# 13
# Tilt to the Opposite Metric

Physicists usually go from $C\ell_{1,3} \simeq \mathrm{Mat}(2, \mathbb{H})$ to its opposite algebra $C\ell_{3,1} \simeq \mathrm{Mat}(4, \mathbb{R})$ by replacing $\gamma_\mu$ by $i\gamma_\mu$ [within $\mathrm{Mat}(4, \mathbb{C})$]. However, such a transition to the opposite metric does not make sense within the space-time $\mathbb{R}^4$, because it calls for $i\,\mathbb{R}^4$ which is outside of $\mathbb{R}^4$. We will instead regard the linear space $\mathbb{R}^4$ as one and the same space-time, endowed with two different metrics or quadratic structures, $\mathbb{R}^{1,3}$ and $\mathbb{R}^{3,1}$.

Let the basis $\{\gamma_0, \gamma_1, \gamma_2, \gamma_3\}$ of the space-time $\mathbb{R}^{1,3}$ generating $C\ell_{1,3}$ correspond to the basis $\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3\}$ of the space-time $\mathbb{R}^{3,1}$ generating $C\ell_{3,1}$,

$$\mathrm{e}_0^2 = -1, \ \mathrm{e}_1^2 = \mathrm{e}_2^2 = \mathrm{e}_3^2 = 1 \qquad [\mathrm{e}_\mu \neq \pm i\gamma_\mu].$$

So the vectors $A^0\gamma_0 + A^1\gamma_1 + A^2\gamma_2 + A^3\gamma_3$ and $A^0\mathrm{e}_0 + A^1\mathrm{e}_1 + A^2\mathrm{e}_2 + A^3\mathrm{e}_3$ correspond to each other but have opposite squares

$$\begin{aligned}(A^0\gamma_0 + A^1\gamma_1 + A^2\gamma_2 + A^3\gamma_3)^2 &= (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2,\\ (A^0\mathrm{e}_0 + A^1\mathrm{e}_1 + A^2\mathrm{e}_2 + A^3\mathrm{e}_3)^2 &= -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2.\end{aligned}$$

We shall go further and regard $A^0\gamma_0 + A^1\gamma_1 + A^2\gamma_2 + A^3\gamma_3 \in \mathbb{R}^{1,3} \subset C\ell_{1,3}$ and $A^0\mathrm{e}_0 + A^1\mathrm{e}_1 + A^2\mathrm{e}_2 + A^3\mathrm{e}_3 \in \mathbb{R}^{3,1} \subset C\ell_{3,1}$ as one and the same vector $\mathbf{A} \in \mathbb{R}^4$ embedded in two non-isomorphic algebras $C\ell_{1,3}$ and $C\ell_{3,1}$ which are identified as linear spaces by the correspondences

| $C\ell_{1,3}$ | $C\ell_{3,1}$ |
|---|---|
| $1$ | $1$ |
| $\gamma_\mu$ | $\mathrm{e}_\mu$ |
| $\gamma_\mu \wedge \gamma_\nu$ | $\mathrm{e}_\mu \wedge \mathrm{e}_\nu \quad (= \mathrm{e}_\mu\mathrm{e}_\nu \ \text{ for } \ \mu \neq \nu)$ |

[Note that $\gamma_0 = \gamma^0$ and $A_0 = A^0$ in $C\ell_{1,3}$ whereas in $C\ell_{3,1}$ we have $\mathrm{e}_0 = -\mathrm{e}^0$ and $A_0 = -A^0$ and that the numerical values of $A^0$ are the same in $C\ell_{1,3}$ and $C\ell_{3,1}$.]

The products in the Clifford algebras $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{3,1}$ are related to each other by (all the terms in this table are computed in $\mathcal{C}\ell_{1,3}$)

| $\mathcal{C}\ell_{1,3}$ | $\mathcal{C}\ell_{3,1}$ |
|---|---|
| $ab$ | $b_0a_0 + b_0a_1 + b_1a_0 - b_1a_1$ |

where $a_0 = \text{even}(a)$ and $a_1 = \text{odd}(a)$. For $a \in \mathcal{C}\ell_{1,3}$ we may sometimes emphasize that $\text{opp}[a] \in \mathcal{C}\ell_{3,1}$ (or the other way round). In this notation, the products in the (graded) opposite algebras are related by [1]

$$\text{opp}[ab] = b_0a_0 + b_0a_1 + b_1a_0 - b_1a_1.$$

In this chapter we shall study the Maxwell equations and the Dirac equation in opposite metrics, in the quadratic spaces $\mathbb{R}^{1,3}$ and $\mathbb{R}^{3,1}$. In particular, we do not consider curved space-times, only flat space-times. In a flat space-time it is also possible to differentiate multivector fields, not only differential forms; we will focus on differentiating multivector fields.

### The Maxwell Equations in Opposite Metrics

There are a few changes of sign in the Maxwell equations in the quadratic spaces $\mathbb{R}^{3,1}$ and $\mathbb{R}^{1,3}$.

