2

Geometric algebra in two and three dimensions

Geometric algebra was introduced in the nineteenth century by the English mathematician William Kingdon Clifford (figure 2.1). Clifford appears to have been one of the small number of mathematicians at the time to be significantly influenced by Grassmann's work. Clifford introduced his *geometric algebra* by uniting the inner and outer products into a single *geometric* product. This is associative, like Grassmann's product, but has the crucial extra feature of being *invertible*, like Hamilton's quaternion algebra. Indeed, Clifford's original motivation was to unite Grassmann's and Hamilton's work into a single structure. In the mathematical literature one often sees this subject referred to as *Clifford algebra*. We have chosen to follow the example of David Hestenes, and many other modern researchers, by returning to Clifford's original choice of name — *geometric algebra*. One reason for this is that the first published definition of the geometric product was due to Grassmann, who introduced it in the second *Ausdehnungslehre*. It was Clifford, however, who realised the great potential of this product and who was responsible for advancing the subject.

In this chapter we introduce the basics of geometric algebra in two and three dimensions in a way that is intended to appear natural and geometric, if somewhat informal. A more formal, axiomatic approach is delayed until chapter 4, where geometric algebra is defined in arbitrary dimensions. The meaning of the various terms in the algebra we define will be illustrated with familiar examples from geometry. In so doing we will also uncover how Hamilton's quaternions fit into geometric algebra, and understand where it was that Hamilton and his followers went wrong in their treatment of three-dimensional geometry. One of the most powerful applications of geometric algebra is to rotations, and these are considered in some detail in this chapter. It is well known that rotations in a plane can be efficiently handled with complex numbers. We will see how to extend this idea to rotations in three-dimensional space. This representation has many applications in classical and quantum physics.

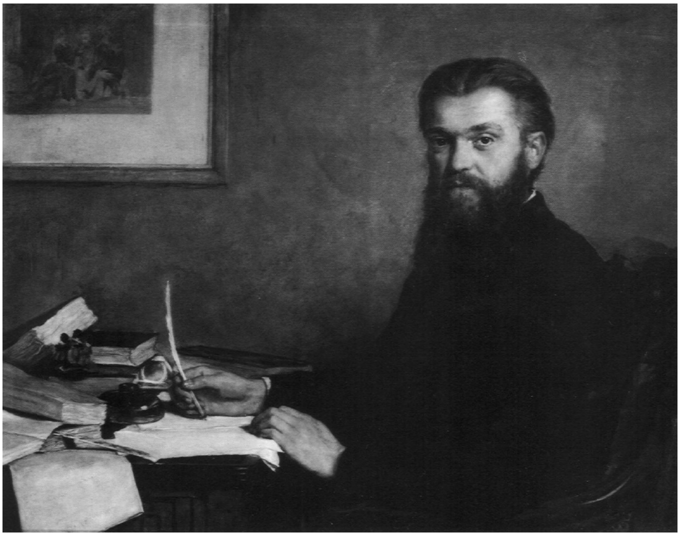

Figure 2.1 *William Kingdon Clifford 1845–1879.* Born in Exeter on 4 May 1845, his father was a justice of the peace and his mother died early in his life. After school he went to King's College, London and then obtained a scholarship to Trinity College, Cambridge, where he followed the likes of Thomson and Maxwell in becoming Second Wrangler. There he also achieved a reputation as a daring athlete, despite his slight frame. He was recommended for a fellowship at Trinity College by Maxwell, and in 1871 took the Professorship of Applied Mathematics at University College, London. He was made a Fellow of the Royal Society at the extremely young age of 29. He married Lucy in 1875, and their house became a fashionable meeting place for scientists and philosophers. As well as being one of the foremost mathematicians of his day, he was an accomplished linguist, philosopher and author of children's stories. Sadly, his insatiable appetite for physical and mental exercise was not matched by his physique, and in 1878 he was instructed to stop work and leave England for the Mediterranean. He returned briefly, only for his health to deteriorate further in the English climate. He left for Madeira, where he died on 3 March 1879 at the age of just 33. Further details of his life can be found in the book *Such Silver Currents* (Chisholm, 2002). Portrait by John Collier (©The Royal Society).

2.1 A new product for vectors

In chapter 1 we studied various products for vectors, including the symmetric scalar (or inner) product and the antisymmetric exterior (or outer) product. In two dimensions, we showed how to interpret the result of the complex product zw^* (section 1.3). The scalar term is the inner product of the two vectors representing the points in the complex plane, and the imaginary term records their directed area. Furthermore, the scalar term is symmetric, and the imaginary term is antisymmetric in the two arguments. Clifford's powerful idea was to generalise this product to arbitrary dimensions by replacing the imaginary term with the outer product. The result is the *geometric product* and is written simply as ab. The result is the sum of a scalar and a bivector, so

$$ab = a\cdot b + a\wedge b. \tag{2.1}$$

This sum of two distinct objects — a scalar and a bivector — looks strange at first and goes against the rule that one should only add like objects. This is the feature of geometric algebra that initially causes the greatest difficulty, in much the same way that $i^2 = -1$ initially unsettles most school children. So how is the sum on the right-hand side of equation (2.1) to be viewed? The answer is that it should be viewed in precisely the same way as the addition of a real and an imaginary number. The result is neither purely real nor purely imaginary — it is a mixture of two different objects which are combined to form a single complex number. Similarly, the addition of a scalar to a bivector enables us to keep track of the separate components of the product ab. The advantages of this are precisely the same as the advantages of complex arithmetic over working with the separate real and imaginary parts. This analogy between *multivectors* in geometric algebra and complex numbers is more than a mere pedagogical device. As we shall discover, geometric algebra encompasses both complex numbers and quaternions. Indeed, Clifford's achievement was to generalise complex arithmetic to spaces of arbitrary dimensions.

From the symmetry and antisymmetry of the terms on the right-hand side of equation (2.1) we see that

$$ba = b\cdot a + b\wedge a = a\cdot b - a\wedge b. \tag{2.2}$$

It follows that

$$a\cdot b = \tfrac{1}{2}(ab + ba) \tag{2.3}$$

and

$$a\wedge b = \tfrac{1}{2}(ab - ba). \tag{2.4}$$

We can thus define the inner and outer products in terms of the geometric product. This forms the starting point for an axiomatic development of geometric algebra, which is presented in chapter 4.

If we form the product of a and the parallel vector λa we obtain

$$a(\lambda a) = \lambda a\cdot a + \lambda a\wedge a = \lambda a\cdot a, \tag{2.5}$$

which is therefore a pure scalar. It follows similarly that a^2 is a scalar, so we can write $a^2 = |a|^2$ for the square of the length of a vector. If instead a and b

are perpendicular vectors, their product is

$$ab = a \cdot b + a \wedge b = a \wedge b \tag{2.6}$$

and so is a pure bivector. We also see that

$$ba = b \cdot a + b \wedge a = -a \wedge b = -ab, \tag{2.7}$$

which shows us that *orthogonal vectors anticommute*. The geometric product between general vectors encodes the relative contributions of both their parallel and perpendicular components, summarising these in the separate scalar and bivector terms.

2.2 An outline of geometric algebra

Clifford went further than just allowing scalars to be added to bivectors. He defined an algebra in which elements of any type could be added or multiplied together. This is what he called a *geometric algebra*. Elements of a geometric algebra are called *multivectors* and these form a linear space — scalars can be added to bivectors, and vectors, etc. Geometric algebra is a *graded* algebra, and elements of the algebra can be broken up into terms of different *grade*. The scalar objects are assigned grade-0, the vectors grade-1, the bivectors grade-2 and so on. Essentially, the grade of the object is the dimension of the hyperplane it specifies. The term 'grade' is preferred to 'dimension', however, as the latter is regularly employed for the size of a linear space. We denote the operation of projecting onto the terms of a chosen grade by $\langle \, \rangle_r$, so $\langle ab \rangle_2$ denotes the grade-2 (bivector) part of the geometric product ab. That is,

$$\langle ab \rangle_2 = a \wedge b. \tag{2.8}$$

The subscript 0 on the scalar term is usually suppressed, so we also have

$$\langle ab \rangle_0 = \langle ab \rangle = a \cdot b. \tag{2.9}$$

Arbitrary multivectors can also be multiplied together with the geometric product. To do this we first extend the geometric product of two vectors to an arbitrary number of vectors. This is achieved with the additional rule that the geometric product is *associative*:

$$a(bc) = (ab)c = abc. \tag{2.10}$$

The associativity property enables us to remove the brackets and write the product as abc. Arbitrary multivectors can now be written as sums of products of vectors. The geometric product of multivectors therefore inherits the two main properties of the product for vectors, which is to say it is associative:

$$A(BC) = (AB)C = ABC, \tag{2.11}$$

and distributive over addition:

$$A(B + C) = AB + AC. \tag{2.12}$$

Here $A, B, \ldots, C$ denote multivectors containing terms of arbitrary grade.

The associativity property ensures that it is now possible to divide by vectors, thus realising Hamilton's goal. Suppose that we know that $ab = C$, where C is some combination of a scalar and bivector. We find that

$$Cb = (ab)b = a(bb) = ab^2, \tag{2.13}$$

so we can define $b^{-1} = b/b^2$, and recover a from

$$a = Cb^{-1}. \tag{2.14}$$

This ability to divide by vectors gives the algebra considerable power.

As an example of these axioms in action, consider forming the square of the bivector $a \wedge b$. The properties of the geometric product allow us to write

$$\begin{aligned}(a\wedge b)(a\wedge b) &= (ab - a\cdot b)(a\cdot b - ba) \\ &= -ab^2 a - (a\cdot b)^2 + a\cdot b(ab + ba) \\ &= (a\cdot b)^2 - a^2 b^2 \\ &= -a^2 b^2 \sin^2(\theta),\end{aligned} \tag{2.15}$$

where we have assumed that $a\cdot b = |a|\,|b|\,\cos(\theta)$. The magnitude of the bivector $a\wedge b$ is therefore equal to the area of the parallelogram with sides defined by a and b. Manipulations such as these are commonplace in geometric algebra, and can provide simplified proofs of a number of useful results.

