8
Electromagnetism

The Maxwell equations can be formulated with vectors or more advanced notation like tensors, differential forms or Clifford bivectors. In these advanced formalisms the Maxwell equations become more uniform and easier to manipulate; for instance, relativistic covariance is more apparent. However, the cost of the convenience is that one has to master new concepts in addition to scalars and vectors; and antisymmetric tensors have to be untangled for physical interpretation.

8.1 The Maxwell equations

The electric field $\vec{E}$ and the magnetic induction $\vec{B}$ act on a charge q moving at velocity $\vec{v}$ by the Lorentz force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

The electric displacement $\vec{D}$ and the magnetic intensity $\vec{H}$ are related to $\vec{E}$ and $\vec{B}$ by the constitutive relations

$$\vec{D} = \varepsilon \vec{E}, \qquad \vec{B} = \mu \vec{H}.$$

J. C. Maxwell brought together the following four equations in 1864:

$$\begin{array}{lll}
\nabla \cdot \vec{D} = \rho & \displaystyle\oint_S \vec{D} \cdot d\vec{s} = Q & \text{Gauss' law} \\
\nabla \times \vec{H} = \vec{J} & \displaystyle\oint_C \vec{H} \cdot d\vec{\ell} = I & \text{Ampère's law} \\
\nabla \cdot \vec{B} = 0 & \displaystyle\oint_S \vec{B} \cdot d\vec{s} = 0 & \text{no magnetic sources} \\
\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \displaystyle\oint_C \vec{E} \cdot d\vec{\ell} = -\dfrac{d\Phi}{dt} & \text{Faraday's law}
\end{array}$$

Maxwell also complemented Ampère's law by a new term, which observed time-dependence. Ampère had developed a mathematical formulation for producing magnetism by electricity, a phenomenon detected by Ørsted [1] in 1820, but his law is not valid in a time-varying situation: take the divergence of both sides to obtain

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J}$$

which violates charge conservation. [2] Maxwell corrected this equation into the form

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t},$$

applied Gauss' law, and got

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

This predicted the existence of a displacement current $\partial \vec{D}/\partial t$, which was first detected experimentally by H. Hertz in 1888, when he radiated electromagnetic waves by a dipole antenna. The electromagnetic field is now described by the *Maxwell equations* [3]

$$\boxed{\begin{array}{ll} \nabla \cdot \vec{D} = \rho & \nabla \times \vec{H} - \dfrac{\partial \vec{D}}{\partial t} = \vec{J} \\ \nabla \cdot \vec{B} = 0 & \nabla \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

These equations are linear, and the last two equations with a vanishing right-hand side are *homogeneous*.

If ε, μ are constants, so that they do not depend on position, then the medium is *homogeneous*. If ε, μ are scalars, and not matrices or tensors, then the medium is *isotropic*. [4] In a medium that is uniform in space, i.e. homogeneous and isotropic, and stationary [5] in time, the Maxwell equations can be

1 In the paper of 1820, Ørsted's name is printed as Örsted, because the printer had no Ø.

2 Charge conservation requires that the continuity equation

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

holds for the charge density ρ and the current density $\vec{J}$ in $\mathbb{R}^3$.

3 We use SI units: $[\vec{E}] = \frac{V}{m}$, $[\vec{B}] = \frac{Vs}{m^2}$, $[\vec{D}] = \frac{C}{m^2}$, $[\vec{H}] = \frac{A}{m}$.

4 In the case that the material is non-isotropic, $D_i = \varepsilon_{ij} E_j$, $B_i = \mu_{ij} H_j$, where the matrices are symmetric $\varepsilon_{ij} = \varepsilon_{ji}$, $\mu_{ij} = \mu_{ji}$.

5 Stationary means that ε and μ do not depend on time. In an explosion ε and μ are time dependent.

expressed in terms of $\vec{E}$ and $\vec{B}$ alone:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}, \qquad \nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu \vec{J},$$
$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0,$$

where $1/c^2 = \varepsilon\mu$. These equations hold *in a vacuum*. In a vacuum it is customary to set $\varepsilon = 1$, $\mu = 1$.

8.2 The Minkowski space-time $\mathbb{R}^{3,1}$

The electromagnetic quantities depend on time $t \in \mathbb{R}$ and position $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \in \mathbb{R}^3$. Position and time can be united into a single entity

$$x = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + ct\mathbf{e}_4,$$

a vector in a 4-dimensional real linear space $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. In this linear space we introduce a metric (or a quadratic form)

$$x_1^2 + x_2^2 + x_3^2 - c^2t^2$$

which makes it a quadratic space, called the *Minkowski space-time* $\mathbb{R}^{3,1}$.

In the Minkowski space-time it is customary to set $x^4 = ct = -x_4$ and agree that the indices are raised and lowered as follows:

$$x^1 = x_1, \; x^2 = x_2, \; x^3 = x_3 \quad \text{and} \quad x^4 = -x_4.$$

With this convention the quadratic form $x_1^2 + x_2^2 + x_3^2 - c^2t^2$ becomes

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = x^1x_1 + x^2x_2 + x^3x_3 + x^4x_4 = x^\alpha x_\alpha$$

where in the last step we have used the summation convention.

Examples. 1. The two densities ρ and $\vec{J}$ can be combined into a single quantity

$$\mathbf{J} = \vec{J} + c\rho\mathbf{e}_4 \quad \text{in} \quad \mathbb{R}^{3,1}$$

with four components J^1, J^2, J^3, $J^4 = c\rho = -J_4$ and the quadratic form $J_1^2 + J_2^2 + J_3^2 - J_4^2$.

