10

The Dirac Equation

The Schrödinger equation describes all atomic phenomena except those involving magnetism and relativity. The Schrödinger-Pauli equation takes care of magnetism by including the spin of the electron.

The relativistic phenomena can be taken into consideration by starting from the equation $E^2/c^2 - \vec{p}^2 = m^2c^2$. Inserting energy and momentum operators into this equation, results in the *Klein-Gordon equation*

$$\hbar^2\Big(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\Big)\psi = m^2c^2\psi,$$

which treats time and space on an equal footing. Dirac 1928 linearized the Klein-Gordon equation, or replaced it by a first-order equation,

$$i\hbar\Big(\gamma_0\frac{1}{c}\frac{\partial}{\partial t}+\gamma_1\frac{\partial}{\partial x_1}+\gamma_2\frac{\partial}{\partial x_2}+\gamma_3\frac{\partial}{\partial x_3}\Big)\psi = mc\psi.$$

The above *Dirac equation* implies the Klein-Gordon equation provided the symbols γ_μ satisfy the relations

$$\gamma_0^2 = I, \quad \gamma_1^2=\gamma_2^2=\gamma_3^2=-I,$$
$$\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu \quad \text{for} \quad \mu\neq\nu.$$

Dirac found a set of 4×4-matrices satisfying these relations, namely, the following *Dirac matrices*:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix},$$
$$\gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},\quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i\\ 0 & 0 & -i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix},\quad \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}.$$

In terms of the Pauli spin-matrices σ_k the Dirac gamma-matrices γ_μ can be expressed as [1]

$$\gamma_0 = \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_k = -\gamma^k = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}.$$

Writing $x_0 = ct$, the Dirac equation can be condensed into the form

$$i\hbar\gamma^\mu \partial_\mu \psi = mc\psi$$

where $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$. [2] An interaction with the electromagnetic field $F^{\mu\nu}$ is included via the space-time potential $(A^0, A^1, A^2, A^3) = (\frac{1}{c}V, A_x, A_y, A_z)$ of $F^{\mu\nu}$ by employing the replacement $i\hbar\partial^\mu \to i\hbar\partial^\mu - eA^\mu$. This leads to the conventional *Dirac equation*

$$\boxed{\gamma_\mu(i\hbar\partial^\mu - eA^\mu)\psi = mc\psi}$$

where the wave function is a *column spinor*, that is,

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4 \quad \text{with} \quad \psi_\alpha \in \mathbb{C}.$$

The Dirac equation takes into account the relativistic phenomena and also spin; it describes spin-$\frac{1}{2}$ particles, like the electron.

10.1 Bilinear covariants

The *Dirac adjoint* [3] of a column spinor $\psi \in \mathbb{C}^4$ is a row matrix

$$\psi^\dagger\gamma_0 = (\,\psi_1^* \quad \psi_2^* \quad -\psi_3^* \quad -\psi_4^*\,).$$

A column spinor $\psi(\mathbf{x})$ and its Dirac adjoint $\psi^\dagger(\mathbf{x})\gamma_0$ can be used to define four real valued functions

$$J^\mu(\mathbf{x}) = \psi^\dagger(\mathbf{x})\gamma_0\gamma^\mu\psi(\mathbf{x})$$

which are components of a space-time vector, the Dirac current,

$$\mathbf{J}(\mathbf{x}) = \gamma_\mu J^\mu(\mathbf{x}).$$

Under a Lorentz transformation

$$\mathbf{x}' = s\mathbf{x}s^{-1}, \quad s \in \mathbf{Spin}_+(1,3),$$

1 The above matrix representation is called the *Pauli-Dirac representation* (although it should be called the Pauli-Dirac basis).

2 Note that $\gamma^\mu\partial_\mu = \gamma_\mu\partial^\mu$ where $\partial^\mu = \frac{\partial}{\partial x_\mu}$.

3 The Dirac adjoint of ψ is often denoted by $\bar{\psi}$, but we have reserved this bar-notation for the Clifford-conjugation.

the Dirac spinor transforms according to

$$\psi' = s^{-1}\psi \quad \text{or} \quad \psi'(\mathbf{x}') = s^{-1}\psi(s\mathbf{x}s^{-1})$$

and the Dirac current according to

$$\mathbf{J}' = s^{-1}\mathbf{J}s \quad \text{or} \quad \mathbf{J}'(\mathbf{x}') = s^{-1}\mathbf{J}(s\mathbf{x}s^{-1})s.$$

Thus, the Dirac current is covariant under the Lorentz transformations. The components $J^\mu = \psi^\dagger\gamma_0\gamma^\mu\psi$ are called *bilinear* [4] *covariants*.

The physical state of the electron is determined by the following 16 bilinear covariants:

$$\begin{aligned}
&\Omega_1 = \psi^\dagger\gamma_0\psi = \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4, \\
&J^\mu = \psi^\dagger\gamma_0\gamma^\mu\psi, \\
&S^{\mu\nu} = \psi^\dagger\gamma_0 i\gamma^{\mu\nu}\psi, \qquad && \gamma^{\mu\nu} = \gamma^\mu\gamma^\nu \neq i\gamma^\mu\gamma^\nu, \\
&K^\mu = \psi^\dagger\gamma_0 i\gamma^{0123}\gamma_\mu\psi, \\
&\Omega_2 = \psi^\dagger\gamma_0\gamma^{0123}\psi, \qquad && \gamma^{0123} = \gamma^0\gamma^1\gamma^2\gamma^3.
\end{aligned}$$

Their integrals over space give expectation values of the physical observables.

