20
Hypercomplex Analysis

Complex analysis has applications in the theory of heat, fluid dynamics and electrostatics. Such versatility gives occasion to explore whether function theory of complex variables can be generalized from the plane to higher dimensions. Are there hypercomplex number systems which could provide a higher-dimensional analog for complex analytic functions?

Function theory can be generalized to higher dimensions in several different ways, for instance, to quasiconformal mappings, several complex variables or to hypercomplex analysis. Clearly, these generalizations cannot maintain all the features of complex analysis.

In the theory of quasiconformal mappings one retains some geometric features, related to similar appearance of images, and renounces some algebraic features, like multiplication of complex numbers. In the theory of quasiconformal mappings one does not multiply vectors in $\mathbb{R}^n$.

The starting point of hypercomplex analysis is the introduction of a suitable multiplication of vectors in $\mathbb{R}^n$. In contrast to the theory of several complex variables, which commute, hypercomplex analysis is a one-variable theory – the argument being in higher dimensions, where orthogonal vectors anticommute.

20.1 Formulation of complex analysis in $\mathcal{C}\ell_2$

For a complex valued function $u+iv = f(x+iy)$ of the complex variable $x+iy$ the *Cauchy-Riemann equations* are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The second equation tells us that the vector $(u, -v)$ is the gradient of a function $\phi : \mathbb{R}^2 \to \mathbb{R}$:

$$u = \frac{\partial \phi}{\partial x}, \qquad -v = \frac{\partial \phi}{\partial y}. \tag{1}$$

Using the first relation of the Cauchy-Riemann equations we obtain

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$

that is, ϕ is a harmonic function, $\nabla^2 \phi = 0$. Conversely, if ϕ is harmonic, then u and v defined by the relation (1) satisfy the Cauchy-Riemann equations.

The Cauchy-Riemann equations can be condensed into a single equation as follows:

$$\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)(u + iv) = 0.$$

Recall that $i = \mathbf{e}_1\mathbf{e}_2$ and multiply this equation on the left and on the right by $\mathbf{e}_1$, then use associativity and anticommutativity to get

$$\Big(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\Big)(\mathbf{e}_1 u - \mathbf{e}_2 v) = 0.$$

As we know, this relation holds if and only if the vector $(u, -v)$ is the gradient of a harmonic function. It follows that

$$\Big(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\Big)(\mathbf{e}_1 u + \mathbf{e}_2 v) = 0 \tag{2}$$

if and only if (u, v) is the gradient of a harmonic function.

There are three possible ways to formulate the Cauchy-Riemann equations employing the Clifford algebra $\mathcal{C}\ell_2$ (these possibilities will be generalized to higher dimensions in three different ways).

1) Firstly, we may consider the Cauchy-Riemann equations to be a condition on *vector fields*, sending a vector $x\mathbf{e}_1 + y\mathbf{e}_2$ in $\mathbb{R}^2$ to a vector $u\mathbf{e}_1 + v\mathbf{e}_2$ in $\mathbb{R}^2$. The above condition (2),

$$(i) \qquad \Big(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\Big)(u\mathbf{e}_1 + v\mathbf{e}_2) = 0,$$

gives us Cauchy-Riemann equations up to sign and results in those conformal maps which reverse the orientation of $\mathbb{R}^2$. In higher dimensions this alternative means the study of those vector fields, that is, mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$, which are gradients of harmonic functions, mappings from $\mathbb{R}^n$ to $\mathbb{R}$.

2) Secondly, we may reformulate the Cauchy-Riemann equations as a condition

on the *even fields* sending a vector $x\mathbf{e}_1 + y\mathbf{e}_2$ in $\mathbb{R}^2$ to an even element in $\mathcal{C}\ell_2^+ = \{u + v\mathbf{e}_1\mathbf{e}_2 \mid u, v \in \mathbb{R}\} \simeq \mathbb{C}$ of the Clifford algebra $\mathcal{C}\ell_2$. The condition

$$(ii) \qquad \Big(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\Big)(u + v\mathbf{e}_{12}) = 0$$

gives us the Cauchy-Riemann equations. This alternative has non-trivial generalizations in higher dimensions sending the vector space $\mathbb{R}^n$ to the even subalgebra $\mathcal{C}\ell_n^+$ of the Clifford algebra $\mathcal{C}\ell_n$.

3) Thirdly, we may focus our attention on *spinor fields* sending the vector plane $\mathbb{R}^2$ to a minimal left ideal of $\mathcal{C}\ell_2$. Before studying this alternative closer, let us recall that the Clifford algebra $\mathcal{C}\ell_2$ is isomorphic to the matrix algebra of real 2×2-matrices $\mathrm{Mat}(2,\mathbb{R})$. The isomorphism is seen by the correspondences

$$\begin{aligned}
&1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&\mathbf{e}_1 \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{e}_2 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\mathbf{e}_{12} \simeq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

In this case one sends a vector $x\mathbf{e}_1 + y\mathbf{e}_2$ in $\mathbb{R}^2$ to a spinor

$$uf_1 + vf_2 \simeq \begin{pmatrix} u & 0 \\ v & 0 \end{pmatrix},$$

where

$$f_1 = \frac{1}{2}(1 + \mathbf{e}_1) \simeq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f_2 = \frac{1}{2}(\mathbf{e}_2 - \mathbf{e}_{12}) \simeq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here $f_1^2 = f_1$, so f_1 is an *idempotent*, and the spinor space $S = \mathcal{C}\ell_2 f_1 = \{af_1 \mid a \in \mathcal{C}\ell_2\}$ is a *left ideal* of $\mathcal{C}\ell_2$, for which $a\psi \in S$ for all $a \in \mathcal{C}\ell_2$ and $\psi \in S$. Since

$$\begin{aligned}
\mathbf{e}_1 f_1 &= f_1, & \mathbf{e}_1 f_2 &= -f_2 \\
\mathbf{e}_2 f_1 &= f_2, & \mathbf{e}_2 f_2 &= f_1
\end{aligned}$$

one verifies that

$$(iii) \qquad \Big(\mathbf{e}_1\frac{\partial}{\partial x} + \mathbf{e}_2\frac{\partial}{\partial y}\Big)(uf_1 + vf_2) = 0$$

is equivalent to (i).

To summarize, there are three alternatives for the 2-dimensional target:

(i) the Euclidean vector space itself $\mathbb{R}^2$,
(ii) the even subalgebra $\mathcal{C}\ell_2^+$ of the Clifford algebra $\mathcal{C}\ell_2$,
(iii) the spinor space $S = \mathcal{C}\ell_2 f_1$.

In the next section, we will generalize, as a preliminary construction, the first alternative to higher dimensions.

20.2 Vector fields

The Dirac operator. It is possible to extract a certain kind of square root of the n-dimensional Laplace-operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and consider instead a first-order differential operator

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n}$$

called the *Dirac operator*. Since the Dirac operator applied twice equals the Laplace operator, the elements $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are subject to the relations

$$\begin{aligned} \mathbf{e}_i^2 &= 1, & i &= 1, 2, \ldots, n, \\ \mathbf{e}_i \mathbf{e}_j &= -\mathbf{e}_j \mathbf{e}_i, & i &< j. \end{aligned}$$

The linear combinations $\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \cdots + x_n \mathbf{e}_n$ can be considered as vectors building up an n-dimensional vector space $\mathbb{R}^n$ with quadratic form $\mathbf{x}^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. The above relations generate an associative algebra of dimension 2^n, the Clifford algebra $\mathcal{C}\ell_n$ of $\mathbb{R}^n$ [or of dimension $\frac{1}{2}2^n$, isomorphic to an ideal $\frac{1}{2}(1 \pm \mathbf{e}_{12\ldots n})\mathcal{C}\ell_n$ of $\mathcal{C}\ell_n$].

Operating on a vector field $\mathbf{f}$ with ∇ gives

$$\nabla \mathbf{f} = \nabla \cdot \mathbf{f} + \nabla \wedge \mathbf{f}$$

where $\nabla \cdot \mathbf{f}$ is the *divergence* of $\mathbf{f}$ and $\nabla \wedge \mathbf{f}$ is the *curl*, which in this approach is bivector valued.