2. We can combine the two potentials V and $\vec{A}$ in $\mathbb{R}^3$ into a single quantity with four components A^1, A^2, A^3 and $A^4 = \frac{1}{c}V = -A_4$, a space-time vector

$$\mathbf{A} = \vec{A} + \frac{1}{c}V\mathbf{e}_4 \quad \text{in} \quad \mathbb{R}^{3,1}$$

with a quadratic form $A_1^2 + A_2^2 + A_3^2 - A_4^2$.

8.3 Antisymmetric tensor of the electromagnetic field

H. Minkowski combined the two vectors $\vec{E}$ and $\vec{B}$ into a single quantity, a 4×4-matrix with entries $F^{\alpha\beta}$ given by

$$(F^{14}, F^{24}, F^{34}) = (\tfrac{1}{c}E_1, \tfrac{1}{c}E_2, \tfrac{1}{c}E_3),$$
$$(F^{23}, F^{31}, F^{12}) = (-B_1, -B_2, -B_3)$$

and antisymmetry, $F^{\alpha\beta} = -F^{\beta\alpha}$, so that

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -B_3 & B_2 & \frac{1}{c}E_1 \\ B_3 & 0 & -B_1 & \frac{1}{c}E_2 \\ -B_2 & B_1 & 0 & \frac{1}{c}E_3 \\ -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 & 0 \end{pmatrix}.$$

The matrix entries $F^{\alpha\beta}$ are coordinates of an antisymmetric tensor of rank 2, namely the electromagnetic field in space-time $\mathbb{R}^{3,1}$.

With this change of notation from $\vec{E}, \vec{B}$ to $F^{\alpha\beta}$ we can write the Maxwell equations in a vacuum:

$$\frac{\partial F^{14}}{\partial x^1} + \frac{\partial F^{24}}{\partial x^2} + \frac{\partial F^{34}}{\partial x^3} = \frac{\rho}{c\varepsilon}, \quad \left(\frac{\partial F^{21}}{\partial x^2} - \frac{\partial F^{13}}{\partial x^3}\right) - \frac{\partial F^{14}}{\partial x^4} = \mu J^1, \ \dots,$$
$$\frac{\partial F^{32}}{\partial x^1} + \frac{\partial F^{13}}{\partial x^2} + \frac{\partial F^{21}}{\partial x^3} = 0, \quad \left(\frac{\partial F^{34}}{\partial x^2} - \frac{\partial F^{24}}{\partial x^3}\right) + \frac{\partial F^{32}}{\partial x^4} = 0, \ \dots .$$

The last displayed equation can also be written as

$$\left(\frac{\partial F^{34}}{\partial x_2} + \frac{\partial F^{42}}{\partial x_3}\right) + \frac{\partial F^{23}}{\partial x_4} = 0$$

by employing antisymmetry and the lowering convention $x_4 = -x^4$.

Using the summation convention the Maxwell equations for $F^{\alpha\beta}$ can be condensed to

$$\frac{\partial F^{\alpha\beta}}{\partial x^\alpha} = \mu J^\beta,$$
$$\frac{\partial F^{\beta\gamma}}{\partial x_\alpha} + \frac{\partial F^{\gamma\alpha}}{\partial x_\beta} + \frac{\partial F^{\alpha\beta}}{\partial x_\gamma} = 0,$$

and further adopting the notations $\partial_\alpha = \dfrac{\partial}{\partial x^\alpha}$ and $\partial^\alpha = \dfrac{\partial}{\partial x_\alpha}$ to

$$\partial_\alpha F^{\alpha\beta} = \mu J^\beta,$$
$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$$

Similarly, $\vec{D}$ and $\vec{H}$ can be combined to a second-rank antisymmetric tensor

$$(G^{\alpha\beta}) = \begin{pmatrix} 0 & -H_3 & H_2 & cD_1 \\ H_3 & 0 & -H_1 & cD_2 \\ -H_2 & H_1 & 0 & cD_3 \\ -cD_1 & -cD_2 & -cD_3 & 0 \end{pmatrix}.$$

Using $G^{\alpha\beta}$ the general Maxwell equations (non-homogeneous, non-isotropic, time-varying) can be written in tensor/index form, due to Minkowski:

$$\boxed{\begin{aligned} &\partial_\alpha G^{\alpha\beta} = J^\beta \\ &\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \end{aligned}}$$

Exercises 1ab,2ab,3a

8.4 Electromagnetic potentials

Because of $\nabla \cdot \vec{B} = 0$ there exists, in a contractible region, a vector-potential $\vec{A}$ such that

$$\vec{B} = \nabla \times \vec{A}.$$

If this equation is substituted into Faraday's law, we get

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$$

or

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0.$$

This curl-free quantity is up to a sign the gradient of a scalar, called the electric potential V,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V.$$

We have shown that $\vec{E}$ and $\vec{B}$ can be expressed in terms of the potentials V and $\vec{A}$ as follows:

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A}.$$

Combine the two potentials V and $\vec{A}$ in $\mathbb{R}^3$ into a single quantity with four components A^1, A^2, A^3 and $A^4 = \frac{1}{c}V$, a space-time vector

$$\mathbf{A} = \vec{A} + \frac{1}{c}V\mathbf{e}_4 \in \mathbb{R}^{3,1}.$$

The above equations mean that $F^{\alpha\beta}$ can be expressed in terms of the potential A^{α} as follows:

$$F^{14} = -\frac{\partial A^4}{\partial x^1} - \frac{\partial A^1}{\partial x^4} = \frac{\partial A^1}{\partial x_4} - \frac{\partial A^4}{\partial x_1}, \cdots ,$$
$$F^{32} = \frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3} = \frac{\partial A^3}{\partial x_2} - \frac{\partial A^2}{\partial x_3}, \cdots ,$$

which can be condensed to

$$F^{\alpha\beta} = -(\partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}).$$