The quantity $J^0 = \psi^\dagger\psi$, $J^0 \geq 0$, integrated over a space-like domain gives the probability of finding the electron in that domain. [5] The quantities $J^k = \psi^\dagger\gamma_0\gamma^k\psi$ $(k = 1, 2, 3)$ give the current of probability $\vec{J} = \gamma_k J^k$; they satisfy the continuity equation

$$\frac{1}{c}\frac{\partial J^0}{\partial t} + \frac{\partial J^k}{\partial x^k} = 0.$$

The Dirac current $\mathbf{J}$ is a future-oriented vector, $\mathbf{J}^2 \geq 0$. [6] The time-component $u_0 = \gamma_0 \cdot \mathbf{u}$ of the unit vector $\mathbf{u} = \mathbf{J}/\sqrt{\mathbf{J}^2}$, $\mathbf{J}^2 \neq 0$, gives the probable velocity of the electron,

$$u_0 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

The bivector $\mathbf{S} = \frac{1}{2}S^{\mu\nu}\gamma_{\mu\nu}$ [7] is usually interpreted as the electromagnetic moment density, while it gives the probability density of the electromagnetic moment of the electron.

4 The quantities $\psi^\dagger\gamma_0\gamma^\mu\psi$ are actually quadratic in ψ. Also their polarized forms $\psi^\dagger\gamma_0\gamma^\mu\varphi$ are not bilinear but rather sesquilinear, while anti-linear in ψ.

5 Or rather the probability multiplied by the (negative of the) charge of the electron. In the case of a large number of particles J^0 can be interpreted as the charge density.

6 Recall that $\mathbf{J}^2 = (J^0)^2 - (J^1)^2 - (J^2)^2 - (J^3)^2$.

7 This is a shorthand notation for $\mathbf{S} = \frac{1}{2}\sum_{\mu,\nu} S^{\mu\nu}\gamma_{\mu\nu} = \sum_{\mu<\nu} S^{\mu\nu}\gamma_{\mu\nu}$.

The vector $\mathbf{K} = K^\mu \gamma_\mu$ is space-like, and such that $\mathbf{K}^2 = -\mathbf{J}^2$. It is orthogonal to $\mathbf{J}$, $\mathbf{K} \cdot \mathbf{J} = 0$, and gives the direction of the spin of the electron, the spin vector $\frac{1}{2}\hbar\mathbf{K}/\sqrt{-\mathbf{K}^2}$, $\mathbf{K}^2 \neq 0$. Note that $K^\mu = \psi^\dagger \gamma_0 \gamma^\mu i\gamma_{0123}\psi$.

The first and last of the bilinear covariants were combined into a single quantity by de Broglie:

$$\boxed{\Omega = \Omega_1 + \Omega_2 \gamma_{0123}}$$

Note that $\Omega_2 = -\psi^\dagger \gamma_0 \gamma_{0123}\psi$.

Spinors in Ideals

Here we shall take a new view on spinors and regard them as elements of minimal left ideals, [8] first in matrix algebras, then in complexified Clifford algebras, and finally in real Clifford algebras.

10.2 Square matrix spinors

Usually the wave function is a column spinor $\psi \in \mathbb{C}^4$, but we shall also regard it as a 4×4-matrix with only the first column being non-zero; that is, $\psi \in \mathrm{Mat}(4, \mathbb{C})f$ where f is the primitive idempotent [9]

$$f = \frac{1}{2}(1 + \gamma_0)\frac{1}{2}(1 + i\gamma_1\gamma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

More explicitly, a Dirac spinor might appear as a *column spinor* or as a *square matrix spinor*: [10]

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathbb{C}^4 \quad \text{or} \quad \psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}(4, \mathbb{C})f$$

8 We shall reject ideal spinors later in favor of spinor operators.
9 The factors $\frac{1}{2}(1 + \gamma_0)$ and $\frac{1}{2}(1 + i\gamma_{12})$ are energy and spin projection operators.
10 We replace column spinors by square matrix spinors in order to be able to get everything – vectors, rotations and spinors – represented within one mathematical system, namely the Clifford algebra.

where $\psi_\alpha \in \mathbb{C}$. In the latter case $\psi = \psi_1 f_1 + \psi_2 f_2 + \psi_3 f_3 + \psi_4 f_4$ expressed in a basis of the complex linear spinor space $S = (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$,

$$\begin{aligned}
f_1 &= \tfrac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_{012}) &&= f, \\
f_2 &= \tfrac{1}{4}(-\gamma_{13} + i\gamma_{23} - \gamma_{013} + i\gamma_{023}) &&= -\gamma_{13} f, \\
f_3 &= \tfrac{1}{4}(\gamma_3 - \gamma_{03} + i\gamma_{123} - i\gamma_{0123}) &&= -\gamma_{03} f, \\
f_4 &= \tfrac{1}{4}(\gamma_1 - i\gamma_2 - \gamma_{01} + i\gamma_{02}) &&= -\gamma_{01} f.
\end{aligned}$$

We write $\gamma_{12} = \gamma_1\gamma_2$ $[\neq i\gamma_1\gamma_2]$ and $\gamma_{0123} = \gamma_0\gamma_1\gamma_2\gamma_3$.

10.3 Real structures and involutions

Although we have an isomorphism of real algebras $\mathbb{C} \otimes \mathrm{Mat}(4, \mathbb{R}) \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$, the complex conjugations are not the same in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ and $\mathbb{C} \otimes \mathrm{Mat}(4, \mathbb{R}) \simeq \mathrm{Mat}(4, \mathbb{C})$. In the matrix algebra $\mathrm{Mat}(4, \mathbb{C})$ we take complex conjugates of the matrix entries $u^* = (u_{jk})^* = (u_{jk}^*)$, whereas in the complexified Clifford algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ complex conjugation has no effect on the real part $\mathcal{C}\ell_{1,3}$, and we have $u^* = (a + ib)^* = a - ib$ for $a, b \in \mathcal{C}\ell_{1,3}$. Thus there are two different complex conjugations (real parts) in the algebra $\mathbb{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathrm{Mat}(4, \mathbb{C})$. This is referred to by saying that there are two different *real structures* [11] in the same complex algebra.

To make this point more explicit, the following table lists some correspondences of involutions.

