (A single particle) *Let S be a single particle surface, defining the interior as its outside. Then* $\dim N(1+\mathcal{K}) = 6$, *and these null double layer densities coincide with RBM velocities on S, giving zero flow in the interior. The null functions of the adjoint second-kind operator, forming $N(1+\mathcal{K}^*)$, are single layer densities that correspond to RBM velocities on S and are proportional to the resulting tractions outside S. The corresponding Jordan blocks are trivial.*

Proof. Choose $\boldsymbol{u}$ to be an RBM velocity inside the particle surface (*i.e.*, in the exterior) with the corresponding stresses set to zero in 14.17. Rewriting the equation using the double layer operator (noting the change in the direction of the normal) we find:

$$\frac{1}{2}\boldsymbol{u} = -\frac{1}{2}\mathcal{K}(;\boldsymbol{u}) \ , \tag{15.20}$$

which shows that $\boldsymbol{u}$ is a null function of $1+\mathcal{K}$. The integral representation also shows that the zero flow is generated in the interior. To show that there are no more null functions, the homogeneous adjoint problem 15.19 with $\lambda = -1$ is considered. This corresponds to a single layer generated flow $\boldsymbol{V}$, for which the tractions vanish on the exterior side of S. The energy equation 14.14 implies that there is no dissipation in the exterior, and so only RBM is possible there. Since the velocity $\boldsymbol{V}$ is continuous through S, on the outside (interior) there is also single layer generated flow with RBM on S. Because of the uniqueness of Stokes solutions with velocity BCs, the flow field in the interior has maximally six degrees of freedom (dof). Because the base pressure in the interior is set to zero at infinity in 15.19, the stress and outside traction fields have no more dof. The single layer density is proportional to the jump in tractions over S, so that it also has maximally six dof. Since for the adjoint double layer problem six linearly independent null functions are already known, the equal dimension of the null spaces must be exactly six.

Now we show that the Jordan blocks corresponding to these null functions are trivial, *i.e.*, no other density is mapped onto a null function by the same second-kind operator. As the Jordan block structures of an operator and its adjoint are similar, we only consider the null functions in $N(1+\mathcal{K})$, namely, RBM velocities on the surface. Let φ be one such RBM null function. If the Jordan block were not trivial, then $\varphi \in R(1+\mathcal{K}) = N(1+\mathcal{K}^*)^\perp$. But then this RBM velocity φ would be orthogonal to any RBM tractions, in particular to the tractions corresponding to φ as boundary velocity. This means that with that boundary velocity, no work would be done on the boundary and the dissipation should be zero. However, the fluid motion with such boundary velocity clearly is not RBM, and thus there is deformation and nonzero dissipation. This contradiction shows the triviality of the Jordan blocks. ◇

For a proof of a part of this theorem without invoking the single layer, see the exercises. The triviality of the Jordan blocks is where the single layer results are needed here. This theorem has some immediate corollaries. Since the null functions of second-kind operators are continuous, and RBM tractions of a single particle (with a specific choice of the base pressure) are null functions of 15.19, these RBM tractions exist and are continuous. Also the orthogonality of a null space to the range of the adjoint second kind operator can be utilized. The RBM densities as null functions in $N(1+\mathcal{K})$ are the orthogonal complement of $R(1+\mathcal{K}^*)$, and so the single layer in 15.19 can represent exactly such inside flow fields whose tractions are orthogonal to RBM velocities on S, or equivalently correspond to no total force or torque on the inside of S. This is not much of a restriction, since according to Lorentz reciprocal theorem any Stokes flow inside satisfies these conditions. On the other hand, $N(1+\mathcal{K}^*)$ consists of the RBM tractions on the outside of S. By the Lorentz reciprocal theorem (assuming that the integrals in it exist) the natural inner product on S of any outside velocity field $\boldsymbol{u}$ with these RBM tractions gives components of the total force and torque corresponding to $\boldsymbol{u}$ on S. If the limiting tractions of a Stokes solution do not exist and allow integration to get the total force and torque, components of the total force and torque are defined by the inner products of the surface velocity field and suitable RBM tractions. Since the RBM tractions are smooth and so in L_2, these inner products are well defined for any $\boldsymbol{u} \in L_2$. The outside velocity fields $\boldsymbol{W} \in L_2$ generated by a double layer are exactly those that in this sense correspond to no total force or torque on S. Physically this result is understood, since the force doublets and sources in a double layer can give no total force or torque on a surface enclosing the particle surface, and the total force and torque are transmitted unchanged by Stokes flow. The Lorentz reciprocal theorem could be applied above, neglecting the surface at infinity, because the single layer generated velocity field has no generation of fluid inside the particle surface, and the base pressure corresponding to the RBM tractions given by 15.19 is such that the stress field is decaying.