We can now verify that $\partial^{\alpha} F^{\beta\gamma} + \partial^{\beta} F^{\gamma\alpha} + \partial^{\gamma} F^{\alpha\beta} = 0$ by computing

$$\partial^{\alpha}(\partial^{\beta} A^{\gamma} - \partial^{\gamma} A^{\beta}) + \partial^{\beta}(\partial^{\gamma} A^{\alpha} - \partial^{\alpha} A^{\gamma}) + \partial^{\gamma}(\partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}) = 0.$$

Exercises 2c,3b

8.5 Gauge transformations

The vector-potential $\vec{A}$ is not unique, since we can add to it, without changing physics, any vector with a vanishing curl. Adding to $\vec{A}$ a curl-free vector, the gradient of a scalar Φ, gives us $\vec{A}' = \vec{A} + \nabla\Phi$. In order to keep $\vec{E} = -\nabla V - \partial\vec{A}/\partial t$ we also change V to V',

$$\begin{aligned} \vec{E} &= -\nabla V' - \frac{\partial \vec{A}'}{\partial t} \\ &= -\nabla V' - \frac{\partial}{\partial t}(\vec{A} + \nabla\Phi) \\ &= -\nabla\left(V' + \frac{\partial\Phi}{\partial t}\right) - \frac{\partial\vec{A}}{\partial t}, \end{aligned}$$

which implies $V' = V - \partial\Phi/\partial t$. The change of potentials

$$\begin{aligned} \vec{A}' &= \vec{A} + \nabla\Phi, \\ V' &= V - \frac{\partial\Phi}{\partial t} \end{aligned}$$

is called a *gauge transformation*. In coordinate form this means

$$A'^{\alpha} = A^{\alpha} + \frac{\partial\Phi}{\partial x_{\alpha}} = A^{\alpha} + \partial^{\alpha}\Phi$$

or swapping the sign of the time component

$$A'_{\alpha} = A_{\alpha} + \frac{\partial\Phi}{\partial x^{\alpha}} = A_{\alpha} + \partial_{\alpha}\Phi.$$

The fact that $\vec{E}, \vec{B}$ remain unchanged in a gauge transformation is called *gauge invariance.* In quantum electrodynamics gauge invariance is used to deduce the existence of a zero-mass carrier for the electromagnetic field.

8.6 The Lorenz condition for potentials

The two homogeneous Maxwell equations guaranteed existence of potentials for the electromagnetic field. Now we shall find out conditions imposed on the potentials by the remaining Maxwell equations. Substitute $\vec{E} = -\nabla V - \partial\vec{A}/\partial t$ into $\nabla \cdot \vec{E} = \rho/\varepsilon$ to obtain

$$-\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \frac{\rho}{\varepsilon}$$

in a vacuum. Substitute $\vec{E} = -\nabla V - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla \times \vec{A}$ into $\nabla \times \vec{B} - \frac{1}{c^2}\partial\vec{E}/\partial t = \mu\vec{J}$, and use the identity $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A}$, to obtain

$$\nabla\Big(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\Big) - \nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \mu\vec{J}.$$

The last two displayed equations couple V and $\vec{A}$.

Although the curl of $\vec{A}$ is designated to $\vec{B}$, we are still at liberty to choose the divergence of $\vec{A}$, which ensures the choice

$$\boxed{\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0}$$

called the *Lorenz condition.* [6] In coordinate form,

$$\frac{\partial A^\alpha}{\partial x^\alpha} = 0 \quad \text{or} \quad \partial_\alpha A^\alpha = 0.$$

When the Lorenz condition is satisfied, the above two second-order differential equations, which coupled V and $\vec{A}$, can be decoupled

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon},$$
$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{J}$$

into wave equations with the d'Alembert operator $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \partial^\alpha\partial_\alpha$.

6 The Lorenz condition/gauge was discovered by the Danish physicist Ludwig Lorenz in 1867, and not by the Dutch physicist H. A. Lorentz, who demonstrated covariance of the Maxwell equations under Lorentz transformations in 1903. See J. van Bladel: Lorenz or Lorentz? *IEEE Antennas and Propagation Magazine* **33** (1991) p. 69 and *The Radioscientist* **2** (1991) p. 55.

ELECTROMAGNETISM IN CLIFFORD ALGEBRAS

In the rest of this chapter we shall discuss electromagnetism in terms of the Clifford algebras. Clifford algebras automatically take care of the manipulation of indices. The Clifford algebra approach allows various degrees of abstraction which gradually become more and more distant from classical vector analysis.

We reformulate the Maxwell equations first in terms of the Clifford algebras $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2,\mathbb{C})$ of the Euclidean space $\mathbb{R}^3$ and then in terms of the Clifford algebra $\mathcal{C}\ell_{3,1} \simeq \mathrm{Mat}(4,\mathbb{R})$ of the Minkowski space $\mathbb{R}^{3,1}$. In the Euclidean space $\mathbb{R}^3$ we shall deal with the vector $\vec{E}$ and the bivector $\vec{B}\mathbf{e}_{123}$, and in the Minkowski space $\mathbb{R}^{3,1}$ we shall deal with the bivector

$$\mathbf{F} = \frac{1}{c}\vec{E}\mathbf{e}_4 - \vec{B}\mathbf{e}_{123}.$$

8.7 The vector $\vec{E}$ and the bivector $\vec{B}\mathbf{e}_{123}$

The work W done by an electric field $\vec{E}$ in moving a charge q along a path C is given by the line integral

$$W = q\int_C \vec{E}\cdot d\vec{\ell}.$$

We conclude that the electric field $\vec{E}$ is a vector, because it is integrated along a path.

