Final Report MAP592: Parking 4-sphere swimmer

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Abstract

This article is about the parking 4-sphere swimmer (SPr4). This is a low-Reynolds number swimmer composed of four balls of equal radii. The four balls can move along the four axes passing through the four vertices of a tetrahedron and its midpoint. The balls do not rotate around their axes such that the shape of the swimmer is characterized by the length of the four arms, measured from the midpoint to the center of each ball. Yet, the swimmer may rotate freely around its center of mass. The governing dynamical system is presented and its geometric structure is displayed. Then it is shown that, in the first order range of small strokes, optimal periodic strokes for planar displacements with an additional rotation about the axis orthogonal to the plane of movement are ellipses embedded in $4d$ space, i.e. closed curves of the form $t \in [0, 2\pi] \mapsto (\cos t)a + (\sin t)b$ for suitable vectors $a, b \in \mathbb{R}^4$. A simple analytic expression for the vectors a and b is derived. Eventually, a conjecture about the general case is made.

1. Introduction

In his novel paper [13], Purcell treats for the first time the issue of swimming on a microscopic level and the principal problems linked to it. He especially illustrates, why any micro-organism trying to swim using a reciprocal movement like the one of a scallop, i.e. swimming by opening and closing a valve, cannot move. This observation, also known as the *scallop theorem*[1] entails the problem of finding the simplest swimming mechanism at microscopic scales; that is, the capacity to advance using a periodic change of shape - a swimming *stroke* - in the absence of external forces. A variety of such mechanisms has already been proposed and analyzed, see e.g. [2], [12], [13].

The principal mathematical challenge of this problem stems from the low value of the Reynolds number $\mathrm{Re} = \rho u L/\mu$ which gives an estimate of the relative importance of inertial to viscous forces for an object of characteristic length scale L moving at speed u through a Newtonian fluid of density ρ and dynamic viscosity μ. In the low Reynolds number regime, i.e. $\mathrm{Re} \ll 1$, the inertial forces become irrelevant and consequently, micro-swimmers can only utilize the viscous resistance of the surrounding fluid to move. In mathematical terms, the micro-swimmers are governed by the steady Stokes equations, which are linear and symmetric under time reversal. In the case of the scallop, this means that whatever forward motion is caused by closing its valves, it will exactly be compensated by the movement produced by reopening them, regardless of the speed of these two processes.

Let us formulate the basic problem of swimming: given a periodic record of shape changes of a swimmer, predict the corresponding history of positions and orientations in space. In more mathematical terms, this is a question of *controllability*; that is, whether it is possible to achieve any prescribed position and orientation in space starting from an arbitrary initial position and orientation using an appropriate sequence of shape deformations. In fact, the peculiarity of swimming at low values of Re stems from the fact that reciprocal shape changes cannot contribute to the net displacement as inertial forces are negligible. This especially becomes an issue when we only have few control variables at our disposal. Indeed, the scallop theorem actually shows that swimmers with only one control variable are not controllable.

[1] For a proof as well as an elementary introduction to the topic we refer to the encyclopedia article [6].

Once the controllability of a swimmer is assured, the natural follow-up question is to ask which swimming strokes achieve a prescribed net displacement with the lowest energy consumption. Mathematically speaking, we face an *optimal control problem.* From the point of view of biology, this is relevant in the light of *natural selection* among micro-organisms, whereas from the engineering standpoint, this is crucial since, as it is pointed out in [5], if a micro-swimmer should have an effect on macroscopic scales, it has to swim faster than bacteria and therefore consumes 10^4 times more energy than a bacterium. Hence, it is desirable to know the most effective swimming strokes for a micro-swimmer.

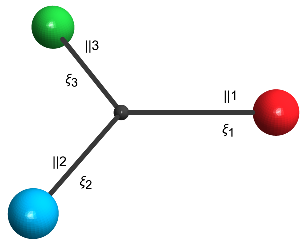

Figure 1: The parking 3-sphere swimmer (SPR3) analyzed in [4].

In [2], a whole class of controllable micro-swimmer is presented. The said paper also puts forward a numerical method to address the problem of optimal swimming. However, their explicit dynamics as well as the structure of optimal swimming strokes remain largely unknown. In this paper, we will analyze further the swimmer SPR4 from [2] and shed a light on the latter aspects. The analysis will take place very much in the spirit of the treatment of the swimmer SPR3 in [4], c.f. figure 1, which originally had been presented in [2] as well. In fact, the swimmer SPR4 is a natural generalization of the swimmer SPR3, capable of moving in the entire $3d$ space instead of just a plane. Although the principal techniques used in this paper are in close analogy to the ones in [4], the more complex geometry of both the position and the shape space cause the analysis to be more involved.

Aim of this paper is to *analytically* address the optimal control problem for SPR4 in the range of *small* strokes.

The rest of the paper is organized as follows: in section 2, we give both a geometric and a kinematic description of parking 4-sphere swimmer (SPR4). Next, we introduce the control system treated in this paper. In section 3, we study the geometric structure of the control system taking advantage of the symmetries it has to satisfy due to the underlying Stokes equations. In section 4, we unravel the properties of the control system in the range of small strokes. Eventually, section 5 addresses the characterization of energy minimizing strokes for a special class of prescribed net displacements.

2. Modeling of the swimmer as a control problem

We restrict ourselves to considering the swimmer SPR4 proposed in [2]. Let (S_1, S_2, S_3, S_4) be a regular reference tetrahedron centered at $c \in \mathbb{R}^3$ such that $\operatorname{dist}(c, S_i) = 1$ for all $i \in \mathbb{N}_4$. Then the swimmer consists of four balls $(B_i)_{i\in\mathbb{N}_4}$ of $\mathbb{R}^3$ centered at $b_i \in \mathbb{R}^3$, all of radius $a > 0$, such that the ball B_i can move along the ray starting at c and passing through S_i, see figure 2.

This reflects the situation where the balls are linked together by thin jacks that are able to elongate and retract. However, the viscous resistance of these jacks is neglected and therefore the fluid is assumed to permeate the entire open set $\mathbb{R}^3 \setminus \bigcup_{i=1}^4 \overline{B}_i$. The balls do not rotate around their arms which implies that the shape of the swimmer is completely determined by the four lengths $\xi_1, \xi_2, \xi_3, \xi_4$ of its arms, measured from c to the center of each ball b_i. However, there are no restrictions for the rotation of the swimmer around the center c, i.e. for fixed arm lengths, the swimmer is considered to be a rigid body in a Stokesian

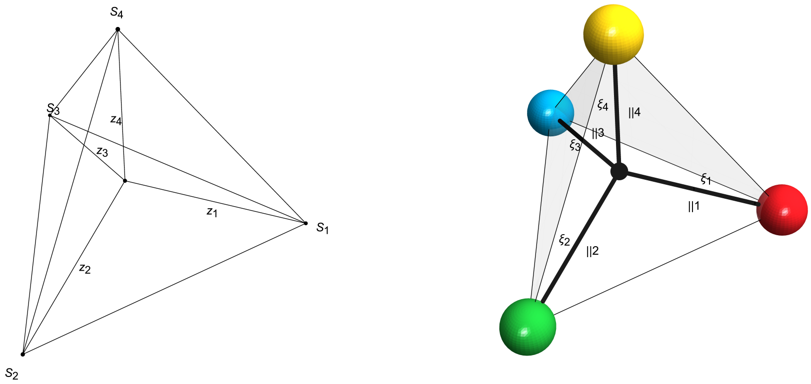

Figure 2: The reference tetrahedron and the parking 4-sphere swimmer (SPR4).

fluid. Hence, the geometrical configuration of the swimmer can be described by two sets of variables:

(i) The vector of *shape variables* $\xi := (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathcal{M} := (\sqrt{\frac{3}{2}}a, +\infty)^4 \subseteq \mathbb{R}^4_+$, from which one obtains the relative distances $(b_{ij})_{i,j\in\mathbb{N}_4}$ between the balls, where the lower bound in the open intervals is chosen such that the balls cannot overlap.

(ii) The vector of *position variables* $p = (c, R) \in \mathcal{P} := \mathbb{R}^3 \times \mathrm{SO}(3)$, which encodes the global position and orientation in space of the swimmer.

To be more precise, we consider the reference tetrahedron convexly spanned by the four unit vectors $z_1 := (2\sqrt{2}/3, 0, -1/3)$, $z_2 := (-\sqrt{2}/3, -\sqrt{2/3}, -1/3)$, $z_3 := (-\sqrt{2}/3, \sqrt{2/3}, -1/3)$ and $z_4 := (0, 0, 1)$. Position and orientation in $\mathbb{R}^3$ are then described by the coordinates of the center $c \in \mathbb{R}^3$ and the rotation $R \in SO(3)$ of the swimmer with respect to the reference orientation induced by the reference tetrahedron, i.e. if the arms are aligned with the reference tetrahedron, then this corresponds to the identity matrix $I \in \mathrm{SO}(3)$. Thus, we set $b_i := c + \xi_i R z_i$ for the center of the ball B_i.

The swimmer is completely described by the parameters $(\xi, p) \in \mathcal{M} \times \mathcal{P}$. Indeed, if we denote by B_a the ball in $\mathbb{R}^3$ of radius a centered at the origin, then for any $r \in \partial B_a$, the position of the current point on the i-th sphere of the swimmer in the state (ξ, p) is given, for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$, by the function

$$r_i(\xi, p, r) := c + R(\xi_i z_i + r). \tag{2.1}$$

Note that the functions $(r_i)_{i\in\mathbb{N}_4}$ are analytic in $\mathcal{M}\times\mathcal{P}$ and thus we can use them to calculate the instantaneous velocity on the i-th sphere B_i, which for any $(\xi, p, r) \in \mathcal{M} \times \mathcal{P} \times \partial B_a$ and every $i \in \mathbb{N}_4$ is given by

$$u_i(\xi, p, r) = \dot{c} + \omega \times (\xi_i z_i + r) + R z_i \dot{\xi}_i, \tag{2.2}$$

where ω is the axial vector associated with the skew matrix $\dot{R}R^T$.

In [2] it is shown that the system SPR4, i.e. both the shape ξ and the position p, is controllable only using the rate of change $\dot{\xi}$ of the shape. To do so, we have to understand how p responds to a variation in $\dot{\xi}$. To that end, the assumptions of *self-propulsion* and *negligible inertia of the swimmer* (which is equivalent to assuming a very low Reynolds number) are made. They imply that the total viscous force and torque exerted by the surrounding fluid on the swimmer must vanish. More precisely, for details see [2], the system can be written as

$$\dot{p} = F(R, \xi)\dot{\xi} := \begin{pmatrix} F_c(R, \xi) \\ F_\theta(R, \xi) \end{pmatrix} \dot{\xi}, \tag{2.3}$$

where $\dot{c} = F_c(R, \xi)\dot{\xi}$ and $\dot{R} = F_\theta(R, \xi)\dot{\xi}$.

In preparation for what follows, let us note that if we denote by $T_p\mathcal{P}$ the tangent space of the smooth manifold $\mathcal{P}$ at the point p, we have $F(R,\xi) \in \mathcal{L}(\mathbb{R}^4, T_p\mathcal{P})$ for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$, where $\mathcal{L}(V,W)$ denotes the set ofs linear maps between two vector spaces V and W. We quickly recall the fact that at any point $R \in \mathrm{SO}(3)$, see e.g. [7] for details, we have

$$T_R\,\mathrm{SO}(3) = \{RM \mid M \in \mathrm{Skew}_3(\mathbb{R})\}, \tag{2.4}$$

where $\mathrm{Skew}_n(\mathbb{R})$ denotes the set of skew-symmetric real matrices of size $n \times n$. Hence, we have in particular that for any $R \in \mathrm{SO}(3)$ and $\xi \in \mathbb{R}^4$

$$F_c(R,\xi) \in \mathcal{L}(\mathbb{R}^4, \mathbb{R}^3) \text{ and } F_\theta(R,\xi) \in \mathcal{L}(\mathbb{R}^4, T_R\,\mathrm{SO}(3)) \tag{2.5}$$

and therefore we can express both $F_c(R,\xi)$ and $F_\theta(R,\xi)$ as real matrices of size 3×4 once we have chosen a basis for the corresponding tangent spaces. Indeed, one verifies quickly that $\mathrm{Skew}_3(\mathbb{R})$ is a three-dimensional vector space over $\mathbb{R}$.

In analogy to [4], it is important to note here that the control system F is independent of c due to the translational invariance of the Stokes equations. However, the translational invariance is not the only symmetry property that SPr4 satisfies. The goal of the following section is to examine the structure of the control system F in consequence of the symmetries it must fulfill being driven by the Stokes equations.

3. Symmetries

For any initial condition $p_0 = (c_0, R_0) \in \mathcal{P}$ and any control curve $\xi : I \subseteq \mathbb{R} \to \mathcal{M}$, with I a neighborhood of zero, we denote by $\gamma(c_0, R_0, \xi) : I \to \mathcal{P}$ the solution associated to the dynamical system

$$\dot{p} = F(R,\xi)\dot{\xi}, \quad p(0) := p_0, \tag{3.1}$$

as well as $\gamma_c(c_0, R_0, \xi)$ and $\gamma_\theta(c_0, R_0, \xi)$ its projections on $\mathbb{R}^3$ and SO(3), respectively, such that for any $t \in I$

$$\dot{\gamma}(c_0, R_0, \xi)(t) = F(\gamma_\theta(c_0, R_0, \xi)(t), \xi(t))\dot{\xi}(t). \tag{3.2}$$

3.1 Rotational invariance

Rotational invariance of the Stokes equations expresses the fact that solution of the dynamical system (3.1) is invariant under rotations, i.e., that for any rotation $R \in \mathrm{SO}(3)$ we have for the spatial part of the solution

$$\gamma_c(c_0, RR_0, \xi)(t) = R\gamma_c(c_0, R_0, \xi)(t) + (I - R)c_0 \tag{3.3}$$

and for the angular part of the solution

$$\gamma_\theta(c_0, RR_0, \xi)(t) = R\gamma_\theta(c_0, R_0, \xi)(t) \tag{3.4}$$

at any point in time $t \in I$. Eventually, we can rigorously state the following symmetry property of the control system (3.1) with respect to rotations:

Condition 1 (Rotational invariance)**.** *If $\gamma(c_0, R_0, \xi)$ is a solution of the control system (3.1), then so is $\gamma(c_0, RR_0, \xi)$ and (3.3) and (3.4) hold.*

Remark. To follow the reasoning of [4], the symmetry relations satisfied by SPr4 are stated as hypotheses on the solution γ. In so doing, the results work for any control system of the form (2.3) and satisfying the hypotheses we state, e.g. rotational invariance, independently of these hypotheses being guaranteed by the invariance of the Stokes equations under a certain group of transformations.