	$\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$	$\mathrm{Mat}(4, \mathbb{C})$	
complex conjugate	u^*	$\gamma_{013} u^* \gamma_{013}^{-1}$	
	$\gamma_{013} u^* \gamma_{013}^{-1}$	u^*	complex conjugate
grade involute	$\hat{u}$	$\gamma_{0123} u \gamma_{0123}^{-1}$	
reverse	$\tilde{u}$	$\gamma_{13} u^\top \gamma_{13}^{-1}$	
Clifford-conjugate	$\bar{u}$	$\gamma_{02} u^\top \gamma_{02}^{-1}$	
	$\gamma_{13} \tilde{u} \gamma_{13}^{-1}$	$u^\top$	transpose
	$\gamma_0 \tilde{u}^* \gamma_0^{-1}$	$u^\dagger = u^{*\top}$	Hermitian conjugate
	$\tilde{u}^*$	$\gamma_0 u^\dagger \gamma_0^{-1}$	Dirac adjoint

An element $u = \langle u \rangle_0 + \langle u \rangle_1 + \langle u \rangle_2 + \langle u \rangle_3 + \langle u \rangle_4 \in \mathcal{C}\ell_{1,3}$, decomposed in

11 Not to be confused with the *complex structure* of an even-dimensional real linear space, a real linear transformation J such that $J^2 = -I$.

dimension degrees $\langle u\rangle_k \in \bigwedge^k \mathbb{R}^{1,3}$, has three important involutions:

$$\begin{aligned}
\hat{u} &= \langle u\rangle_0 - \langle u\rangle_1 + \langle u\rangle_2 - \langle u\rangle_3 + \langle u\rangle_4, && \text{grade involution,}\\
\tilde{u} &= \langle u\rangle_0 + \langle u\rangle_1 - \langle u\rangle_2 - \langle u\rangle_3 + \langle u\rangle_4, && \text{reversion,}\\
\bar{u} &= \langle u\rangle_0 - \langle u\rangle_1 - \langle u\rangle_2 + \langle u\rangle_3 + \langle u\rangle_4, && \text{Clifford-conjugation.}
\end{aligned}$$

The reversion and Clifford-conjugation are anti-automorphisms satisfying $\widetilde{uv} = \tilde{v}\tilde{u}$, $\overline{uv} = \bar{v}\bar{u}$, whereas the grade involution is an automorphism $\widehat{uv} = \hat{u}\hat{v}$. These three involutions are extended to $\mathbb{C}\otimes \mathcal{C}\ell_{1,3}$ as complex linear functions, that is, for $\lambda \in \mathbb{C}$ and $u \in \mathcal{C}\ell_{1,3}$ we have $(\lambda u)\hat{} = \lambda\hat{u}$, $(\lambda u)\tilde{} = \lambda\tilde{u}$, $(\lambda u)^- = \lambda\bar{u}$, whereas the complex conjugation is by definition anti-linear: $(\lambda u)^* = \lambda^* u$. Complex conjugation is of course an automorphism, $(uv)^* = u^*v^*$ for $u, v \in \mathbb{C}\otimes \mathcal{C}\ell_{1,3}$.

10.4 Comparison of real parts/structures

Note that the real part and the complex conjugate of a Dirac spinor depend on the decomposition (in the real structure) singling out the real part. For $\psi \in \mathrm{Mat}(4,\mathbb{C})f$:

$$\mathrm{Re}(\psi) = \begin{pmatrix} \mathrm{Re}(\psi_1) & 0 & 0 & 0\\ \mathrm{Re}(\psi_2) & 0 & 0 & 0\\ \mathrm{Re}(\psi_3) & 0 & 0 & 0\\ \mathrm{Re}(\psi_4) & 0 & 0 & 0 \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} \psi_1^* & 0 & 0 & 0\\ \psi_2^* & 0 & 0 & 0\\ \psi_3^* & 0 & 0 & 0\\ \psi_4^* & 0 & 0 & 0 \end{pmatrix}.$$

For $\psi \in (\mathbb{C}\otimes \mathcal{C}\ell_{1,3})f$ [viewed as a matrix]:

$$\mathrm{Re}(\psi) = \frac{1}{2}\begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0\\ \psi_2 & \psi_1^* & 0 & 0\\ \psi_3 & \psi_4^* & 0 & 0\\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} 0 & -\psi_2^* & 0 & 0\\ 0 & \psi_1^* & 0 & 0\\ 0 & \psi_4^* & 0 & 0\\ 0 & -\psi_3^* & 0 & 0 \end{pmatrix}.$$

The Dirac spinor ψ might appear as a column spinor $\psi \in \mathbb{C}^4$ or else as a square matrix spinor $\psi \in \mathrm{Mat}(4,\mathbb{C})f$ or as a *Clifford algebraic spinor* $\psi \in (\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$ where the last two differ in their real structures.

Important Note. To indicate in what real structure the real part and the complex conjugate are taken we write

$$\mathrm{Re}(\psi) \text{ in } \mathrm{Mat}(4,\mathbb{C})f \qquad \text{and} \qquad \psi^* \text{ in } \mathrm{Mat}(4,\mathbb{C})f$$

or

$$\mathrm{Re}(\psi) \text{ in } \mathbb{C}\otimes\mathcal{C}\ell_{1,3} \qquad \text{and} \qquad \psi^* \text{ in } \mathbb{C}\otimes\mathcal{C}\ell_{1,3}.$$

Other contextual indicators are the Hermitian conjugation [either $\psi^\dagger\gamma_0$ is a row spinor or else it is in $\mathrm{Mat}(4,\mathbb{C})$] and for instance the reversion [the composite of the reversion and the complex conjugation $\tilde{\psi}^*$ in $\mathbb{C}\otimes \mathcal{C}\ell_{1,3}$ corresponds to $\psi^\dagger\gamma_0$ in $\mathrm{Mat}(4,\mathbb{C})$].

The reader should also observe that the real part $\mathrm{Re}(\psi)$ in $\mathbb{C}\otimes \mathcal{C}\ell_{1,3}$ carries the same information as the original Dirac spinor $\psi \in \mathbb{C}^4$ [in contrast to $\mathrm{Re}(\psi)$ in $\mathrm{Mat}(4,\mathbb{C})f$]. ∎

Exercises 1,2,3,4,5

10.5 Bilinear covariants via algebraic spinors

For a column spinor $\psi \in \mathbb{C}^4$ the Dirac adjoint is a row matrix

$$\psi^\dagger\gamma_0 = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \end{pmatrix}$$

but for a square matrix spinor $\psi \in \mathrm{Mat}(4,\mathbb{C})f$ the Dirac adjoint is a square matrix

$$\psi^\dagger\gamma_0 = \gamma_0\psi^\dagger\gamma_0^{-1} = \begin{pmatrix} \psi_1^* & \psi_2^* & -\psi_3^* & -\psi_4^* \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with only the first row being non-zero.