Similarly, the magnetic induction $\vec{B}$ is integrated over a surface S in order to get the magnetic flux:

$$\Phi = \int_S \vec{B}\cdot d\vec{s}.$$

Since we are integrating over a surface, we conclude that we are actually dealing with the bivector $\vec{B}\mathbf{e}_{123} = B_1\mathbf{e}_{23} + B_2\mathbf{e}_{31} + B_3\mathbf{e}_{12}$, rather than the vector $\vec{B} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3$.

8.8 Differentiating vectors and bivectors

Differentiate the vector $\vec{E}$, in $\mathbb{R}^3 \subset \mathcal{C}\ell_3$, to find

$$\begin{aligned}\nabla\vec{E} &= \nabla\cdot\vec{E} + \nabla\wedge\vec{E}\\ &= \nabla\cdot\vec{E} + \mathbf{e}_{123}(\nabla\times\vec{E})\end{aligned}$$

where $\nabla \times \vec{E} = -\mathbf{e}_{123}(\nabla \wedge \vec{E})$. Differentiate the bivector $\vec{B}\mathbf{e}_{123}$ to find $\nabla(\vec{B}\mathbf{e}_{123}) = \nabla \wedge (\vec{B}\mathbf{e}_{123}) + \nabla \rfloor (\vec{B}\mathbf{e}_{123})$ where

$$\begin{aligned}\nabla \wedge (\vec{B}\mathbf{e}_{123}) &= \mathbf{e}_{123}(\nabla \cdot \vec{B}),\\ \nabla \rfloor (\vec{B}\mathbf{e}_{123}) &= \mathbf{e}_{123}(\nabla \wedge \vec{B}) = -\nabla \times \vec{B}.\end{aligned}$$

8.9 Single equation in $\mathcal{C}\ell_3$

Recall the Maxwell equations in a vacuum:

$$\begin{aligned}&\nabla \cdot \vec{E} = \rho,\\ &\frac{\partial \vec{E}}{\partial t} - \nabla \times \vec{B} = -\vec{J},\\ &\frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0,\\ &\nabla \cdot \vec{B} = 0.\end{aligned}$$

Multiply the last two equations by $\mathbf{e}_{123}$, use the following replacements $\nabla \wedge \vec{E} = (\nabla \times \vec{E})\mathbf{e}_{123}$, $\nabla \rfloor (\vec{B}\mathbf{e}_{123}) = -\nabla \times \vec{B}$ and $\nabla \wedge (\vec{B}\mathbf{e}_{123}) = (\nabla \cdot \vec{B})\mathbf{e}_{123}$, and you will get

$$\begin{array}{ll} 0 & \nabla \cdot \vec{E} = \rho \\ 1 & \dfrac{\partial \vec{E}}{\partial t} + \nabla \rfloor (\vec{B}\mathbf{e}_{123}) = -\vec{J} \\ 2 & \dfrac{\partial}{\partial t}(\vec{B}\mathbf{e}_{123}) + \nabla \wedge \vec{E} = 0 \\ 3 & \nabla \wedge (\vec{B}\mathbf{e}_{123}) = 0. \end{array}$$

The numbers on the left indicate the dimension degrees of the equations. Summing up these four equations we get (use $\nabla \vec{E} = \nabla \cdot \vec{E} + \nabla \wedge \vec{E}$)

$$\frac{\partial}{\partial t}(\vec{E} + \vec{B}\mathbf{e}_{123}) + \nabla \vec{E} + \nabla \rfloor (\vec{B}\mathbf{e}_{123}) + \nabla \wedge (\vec{B}\mathbf{e}_{123}) = \rho - \vec{J}.$$

Use $\nabla F = \nabla \rfloor F + \nabla \wedge F$ to find

$$(\frac{\partial}{\partial t} + \nabla)(\vec{E} + \vec{B}\mathbf{e}_{123}) = \rho - \vec{J},$$

and we have condensed all the Maxwell equations into a single equation in terms of the Clifford algebra $\mathcal{C}\ell_3$. Taking the grade involute of both sides results in

$$(\frac{\partial}{\partial t} - \nabla)(-\vec{E} + \vec{B}\mathbf{e}_{123}) = \rho + \vec{J}.$$

The potentials V and $\vec{A}$, a scalar and a vector, can be united into a paravector $V + \vec{A}$. Differentiate the paravector $V + \vec{A}$ by the paravector differential operator, $\frac{\partial}{\partial t} + \nabla$,

$$(\frac{\partial}{\partial t} + \nabla)(V + \vec{A}) = \frac{\partial V}{\partial t} + \frac{\partial \vec{A}}{\partial t} + \nabla V + \nabla \vec{A},$$

where $\nabla \vec{A} = \nabla \cdot \vec{A} + (\nabla \times \vec{A})\mathbf{e}_{123}$, and you will get

$$(\frac{\partial}{\partial t} + \nabla)(V + \vec{A}) = -\vec{E} + \vec{B}\mathbf{e}_{123}.$$

Taking the grade involute of both sides results in

$$(\frac{\partial}{\partial t} - \nabla)(V - \vec{A}) = \vec{E} + \vec{B}\mathbf{e}_{123}.$$

8.10 The use of the Clifford algebra $\mathcal{C}\ell_{3,1}$

Consider the Clifford algebra $\mathcal{C}\ell_3 \simeq \mathrm{Mat}(2, \mathbb{C})$ as a subalgebra of the Clifford algebra $\mathcal{C}\ell_{3,1}$ generated by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$ with the relations

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1, \ \mathbf{e}_4^2 = -1,$$
$$\mathbf{e}_\alpha \mathbf{e}_\beta = -\mathbf{e}_\beta \mathbf{e}_\alpha \quad \text{for} \quad \alpha \neq \beta.$$