Theorem 2 (A container) *Let S be a single closed container surface, defining the interior as its inside. Then* $\dim N(1+\mathcal{K}) = 1$, *and these null double layer densities are nonorthogonal to the surface normal on S and generate zero flow in the interior. The null functions of the adjoint second kind operator, forming $N(1+\mathcal{K}^*)$, are single layer densities proportional to the surface normal. The corresponding Jordan blocks are trivial.*

Proof. Consider the single layer outside problem given by 15.19 with $\lambda = -1$. The null functions are such that the tractions on the outside vanish. Then the dissipation on the outside is zero, and there can only be RBM. Since the single layer generated velocity field $\boldsymbol{V}$ is decaying, it must be zero on the outside. The velocity $\boldsymbol{V}$ is continuous through S, and so vanishes also on the inside. Then, by the uniqueness theorem (which can be used with single layer generated flow fields), there is zero motion everywhere inside.

The corresponding stress field is a constant pressure, and so the tractions on the inside are a constant multiple of the surface normal. Now the single layer null density must also be a multiple of the surface normal, since it is proportional to the traction jump over S. That such single layer density really is a null density can be seen by applying 14.15 to zero velocity field with constant pressure and $\boldsymbol{y}$ in the exterior. This shows that the generated velocity field is zero in the exterior, and since the chosen single layer stresses in Equation 15.19 are decaying, the stresses and tractions on the outside really are zero. Thus it has been shown that the equal dimensions of the null spaces are one.

As the null double layer density gives zero velocity on the container boundary and has continuous tractions, the uniqueness theorem for Stokes flows implies that it gives zero flow in all of the interior. (This is where we require the class Bh smoothness of the boundaries, as defined by Odqvist. Without it the uniqueness theorem would be inapplicable.)

To show that the corresponding Jordan block is trivial (since the null space has dimension one, there is only one Jordan block), assume to the contrary that there exists a single layer density that gives the null function found. This density should give outside tractions that are a multiple of the surface normal. Since the velocity field $\boldsymbol{V}$ is mass-conserving on the inside and continuous through S, it is orthogonal to the surface normal. Thus again from Equation 14.14 zero dissipation is found in the outside. The rest of the argument is as above, and it is found that principal functions other than the null function should coincide with the null function. This contradiction shows that the Jordan block is trivial.

Since the surface normal is not in $R(1+\mathcal{K}^*)$, it necessarily has a component in the orthogonal complement of this closed subspace, namely, in $N(1+\mathcal{K}) = R(1+\mathcal{K}^*)^\perp$. The null space $N(1+\mathcal{K})$ has dimension one, and therefore all (nonzero) functions in it are nonorthogonal to the surface normal. $\diamondsuit$

For another way of showing that the container null functions in $N(1+\mathcal{K})$ are nonorthogonal to the surface normal, see the exercises.

This theorem again has a corollary following similarly as with the previous theorem. The double layer can represent any inside velocity field that is orthogonal to the surface normal (*i.e.*, mass-conserving) and only these. Again this restriction is not too severe, since all Stokes flows in a container with rigid walls satisfy mass-conservation. The single layer necessarily has a deficiency of similar dimension and cannot represent arbitrary tractions, its restriction being the decay condition. From this it can be immediately inferred that the surface normal (on the outside of a surface) is not in the range of a single layer.

15.4 The Completion Method of Power and Miranda for a Single Particle

Although Odqvist's work was published as early as 1930, it has received little attention to date. This holds particularly in connection with numerical applications. In fact as his work has partly been summarized by Ladyzhenskaya, and the original was published in German, sometimes his results have been attributed to Ladyzhenskaya.