The components of the Dirac current can be computed as follows for column spinors, square matrix spinors and Clifford algebraic spinors

$$\begin{aligned} J_\mu &= \psi^\dagger\gamma_0\gamma_\mu\psi && \psi \in \mathbb{C}^4 \\ &= \mathrm{trace}(\psi^\dagger\gamma_0\gamma_\mu\psi) && \psi \in \mathrm{Mat}(4,\mathbb{C})f \\ &= \mathrm{trace}(\gamma_\mu\psi\psi^\dagger\gamma_0) = 4\langle\gamma_\mu\psi\psi^\dagger\gamma_0\rangle_0 && \\ &= 4\langle\gamma_\mu\psi\tilde{\psi}^*\rangle_0 && \psi \in (\mathbb{C}\otimes \mathcal{C}\ell_{1,3})f \end{aligned}$$

where the factor 4 appeared because

$$f = \frac{1}{4}(1+\gamma_0+i\gamma_{12}+i\gamma_{012}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

has scalar part $\frac{1}{4}$, that is, $\langle f\rangle_0 = \frac{1}{4}$, while $\mathrm{trace}(f) = 1$. The current vector is the resultant

$$\begin{aligned} \mathbf{J} = \gamma^\mu J_\mu &= \gamma^\mu 4\langle\gamma_\mu\psi\psi^\dagger\gamma_0\rangle_0 && \psi \in \mathrm{Mat}(4,\mathbb{C})f \\ &= \gamma^\mu\langle\gamma_\mu \,\lrcorner\, (4\psi\psi^\dagger\gamma_0)\rangle_0 = \gamma^\mu {<}\gamma_\mu, 4\psi\psi^\dagger\gamma_0{>} && \\ &= \langle 4\psi\psi^\dagger\gamma_0\rangle_1 && J_\mu = \gamma_\mu\cdot\mathbf{J} \\ &= \langle 4\psi\tilde{\psi}^*\rangle_1 && \psi \in (\mathbb{C}\otimes \mathcal{C}\ell_{1,3})f. \end{aligned}$$

Similarly $\psi \in \mathbb{C}^4$ carries a real bivector $\mathbf{S}$ with components

$$S_{\mu\nu} = \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi \qquad (\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu \neq i\gamma_\mu \gamma_\nu)$$

for which $S_{\mu\nu} = -\gamma_{\mu\nu} \rfloor \mathbf{S}$ and $\mathbf{S} = \frac{1}{2} S_{\mu\nu} \gamma^{\mu\nu}$. In various formalisms

$$\begin{aligned}
S_{\mu\nu} &= \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi && \psi \in \mathbb{C}^4 \\
&= \operatorname{trace}(\psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi) && \psi \in \operatorname{Mat}(4, \mathbb{C}) f \\
&= \operatorname{trace}(i\gamma_{\mu\nu} \psi \psi^\dagger \gamma_0) = 4\langle i\gamma_{\mu\nu} \psi \psi^\dagger \gamma_0 \rangle_0 && \\
&= 4\langle i\gamma_{\mu\nu} \psi \tilde{\psi}^* \rangle_0 && \psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) f
\end{aligned}$$

$$\begin{aligned}
\mathbf{S} = \tfrac{1}{2} \gamma^{\mu\nu} S_{\mu\nu} &= \tfrac{1}{2} \gamma^{\mu\nu} 4\langle i\gamma_{\mu\nu} \psi \psi^\dagger \gamma_0 \rangle_0 && \psi \in \operatorname{Mat}(4, \mathbb{C}) f \\
&= \tfrac{1}{2} \gamma^{\mu\nu} \langle i\gamma_{\mu\nu} \rfloor (4\psi\psi^\dagger \gamma_0) \rangle_0 && S_{\mu\nu} = -\gamma_{\mu\nu} \rfloor \mathbf{S} \\
&= \tfrac{1}{2} \gamma^{\mu\nu} {<} -i\gamma_{\mu\nu}, 4\psi\psi^\dagger \gamma_0 {>} && {<}u, v{>} = \langle \tilde{u} \rfloor v \rangle_0 \\
&= \langle -i4\psi\psi^\dagger \gamma_0 \rangle_2 = -i\langle 4\psi\psi^\dagger \gamma_0 \rangle_2 && \\
&= -i\langle 4\psi\tilde{\psi}^* \rangle_2 && \psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) f.
\end{aligned}$$

The Dirac adjoint $\psi^\dagger \gamma_0$ of a column spinor $\psi \in \mathbb{C}^4$ corresponds to $\tilde{\psi}^*$ of an algebraic spinor $\psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) f$. The current vector $\mathbf{J}$ and the bivector $\mathbf{S}$ are examples of *bilinear covariants* listed below for a column spinor $\psi \in \mathbb{C}^4$ and for an algebraic spinor $\psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) f$.

$$\begin{aligned}
\Omega_1 &= \psi^\dagger \gamma_0 \psi = 4\langle \tilde{\psi}^* \psi \rangle_0 = 4\langle \psi \tilde{\psi}^* \rangle_0 && \\
J_\mu &= \psi^\dagger \gamma_0 \gamma_\mu \psi = 4\langle \tilde{\psi}^* \gamma_\mu \psi \rangle_0 && \\
S_{\mu\nu} &= \psi^\dagger \gamma_0 i \gamma_{\mu\nu} \psi = 4\langle \tilde{\psi}^* i \gamma_{\mu\nu} \psi \rangle_0 && \gamma_{\mu\nu} = \gamma_\mu \gamma_\nu \neq i\gamma_\mu \gamma_\nu \\
K_\mu &= \psi^\dagger \gamma_0 i \gamma_{0123} \gamma_\mu \psi = 4\langle \tilde{\psi}^* i \gamma_{0123} \gamma_\mu \psi \rangle_0 && \mathbf{K} = K_\mu \gamma^\mu \\
\Omega_2 &= -\psi^\dagger \gamma_0 \gamma_{0123} \psi = -4\langle \tilde{\psi}^* \gamma_{0123} \psi \rangle_0 && \gamma_{0123} = \gamma_0 \gamma_1 \gamma_2 \gamma_3.
\end{aligned}$$

Later we shall need the following aggregate of bilinear covariants $Z = \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2 \gamma_{0123}$.