We then have

Proposition 2. *Let $\xi_0 := \xi(0) \in \mathcal{M}$ denote the initial state of the control parameters and by $T_\xi\mathcal{M}$ the tangent space of $\mathcal{M}$ at ξ. If the control system (3.1) is invariant under rotations and for every $\xi \in \mathcal{M}$ it holds that $T_\xi\mathcal{M} \simeq \mathbb{R}^4$, then*

$$F_c(R,\xi) = RF_c(\xi) \text{ and } F_\theta(R,\xi) = RF_\theta(R,\xi), \tag{3.5}$$

for every $(R,\xi) \in \mathrm{SO}(3) \times \mathcal{M}$, where $F_c(\xi) := F_c(I,\xi)$ and $F_\theta(\xi) := F_\theta(I,\xi)$.

Proof. On the one hand, we have by definition of the dynamical system (2.3) that

$$\dot{\gamma}_c(c_0, R, \xi) = F_c(\gamma_\theta(c_0, R, \xi), \xi)\dot{\xi}. \tag{3.6}$$

On the other hand, using equation (3.3) and once more the definition of the dynamical system (3.1), we obtain

$$\dot{\gamma}_c(c_0, R, \xi) = R\dot{\gamma}_c(c_0, I, \xi) = RF_c(\gamma_\theta(c_0, I, \xi), \xi)\dot{\xi}. \tag{3.7}$$

Therefore, $F_c(\gamma_\theta(c_0, R, \xi), \xi)\dot{\xi} = RF_c(\gamma_\theta(c_0, I, \xi), \xi)\dot{\xi}$ for every $R \in \mathrm{SO}(3)$. Since $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$, evaluation of the preceding expression at $t = 0$ yields $F_c(R, \xi_0) = RF_c(I, \xi_0)$, as desired. The proof for F_θ is completely analogous and thus is omitted. □

3.2 Permutation of two arms

In this section, we investigate the effect of a swap of two arms on the generic solution of the dynamical system (3.1). To that end, let $P_{ij} \in M_{4\times 4}(\mathbb{R})$ denote the permutation matrix that interchanges the i-th and j-th index of a vector, which corresponds to the swap of the arms $||i$ and $||j$, denoted by $(||i \leftrightsquigarrow ||j)$, if applied to the shape space $\mathcal{M}$. In addition, let S_{ij} denote the reflection of $\mathbb{R}^3$ sending arm $||i$ onto arm $||j$ in the reference orientation I. Geometrical inspection of the reference tetrahedron shows that S_{ij} is always a reflection at a plane containing the remaining arms $||k$ and $||l$.

Before we formulate the symmetry conditions for the interchanging of two arms, we recall some results about how rotations behave under reflections. So far, we have only regarded the orientation of SPr4 as a rotation matrix in SO(3). However, by Euler's rotation theorem to every such rotation matrix $R \in \mathrm{SO}(3)$ there exists a corresponding rotation vector $\omega \in \mathbb{R}^3$ which is collinear to the unique axis of rotation defined by R, i.e. ω is an eigenvector associated to the eigenvalue 1 of R. Its length is given by the angle of rotation around this axis. The rotation vector ω is then directly related to the rotation matrix R via the map $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$, where $\mathfrak{so}(3) = T_I\,\mathrm{SO}(3) = \mathrm{Skew}_3(\mathbb{R})$ denotes the Lie algebra over SO(3), which we will illustrate in the following paragraphs.

It is clear that $\dim \mathrm{Skew}_3(\mathbb{R}) = 3$. In particular, if $R_1(\theta), R_2(\theta)$ and $R_3(\theta)$ denote the simple rotations around the $\hat{e}_1$-, $\hat{e}_2$- and $\hat{e}_3$-axis, where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ denote the canonical basis vectors of $\mathbb{R}^3$, then the matrices

$$L_1 = \frac{\mathrm{d}}{\mathrm{d}\theta}R_1(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.8}$$

$$L_2 = \frac{\mathrm{d}}{\mathrm{d}\theta}R_2(\theta)_{|\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

$$L_3 = \frac{\mathrm{d}}{\mathrm{d}\theta}R_3(\theta)_{|\theta=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

form a basis of $\mathfrak{so}(3)$, denoted by $\mathcal{L}$, consisting of the infinitesimal rotations around the corresponding axes. A trivial computation then shows that $R = \exp\left(\sum_{k\in\mathbb{N}_3} \omega_k L_k\right)$.

However, to clearly state the relations between reflections and orientations, let us first fix some notation. We denote by $\mathcal{E} := (e_1, e_2, e_3, e_4)$ the canonical basis for $\mathbb{R}^4$. Then we denote the matrix representing an arbitrary linear map $T : V \to W$ between two vector spaces V and W with respect to two bases $\mathcal{B}$ and $\mathcal{C}$ by $[T]^{\mathcal{C}}_{\mathcal{B}}$. Subsequently, we will especially make use of the adjoint map on $\mathfrak{so}(3)$ associated to an orthogonal transformation Q of $\mathbb{R}^3$. More precisely, if Q is an orthogonal transformation of $\mathbb{R}^3$, we define the adjoint map $\mathrm{ad}_Q : \mathfrak{so}(3) \to \mathfrak{so}(3)$ by

$$\mathrm{ad}_Q(M) := QMQ^T, \tag{3.11}$$

which is clearly a linear map. Then we have the following result about orthogonal transformations and their adjoint maps, which we state in a rather general fashion as this will be beneficial at a later stage.

Lemma 3. *Let $S, Q : \mathbb{R}^3 \to \mathbb{R}^3$ be a reflection at a plane and a rotation of $\mathbb{R}^3$, respectively. Then the representation matrices of their adjoint maps are given by*

$$[\mathrm{ad}_S]_{\mathcal{L}} = -S \quad \textit{and} \quad [\mathrm{ad}_Q]_{\mathcal{L}} = Q. \tag{3.12}$$

In particular, if $R \in \mathrm{SO}(3)$ characterizes the orientation of a rigid body, the orientation $\tilde{R}$ of its mirror image under a reflection S is characterized by $\tilde{R} = SRS$.

Proof. Let us denote by S_i the reflection of the $\hat{e}_i$ axis. Then one finds by a straightforward computation that the first statement is true for S_i and $R_i(\theta)$ for all $\theta \in \mathbb{R}$ and $i \in \mathbb{N}_3$. Now the statement follows by decomposing any rotation $Q \in \mathrm{SO}(3)$ into its Euler angles and any reflection S into $S = QS_iQ^T$ for some rotation Q and one of the elementary reflections S_i.

For the second part, let S be an arbitrary reflection in $\mathbb{R}^3$ and let us denote by $\tilde{\omega}$ the rotation vector of the mirror image. Since the rotation vector of a rigid body is a pseudovector, we have $\tilde{\omega} = -S\omega$. In other words, we reflect the rotation vector and change its sense of rotation. Finally, we have by the first part

$$\sum_{k \in \mathbb{N}_3} \tilde{\omega}_k L_k = \sum_{k \in \mathbb{N}_3} (-S\omega)_k L_k = S \left(\sum_{k \in \mathbb{N}_3} \omega_k L_k \right) S, \tag{3.13}$$

and thus

$$\tilde{R} = \exp(\sum_{k \in \mathbb{N}_3} \tilde{\omega}_k L_k) = \exp \left(S \left[\sum_k \omega_k L_k \right] S \right) = SRS, \tag{3.14}$$

as desired. □

With this lemma at hand, Stokes' equations allow us to state the following

Condition 4 (Swap ($||i \leftrightsquigarrow ||j$))**.** *Let the initial position be $p_0 := (c_0, I)$. If $\gamma(c_0, I, P_{ij}\xi)$ is a solution of the control system (2.3), then so is $\gamma(S_{ij}c_0, I, \xi)$ and the following relations hold*

$$\gamma_c(c_0, I, P_{ij}\xi) = S_{ij}\gamma_c(S_{ij}c_0, I, \xi) \tag{3.15}$$

and

$$\gamma_\theta(c_0, I, P_{ij}\xi) = S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{3.16}$$

Remark. In physical terms, the previous condition stems from the invariance of Stokes equations with respect to the observation point, see figure 3. In fact, an observer watching the dynamics of $\gamma(S_{ij}c_0, I, \xi)$ of SPR4 in a mirror in the reflection plane of S_{ij} sees the dynamics $\gamma(c_0, I, P_{ij}\xi)$ of a micro-swimmer obtained from SPR4 by swapping arms $||i$ and $||j$.

To avoid chaos in our notation, we treat the spatial and angular parts now separately. For the spatial part, we find

Proposition 5. *If the control system (2.3) is invariant under the swap ($||i \leftrightsquigarrow ||j$) and $T_\xi\mathcal{M} \simeq \mathbb{R}^4$ for all $\xi \in \mathcal{M}$, then for all $\xi \in \mathcal{M}$*

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij}. \tag{3.17}$$

Proof. Let $\gamma_c(c_0, R_0, P_{ij}\xi)$ be the spatial part of any solution of the control problem (3.1). The hypothesis of rotational invariance, i.e. (3.3), implies that

$$\gamma_c(c_0, R_0, P_{ij}\xi) = R_0\gamma_c(c_0, I, P_{ij}\xi) + (I - R_0)c_0. \tag{3.18}$$

From condition 4, we then get

$$\gamma_c(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\gamma_c(S_{ij}c_0, I, \xi) + (I - R_0)c_0. \tag{3.19}$$

As both $\gamma_c(c_0, R_0, P_{ij}\xi)$ and $\gamma_c(S_{ij}c_0, I, \xi)$ are spatial parts of solutions of the control system (3.1), we have on the one hand using Proposition 2

$$\dot{\gamma_c}(c_0, R_0, P_{ij}\xi) = \gamma_\theta(c_0, R_0, P_{ij}\xi)F_c(P_{ij}\xi)P_{ij}\dot{\xi}, \tag{3.20}$$

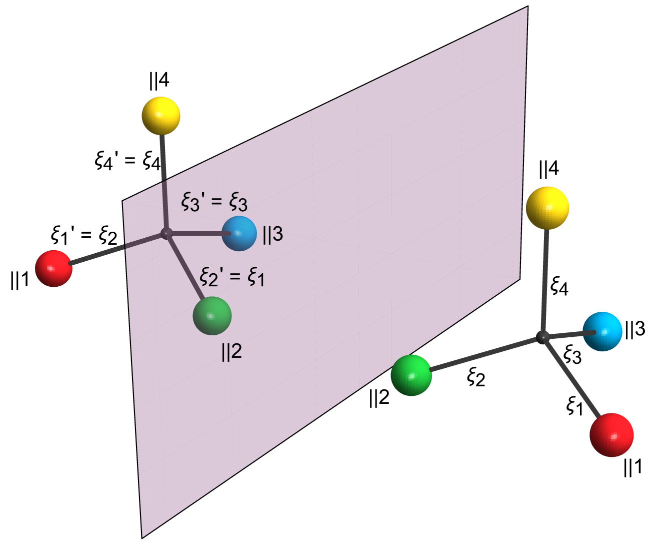

Figure 3: The reflection S_{12} applied to SPR4 in the reference orientation corresponding to the swap $(||1 \leftrightsquigarrow ||2)$

and on the other hand using (3.19) and once more (3.1) together with Proposition 2

$$\dot{\gamma}_c(c_0, R_0, P_{ij}\xi) = R_0 S_{ij} \dot{\gamma}_c(S_{ij}c_0, I, \xi) = R_0 S_{ij} \gamma_\theta(S_{ij}c_0, I, \xi) F_c(\xi)\dot{\xi}. \tag{3.21}$$

Equating (3.20) and (3.21) at $t = 0$ yields $F_c(P_{ij}\xi_0) = S_{ij}F_c(\xi_0)P_{ij}$, since by hypothesis $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$. As ξ_0 was arbitrary, we conclude. □

For the angular part, we have now the following result, in the proof of which we exploit the first, more general statement of Lemma 3:

Proposition 6. *If the control system (3.1) is invariant under the swap $(||i \leftrightsquigarrow ||j)$ and $T_\xi\mathcal{M} \simeq \mathbb{R}^4$ for all $\xi \in \mathcal{M}$, then for all $\xi \in \mathcal{M}$*

$$[F_\theta(P_{ij}\xi)]^{\mathcal{L}}_{\mathcal{E}} = -S_{ij}[F_\theta(\xi)]^{\mathcal{L}}_{\mathcal{E}} P_{ij}. \tag{3.22}$$

Proof. Let $\gamma_\theta(c_0, R_0, P_{ij}\xi)$ be the angular part of any solution of the control problem (3.1). By the rotational invariance hypothesis, i.e. (3.4), we have

$$\gamma_\theta(c_0, R_0, P_{ij}\xi) = R_0\gamma_\theta(c_0, I, \xi). \tag{3.23}$$

Then Condition 4 implies that

$$\gamma_\theta(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)S_{ij}. \tag{3.24}$$

Since both $\gamma_\theta(c_0, R_0, P_{ij}\xi)$ and $\gamma_\theta(S_{ij}c_0, I, \xi)$ are the angular parts of solutions of the control problem (3.1), we obtain with Proposition 2 on the one hand

$$\dot{\gamma}_\theta(c_0, R_0, P_{ij}\xi) = \gamma_\theta(c_0, R_0, P_{ij}\xi)F_\theta(P_{ij}\xi)P_{ij}\dot{\xi} \tag{3.25}$$

and on the other hand using (3.24) and once more Proposition 2

$$\dot{\gamma}_\theta(c_0, R_0, P_{ij}\xi) = R_0 S_{ij}\dot{\gamma}_\theta(S_{ij}c_0, I, \xi)S_{ij} = R_0 S_{ij}\gamma_\theta(S_{ij}c_0, I, \xi)F_\theta(\xi)\dot{\xi}S_{ij}. \tag{3.26}$$

Imposing equality of (3.25) and (3.26) at $t = 0$ yields

$$F_\theta(P_{ij}\xi_0)P_{ij}\dot{\xi}(0) = S_{ij}F_\theta(\xi_0)\dot{\xi}(0)S_{ij}. \tag{3.27}$$

By choice of the canonical basis for $\mathbb{R}^4$ we clearly have $[P_{ij}]_{\mathcal{E}}^{\mathcal{E}} = P_{ij}$. Therefore, by Lemma 3, we have

$$[F_\theta(P_{ij}\xi_0)]_{\mathcal{E}}^{\mathcal{L}} P_{ij}[\dot{\xi}(0)]_{\mathcal{E}} = [S_{ij}F_\theta(\xi_0)\dot{\xi}(0)S_{ij}]_{\mathcal{L}} = -S_{ij}[F_\theta(\xi_0)]_{\mathcal{E}}^{\mathcal{L}}[\dot{\xi}(0)]_{\mathcal{E}}. \tag{3.28}$$

Recalling that $T_{\xi_0}\mathcal{M} \simeq \mathbb{R}^4$ as well as the arbitrariness of ξ_0 finish the proof. □

In the following sections, we will always understand $F_\theta(\xi)$ as a matrix of size 3×4 and thus, since no confusion may arise, we will abandon the slightly cumbersome notation and identify $[F_\theta(\xi)]_{\mathcal{E}}^{\mathcal{L}}$ with $F_\theta(\xi)$.