2.3 Geometric algebra of the plane

The easiest way to understand the geometric product is by example, so consider a two-dimensional space (a plane) spanned by two orthonormal vectors e_1 and e_2. These basis vectors satisfy

$$\mathsf{e}_1{}^2 = \mathsf{e}_2{}^2 = 1, \qquad \mathsf{e}_1\cdot\mathsf{e}_2 = 0. \tag{2.16}$$

The final entity present in the algebra is the bivector $\mathsf{e}_1 \wedge \mathsf{e}_2$. This is the highest grade element in the algebra, since the outer product of a set of dependent vectors is always zero. The highest grade element in a given algebra is usually called the *pseudoscalar*, and its grade coincides with the dimension of the underlying vector space.

The full algebra is spanned by the basis set

$$\begin{array}{ccc} 1 & \{\mathsf{e}_1, \mathsf{e}_2\} & \mathsf{e}_1 \wedge \mathsf{e}_2 \\ \text{1 scalar} & \text{2 vectors} & \text{1 bivector} \end{array}. \tag{2.17}$$

We denote this algebra $\mathcal{G}_2$. Any multivector can be decomposed in this basis, and sums and products can be calculated in terms of this basis. For example, suppose that the multivectors A and B are given by

$$A = \alpha_0 + \alpha_1 \mathsf{e}_1 + \alpha_2 \mathsf{e}_2 + \alpha_3 \mathsf{e}_1 \wedge \mathsf{e}_2,$$
$$B = \beta_0 + \beta_1 \mathsf{e}_1 + \beta_2 \mathsf{e}_2 + \beta_3 \mathsf{e}_1 \wedge \mathsf{e}_2,$$

then their sum $S = A + B$ is given by

$$S = (\alpha_0 + \beta_0) + (\alpha_1 + \beta_1)\mathsf{e}_1 + (\alpha_2 + \beta_2)\mathsf{e}_2 + (\alpha_3 + \beta_3)\mathsf{e}_1 \wedge \mathsf{e}_2. \tag{2.18}$$

This result for the addition of multivectors is straightforward and unsurprising. Matters become more interesting, however, when we start forming products.

2.3.1 The bivector and its products

To study the properties of the bivector $\mathsf{e}_1 \wedge \mathsf{e}_2$ we first recall that for orthogonal vectors the geometric product is a pure bivector:

$$\mathsf{e}_1 \mathsf{e}_2 = \mathsf{e}_1 \cdot \mathsf{e}_2 + \mathsf{e}_1 \wedge \mathsf{e}_2 = \mathsf{e}_1 \wedge \mathsf{e}_2, \tag{2.19}$$

and that orthogonal vectors anticommute:

$$\mathsf{e}_2 \mathsf{e}_1 = \mathsf{e}_2 \wedge \mathsf{e}_1 = -\mathsf{e}_1 \wedge \mathsf{e}_2 = -\mathsf{e}_1 \mathsf{e}_2. \tag{2.20}$$

We can now form products in which $\mathsf{e}_1 \mathsf{e}_2$ multiplies vectors from the left and the right. First from the left we find that

$$(\mathsf{e}_1 \wedge \mathsf{e}_2)\mathsf{e}_1 = (-\mathsf{e}_2 \mathsf{e}_1)\mathsf{e}_1 = -\mathsf{e}_2 \mathsf{e}_1 \mathsf{e}_1 = -\mathsf{e}_2 \tag{2.21}$$

and

$$(\mathsf{e}_1 \wedge \mathsf{e}_2)\mathsf{e}_2 = (\mathsf{e}_1 \mathsf{e}_2)\mathsf{e}_2 = \mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_2 = \mathsf{e}_1. \tag{2.22}$$

If we assume that e_1 and e_2 form a right-handed pair, we see that left-multiplication by the bivector rotates vectors 90° clockwise (i.e. in a negative sense). Similarly, acting from the right

$$\mathsf{e}_1(\mathsf{e}_1 \mathsf{e}_2) = \mathsf{e}_2, \qquad \mathsf{e}_2(\mathsf{e}_1 \mathsf{e}_2) = -\mathsf{e}_1. \tag{2.23}$$

So right multiplication rotates 90° anticlockwise — a positive sense.

The final product in the algebra to consider is the square of the bivector $\mathsf{e}_1 \wedge \mathsf{e}_2$:

$$(\mathsf{e}_1 \wedge \mathsf{e}_2)^2 = \mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_1 \mathsf{e}_2 = -\mathsf{e}_1 \mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_2 = -1. \tag{2.24}$$

Geometric considerations have led naturally to a quantity which squares to -1. This fits with the fact that two successive left (or right) multiplications of a vector by $\mathsf{e}_1 \mathsf{e}_2$ rotates the vector through 180°, which is equivalent to multiplying by -1. The fact that we now have a firm geometric picture for objects whose algebraic square is -1 opens up the possibility of providing a geometric interpretation for the unit imaginary employed throughout physics, a theme which will be explored further in this book.

2.3.2 Multiplying multivectors

Now that all of the individual products have been found, we can compute the product of the two general multivectors A and B of equation (2.18),

$$AB = M = \mu_0 + \mu_1 \mathsf{e}_1 + \mu_2 \mathsf{e}_2 + \mu_3 \mathsf{e}_1 \mathsf{e}_2, \tag{2.25}$$

where

$$\begin{aligned} \mu_0 &= \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3, \\ \mu_1 &= \alpha_0\beta_1 + \alpha_1\beta_0 + \alpha_3\beta_2 - \alpha_2\beta_3, \\ \mu_2 &= \alpha_0\beta_2 + \alpha_2\beta_0 + \alpha_1\beta_3 - \alpha_3\beta_1, \\ \mu_3 &= \alpha_0\beta_3 + \alpha_3\beta_0 + \alpha_1\beta_2 - \alpha_2\beta_1. \end{aligned} \tag{2.26}$$

The full product shown here is actually rarely used, but writing it out explicitly does emphasise some of its key features. The product is always well defined, and the algebra is closed under it. Indeed, the product could easily be made an intrinsic part of a computer language, in the same way that complex arithmetic is already intrinsic to some languages. The basis vectors can also be represented with matrices, for example

$$\boldsymbol{E}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.27}$$

(Verifying that these satisfy the required algebraic relations is left as an exercise.) Geometric algebras in general are associative algebras, so it is always possible to construct a matrix representation for them. The problem with this is that the matrices hide the geometric content of the elements they represent. Much of the mathematical literature does focus on matrix representations, and for this work the term *Clifford algebra* is appropriate. For the applications in this book, however, the underlying geometry is the important feature of the algebra and matrix representations are usually redundant. *Geometric algebra* is a much more appropriate name for this subject.

2.3.3 Connection with complex numbers

It is clear that there is a close relationship between geometric algebra in two dimensions and the algebra of complex numbers. The unit bivector squares to -1 and generates rotations through $90°$. The combination of a scalar and a bivector, which is formed naturally via the geometric product, can therefore be viewed as a complex number. We write this as

$$Z = u + v\mathsf{e}_1\mathsf{e}_2 = u + Iv, \tag{2.28}$$

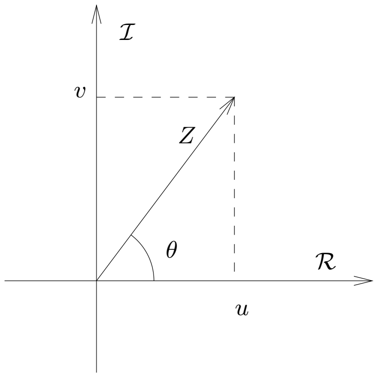

Figure 2.2 *The Argand diagram.* The complex number $Z = u + iv$ represents a vector in the complex plane, with Cartesian components u and v. The polar decomposition into $|Z|\exp(i\theta)$ can alternatively be viewed as an instruction to rotate 1 through θ and dilate by $|Z|$.

where

$$I = \mathsf{e}_1 \wedge \mathsf{e}_2, \qquad I^2 = -1. \tag{2.29}$$

Throughout we employ the symbol I for the pseudoscalar of the algebra of interest. That is why we have used it here, rather than the tempting alternative i. The latter is seen often in the literature, but the i symbol has the problem of suggesting an element which commutes with all others, which is not necessarily a property of the pseudoscalar.

Complex numbers serve a dual purpose in two dimensions. They generate rotations and dilations through their polar decomposition $|Z|\exp(i\theta)$, and they also represent vectors as points on the Argand diagram (see figure 2.2). But in the geometric algebra $\mathcal{G}_2$ complex numbers are replaced by scalar + bivector combinations, whereas vectors are grade-1 objects,

$$x = u\mathsf{e}_1 + v\mathsf{e}_2. \tag{2.30}$$

Is there a natural map between x and the multivector Z? The answer is simple — pre-multiply by e_1,

$$\mathsf{e}_1 x = u + v\mathsf{e}_1\mathsf{e}_2 = u + Iv = Z. \tag{2.31}$$

That is all there is to it! The role of the preferred vector e_1 is clear — it is the real axis. Using this product vectors in a plane can be interchanged with complex numbers in a natural manner.

If we now consider the complex conjugate of Z, $Z^\dagger = u - iv$, we see that

$$Z^\dagger = u + v\mathsf{e}_2\mathsf{e}_1 = x\mathsf{e}_1, \tag{2.32}$$

which has simply reversed the order of the geometric product of x and e_1. This operation of reversing the order of products is one of the fundamental operations performed in geometric algebra, and is called *reversion* (see section 2.5). Suppose now that we introduce a second complex number W, with vector equivalent y:

$$W = \mathsf{e}_1 y. \tag{2.33}$$

The complex product $ZW^\dagger = W^\dagger Z$ now becomes

$$W^\dagger Z = y\mathsf{e}_1\mathsf{e}_1 x = yx, \tag{2.34}$$

which returns the geometric product yx. This is as expected, as the complex product was used to suggest the form of the geometric product.