Sourceless and irrotational vector fields. Consider a steady motion of incompressible fluid in an n-dimensional Euclidean space $\mathbb{R}^n$. Represent the velocity of the flow by the vector field $\mathbf{f}$. The integral

$$\Psi = \int_S d\mathbf{S} \wedge \mathbf{f}$$

over an orientable hypersurface S, $\dim S = n - 1$, is the *stream* across S. We regard $d\mathbf{S}$ as a tangent $(n-1)$-vector measure, rather than the normal vector measure; this makes the stream n-vector valued. If a vector field $\mathbf{f}$ is *sourceless*, $\nabla \cdot \mathbf{f} = 0$, its stream across S depends in a contractible domain only on the boundary ∂S of S. In particular, no stream emerges through a closed hypersurface. If $n \geq 2$, a sourceless vector field $\mathbf{f}$ has a bivector valued potential $\mathbf{v}$ such that $\mathbf{f} = \nabla \rfloor \mathbf{v}$. If $n \geq 3$, the bivector potential can be subjected to a supplementary condition $\nabla \wedge \mathbf{v} = 0$, in which case $\mathbf{f} = \nabla \mathbf{v}$.

The *circulation* of the vector field $\mathbf{f}$ around a closed path C is given by the line integral

$$\int_C d\mathbf{x} \cdot \mathbf{f}.$$

If a vector field $\mathbf{f}$ is *irrotational*, $\nabla \wedge \mathbf{f} = 0$, the circulation vanishes in a simply connected domain, and the line integral

$$u(\mathbf{x}) = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{f} \cdot d\mathbf{x}$$

is independent of path. The function u is called the scalar potential of $\mathbf{f}$. The irrotational vector field $\mathbf{f}$ is the gradient of its scalar potential u, $\mathbf{f} = -\nabla u$.

If a vector field $\mathbf{f}$ is sourceless and irrotational, that is $\nabla \cdot \mathbf{f} = 0$ and $\nabla \wedge \mathbf{f} = 0$, its scalar potential is harmonic, $\nabla^2 u = 0$. A vector field $\mathbf{f}$ is called *monogenic*, if $\nabla \mathbf{f} = 0$. For a monogenic vector field $\nabla \mathbf{f} = \nabla \cdot \mathbf{f} + \nabla \wedge \mathbf{f} = 0$, and so it is sourceless and irrotational. A monogenic vector field has a potential which is a sum or complex of the scalar and bivector potentials: $w = u + \mathbf{v}$. The complex of potentials is also monogenic, $\nabla w = 0$.

Example. A monogenic vector field $\mathbf{f}$, homogeneous of degree ℓ, has a scalar potential

$$u = -\frac{\mathbf{x} \cdot \mathbf{f}}{\ell + 1}, \quad \ell \neq -1,$$

and a bivector potential

$$\mathbf{v} = \frac{\mathbf{x} \wedge \mathbf{f}}{\ell + n - 1}, \quad \ell \neq -(n-1).$$

In the singular case, a monogenic vector field $\mathbf{f}$, homogeneous of degree $\ell = -(n-1)$, might still have a bivector/complex potential. For instance, the Cauchy kernel $\mathbf{q}(\mathbf{x}) = \mathbf{x}/r^n$, $r = |\mathbf{x}|$, has a complex potential

$$\begin{aligned} -\log r + \mathbf{i}\theta \quad &\text{for} \quad n = 2, \\ \frac{1}{r}(1 + \mathbf{i} \tan \frac{\theta}{2}) \quad &\text{for} \quad n = 3, \end{aligned}$$

where θ is the angle between $\mathbf{x}$ and a fixed direction $\mathbf{a}$, and $\mathbf{i}$ is the imaginary unit of the plane $\mathbf{x} \wedge \mathbf{a}$, given by $\mathbf{i} = \mathbf{x} \wedge \mathbf{a}/|\mathbf{x} \wedge \mathbf{a}|$. ∎

The plane case. In the plane the stream is a bivector valued line integral. If the plane vector field is sourceless, its stream across any line in a simply connected domain depends only on the two end points of the line. Integrating from a fixed point $\mathbf{x}_0$ to a variable point $\mathbf{x}$ the stream becomes a bivector valued function, called the stream function,

$$\psi = \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x} \wedge \mathbf{f},$$

which can serve as the bivector potential of $\mathbf{f} = \nabla \rfloor \psi$ $[\mathbf{v} = \psi]$.

If $\mathbf{f}$ is monogenic, $\nabla \mathbf{f} = 0$, then there is an even valued function

$$w = u + \psi = -\int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x} \cdot \mathbf{f} + \int_{\mathbf{x}_0}^{\mathbf{x}} d\mathbf{x} \wedge \mathbf{f} = -\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{f}\, d\mathbf{x},$$

which serves as the complex potential of $\mathbf{f}$. This complex potential is also monogenic, $\nabla w = 0$. (In higher dimensions there is no correspondence for such line integrals representing complex potentials – unless one is confined to axially symmetric vector fields.) ∎

Even fields. Instead of vector fields, we could instead examine the even fields

$$\mathbb{R}^n \to \mathcal{C}\ell_n^+, \quad \mathbf{x} \to f(\mathbf{x}).$$

Here we replace the target $\mathbb{R}^n$ of dimension n by a wider target $\mathcal{C}\ell_n^+$ of dimension $\frac{1}{2}2^n$. The even subalgebra $\mathcal{C}\ell_n^+$ is a direct sum of the k-vector spaces $\bigwedge^k \mathbb{R}^n$ with even k. If we require the even functions to be monogenic, $\nabla f(\mathbf{x}) = 0$, then we have a system of coupled equations:

$$\nabla \wedge f_{k-2} + \nabla \rfloor f_k = 0, \quad \nabla \wedge f_k + \nabla \rfloor f_{k+2} = 0,$$

where f_k is the homogeneous part of degree k of $f = f(\mathbf{x})$. These equations are invariant under the rotation group $SO(n)$.

Irreducible fields. Instead of vector fields or even fields, we could examine functions with values in an irreducible representation of $SO(n)$ or $\mathbf{Spin}(n)$. In the Clifford algebra realm this would mean studying k-vector fields or spinor fields. Important physical fields fall into this category: the Maxwell equations are of the form $\nabla \wedge \mathbf{F} = 0$, $\nabla \rfloor \mathbf{F} = \mathbf{J}$, where $\mathbf{F} \in \bigwedge^2 \mathbb{R}^{3,1}$, and the Dirac field has as its target the spinor space, a minimal left ideal of $\mathbb{C} \otimes \mathcal{C}\ell_{3,1}$. It should be emphasized though that in modern treatment of the Dirac theory the spinor space is replaced by the even subalgebra $\mathcal{C}\ell_{3,1}^+$. ∎

20.3 Tangential integration

Here we consider integration over surfaces, that is, smooth manifolds embedded in a linear space $\mathbb{R}^n$. Our surface S is compact, connected, orientable and contractible. The surface S is k-dimensional if there are k linearly independent vectors tangent to S at each point $\mathbf{x}$ of S. The tangent vectors span a tangent space $T_{\mathbf{x}}$.

In a Euclidean space $\mathbb{R}^n$ the tangent space $T_{\mathbf{x}}$ generates a tangent algebra $\mathcal{C}\ell(T_{\mathbf{x}})$ isomorphic to $\mathcal{C}\ell_k$. A multivector field on S is a smooth function $f : S \to \mathcal{C}\ell_n$; it is *tangential* if $f(\mathbf{x}) \in \mathcal{C}\ell(T_{\mathbf{x}})$ for each $\mathbf{x} \in S$.

There are exactly two continuous tangential unit k-vector fields on an orientable k-dimensional surface S, each corresponding to one of the two orientations attached to S. So tangent to each point $\mathbf{x}$ of an oriented k-dimensional surface S there is a unique unit k-vector $\tau(\mathbf{x})$ characterizing the orientation of S at $\mathbf{x} \in S$. The value of the map τ at $\mathbf{x}$ is called the *tangent* of S at $\mathbf{x}$.

Consider a multivector field $f : S \to \mathcal{C}\ell_n$ on a k-dimensional surface $S \subset \mathbb{R}^n$, $1 \leq k \leq n$. Define the *tangential integral* of f over S by

$$\int_S d\mathbf{S}\, f(\mathbf{x}) = \int_S \tau(\mathbf{x}) f(\mathbf{x})\, dV,$$

where dV is the usual scalar measure of the k-dimensional volume element of S and $d\mathbf{S}$ is the k-vector valued *tangential measure*,

$$d\mathbf{S} = \tau(\mathbf{x})\, dV.$$

So the tangential integral of $f(\mathbf{x})$ is equivalent to the usual (Riemann) integral of $\tau(\mathbf{x})f(\mathbf{x})$ over S.