The Clifford algebra $\mathcal{C}\ell_{3,1}$ is isomorphic, as an associative algebra, with the algebra of real 4×4-matrices $\mathrm{Mat}(4, \mathbb{R})$. In the Clifford algebra $\mathcal{C}\ell_{3,1}$ we consider the electromagnetic bivector [7]

$$\mathbf{F} = \frac{1}{c}\vec{E}\mathbf{e}_4 - \vec{B}\mathbf{e}_{123} \in \bigwedge^2 \mathbb{R}^{3,1}$$

and the space-time current vector

$$\mathbf{J} = \vec{J} + c\rho\mathbf{e}_4 \in \mathbb{R}^{3,1}.$$

From $\mathbf{F}$ we can find $\vec{E}$ by $\vec{E} = c\mathbf{e}_4 \lrcorner \mathbf{F}$, and from $\mathbf{J}$ we can find $\vec{J}$ by $\vec{J} = (\mathbf{J} \wedge \mathbf{e}_4)\mathbf{e}_4^{-1}$.

We introduce the differential operator

$$\partial = \nabla - \mathbf{e}_4 \frac{1}{c} \frac{\partial}{\partial t}.$$

For a function $f : \mathbb{R}^{3,1} \to \mathcal{C}\ell_{3,1}$ we have $\partial f = \partial \wedge f + \partial \lrcorner f$, where $\partial \wedge f$ is the *raising differential* and $\partial \lrcorner f$ is the *lowering differential*.

7 If we use the orthonormal basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with $\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = 1$ and $\mathbf{e}_0^2 = -1$, that is, $\mathbf{e}_0 = \mathbf{e}_4$, then we find by reordering the indices that $\mathbf{e}_{0123} = -\mathbf{e}_{1234}$ and $\mathbf{F} = \frac{1}{c}\vec{E}\mathbf{e}_4 - (\vec{B}\mathbf{e}_4)\mathbf{e}_{1234} = \frac{1}{c}\vec{E}\mathbf{e}_0 + (\vec{B}\mathbf{e}_0)\mathbf{e}_{0123}$.

Compute the raising differential

$$\partial \wedge \mathbf{F} = \left(\nabla - \mathrm{e}_4 \frac{1}{c}\frac{\partial}{\partial t}\right) \wedge \left(\frac{1}{c}\vec{E}\mathrm{e}_4 - \vec{B}\mathrm{e}_{123}\right)$$
$$= \frac{1}{c}\mathrm{e}_{123}(\nabla \times \vec{E})\mathrm{e}_4 - \mathrm{e}_{123}(\nabla \cdot \vec{B}) - \mathrm{e}_{1234}\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0.$$

Define $\mathbf{G} = c\vec{D}\mathrm{e}_4 - \vec{H}\mathrm{e}_{123}$ and compute the lowering differential

$$\partial \lrcorner \mathbf{G} = \left(\nabla - \mathrm{e}_4 \frac{1}{c}\frac{\partial}{\partial t}\right) \lrcorner (c\vec{D}\mathrm{e}_4 - \vec{H}\mathrm{e}_{123})$$
$$= c(\nabla \cdot \vec{D})\mathrm{e}_4 + \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = c\rho\mathrm{e}_4 + \vec{J}.$$

The Maxwell equations now have a particularly succinct form [8]

$$\boxed{\begin{aligned} \partial \lrcorner \mathbf{G} &= \mathbf{J} \\ \partial \wedge \mathbf{F} &= 0 \end{aligned}}$$

corresponding to

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho, & -\nabla \lrcorner (\vec{H}\mathrm{e}_{123}) - \frac{\partial \vec{D}}{\partial t} &= \vec{J}, \\ -\nabla \wedge (\vec{B}\mathrm{e}_{123}) &= 0, & \nabla \wedge \vec{E} + \frac{\partial}{\partial t}(\vec{B}\mathrm{e}_{123}) &= 0. \end{aligned}$$

8.11 Single equation in a vacuum, $\mathcal{C}\ell_{3,1}$

In a vacuum the Maxwell equations can be further compressed into a single equation

$$\partial \mathbf{F} = \mathbf{J},$$

which decomposes into two parts, $\partial \wedge \mathbf{F} = 0$ and $\partial \lrcorner \mathbf{F} = \mathbf{J}$. Also, $\partial \wedge \mathbf{A} = -\mathbf{F}$ and the Lorenz condition $\partial \cdot \mathbf{A} = 0$ imply

$$\partial \mathbf{A} = -\mathbf{F}.$$

8.12 The energy-momentum tensor

Marcel Riesz in 1947 wrote the energy-momentum tensor in the form

$$T_{\mu\nu} = -\frac{1}{2}\langle \mathrm{e}_\mu \mathbf{F} \mathrm{e}_\nu \mathbf{F} \rangle_0.$$

8 The 3D formulation differs from this 4D formulation in the sense that $\mathbf{G}$ and $\mathbf{F}$ are bivectors in $\bigwedge^2 \mathbb{R}^{3,1}$.

D. Hestenes 1966 p. 31 introduced the vectors

$$\mathbf{T}_\mu = -\frac{1}{2}\mathbf{F}\mathbf{e}_\mu\mathbf{F}$$

for which $T_{\mu\nu} = \mathbf{T}_\mu \cdot \mathbf{e}_\nu = \mathbf{e}_\mu \cdot \mathbf{T}_\nu$, and also the mapping $T\mathbf{x} = -\frac{1}{2}\mathbf{F}\mathbf{x}\mathbf{F}$ where $(T\mathbf{x})^\mu = T^\mu{}_\nu x^\nu$. [9] The energy-momentum tensor is symmetric, that is, $T_{\mu\nu} = T_{\nu\mu}$ or $gT^\top g^{-1} = T$, where $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$, and traceless, that is, $T^\mu{}_\mu = 0$. [10]

Note that the Poynting vector $\mathbf{T}_4 = \vec{E} \times \vec{B} + \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\mathbf{e}_4$ is not a space-time vector, in the sense that it does not transform properly under Lorentz transformations, but rather it is just the last column of the energy-momentum matrix $T = (T^\mu{}_\nu)$ which transforms as $T' = LTL^{-1}$.