2.3.4 Rotations

Since we know how to rotate complex numbers, we can use this to find a formula for rotating vectors in a plane. We know that a positive rotation through an angle ϕ for a complex number Z is achieved by

$$Z \mapsto Z' = \mathrm{e}^{i\phi} Z, \tag{2.35}$$

where i is the standard unit imaginary (see figure 2.3). Again, we now view Z as a combination of a scalar and a pseudoscalar in $\mathcal{G}_2$ and so replace i with I. The exponential of $I\phi$ is defined by power series in the normal way, so we still have

$$\mathrm{e}^{I\phi} = \sum_{n=0}^{\infty} \frac{(I\phi)^n}{n!} = \cos\phi + I\,\sin\phi. \tag{2.36}$$

Suppose that Z' has the vector equivalent x',

$$x' = \mathsf{e}_1 Z'. \tag{2.37}$$

We now have a means of rotating the vector directly by writing

$$x' = \mathsf{e}_1\mathrm{e}^{I\phi} Z = \mathsf{e}_1\mathrm{e}^{I\phi}\mathsf{e}_1 x. \tag{2.38}$$

But

$$\begin{aligned}\mathsf{e}_1\mathrm{e}^{I\phi}\mathsf{e}_1 &= \mathsf{e}_1(\cos\phi + I\,\sin\phi)\mathsf{e}_1 \\ &= \cos\phi - I\,\sin\phi = \mathrm{e}^{-I\phi},\end{aligned} \tag{2.39}$$

where we have employed the result that I *anticommutes* with vectors. We therefore arrive at the formulae

$$x' = \mathrm{e}^{-I\phi} x = x\mathrm{e}^{I\phi}, \tag{2.40}$$

which achieve a rotation of the vector x in the I plane, through an angle ϕ. In section 2.7 we show how to extend this idea to arbitrary dimensions. The

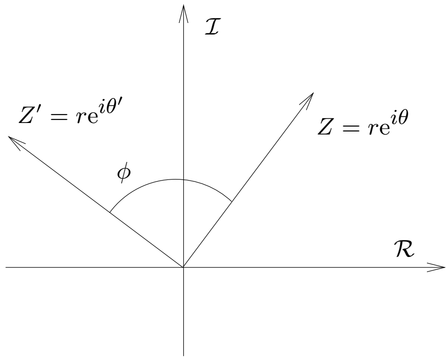

Figure 2.3 *A rotation in the complex plane.* The complex number Z is multiplied by the phase term $\exp(I\phi)$, the effect of which is to replace θ by $\theta' = \theta + \phi$.

change of sign in the exponential acting from the left and right of the vector x is to be expected. We saw earlier that left-multiplication by I generated left-handed rotations, and right-multiplication generated right-handed rotations. As the overall rotation is right-handed, the sign of I must be negative when acting from the left.

This should illustrate that geometric algebra fully encompasses complex arithmetic, and we will see later that complex analysis is fully incorporated as well. The beauty of the geometric algebra formulation is that it shows immediately how to extend the ideas of complex analysis to higher dimensions, a problem which had troubled mathematicians for many years. The key to this is the separation of the two roles of complex numbers by treating vectors as grade-1 objects, and the quantities acting on them (the complex numbers) as combinations of grade-0 and grade-2 objects. These two roles generalise differently in higher dimensions and, once one sees this, extending complex analysis becomes straightforward.

2.4 The geometric algebra of space

The geometric algebra of three-dimensional space is a remarkably powerful tool for solving problems in geometry and classical mechanics. It describes vectors, planes and volumes in a single algebra, which contains all of the familiar vector operations. These include the vector cross product, which is revealed as a disguised form of bivector. The algebra also provides a very clear and compact method for encoding rotations, which is considerably more powerful than working with matrices.

We have so far constructed the geometric algebra of a plane. We now add a third vector e_3 to our two-dimensional set $\{e_1, e_2\}$. All three vectors are assumed to be orthonormal, so they all *anticommute*. From these three basis vectors we generate the independent bivectors

$$\{e_1e_2, e_2e_3, e_3e_1\}.$$

This is the expected number of independent planes in space. There is one further term to consider, which is the product of all three vectors:

$$(e_1e_2)e_3 = e_1e_2e_3. \tag{2.41}$$

This results in a grade-3 object, called a *trivector*. It corresponds to sweeping the bivector $e_1 \wedge e_2$ along the vector e_3, resulting in a three-dimensional volume element (see section 2.4.3). The trivector represents the unique volume element in three dimensions. It is the highest grade element and is unique up to scale (or volume) and handedness (sign). This is again called the *pseudoscalar* for the algebra.

In three dimensions there are no further directions to add, so the algebra is spanned by

$$\begin{array}{cccc} 1 & \{e_i\} & \{e_i \wedge e_j\} & e_1e_2e_3 \\ \text{1 scalar} & \text{3 vectors} & \text{3 bivectors} & \text{1 trivector} \end{array} \tag{2.42}$$

This basis defines a graded linear space of total dimension $8 = 2^3$. We call this algebra $\mathcal{G}_3$. Notice that the dimensions of each subspace are given by the binomial coefficients.

2.4.1 Products of vectors and bivectors

Our expanded algebra gives us a number of new products to consider. We start by considering the product of a vector and a bivector. We have already looked at this in two dimensions, and found that a normalised bivector rotates vectors in its plane by 90°. Each of the basis bivectors in equation (2.42) shares the properties of the single bivector studied previously for two dimensions. So

$$(e_1e_2)^2 = (e_2e_3)^2 = (e_3e_1)^2 = -1 \tag{2.43}$$

and each bivector generates 90° rotations in its own plane.

The geometric product for vectors extends to all objects in the algebra, so we can form expressions such as aB, where a is a vector and B is a bivector. Now that our algebra contains a trivector $e_1(e_2 \wedge e_3)$, we see that the result of the product aB can contain both vector and trivector terms, the latter arising if a does not lie fully in the B plane. To understand the properties of the product aB we first decompose a into terms in and out of the plane,

$$a = a_{\parallel} + a_{\perp}, \tag{2.44}$$

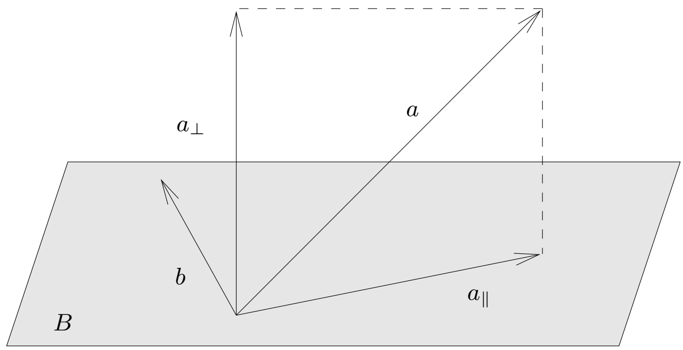

Figure 2.4 *A vector and a bivector.* The vector a can be written as the sum of a term in the plane B and a term perpendicular to the plane, so that $a = a_\parallel + a_\perp$. The bivector B can be written as $a_\parallel \wedge b$, where b is perpendicular to $a_\parallel$.

as shown in figure 2.4. We can now write $aB = (a_\parallel + a_\perp)B$. Suppose that we also write

$$B = a_\parallel \wedge b = a_\parallel b, \tag{2.45}$$

where b is orthogonal to $a_\parallel$ in the B plane. It is always possible to find such a vector b. We now see that

$$a_\parallel B = a_\parallel (a_\parallel b) = {a_\parallel}^2 b \tag{2.46}$$

and so is a vector. This is clear in that the product of a plane with a vector in the plane must remain in the plane. On the other hand

$$a_\perp B = a_\perp (a_\parallel \wedge b) = a_\perp a_\parallel b, \tag{2.47}$$

which is the product of three orthogonal (anticommuting) vectors and so is a trivector. As expected, the product of a vector and a bivector will in general contain vector and trivector terms.

To explore this further let us form the product of the vector a with the bivector $b \wedge c$. From the associative and distributive properties of the geometric product we have

$$a(b \wedge c) = a\tfrac{1}{2}(bc - cb) = \tfrac{1}{2}(abc - acb). \tag{2.48}$$

We now use the rearrangement

$$ab = 2a \cdot b - ba \tag{2.49}$$

to write

$$\begin{aligned} a(b\wedge c) &= (a\cdot b)c - (a\cdot c)b - \tfrac{1}{2}(bac - cab) \\ &= 2(a\cdot b)c - 2(a\cdot c)b + \tfrac{1}{2}(bc - cb)a, \end{aligned} \tag{2.50}$$

so that

$$a(b\wedge c) - (b\wedge c)a = 2(a\cdot b)c - 2(a\cdot c)b. \tag{2.51}$$

The right-hand side of this equation is a vector, so the antisymmetrised product of a vector with a bivector is another vector. Since this operation is grade-lowering, we give it the dot symbol again and write

$$a\cdot B = \tfrac{1}{2}(aB - Ba), \tag{2.52}$$

where B is an arbitrary bivector. The preceding rearrangement means that we have proved one of the most useful results in geometric algebra,

$$a\cdot(b\wedge c) = a\cdot b\, c - a\cdot c\, b. \tag{2.53}$$

Returning to equation (2.46) we see that we must have

$$a\cdot B = a_{\|}B = a_{\|}\cdot B. \tag{2.54}$$

So the effect of taking the inner product of a vector with a bivector is to project onto the component of the vector in the plane, and then rotate this through 90° and dilate by the magnitude of B. We can also confirm that

$$a\cdot B = {a_{\|}}^2 b = -(a_{\|}b)a_{\|} = -B\cdot a, \tag{2.55}$$

as expected.