Spinors as Operators

Here we shall view spinors as new kinds of objects: rather than being something which are operated upon they are regarded as active operators. The big advantage is that the physical observables, which were earlier calculated component-wise, can now be obtained at one stroke.

10.6 Spinor operators $\Psi \in C\ell^+_{1,3}$

We will associate to a Clifford algebraic spinor $\psi \in (\mathbb{C} \otimes C\ell_{1,3})f$ [viewed here as a matrix] the *mother spinor* [this will be the mother of all real spinors]

$$\Phi = 4\,\mathrm{Re}(\psi) = 2\begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0 \\ \psi_2 & \psi_1^* & 0 & 0 \\ \psi_3 & \psi_4^* & 0 & 0 \\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix}$$

and the *spinor operator*

$$\Psi = \mathrm{even}(\Phi) = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}.$$

From the mother spinor $\Phi \in C\ell_{1,3}\frac{1}{2}(1+\gamma_0)$ we may reobtain the original Dirac spinor

$$\psi = \Phi\frac{1}{4}(1+i\gamma_{12}) \in (\mathbb{C} \otimes C\ell_{1,3})f$$

[that is, the square matrix spinor, not the column spinor], and from the spinor operator Ψ we may reobtain the mother spinor $\Phi = \Psi(1+\gamma_0)$ and the original Dirac spinor

$$\psi = \Psi\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12}) \in (\mathbb{C} \otimes C\ell_{1,3})f.$$

Note that the spinor operator is invertible if

$$|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 \neq 0 \quad \text{and} \quad 2\,\mathrm{Im}(\psi_1^*\psi_3 + \psi_2^*\psi_4) \neq 0,$$

or equivalently $\Psi\tilde{\Psi} \neq 0$, the inverse being

$$\Psi^{-1} = \frac{\tilde{\Psi}}{\Psi\tilde{\Psi}}.$$

Multiplication by $i = \sqrt{-1}$ corresponds to right multiplication by the bivector

$$\gamma_2\gamma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix},$$

that is, $i\psi = \psi\gamma_2\gamma_1$ for $\psi \in \mathbb{C}^4$. In other words, the real part of $i\psi$, $\psi \in (\mathbb{C}\otimes C\ell_{1,3})f$, is the mother spinor $\Phi\gamma_2\gamma_1$ whose even part is the spinor operator $\Psi\gamma_2\gamma_1$, $4\,\mathrm{Re}(\mathrm{even}(i\psi)) = \Psi\gamma_2\gamma_1$.

Decompose the mother spinor $\Phi \in C\ell_{1,3}\frac{1}{2}(1+\gamma_0)$ into even and odd parts

$\Phi = \Phi_0 + \Phi_1 = (\Phi_0 + \Phi_1)\frac{1}{2}(1+\gamma_0) = \frac{1}{2}(\Phi_0 + \Phi_1\gamma_0) + \frac{1}{2}(\Phi_1 + \Phi_0\gamma_0)$. It follows that $\Phi_0 = \Phi_1\gamma_0$ and $\Phi_1 = \Phi_0\gamma_0$. Taking the real part [in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$] of the Dirac equation $(i\partial - e\mathbf{A})\psi = m\psi$ results in

$$\partial\Phi\gamma_2\gamma_1 - e\mathbf{A}\Phi = m\Phi,$$

which decomposes into even and odd parts [$\Phi_0 = \text{even}(\Phi)$, $\Phi_1 = \text{odd}(\Phi)$]

$$\begin{aligned} \partial\Phi_0\gamma_{21} - e\mathbf{A}\Phi_0 &= m\Phi_1 \qquad [\Phi_1 = \Phi_0\gamma_0], \\ \partial\Phi_1\gamma_{21} - e\mathbf{A}\Phi_1 &= m\Phi_0 \qquad [\Phi_0 = \Phi_1\gamma_0]. \end{aligned}$$

Therefore, the even part of the mother spinor, the spinor operator, satisfies the equation [12]

$$\boxed{\partial\Psi\gamma_{21} - e\mathbf{A}\Psi = m\Psi\gamma_0}$$

where $\Psi : \mathbb{R}^{1,3} \to \mathcal{C}\ell^+_{1,3}$. In this *Dirac-Hestenes equation* the role of the Dirac column spinors is taken over by real even multivectors, which are not in any proper left ideal of the Clifford algebra $\mathcal{C}\ell_{1,3}$.

Comments. 1. Under a Lorentz transformation $\mathbf{x} \to s\mathbf{x}\hat{s}^{-1}$, $s \in \mathbf{Pin}(1,3)$, $\mathbf{x} \in \mathbb{R}^{1,3}$, a Dirac spinor $\psi \in \text{Mat}(4,\mathbb{C})f \simeq (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$ transforms according to $\psi \to s\psi$, and a spinor operator $\Psi \in \mathcal{C}\ell^+_{1,3}$ transforms like this:

$$\begin{aligned} &\Psi \to s\Psi && \text{when} \quad s \in \mathbf{Spin}(1,3), \\ &\Psi \to s\Psi\gamma_0 && \text{when} \quad s \in \mathbf{Pin}(1,3)\backslash\mathbf{Spin}(1,3). \end{aligned}$$

This can be seen by the definition $\Psi = 4\,\text{Re}(\text{even}(\psi))$ and using $\psi = \psi f$, $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})$.

Note that the so-called Wigner time-reversal is not represented by any $s \in \mathbf{Pin}(1,3)\backslash\mathbf{Spin}(1,3)$.