Electromagnetism in Differential Forms

Electromagnetism can also be formulated with differential forms, based on Grassmann's exterior algebra. In this context it is customary to invoke the dual space

$$V^* = \{f : V \to \mathbb{R} \mid f \text{ linear}\}$$

of the real linear space $V = \mathbb{R}^{3,1}$. Instead of vectors and bivectors, in V and $\bigwedge^2 V$, one considers 1-forms and 2-forms, in V^* and $\bigwedge^2 V^*$.

In theoretical physics one applies differential forms to electromagnetism, but in electrical engineering one uses almost exclusively the vector analysis of Gibbs and Heaviside. [11] Electrical engineers are not interested in transformation laws, [12] and so it is convenient for them to place all vectors in $V = \mathbb{R}^{3,1}$ [and disregard the dual space V^*]. However, a theory without the dual space V^* cannot be generalized to curved space-times. In a curved space-time it is not possible to differentiate vector valued functions, only differential forms can be differentiated [in general relativity vectors are differentiated covariantly].

Although differential forms are not of practical value for electrical engineers,

9 Juvet & Schidlof 1932 p. 141 gave $T_{\mu\nu} = F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ but did not consider $T\mathbf{x} = -\frac{1}{2}\mathbf{F}\mathbf{x}\mathbf{F}$, compare this to Bolinder p. 469 in Chisholm & Common (eds.) 1986.

10 The tracelessness of $T^\mu{}_\nu = -\frac{1}{2}\langle\mathbf{e}^\mu\mathbf{F}\mathbf{e}_\nu\mathbf{F}\rangle_0$ is an accident in dimension 4, since $\mathbf{e}^\mu\mathbf{F}\mathbf{e}_\mu = 0$, and in general $\mathbf{e}^\mu\mathbf{a}\mathbf{e}_\mu = (n-2k)\hat{\mathbf{a}}$ for $\mathbf{a} \in \bigwedge^k \mathbb{R}^n$.

11 As far as the author knows the only university where electrical engineers have used differential forms in teaching is Helsinki University of Technology, see lecture notes Lindell & Lounesto 1995.

12 For instance, the space-time position $\mathbf{x} = \vec{x} + ct\mathbf{e}_4$ and the current density $\mathbf{J} = \vec{J} + c\rho\mathbf{e}_4$ transform differently under the Lorentz group; one transforms contravariantly and the other covariantly. In tensor calculus elements of V are called vectors and elements of the dual space V^* are called covectors.

we shall close this chapter with a short discussion on the formulation of electromagnetism with differential forms, see Lindell 1995. But first some observations about functions with values in $\bigwedge V = \mathcal{C}\ell_{3,1}$.

8.13 Using only raising or lowering differentials

Since the current density $\vec{J}$ integrates over a surface S,

$$\oint_S \vec{J} \cdot d\vec{s} = -I,$$

we can replace it by a bivector $\vec{J}\mathbf{e}_{123}$, and since the charge density ρ integrates over a 3-volume, we can replace it by a 3-vector $\rho\mathbf{e}_{123}$. Similarly, we can regard $\vec{H}$ as a vector, but replace the vector $\vec{D}$ by a bivector $\vec{D}\mathbf{e}_{123}$.

The two Maxwell equations with a source-term on the right hand side can be rewritten in the form

$$\nabla \wedge (\vec{D}\mathbf{e}_{123}) = \rho\mathbf{e}_{123}, \qquad -\nabla \wedge \vec{H} - \frac{\partial}{\partial t}(\vec{D}\mathbf{e}_{123}) = \vec{J}\mathbf{e}_{123}.$$

Take the Hodge dual

$$\begin{aligned} \star\mathbf{G} = \tilde{\mathbf{G}}\mathbf{e}_{1234} = -c\vec{D}\mathbf{e}_{123} - \vec{H}\mathbf{e}_4 \quad \text{and} \\ \star\mathbf{J} = \tilde{\mathbf{J}}\mathbf{e}_{1234} = c\rho\mathbf{e}_{123} + (\vec{J}\mathbf{e}_{123})\mathbf{e}_4, \end{aligned}$$

and compute the raising differential

$$\begin{aligned} \partial \wedge \star\mathbf{G} &= \left(\nabla - \mathbf{e}_4\frac{1}{c}\frac{\partial}{\partial t}\right) \wedge (-c\vec{D}\mathbf{e}_{123} - \vec{H}\mathbf{e}_4) \\ &= -c(\nabla \cdot \vec{D})\mathbf{e}_{123} - \mathbf{e}_{123}(\nabla \times \vec{H})\mathbf{e}_4 + \frac{\partial \vec{D}}{\partial t}\mathbf{e}_{1234}. \end{aligned}$$

The Maxwell equations can now be expressed in terms of the raising differential alone:

$$\begin{aligned} \partial \wedge \star\mathbf{G} &= -\star\mathbf{J}, \\ \partial \wedge \mathbf{F} &= 0. \end{aligned}$$

Dually, we can write down the Maxwell equations using only the lowering differential:

$$\begin{aligned} \partial \,\lrcorner\, \mathbf{G} &= \mathbf{J}, \\ \partial \,\lrcorner\, \star\mathbf{F} &= 0. \end{aligned}$$

These equations are invariant under the general linear group $GL(4, \mathbb{R})$, and the solutions are independent of the choice of metric. [13]

13 In the absence of a metric it is customary to invoke the dual algebra $\bigwedge V^*$ of the exterior algebra $\bigwedge V$ and take exterior differentials of differential forms rather than differentials of multivector valued functions.