The remaining part of the product of a vector and a bivector returns a grade-3 trivector. This product is denoted with a wedge since it is grade-raising, so

$$a\wedge(b\wedge c) = \tfrac{1}{2}\big(a(b\wedge c) + (b\wedge c)a\big). \tag{2.56}$$

A few lines of algebra confirm that this outer product is associative,

$$\begin{aligned} a\wedge(b\wedge c) &= \tfrac{1}{2}\big(a(b\wedge c) + (b\wedge c)a\big) \\ &= \tfrac{1}{4}\big(abc - acb + bca - cba\big) \\ &= \tfrac{1}{4}\big(2(a\wedge b)c + bac + bca + 2c(a\wedge b) - cab - acb\big) \\ &= \tfrac{1}{2}\big((a\wedge b)c + c(a\wedge b) + b(c\cdot a) - (c\cdot a)b\big) \\ &= (a\wedge b)\wedge c, \end{aligned} \tag{2.57}$$

so we can unambiguously write the result as $a \wedge b \wedge c$. The product $a \wedge b \wedge c$ is therefore associative and antisymmetric on all pairs of vectors, and so is precisely Grassmann's exterior product (see section 1.6). This demonstrates that

Grassmann's exterior product sits naturally within geometric algebra. From equation (2.47) we have

$$a\wedge B = a_\perp B = a_\perp \wedge B, \tag{2.58}$$

so the effect of the exterior product with a bivector is to project onto the component of the vector perpendicular to the plane, and return a volume element (a trivector). We can confirm simply that this product is symmetric in its vector and bivector arguments:

$$a\wedge B = a_\perp \wedge a_\parallel \wedge b = -a_\parallel \wedge a_\perp \wedge b = a_\parallel \wedge b \wedge a_\perp = B\wedge a. \tag{2.59}$$

The full product of a vector and a bivector can now be written as

$$aB = a\cdot B + a\wedge B, \tag{2.60}$$

where the dot is generalised to mean the *lowest* grade part of the product, while the wedge means the *highest* grade part of the product. In a similar manner to the geometric product of vectors, the separate dot and wedge products can be written in terms of the geometric product as

$$\begin{aligned} a\cdot B &= \tfrac{1}{2}(aB - Ba), \\ a\wedge B &= \tfrac{1}{2}(aB + Ba). \end{aligned} \tag{2.61}$$

But pay close attention to the signs in these formulae, which are the opposite way round to the case of two vectors. The full product of a vector and a bivector wraps up the separate vector and trivector terms in the single product aB. The advantage of this is again that the full product is invertible.

2.4.2 The bivector algebra

Our three independent bivectors also give us another new product to consider. We already know that squaring a bivector results in a scalar. But if we multiply together two bivectors representing orthogonal planes we find that, for example,

$$(\mathsf{e}_1\wedge \mathsf{e}_2)(\mathsf{e}_2\wedge \mathsf{e}_3) = \mathsf{e}_1\mathsf{e}_2\mathsf{e}_2\mathsf{e}_3 = \mathsf{e}_1\mathsf{e}_3, \tag{2.62}$$

resulting in a third bivector. We also find that

$$(\mathsf{e}_2\wedge \mathsf{e}_3)(\mathsf{e}_1\wedge \mathsf{e}_2) = \mathsf{e}_3\mathsf{e}_2\mathsf{e}_2\mathsf{e}_1 = \mathsf{e}_3\mathsf{e}_1 = -\mathsf{e}_1\mathsf{e}_3, \tag{2.63}$$

so the product of orthogonal bivectors is antisymmetric. The symmetric contribution vanishes because the two planes are perpendicular.

If we introduce the following labelling for the basis bivectors:

$$B_1 = \mathsf{e}_2\mathsf{e}_3, \quad B_2 = \mathsf{e}_3\mathsf{e}_1, \quad B_3 = \mathsf{e}_1\mathsf{e}_2, \tag{2.64}$$

we find that their product satisfies

$$B_iB_j = -\delta_{ij} - \epsilon_{ijk}B_k. \tag{2.65}$$

There is a clear analogy with the geometric product of vectors here, in that the symmetric part is a scalar, whereas the antisymmetric part is a bivector. In higher dimensions it turns out that the symmetrised product of two bivectors can have grade-0 and grade-4 terms (which we will ultimately denote with the dot and wedge symbols). The antisymmetrised product is always a bivector, and bivectors form a closed algebra under this product.

The basis bivectors satisfy

$$B_1{}^2 = B_2{}^2 = B_3{}^2 = -1 \tag{2.66}$$

and

$$B_1 B_2 = -B_2 B_1, \quad \text{etc.} \tag{2.67}$$

These are the properties of the generators of the quaternion algebra (see section 1.4). This observation helps to sort out some of the problems encountered with the quaternions. Hamilton attempted to identify pure quaternions (null scalar part) with vectors, but we now see that they are actually *bivectors*. This causes problems when looking at how objects transform under reflections. Hamilton also imposed the condition $ijk = -1$ on his unit quaternions, whereas we have

$$B_1 B_2 B_3 = \mathsf{e}_2\mathsf{e}_3\mathsf{e}_3\mathsf{e}_1\mathsf{e}_1\mathsf{e}_2 = +1. \tag{2.68}$$

To set up an isomorphism we must flip a sign somewhere, for example in the y component:

$$i \leftrightarrow B_1, \quad j \leftrightarrow -B_2, \quad k \leftrightarrow B_3. \tag{2.69}$$

This shows us that the quaternions are a *left-handed* set of bivectors, whereas Hamilton and others attempted to view the i, j, k as a right-handed set of vectors. Not surprisingly, this was a potential source of great confusion and meant one had to be extremely careful when applying quaternions in vector algebra.

2.4.3 The trivector

Given three vectors, a, b and c, the trivector $a \wedge b \wedge c$ is formed by sweeping $a \wedge b$ along the vector c (see figure 2.5). The result can be represented pictorially as an oriented parallelepiped. As with bivectors, however, the picture should not be interpreted too literally. The trivector $a \wedge b \wedge c$ does not contain any shape information. It just records a volume and an orientation.

The various algebraic properties of trivectors have straightforward geometric interpretations. The same oriented volume is obtained by sweeping $a \wedge b$ along c or $b \wedge c$ along a. The mathematical expression of this is that the outer product is associative, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$. The trivector $a \wedge b \wedge c$ changes sign under interchange of any pair of vectors, which follows immediately from the

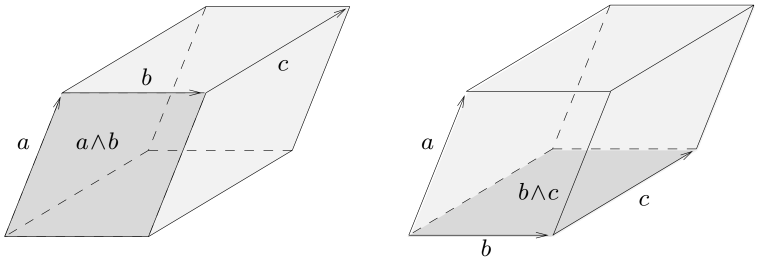

Figure 2.5 *The trivector.* The trivector $a \wedge b \wedge c$ can be viewed as the oriented parallelepiped obtained from sweeping the bivector $a \wedge b$ along the vector c. In the left-hand diagram the bivector $a \wedge b$ is swept along c. In the right-hand one $b \wedge c$ is swept along a. The result is the same in both cases, demonstrating the equality $a \wedge b \wedge c = b \wedge c \wedge a$. The associativity of the outer product is also clear from such diagrams.

antisymmetry of the exterior product. The geometric picture of this is that swapping any two vectors reverses the orientation by which the volume is swept out. Under two successive interchanges of pairs of vectors the trivector returns to itself, so

$$a \wedge b \wedge c = c \wedge a \wedge b = b \wedge c \wedge a. \tag{2.70}$$

This is also illustrated in figure 2.5.

The unit right-handed pseudoscalar for space is given the standard symbol I, so

$$I = \mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_3, \tag{2.71}$$

where the $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ are any right-handed frame of orthonormal vectors. If a left-handed set of orthonormal vectors is multiplied together the result is $-I$. Given an arbitrary set of three vectors we must have

$$a \wedge b \wedge c = \alpha I, \tag{2.72}$$

where α is a scalar. It is not hard to show that $|\alpha|$ is the volume of the parallelepiped with sides defined by a, b and c. The sign of α encodes whether the set $\{a, b, c\}$ forms a right-handed or left-handed frame. In three dimensions this fully accounts for the information in the trivector.

Now consider the product of the vector e_1 and the pseudoscalar,

$$\mathsf{e}_1 I = \mathsf{e}_1(\mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_3) = \mathsf{e}_2 \mathsf{e}_3. \tag{2.73}$$

This returns a bivector — the plane perpendicular to the original vector (see figure 2.6). The product of a grade-1 vector with the grade-3 pseudoscalar is therefore a grade-2 bivector. Multiplying from the left we find that

$$I \mathsf{e}_1 = \mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_3 \mathsf{e}_1 = -\mathsf{e}_1 \mathsf{e}_2 \mathsf{e}_1 \mathsf{e}_3 = \mathsf{e}_2 \mathsf{e}_3. \tag{2.74}$$

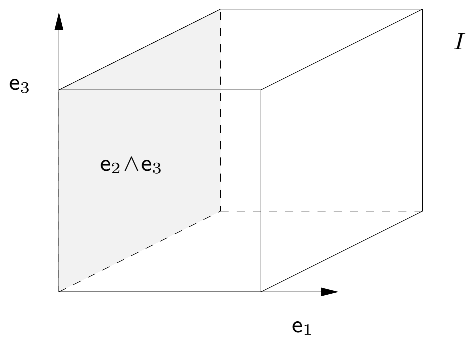

Figure 2.6 *A vector and a trivector.* The result of multiplying the vector e_1 by the trivector I is the plane $e_1(e_1e_2e_3) = e_2e_3$. This is the plane perpendicular to the e_1 vector.