Since multiplication of multivectors is not commutative, the above equation is not the most general form for a tangential integral. The appropriate generalization is the following:

$$\int_S g(\mathbf{x})\, d\mathbf{S}\, f(\mathbf{x}) = \int_S g(\mathbf{x})\tau(\mathbf{x}) f(\mathbf{x})\, dV.$$

Consider an oriented k-dimensional surface S with boundary ∂S of dimension $k-1$. Set at the point $\mathbf{x} \in \partial S$ a tangent $\tau_{\partial S}(\mathbf{x})$ of the boundary ∂S and a tangent $\tau_S(\mathbf{x})$ of the surface S. Then the expression $(\tau_S(\mathbf{x}))^{-1}\tau_{\partial S}(\mathbf{x})$ is a vector normal to the boundary ∂S. There are two alternatives: the normal vector points inwards, in which case it is tangent to the surface S, or outwards, in which case it is opposite to the inward tangent of S. The orientations of S and ∂S are *compatible*, when the vector $(\tau_S(\mathbf{x}))^{-1}\tau_{\partial S}(\mathbf{x})$ points outwards. Define the *normal integral* of f over the boundary ∂S by

$$\int_{\partial S} \partial \mathrm{s}\, f(\mathbf{x}) = \int_{\partial S} (\tau_S(\mathbf{x}))^{-1}\, d\mathrm{s}\, f(\mathbf{x})$$

where $d\mathrm{s}$ is the $(k-1)$-vector valued tangential measure on the boundary ∂S, and $\partial\mathrm{s}$ is an outward pointing vector normal to ∂S,

$$\partial \mathrm{s} = (\tau_S(\mathbf{x}))^{-1}\, d\mathrm{s} \quad \text{for} \quad \mathbf{x} \in \partial S.$$

The Dirac operator without coordinates. Take a k-dimensional surface S which is contractible to a point $\mathbf{x} \in S$ in such a way that the tangent of S at $\mathbf{x}$ remains a fixed k-vector τ. Define a differential operator

$$\nabla_\tau f(\mathbf{x}) = \lim_{\mathrm{d}(S)\to 0} \frac{1}{\mathrm{vol}(S)} \int_{\partial S} \partial \mathrm{s}\, f(\mathbf{x}),$$

where $\mathrm{d}(S)$ is the diameter of S and $\mathrm{vol}(S)$ is the scalar volume of S.

For instance, when τ is a 1-vector, the partial derivative ∂_τ in the direction of τ could be expressed as $\partial_\tau = \tau\nabla_\tau$.

The case when τ is an oriented volume element is important. The same result is obtained for $+\mathbf{e}_{12\ldots n}$ and $-\mathbf{e}_{12\ldots n}$. So it is convenient to drop the subscript and write ∇. The differential operator ∇ is called the *Dirac operator*. Applying an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ of $\mathbb{R}^n$ the Dirac operator is seen to be

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \cdots + \mathbf{e}_n \frac{\partial}{\partial x_n},$$

where $\dfrac{\partial}{\partial x_i} = \partial_{\mathbf{e}_i}$.

The relation of ∇_τ to the Dirac operator ∇ is obtained by computing

$$\nabla = \tau^{-1}\tau\nabla = \tau^{-1}(\tau \llcorner \nabla + \tau \wedge \nabla)$$

where

$$\nabla_\tau = \tau^{-1}(\tau \llcorner \nabla).$$

20.4 Stokes' theorem

Consider a compact, contractible and oriented surface $S \subset \mathbb{R}^n$ with boundary ∂S and a real differentiable function $f : S \to \mathcal{C}\ell_n$. Stokes' theorem relates tangential integrals over S and ∂S, with compatible orientations,

$$\int_S d\mathbf{S}\, \nabla_\tau f = \int_{\partial S} d\mathrm{s}\, f,$$

where the left hand side becomes, using $\nabla_\tau = \tau^{-1}(\tau \llcorner \nabla)$,

$$\int_S (d\mathbf{S} \llcorner \nabla) f = \int_{\partial S} d\mathbf{s}\, f.$$

Here $d\mathbf{S}$ and $d\mathbf{s}$ are of dimension degree k and $k-1$, respectively.

Examples. 1. Consider a 2-dimensional surface S in $\mathbb{R}^3$, and a vector field $\mathbf{f}$ on S. Then the scalar part of Stokes' theorem says

$$\int_S (d\mathbf{S} \llcorner \nabla) \cdot \mathbf{f} = \int_{\partial S} d\mathbf{s} \cdot \mathbf{f}.$$

Use a vector measure $d\mathbf{A} = \mathbf{e}_{123}^{-1} d\mathbf{S}$, normal to the surface S, to write the left hand side as

$$\int_S (d\mathbf{S} \llcorner \nabla) \cdot \mathbf{f} = \int_S (d\mathbf{A} \times \nabla) \cdot \mathbf{f},$$

then use the interchange rule (of dot and cross) to get the usual *Stokes' theorem*

$$\int_S d\mathbf{A} \cdot (\nabla \times \mathbf{f}) = \int_{\partial S} d\mathbf{s} \cdot \mathbf{f},$$

where $d\mathbf{s}$ and $d\mathbf{A}$ form a right-hand system.
2. Consider a 2-dimensional surface $S \in \mathbb{R}^n$, with bounding line $C = \partial S$, and a circulation of vector field $\mathbf{f}$ around C. First, convert the line integral to a surface integral by Stokes' theorem,

$$\int_C d\mathbf{x} \cdot \mathbf{f} = \int_S (d\mathbf{S} \llcorner \nabla) \cdot \mathbf{f},$$

and then compare the homogeneous components of degree 0 to obtain

$$\int_S (d\mathbf{S} \llcorner \nabla) \cdot \mathbf{f} = \int_S d\mathbf{S} \llcorner (\nabla \wedge \mathbf{f}).$$

This shows that in a simply connected domain the circulation vanishes if the divergence vanishes. ∎

By convention ∇ differentiates only quantities to its right, unless otherwise indicated. Because of non-commutativity of multiplication, it is good to have a notation indicating differentiation both to the right and to the left, when desired. Accordingly, we have, for instance, the Leibniz rule,

$$\dot{g}\dot{\nabla}\dot{f} = g\dot{\nabla}\dot{f} + \dot{g}\dot{\nabla} f,$$

where the dots indicate where the differentiation is applied. Stokes' theorem is now generalized to the form

$$\int_{\partial S} g\, d\mathbf{s}\, f = \int_S g(d\mathbf{S} \llcorner \dot{\nabla})\dot{f} - (-1)^k \int_S \dot{g}(\dot{\nabla} \lrcorner d\mathbf{S}) f.$$

Here $\dim S = k \in \{1, 2, ..., n\}$. The minus sign on the last term comes from $\mathbf{a} \rfloor \mathbf{b} = -(-1)^k \mathbf{b} \llcorner \mathbf{a}$ for $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{b} \in \bigwedge^k \mathbb{R}^n$.

20.5 Positive and negative definite metrics

The monogenic homogeneous polynomials play a part in hypercomplex analysis similar to that of powers of a complex variable in the classical function theory. In constructing an explicit basis for the function space of monogenic polynomials it is customary to single out a special direction, say $\mathbf{e}_n$, in an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}, \mathbf{e}_n\}$ of the Euclidean space $\mathbb{R}^n$. The unit bivectors

$$\mathbf{i}_k = \mathbf{e}_n \mathbf{e}_k, \quad k = 1, 2, \ldots, n-1,$$

generate the even subalgebra $\mathcal{C}\ell_n^+$; they anticommute and square up to -1. Thus, they form an orthonormal basis $\{\mathbf{i}_1, \mathbf{i}_2, \ldots, \mathbf{i}_{n-1}\}$ of a negative definite quadratic space $\mathbb{R}^{0,n-1}$ generating a Clifford algebra $\mathcal{C}\ell_{0,n-1} \simeq \mathcal{C}\ell_n^+$. A closer contact with the classical function theory is obtained if a vector

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \ldots + x_{n-1} \mathbf{e}_{n-1} + x_n \mathbf{e}_n$$

in the Euclidean space $\mathbb{R}^n$ is replaced by a sum of a vector and a scalar, a paravector,

$$z = x_1 \mathbf{i}_1 + x_2 \mathbf{i}_2 + \ldots + x_{n-1} \mathbf{i}_{n-1} + y, \quad y = x_n,$$

in $\mathbb{R}^{0,n-1} \oplus \mathbb{R}$ [the special direction is the scalar/real part y, also denoted by $x_0 = x_n$]. By the above correspondence $z \leftrightarrow \mathbf{e}_n \mathbf{x}$, we have established a correspondence between the following two mappings:

$$\mathbb{R}^n \to \mathcal{C}\ell_n^+, \quad \mathbf{x} \to f(\mathbf{x}),$$
$$\mathbb{R} \oplus \mathbb{R}^{0,n-1} \to \mathcal{C}\ell_{0,n-1}, \quad z \to f(z);$$

both are denoted for convenience by f.