2. The Dirac-Hestenes equation has been criticized on the basis that it is not Lorentz covariant because of an explicit appearance of the two basis elements γ_0 and γ_{12}. This criticism does not hold. The Dirac-Hestenes equation is Lorentz covariant in two different ways: first, we can regard γ_0 and γ_{12} as constants and transform Ψ to $s\Psi$; secondly, we can transform γ_0, γ_{12} to $s\gamma_0 s^{-1}$, $s\gamma_{12}s^{-1}$ and Ψ to $s\Psi s^{-1}$, $s \in \mathbf{Spin}_+(1,3)$.

3. In curved space-times spinor fields/bundles [functions with values in a minimal left ideal of a Clifford algebra] exist globally only under certain topological conditions: the space-time must be a spin manifold [have a spinor structure]. It has been argued that since even multivector functions exist on all oriented manifolds, the theory of spin manifolds is superfluous. This argument is misplaced since γ_0 and γ_{12} do not exist globally.

12 Note that $4\,\text{Re}(\text{even}(\mathbf{x}\psi)) = \mathbf{x}\Psi\gamma_0$ for $\mathbf{x} \in \mathbb{R}^{1,3}$.

However, the physical justification of the theory of spin manifolds could be questioned on the following basis: why should we need to know the global properties of the universe if we want to explore the local properties of a single electron?
4. The explicit occurrence of γ_0 and γ_{12} is due to our injection $\mathbb{C}^4 \to S = (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$. In curved manifolds it is more appropriate to use abstract representation modules as spinor spaces and not minimal left ideals [nor the even subalgebras] of Clifford algebras. The injection ties these spaces together in a manner that singles out special directions in $\mathbb{R}^{1,3}$. ∎

10.7 Bilinear covariants via spinor operators

Write as before $\Psi = 4\,\mathrm{Re}(\mathrm{even}(\psi))$ [real part taken in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$]. Because of the identities [13]

$$\begin{aligned}
&\Psi\tilde{\Psi} = \Omega_1 + \Omega_2\gamma_{0123}, \\
&\Psi\gamma_0\tilde{\Psi} = \mathbf{J}, \\
&\Psi\gamma_{12}\tilde{\Psi} = \mathbf{S}, && \Psi\gamma_{03}\tilde{\Psi} = -\mathbf{S}\gamma_{0123}, \\
&\Psi\gamma_3\tilde{\Psi} = \mathbf{K}, && \Psi\gamma_{012}\tilde{\Psi} = \mathbf{K}\gamma_{0123},
\end{aligned}$$

we call Ψ a *spinor operator*. In the non-null case $\Psi\tilde{\Psi} \neq 0$ the element Ψ operates like a Lorentz transformation composed with a dilation [and a duality transformation]. In coordinate form

$$\begin{aligned}
&\Omega_1 + \Omega_2\gamma_{0123} = \Psi\tilde{\Psi} = \tilde{\Psi}\Psi, \\
&J_\mu = \langle\tilde{\Psi}\gamma_\mu\Psi\gamma_0\rangle_0 = (\tilde{\Psi}\gamma_\mu\Psi)\cdot\gamma_0, \\
&S_{\mu\nu} = -\langle\tilde{\Psi}\gamma_{\mu\nu}\Psi\gamma_{12}\rangle_0 = (\tilde{\Psi}\gamma_{\mu\nu}\Psi)\cdot\gamma_{12}, \\
&K_\mu = \langle\tilde{\Psi}\gamma_\mu\Psi\gamma_3\rangle_0 = (\tilde{\Psi}\gamma_{0123}\gamma_\mu\Psi)\cdot\gamma_{012}.
\end{aligned}$$

For later convenience we introduce $P = \Omega + \mathbf{J}$, $\Omega = \Omega_1 + \Omega_2\gamma_{0123}$, and $Q = \mathbf{S} + \mathbf{K}\gamma_{0123}$. We have the following identities:

$$\begin{aligned}
&\Psi(1+\gamma_0)\tilde{\Psi} = P, && \Psi(1+\gamma_0)\gamma_{12}\tilde{\Psi} = Q, \\
&\Psi(1+\gamma_0)\gamma_{ij}\tilde{\Psi} = Q_k && (ijk \text{ cycl.},\ Q_3 = Q), \\
&\Psi(1+\gamma_0)(1+i\gamma_{12})\tilde{\Psi} = Z && [= P + iQ].
\end{aligned}$$

Hestenes 1986 p. 334 gives $P, -Q$ and Z in (2.26), (2.27) and (2.28).

Exercises 6,7,8

13 The Dirac-Hestenes equation $\partial\Psi\gamma_{21} - e\mathbf{A}\Psi = m\Psi\gamma_0$ contains γ_{21}, γ_0 explicitly. It follows that γ_0, γ_{12} must be explicit in $\mathbf{J} = \Psi\gamma_0\tilde{\Psi}$, $\mathbf{S} = \Psi\gamma_{12}\tilde{\Psi}$.

10.8 Higher-dimensional analogies for spinor operators

In the case of the Minkowski space-time $\mathbb{R}^{1,3}$ the spinor space is a minimal left ideal $(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$ induced by the primitive idempotent $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})$. In the primitive idempotent we have projection operators for energy $\frac{1}{2}(1+\gamma_0)$ and spin $\frac{1}{2}(1+i\gamma_{12})$. In other words, spin is quantized in the γ_3-direction or more precisely in the $\gamma_1\gamma_2$-plane.