4. The small stroke regime

Let us return to the control equations for SPr4 given by (3.1). The response of the control system is characterized by the two matrix valued functions $F_c, F_\theta : \mathrm{SO}(3) \times \mathbb{R}^4 \to M_{3\times 4}(\mathbb{R})$ which can by Proposition 2 can be factorized as:

$$F_c(R, \zeta) = RF_c(\zeta) \text{ and } F_\theta(R, \xi) = RF_\theta(\zeta), \tag{4.1}$$

with $F_c(\zeta) := F_c(I, \zeta)$ and $F_\theta(\zeta) := F_\theta(I, \zeta)$. Hereinafter, we suppose that $\zeta := \xi_0 + \xi$ with $\xi_0 \in \mathcal{M}$ having all its components equal. Furthermore, we set $F_{c,\xi_0}(\xi) := F_c(\xi_0 + \xi)$ and analogously $F_{\theta,\xi_0}(\xi) := F_\theta(\xi_0 + \xi)$. It has been shown in [2] that F and thus both F_{c,ξ_0} and F_{θ,ξ_0} are analytic functions. Therefore, we can consider their first order expansions in ξ:

$$F_{c,\xi_0}(\xi)\eta = F_{c,0}\eta + \mathcal{H}_{c,0}(\xi \otimes \eta) + \mathcal{O}(|\xi|)\eta \tag{4.2}$$

$$F_{\theta,\xi_0}(\xi)\eta = F_{\theta,0}\eta + \mathcal{H}_{\theta,0}(\xi \otimes \eta) + \mathcal{O}(|\xi|)\eta, \tag{4.3}$$

where $F_{c,0} := F_c(\xi_0) \in M_{3\times 4}(\mathbb{R})$, $\mathcal{H}_{c,0} \in \mathcal{L}(\mathbb{R}^4 \otimes \mathbb{R}^4, \mathbb{R}^3)$ represents the first order derivative of F_{c,ξ_0} at $\xi = 0$ and for $F_{\theta,\xi}$ the analogous definitions are made. The purpose of this section is to reveal the structure of the different terms in the expansions (4.2) and (4.3) in light of the symmetry properties fulfilled by F_c and F_θ due to Propositions 5 and 6 , i.e.

$$F_c(P_{ij}\xi) = S_{ij}F_c(\xi)P_{ij} \qquad \text{and} \qquad F_\theta(P_{ij}\xi) = -S_{ij}F_\theta(\xi)P_{ij} \qquad \forall \xi \in \mathcal{M}. \tag{4.4}$$

The following slightly generalized statement of Lemma 9 in [4] proves useful in our case as well. However, the proof, being exactly the same, is omitted.

Lemma 7. *Let $G : \mathbb{R}^n \to M_{m\times n}(\mathbb{R})$ be an analytic function and $S \in M_{m\times m}(\mathbb{R})$ and $P \in M_{n\times n}(\mathbb{R})$ matrices such that $G(P\xi) = SG(\xi)P$ for every $\xi \in \mathbb{R}^n$. For $\xi_0 \in \mathbb{R}^n$ with all components equal, set $G_{\xi_0}(\xi) := G(\xi_0 + \xi)$ and write the first order expansion*

$$G_{\xi_0}(\xi)\eta = G_0\eta + \mathcal{H}_0(\xi \otimes \eta) + \mathcal{O}(|\xi|)\eta. \tag{4.5}$$

Then we have

$$G_0 = SG_0P, \tag{4.6}$$

and

$$\mathcal{H}_0((P\xi) \otimes \eta) = S\mathcal{H}_0(\xi \otimes (P\eta)) \qquad \forall \xi, \eta \in \mathbb{R}^n. \tag{4.7}$$

4.1 The zeroth order terms $F_{c,0}$ and $F_{\theta,0}$

One could directly use Lemma 7 to obtain the terms of order zero up to a scalar by solving a system of two matrix equations as it was done in [4]. Yet, there is a direct geometric argument, which incidentally also works for SPr3 in [4], that reveals their structure. Let us denote by $A^{(i)}$ the i-th column of a matrix $A \in M_{m\times n}(\mathbb{R})$. Then we note that for the spatial term $F_{c,0}$ and any $i \in \mathbb{N}_4$ we obtain from Lemma 7 and Proposition 5

$$F_{c,0}^{(i)} = F_{c,0}e_i = S_{ij}F_{c,0}P_{ij}e_i = S_{ij}F_{c,0}e_j = S_{ij}F_{c,0}^{(j)}. \tag{4.8}$$

Similarly, for any $k \in \mathbb{N}_4$ one has

$$F_{c,0}^{(k)} = S_{ij}F_{c,0}P_{ij}e_k = S_{ij}F_{c,0}e_k = S_{ij}F_{c,0}^{(k)}. \tag{4.9}$$

Recall that S_{ij} is the reflection that maps the arm $||i$ onto arm $||j$ and vice-versa in the reference orientation, i.e. where $||i$ is collinear to z_i, the i-th arm of the reference tetrahedron defined in section 2. In particular, the plane of reflection is defined by the origin and the remaining arms of the reference tetrahedron, i.e. by the vectors z_k and z_l. Thus, equation (4.9) implies that $F_{c,0}^{(k)} \in \text{span}\{z_k, z_l\}$ as $F_{c,0}^{(k)}$ apparently is an eigenvector associated to the eigenvalue 1 of S_{ij}. Yet, the same argument shows that $F_{c,0}^{(k)} \in \text{span}\{z_k, z_i\}$ and $F_{c,0} \in \text{span}\{z_k, z_j\}$ and the vectors z_i, z_j and z_l being linearly independent implies that $F_{c,0}^{(k)} = a_k z_k$ for some $a_k \in \mathbb{R}$. Due to equation (4.8) and the fact that S_{ij} is orthogonal, we have $|a_i| = |a_j| = |a_k| = |a_l|$. Finally, we note that the quantity $a_k z_k \cdot z_k$ stays constant under change of indices again in consequence of the symmetry conditions from Lemma 7. Hence, we have $a_1 = a_2 = a_3 = a_4 := \mathfrak{a}$ and therefore

$$F_{c,0} = \mathfrak{a}(z_1|z_2|z_3|z_4). \tag{4.10}$$

In the following sections, we will exploit the orthonormal basis of $\mathbb{R}^4$ consisting of the vectors $\tau_1 := \frac{1}{\sqrt{6}}(-2,1,1,0)^T$, $\tau_2 := \frac{1}{\sqrt{2}}(0,1,-1,0)^T$, $\tau_3 := \frac{1}{2\sqrt{3}}(1,1,1,-3)^T$ and $\tau_4 := \frac{1}{2}(1,1,1,1)^T$, in terms of which $F_{c,0}$ can be written as $F_{c,0} = -3\sqrt{3}\mathfrak{a}[\tau_1|\tau_2|\tau_3]^T$.

For the angular term $F_{\theta,0}$ we find by Lemma 7, Proposition 6 and a similar argument for any $k \in \mathbb{N}_4$ that

$$F_{\theta,0}^{(k)} = -S_{ij} F_{\theta,0}^{(k)}. \tag{4.11}$$

This means that in this case $F_{\theta,0}^{(k)}$ is an eigenvector to the eigenvalue -1 and therefore must be orthogonal to the plane of reflection, i.e. $\text{span}\{z_k, z_l\}$. The same is true for the reflections S_{il} and S_{jl} and hence $F_{c,0}^{(k)}$ is in particular orthogonal to z_i, z_j, z_l. This eventually implies that $F_{\theta,0} = 0$ since the latter three vectors form a basis of $\mathbb{R}^3$.

Remark. First, we observe that the upper left corner of $F_{c,0}$ corresponding to the arms $||1, ||2$ and $||3$ up to multiplication by a scalar is the same as for SPr3 in [4]. Furthermore, one notes that physically it is clear that $F_{\theta,0}$ must vanish since by hypothesis ξ_0 has all its components equal and thus the swimmer is in a symmetric shape at $\xi = 0$. Therefore, the balls moving along their axes cannot create any torque.

4.2 The first order terms $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$

Following the approach in [4], we evaluate the tensors $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$ on the basis $(e_i \otimes e_j)_{i,j\in\mathbb{N}_4}$. Setting $A_k := (\mathcal{H}_{c,0}(e_i \otimes e_j) \cdot \hat{e}_k)_{i,j\in\mathbb{N}_4}$ and $B_k := (\mathcal{H}_{\theta,0}(e_i \otimes e_j) \cdot L_k)_{i,j\in\mathbb{N}_4}$ for $k \in \mathbb{N}_3$, we can write the vectors $\mathcal{H}_{c,0}(\xi \otimes \eta), \mathcal{H}_{\theta,0}(\xi \otimes \eta) \in \mathbb{R}^3$ for any $\xi, \eta \in \mathbb{R}^3$ as

$$\mathcal{H}_{c,0}(\xi \otimes \eta) = \sum_{k\in\mathbb{N}_3} (A_k \eta \cdot \xi)\hat{e}_k, \tag{4.12}$$

and similarly

$$\mathcal{H}_{\theta,0}(\xi \otimes \eta) = \sum_{k\in\mathbb{N}_3} (B_k \eta \cdot \xi) L_k. \tag{4.13}$$

We could pursue the approach of [4] and directly calculate the matrices A_k and B_k. However, as we shall see later, the dynamics of SPr4, up to higher order terms in the norm of the control curve ξ, will only be governed by their skew symmetric parts. Thus, we will evade this strenuous task and we determine the skew symmetric matrices $M_k := \frac{1}{2}(A_k - A_k^T)$ and $M_{k+3} := \frac{1}{2}(B_k - B_k^T)$ for $k \in \mathbb{N}_3$, up to two scalar parameters, using a geometric argument similar to the one used in the previous section. Nevertheless, it is possible to calculate the symmetric parts of the matrices A_k and B_k using similar geometric arguments, c.f. appendix, to obtain a complete description of the dynamics.

To that end, we notice that Lemma 7 together with the fact that $(P_{ij})^2 = I$ yields for all $i, j \in \mathbb{N}_4$ and for all $\xi, \eta \in \mathbb{N}_4$

$$S_{ij}\mathcal{H}_{c,0}(P_{ij}\xi \otimes P_{ij}\eta) = \mathcal{H}_{c,0}(\xi \otimes \eta), \tag{4.14}$$

as well as

$$-S_{ij}\mathcal{H}_{\theta,0}(P_{ij}\xi \otimes P_{ij}\eta) = \mathcal{H}_{\theta,0}(\xi \otimes \eta). \tag{4.15}$$

Next, we define $\mathcal{K}_{c,0}(\xi \otimes \eta) := \frac{1}{2}[\mathcal{H}_{c,0}(\xi \otimes \eta) - \mathcal{H}_{c,0}(\eta \otimes \xi)]$ and similarly $\mathcal{K}_{\theta,0}$ such that

$$M_k = (\mathcal{K}_{c,0}(e_i \otimes e_j) \cdot \hat{e}_k)_{i,j \in \mathbb{N}_4} \text{ and } M_{k+3} = (\mathcal{K}_{\theta,0}(e_i \otimes e_j) \cdot L_k)_{i,j \in \mathbb{N}_4}, \quad k \in \mathbb{N}_3. \tag{4.16}$$

In particular, it is clear that $\mathcal{K}_{c,0}$ and $\mathcal{K}_{\theta,0}$ satisfy the same symmetry relations as $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$, respectively, and additionally, we have $\mathcal{K}_{c,0}(e_i \otimes e_j) = -\mathcal{K}_{c,0}(e_j \otimes e_i)$ and $\mathcal{K}_{\theta,0}(e_i \otimes e_j) = -\mathcal{K}_{\theta,0}(e_j \otimes e_i)$.

For the spatial part, we deduce from the symmetry properties above that for all $i, j \in \mathbb{N}_4$

$$\mathcal{K}_{c,0}(e_i \otimes e_j) = S_{ij}\mathcal{K}_{c,0}(e_j \otimes e_i) = -S_{ij}\mathcal{K}_{c,0}(e_i \otimes e_j) \tag{4.17}$$

and therefore $\mathcal{K}_{c,0}(e_i \otimes e_j)$ is an eigenvector associated to the eigenvalue -1 of the reflection S_{ij}. The reflection S_{ij} taking place at the plane passing through the two remaining arms of the reference tetrahedron z_k and z_l, implies that $\mathcal{K}_{c,0}(e_i \otimes e_j) = \alpha_{ij}(z_k \times z_l)$ for some scalar $\alpha_{ij} \in \mathbb{R}$. Additionally, we have $\mathcal{K}_{c,0}(e_i \otimes e_j) = S_{jk}\mathcal{K}_{c,0}(e_i \otimes e_k) = \alpha_{ik}(z_j \times z_l)$ and since S_{jk} is orthogonal, we have $|\alpha_{ij}| = |\alpha_{ik}|$ as the vectors z_i are normalized. Eventually, one quickly verifies that the quantity

$$\mathcal{K}_{c,0}(e_i \otimes e_j) \cdot \operatorname{sgn}(ijkl)(z_k \times z_l), \tag{4.18}$$

where $\operatorname{sgn}(ijkl)$ denotes the parity of the permutation $(ijkl)$ of $\mathbb{N}_4$, stays constant under any permutation of the indices as well as any symmetry condition. Hence, we may conclude that

$$\mathcal{K}_{c,0}(e_i \otimes e_j) = \alpha \operatorname{sgn}(ijkl)(z_k \times z_l), \tag{4.19}$$

for all $i \neq j \in \mathbb{N}_4$ and some scalar $\alpha \in \mathbb{R}$. Clearly, the symmetry conditions imply that $\mathcal{K}_{c,0}(e_i \otimes e_i) = 0$ for all $i \in \mathbb{N}_4$. Thus, we have determined the matrices M_1, M_2, M_3 up to one scalar parameter. By explicitly calculating the cross products $z_i \times z_j$, we find

$$M_1 = \alpha \begin{pmatrix} 0 & 3 & 3 & 2 \\ -3 & 0 & 0 & -1 \\ -3 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{pmatrix}, \qquad M_2 = \sqrt{3}\alpha \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -2 & -1 \\ 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}, \tag{4.20}$$

and

$$M_3 = 2\sqrt{2}\alpha \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.21}$$

Similarly, for the angular part, we find that

$$\mathcal{K}_{\theta,0}(e_i \otimes e_j) = -S_{kl}\mathcal{K}_{\theta,0}(e_i \otimes e_j) \tag{4.22}$$

and therefore $\mathcal{K}_{\theta,0}(e_i \otimes e_j) = \delta_{ij} e_i \times e_j$. By noticing that this time the quantity $\mathcal{K}_{\theta,0}(e_i \otimes e_j) \cdot (z_i \times z_j)$ stays constant, a similar argument to the one above shows that

$$\mathcal{K}_{\theta,0}(e_i \otimes e_j) = \delta(z_i \times z_j), \tag{4.23}$$

for all $i \neq j \in \mathbb{N}_4$. Again, we have $\mathcal{K}_{\theta}(e_i \otimes e_i) = 0$ for all $i \in \mathbb{N}_4$. A calculation similar to the one above now yields

$$M_4 = \delta \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & -2 & 3 \\ 1 & 2 & 0 & -3 \\ 0 & -3 & 3 & 0 \end{pmatrix}, \qquad M_5 = \sqrt{3}\delta \begin{pmatrix} 0 & -1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ -2 & 1 & 1 & 0 \end{pmatrix}, \tag{4.24}$$

and

$$M_6 = 2\sqrt{2}\delta \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.25}$$

Remark. At this point, let us point out the apparent similarity between the matrices M_1, M_2 and M_6 with their corresponding matrices in [4]. In fact, in the upper left corner, i.e. the entries that relate the first three arms to each other, we retrieve the same matrices as in [4] up to rescaling α and δ, which very well reflects the similarity between SPR3 and SPR4. However, the fact that it is the first three arms that corresponds to the three arms in SPR3 merely stems from our choice of the reference orientation.