The result is therefore independent of order, and this holds for any basis vector. It follows that the pseudoscalar commutes with all vectors in three dimensions:

$$Ia = aI. \tag{2.75}$$

This is always the case for the pseudoscalar in spaces of odd dimension. In even dimensions, the pseudoscalar anticommutes with all vectors, as we have already seen in two dimensions.

We can now express each of our basis bivectors as the product of the pseudoscalar and a *dual* vector:

$$e_1e_2 = Ie_3, \qquad e_2e_3 = Ie_1, \qquad e_3e_1 = Ie_2. \tag{2.76}$$

This operation of multiplying by the pseudoscalar is called a *duality* transformation and was originally introduced by Grassmann. Again, we can write

$$aI = a{\cdot}I \tag{2.77}$$

with the dot used to denote the lowest grade term in the product. The result of this can be understood as a projection — projecting onto the component of I perpendicular to a.

We next form the square of the pseudoscalar:

$$I^2 = e_1e_2e_3e_1e_2e_3 = e_1e_2e_1e_2 = -1. \tag{2.78}$$

So the pseudoscalar commutes with all elements and squares to -1. It is therefore a further candidate for a unit imaginary. In some physical applications this is the correct one to use, whereas for others it is one of the bivectors. The properties of I in three dimensions make it particularly tempting to replace it with the symbol i, and this is common practice in much of the literature. This convention can still lead to confusion, however, and is not adopted in this book.

Finally, we consider the product of a bivector and the pseudoscalar:

$$I(\mathsf{e}_1 \wedge \mathsf{e}_2) = I\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3\mathsf{e}_3 = II\mathsf{e}_3 = -\mathsf{e}_3. \tag{2.79}$$

So the result of the product of I with the bivector formed from e_1 and e_2 is $-\mathsf{e}_3$, that is, minus the vector perpendicular to the $\mathsf{e}_1 \wedge \mathsf{e}_2$ plane. This provides a definition of the vector cross product as

$$a \times b = -I(a \wedge b). \tag{2.80}$$

The vector cross product is largely redundant now that we have the exterior product and duality at our disposal. For example, consider the result for the double cross product. We form

$$\begin{aligned} a \times (b \times c) &= -Ia \wedge (-I(b \wedge c)) \\ &= \tfrac{1}{2} I\big(aI(b \wedge c) - (b \wedge c)Ia\big) \\ &= -a \cdot (b \wedge c). \end{aligned} \tag{2.81}$$

We have already calculated the expansion of the final line, which turns out to be the first example of a much more general, and very useful, formula.

Equation (2.80) shows how the cross product of two vectors is a disguised bivector, the bivector being mapped to a vector by a duality operation. It is now clear why the product only exists in three dimensions — this is the only space for which the dual of a bivector is a vector. We will have little further use for the cross product and will rarely employ it from now on. This means we can also do away with the awkward distinction between polar and axial vectors. Instead we just talk in terms of vectors and bivectors. Both may belong to three-dimensional linear spaces, but they are quite different objects with distinct algebraic properties.

2.4.4 The Pauli algebra

The full geometric product for vectors can be written

$$\mathsf{e}_i\mathsf{e}_j = \mathsf{e}_i \cdot \mathsf{e}_j + \mathsf{e}_i \wedge \mathsf{e}_j = \delta_{ij} + I\epsilon_{ijk}\mathsf{e}_k. \tag{2.82}$$

This may be familiar to many — it is the Pauli algebra of quantum mechanics! The Pauli matrices therefore form a matrix representation of the geometric algebra of space. The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.83}$$

These matrices satisfy

$$\sigma_i\sigma_j = \delta_{ij}\mathsf{I} + i\epsilon_{ijk}\sigma_k, \tag{2.84}$$

where I is the 2×2 identity matrix. Historically, these matrices were discovered by Pauli in his investigations of the quantum theory of spin. The link with geometric algebra ('Clifford algebra' in the quantum theory textbooks) was only made later.

Surprisingly, though the link with the geometric algebra of space is now well established, one seldom sees the Pauli matrices referred to as a representation for the algebra of a set of vectors. Instead they are almost universally referred to as the components of a single vector in 'isospace'. A handful of authors (most notably David Hestenes) have pointed out the curious nature of this interpretation. Such discussion remains controversial, however, and will only be touched on in this book. As with all arguments over interpretations of quantum mechanics, how one views the Pauli matrices has little effect on the predictions of the theory.

The fact that the Pauli matrices form a matrix representation of $\mathcal{G}_3$ provides an alternative way of performing multivector manipulations. This method is usually slower, but can sometimes be used to advantage, particularly in programming languages where complex arithmetic is built in. Working directly with matrices does obscure geometric meaning, and is usually best avoided.

2.5 Conventions

A number of conventions help to simplify expressions in geometric algebra. For example, expressions such as $(a \cdot b)c$ and $I(a \wedge b)$ demonstrate that it would be useful to have a convention which allows us to remove the brackets. We thus introduce the operator ordering convention that in the absence of brackets, *inner and outer products are performed before geometric products.* This can remove significant numbers of unnecessary brackets. For example, we can safely write

$$I(a\wedge b) = I\, a\wedge b. \tag{2.85}$$

and

$$(a\cdot b)c = a\cdot b\, c. \tag{2.86}$$

In addition, unless brackets specify otherwise, inner products are performed before outer products,

$$a\cdot b\, c\wedge d = (a\cdot b)c\wedge d. \tag{2.87}$$

A simple notation for the result of projecting out the elements of a multivector that have a given grade is also invaluable. We denote this with angled brackets $\langle\,\rangle_r$, where r is the grade onto which we want to project. With this notation we can write, for example,

$$a\wedge b = \langle a\wedge b\rangle_2 = \langle ab\rangle_2. \tag{2.88}$$

The final expression holds because $a \wedge b$ is the sole grade-2 component of the geometric product ab. This notation can be extremely useful as it often enables inner and outer products to be replaced by geometric products, which are usually simpler to manipulate. The operation of taking the scalar part of a product is often needed, and it is conventional for this to drop the subscript zero and simply write

$$\langle M \rangle = \langle M \rangle_0. \tag{2.89}$$

The scalar part of any pair of multivectors is symmetric:

$$\langle AB \rangle = \langle BA \rangle. \tag{2.90}$$

It follows that the scalar part satisfies the cyclic reordering property

$$\langle AB \cdots C \rangle = \langle B \cdots CA \rangle, \tag{2.91}$$

which is frequently employed in manipulations.

An important operation in geometric algebra is that of *reversion*, which reverses the order of vectors in any product. There are two conventions for this in common usage. One is the dagger symbol, $A^\dagger$, used for Hermitian conjugation in matrix algebra. The other is to use a tilde, $\tilde{A}$. In three-dimensional applications the dagger symbol is often employed, as the reverse operation returns the same result as Hermitian conjugation of the Pauli matrix representation of the algebra. In spacetime physics, however, the tilde symbol is the better choice as the dagger is reserved for a different (frame-dependent) operation in relativistic quantum mechanics. For the remainder of this chapter we will use the dagger symbol, as we will concentrate on applications in three dimensions.

Scalars and vectors are invariant under reversion, but bivectors change sign:

$$(\mathsf{e}_1\mathsf{e}_2)^\dagger = \mathsf{e}_2\mathsf{e}_1 = -\mathsf{e}_1\mathsf{e}_2. \tag{2.92}$$

Similarly, we see that

$$I^\dagger = \mathsf{e}_3\mathsf{e}_2\mathsf{e}_1 = \mathsf{e}_1\mathsf{e}_3\mathsf{e}_2 = -\mathsf{e}_1\mathsf{e}_2\mathsf{e}_3 = -I. \tag{2.93}$$

A general multivector in $\mathcal{G}_3$ can be written

$$M = \alpha + a + B + \beta I, \tag{2.94}$$

where a is a vector, B is a bivector and α and β are scalars. From the above we see that the reverse of M, $M^\dagger$, is

$$M^\dagger = \alpha + a - B - \beta I. \tag{2.95}$$

As stated above, this operation has the same effect as Hermitian conjugation applied to the Pauli matrices.

We have now introduced a number of terms, some of which have overlapping meaning. It is useful at this point to refer to multivectors which only contain terms of a single grade as *homogeneous*. The term *inner* product is reserved for the lowest grade part of the geometric product of two homogeneous multivectors. For two homogeneous multivectors of the same grade the inner product and scalar product reduce to the same thing. The terms *exterior* and *outer* products are interchangeable, though we will tend to prefer the latter for its symmetry with the inner product. The inner and outer products are also referred to colloquially as the *dot* and *wedge* products. We have followed convention in referring to the highest grade element in a geometric algebra as the *pseudoscalar*. This is a convenient name, though one must be wary that in tensor analysis the term can mean something subtly different. Both *directed volume element* and *volume form* are good alternative names, but we will stick with *pseudoscalar* in this book.

2.6 Reflections

The full power of geometric algebra begins to emerge when we consider reflections and rotations. We start with an arbitrary vector a and a unit vector n ($n^2 = 1$), and resolve a into parts parallel and perpendicular to n. This is achieved simply by forming

$$\begin{aligned} a &= n^2 a \\ &= n(n\cdot a + n\wedge a) \\ &= a_{\|} + a_{\perp}, \end{aligned} \tag{2.96}$$

where

$$a_{\|} = a\cdot n\, n, \qquad a_{\perp} = n\, n\wedge a. \tag{2.97}$$

The formula for $a_{\|}$ is certainly the projection of a onto n, and the remaining term must be the perpendicular component (sometimes called the rejection). We can check that $a_{\perp}$ is perpendicular to n quite simply:

$$n\cdot a_{\perp} = \langle nn\, n\wedge a\rangle = \langle n\wedge a\rangle = 0. \tag{2.98}$$

This is a simple example of how using the projection onto grade operator to replace inner and outer products with geometric products can simplify derivations.