8.14 The constitutive relations

The constitutive relations of the medium are

$$\vec{D} = \varepsilon \vec{E} + \alpha \vec{B},$$
$$\vec{H} = \beta \vec{E} + \mu^{-1} \vec{B}.$$

Here $\varepsilon, \alpha, \beta$ and μ^{-1} are 3×3-matrices. To find the rules imposed on them, write the above equations in coordinate form:

$$G^{\kappa\lambda} = \tfrac{1}{2}\chi^{\kappa\lambda\mu\nu} F_{\mu\nu}.$$

Then, if $\chi^{\kappa\lambda\mu\nu}$ is an irreducible tensor, [14] we must have

$$\chi^{\kappa\lambda\mu\nu} = -\chi^{\lambda\kappa\mu\nu}, \quad \chi^{\kappa\lambda\mu\nu} = -\chi^{\kappa\lambda\nu\mu},$$
$$\chi^{\kappa\lambda\mu\nu} = \chi^{\mu\nu\lambda\kappa},$$
$$\chi^{[\kappa\lambda\mu\nu]} = 0,$$

where the brackets [ ] mean complete alternation of indices. The second relation implies $\varepsilon^{\mathsf{T}} = \varepsilon$, $\mu^{\mathsf{T}} = \mu$ and $\alpha = -\beta^{\mathsf{T}}$ and the third relation implies $\mathrm{trace}(\alpha) = \mathrm{trace}(\beta)$, which together with the former implies $\mathrm{trace}(\alpha) = \mathrm{trace}(\beta) = 0$. These considerations can be condensed into saying that the indices of the constitutive tensor $\chi^{\kappa\lambda\mu\nu} = \chi^{\kappa\mu}_{\lambda\nu}$ can be arranged into a Young tableau

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array}$$

The irreducible tensor χ has 20 components, where $20 = \frac{1}{12}n^2(n^2-1)$ for $n = 4$. In chiral media the tensor χ need not be irreducible, and the number of components may rise to 36.

8.15 The derivative and the exterior differential

Let U and V be real linear spaces with norms. The derivative of $f : U \to V$ at $\mathbf{x} \in U$ is a linear function

$$f'(\mathbf{x}) : U \to V, \ \mathbf{h} \to f'(\mathbf{x})\mathbf{h}$$

such that

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = f'(\mathbf{x})\mathbf{h} + \|\, \mathbf{h} \,\|\, \varepsilon(\mathbf{x},\mathbf{h})$$

where $\varepsilon(\mathbf{x},\mathbf{h}) \to 0$ as $\mathbf{h} \to 0$. The linear function $f'(\mathbf{x}) : U \to V$ can be identified with an element of $U^* \otimes V$.

14 The factor χ need not be a tensor. For instance, magnetic saturation and hysteresis are not expressible with a tensor χ.

Consider now a function $f : V \to \bigwedge V^*$. Its derivative at $\mathbf{x}$,

$$f'(\mathbf{x}) \in V^* \otimes \bigwedge V^*,$$

is no longer an element of the dual exterior algebra $\bigwedge V^* \subset \otimes V^*$. The alternation, which antisymmetrizes tensor product of vectors, is a linear function projecting $\bigwedge V^*$ out of $\otimes V^*$ so that

$$u \wedge u = Alt(u \otimes v) \quad \text{for} \quad u, v \in \bigwedge V^*.$$

We define the *exterior differential* of $f : V \to \bigwedge V^*$ at $\mathbf{x}$ by [15]

$$d \wedge f(\mathbf{x}) = Alt(f'(\mathbf{x})).$$

Next, we will replace vector valued functions $V \to V$ by 1-forms $V \to V^*$, and bivector valued functions $V \to \bigwedge^2 V$ by 2-forms $V \to \bigwedge^2 V^*$. The electromagnetic bivectors $\mathbf{F}$ and $\mathbf{G}$ are replaced by 2-forms F and G. The current vector $\mathbf{J}$ is replaced by a 1-form J.

The exterior differential raises the degree. The dual of the exterior differential, called the *contraction differential* $d \lrcorner f = \star^{-1} d \wedge \star f$, [16] lowers the degree. In differential forms the Maxwell equations look like

$$d \lrcorner G = J,$$
$$d \wedge F = 0.$$

8.16 General linear covariance of the Maxwell equations

Using the differential forms we may find the most general expression of the Maxwell equations:

$$\boxed{\begin{aligned} &d \wedge \star G = - \star J \\ &d \wedge F = 0 \end{aligned}}$$

These equations include only the exterior differential, and no contraction differential, so that a metric is not involved. This makes the equations independent of any coordinate system. The metric gets involved by the constitutive relations of the medium

$$\boxed{G = \chi(F)}$$

and the Hodge dual.

This form of the Maxwell equations is not only relativistically covariant,

15 The exterior differential is usually denoted by df.
16 The contraction differential is commonly called the co-differential and denoted by δf.

under the Lorentz group $O(3,1)$, [17] but also covariant under any linear transformation of space-time coordinates, that is, under the general linear group $GL(4,\mathbb{R})$. This general linear covariance of the Maxwell equations, and their independence of metric/medium, were recognized by Weyl 1921, Cartan 1926 and van Dantzig 1934.