In the case of a Euclidean space $\mathbb{R}^n$, the Dirac operator is homogeneous,

$$\nabla = \mathbf{e}_1 \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \ldots + \mathbf{e}_{n-1} \frac{\partial}{\partial x_{n-1}} + \mathbf{e}_n \frac{\partial}{\partial x_n},$$

but it is replaced by a differential operator (inhomogeneous in the dimension degrees)

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}_1 \frac{\partial}{\partial x_1} + \mathbf{i}_2 \frac{\partial}{\partial x_2} + \ldots + \mathbf{i}_{n-1} \frac{\partial}{\partial x_{n-1}},$$

$D = \dfrac{\partial}{\partial x_0} + \nabla$, in the paravector space $\mathbb{R} \oplus \mathbb{R}^{0,n-1}$.

20.6 Cauchy's integral formula

Consider a region $S \subset \mathbb{R}^n$ of dimension n with boundary ∂S of dimension $n-1$ and multivector function $f : S \to C\ell_n$. In this case Stokes' theorem is of the form

$$\int_S d\mathbf{S}\,\nabla f = \int_{\partial S} d\mathbf{s}\, f.$$

If a multivector function f is monogenic, $\nabla f = 0$, then

$$\int_{\partial S} d\mathbf{s}\, f = 0 \quad \text{or equivalently} \quad \int_{\partial S} \partial\mathbf{s}\, f = 0,$$

which means that the 'stream' of a monogenic function across any closed hypersurface vanishes. This is *Cauchy's theorem*.

In the following we need the *Cauchy kernel*

$$\mathbf{q}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^n},$$

which is both left monogenic, $\nabla \mathbf{q} = 0$, and right monogenic, $\dot{\mathbf{q}}\dot{\nabla} = 0$, at $\mathbf{x} \neq 0$. Substitute the Cauchy kernel $g(\mathbf{x}) = \mathbf{q}(\mathbf{x}-\mathbf{a})$ and a left monogenic function $f(\mathbf{x})$, $\nabla f(\mathbf{x}) = 0$, into Stokes' theorem. On the right hand side the first term vanishes, and the second term can be evaluated by a limiting process. One obtains *Cauchy's integral formula*

$$\int_{\partial S} \mathbf{q}(\mathbf{x}-\mathbf{a})\, d\mathbf{s}\, f(\mathbf{x}) = -(-1)^n \mathbf{e}_{12\ldots n}\, n\omega_n f(\mathbf{a}),$$

where $\omega_n = \pi^{n/2}/(n/2)!$ and the sign in $-(-1)^n\mathbf{e}_{12\ldots n}$ comes from the choice of orientation $\tau_S = \mathbf{e}_{12\ldots n}$ for S.

Example. In the special case $n = 2$ the above formula is

$$\int_{\partial S} \frac{\mathbf{x}-\mathbf{a}}{|\mathbf{x}-\mathbf{a}|^2}\, d\mathbf{s}\, f(\mathbf{x}) = -\mathbf{e}_{12} 2\pi f(\mathbf{a}),$$

since according to our convention $\mathbf{e}_{12}$ is compatible with the clockwise orientation. The classical formula corresponds to both the special cases $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to C\ell_2^+$. [A better matching with the classical case would be obtained by mappings $f(x + y\mathbf{e}_1) = u + v\mathbf{e}_1$, where $\mathbf{e}_1^2 = -1$, in the Clifford algebra $C\ell_{0,1} \simeq \mathbb{C}$ of $\mathbb{R}^{0,1}$.] ∎

Cauchy's integral formula can also be written in the form

$$\int_{\partial S} \mathbf{q}(\mathbf{x}-\mathbf{a})\, \partial\mathbf{s}\, f(\mathbf{x}) = n\omega_n f(\mathbf{a}) \quad \text{for} \quad f : \mathbb{R}^n \to C\ell_n^+,$$

and in the form

$$\int_{\partial S} q(z-a)\,\partial s\, f(z) = n\omega_n f(a) \quad \text{for} \quad f:\mathbb{R}\oplus\mathbb{R}^{0,n-1} \to \mathcal{C}\ell_{0,n-1},$$

where the Cauchy kernel is

$$q(z) = \frac{\bar{z}}{|z|^n} = \frac{z^{-1}|z|^2}{|z|^n}.$$

The paravector $z \in \mathbb{R}\oplus\mathbb{R}^{0,n-1}$ has a norm $|z| = \sqrt{z\bar{z}}$, and an arbitrary element $u \in \mathcal{C}\ell_{0,n-1}$ has a norm given by $|u|^2 = \langle u\bar{u}\rangle_0$.

As in the classical case, we conclude that in a simply connected domain the values of a left monogenic function are determined by its values on the boundary.

20.7 Monogenic homogeneous functions

A function $f:\mathbb{R}^n \to \mathcal{C}\ell_n$ homogeneous of degree ℓ satisfies

$$f(\lambda\mathbf{x}) = \lambda^\ell f(\mathbf{x}) \quad \text{for} \quad \lambda \in \mathbb{R},$$

which implies Euler's formula

$$r\frac{\partial}{\partial r} f(\mathbf{x}) = \ell f(\mathbf{x}), \quad \text{where} \quad r = |\mathbf{x}|.$$

If a multivector function $f(\mathbf{x})$ is monogenic, $\nabla f(\mathbf{x}) = 0$, then also $\mathbf{x}\nabla f(\mathbf{x}) = (\mathbf{x}\cdot\nabla + \mathbf{x}\wedge\nabla)f(\mathbf{x}) = 0$, and so

$$(r\frac{\partial}{\partial r} + \mathbf{L})f(\mathbf{x}) = 0$$

where

$$r\frac{\partial}{\partial r} f(\mathbf{x}) = \mathbf{x}\cdot\nabla = \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}, \quad r = |\mathbf{x}|,$$

and

$$\mathbf{L} = \mathbf{x}\wedge\nabla = \sum_{i<j} \mathbf{e}_{ij}\Big(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\Big).$$

If a multivector function f is monogenic, $\nabla f(\mathbf{x}) = 0$, and homogeneous of degree ℓ, that is $r\frac{\partial}{\partial r} f = \ell f$, then $\mathbf{L}f = \kappa f$, where $\kappa = -\ell$. If a multivector function $f:\mathbb{R}^n \to \mathcal{C}\ell_n$ is harmonic, $\nabla^2 f = 0$, and homogeneous of degree ℓ, that is $r\dfrac{\partial}{\partial r}f = \ell f$, then $\mathbf{L}f = \kappa f$, where

$$\begin{aligned} &\kappa = \ell + n - 2 && \text{spin up} \\ &\kappa = -\ell && \text{spin down.} \end{aligned}$$

This can be seen by writing the Laplace operator ∇^2 as the square of the Dirac operator $\nabla = \mathbf{x}^{-1}(r\dfrac{\partial}{\partial r} + \mathbf{L})$ and factoring it by the relations

$$r\frac{\partial}{\partial r}\mathbf{x} - \mathbf{x}\, r\frac{\partial}{\partial r} = \mathbf{x}$$
$$\mathbf{L}\mathbf{x} + \mathbf{x}\mathbf{L} = (n-1)\mathbf{x}$$

as follows:

$$\nabla^2 = \mathbf{x}^{-2}\Big(r\frac{\partial}{\partial r} + n - 2 - \mathbf{L}\Big)\Big(r\frac{\partial}{\partial r} + \mathbf{L}\Big).$$

If a multivector function $f(\mathbf{x})$ is monogenic, $\nabla f(\mathbf{x}) = 0$, then $h(\mathbf{x}) = \mathbf{x}f(\mathbf{x})$ is harmonic, $\nabla^2 h(\mathbf{x}) = 0$.