4.3 The linearized control equations

Herein, we denote by J the closed interval $[0, 2\pi] \subset \mathbb{R}$ and for every $f \in L^2_\sharp(J, \mathbb{R}^4)$ we denote by $\langle f \rangle := (2\pi)^{-1} \int_J f(s)\mathrm{d}s$ the average of f on J. We define the so-called *strokes space* as $\dot{H}^1_\sharp(J, \mathbb{R}^4)$, i.e. the Sobolev space of 2π-periodic vector valued functions of $L^2_\sharp(J, \mathbb{R}^4)$ having first order weak derivative in $L^2_\sharp(\mathbb{R}^4)$ and average zero. Indeed, we can assume a control curve $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ without loss of generality since we can add any non-zero average to the initial shape ξ_0.

In the previous section, we have seen that the control system governing the evolution of SPR4 under the action of the control parameters $\zeta \in \mathcal{M}$ is given by

$$\begin{cases} \dot{c} &= RF_c(\zeta)\dot{\zeta} \\ \dot{R} &= RF_\theta(\zeta)\dot{\zeta}, \end{cases} \tag{4.26}$$

where $(c, R) \in \mathcal{P} = \mathbb{R}^3 \times \mathrm{SO}(3)$, the systems $F_c, F_\theta : \mathcal{M} \to M_{3\times 4}(\mathbb{R})$ are given by (4.1) and $\dot{\zeta} \in T_\zeta\mathcal{M}$. Furthermore, we have seen previously, c.f. (4.4), (4.12) and (4.13), that if we set $\zeta = \xi_0 + \xi$, the response of the system around $\xi = 0$, up to higher order terms, simplifies to

$$\begin{cases} \dot{c} &= RF_{c,0}\dot{\xi} + R\sum_{k\in\mathbb{N}_3}(A_k\dot{\xi}\cdot\xi)\hat{e_k} \\ \dot{R} &= R\sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi)L_k. \end{cases} \tag{4.27}$$

In particular, if we fix $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ and define $\Gamma := \sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi) : J \to \mathfrak{so}(3)$, then the dynamics of R can be written as an ordinary differential equation on the Lie group SO(3):

$$\begin{cases} \dot{R}(t) = R(t)\Gamma(t) \\ R(0) := R_0. \end{cases} \tag{4.28}$$

To simplify equations (4.27) further, we are interested in the solution of (4.28) in the regime of a small stroke $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ or equivalently in the regime of a small vector field Γ. Intuitively, the solution R of (4.28) should not deviate too much from the initial value R_0 if the vector field Γ driving the differential equation is small. The solution of (4.28) and its relation to a small vector field Γ as well as the notion of smallness for the vector field Γ in the first place, are mathematically formalized by the concept of *chronological calculus.* For details on the topic, we refer to [1] but essentially it works as follows: First, one identifies any smooth manifold M with the space $C^\infty(M)$, on which one defines a certain metric topology, the Whitney topology. Then, the solution of a differential equation $\dot{q}(t) = q(t)V(t)$ for V a non-autonomous vector field on M is given by

$$q(t) = q(0)\overrightarrow{\exp}\int_0^t V(s)\mathrm{d}s, \tag{4.29}$$

where $\overrightarrow{\exp}$ is a special operator called the *right chronological exponential.* It is defined as a limit of an iterated integral, see [1]. For the series expansion

$$S_m(t) := I + \sum_{n=1}^{m-1}\int_{\Delta_n(t)}\cdots\int V(s_n)\circ\cdots\circ V(s_1)\mathrm{d}s_n\cdots\mathrm{d}s_1, \tag{4.30}$$

with $\Delta_n(t) = \{(s_1, \ldots, s_n) \in \mathbb{R}^n | 0 \le s_n \le \cdots \le s_1 \le t\}$, we have

$$q(t) = S_m(t) + \mathcal{O}(t^m), \qquad t \downarrow 0. \tag{4.31}$$

In particular, for an equation of the form $\dot{q}(t) = q(t)\varepsilon V(t)$, it is shown in [1], that

$$q(t) = S^{\varepsilon}_m(t) + \mathcal{O}(\varepsilon^m), \qquad \varepsilon \downarrow 0, \tag{4.32}$$

where S^{ε}_m denotes the series expansion (4.30) for the vector field εV.

With the estimate (4.32) at hand, let $\hat{\xi} \in H^1_\sharp(J, \mathbb{R}^4)$ be a normalized stroke, i.e. $||\hat{\xi}||_{H^1_\sharp} = 1$, and $\varepsilon > 0$. Set $\xi := \varepsilon\hat{\xi}$ as well as $\Gamma_\varepsilon := \sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi)L_k$ such that $\Gamma_1 = \sum_{k\in\mathbb{N}_3}(B_k\dot{\hat{\xi}}\cdot\hat{\xi})L_k$ and $\Gamma_\varepsilon = \varepsilon^2\Gamma_1$. Writing S^{ϵ}_m for the expansion (4.30) of the vector field Γ_ε, we find by (4.32)

$$R(t) = R_0\left(I + \int_0^t \Gamma_\varepsilon(\tau)\mathrm{d}\tau\right) + \mathcal{O}(\epsilon^4), \qquad \varepsilon \downarrow 0. \tag{4.33}$$

Hence, choosing $R_0 = I$, we have in particular the following approximations for any $t \in J$

$$\begin{cases} \dot{c} &= \left(I + \int_0^t \Gamma_\varepsilon(\tau)\mathrm{d}\tau\right)\left(F_{c,0}\dot{\xi} + \sum_{k\in\mathbb{N}_3}(A_k\dot{\xi}\cdot\xi)\hat{e}_k\right) + \mathcal{O}(\varepsilon^4) \\ \dot{R} &= \left(I + \int_0^t \Gamma_\varepsilon(\tau)\mathrm{d}\tau\right)\sum_{k\in\mathbb{N}_3}(B_k\dot{\xi}\cdot\xi)L_k + \mathcal{O}(\varepsilon^4), \end{cases} \tag{4.34}$$

for $\varepsilon \downarrow 0$. By integrating the previous two relations over J, we find an estimate of the net displacement undergone by the center c of SPr4 as well as its orientation R after a small stroke. Moreover, with equations (4.34) we can express the net displacements δc and δR as maps $H^1_\sharp(J, \mathbb{R}^4) \to \mathbb{R}^3$ and $H^1_\sharp(J, \mathbb{R}^4) \to \mathfrak{so}(3)$, respectively, given by $\xi \mapsto 2\pi\langle\dot{c}(\xi)\rangle$ and $\xi \mapsto 2\pi\langle\dot{R}(\xi)\rangle$, respectively. Consequently, let us prove that

Proposition 8. *For any $\xi \in H^1_\sharp(J, \mathbb{R}^4)$, in a neighborhood of $0 \in H^1_\sharp(J, \mathbb{R}^4)$, the following estimates hold*

$$\begin{aligned} \delta c(\xi) &= 2\pi \sum_{k\in\mathbb{N}_3}\langle A_k\dot{\xi}\cdot\xi\rangle\hat{e}_k + \mathcal{O}(||\xi||^3_{H^1_\sharp}), \\ \delta R(\xi) &= 2\pi \sum_{k\in\mathbb{N}_3}\langle B_k\dot{\xi}\cdot\xi\rangle L_k + \mathcal{O}(||\xi||^4_{H^1_\sharp}). \end{aligned} \tag{4.35}$$

Proof. First, let us note that the term $\langle F_{c,0}\dot{\xi}\rangle$ vanishes due to the periodicity of the stroke ξ. Next, we observe that it suffices to prove that the scalar quantities of the form

$$\left\langle \left(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau\right) A_l\dot{\hat{\xi}}\cdot\hat{\xi}\right\rangle, \qquad k, l \in \mathbb{N}_3, \tag{4.36}$$

as well as $\langle(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau)\dot{\hat{\xi}}_i\rangle$, $i \in \mathbb{N}_4$ are bounded, where we again set $\xi = \varepsilon\hat{\xi}$. We focus on the terms of the latter form, since the others can be treated in the same manner. We have

$$\begin{aligned} \left|\int_J \left(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau\right)\dot{\hat{\xi}}_i(t)\mathrm{d}t\right| &= \left|\int_0^{2\pi} B_k\dot{\hat{\xi}}(t)\cdot\hat{\xi}(t)\int_t^{2\pi}\dot{\hat{\xi}}_i(s)\mathrm{d}s\mathrm{d}t\right| \\ &\le ||B_k||_{op}\int_J |\dot{\hat{\xi}}(t)|\cdot|\hat{\xi}(t) - \hat{\xi}(0)|^2\mathrm{d}t \end{aligned} \tag{4.37}$$

The Sobolev-Morrey embedding $H^1_\sharp(J, \mathbb{R}^4) \subseteq L^\infty_\sharp(J, \mathbb{R}^4)$ guarantees the existence of a $c_S > 0$ such that $||\xi||_\infty \le c_S||\xi||_{H^1_\sharp}$ for every $\xi \in H^1_\sharp(J, \mathbb{R}^4)$. Hence, we have

$$\begin{aligned} \left|\left\langle\left(\int_0^t B_k\dot{\hat{\xi}}(\tau)\cdot\hat{\xi}(\tau)\mathrm{d}\tau\right)\dot{\hat{\xi}}_i\right\rangle\right| &\le ||B_k||_{op}||\hat{\xi}||^2_\infty||||_{H^1_\sharp} \\ &\le c_S||B_k||_{op}||\hat{\xi}||^3_{H^1_\sharp} = c_S||B_k||_{op}, \end{aligned} \tag{4.38}$$

which is clearly bounded. This finishes the proof. □

To end this section, we note that on the one hand, we have $\langle A_k\dot{\xi}\cdot\xi\rangle = \langle M_k\dot{\xi}\cdot\xi\rangle$ and $\langle B_k\dot{\xi}\cdot\xi\rangle = \langle M_{k+3}\dot{\xi}\cdot\xi\rangle$ for all $k \in \mathbb{N}_3$. Indeed, if A is a symmetric matrix, we have by integration by parts that $\langle A\xi\cdot\dot{\xi}\rangle = \langle A\dot{\xi}\cdot\xi\rangle = -\langle A\xi\cdot\dot{\xi}\rangle$ and thus only the skew-symmetric parts of the matrices A_k and B_k contribute to the net displacement. Furthermore, similarly to [4], we can represent the terms $M_k\dot{\xi}\cdot\xi$ in terms of certain operations of the orthonormal basis $\{\tau_i\}_{i\in\mathbb{N}_4}$ of $\mathbb{R}^4$. In fact, we find by straightforward calculation using that

$$M_k\dot{\xi}\cdot\xi = -2\sqrt{6}\,\alpha\det(\xi|\dot{\xi}|\tau_{k+1}|\tau_{k+2}), \qquad k\in\mathbb{N}_3 \tag{4.39}$$

$$M_{3+k}\dot{\xi}\cdot\xi = -2\sqrt{6}\,\delta\det(\xi|\dot{\xi}|\tau_k|\tau_4), \qquad k\in\mathbb{N}_3, \tag{4.40}$$

where $\det(\xi|\dot{\xi}|\tau_j|\tau_k)$ denotes the determinant of the matrix $(\xi|\dot{\xi}|\tau_j|\tau_k)$ and the index k is reduced mod 3 to simplify the notation. Ultimately, using that $\mathbb{R}^3\times\mathfrak{so}(3)\simeq\mathbb{R}^6$, we can write the net displacement in position and orientation simultaneously as

$$\frac{\delta p}{2\pi} = -2\sqrt{6}\alpha\sum_{k\in\mathbb{N}_3}\langle\det(\xi|\dot{\xi}|\tau_{k+1}|\tau_{k+2})\rangle f_k - 2\sqrt{6}\delta\sum_{k\in\mathbb{N}_3}\langle\det(\xi|\dot{\xi}|\tau_k|\tau_4)\rangle f_{k+3}, \tag{4.41}$$

where $\{f_i\}_{i\in\mathbb{N}_6}$ denotes the canonical Basis of $\mathbb{R}^6$ and the index k is once more reduced mod 3. This representation will prove particularly useful in the following section.