It was only as recently as 1987 that Power and Miranda showed how the weaknesses of the double layer, namely, inability to represent net force and torque, and indeterminacy of the density, could be utilized beneficially. As Odqvist had restricted his attention to single closed boundaries, Power and Miranda also considered just a single particle. Further, as they chose to couple the total force and torque acting on the particle with the known null functions, they did not consider the direct solution of mobility problems. In fact for a single particle it does not make a whole lot of difference if one is able to solve mobility problems directly, but with multiparticle systems forming the whole resistance matrix needed to solve a mobility problem becomes prohibitively expensive computationally. Later on it will be seen how the mobility problems actually are computationally easier than the resistance problems, in that they allow the iterative solution of the discretized system. This single fact accounts for significant computational savings with large linear systems. In the following the main results of Power and Miranda are briefly explained, in a somewhat different fashion from their original derivation.

Consider the outside problem for a single surface, the outside now being the interior fluid domain where the surface normal points. Power and Miranda [57] observed that although the double layer representation is able to represent only those flow fields that correspond to a force- and torque-free surface, the representation may be completed by adding terms that give arbitrary total force and torque in suitable linear combination. They chose to use a Stokeslet $\boldsymbol{F} \cdot \mathcal{G}$ and a rotlet $-\frac{1}{2}(\boldsymbol{T} \times \nabla) \cdot \mathcal{G}$ positioned at the origin chosen inside the particle. Thus their equation is

$$\boldsymbol{u}(\boldsymbol{\eta}) = \boldsymbol{\varphi}(\boldsymbol{\eta}) + \mathcal{K}\boldsymbol{\varphi}(\boldsymbol{\eta}) + (\boldsymbol{F} - \frac{1}{2}\boldsymbol{T} \times \nabla) \cdot \mathcal{G}(\boldsymbol{\eta})/(8\pi\mu) \quad \text{for } \boldsymbol{\eta} \in S. \qquad (15.21)$$

Note again that, although this is the outside problem, $\boldsymbol{W}_{(i)}$ of Equation 15.13 is used, since the fluid domain of consideration is called the interior and the surface normal points in that direction. The total force and torque are fully determined by the square integrable $\boldsymbol{u}$ on S so that $\boldsymbol{F}$ and $\boldsymbol{T}$ are fixed, see the discussion after Theorem 1. Since the double layer part here has null functions (six linearly independent), the resulting system is indeterminate. Power and Miranda chose to associate these null functions with the added six parameters, namely, three components of force and torque each, thus making the density fully determined. Denoting the null functions by $\boldsymbol{\varphi}_i$, where $i = 1, \ldots, 6$ and using the natural inner product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \oint_S \boldsymbol{a}(\boldsymbol{\xi}) \cdot \boldsymbol{b}(\boldsymbol{\xi})\, dS(\boldsymbol{\xi}) \tag{15.22}$$

for vector surface fields, the association was done by

$$\left.\begin{array}{rcl} F_i & = & \langle \boldsymbol{\varphi}, \boldsymbol{\varphi}_i \rangle \\ T_i & = & \langle \boldsymbol{\varphi}, \boldsymbol{\varphi}_{i+3} \rangle \end{array}\right\} \text{ for } i \in \{1, 2, 3\}. \tag{15.23}$$

Substituting these in the equation for $\boldsymbol{\varphi}$ above gives an ordinary IE of the second kind. Since the double layer is now fully determined, the *Fredholm alternative* implies that a solution always exists for any given $\boldsymbol{u}$. This corresponds to the result of linear algebra, that a square linear system either always has a solution and it is unique, or that solutions exist only sometimes but then they are not unique.