In the case of a higher-dimensional space-time, say $\mathbb{R}^{1,5}$ with an orthonormal basis $\{\gamma_0, \gamma_1, \ldots, \gamma_5\}$, the spinor space is a minimal left ideal $(\mathbb{C}\otimes\mathcal{C}\ell_{1,5})f$ induced, for instance, by the primitive idempotent

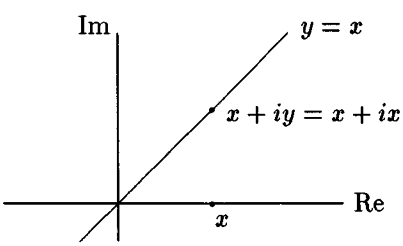
$$f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})\frac{1}{2}(1+i\gamma_{34}).$$

The spin is quantized in the $\gamma_1\gamma_2$-plane and the $\gamma_3\gamma_4$-plane. The procedure of taking the real part and the even part does not result in an invertible operator, since $4\,\mathrm{Re}(\mathrm{even}(f)) = \frac{1}{2}(1-\gamma_{1234})$. In other words, for a spinor in a minimal left ideal $\psi\in(\mathbb{C}\otimes\mathcal{C}\ell_{1,5})f$ the 'spinor operator' is also in a left ideal, $\Psi = 4\,\mathrm{Re}(\mathrm{even}(\psi)) \in \mathcal{C}\ell^{+}_{1,5}\frac{1}{2}(1-\gamma_{1234})$. We conclude that there is no analogy for spinor operators in higher dimensions.

Appendix 1: Discussion on the role of $i = \sqrt{-1}$ in QM

Are there superfluous complex numbers in the present formulation of quantum mechanics? Is it possible to get rid of *some* complex numbers in QM? To answer these questions, we present analogies which become step by step closer to the present situation in quantum mechanics.

Analogy # 1. Consider someone who uses only the line $y = x$ in the complex plane $\mathbb{C}$, that is, someone who does not use all the complex numbers $z = x+iy$, but instead restricts himself to complex numbers of the form $x+ix$. This person could equally well restrict himself to the real axis and consider instead only the real part $x = \mathrm{Re}(x+ix)$. In terms of the picture



while $y = x$

this would mean a projection from the line $y = x$ onto the real axis $y = 0$ with no information lost.

This analogy/picture could be criticized by arguing that the product w of two complex numbers of the form $x + ix$ is not of the same type, that is, $\mathrm{Re}(w) \neq \mathrm{Im}(w)$. ∎

Analogy # 2. The sums and products of matrices of type

$$X = \frac{1}{2}\begin{pmatrix} x & x \\ x & x \end{pmatrix}$$

provide an isomorphic image for addition and multiplication of the reals $x \in \mathbb{R}$. It is of course more economical to use just the real numbers $x \in \mathbb{R}$ instead of the real 2×2-matrices $X \in \mathrm{Mat}(2, \mathbb{R})$. ∎

Analogy # 3. If we have a complex matrix

$$S = \frac{1}{2}\begin{pmatrix} x+iy & -y+ix \\ y-ix & x+iy \end{pmatrix}$$

then the real part, multiplied by two, $Z = 2\,\mathrm{Re}(S)$, that is,

$$Z = \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

obeys the same addition and multiplication rules as S and carries the same information as S [contained in the pair (x, y)]. Note that for a complex number $z = x + iy$ we have $S = \frac{z}{2}(I - \sigma_2)$, where the matrix $f = \frac{1}{2}(I - \sigma_2)$ is an idempotent satisfying $f^2 = f$. ∎

Situation in QM. In the present formulation of quantum mechanics one uses column spinors $\psi \in \mathbb{C}^4$, which could be replaced without loss of generality by spinors in a minimal left ideal of the complex Clifford algebra $\psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$, $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})$. Spinors in minimal left ideals $\psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$ can be replaced without reduction of information by spinor operators $\Psi = 4\,\mathrm{Re}(\mathrm{even}(\psi)) \in \mathcal{C}\ell^+_{1,3}$. No information is lost in this replacement, because the original spinor can be recovered as $\psi = \Psi f$, $f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})$. ∎

Appendix 2: Real ideal spinors $\phi \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03})$

This appendix is included mainly for historical reasons. The concept of a spinor operator $\Psi \in \mathcal{C}\ell^+_{1,3}$ was introduced by Hestenes 1966. In his invention he used as an intermediate step the *real ideal spinor*

$$\phi = \Phi\,\frac{1}{2}(1-\gamma_{03}) \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03})$$

and not the mother spinor $\Phi \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1+\gamma_0)$, $\Phi = 4\,\mathrm{Re}(\psi)$, $\psi \in (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f$. The ideal spinor contains the same information as the mother spinor, since

$\Phi = \phi(1+\gamma_0)$. Note that $\phi = \Phi\frac{1}{2}(1-\gamma_{03})$ or $\phi = 2\,\mathrm{Re}(\psi)(1-\gamma_{03})$ which implies $\phi\gamma_2\gamma_1 = 2\,\mathrm{Re}(i\psi)(1-\gamma_{03})$, and so the Dirac equation has the form

$$\partial\phi\gamma_{21} = (e\mathbf{A}+m)\phi, \qquad \phi \in \mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03}).$$

In contrast to the mother spinor Φ, the real ideal spinor ϕ satisfies $\phi\gamma_{0123} = \phi\gamma_{21}$, and so we could rewrite the Dirac equation in the same way as Hestenes 1966:

$$\partial\phi\gamma_{0123} = (e\mathbf{A}+m)\phi.$$

Comments. The ideal spinors might be useful in conjunction with conformal transformations of the Dirac equation. Decompose the ideal spinor $\phi = \phi_0+\phi_1$ into its even and odd parts and separate the parts,

$$\begin{aligned}\partial\phi_0\gamma_{0123} &= e\mathbf{A}\phi_0 + m\phi_1, \quad \phi_0 = \mathrm{even}(\phi) \in \mathcal{C}\ell^+_{1,3}\,\tfrac{1}{2}(1-\gamma_{03}),\\ \partial\phi_1\gamma_{0123} &= e\mathbf{A}\phi_1 + m\phi_0, \quad \phi_1 = \mathrm{odd}(\phi) \in \mathcal{C}\ell^-_{1,3}\,\tfrac{1}{2}(1-\gamma_{03}),\end{aligned}$$

which can be put into the matrix form

$$\begin{pmatrix}\gamma_{0123} & 0\\ 0 & -\gamma_{0123}\end{pmatrix}\begin{pmatrix}0 & \partial\\ -\partial & 0\end{pmatrix}\begin{pmatrix}\phi_0 & 0\\ \phi_1 & 0\end{pmatrix}$$
$$= e\begin{pmatrix}0 & \mathbf{A}\\ -\mathbf{A} & 0\end{pmatrix}\begin{pmatrix}\phi_0 & 0\\ \phi_1 & 0\end{pmatrix} + m\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\phi_0 & 0\\ \phi_1 & 0\end{pmatrix},$$