5. Energy minimizing strokes

In the spirit of [2] and [4], we follow the notion of swimming efficiency suggested by Lighthill [10] and we adopt the following notion of optimality: energy minimizing strokes are the ones that minimize the kinematic energy dissipated while trying to reach a given net displacement $\delta p\in\mathbb{R}^3\times\mathfrak{so}(3)\simeq\mathbb{R}^6$. Mathematically speaking, the total energy dissipation due to a stroke $\xi\in H^1_\sharp(J,\mathbb{R}^4)$ can be evaluated through an adequate quadratic energy functional, c.f. [2],

$$\mathcal{G}(\xi) := \int_J \mathfrak{g}(\xi(t))\dot{\xi}(t)\cdot\dot{\xi}(t)\mathrm{d}t, \tag{5.1}$$

where the energy density $\mathfrak{g}\in C^1(\mathbb{R}^4)$ is a function with values in the space of symmetric and positive definite matrices $M_{4\times4}(\mathbb{R})$. In other words, $\mathfrak{g}$ defines a continuous Riemannian metric on $\mathcal{M}$. In the small stroke regime, we can approximate the energy density by $\mathfrak{g}(\xi) = \mathfrak{g}(0)+o(1)$, where $\mathfrak{g}(0)\in M_{4\times4}(\mathbb{R})$ is symmetric and positive definite. More precisely,

$$\mathcal{G}(\xi) := \int_J Q_{\mathfrak{g}}(\dot{\xi}(t))\mathrm{d}t, \tag{5.2}$$

with $Q_{\mathfrak{g}}(\eta) := \mathfrak{g}(0)\eta\cdot\eta$. For the same symmetry reasons as discussed in section 3, we necessarily have for all $\eta\in\mathbb{R}^4$

$$Q_{\mathfrak{g}}(P_{ij}\eta) = Q_{\mathfrak{g}}(\eta),\ i,j\in\mathbb{N}_4, \tag{5.3}$$

where P_{ij} denotes the permutation matrix swapping the i-th and j-th entries. By direct computation, on finds that the symmetric positive matrix G representing the quadratic form $Q_{\mathfrak{g}}$ is of the form

$$G = \begin{pmatrix} \kappa & h & h & h \\ h & \kappa & h & h \\ h & h & \kappa & h \\ h & h & h & \kappa \end{pmatrix}, \tag{5.4}$$

for two parameters h and $\kappa > \max(h,-3h)$. In particular, we observe that $G\tau_k = (\kappa-h)\tau_k$ for $k\in\mathbb{N}_3$ and $G\tau_4 = (\kappa+3h)\tau_4$. In the following, we denote by $\mathfrak{g}_1 := \mathfrak{g}_2 := \mathfrak{g}_3 := \kappa-h$ and $\mathfrak{g}_4 := \kappa+3h$ the eigenvalues of G. Furthermore, the eigenvalues $(\mathfrak{g}_i)_{i\in\mathbb{N}_4}$ allow us to diagonalize G as

$$G = U\Lambda_{\mathfrak{g}}U^T,\quad U := [\tau_1|\tau_2|\tau_3|\tau_4],\quad \Lambda_{\mathfrak{g}} := \mathrm{diag}(\mathfrak{g}_i). \tag{5.5}$$

The goal of this section is the minimization of $\mathcal{G}$ in $H^1_\sharp(J,\mathbb{R}^4)$ subject to a prescribed net displacement $\delta p\in\mathbb{R}^6$, i.e. subject to the constraint (c.f. (4.41))

$$\begin{aligned}\delta p = \mathfrak{h}_c &\sum_{k\in\mathbb{N}_3}\left(\int_J \det(\xi(t)|\dot{\xi}(t)|\tau_{k+1}|\tau_{k+2})\mathrm{d}t\right) f_k \\ +\mathfrak{h}_\theta &\sum_{k\in\mathbb{N}_3}\left(\int_J \det(\xi(t)|\dot{\xi}(t)|\tau_k|\tau_4)\mathrm{d}t\right) f_{k+3},\end{aligned} \tag{5.6}$$

with $\mathfrak{h}_c = -2\sqrt{6}\alpha$ and $\mathfrak{h}_\theta = -2\sqrt{6}\delta$. The existence of such solutions follows readily by the direct method of variational calculus.

5.1 Bivectors in four dimensions

Let us recall in this section the basic definitions around the notion of *bivectors*, where we refer to [11] for details. As the abstract definition of general k-vectors is not very useful for our purposes, we merely illustrate them in $\mathbb{R}^3$, which then generalizes easily to higher dimensions. In $\mathbb{R}^3$, a bivector is an oriented plane segment; that is, a small piece of surface having a magnitude given by the area of the surface element as well as a direction given by the attitude of the plane the surface element lies in as well as a sense of rotation. Together, they form the vector space $\bigwedge^2 \mathbb{R}^3$. We can represent a bivector $\omega \in \bigwedge^2 \mathbb{R}^3$ as a small parallelogram which suggests that we can think of it as some product of the two vectors along its sides. This is realized by the *exterior product*, also called *wedge product*, $u \wedge v$ of two vectors u and v. The product $u \wedge v$ then represents the bivector obtained by sweeping v along u. This operation yields a direct link between $\mathbb{R}^3$ and the vector space $\bigwedge^2 \mathbb{R}^3$ of bivectors, a basis of which is given by

$$\{\hat{e}_1 \wedge \hat{e}_2, \hat{e}_1 \wedge \hat{e}_3, \hat{e}_2 \wedge \hat{e}_3\}, \tag{5.7}$$

if $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ is a basis of $\mathbb{R}^3$. In fact, the standard scalar product on $\mathbb{R}^3$ extends to a scalar product on $\bigwedge^2 \mathbb{R}^3$ by

$$(u_1 \wedge u_2, v_1 \wedge v_2) = \det \begin{pmatrix} u_1 \cdot v_1 & u_1 \cdot v_2 \\ u_2 \cdot v_1 & u_2 \cdot v_2 \end{pmatrix}. \tag{5.8}$$

In particular, $(u \wedge v, u \wedge v) = |u|^2 |v|^2 \sin^2 \psi$, where ψ is the angle between the vectors u and v. Eventually, the norm of a bivector $\omega = \omega_{12}\hat{e}_1 \wedge \hat{e}_2 + \omega_{13}\hat{e}_1 \wedge \hat{e}_3 + \omega_{23}\hat{e}_2 \wedge \hat{e}_3$ is given by

$$|\omega| = \sqrt{\omega_{12}^2 + \omega_{13}^2 + \omega_{23}^2}. \tag{5.9}$$

These definitions then extend naturally to all higher dimensions. In particular, we note that if $\{e_1, e_2, e_3, e_4\}$ again denotes the canonical basis of $\mathbb{R}^4$, a basis of the space $\bigwedge^2 \mathbb{R}^4$ is given by

$$\{e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}, \tag{5.10}$$

where we write $e_{ij} := e_i \wedge e_j$ to simplify notation.

To finish this section, we will point out some properties of the space $\bigwedge^2 \mathbb{R}^4$ and how it differs from $\bigwedge^2 \mathbb{R}^3$, which at a later point will illustrate why the optimal control curves for SPr3 and SPr4 in general do not have the same structure. Nevertheless, we will be able treat a simplified situation for SPr4 in the same manner as SPr3 in [4].

As a matter of fact, one peculiarity of the bivectors in $\mathbb{R}^3$ is that they are isomorphic to the space $\mathbb{R}^3$ itself. This is realized by the so-called *Hodge dual operator* $\star$, c.f. [11] p. 38, which is defined in a way such that for any two vectors $u, v \in \mathbb{R}^3$ one has

$$u \wedge v = \star(u \times v), \tag{5.11}$$

where $\times$ denotes the usual cross product. This entails two things: First, every bivector in $\mathbb{R}^3$ is *simple*; that is, it can be written as the wegde product of two vectors. Second, every bivector in $\mathbb{R}^3$ defines one unique plane in $\mathbb{R}^3$. It is due to this underlying geometrical fact that one can always reduce the Fourier coefficients of an optimal control curve to one pair of vectors, by which one attains the same net displacement at the same energy consumption in [4], c.f. Proposition 15. Furthermore, this implies that the optimal control curves are situated in a certain plane defined by the vector of the net displacement.

Yet, in $\mathbb{R}^4$, the geometry of its bivectors proves to be more involved. As $\dim \bigwedge^2 \mathbb{R}^4 = 6$, it is clear that the bivectors in $\mathbb{R}^4$ are not isomorphic to $\mathbb{R}^4$ itself. In particular, in $\mathbb{R}^4$ not all bivectors are simple. Indeed, the bivector $e_1 \wedge e_2 + e_3 \wedge e_4 \in \bigwedge^2 \mathbb{R}^4$ cannot be written as an exterior product of just two vectors in $\mathbb{R}^4$. Nevertheless, any bivector in $\mathbb{R}^4$ can be written as the sum of two orthogonal simple bivectors [11]. Moreover, we have the following criterion to determine whether a bivector is simple:

Lemma 9. *A bivector $\omega \in \bigwedge^2 \mathbb{R}^4$ is simple if and only if $\omega \wedge \omega = 0$.*

Proof. If $\omega \in \bigwedge^2 \mathbb{R}^4$ is a simple bivector, i.e. if there are vectors $u, v \in \mathbb{R}^4$ such that $\omega = u \wedge v$, then it is clear from the anticommutativity and associativity of the wedge product that

$$\omega \wedge \omega = (u \wedge v) \wedge (u \wedge v) = -u \wedge u \wedge v \wedge v = 0. \tag{5.12}$$

The inverse requires a rather lengthy proof by induction, see the lecture notes on projective geometry by Nigel Hitchin, chapter 3, p.48 [8]. □

In what follows, we will find that the net displacement actually identifies with a bivector of $\mathbb{R}^4$ and the solution to the optimization problem in [4] then suggests that we should be able to find a similar solution for our optimization problem, at least in the case where the net displacement is a simple bivector. The preceding lemma then serves us to identify certain subspaces of $\bigwedge^2 \mathbb{R}^4$ consisting only of simple bivectors.

5.2 G-Orthogonalization

We begin by rewriting the energy functional (5.2) and the constraint (5.6) in terms of the orthonormal basis of eigenvectors $(\tau_i)_{i\in\mathbb{N}_4}$ of the matrix G. The change of variable $\eta(t) := U^T\xi(t) \in H^1_\sharp(J, \mathbb{R}^4)$, allows us to write

$$\mathcal{G}_U(\eta) = \int_J \Lambda_{\mathfrak{g}}\dot{\eta}(t) \cdot \dot{\eta}(t)\mathrm{d}t, \tag{5.13}$$

with $\mathcal{G}_U(\eta) := \mathcal{G}(\xi) = \mathcal{G}(U\eta)$. For the constraint, we note that

$$\det(\xi|\dot{\xi}|\tau_i|\tau_j) = \det U \det(\eta|\dot{\eta}|e_i|e_j) = \det(\dot{\eta}|\eta|e_i|e_j), \tag{5.14}$$

since $\det U = -1$. Eventually, we can express the determinants more elegantly in terms of exterior products. In fact, by direct calculation one obtains $\det(\dot{\eta}|\eta|e_k|e_4) = (\dot{\eta}\wedge\eta, e_{k+1}\wedge e_{k+2})$ and $\det(\dot{\eta}|\eta|e_{k+1}|e_{k+2}) = (\dot{\eta} \wedge \eta, e_k \wedge e_4)$, for $k \in \mathbb{N}_3$ taken mod 3. Then, the isomorphism sending the standard basis $\{f_i\}_{i\in\mathbb{N}_6}$ of $\mathbb{R}^6$ onto the ordered basis

$$(e_{14}, e_{24}, e_{34}, e_{23}, e_{31}, e_{12}) \tag{5.15}$$

of $\bigwedge^2 \mathbb{R}^4$, where we write $e_{ij} := e_i \wedge e_j$, allows us to rewrite (5.6) as

$$\Lambda_{\mathfrak{h}}^{-1}\delta p = \int_J \dot{\eta}(t) \wedge \eta(t)\mathrm{d}t, \tag{5.16}$$

with $\Lambda_{\mathfrak{h}} := \mathrm{diag}(\mathfrak{h}_c, \mathfrak{h}_c, \mathfrak{h}_c, \mathfrak{h}_\theta, \mathfrak{h}_\theta, \mathfrak{h}_\theta)$.

5.3 Fourier transformation of the minimization problem

We denote by $\ell^2(\mathbb{R}^4)$ the space of sequences $\mathbf{u} := (u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^4$ such that the norm

$$||\mathbf{u}||_{\ell^2(\mathbb{R}^4)} := \sqrt{\sum_{n\in\mathbb{N}} |u_n|^2} \tag{5.17}$$

is finite. Consequently, we denote by $\dot{\ell}^2(\mathbb{R}^4)$ the Hilbert space of sequences $\mathbf{u} = (u_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{R}^4)$ such that $(nu_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{R}^4)$. As the elements in $H^1_\sharp(J, \mathbb{R}^4)$ are 2π-periodic, we can express η in terms of its Fourier series as

$$\eta(t) := \frac{1}{2}a_0 + \sum_{n\in\mathbb{N}} \cos(nt)a_n + \sin(nt)b_n, \tag{5.18}$$

with $(a_n, b_n)_{n\in\mathbb{N}} \in \dot{\ell}^2(\mathbb{R}^4) \times \dot{\ell}^2(\mathbb{R}^4)$. Substitution of the Fourier series of $\dot{\eta}$ into the energy functional (5.13) yields due to Parseval's equality

$$\mathcal{G}_U(\eta) := \int_J \Lambda_{\mathfrak{g}}\dot{\eta}(t) \cdot \dot{\eta}dt = \pi \sum_{n\in\mathbb{N}} n^2(\Lambda_{\mathfrak{g}}a_n \cdot a_n + \Lambda_{\mathfrak{g}}b_n \cdot b_n) \tag{5.19}$$

$$= \frac{1}{2}||\mathbf{u}||^2_{\ell^2(\mathbb{R}^4)} + \frac{1}{2}||\mathbf{v}||^2_{\ell^2(\mathbb{R}^4)}, \tag{5.20}$$

where we have set

$$\mathbf{u} := (u_n)_{n\in\mathbb{N}} := \sqrt{2\pi\Lambda_{\mathfrak{g}}}(na_n)_{n\in\mathbb{N}} \text{ and } \mathbf{v} := (v_n)_{n\in\mathbb{N}} := \sqrt{2\pi\Lambda_{\mathfrak{g}}}(nb_n)_{n\in\mathbb{N}}. \tag{5.21}$$

Clearly, we have $(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$. As a result of the $L^2_\sharp(J, \mathbb{R}^4)$-orthogonality of the Fourier trigonometric system, we can express the constraint (5.16) in terms of Fourier coefficients as $2\pi \sum_{n\in\mathbb{N}} \frac{1}{n}(nb_n) \wedge (na_n) = \Lambda_{\mathfrak{h}}^{-1}\delta p$. Returning to our old notation for a moment, we note that for any $n \in \mathbb{N}$, we have

$$2\pi n^2 \det(b_n|a_n|e_i|e_j) = \frac{\sqrt{\mathfrak{g}_i\mathfrak{g}_j}}{\sqrt{\det\Lambda_{\mathfrak{g}}}} \det(v_n|u_n|e_i|e_j), \tag{5.22}$$

and hence by setting $\tilde{\Lambda}_{\mathfrak{g}} := \operatorname{diag}(\mathfrak{g}_c, \mathfrak{g}_c, \mathfrak{g}_c, \sqrt{\mathfrak{g}_c\mathfrak{g}_\theta}, \sqrt{\mathfrak{g}_c\mathfrak{g}_\theta}, \sqrt{\mathfrak{g}_c\mathfrak{g}_\theta})$, with $\mathfrak{g}_1 := \mathfrak{g}_2 := \mathfrak{g}_3 := \mathfrak{g}_c$ and $\mathfrak{g}_4 := \mathfrak{g}_\theta$, we eventually find

$$\sqrt{\det\Lambda_{\mathfrak{g}}}(\Lambda_{\mathfrak{h}}\tilde{\Lambda}_{\mathfrak{g}})^{-1}\delta p = \sum_{n\in\mathbb{N}} \frac{v_n \wedge u_n}{n} \tag{5.23}$$

We thus have proved the following

Proposition 10. *The $H^1_\sharp(J, \mathbb{R}^4)$ minimization of the functional $\mathcal{G}_U$ given by (5.13) under the constraint (5.16) is equivalent to the minimization of the functional*

$$\mathcal{F}(\mathbf{u}, \mathbf{v}) := \frac{1}{2}||\mathbf{u}||^2_{\ell^2(\mathbb{R}^4)} + \frac{1}{2}||\mathbf{v}||^2_{\ell^2(\mathbb{R}^4)}, \tag{5.24}$$

defined in the product Hilbert space $\ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$ and subject to the constraint

$$\sum_{n\in\mathbb{N}} \frac{1}{n} v_n \wedge u_n = \omega \text{ with } \omega := \sqrt{\det\Lambda_{\mathfrak{g}}}(\Lambda_{\mathfrak{h}}\tilde{\Lambda}_{\mathfrak{g}})^{-1}\delta p, \tag{5.25}$$

where $\delta p \in \mathbb{R}^3 \times \mathfrak{so}(3)$ is a prescribed net displacement of position and orientation.