The result of reflecting a in the plane orthogonal to n is the vector $a' = a_{\perp} - a_{\|}$ (see figure 2.7). This can be written

$$\begin{aligned} a' = a_{\perp} - a_{\|} &= n\, n\wedge a - a\cdot n\, n \\ &= -n\cdot a\, n - n\wedge a\, n \\ &= -nan. \end{aligned} \tag{2.99}$$

This formula is already more compact than can be written down without the geometric product. The best one can do with just the inner product is the

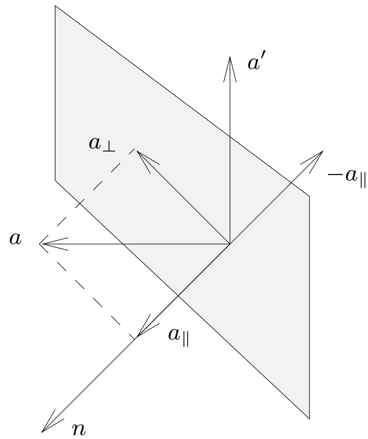

Figure 2.7 *A reflection.* The vector a is reflected in the (hyper)plane perpendicular to n. This is the way to describe reflections in arbitrary dimensions. The result a' is formed by reversing the sign of $a_\|$, the component of a in the n direction.

equivalent expression

$$a' = a - 2a\cdot n\, n. \tag{2.100}$$

The compression afforded by the geometric product becomes increasingly impressive as reflections are compounded together. The formula

$$a' = -nan \tag{2.101}$$

is valid is spaces of any dimension — it is a quite general formula for a reflection.

We should check that our formula for the reflection has the desired property of leaving lengths and angles unchanged. To do this we need only verify that the scalar product between vectors is unchanged if both are reflected, which is achieved with a simple rearrangement:

$$(-nan)\cdot(-nbn) = \langle(-nan)(-nbn)\rangle = \langle nabn\rangle = \langle abnn\rangle = a\cdot b. \tag{2.102}$$

In this manipulation we have made use of the cyclic reordering property of the scalar part of a geometric product, as defined in equation (2.91).

2.6.1 Complex conjugation

In two dimensions we saw that the vector x is mapped to a complex number Z by

$$Z = \mathsf{e}_1 x, \quad x = \mathsf{e}_1 Z. \tag{2.103}$$

The complex conjugate $Z^\dagger$ is the reverse of this, $Z^\dagger = x\mathsf{e}_1$, so maps to the vector

$$x' = \mathsf{e}_1 Z^\dagger = \mathsf{e}_1 x \mathsf{e}_1. \tag{2.104}$$

This can be converted into the formula for a reflection if we remember that the two-dimensional pseudoscalar $I = \mathsf{e}_1\mathsf{e}_2$ anticommutes with all vectors and squares to -1. We therefore have

$$x' = -\mathsf{e}_1 I I x \mathsf{e}_1 = -\mathsf{e}_1 I x \mathsf{e}_1 I = -\mathsf{e}_2 x \mathsf{e}_2. \tag{2.105}$$

This is precisely the expected relation for a reflection in the line perpendicular to e_2, which is to say a reflection in the real axis.

2.6.2 Reflecting bivectors

Now suppose that we form the bivector $B = a \wedge b$ and reflect both of these vectors in the plane perpendicular to n. The result is

$$B' = (-nan)\wedge(-nbn). \tag{2.106}$$

This simplifies as follows:

$$\begin{aligned}(-nan)\wedge(-nbn) &= \tfrac{1}{2}(nannbn - nbnnan)\\ &= \tfrac{1}{2}n(ab - ba)n\\ &= nBn.\end{aligned} \tag{2.107}$$

The effect of sandwiching a multivector between a vector, nMn, always preserves the grade of the multivector M. We will see how to prove this in general when we have derived a few more results for manipulating inner and outer products. The resulting formula nBn shows that bivectors are subject to the same transformation law as vectors, *except for a change in sign*. This is the origin of the conventional distinction between polar and axial vectors. Axial vectors are usually generated by the cross product, and we saw in section 2.4.3 that the cross product generates a bivector, and then dualises it back to a vector. But when the two vectors in the cross product are reflected, the bivector they form is reflected according to (2.107). The dual vector IB is subject to the same transformation law, since

$$I(nBn) = n(IB)n, \tag{2.108}$$

and so does not transform as a (polar) vector. In many texts this can be a source of much confusion. But now we have a much healthier alternative: banish all talk of axial vectors in favour of bivectors. We will see in later chapters that all of the main examples of 'axial' vectors in physics (angular velocity, angular momentum, the magnetic field etc.) are better viewed as bivectors.

2.6.3 Trivectors and handedness

The final object to try reflecting in three dimensions is the trivector $a \wedge b \wedge c$. We first write

$$(-nan)\wedge(-nbn)\wedge(-ncn) = \langle(-nan)(-nbn)(-ncn)\rangle_3 \\ = -\langle nabcn\rangle_3, \tag{2.109}$$

which follows because the only way to form a trivector from the geometric product of three vectors is through the exterior product of all three. Now the product abc can only contain a vector and trivector term. The former cannot give rise to an overall trivector, so we are left with

$$(-nan)\wedge(-nbn)\wedge(-ncn) = -\langle na\wedge b\wedge cn\rangle_3. \tag{2.110}$$

But any trivector in three dimensions is a multiple of the pseudoscalar I, which commutes with all vectors, so we are left with

$$(-nan)\wedge(-nbn)\wedge(-ncn) = -a\wedge b\wedge c. \tag{2.111}$$

The overall effect is simply to flip the sign of the trivector, which is a way of stating that reflections have determinant -1. This means that if all three vectors in a right-handed triplet are reflected in some plane, the resulting triplet is left handed (and vice versa).

2.7 Rotations

Our starting point for the treatment of rotations is the result that *a rotation in the plane generated by two unit vectors m and n is achieved by successive reflections in the (hyper)planes perpendicular to m and n.* This is illustrated in figure 2.8. Any component of a perpendicular to the $m\wedge n$ plane is unaffected, and simple trigonometry confirms that the angle between the initial vector a and the final vector c is twice the angle between m and n. (The proof of this is left as an exercise.) The result of the successive reflections is therefore to rotate through 2θ in the $m\wedge n$ plane, where $m\cdot n = \cos(\theta)$.

So how does this look using geometric algebra? We first form

$$b = -mam \tag{2.112}$$

and then perform a second reflection to obtain

$$c = -nbn = -n(-mam)n = nmamn. \tag{2.113}$$

This is starting to look extremely simple! We define

$$R = nm, \tag{2.114}$$

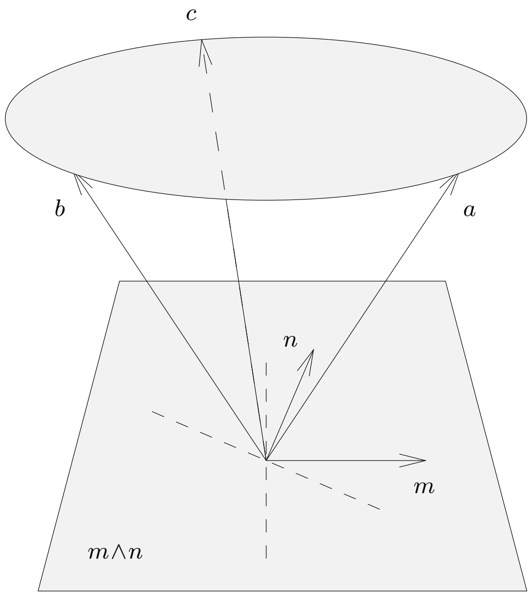

Figure 2.8 *A rotation from two reflections.* The vector b is the result of reflecting a in the plane perpendicular to m, and c is the result of reflecting b in the plane perpendicular to n.

so that we can now write the result of the rotation as

$$c = RaR^{\dagger}. \tag{2.115}$$

This transformation $a \mapsto RaR^{\dagger}$ is a totally general way of handling rotations. In deriving this transformation the dimensionality of the space of vectors was never specified, so the transformation law must work in all spaces, *whatever their dimension*. The rule also works for *any grade* of multivector!

2.7.1 Rotors

The quantity $R = nm$ is called a *rotor* and is one of the most important objects in applications of geometric algebra. Immediately, one can see the importance of the *geometric* product in both (2.114) and (2.115), which tells us that rotors provide a way of handling rotations that is unique to geometric algebra. To study the properties of the rotor R we first write

$$R = nm = n\cdot m + n\wedge m = \cos(\theta) + n\wedge m. \tag{2.116}$$

We already calculated the magnitude of the bivector $m \wedge n$ in equation (2.15), where we obtained

$$(n\wedge m)(n\wedge m) = -\sin^2(\theta). \tag{2.117}$$

We therefore define the *unit* bivector B in the $m\wedge n$ plane by

$$B = \frac{m\wedge n}{\sin(\theta)}, \qquad B^2 = -1. \tag{2.118}$$

The reason for this choice of orientation ($m \wedge n$ rather than $n \wedge m$) is to ensure that the rotation has the orientation specified by the generating bivector, as can be seen in figure 2.8. In terms of the bivector B we now have

$$R = \cos(\theta) - B\sin(\theta), \tag{2.119}$$

which is simply the polar decomposition of a complex number, with the unit imaginary replaced by the unit bivector B. We can therefore write

$$R = \exp(-B\theta), \tag{2.120}$$

with the exponential defined in terms of its power series in the normal way. (The power series for the exponential is absolutely convergent for any multivector argument.)