If a multivector function $f : \mathbb{R}^n \to \mathcal{C}\ell_n$ is monogenic at $\mathbf{x} \neq 0$, that is $(r\dfrac{\partial}{\partial r} + \mathbf{L})f(\mathbf{x}) = 0$, then $f(\mathbf{x}^{-1})$ satisfies $(-r\dfrac{\partial}{\partial r} + \mathbf{L})f(\mathbf{x}) = 0$, and the function

$$g(\mathbf{x}) = \mathbf{q}(\mathbf{x})f(\mathbf{x}^{-1}), \quad \mathbf{q}(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|^n},$$

is monogenic,

$$\Big(r\frac{\partial}{\partial r} + \mathbf{L}\Big)g(\mathbf{x}) = 0.$$

The plane case. Consider conformal mappings sending $\mathbf{x} \in \mathbb{R}^2$ to $f(\mathbf{x}) \in \mathbb{R}^2$. For sense-preserving conformal mappings $(-r\dfrac{\partial}{\partial r} + \mathbf{L})f = 0$, and for sense-reversing conformal mappings $(r\dfrac{\partial}{\partial r} + \mathbf{L})f = 0$. ∎

Example. Using the vector identity $\mathbf{a} \rfloor (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - \mathbf{b}(\mathbf{a}\cdot\mathbf{c})$ we find that

$$\begin{aligned} \nabla \rfloor (\mathbf{x} \wedge \mathbf{f}) &= (\nabla\cdot\mathbf{x})\mathbf{f} - \mathbf{x}(\nabla\cdot\mathbf{f}) + (\mathbf{x}\cdot\dot{\nabla})\dot{\mathbf{f}} - \dot{\mathbf{x}}(\dot{\nabla}\cdot\mathbf{f}) \\ &= n\mathbf{f} - \mathbf{x}(\nabla\cdot\mathbf{f}) + r\frac{\partial}{\partial r}\mathbf{f} - \mathbf{f} \end{aligned}$$

for a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$.

If the vector field $\mathbf{f}$ is sourceless, $\nabla\cdot\mathbf{f} = 0$, and homogeneous of degree ℓ, that is $r\dfrac{\partial}{\partial r}\mathbf{f} = \ell\mathbf{f}$, then

$$\nabla \rfloor (\mathbf{x} \wedge \mathbf{f}) = (\ell + n - 1)\mathbf{f},$$

which shows that

$$\mathbf{v} = \frac{1}{\ell + n - 1} \mathbf{x} \wedge \mathbf{f}, \qquad \ell \neq -(n-1),$$

is a bivector potential for $\mathbf{f}$.

If the vector field $\mathbf{f}$ is irrotational, $\nabla \wedge \mathbf{f} = 0$, and homogeneous of degree ℓ, that is $r \dfrac{\partial}{\partial r} \mathbf{f} = \ell \mathbf{f}$, then

$$\nabla(\mathbf{x} \cdot \mathbf{f}) = (\ell + 1)\mathbf{f},$$

which shows that

$$u = -\frac{1}{\ell + 1} \mathbf{x} \wedge \mathbf{f}, \qquad \ell \neq -1,$$

is a scalar potential for $\mathbf{f}$. ∎

20.8 A basis for monogenic homogeneous polynomials

A monogenic function is real analytic, that is, a power series of the components of the argument. The homogeneous part of degree 1 of the Taylor series expansion

$$f'(a)z = x_1 \frac{\partial f}{\partial x_1}\bigg|_a + x_2 \frac{\partial f}{\partial x_2}\bigg|_a + \ldots + x_{n-1} \frac{\partial f}{\partial x_{n-1}}\bigg|_a + y \frac{\partial f}{\partial y}\bigg|_a$$

can be written, in virtue of a monogenic f, as follows:

$$f'(a)z = (x_1 - y\mathbf{i}_1) \frac{\partial f}{\partial x_1}\bigg|_a + (x_2 - y\mathbf{i}_2) \frac{\partial f}{\partial x_2}\bigg|_a + \ldots + (x_{n-1} - y\mathbf{i}_{n-1}) \frac{\partial f}{\partial x_{n-1}}\bigg|_a.$$

The functions

$$z_1 = x_1 - y\mathbf{i}_1,\ z_2 = x_2 - y\mathbf{i}_2,\ \ldots,\ z_{n-1} = x_{n-1} - y\mathbf{i}_{n-1}$$

are monogenic; note that $z_k = x_k + y\mathbf{e}_k\mathbf{e}_n$, $k = 1, 2, \ldots, n-1$. Write $\underline{l} = (l_1, l_2, \ldots, l_{n-1})$, and define the symmetrized polynomials

$$p_{\underline{l}}(z) = \frac{1}{l!} \sum_{\pi \in S_l} z_{\pi(1)} z_{\pi(2)} \cdots z_{\pi(l)},$$

each term being homogeneous of degree l_k with respect to z_k, and homogeneous of degree $l = l_1 + l_2 + \ldots + l_{n-1}$ with respect to z. The symmetric polynomials are monogenic, $Dp_{\underline{l}}(z) = 0$; they appear as multipliers of partial derivatives

$$\partial_{\underline{l}} f|_a = \frac{\partial^l f}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_{n-1}^{l_{n-1}}}\bigg|_a$$

in the Taylor series expansion of $f(a + z)$.

Examples. 1. For $\underline{l} = (2, 1, 0, 0, \ldots, 0)$ we have

$$\begin{aligned} p_{\underline{l}}(z) &= \tfrac{1}{3}(z_1^2 z_2 + z_1 z_2 z_1 + z_2 z_1^2) \\ &= (x_1^2 - y^2)x_2 - 2x_1 x_2 y \mathbf{i}_1 - (x_1^2 - \tfrac{y^2}{3}) y \mathbf{i}_2 \\ &= (x_1^2 - y^2)x_2 + 2x_1 x_2 y \mathbf{e}_1 \mathbf{e}_n + (x_1^2 - \tfrac{y^2}{3}) y \mathbf{e}_2 \mathbf{e}_n. \end{aligned}$$

2. The functions $q_{\underline{l}}(z) = \partial_{\underline{l}} q(z)$ are homogeneous of degree $-(l + n - 1)$ and monogenic when $z \neq 0$. ∎

For all $z \in \mathbb{R} \oplus \mathbb{R}^{0,n-1}$, we also have $p_{\underline{l}}(z) \in \mathbb{R} \oplus \mathbb{R}^{0,n-1}$.

The monogenic polynomials homogeneous of degree l span a right module over the ring $\mathcal{C}\ell_{0,n-1}$; the polynomials $p_{\underline{l}}(z)$ form a basis of this module of dimension $\binom{l+n-2}{l}$. Harmonic polynomials homogeneous of degree l form a module over the rotation group $SO(n)$ of $\mathbb{R} \oplus \mathbb{R}^{0,n-1}$, namely the irreducible module of traceless symmetric tensors of degree l, the dimension of this module being

$$\binom{l+n-1}{l} - \binom{l+n-3}{l-2} = \binom{l+n-2}{l} + \binom{l+n-3}{l-1}.$$

The Laurent series expansion is formulated as follows:

$$f(z) = \sum_{l=0}^{\infty} \sum_{\underline{l}} \Big[p_{\underline{l}}(z-a) b_{\underline{l}} + q_{\underline{l}}(z-a) c_{\underline{l}} \Big],$$

where

$$b_{\underline{l}} = \frac{1}{n\omega_n} \int_{\partial S} q_{\underline{l}}(z-a)\, \partial s\, f(z),$$
$$c_{\underline{l}} = \frac{1}{n\omega_n} \int_{\partial S} p_{\underline{l}}(z-a)\, \partial s\, f(z),$$

when $f(z)$ is monogenic in a region $S \subset \mathbb{R} \oplus \mathbb{R}^{0,n-1}$ except at $a \in S$.