The point to note here is that the association of the null space with the components of force and torque is completely arbitrary (the basis for the null space can be chosen and permuted freely), although Power and Miranda did not mention this. Their method is mathematically nice because it leads to an ordinary IE of the second kind without any extra terms, but after solving for $\boldsymbol{\varphi}$, $\boldsymbol{F}$ and $\boldsymbol{T}$ have to be separately computed. Their mathematical argument, however, is much more elaborate than the one given here and does not easily allow extension to multiparticle systems or bounded domains. It may be observed that this method is related to Wielandt's deflation, see Bodewig [5], in that the original integral operator is modified by adding some tensor products. In fact, we will show in Chapter 17 that such deflations can be carried out for the mobility problem. The end result is a fast iterative algorithm for the solution of mobility problems.

Finally, we note that the flow fields of the Stokeslet and rotlet are infinitely differentiable on the boundary surface, and therefore subtraction of these terms from given sufficiently smooth boundary conditions gives again equally smooth boundary conditions, representable by the double layer alone when $\boldsymbol{F}$ and $\boldsymbol{T}$ are chosen correctly.

Exercises

Exercise 15.1 Regarding Theorem 1.

Show by using the double layer operator $\mathcal{K}$ (without invoking the adjoint operator) that $\dim \mathcal{N}(1 + \mathcal{K}) \leq 6$.

Hint: The chain of deduction could be [the double layer null functions give zero outside flow] — [the tractions on both sides are zero] — [the inside flow is RBM] — [apply the jump condition].

Note: The advantage of the earlier proof in this chapter is that it can be used when requiring only Lyapunov-smoothness.

Exercise 15.2 Regarding Theorem 2.

Show directly with the double layer operator that the container null function is

nonorthogonal to the surface normal.
Hint: [zero flow on the inside implies constant pressure stress field] — [also on the outside the tractions are a multiple of the surface normal] — [orthogonality would imply no dissipation in the outside due to energy relation and jump condition] — [zero flow on the outside due to decay condition] — [jump condition gives zero double layer density, a contradiction].

Exercise 15.3 Double Layer Representation and Solution of an Exterior Traction Problem.
Suppose $\boldsymbol{u}$ is a velocity field representable by a double layer. Show how to construct its double layer representation assuming that the tractions $\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma}(;\boldsymbol{u})$ are known.
Hint: In the exterior construct a single layer generated field $\boldsymbol{V}$ with tractions equal to the interior tractions of $\boldsymbol{u}$. This is possible when the tractions of $\boldsymbol{u}$ correspond to no total force or torque. Use the integral representation 14.15 in the interior both for $\boldsymbol{u}$ and for $\boldsymbol{V}$ (the latter having zero LHS), noting that the single layer terms are equal.
Note: We know that the completed double layer equations (either bordered or deflated) have a unique solution. But is this solution the one we are seeking? We have a uniqueness theorem for Stokes flows with nice smooth tractions, but that cannot be applied to flows generated by a double layer without smoothness restrictions for both the surface shapes and the density. Using the technique above it can be shown that if a solution with nice smooth tractions exists, then the completed equations will give just that solution [37] .

Exercise 15.4 The Container Null Function on a Sphere.
Show that on a sphere the container null function is the surface normal.
Hint: The dimension of that null space is one. Because of symmetry any rotation about the center of the sphere, or reflection through an equatorial plane, would transform a null function to a null function again. The only null function must be invariant with respect to these, which easily implies the result desired.

Exercise 15.5 Green's Functions Expressed with Single or Double Layers.
In general a Green's function (in our case tensor) is such that it satisfies some homogeneous boundary conditions, so that several such functions can be superposed while the boundary conditions remain the same. Also the Green's function is an inhomogeneous solution of the governing linear partial differential equations, corresponding to a delta function as the driving force term that makes the equations inhomogeneous. In our ordinary Green's function $\mathcal{G}$, the homogeneous boundary conditions are satisfied at infinity. Replace these by BCs requiring zero velocity on a container surface S_c. Show that the resulting Green's function can be expressed as the sum of $\mathcal{G}$ and a double layer generated flow field in the interior (inside S_c). As a more difficult problem, show that the double layer "correction" can be replaced by a single layer "correction" to $\mathcal{G}$.

Hint: All you really need to do is cancel $\mathcal{G}$ on the boundary S_c. With the single layer first generate an exterior flow field that matches the tractions of $\mathcal{G}$ (with some choice of the base pressure). Deduce that the velocity fields are the same in the exterior, and use continuity of the single layer generated velocity field through the surface.
Note: The Green's functions and their representations with single and double layers can be generalized to more complicated geometries [37].