Now recall that our formula was for a rotation through 2θ. If we want to rotate through θ, the appropriate rotor is

$$R = \exp(-B\theta/2), \tag{2.121}$$

which gives the formula

$$a \mapsto a' = \mathrm{e}^{-B\theta/2} a \mathrm{e}^{B\theta/2} \tag{2.122}$$

for a rotation through θ in the B plane, with handedness determined by B (see figure 2.9). This description encourages us to think of rotations taking place *in a plane*, and as such gives equations which are valid in any dimension. The more traditional idea of rotations taking place around an axis is an entirely three-dimensional concept which does not generalise.

Since the rotor R is a geometric product of two unit vectors, we see immediately that

$$RR^\dagger = nm(nm)^\dagger = nmmn = 1 = R^\dagger R. \tag{2.123}$$

This provides a quick proof that our formula has the correct property of preserving lengths and angles. Suppose that $a' = RaR^\dagger$ and $b' = RbR^\dagger$, then

$$\begin{aligned} a'\cdot b' &= \tfrac{1}{2}(RaR^\dagger RbR^\dagger + RbR^\dagger RaR^\dagger) \\ &= \tfrac{1}{2}R(ab+ba)R^\dagger \\ &= a\cdot b\, RR^\dagger \\ &= a\cdot b. \end{aligned} \tag{2.124}$$

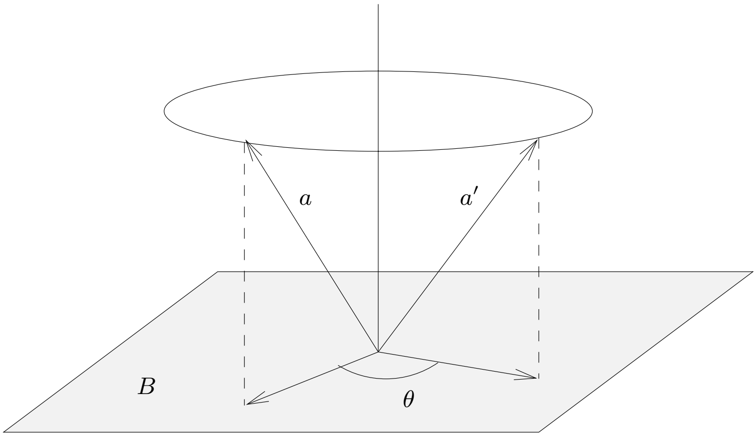

Figure 2.9 *A rotation in three dimensions.* The vector a is rotated to $a' = RaR^\dagger$. The rotor R is defined by $R = \exp(-B\theta/2)$, which describes the rotation directly in terms of the plane and angle. The rotation has the orientation specified by the bivector B.

We can also see that the inverse transformation is given by

$$a = R^\dagger a' R. \tag{2.125}$$

The proof is straightforward:

$$R^\dagger a' R = R^\dagger R a R^\dagger R = a. \tag{2.126}$$

The usefulness of rotors provides ample justification for adding up terms of different grades. The rotor R on its own has no geometric significance, which is to say that no meaning should be attached to the separate scalar and bivector terms. When R is written in the form $R = \exp(-B\theta/2)$, however, the bivector B has clear geometric significance, as does the vector formed from $RaR^\dagger$. This illustrates a central feature of geometric algebra, which is that both geometrically meaningful objects (vectors, planes etc.) and the elements that act on them (in this case rotors) are represented in the same algebra.

2.7.2 Constructing a rotor

Suppose that we wish to rotate the unit vector a into another unit vector b, leaving all vectors perpendicular to a and b unchanged. This is accomplished by a reflection perpendicular to the unit vector n half-way between a and b followed by a reflection in the plane perpendicular to b (see figure 2.10). The vector n is

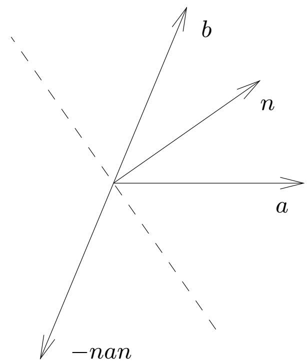

Figure 2.10 *A rotation from a to b.* The vector a is rotated onto b by first reflecting in the plane perpendicular to n, and then in the plane perpendicular to b. The vectors a, b and n all have unit length.

given by

$$n = \frac{(a+b)}{|a+b|}, \tag{2.127}$$

which reflects a into $-b$. Combining this with the reflection in the plane perpendicular to b we arrive at the rotor

$$R = bn = \frac{1+ba}{|a+b|} = \frac{1+ba}{\sqrt{2(1+b\cdot a)}}, \tag{2.128}$$

which represents a simple rotation in the $a\wedge b$ plane. This formula shows us that

$$Ra = \frac{a+b}{\sqrt{2(1+b\cdot a)}} = a\frac{1+ab}{\sqrt{2(1+b\cdot a)}} = aR^\dagger. \tag{2.129}$$

It follows that we can write

$$RaR^\dagger = R^2 a = aR^{\dagger 2}. \tag{2.130}$$

This is always possible for vectors in the plane of rotation. Returning to the polar form $R = \exp(-B\theta/2)$, where B is the $a \wedge b$ plane, we see that

$$R^2 = \exp(-B\theta), \tag{2.131}$$

so we can rotate a onto b with the formula

$$b = \mathrm{e}^{-B\theta}a = a\mathrm{e}^{B\theta}. \tag{2.132}$$

This is precisely the form found in the plane using complex numbers, and was the source of much of the confusion over the use of quaternions for rotations. Hamilton thought that a single-sided transformation law of the form $a \mapsto Ra$ should be the correct way to encode a rotation, with the full angle appearing in the exponential. He thought that this was the natural generalisation of the complex number representation. But we can see now that this formula only works for vectors *in the plane of rotation*. The correct formula for all vectors is the double-sided, half-angle formula $a \mapsto RaR^\dagger$. This formula ensures that given a vector c perpendicular to the $a \wedge b$ plane we have

$$Rc = c\frac{1+ba}{\sqrt{2(1+b\cdot a)}} = \frac{1+ba}{\sqrt{2(1+b\cdot a)}}c = Rc, \tag{2.133}$$

so that

$$RcR^\dagger = cRR^\dagger = c, \tag{2.134}$$

and the vector is unrotated. The single-sided law does not have this property. Correctly identifying the double-sided transformation law means that unit bivectors such as

$$\mathsf{e}_1\mathsf{e}_2 = \mathrm{e}^{\mathsf{e}_1\mathsf{e}_2\pi/2} \tag{2.135}$$

are generators of rotations through π, and not $\pi/2$. The fact that unit bivectors square to -1 is consistent with this because, acting double sidedly, the rotor -1 is the identity operation. More generally, R and $-R$ generate the same rotation, so there is a two-to-one map between rotors and rotations. (Mathematicians talk of the rotors providing a *double-cover* representation of the rotation group.)

2.7.3 Rotating multivectors

Suppose that the two vectors forming the bivector $B = a\wedge b$ are both rotated. What is the expression for the resulting bivector? To find this we form

$$\begin{aligned} B' = a'\wedge b' &= \tfrac{1}{2}(RaR^\dagger RbR^\dagger - RbR^\dagger RaR^\dagger) \\ &= \tfrac{1}{2}R(ab-ba)R^\dagger \\ &= Ra\wedge bR^\dagger \\ &= RBR^\dagger, \end{aligned} \tag{2.136}$$

where we have used the rotor normalisation formula $R^\dagger R = 1$. Bivectors are rotated using precisely the same formula as vectors! The same turns out to be true for all geometric multivectors, and this is one of the most attractive features of geometric algebra. In section 4.2 we prove that the transformation $A \mapsto RAR^\dagger$ preserves the grade of the multivector on which the rotors act. For applications in three dimensions we only need check this result for the trivector case, as we have already demonstrated it for vectors and bivectors. The pseudoscalar in three dimensions, I, commutes with all other terms in the algebra, so we have

$$RIR^\dagger = IRR^\dagger = I, \tag{2.137}$$

which is certainly grade-preserving. This result is one way of saying that rotations have determinant +1. We now have a means of rotating all geometric objects in three dimensions. In chapter 3 we will take full advantage of this when studying rigid-body dynamics.

2.7.4 Rotor composition law

Having seen how individual rotors are used to represent rotations, we now look at their composition law. Let the rotor R_1 transform the vector a into a vector b:

$$b = R_1 a R_1^\dagger. \tag{2.138}$$

Now rotate b into another vector c, using a rotor R_2. This requires

$$c = R_2 b R_2^\dagger = R_2 R_1 a R_1^\dagger R_2^\dagger = R_2 R_1 a (R_2 R_1)^\dagger, \tag{2.139}$$

so that if we write

$$c = R a R^\dagger, \tag{2.140}$$

then the composite rotor is given by

$$R = R_2 R_1. \tag{2.141}$$

This is the *group combination rule* for rotors. Rotors form a group because the product of two rotors is a third rotor, as can be checked from

$$R_2 R_1 (R_2 R_1)^\dagger = R_2 R_1 R_1^\dagger R_2^\dagger = R_2 R_2^\dagger = 1. \tag{2.142}$$

In three dimensions the fact that the multivector R contains only even-grade elements and satisfies $RR^\dagger = 1$ is sufficient to ensure that R is a rotor. The fact that rotors form a continuous group (called a *Lie group*) is a subject we will return to later in this book.

Rotors are the exception to the rule that all multivectors are subject to a double-sided transformation law. Rotors are already mixed-grade objects, so multiplying on the left (or right) by another rotor does not take us out of the space of rotors. All geometric entities, such as lines and planes, are single-grade objects, and their grades cannot be changed by a rotation. They are therefore all subject to a double-sided transformation law. Again, this brings us back to the central theme that both geometric objects and the operators acting on them are contained in a single algebra.