Axial Vector Fields

Single out a distinguished direction or *axis* $\mathbf{a}$ in $\mathbb{R}^n$, say

$$\mathbf{a} = \mathbf{e}_n.$$

Write $y = x_n$ and $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_{n-1}\mathbf{e}_{n-1}$ so that

$$\mathbf{r} = \mathbf{x} + y\mathbf{a} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_{n-1}\mathbf{e}_{n-1} + x_n\mathbf{e}_n.$$

Write $x = |\mathbf{x}|$ and let

$$\mathbf{i} = \frac{\mathbf{r} \wedge \mathbf{a}}{|\mathbf{r} \wedge \mathbf{a}|} = \frac{\mathbf{xa}}{x}$$

be the unit bivector of the plane determined by $\mathbf{a}$ and $\mathbf{r}$. Since $\nabla(\mathbf{ra}) = (\nabla\mathbf{r})\mathbf{a} = n\mathbf{a}$ and $\nabla(\mathbf{ar}) = -(n-2)\mathbf{a}$, it is evident that the polynomial

$$w = \frac{1}{2}[n\mathbf{ar} + (n-2)\mathbf{ra}] = (n-1)\mathbf{a}\cdot\mathbf{r} + \mathbf{a}\wedge\mathbf{r} = (n-1)y - x\mathbf{i}$$

satisfies the equation $\nabla w = 0$; such a w is called *monogenic*. The polynomial w is a complex of its scalar part $u = (n-1)\mathbf{a}\cdot\mathbf{r} = (n-1)y$ and its bivector part $\mathbf{v} = \mathbf{a}\wedge\mathbf{r} = -x\mathbf{i}$. The vector field

$$\mathbf{f} = -(n-1)\mathbf{a}$$

is irrotational, that is $\nabla\wedge\mathbf{f} = 0$, with a (scalar) potential $u = (n-1)y$, such that $\mathbf{f} = -\nabla u$, and also sourceless, that is $\nabla\cdot\mathbf{f} = 0$, with a *stream function* $\psi = -x^{n-1}$ and a *bivector potential* $\mathbf{v} = -x\mathbf{i}$,such that $\mathbf{f} = \nabla\lrcorner\mathbf{v}$ and $\nabla\wedge\mathbf{v} = 0$. In fact, $\mathbf{f}$ is monogenic, $\nabla\mathbf{f} = \nabla\cdot\mathbf{f} + \nabla\wedge\mathbf{f} = 0$, with a complex potential $w = u + v\mathbf{i} = (n-1)y - x\mathbf{i}$, which is also monogenic, $\nabla w = 0$.

The vector function $\mathbf{q}(\mathbf{r}) = \mathbf{r}/r^n$, where $r = |\mathbf{r}|$, is called the *Cauchy kernel*, and

$$\mathbf{q}_1(\mathbf{r}) = \nabla[\mathbf{a}\cdot\mathbf{q}(\mathbf{r})] = \frac{1}{r^n}(\mathbf{a} - n(\mathbf{a}\cdot\mathbf{r})\mathbf{r}^{-1})$$

is the field of an n-dimensional *dipole*. These vector fields can be used to reproduce the complex potential

$$w = -\mathbf{q}(\mathbf{r})\mathbf{q}_1(\mathbf{r}^{-1})$$

of the vector field $\mathbf{f} = -(n-1)\mathbf{a}$.

Axial monogenic polynomials of degree 2. In the above we just found that

$$\begin{aligned}\nabla(\mathbf{ar}) &= -(n-2)\mathbf{a}\\ \nabla\mathbf{ra} &= n\mathbf{a}.\end{aligned}$$

Next, we differentiate the second powers of $\mathbf{r}$. The product $\mathbf{ara} = (\mathbf{ar}+\mathbf{ra})\mathbf{a} - \mathbf{a}^2\mathbf{r} = 2(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - a^2\mathbf{r}$ is a vector in the plane determined by $\mathbf{a}$ and $\mathbf{r}$. So

$$\begin{aligned}(\mathbf{ar})^2 &= 2(\mathbf{a}\cdot\mathbf{r})\mathbf{ar} - a^2r^2\\ (\mathbf{ra})^2 &= 2(\mathbf{a}\cdot\mathbf{r})\mathbf{ra} - a^2r^2\end{aligned}$$

and $(\mathbf{ar})(\mathbf{ra}) = a^2r^2$. Since

$$\begin{aligned}\nabla(\mathbf{ar})^2 &= -2(n-2)(\mathbf{a}\cdot\mathbf{r})\mathbf{a}\\ \nabla(\mathbf{ar})(\mathbf{ra}) &= 2a^2\mathbf{r}\\ \nabla(\mathbf{ra})^2 &= 2(n+2)(\mathbf{a}\cdot\mathbf{r})\mathbf{a} - 4a^2\mathbf{r}\end{aligned}$$

it is evident that the polynomial

$$\begin{aligned} p_2(\mathbf{r}) &= \tfrac{n}{4}[(n+2)(\mathbf{ar})^2 + 2(n-2)(\mathbf{ar})(\mathbf{ra}) + (n-2)(\mathbf{ra})^2] \\ &= \tfrac{n}{2}[n(\mathbf{a}\cdot\mathbf{r})^2 - a^2r^2 + 2(\mathbf{a}\cdot\mathbf{r})(\mathbf{a}\wedge\mathbf{r})] \end{aligned}$$

is monogenic, $\nabla p_2(\mathbf{r}) = 0$. In cylindrical coordinates $y, \mathbf{x}$, the polynomial

$$p_2(\mathbf{r}) = \frac{n}{2}[(n-1)y^2 - x^2 - 2yx\mathbf{i}]$$

is a complex potential of the monogenic vector field $\mathbf{f} = n(\mathbf{x} - (n-1)y\mathbf{a})$ which has a stream function $\psi = -nyx^{n-1}$.

20.9 Axial monogenic polynomials of homogeneous degree

In the following, monogenic polynomials, homogeneous of degree l in the factors $\mathbf{ar}$ and $\mathbf{ra}$ (and then also in $\mathbf{r}$ or in y and $\mathbf{x}$), will be introduced. First,

$$\begin{aligned} \nabla(\mathbf{ar})^l = -(n-2)\mathbf{a}[(\mathbf{ar})^{l-1} + (\mathbf{ar})^{l-2}(\mathbf{ra}) + \\ \cdots + (\mathbf{ar})(\mathbf{ra})^{l-2} + (\mathbf{ra})^{l-1}], \end{aligned} \tag{3}$$

$$\begin{aligned} \nabla(\mathbf{ra})^l = (n-2)\mathbf{a}[(\mathbf{ar})^{l-1} + (\mathbf{ar})^{l-2}(\mathbf{ra}) + \\ \cdots + (\mathbf{ar})(\mathbf{ra})^{l-2} + (\mathbf{ra})^{l-1}] + 2l\mathbf{a}(\mathbf{ra})^{l-1}. \end{aligned} \tag{4}$$

To verify these, observe that $\mathbf{e}_1\mathbf{f}\mathbf{e}_1 + \cdots + \mathbf{e}_n\mathbf{f}\mathbf{e}_n = -(n-2)\mathbf{f}$ for any vector $\mathbf{f}$. Then

$$\begin{aligned} \sum_{i=1}^{n} \mathbf{e}_i(\mathbf{ar})^{j-1}\mathbf{a}\mathbf{e}_i(\mathbf{ar})^{l-j} &= -(n-2)(\mathbf{ar})^{j-1}\mathbf{a}(\mathbf{ar})^{l-j} \\ &= -(n-2)\mathbf{a}(\mathbf{ra})^{j-1}(\mathbf{ar})^{l-j} \\ &= -(n-2)\mathbf{a}(\mathbf{ar})^{l-j}(\mathbf{ra})^{j-1} \end{aligned}$$

because $(\mathbf{ar})^{j-1}\mathbf{a}$ is a vector in the plane determined by $\mathbf{a}$ and $\mathbf{r}$, as can be seen by inspection on $(\mathbf{ra})^j\mathbf{r} = \mathbf{r}((\mathbf{ar})^{j-1}\mathbf{a})\mathbf{r}$ and induction on j. The equation (3) is now proved. Similarly, observe that $\sum_{i=1}^{n}\mathbf{e}_i\mathbf{u}\mathbf{e}_i = (-1)^l(n-2l)\mathbf{u}$ for any element $\mathbf{u}$, homogeneous of grade l. Then

$$\begin{aligned} \sum_{i=1}^{n}\mathbf{e}_i(\mathbf{ra})^j\mathbf{e}_i &= \sum_{i=1}^{n}\tfrac{1}{2}\mathbf{e}_i[((\mathbf{ra})^j + (\mathbf{ar})^j) + ((\mathbf{ra})^j - (\mathbf{ar})^j)]\mathbf{e}_i \\ &= \tfrac{n}{2}[(\mathbf{ra})^j + (\mathbf{ar})^j] + \tfrac{n-4}{2}[(\mathbf{ra})^j - (\mathbf{ar})^j] \\ &= (n-2)(\mathbf{ra})^j + 2(\mathbf{ar}^j), \end{aligned}$$

because $(\mathbf{ra})^j + (\mathbf{ar})^j$ is a scalar of grade 0 and $(\mathbf{ra})^j - (\mathbf{ar})^j$ is a bivector of grade 2. It follows that

$$\sum_{i=1}^{n} \mathbf{e}_i(\mathbf{ra})^j \mathbf{e}_i \mathbf{a} = (n-2)\mathbf{a}(\mathbf{ar})^j + 2\mathbf{a}(\mathbf{ra})^j,$$

which multiplied by $(\mathbf{ra})^{l-j-1}$ on the right gives the equation (4) after summing up the terms $j = 0, 1, \cdots, l-1$.