Exercise 15.6 Second-Kind IEs from Other Green's Functions.
Let $\boldsymbol{G} = \mathcal{G} + \mathcal{K}\varphi$ be the Green's function relative to some fixed finite measurable boundaries S_0, over which the ordinary double layer integral operator $\mathcal{K}$ is defined. In the interior fluid domain insert some new closed finite boundaries, denoted by S, and denote the new smaller fluid domain between S and S_0 by Q. For $\boldsymbol{u}$ vanishing on S_0 derive an integral representation similar to 14.15 by using the Green's formula (*i.e.*, the Lorentz reciprocal theorem with the volume integrals retained) with $\boldsymbol{G}$ instead of $\mathcal{G}$. Inspect the resulting terms (integrals over S) separately. Clearly the new single layer term gives a Stokes velocity field vanishing on S_0, with no velocity jump over S. Show that the new double layer term also generates a Stokes velocity field that vanishes on S_0 (provided, in case Q is bounded, that the double layer density is orthogonal to $\hat{\boldsymbol{n}}$ on S).
Hint: With the completed double layer representation on $S + S_0$ we can generate a Stokes field $\boldsymbol{u}$ vanishing on S_0 and coinciding with the on S given double layer density (used with the "new" kernel). Then the new integral representation for $\boldsymbol{u}$ verifies the claim.
Exercise continued: Note that the new integral representation gets value zero outside Q, and use a symmetry argument with the new double layer to get an IE of the second kind for velocity BCs on S (with zero velocity on S_0 assumed).
Note: In some cases the Green's function is available in analytic form. Such is the case of half-space restricted by an infinite plane boundary, as shown in Chapter 12. The new Green's function $\boldsymbol{G}$ is found by adding the Lorentz image to $\mathcal{G}$. In this way some problems with infinite boundaries can be treated with compact integral operators (this in spite of the fact that in the exercise above we require S_0 to be finite).

Exercise 15.7 Constant Pressure Solutions.
Take the pressure representation corresponding to the completed double layer representation for a single particle. By order of magnitude analysis of the decay rates show that if the pressure is constant, the total force has to vanish.
Note: This exercise was inspired by [16]. Also Lamb's general solution could be used to solve this problem.

Exercise 15.8 Another Variational Principle for Mobility Problems; Towards a Convergence Proof of the Method of Reflections.
Consider a mobility problem with rigid particles (forces and torques given, translation and rotation or angular velocities to be determined). All the flow fields

complying with the supplied data (but not necessarily corresponding to rigid particles) constitute the set

$$\{\sum_{\alpha}(\boldsymbol{F}_\alpha - \frac{1}{2}\boldsymbol{T}_\alpha \times \nabla) \cdot \frac{\mathcal{G}(\boldsymbol{x}-\boldsymbol{x}_\alpha)}{8\pi\mu} + (\mathcal{K}\boldsymbol{\varphi})(\boldsymbol{x}) \mid \boldsymbol{\varphi} \text{ smooth.}\}$$

Show that the mobility solution, corresponding to rigid-body motion and found within this set, minimizes the energy dissipation among all the flow fields considered here, *i.e.*, having the specified forces and torques.

Hint: This again can be considered an application of the Pythagorean theorem, as in Chapter 2. To prove orthogonality, just observe that the variations in the flow field must be orthogonal to RBM tractions, since they correspond to no total force or torque on any of the particles.

Note: A similar result holds for the "suboptimization problem" where the double layer on just one of the particles (at a time) is adjusted so as to make the surface velocity on that particle an RBM. Performing such suboptimizations cyclically, stepping through each of the particles at each cycle, is precisely the method of reflections for mobility problems, as described in Part III. Using results from functional analysis, Luke [46] has shown that for *mobility problems* this type of method of reflections always converges (provided the particles are smooth and nontouching). Luke's proof is not completely rigorous for unbounded domains, and he actually suggests using integral equation formulations for extending the validity to the cases without a container.