We observe that we are in a very similar situation as in [4] with the fundamental difference however that this time the constraint is a bivector. Nevertheless, it is natural to try to generalize the approach in [4], which in fact is true at least in the case of ω being simple.

5.4 The simple case

With the remarks from section 5.1, we are able to solve the constrained minimization problem of Proposition 10 in a similar manner to [4] whenever the net displacement is a simple bivector. In fact, we retrieve essentially the same result, i.e. that the optimal control curves are ellipses in a certain plane defined by the net displacement. Let us prove

Proposition 11. *If ω is a simple bivector, then for any $(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$ such that the constraint (5.25) holds, there exist two vectors $u, v \in \mathbb{R}^4$ such that for the sequences $\mathbf{u}_\star := \mathbf{e}_1 u$ and $\mathbf{v}_\star := \mathbf{e}_1 v \in \ell^2(\mathbb{R}^4)$ one has*

$$\mathcal{F}(\mathbf{u}_\star, \mathbf{v}_\star) = \mathcal{F}(\mathbf{u}, \mathbf{v}) \text{ and } v \wedge u = \omega. \tag{5.26}$$

Proof. If $\omega = 0$, then the proof is trivial. Thus, let us denote by $\hat{\omega}$ the unit bivector associated to ω. For a couple $(\mathbf{u}, \mathbf{v}) \in \ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$, we then choose $u, v \in \mathbb{R}^4$ such that the following relations hold:

$$|u| = ||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}, \quad |v| = ||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}, \quad \frac{u \wedge v}{|u \wedge v|} = \hat{\omega}. \tag{5.27}$$

The latter is possible since $\hat{\omega}$ is a simple bivector by hypothesis. Hence, there exist $x, y \in \mathbb{R}^4$ such that $\omega = x \wedge y$. Then we have for all $x', y' \in \operatorname{span}\{x, y\}$ such that $x' \wedge y' \neq 0$ that $x' \wedge y'/|x' \wedge y'| = \hat{\omega}$. Furthermore, we have $u \wedge v = ||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}(\sin\psi)\hat{\omega}$, where ψ is the angle between u and v. Therefore, the equality $u \wedge v = \omega$ can be satisfied by choosing the angle $\psi \in (0, \pi)$ such that

$$\sin\psi = \frac{|\omega|}{||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}} \tag{5.28}$$

This is possible under the condition that the right hand side of the previous equation is not greater than one. In fact, we have using the Cauchy-Schwarz inequality

$$|\omega| \leq \sum_{n\in\mathbb{N}} \frac{1}{n}|v_n \wedge u_n| \leq \sum_{n\in\mathbb{N}} |v_n||u_n| \leq ||\mathbf{u}||_{\ell^2(\mathbb{R}^4)}||\mathbf{v}||_{\ell^2(\mathbb{R}^4)}. \tag{5.29}$$

Finally, from (5.27) we obtain

$$\mathcal{F}(\mathbf{u}_\star, \mathbf{v}_\star) = \frac{1}{2}|u|^2 + \frac{1}{2}|v|^2 = \frac{1}{2}||\mathbf{u}||^2_{\ell^2(\mathbb{R}^4)} + \frac{1}{2}||\mathbf{v}||^2_{\ell^2(\mathbb{R}^4)}, \tag{5.30}$$

which concludes the proof. □

We immediately have

Corollary 12. *If ω is a simple bivector, the minimization problem for $\mathcal{F}$ in $\ell^2(\mathbb{R}^4)\times\ell^2(\mathbb{R}^4)$, under the constraint (5.25), is equivalent to the minimization in $\mathbb{R}^4 \times \mathbb{R}^4$ of the function*

$$f(u, v) := \frac{1}{2}|u|^2 + \frac{1}{2}|v|^2 \tag{5.31}$$

under the constraint

$$v \wedge u = \omega. \tag{5.32}$$

Proof. It suffices to observe that if $\mathcal{V}_\omega$ denotes the subset of $\ell^2(\mathbb{R}^4) \times \ell^2(\mathbb{R}^4)$ satisfying the constraint (5.25) and by V_ω the subset of $(u, v) \in \mathbb{R}^4 \times \mathbb{R}^4$ such that $u \wedge v = \omega$, then Proposition 11 yields

$$\min_{(\mathbf{u},\mathbf{v})\in\mathcal{V}_\omega} \mathcal{F}(\mathbf{u}, \mathbf{v}) = \min_{(u,v)\in V_\omega} \mathcal{F}(\mathbf{e}_1 u, \mathbf{e}_1 v) = \min_{(u,v)\in V_\omega} f(u, v). \tag{5.33}$$

□

Let us now prove the following

Proposition 13. *Any couple of vectors $(u_\star, v_\star) \in \mathbb{R}^4 \times \mathbb{R}^4$ minimizing the function f given in (5.31) and subject to the constraint (5.32) with $\omega = x\wedge y$ a simple bivector, is characterized by the following conditions:*

$$|u_\star|^2 = |v_\star|^2 = |\omega|, \quad u_\star \cdot v_\star = 0. \tag{5.34}$$

Therefore, any two vectors $\sigma, \mu \in \operatorname{span}\{x, y\}$ such that $|\sigma|^2 = |\mu|^2 = |\omega|$ and $\sigma \cdot \mu = 0$, the couple $(\sigma, \mu) \in \mathbb{R}^4 \times \mathbb{R}^4$ is a (global) constrained minimizer for f.

Remark. To construct such a couple, it suffices to scale x to get u and then find v by Gram-Schmidt orthogonalization.

Proof. Note that to find the minimizers of the problem (5.31) - (5.32), the constraint $u\wedge v = \omega$ implies the existence of a $\psi \in (0, \pi)$ such that $|u||v|\sin\psi = |\omega|$. Hence, the constrained minimization for f is equivalent to the unconstrained minimization of the function $\hat{f}$: $\mathbb{R}^4 \times (0, \pi) \to \mathbb{R}$ defined by

$$(u, \psi) \mapsto \frac{1}{2}|u|^2 + \frac{1}{2}\frac{|\omega|^2}{|u|^2 \sin^2\psi}, \tag{5.35}$$

whose stationary satisfy $\psi_\star = \frac{\pi}{2}$ and $|u_\star|^2 = |\omega|$. This shows the necessity of the conditions stated in (5.34). To show sufficiency of the condition, one observes that for any such points one has $\hat{f}(u_\star, \psi_\star) = |\omega|$. Indeed, for any $(u, \psi) \in D_{ijk,\psi} \in \mathbb{R}^4 \times (0, \pi)$ we have

$$\hat{f}(u, \psi) \geq \frac{1}{2}\frac{|u|^4 + |\omega|^2}{|u|^2} = |\omega| + \frac{1}{2}\frac{(|\omega| - |u|^2)^2}{|u|^2} \geq |\omega| = \hat{f}(u_\star, \psi_\star). \tag{5.36}$$

A straightforward calculation shows then that for such σ and μ, one has $\sigma \wedge \mu \mid\mid \hat{\omega}$ since they are in the plane spanned by the vectors x and y. By construction, one has $|\sigma \wedge \mu| = |\sigma||\mu| = |\omega|$. □

Pasting everything worked out above together leads to the final result of this section. We have

Theorem 14. *Let $\delta p \in \mathbb{R}^3 \times \mathfrak{so}(3) \simeq \bigwedge^2 \mathbb{R}^4$ be a prescribed net displacement. Moreover, assume that $\delta p \simeq x \wedge y$ identifies with a simple bivector. Then, any minimizer $\xi \in H^1_\sharp(J, \mathbb{R}^4)$ of the energy functional (5.2) subject to the constraint (5.6) is of the form*

$$\xi(t) := (\cos t)a + (\sin t)b, \tag{5.37}$$

i.e. an ellipse of $\mathbb{R}^4$ centered at the origin and contained in the plane spanned by the vectors a and b. The vectors $a, b \in \mathbb{R}^4$ are obtained as follows:

(i) We compute the vector ω via the relation

$$\omega := \operatorname{diag}\left(\frac{\sqrt{\mathfrak{g}_c \mathfrak{g}_\theta}}{\mathfrak{h}_c}, \frac{\sqrt{\mathfrak{g}_c \mathfrak{g}_\theta}}{\mathfrak{h}_c}, \frac{\sqrt{\mathfrak{g}_c \mathfrak{g}_\theta}}{\mathfrak{h}_c}, \frac{\mathfrak{g}_c}{\mathfrak{g}_\theta}, \frac{\mathfrak{g}_c}{\mathfrak{g}_\theta}, \frac{\mathfrak{g}_c}{\mathfrak{g}_\theta} \right) \delta p \simeq \tilde{x} \wedge \tilde{y}. \tag{5.38}$$

Then we consider two vectors $u, v \in \operatorname{span}\{\tilde{x}, \tilde{y}\}$ such that

$$|u|^2 = |v|^2 = |\omega| \text{ and } u \cdot v = 0. \tag{5.39}$$

(ii) We set $\hat{\omega} := \omega / |\omega|$ and we calculate the vectors a and b via the relations

$$a := \frac{U \Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{2\pi}} u, \quad b := \frac{U \Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{2\pi}} v. \tag{5.40}$$

We then have $v \wedge u = \omega$ and the minimum value of $\mathcal{G}$ is equal to $|\omega|$.

In addition, the vectors a and b are $\mathfrak{g}$-orthogonal, i.e. with respect to the inner product defined for every $x, y \in \mathbb{R}^4$ by $(x, y)_{\mathfrak{g}} := 2\pi \Lambda_{\mathfrak{g}} x \cdot y$, and have the same $\mathfrak{g}$-norm $|a|^2_{\mathfrak{g}} = |b|^2_{\mathfrak{g}} = |\omega|$.

Proof. From Proposition 13, Corollary 12 and then Proposition 10, we get that any $u, v \in \operatorname{span}\{\tilde{x}, \tilde{y}\}$ satisfying the relations

$$u \cdot v = 0, \quad |u|^2 = |v|^2 = |\omega|, \quad \omega := \sqrt{\det \Lambda_{\mathfrak{g}}} (\Lambda_{\mathfrak{h}} \tilde{\Lambda}_{\mathfrak{g}})^{-1} \delta p, \tag{5.41}$$

is associated to a (global) constrained minimizer for $\mathcal{G}_U$, via the curve $\eta(t) := (\cos t)\tilde{a} + (\sin t)\tilde{b}$, where the Fourier coefficients $\tilde{a}, \tilde{b} \in \mathbb{R}^4$ are related to ω (c.f. 5.21) by $(\sqrt{2\pi\Lambda_{\mathfrak{g}}})\tilde{a} = u$ and $(\sqrt{2\pi\Lambda_{\mathfrak{g}}})\tilde{b} = v$. The minimum value of the energy is then $\mathcal{G}_U(\eta) = |\omega|$.

Finally, in the $\mathfrak{g}$-orthogonal reference frame, the inner product is defined by $(x, y)_{\mathfrak{g}} := 2\pi \Lambda_{\mathfrak{g}} x \cdot y$ for $x, y \in \mathbb{R}^4$. Let us denote by $|\cdot|_{\mathfrak{g}}$ the associated norm. Then we have the following relations:

$$|\tilde{a}|^2_{\mathfrak{g}} = |\tilde{b}|^2_{\mathfrak{g}} = |\omega| \text{ and } (\tilde{a}, \tilde{b})_{\mathfrak{g}} = 0. \tag{5.42}$$

Applying the orthogonal map U to $\tilde{a}$ and $\tilde{b}$ finishes the proof. □

5.5 Some examples of simple net displacements

In light of Theorem 14 presented above, one might ask whether there are concrete cases in which the net displacement happens to be a simple bivector. It turns out that there is convenient correspondence for engineering purposes between certain subspaces of $\bigwedge^2 \mathbb{R}^4$ consisting only of simple bivectors and certain net displacements. Indeed, note that the condition in Lemma 9 is in particular satisfied if all coefficients corresponding to a certain index are zero, e.g. $\omega_{i4} = 0$ for $i \in \mathbb{N}_3$. This yields four subspaces D^*_{ijk} of $\bigwedge^2 \mathbb{R}^4$ consisting only of simple bivectors. Then, by inspection of the basis of $\bigwedge^2 \mathbb{R}^4$, we have the following correspondences:

$D^*_{123} \longleftrightarrow$ rotations around all three axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$
$D^*_{124} \longleftrightarrow$ translation in the $\hat{e}_1\hat{e}_2$-plane, rotation around the $\hat{e}_3$-axis
$D^*_{134} \longleftrightarrow$ translation in the $\hat{e}_1\hat{e}_3$-plane, rotation around the $\hat{e}_2$-axis
$D^*_{234} \longleftrightarrow$ translation in the $\hat{e}_2\hat{e}_3$-plane, rotation around the $\hat{e}_1$-axis

By comparison, the non-simple bivector $e_{12} + e_{34}$ corresponds to the net displacement $e_3 + L_3$, i.e. a screw motion. This kind of movement requires a solution to the general problem.