To calculate $\nabla(\mathbf{ar})^{l-j}(\mathbf{ra})^j$ note that $(\mathbf{ar})^{l-j}(\mathbf{ra})^j = r^{2j}a^{2j}(\mathbf{ar})^{l-2j}$ when $l \geq 2j$ and $(\mathbf{ar})^{l-j}(\mathbf{ra})^j = r^{2(l-j)}(\mathbf{ra})^{2j-1}$ when $l \leq 2j$. Use results (3), (4) and

$$\nabla \mathbf{r}^l = \begin{cases} l\mathbf{r}^{l-1} & \text{for} \quad l \text{ even} \\ (l+n-1)r^{l-1} & \text{for} \quad l \text{ odd.} \end{cases}$$

Then

$$\nabla(\mathbf{ar})^{l-j}(\mathbf{ra})^j = \sum_{k=1}^{l} m_{jk}\mathbf{a}(\mathbf{ar})^{l-k}(\mathbf{ra})^k$$

where the $(l+1) \times l$-matrix m_{jk} is

$$\begin{pmatrix}
-(n-2) & -(n-2) & -(n-2) \cdots & -(n-2) & -(n-2) & -(n-2) \\
2 & -(n-2) & -(n-2) \cdots & -(n-2) & -(n-2) & 0 \\
0 & 4 & -(n-2) \cdots & -(n-2) & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots \quad \vdots & \vdots & \vdots & \vdots \\
0 & 0 & (n-2) \cdots & (n+2(l-3)) & 0 & 0 \\
0 & (n-2) & (n-2) \cdots & (n-2) & (n+2(l-2)) & 0 \\
(n-2) & (n-2) & (n-2) \cdots & (n-2) & (n-2) & (n+2(l-1))
\end{pmatrix}$$

To get the coefficients $p_{l,j}$ in

$$p_l(\mathbf{r}) = \sum_{j=0}^{l} p_{l,j}(\mathbf{ar})^{l-j}(\mathbf{ra})^j$$

such that $\nabla p_l(\mathbf{r}) = 0$, multiply the rows of m_{jk} by the corresponding coefficients $p_{l,j}$ and determine $p_{l,j}$ so that the sum of the resulting elements in each column is zero. To calculate the coefficients $p_{l,j}$ one has useful algorithms such as [1]

$$p_{l,j} = \frac{1}{l}\left(\frac{n-2}{2} + l - j\right)(p_{l-1,j} + p_{l-1,l-j})$$

1 It is worth noting that the formula (5) for the coefficients $p_{l,j}$ is valid for all signatures and not only for positive definite quadratic forms.

which gives

$$p_{l,j} = \binom{\frac{n-2}{2}+l-j}{l-j}\binom{\frac{n-4}{2}+j}{j}. \tag{5}$$

Example. Let $n = 3$ and $l = 2$. Then

$$\begin{array}{l|l} \nabla(\mathbf{ar})^2 = -\mathbf{a}(\mathbf{ar}) - \mathbf{a}(\mathbf{ra}) & 5 \\ \nabla(\mathbf{arra}) = 2\mathbf{a}(\mathbf{ar}) & 2 \\ \nabla(\mathbf{ra})^2 = \mathbf{a}(\mathbf{ar}) + 5\mathbf{a}(\mathbf{ra}) & 1 \end{array}$$

where the right-hand sides of the identities give the matrix

$$\begin{pmatrix} -1 & -1 \\ 2 & 0 \\ 1 & 5 \end{pmatrix}$$

and the right column gives

$$p_2(\mathbf{r}) = 5(\mathbf{ar})^2 + 2(\mathbf{arra}) + (\mathbf{ra})^2$$

with coefficients $p_{2,0} = 5$, $p_{2,1} = 2$ and $p_{2,2} = 1$. ∎

In some cases it is worth knowing the smallest integer coefficients, which in a few lower-dimensional cases are as follows:

	$n=3$					$n=4$					$n=5$				
$l \setminus j$	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
1	3	1				2	1				5	3			
2	5	2	1			3	2	1			7	6	3		
3	35	15	9	5		4	3	2	1		21	21	15	7	
4	63	28	18	12	7	5	4	3	2	1	33	36	30	20	9

20.10 Differential equations in cylindrical coordinates

Consider an axially symmetric vector field $\mathbf{f}$ in the cylindrical coordinates y, $\mathbf{x}$. Write $\mathbf{r} = \mathbf{x} + y\mathbf{a}$, $\nabla = \nabla_{\mathbf{x}} + \mathbf{a}(\partial/\partial y)$ and $\mathbf{f} = \mathbf{g} + h\mathbf{a}$ with $\mathbf{g} = (\mathbf{x}/x)g$. Then

$$\nabla \cdot \mathbf{f} = \nabla_{\mathbf{x}} \cdot \mathbf{g} + \frac{\partial h}{\partial y} = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} + (n-2)\frac{g}{x}$$

$$\nabla \wedge \mathbf{f} = \frac{\mathbf{x}}{x}\frac{\partial h}{\partial x}\mathbf{a} + \mathbf{a}\frac{\partial \mathbf{g}}{\partial y} = \mathbf{i}\Big(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\Big).$$

If now $\mathbf{f}$ is monogenic, $\nabla\mathbf{f} = 0$, then there is a complex potential $w = u + v\mathbf{i} = u + \mathbf{v}$ such that $\mathbf{f} = -\nabla u = \nabla \rfloor \mathbf{v}$, $\nabla \wedge \mathbf{v} = 0$. The condition to be monogenic,

$\nabla w = 0$, means that

$$\left(\nabla_{\mathbf{x}} + \mathbf{a}\frac{\partial}{\partial y}\right)\left(u + v\frac{\mathbf{xa}}{x}\right)$$
$$= \frac{\mathbf{x}}{x}\frac{\partial u}{\partial x} + \frac{\mathbf{x}}{x}\frac{\partial v}{\partial x}\frac{\mathbf{xa}}{x} + v(n-2)\frac{\mathbf{a}}{x} + \mathbf{a}\frac{\partial u}{\partial y} + \mathbf{a}\frac{\partial v}{\partial y}\frac{\mathbf{xa}}{x} = 0,$$

which decomposed gives an n-dimensional analog of the Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} + (n-2)\dfrac{v}{x} = 0. \end{cases} \tag{6}$$

The components of $\mathbf{f} = \mathbf{g} + h\mathbf{a}$ are then expressed as

$$\begin{cases} g = -\dfrac{\partial u}{\partial x} = -\dfrac{\partial v}{\partial y} \\ h = -\dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x} + (n-2)\dfrac{v}{x}. \end{cases}$$

Of course, u is harmonic,

$$\nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{n-2}{x}\frac{\partial}{\partial x}\right) u = 0,$$

and also $\mathbf{v}$ is harmonic,

$$\nabla^2 \mathbf{v} = \nabla^2(v\mathbf{i}) = \mathbf{i}\left\{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{n-2}{x}\left(\frac{\partial v}{\partial x} - \frac{v}{x}\right)\right\} = 0,$$

and so

$$\nabla^2 v = \frac{n-2}{x^2} v.$$

As an axially symmetric and sourceless vector field $\mathbf{f}$ has a *stream function* $\psi = x^{n-2}v$, satisfying

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{n-2}{x}\frac{\partial}{\partial x}\right)\psi = 0,$$

such that

$$\frac{\partial \psi}{\partial x} = x^{n-2}h, \quad \frac{\partial \psi}{\partial y} = -x^{n-2}g.$$

The stream function can be expressed by a line integral

$$\psi - \psi_0 = \int_{P_0}^{P} x^{n-2}(g\,dy - h\,dx)$$

independent of path in a fixed plane containing the symmetry axis $\mathbf{a}$. Let the path of integration sweep around the symmetry axis $\mathbf{a}$ and form an axially symmetric hypersurface S. The stream

$$\Psi = \int_S d\mathbf{S} \wedge \mathbf{f}$$

across S is $\Psi = \mathbf{e}_{12\ldots n}(n-1)\omega_{n-1}(\psi - \psi_0)$ where $(n-1)\omega_{n-1}$ is the measure of the unit sphere S^{n-2} in the $(n-1)$-dimensional space (orthogonal to the axis $\mathbf{a} \in \mathbb{R}^n$).