Historical Survey

The Maxwell equations have been condensed into a single equation using complex vectors (Silberstein 1907), complex quaternions (Silberstein 1912/1914, Lanczos 1919), spinors (Laporte & Uhlenbeck 1931, Bleuler & Kustaanheimo 1968) and using Clifford algebras (Juvet & Schidlof 1932, Mercier 1935, M. Riesz 1958). Marcel Riesz 1947 wrote the energy-momentum tensor in the form $T_{\mu\nu} = -\frac{1}{2}\langle \mathbf{e}_\mu \mathbf{F} \mathbf{e}_\nu \mathbf{F}\rangle_0$.

Exercises

Metric $x_1^2 + x_2^2 + x_3^2 - x_4^2$:

1. Recall that $(F^{14}, F^{24}, F^{34}) = (\frac{1}{c}E_1, \frac{1}{c}E_2, \frac{1}{c}E_3)$ and $(F^{23}, F^{31}, F^{12}) = (-B_1, -B_2, -B_3)$. Compute the matrices
a) $F^{\alpha}{}_{\beta}$, b) $F_{\alpha}{}^{\beta}$, and the vector
c) $v^\alpha F_\alpha{}^\beta$ for $(v^1, v^2, v^3, v^4) = (v_1, v_2, v_3, c)$. [18]

Metric $-x_0^2 + x_1^2 + x_2^2 + x_3^2$:

In this metric $\partial^\alpha = (-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

2. Replace $\vec{E}$ and $\vec{B}$ by $F^{\alpha\beta} = -F^{\beta\alpha}$ so that $(F^{01}, F^{02}, F^{03}) = (-\frac{1}{c}E_1, -\frac{1}{c}E_2, -\frac{1}{c}E_3)$ and $(F^{23}, F^{31}, F^{12}) = (-B_1, -B_2, -B_3)$, and determine
a) the antisymmetric matrix $F^{\alpha\beta}$,
b) the Maxwell equations in terms of $F^{\alpha\beta}$,
c) $F^{\alpha\beta}$ in terms of A^α,

17 The Maxwell equations describe massless particles, photons, and as such they are conformally covariant, as was demonstrated by Cunningham and Bateman in 1910. The conformal transformations are not linear in general, that is, they are not in $GL(4,\mathbb{R})$.

18 For simplicity we have omitted the factor

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},$$

which makes both sides of the equation of the Lorentz force, $f^\beta = u^\alpha F_\alpha{}^\beta$, $u^\alpha = \gamma v^\alpha$, properly transforming space-time vectors.

d) $v^\alpha F_\alpha{}^\beta$ for $(v^0, v^1, v^2, v^3) = (c, v_1, v_2, v_3)$.

Metric $x_0^2 - x_1^2 - x_2^2 - x_3^2$:

In this metric $\partial^\alpha = (\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z})$.

3. Replace $\vec{E}$ and $\vec{B}$ by $F^{\alpha\beta} = -F^{\beta\alpha}$ so that $(F^{01}, F^{02}, F^{03}) = (-\frac{1}{c}E^1, -\frac{1}{c}E^2, -\frac{1}{c}E^3)$ and $(F^{23}, F^{31}, F^{12}) = (-B^1, -B^2, -B^3)$, and determine

a) the Maxwell equations in terms of $F^{\alpha\beta}$,

b) $F^{\alpha\beta}$ in terms of A^α.
[Note that $A^1 = A_x = -A_1$, $A^2 = A_y = -A_2$, $A^3 = A_z = -A_3$ but $A^0 = \frac{1}{c}V = A_0$.]

4. Electrical engineers use the pairs $\vec{E}, \vec{H}$ and $\vec{D}, \vec{B}$. The constitutive relations sending $\vec{E}, \vec{H}$ to $\vec{D}, \vec{B}$ are then

$$\vec{D} = \breve{\varepsilon}\vec{E} + \breve{\alpha}\vec{H},$$
$$\vec{B} = \breve{\beta}\vec{E} + \breve{\mu}\vec{H}.$$

Show that $\breve{\mu} = \mu$, $\breve{\alpha} = \alpha\mu$, $\breve{\beta} = -\mu\beta$ and $\breve{\varepsilon} = \varepsilon - \alpha\mu\beta$.

Solutions

1a.

$$(F^\alpha{}_\beta) = \begin{pmatrix} 0 & -B_3 & B_2 & -\frac{1}{c}E_1 \\ B_3 & 0 & -B_1 & -\frac{1}{c}E_2 \\ -B_2 & B_1 & 0 & -\frac{1}{c}E_3 \\ -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 & 0 \end{pmatrix}$$

b.

$$(F_\alpha{}^\beta) = \begin{pmatrix} 0 & -B_3 & B_2 & \frac{1}{c}E_1 \\ B_3 & 0 & -B_1 & \frac{1}{c}E_2 \\ -B_2 & B_1 & 0 & \frac{1}{c}E_3 \\ \frac{1}{c}E_1 & \frac{1}{c}E_2 & \frac{1}{c}E_3 & 0 \end{pmatrix}$$

c. The space-component is $\vec{E} + \vec{v} \times \vec{B}$ and the time-component $\frac{1}{c}\vec{v} \cdot \vec{E}$.

2a.

$$(F^{\alpha\beta}) = \begin{pmatrix} 0 & -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & 0 & -B_3 & B_2 \\ \frac{1}{c}E_2 & B_3 & 0 & -B_1 \\ \frac{1}{c}E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

b. $\partial_\alpha F^{\alpha\beta} = \mu J^\beta$, $\quad \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$.

c. $F^{\alpha\beta} = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$.

d. The time-component is $\frac{1}{c}\vec{v} \cdot \vec{E}$ and the space-component $\vec{E} + \vec{v} \times \vec{B}$.

3a. $\partial_\alpha F^{\alpha\beta} = \mu J^\beta, \quad \partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0.$

b. $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha.$

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