where we have used the fact that the matrix

$$\begin{pmatrix}\gamma_{0123} & 0\\ 0 & -\gamma_{0123}\end{pmatrix} \text{ commutes with } \begin{pmatrix}0 & \partial\\ -\partial & 0\end{pmatrix} \text{ and } \begin{pmatrix}\phi_0 & 0\\ \phi_1 & 0\end{pmatrix}$$

and takes the role of an overall commuting imaginary unit $\sqrt{-1}$. ∎

Exercises 9,10

Historical survey

Pauli 1927 and Dirac 1928 presented their spinor equations for the description of the electron spin. Juvet 1930 and Sauter 1930 replaced column spinors by square matrix spinors, where only the first column was non-zero. Marcel Riesz 1947 was the first one to consider spinors as elements in a minimal left ideal of a Clifford algebra (although the special case of pure spinors had been considered earlier by Cartan in 1938).

Gürsey 1956-58 rewrote the Dirac equation with 2×2 quaternion matrices in $\mathrm{Mat}(2,\mathbb{H})$ [Lanczos 1929 had used pairs of quaternions, see Gsponer & Hurni 1993]. Kustaanheimo 1964 presented the spinor regularization of the Kepler motion, the KS-transformation, which emphasized the operator aspect of spinors. This led David Hestenes 1966-74 to a reformulation of the Dirac theory, where the role of spinors [in columns $\mathbb{C}^4$ or in minimal left ideals of the complex Clifford algebra $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}\simeq \mathrm{Mat}(4,\mathbb{C})$] was taken over by operators in the even subalgebra $\mathcal{C}\ell^+_{1,3}$ of the real Clifford algebra $\mathcal{C}\ell_{1,3}\simeq \mathrm{Mat}(2,\mathbb{H})$.

Exercises

1. Show that for $u\in\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ the real part $\mathrm{Re}(u)$ corresponds to $\frac{1}{2}(u+\gamma_{013}u^*\gamma_{013}^{-1})\in\mathrm{Mat}(4,\mathbb{C})$.
2. Show that if $u\in\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ satisfies the condition $u=u\,\frac{1}{2}(1+i\gamma_{12})$ then $u=\mathrm{Re}(u)(1+i\gamma_{12})$ and $iu=u\gamma_2\gamma_1$.
3. Show that $\mathrm{Im}(\psi)=\mathrm{Re}(\psi)\gamma_{12}$ in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$.
4. Show that the charge conjugate $\psi_C=-i\gamma_2\psi^*$ of the Dirac spinor $\psi\in\mathrm{Mat}(4,\mathbb{C})f$ corresponds to $\psi_C=\hat{\psi}^*\gamma_1\in(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$.
5. Show that although for $\psi\in\mathrm{Mat}(4,\mathbb{C})f$, $\mathrm{Re}(\psi)\in\mathrm{Mat}(4,\mathbb{C})f$, for a non-zero $\psi\in(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$, $\mathrm{Re}(\psi)\notin(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})f$.
6. Show that in terms of the ideal spinor $\phi=\Phi\,\frac{1}{2}(1-\gamma_{03})\in\mathcal{C}\ell_{1,3}\,\frac{1}{2}(1-\gamma_{03})$, where $\Phi=4\,\mathrm{Re}(\psi)$, the bilinear covariants can be expressed as

$$\begin{aligned}
\Omega_1 &= \langle\tilde{\phi}\phi\gamma_0\rangle_0 = (\tilde{\phi}\phi)\cdot\gamma_3,\\
J_\mu &= \langle\tilde{\phi}\gamma_\mu\phi\gamma_0\rangle_0 = (\tilde{\phi}\gamma_\mu\phi)\cdot\gamma_3,\\
S_{\mu\nu} &= -\langle\tilde{\phi}\gamma_{\mu\nu}\phi\gamma_{123}\rangle_0 = -\langle\tilde{\phi}\gamma_{\mu\nu}\phi\rangle_3\cdot\gamma_{123},\\
K_\mu &= -\langle\tilde{\phi}\gamma_{0123}\gamma_\mu\phi\gamma_{123}\rangle_0,\\
\Omega_2 &= -\langle\tilde{\phi}\gamma_{0123}\phi\gamma_0\rangle_0
\end{aligned}$$

and the aggregates $P=\Omega+\mathbf{J}$ and $Q=\mathbf{S}+\mathbf{K}\gamma_{0123}$ as

$$\begin{aligned}
&\phi\gamma_0\tilde{\phi}=\phi\gamma_3\tilde{\phi}=P,\\
&\phi\gamma_0\bar{\phi}=\phi\gamma_3\bar{\phi}=-Q\gamma_{0123},\qquad Q=\phi\gamma_{123}\tilde{\phi}.
\end{aligned}$$

[Hestenes 1986 p. 334 gives P in (2.26) and $-Q$ in (2.27).]

7. Show that $\phi\tilde{\phi}$, $\phi\bar{\phi}$, $\phi\gamma_1\tilde{\phi}$, $\phi\gamma_1\bar{\phi}$, $\phi\gamma_2\tilde{\phi}$, $\phi\gamma_2\bar{\phi}$ all vanish.
8. Show that for a different choice of sign in $\phi=\Phi\,\frac{1}{2}(1-\gamma_{03})$, namely $\varphi=\Phi\,\frac{1}{2}(1+\gamma_{03})$, we have $\varphi\gamma_0\tilde{\varphi}=P=-\varphi\gamma_3\tilde{\varphi}$ and $\varphi\gamma_3\bar{\varphi}=-Q\gamma_{0123}=-\varphi\gamma_0\bar{\varphi}$.
9. Show that $\phi=\Phi\,\frac{1}{2}(1-\gamma_3)$.
10. Show that $\phi=\Psi\,\frac{1}{2}(1+\gamma_0)(1-\gamma_{03})$.

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