The n-dimensional Cauchy-Riemann equations (6), by a change of variables $x = r\sin\theta$, $y = r\cos\theta$, become

$$\begin{cases} \dfrac{\partial u}{\partial r} + \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + (n-2)\cot\theta\dfrac{v}{r} = 0 \\ \dfrac{\partial v}{\partial r} - \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} + (n-2)\dfrac{v}{r} = 0. \end{cases}$$

If the complex potential $w = u + v\mathbf{i}$ is homogeneous of degree l, then

$$\begin{cases} \dfrac{\partial v}{\partial \theta} + (n-2)\cot\theta v + lu = 0 \\ \dfrac{\partial u}{\partial \theta} = (l+n-2)v. \end{cases}$$

Differentiation with respect to θ and a further change of variable $\mu = \cos\theta$ then result in

$$(1-\mu^2)\frac{d^2v}{d\mu^2} - (n-1)\mu\frac{dv}{d\mu} + \left\{l(l+n-2) - \frac{n-2}{1-\mu^2}\right\}v = 0$$

and

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - (n-1)\mu\frac{du}{d\mu} + l(l+n-2)u = 0.$$

The solutions u/r^l of this last equation can be expressed as hypergeometric series or ultraspherical (Gegenbauer) functions

$${}_2F_1(-l, l+n-2, \frac{n-1}{2}; \frac{1-\mu}{2}) = \frac{l!(n-3)!}{(l+n-3)!}C_l^{((n-2)/2)}(\mu)$$

and $v = [\sqrt{1-\mu^2}/(l+n-2)](du/d\mu)$. The previously introduced monogenic polynomials $p_l(\mathbf{r})$, homogeneous of degree l, are now $p_l(\mathbf{r}) = u + v\mathbf{i}$.

In some lower-dimensional cases the scalar part u of $p_l(\mathbf{r})$ divided by r^l is

n	u/r^l	
2	$\cos l\theta = T_l(\cos\theta)$	Chebyshev
3	$P_l(\cos\theta)$	Legendre
4	$(\frac{l}{2}+1)(\cos l\theta + \cot\theta \sin l\theta) = (\frac{l}{2}+1)\dfrac{\sin(l+1)\theta}{\sin\theta}$	

20.11 Inversion of multipoles in unit sphere

If a function $f(\mathbf{r})$ with values in the Clifford algebra is monogenic, then the function $\mathbf{q}(\mathbf{r})f(\mathbf{r}^{-1})$, obtained by inversion, is also monogenic for $\mathbf{r} \neq 0$. The vector field

$$\mathbf{q}_l(\mathbf{r}) = \frac{\partial^l}{\partial y^l}\mathbf{q}(\mathbf{r})$$

is axially symmetric, monogenic and homogeneous of degree $-(l+n-1)$, and it describes an axial multipole of order 2^l. The previously introduced monogenic complex polynomials are obtained by inversion in the unit sphere,

$$p_l(\mathbf{r}) = \frac{(-1)^l}{l\,!}\mathbf{q}(\mathbf{r})\mathbf{q}_l(\mathbf{r}^{-1}).$$

These axially symmetric monogenic complex polynomials $p_l(\mathbf{r})$ should be distinguished from the monogenic complex polynomials introduced by Haefeli, who defined symmetrized products of the functions $z_i = x_i + y\mathbf{e}_i\mathbf{e}_n$, where $i = 1, 2, \ldots, n-1$.

Example. The polynomial $\frac{1}{3}(z_1^2 z_2 + z_1 z_2 z_1 + z_2 z_1^2) = x_2(x_1^2 - y^2) + 2x_1x_2y\mathbf{e}_1\mathbf{e}_n + y(x_1^2 - (y^2/3))\mathbf{e}_2\mathbf{e}_n$ is such a monogenic symmetrized product. ∎

Haefeli's monogenic symmetrized products form a basis of the right module (over the Clifford algebra) which consists of monogenic polynomials.

Example. The axially symmetric monogenic polynomials, homogeneous of degree 1 and 2, can be expressed in this basis in the forms

$$p_1(\mathbf{r}) = -\sum_{i=1}^{n-1} z_i\mathbf{e}_i\mathbf{e}_n \quad \text{and} \quad p_2(\mathbf{r}) = -\frac{n}{2}\sum_{i=1}^{n-1} z_i^2$$

respectively. ∎

Finally, the complex polynomial $p_l(\mathbf{r})$ is such that its bivector part determines the plane spanned by $\mathbf{a}$ and $\mathbf{r} = \mathbf{x} + y\mathbf{a}$. So the function $\mathbf{r}p_l(\mathbf{r})$ is vector valued in this same plane. Since the complex function $p_l(\mathbf{r})$ is monogenic,

$\nabla p_l(\mathbf{r}) = 0$, the vector function $\mathbf{r}p_l(\mathbf{r})$ is harmonic, $\nabla^2 \mathbf{r}p_l(\mathbf{r}) = 0$. This can be seen from

$$\mathbf{r}^2\nabla^2 = \left(r\frac{\partial}{\partial r} + \frac{n-2}{2}\right)^2 - \left(\mathbf{L} - \frac{n-2}{2}\right)^2,$$

where $\mathbf{r}\nabla = \mathbf{r}\cdot\nabla + \mathbf{r}\wedge\nabla$, $r(\partial/\partial r) = \mathbf{r}\cdot\nabla$ and $\mathbf{L} = \mathbf{r}\wedge\nabla$, which has axially symmetric eigenfunctions $\mathbf{L}p_l(\mathbf{r}) = -lp_l(\mathbf{r})$ and $\mathbf{L}[\mathbf{r}p_l(\mathbf{r})] = (l+n-1)\mathbf{r}p_l(\mathbf{r}.)$

History and survey of research

Hypercomplex analysis attempts to generalize one-variable complex analysis to a higher-dimensional one-variable theory using Clifford algebras of Euclidean spaces. It was first examined by Moisil (in terms of integrals), and rediscovered in quaternion form by Fueter, who introduced the symmetrized polynomials. In quaternion analysis the central result was Cauchy's integral formula in dimension 4. The notion of monogenic functions with values in a Clifford algebra is due to Iftimie and Bosshard. Habetha showed that if an algebra gives rise to Cauchy's integral formula, then it is sufficient that it contains a linear subspace where all non-zero vectors are invertible in the algebra; that is, the algebra is almost a Clifford algebra.

Lounesto & Bergh 1983 initiated a study of axially symmetric functions with values in a Clifford algebra. The research was later taken over by Sommen.

Presently, there are several schools studying hypercomplex analysis with different emphasis: harmonic analysis (J. Ryan, J. Gilbert), functional analysis (R. Delanghe, F. Brackx), and function theory (K. Habetha, R. Gilbert).

Exercises

1. Show that $\nabla\mathbf{x} = n$, $\nabla\mathbf{x}^2 = 2\mathbf{x}$, $\nabla\mathbf{x}^3 = (n+2)\mathbf{x}^2$, $\nabla\mathbf{x}^4 = 4\mathbf{x}^3$.
2. Show that $\nabla\mathbf{x}^k = k\mathbf{x}^{k-1}$ for k even, and $\nabla\mathbf{x}^k = (n+k-1)\mathbf{x}^{k-1}$ for k odd.
3. Show that $\nabla\cos\mathbf{x} = -\sin\mathbf{x}$, $\nabla\sin\mathbf{x} = \cos\mathbf{x} + (n-1)\frac{1}{\mathbf{x}}\sin\mathbf{x}$.
4. Show that $\nabla\exp\mathbf{x} = \exp\mathbf{x} + (n-1)\frac{1}{\mathbf{x}}\sinh\mathbf{x}$,
$$\nabla\log(1+\mathbf{x}) = \frac{1}{1+\mathbf{x}} + (n-1)\frac{1}{\mathbf{x}}\arctan\mathbf{x}.$$

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