The composition rule (2.141) has a surprising consequence. Suppose that the rotor R_1 is kept fixed, and we set $R_2 = \exp(-B\theta/2)$. We now take the vector c on a 2π excursion back to itself. The final rotor R is

$$R = \mathrm{e}^{-B\pi} R_1 = -R_1. \tag{2.143}$$

The rotor has changed sign under a 2π rotation! This is usually viewed as a quantum-mechanical phenomenon related to the existence of fermions. But we can now see that the result is classical and is simply a consequence of our rotor description of rotations. (The relationship between rotors and fermion wavefunctions is discussed in chapter 8.) A geometric interpretation of the distinction between R and $-R$ is provided by the *direction* in which a rotation is performed. Suppose we want to rotate e_1 onto e_2. The rotor to achieve this is

$$R(\theta) = \mathrm{e}^{-\mathsf{e}_1\mathsf{e}_2\theta/2}. \tag{2.144}$$

If we rotate in a positive sense through $\pi/2$ the final rotor is given by

$$R(\pi/2) = \frac{1}{\sqrt{2}}(1 - \mathsf{e}_1\mathsf{e}_2). \tag{2.145}$$

If we rotate in the negative (clockwise) sense, however, the final rotor is

$$R(-3\pi/2) = -\frac{1}{\sqrt{2}}(1 - \mathsf{e}_1\mathsf{e}_2) = -R(\pi/2). \tag{2.146}$$

So, while R and $-R$ define the same absolute rotation (and the same rotation matrix), their different signs can be employed to record information about the handedness of the rotation.

The rotor composition rule provides a simple formula for the compound effect of two rotations. Suppose that we have

$$R_1 = \mathrm{e}^{-B_1\theta_1/2}, \quad R_2 = \mathrm{e}^{-B_2\theta_2/2}, \tag{2.147}$$

where both B_1 and B_2 are unit bivectors. The product rotor is

$$\begin{aligned} R =& \big(\cos(\theta_2/2) - \sin(\theta_2/2)B_2\big)\big(\cos(\theta_1/2) - \sin(\theta_1/2)B_1\big) \\ =& \cos(\theta_2/2)\cos(\theta_1/2) - \big(\cos(\theta_2/2)\sin(\theta_1/2)B_1 + \cos(\theta_1/2)\sin(\theta_2/2)B_2\big) \\ &+ \sin(\theta_2/2)\sin(\theta_1/2)B_1B_2. \end{aligned} \tag{2.148}$$

So if we write $R = R_2R_1 = \exp(-B\theta/2)$, where B is a new unit bivector, we immediately see that

$$\cos(\theta/2) = \cos(\theta_2/2)\cos(\theta_1/2) + \sin(\theta_2/2)\sin(\theta_1/2)\langle B_1B_2\rangle \tag{2.149}$$

and

$$\begin{aligned} \sin(\theta/2)B = \cos(\theta_2/2)\sin(\theta_1/2)B_1 + \cos(\theta_1/2)\sin(\theta_2/2)B_2 \\ - \sin(\theta_2/2)\sin(\theta_1/2)\langle B_1B_2\rangle_2. \end{aligned} \tag{2.150}$$

These half-angle relations for rotations were first discovered by the mathematician Rodriguez, three years before the invention of the quaternions! It is well known that these provide a simple means of calculating the compound effect of two rotations. Numerically, it is usually even simpler to just multiply the rotors directly and not worry about calculating any trigonometric functions.

2.7.5 *Euler angles*

A standard way to parameterise rotations is via the three Euler angles $\{\phi, \theta, \psi\}$. These are defined to rotate an initial set of axes, $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$, onto a new set $\{\mathsf{e}_1', \mathsf{e}_2', \mathsf{e}_3'\}$ (often denoted x, y, z and x', y', z' respectively). First we rotate about the e_3 axis — i.e. in the $\mathsf{e}_1\mathsf{e}_2$ plane — anticlockwise through an angle ϕ. The rotor for this is

$$R_\phi = \mathrm{e}^{-\mathsf{e}_1\mathsf{e}_2\phi/2}. \tag{2.151}$$

Next we rotate about the axis formed by the transformed e_1 axis through an amount θ. The plane for this is

$$IR_\phi \mathsf{e}_1 R_\phi^\dagger = R_\phi \mathsf{e}_2\mathsf{e}_3 R_\phi^\dagger. \tag{2.152}$$

The rotor is therefore

$$R_\theta = \exp\bigl(-R_\phi \mathsf{e}_2\mathsf{e}_3 R_\phi^\dagger \theta/2\bigr) = R_\phi \mathrm{e}^{-\mathsf{e}_2\mathsf{e}_3\theta/2} R_\phi^\dagger. \tag{2.153}$$

The intermediate rotor is now

$$R' = R_\theta R_\phi = \mathrm{e}^{-\mathsf{e}_1\mathsf{e}_2\phi/2}\mathrm{e}^{-\mathsf{e}_2\mathsf{e}_3\theta/2}. \tag{2.154}$$

Note the order! Finally, we rotate about the transformed e_3 axis through an angle ψ. The appropriate plane is now

$$IR'\mathsf{e}_3 R'^\dagger = R'\mathsf{e}_1\mathsf{e}_2 R'^\dagger \tag{2.155}$$

and the rotor is

$$R_\psi = \exp\bigl(-R'\mathsf{e}_1\mathsf{e}_2 R'^\dagger \psi/2\bigr) = R'\mathrm{e}^{-\mathsf{e}_1\mathsf{e}_2\psi/2} R'^\dagger. \tag{2.156}$$

The resultant rotor is therefore

$$R = R_\psi R' = \mathrm{e}^{-\mathsf{e}_1\mathsf{e}_2\phi/2}\mathrm{e}^{-\mathsf{e}_2\mathsf{e}_3\theta/2}\mathrm{e}^{-\mathsf{e}_1\mathsf{e}_2\psi/2}, \tag{2.157}$$

which has decoupled very nicely and is really quite simple — it is much easier to visualise and work with than the equivalent matrix formula! Now that we have geometric algebra at our disposal we will, in fact, have little cause to use the Euler angles in calculations.

2.8 Notes

In this chapter we have given a lengthy introduction to geometric algebra in two and three dimensions. The latter algebra is generated entirely by three basis vectors $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ subject to the rule that $\mathsf{e}_i\mathsf{e}_j + \mathsf{e}_j\mathsf{e}_i = 2\delta_{ij}$. This simple rule generates an algebra of remarkable power and richness which we will explore in following chapters.

There is a large literature on the geometric algebra of three-dimensional space and its applications in physics. The most complete text is *New Foundations* *for Classical Mechanics* by David Hestenes (1999). Hestenes has also written many papers on the subject, most of which are listed in the bibliography at the end of this book. Other introductory papers have been written by Gull, Lasenby and Doran (1993a), Doran et al. (1996a) and Vold (1993a, 1993b). Clifford's *Mathematical Papers* (1882) are also of considerable interest. The use of geometric algebra for handling rotations is very common in the fields of engineering and computer science, though often purely in the guise of the quaternion algebra. Searching one of the standard scientific databases with the keyword 'quaternions' returns too many papers to begin to list here.

2.9 Exercises

2.1 From the properties of the geometric product, show that the symmetrised product of two vectors satisfies the properties of a scalar product, as listed in section 1.2.

2.2 By expanding the bivector $a\wedge b$ in terms of geometric products, prove that it anticommutes with both a and b, but commutes with any vector perpendicular to the $a\wedge b$ plane.

2.3 Verify that the $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ matrices of equation (2.27) satisfy the correct multiplication relations to form a representation of $\mathcal{G}_2$. Use these to verify equations (2.26).

2.4 Construct the multiplication table generated by the orthonormal vectors e_1, e_2 and e_3. Do these generate a (finite) group?

2.5 Prove that all of the following forms are equivalent expressions of the vector cross product:

$$a\times b = -Ia\wedge b = b\cdot(Ia) = -a\cdot(Ib).$$

Interpret each form geometrically. Hence establish that

$$a\times(b\times c) = -a\cdot(b\wedge c) = -(a\cdot b\, c - a\cdot c\, b)$$

and

$$a\cdot(b\times c) = [a,b,c] = a\wedge b\wedge c\, I^{-1}.$$

2.6 Prove that the effect of successive reflections in the planes perpendicular to the vectors m and n results in a rotation through twice the angle between m and n.

2.7 What is the reverse of $RaR^{\dagger}$, where a is a vector? Which objects in three dimensions have this property, and why must the result be another vector?

2.8 Show that the rotor

$$R = \frac{1+ba}{|a+b|}$$

can also be written as $\exp(-B\theta/2)$, where B is the unit bivector in the $a \wedge b$ plane and θ is the angle between a and b.

2.9 The Cayley–Klein parameters are a set of four real numbers α, β, γ and δ subject to the normalisation condition

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

These can be used to paramaterise an arbitrary rotation matrix as follows:

$$\boldsymbol{U} = \begin{pmatrix} \alpha^2 + \beta^2 - \gamma^2 - \delta^2 & 2(\beta\gamma + \alpha\delta) & 2(\beta\delta - \alpha\gamma) \\ 2(\beta\gamma - \alpha\delta) & \alpha^2 - \beta^2 + \gamma^2 - \delta^2 & 2(\gamma\delta + \alpha\beta) \\ 2(\beta\delta + \alpha\gamma) & 2(\gamma\delta - \alpha\beta) & \alpha^2 - \beta^2 - \gamma^2 + \delta^2 \end{pmatrix}.$$

Can you relate the Cayley–Klein parameters to the rotor description?

2.10 Show that the set of all rotors forms a continuous group. Can you identify the group manifold?