5.6 The general case

Let us now address the case of a general net displacement, i.e. $\delta p \in \mathbb{R}^6 \simeq \bigwedge^2 \mathbb{R}^4$ which identifies to a non-simple bivector. The observations from section 5.1 suggest that the optimal curve in the general case consist of two ellipses in certain planes reflecting the fact that δp is the sum of two orthogonal simple bivectors and any simple bivector represents a plane in $\mathbb{R}^4$. We will indeed be able to prove this result. However, we have to return to variational calculus to do so.

5.6.1 The optimization problem in the variational setting

More precisely, we will establish the structure of the optimal control curves using the Euler-Lagrange equation associated with the optimization problem. To that end, let us quickly recast the optimization problem in its original form:

$$\begin{cases} \text{Find } \min_{\xi \in \dot{H}^1_\sharp} \int_J G\dot{\xi}(t) \cdot \dot{\xi}(t)\mathrm{d}t \\ \text{under the constraints} \\ \int_J M_i \xi(t) \cdot \dot{\xi}(t)\mathrm{d}t = \delta p_i, i \in \mathbb{N}_6. \end{cases} \tag{5.43}$$

So, we are in the setting of a variational problem with six isoperimetric constraints. For $i \in \mathbb{N}_6$, denote by $K_i : \dot{H}^1_\sharp(J, \mathbb{R}^4) \times \dot{H}^1_\sharp(J, \mathbb{R}^4) \times J \to \mathbb{R}$ the map

$$K_i(\xi, \eta, t) := M_i \xi(t) \cdot \eta(t). \tag{5.44}$$

Furthermore, denote by $\mathcal{K}_i : \dot{H}^1_\sharp(J, \mathbb{R}^4) \to \mathbb{R}$ the functional

$$\mathcal{K}_i(\xi) := \int_J K_i(\xi, \dot{\xi}, t)\mathrm{d}t. \tag{5.45}$$

Then, the six isoperimetric constraints read $\mathcal{K}_i(\xi) = \delta p_i$ for $i \in \mathbb{N}_6$. Now, let us denote by $\delta\mathcal{G}$ and $\delta\mathcal{K}_i$ the first variations of $\mathcal{G}$ and the $\mathcal{K}_i$, respectively. Then a slight adaptation of Proposition 2.1.3. in [9] shows that $\xi \in \dot{H}^1_\sharp(J, \mathbb{R}^4)$ is a minimizer of (5.43) and if ξ is not critical for the constraints, i.e. $\delta\mathcal{K}_1(\xi), \dots, \delta\mathcal{K}_6(\xi)$ are linearly independent, then ξ satisfies the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\xi}}\left(G\dot{\xi}(t) \cdot \dot{\xi}(t) + \sum_{i \in \mathbb{N}_6} \mu_i K_i(\xi, \dot{\xi}, t)\right) = \frac{\partial}{\partial \xi}\left(G\dot{\xi}(t) \cdot \dot{\xi}(t) + \sum_{i \in \mathbb{N}_6} \mu_i K_i(\xi, \dot{\xi}, t)\right), \tag{5.46}$$

for some $\mu \in \mathbb{R}^6$. To make use use of equation (5.46), let us prove the following

Proposition 15. *Let $\delta p \in \mathbb{R}^6 \simeq \bigwedge^2 \mathbb{R}^4$ be non-simple and $\xi \in \dot{H}^1_\sharp(J, \mathbb{R}^4)$ a minimizer of (5.43). Then the functionals $\delta\mathcal{K}_1(\xi), \dots, \delta\mathcal{K}_6(\xi)$ are linearly independent.*

Proof. Let $\lambda_1, \dots, \lambda_6 \in \mathbb{R}$ be such taht $\sum_{i \in \mathbb{N}_6} \lambda_i \delta\mathcal{K}_i(\xi)$ is the zero functional in $(\dot{H}^1_\sharp(J, \mathbb{R}^4))^*$. Note that by the periodicity of ξ and integration by parts, we have for $h \in \dot{H}^1_\sharp(J, \mathbb{R}^4)$ that

$$\delta\mathcal{K}_i(\xi)h = 2\int_J M_i \dot{\xi}(t) \cdot h(t)\mathrm{d}t. \tag{5.47}$$

Setting $\Omega(\lambda) := \sum_{i \in \mathbb{N}_6} \lambda_i M_i$ for $\lambda \in \mathbb{R}^6$, we have that the functional $h \in \dot{H}^1_\sharp(J, \mathbb{R}^4) \mapsto \int_J \Omega(\lambda)\dot{\xi}(t) \cdot h(t)\mathrm{d}t$ is the zero functional. Let us take for the time being $h = \Omega(\lambda)\dot{\xi}$ but note that h is not necessarily in $\dot{H}^1_\sharp(J, \mathbb{R}^4)$. Then we have

$$0 = \int_J \Omega(\lambda)\dot{\xi}(t) \cdot h(t)\mathrm{d}t = ||h||^2_{L^2}, \tag{5.48}$$

i.e. $h \equiv 0$. This can only happen in the two cases $\xi(t) \in \ker \Omega(\lambda)$ for all $t \in J$ or $\lambda_1 = \dots = \lambda_6 = 0$, in the latter of which we are done. So let us suppose that we are in the former case.

Note that the matrix $\Omega(\lambda)$ is skew symmetric. Hence, we find an orthogonal transformation S such that $\Omega(\lambda) = S\Sigma(\lambda)S^T$ with

$$\Sigma(\lambda) = \operatorname{diag}\left(\begin{pmatrix} 0 & \nu_+(\lambda) \\ -\nu_+(\lambda) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \nu_-(\lambda) \\ -\nu_-(\lambda) & 0 \end{pmatrix}\right). \tag{5.49}$$

Denoting by P and Q the projections $\mathbb{R}^6 \to \mathbb{R}^3$ on the first and the last three coordinates, respectively, the scalars ν_{pm} are given by

$$\nu_\pm = 2\sqrt{3}\sqrt{A \pm \sqrt{A^2 - K}}, \tag{5.50}$$

with $A := \alpha^2|P\lambda|^2 + \delta^2|Q\lambda|^2$ and $K := 4\alpha^2\delta^2|P\lambda \cdot Q\lambda|^2$. Hence, only ν_- can vanish, which implies that $\ker \Omega(\lambda)$ is at most of dimension two. However, this is excluded by the Lemma below, as δp is assumed to be non-simple. Thus, we have $||h||_{L^2} > 0$. Now, approximation of $\dot{\xi}$ by smooth functions shows that $h \in \dot{H}^1_\sharp(J, \mathbb{R}^4) \mapsto \int_J \Omega(\lambda)\dot{\xi}(t) \cdot h(t)\mathrm{d}t$ cannot be the zero functional. Therefore, we must have $\lambda_1 = \cdots = \lambda_6 = 0$, which finishes the proof. □

To complete the proof of Proposition 15, let us prove the following

Lemma 16. *Let $\xi \in \dot{H}^1_\sharp(J, \mathbb{R}^4)$ be a control curve. Suppose that $\xi(t) \in D$ for all $t \in J$, where $D \subset \mathbb{R}^4$ is a plane through the origin. Then, the net displacement due to ξ is a simple bivector.*

Proof. Note that if ξ only takes values in the plane D, then the rescaled Fourier coefficients in relation (5.25) must also lie in a plane D' (not necessarily the same). Hence, the net displacement due to ξ lies in fact in $\bigwedge^2 D'$. Thus, it must be simple as D' is of dimension two. □

5.6.2 Structure of the solutions to the Euler-Lagrange equation

By direct computation, one finds that that after integration and utilizing the fact that ξ has average, that the Euler-Lagrange equation (5.46) reads

$$G\dot{\xi} - \Omega(\mu)\xi = 0, \tag{5.51}$$

where $\Omega(\mu)$ is the same skew symmetric matrix as above. To reveal the structure of the solutions to (5.51), we need to apply two basis transformations: First, setting $\eta := G^{1/2}\xi$ yields

$$\dot{\eta} - \tilde{\Omega}(\mu)\eta = 0, \tag{5.52}$$

where $\tilde{\Omega}(\mu) := \sum_{i\in\mathbb{N}_6} \mu_i G^{-1/2} M_i G^{-1/2}$, which is still a skew symmetric matrix. Hence, we find an orthogonal transformation Q such that $\tilde{\Omega} = Q\tilde{\Sigma}(\mu)Q^T$ with

$$\tilde{\Sigma}(\mu) \operatorname{diag}\left(\begin{pmatrix} 0 & \sigma_1(\mu) \\ -\sigma_1(\mu) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_2(\mu) \\ -\sigma_2(\mu) & 0 \end{pmatrix}\right). \tag{5.53}$$

So setting $\phi := Q\eta$, we have the equation $\dot{\phi} = \tilde{\Sigma}(\mu)\phi$, the solution of which is given by $\phi(t) = \exp\left(\tilde{\Sigma}(\mu)t\right)\phi_0$ with $\phi_0 := QG^{1/2}\xi(0)$. By defining the vectors

$$\phi_1 := (\phi_{0,1}, \phi_{0,2}, 0, 0)^T, \qquad \phi_1' := (\phi_{0,2}, -\phi_{0,1}, 0, 0)^T \tag{5.54}$$

$$\phi_2 := (0, 0, \phi_{0,3}, \phi_{0,4})^T, \qquad \phi_2' := (0, 0, -\phi_{0,4}, \phi_{0,3})^T, \tag{5.55}$$

we can recast the curve ϕ as

$$\phi(t) = \sum_{i\in\mathbb{N}_2} [\cos(\sigma_i(\mu)t)\phi_i + \sin(\sigma_i(\mu)t)\phi_i'], \qquad t \in J. \tag{5.56}$$

Resubstituting the basis transformations, we find that ξ must be of the form

$$\xi(t) = \sum_{i\in\mathbb{N}_2} [\cos(\sigma_i(\mu)t)a_i + \sin(\sigma_i(\mu)t)a_i'], \qquad t \in J. \tag{5.57}$$

Note that the vectors ϕ_1, ϕ_1', ϕ_2 and ϕ_2' are pairwise orthogonal. So, ϕ is in fact a rotation in two completely orthogonal planes of $\mathbb{R}^4$. In particular, this implies that the vectors a_1, a_1', a_2 and a_2' are orthogonal in the reference system of G.

5.6.3 Existence of integer eigenvalues

It is clear that the eigenvalues $\sigma_1(\mu)$ and $\sigma_2(\mu)$ must be integers for the curve ξ in (5.57) to be periodic. Hence, we must be able to choose $\mu \in \mathbb{R}^6$ such that $\sigma_1(\mu), \sigma_2(\mu) \in \mathbb{N}$. This question is addressed in this subsection.

First, note that we cannot have $sigma_1(\mu) = \sigma_2(\mu)$ since then the relation (5.25) would imply that δp is simple. Next, one finds that the eigenvalues are given by

$$\sigma_{1,2}(\mu) = \frac{2\sqrt{3}}{g_c\sqrt{g_\theta}}\sqrt{A \mp \sqrt{A^2 - K}}, \tag{5.58}$$

where this time $A := \alpha^2 g_c|P\mu|^2 + \delta^2 g_\delta|Q\mu|^2 > 0$ and $K := 4\alpha^2\delta^2 g_c g_\theta|P\mu \cdot Q\mu|^2 > 0$. Clearly, we have $\sigma_2(\mu) \geq \sigma_1(\mu)$. So let us find the values for $\mu \in \mathbb{R}^6$ such that

$$\sigma_1(\mu) = k \in \mathbb{N} \text{ and } \sigma_2(\mu) = lk, l \in \mathbb{N}. \tag{5.59}$$

This does not cover all possible pairs of integers. However, we will see in the next subsection that this is sufficient.

Imposing the above equations on the eigenvalues leads to the definition of the quadratic form associated to the positive definite matrix

$$\Gamma := \frac{48}{g_c^2 g_\theta}\begin{pmatrix} \alpha^2 g_c I_3 & 0 \\ 0 & \delta^2 g_\theta I_3 \end{pmatrix}. \tag{5.60}$$

Indeed, one can show that the solution sets to (5.59) are given by $E_+^{k,l} \cup E_-^{k,l}$, where

$$E_+^{k,l} := \{\mu \in \mathbb{R}^6 \mid \Gamma\mu \cdot \mu = \frac{(1+l^2)k^2}{l^2}\} \text{ and } E_-^{k,l} := \{\mu \in \mathbb{R}^6 \mid \Gamma\mu \cdot \mu = (1+l^2)k^2\}, \tag{5.61}$$

i.e. the solutions are the union of two specific ellipsoids in $\mathbb{R}^6$. Note in particular, that this calculation covers the case $\sigma_1(\mu) = 1$ and $\sigma_2(\mu) = 2$. So, we have assured the existence of periodic solutions to the Euler-Lagrange equation and we will show in the following subsection that the latter choice of values for the eigenvalues is indeed optimal.

5.6.4 The optimal control curves

Up to now, the geometric structure of the optimal control curves is clear from (5.57). So, let us now settle their explicit construction from a given non-simple net displacement. To that end, let us transform the Fourier modes a_1, a_1', a_2 and a_2' to the G-orthogonal by setting $\tilde{a}_1 := U^T a_1$ and similarly for the others. Then, substituting the curve into the energy functional G yields

$$\begin{aligned} \mathcal{G}(\xi) &= \sigma_1(\mu)^2[\Lambda_{\mathfrak{g}}\tilde{a}_1 \cdot \tilde{a}_1 + \Lambda_{\mathfrak{g}}\tilde{a}_1' \cdot \tilde{a}_1'] + \sigma_2(\mu)^2[\Lambda_{\mathfrak{g}}\tilde{a}_2 \cdot \tilde{a}_2 + \Lambda_{\mathfrak{g}}\tilde{a}_2' \cdot \tilde{a}_2'] & (5.62)\\ &= \frac{\sigma_1(\mu)}{2\pi}[|u_{\sigma_1}|^2 + |v_{\sigma_1}|^2] + \frac{\sigma_2(\mu)}{2\pi}[|u_{\sigma_2}|^2 + |v_{\sigma_2}|^2], & (5.63) \end{aligned}$$

with

$$u_{\sigma_i} := \sqrt{2\pi\sigma_i(\mu)\Lambda_{\mathfrak{g}}}\tilde{a}_i \text{ and } \qquad v_{\sigma_i} := \sqrt{2\pi\sigma_i(\mu)\Lambda_{\mathfrak{g}}}\tilde{a}_i', \tag{5.64}$$

for $i \in \mathbb{N}_2$. In particular, it follows from relation (5.25) that

$$\sqrt{\det \Lambda_{\mathfrak{g}}}(\Lambda_{\mathfrak{h}}\Lambda_{\mathfrak{g}})^{-1}\delta p = v_{\sigma_1} \wedge u_{\sigma_1} + v_{\sigma_2} \wedge u_{\sigma_2}. \tag{5.65}$$

We observe that the latter relation in the sense of (5.25) is independent of the indices σ_1 and σ_2. So, we can choose $\sigma_1(\mu) = 1$ and $\sigma_2(\mu) = 2$ up to permutation such that $|u_{\sigma_1}|^2 + |v_{\sigma_1}|^2 \geq |u_{\sigma_2}|^2 + |v_{\sigma_2}|^2$.

Moreover[2], note that the vectors u_1, v_1, u_2, and v_2 are pairwise orthogonal due to the G-orthogonality of a_1, a_1', a_2, and a_2'. Eventually, due to the bilinearity of the exterior product, we can always suppose that $|u_1| = |v_1|$ and $|u_2| = |v_2|$, which further minimizes the energy. Finally, we can summarize the argument above in the following

[2]Recall that we have shown in the section 5.1 that the reverse is also true.

Theorem 17. *Let $\delta p \in \mathbb{R}^6 \simeq \bigwedge^2 \mathbb{R}^4$ a net displacement identifying to a non-simple bivector, then the energy minimizing curve which attains δp is given by*

$$\xi(t) := cos(t)a_1 + sin(t)a_1' + cos(2t)a_2 + sin(2t)a_2', t \in J, \tag{5.66}$$

where the vectors $a_1, a_1', a_2, a_2' \in \mathbb{R}^4$ are determined as follows:

(i) First we decompose the still simple bivector $\omega := \sqrt{\det \Lambda_{\mathfrak{g}}}(\Lambda_{\mathfrak{h}}\tilde{\Lambda}_{\mathfrak{g}})^{-1}\delta p$ into the sum of two orthogonal simple bivectors, i.e.

$$\omega = v_1 \wedge u_1 + v_2 \wedge u_2, \tag{5.67}$$

with all four vectors u_1, u_2, v_1, v_2 pairwise orthogonal and $|u_i| = |v_i|$, for $i \in \mathbb{N}_2$.

(ii) If necessary, we permute the indices such that $|u_2|^2 + |v_2|^2 \leq |u_1|^2 + |v_1|^2$.

(iii) We set

$$a_1 := \frac{U\Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{2\pi}}u_1, \qquad a_1' := \frac{U\Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{2\pi}}v_1 \tag{5.68}$$

$$a_2 := \frac{U\Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{4\pi}}u_2, \qquad a_2' := \frac{U\Lambda_{\mathfrak{g}}^{-1/2}}{\sqrt{4\pi}}v_2. \tag{5.69}$$

Then the four vectors a_1, a_1', a_2, a_2' are $\mathfrak{g}$-orthogonal and the minimum value of the energy functional is $\frac{1}{2\pi}[|u_1|^2 + |v_2|^2] + \frac{1}{\pi}[|u_2|^2 + |v_2|^2]$.

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7. Appendix

7.1 Complete calculation of the first order term $\mathcal{H}_0$

In analogy to the calculation of the matrices M_k, we define $\mathcal{T}_{0,c}(\xi\otimes\eta) := \frac{1}{2}[\mathcal{H}_{c,0}(\xi\otimes\eta) + \mathcal{H}_{c,0}(\eta\otimes\xi)]$ and similarly $\mathcal{T}_{0,\theta}(\xi\otimes\eta)$ for $\xi,\eta\in\mathbb{R}^4$ such that $\mathcal{T}_{0,c}(\xi\otimes\eta) = \mathcal{T}_{0,c}(\eta\otimes\xi)$ as well as $\mathcal{T}_{0,\theta}(\xi\otimes\eta) = \mathcal{T}_{0,\theta}(\eta\otimes\xi)$. Clearly, $\mathcal{T}_{c,0}$ and $\mathcal{T}_{\theta,0}$ satisfy the same symmetry relations as $\mathcal{H}_{c,0}$ and $\mathcal{H}_{\theta,0}$ and thus we find for any permutation matrix P_{ij} and the corresponding reflection S_{ij} that

$$S_{ij}\mathcal{T}_{c,0}(P_{ij}\xi\otimes P_{ij}\eta) = \mathcal{T}_{c,0}(\xi\otimes\eta) \tag{7.1}$$

$$-S_{ij}\mathcal{T}_{\theta,0}(P_{ij}\xi\otimes P_{ij}\eta) = \mathcal{T}_{\theta,0}(\xi\otimes\eta) \tag{7.2}$$

We treat the spatial part first. Since the matrix S_{ij} represents the reflection at the plane spanned by the remaining two arms z_k and z_l, equation (8.1) implies that $\mathcal{T}_{c,0}(e_i\otimes e_j)\in\operatorname{span}\{z_k,z_l\}$. Next, we find, using again (8.1) and the fact that the reflection S_{kl} is an orthogonal transformation, that

$$\mathcal{T}_{c,0}(e_i\otimes e_j)\cdot z_k = S_{kl}\mathcal{T}_{c,0}(e_i\otimes e_j)\cdot S_{kl}z_k = \mathcal{T}_{c,0}(e_i\otimes e_j)\cdot z_l, \tag{7.3}$$

from which we deduce that $\mathcal{T}_{c,0}(e_i\otimes e_j) = \beta_{ij}(z_k+z_l)$ for some scalar $\beta_{ij}\in\mathbb{R}$. However, the same holds for $\mathcal{T}_{c,0}(e_i\otimes e_k)$ and we find

$$\beta_{ik}(z_j+z_l) = \mathcal{T}_{c,0}(e_i\otimes e_k) = S_{jk}\mathcal{T}_{c,0}(e_i\otimes e_j) = S_{jk}\beta_{ij}(z_k+z_l) = \beta_{ij}(z_j+z_l). \tag{7.4}$$

Since the vectors z_1,z_2,z_3 and z_4 are normalized and enclose pairwise the same angle, we can conclude that $\beta_{ik}=\beta_{ij}$ or more generally that $\mathcal{T}_{c,0}(e_i\otimes e_j) = \beta(z_k+z_l)$ for all $i\neq j\in\mathbb{N}_4$. Furthermore, for the term $\mathcal{T}_{c,0}(e_i\otimes e_i)$ we find in a similar fashion that $\mathcal{T}_{c,0}(e_i\otimes e_i)\in\operatorname{span}\{z_i,z_j\}$, $\mathcal{T}_{c,0}(e_i\otimes e_i)\in\operatorname{span}\{z_i,z_k\}$ and $\mathcal{T}_{c,0}(e_i\otimes e_i)\in\operatorname{span}\{z_i,z_l\}$. By noting that the line of intersection of these three planes is $\{\lambda z_i|\lambda\in\mathbb{R}\}$, we obtain $\mathcal{T}_{c,0}(e_i\otimes e_i) = \lambda_i z_i$ for some $\lambda_i\in\mathbb{R}$. Again by using the orthogonality of the reflections Sij, we find that

$$\lambda_j = \mathcal{T}_{c,0}(e_j\otimes e_j)\cdot z_j = S_{ij}\mathcal{T}_{c,0}(e_j\otimes e_j)\cdot S_{ij}z_j = \mathcal{T}_{c,0}(e_i\otimes e_i)\cdot z_i = \lambda_i. \tag{7.5}$$

Hence, we have $\mathcal{T}_{c,0}(e_i\otimes e_i) = \lambda z_i$ for all $i\in\mathbb{N}_4$ and some $\lambda\in\mathbb{R}$.

For the rotational part, we observe that equation (8.2) implies that on the one hand we have $\mathcal{T}_{\theta,0}(e_i\otimes e_j) = -S_{ij}\mathcal{T}_{\theta,0}(e_i\otimes e_j)$ and on the other hand that $\mathcal{T}_{\theta,0}(e_i\otimes e_j) = -S_{kl}\mathcal{T}_{\theta,0}(e_i\otimes e_j)$. However, the first equation implies that $\mathcal{T}_{\theta,0}(e_i\otimes e_j)$ is proportional to $z_k\times z_l$, while the second implies that $\mathcal{T}_{\theta,0}(e_i\otimes e_j)$ is proportional to $z_i\times z_j$. Yet, from this we conclude that necessarily $\mathcal{T}_{\theta,0}(e_i\otimes e_j)=0$ since $z_k\times z_l\in\operatorname{span}\{e_i,e_j\}$ and vice-versa. The argument for $\mathcal{T}_{\theta,0}(e_i\otimes e_i)=0$ is very similar and thus omitted.

In conclusion, the matrices $N_k := (\mathcal{T}_{c,0}(e_i\otimes e_j)\cdot\hat{e}_k)_{i,j\in\mathbb{N}_4} = \frac{1}{2}(A_k+A_k^T), k\in\mathbb{N}_3$ and $N_{k+3} := (\mathcal{T}_{\theta,0}(e_i\otimes e_j)\cdot\hat{e}_k)_{i,j\in\mathbb{N}_4} = \frac{1}{2}(B_k+B_k^T), k\in\mathbb{N}_3$ are given by

$$N_1 = \begin{pmatrix} \frac{2\sqrt{2}\lambda}{3} & -\frac{\sqrt{2}\beta}{3} & -\frac{\sqrt{2}\beta}{3} & -\frac{2\sqrt{2}\beta}{3} \\ -\frac{\sqrt{2}\beta}{3} & -\frac{\sqrt{2}\lambda}{3} & \frac{2\sqrt{2}\beta}{3} & \frac{\sqrt{2}\beta}{3} \\ -\frac{\sqrt{2}\beta}{3} & \frac{2\sqrt{2}\beta}{3} & -\frac{\sqrt{2}\lambda}{3} & \frac{\sqrt{2}\beta}{3} \\ -\frac{2\sqrt{2}\beta}{3} & \frac{\sqrt{2}\beta}{3} & \frac{\sqrt{2}\beta}{3} & 0 \end{pmatrix}, \tag{7.6}$$

$$N_2 = \begin{pmatrix} 0 & \sqrt{\frac{2}{3}}\beta & -\sqrt{\frac{2}{3}}\beta & 0 \\ \sqrt{\frac{2}{3}}\beta & -\sqrt{\frac{2}{3}}\lambda & 0 & \sqrt{\frac{2}{3}}\beta \\ -\sqrt{\frac{2}{3}}\beta & 0 & \sqrt{\frac{2}{3}}\lambda & -\sqrt{\frac{2}{3}}\beta \\ 0 & \sqrt{\frac{2}{3}}\beta & -\sqrt{\frac{2}{3}}\beta & 0 \end{pmatrix}, N_3 = \begin{pmatrix} -\frac{\lambda}{3} & \frac{2\beta}{3} & \frac{2\beta}{3} & -\frac{2\beta}{3} \\ \frac{2\beta}{3} & -\frac{\lambda}{3} & \frac{2\beta}{3} & -\frac{2\beta}{3} \\ \frac{2\beta}{3} & \frac{2\beta}{3} & -\frac{\lambda}{3} & -\frac{2\beta}{3} \\ -\frac{2\beta}{3} & -\frac{2\beta}{3} & -\frac{2\beta}{3} & \lambda \end{pmatrix} \tag{7.7}$$

$$N_4 = N_5 = N_6 = 0. \tag{7.8}$$

Bibliography

[1] A. A. Agrachev and Y. L. Sachkov, *Control Theory from the Geometric Viewpoint*. Springer Berlin Heidelberg, 2004. DOI: 10.1007/978-3-662-06404-7.

[2] F. Alouges, A. DeSimone, L. Heltai, A. Lefebvre-Lepot, and B. M. and, "Optimally swimming stokesian robots," *Discrete & Continuous Dynamical Systems - B*, vol. 18, no. 5, pp. 1189–1215, 2013. DOI: 10.3934/dcdsb.2013.18.1189.

[3] F. Alouges and G. D. Fratta, "Parking 3-sphere swimmer: II. the long-arm asymptotic regime," *The European Physical Journal E*, vol. 43, no. 2, Feb. 2020. DOI: 10.1140/epje/i2020-11932-5.

[4] ——, "Parking 3-sphere swimmer. i. energy minimizing strokes," Sep. 2017. DOI: 10.31219/osf.io/7sfbj.

[5] J. E. Avron, O. Gat, and O. Kenneth, "Optimal swimming at low reynolds numbers," *Physical Review Letters*, vol. 93, no. 18, Oct. 2004. DOI: 10.1103/physrevlett.93.186001.

[6] A. DeSimone, F. Alouges, and A. Lefebvre, "Biological fluid dynamics, non-linear partial differential equations," in *Mathematics of Complexity and Dynamical Systems*, R. A. Meyers, Ed. New York, NY: Springer New York, 2011, pp. 26–31, ISBN: 978-1-4614-1806-1. DOI: 10.1007/978-1-4614-1806-1_3. [Online]. Available: https://doi.org/10.1007/978-1-4614-1806-1_3.

[7] B. C. Hall, *Lie Groups, Lie Algebras, and Representations*. Springer International Publishing, 2015. DOI: 10.1007/978-3-319-13467-3.

[8] N. Hitchin. (2003). "Projective geometry - b3 course," [Online]. Available: http://people.maths.ox.ac.uk/hitchin/files/LectureNotes/Projective_geometry/Chapter_3_Exterior.pdf.

[9] H. Kielhöfer, *Calculus of Variations*. Springer International Publishing, 2018. DOI: 10.1007/978-3-319-71123-2.

[10] M. J. Lighthill, "On the squirming motion of nearly spherical deformable bodies through liquids at very small reynolds numbers," *Communications on Pure and Applied Mathematics*, vol. 5, no. 2, pp. 109–118, May 1952. DOI: 10.1002/cpa.3160050201.

[11] P. Lounesto, *Clifford Algebras and Spinors*. Cambridge University Press, Jun. 15, 2006, 352 pp., ISBN: 0521005515. [Online]. Available: https://www.ebook.de/de/product/2991827/pertti_lounesto_clifford_algebras_and_spinors.html.

[12] A. Najafi and R. Golestanian, "Simple swimmer at low reynolds number: Three linked spheres," *Physical Review E*, vol. 69, no. 6, Jun. 2004. DOI: 10.1103/physreve.69.062901.

[13] E. M. Purcell, "Life at low reynolds number," *American Journal of Physics*, vol. 45, no. 1, pp. 3–11, Jan. 1977. DOI: 10.1119/1.10903.