## 13.1 The Maxwell equations in $\mathbb{R}^{3,1}$

We use the following definitions for the potential $\mathbf{A}$, the current $\mathbf{J}$, the differential operator $\partial$, and the electromagnetic field $\mathbf{F}$:

$$\begin{aligned}
&A^\alpha = (\tfrac{1}{c}V, A_x, A_y, A_z), \qquad A_\alpha = (-\tfrac{1}{c}V, A_x, A_y, A_z)\\
&J^\alpha = (\rho c, J_x, J_y, J_z), \qquad J_\alpha = (-\rho c, J_x, J_y, J_z)\\
&\partial^\alpha = (-\tfrac{1}{c}\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}), \qquad \partial_\alpha = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z})\\
&(F^{01}, F^{02}, F^{03}) = (-\tfrac{1}{c}E_x, -\tfrac{1}{c}E_y, -\tfrac{1}{c}E_z)\\
&(F^{23}, F^{31}, F^{12}) = (-B_x, -B_y, -B_z).
\end{aligned}$$

This leads to the d'Alembert operator

$$\partial^\alpha\partial_\alpha = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

and the equation

$$\partial^\alpha\partial_\alpha A^\beta = -\mu J^\beta.$$

1 The symbol 'opp' is not a function of one or two variables; it is rather an indicator signaling that all the computations in the brackets will be computed in the opposite algebra.

The equations

$$\begin{aligned}\partial^0 A^1 - \partial^1 A^0 &= -\tfrac{1}{c}\tfrac{\partial}{\partial t}A_x - \tfrac{\partial}{\partial x}\tfrac{V}{c} = \tfrac{1}{c}E_x, \ \ldots\\ \partial^1 A^2 - \partial^2 A^1 &= \tfrac{\partial}{\partial x}A_y - \tfrac{\partial}{\partial y}A_x = B_z, \ \ldots\end{aligned}$$

give

$$F^{\alpha\beta} = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha),$$

and

$$\begin{aligned}\partial_\alpha F^{\alpha 0} &= \tfrac{\partial}{\partial x}\tfrac{E_x}{c} + \tfrac{\partial}{\partial y}\tfrac{E_y}{c} + \tfrac{\partial}{\partial z}\tfrac{E_z}{c} = \tfrac{1}{c}\nabla\cdot\vec{E} = \tfrac{\rho}{c\varepsilon} = \mu J^0\\ \partial_\alpha F^{\alpha 1} &= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\tfrac{E_x}{c} + \tfrac{\partial}{\partial y}B_z - \tfrac{\partial}{\partial z}B_y = \mu J^1, \ \ldots\end{aligned}$$

give the Maxwell equations

$$\partial_\alpha F^{\alpha\beta} = \mu J^\beta.$$

In the Clifford algebra $\mathcal{C}\ell_{3,1}$ we have

$$\begin{aligned}\partial &= -\mathbf{e}_0\tfrac{1}{c}\tfrac{\partial}{\partial t} + \mathbf{e}_1\tfrac{\partial}{\partial x} + \mathbf{e}_2\tfrac{\partial}{\partial y} + \mathbf{e}_3\tfrac{\partial}{\partial z}\\ \mathbf{F} &= -\tfrac{1}{c}E_x\mathbf{e}_{01} - \tfrac{1}{c}E_y\mathbf{e}_{02} - \tfrac{1}{c}E_z\mathbf{e}_{03} - B_x\mathbf{e}_{23} - B_y\mathbf{e}_{31} - B_z\mathbf{e}_{12}\\ &= \tfrac{1}{c}\vec{E}\mathbf{e}_0 - \vec{B}\mathbf{e}_{123}\\ \mathbf{A} &= \tfrac{1}{c}V\mathbf{e}_0 + A_x\mathbf{e}_1 + A_y\mathbf{e}_2 + A_z\mathbf{e}_3\\ \mathbf{J} &= c\rho\mathbf{e}_0 + J_x\mathbf{e}_1 + J_y\mathbf{e}_2 + J_z\mathbf{e}_3\end{aligned}$$

and the equations

$$\begin{aligned}\partial\mathbf{A} &= \mathbf{e}_{01}(-\tfrac{1}{c}\tfrac{\partial}{\partial t}A_x - \tfrac{\partial}{\partial x}\tfrac{V}{c}) + \ldots + \mathbf{e}_{12}(\tfrac{\partial}{\partial x}A_y - \tfrac{\partial}{\partial y}A_x)\\ \partial\mathbf{F} &= \mathbf{e}_0(\tfrac{\partial}{\partial x}\tfrac{E_x}{c} + \tfrac{\partial}{\partial y}\tfrac{E_y}{c} + \tfrac{\partial}{\partial z}\tfrac{E_z}{c}) + \mathbf{e}_1(-\tfrac{1}{c^2}\tfrac{\partial}{\partial t}E_x - \tfrac{\partial}{\partial z}B_y + \tfrac{\partial}{\partial y}B_z) + \ldots\end{aligned}$$

which lead to

$$\boxed{\partial\mathbf{A} = -\mathbf{F}, \quad \partial\mathbf{F} = \mu\mathbf{J}, \quad \partial^2\mathbf{A} = -\mu\mathbf{J}}$$

The computations can be related to Gibbs' vector algebra as follows (here $c = 1,\ \mu = 1$):

$$\begin{aligned}\partial\mathbf{F} &= (\nabla\cdot\vec{E})\mathbf{e}_0 + (\nabla\wedge\vec{E})\mathbf{e}_0 + \nabla\times\vec{B} - (\nabla\cdot\vec{B})\mathbf{e}_{123} - \mathbf{e}_0\tfrac{\partial\vec{E}}{\partial t}\mathbf{e}_0 - \mathbf{e}_0\tfrac{\partial\vec{B}}{\partial t}\mathbf{e}_{123}\\ &= (\nabla\cdot\vec{E})\mathbf{e}_0 + \nabla\times\vec{B} - \tfrac{\partial\vec{E}}{\partial t} - (\nabla\cdot\vec{B})\mathbf{e}_{123} + (\nabla\times\vec{E})\mathbf{e}_{0123} + \tfrac{\partial\vec{B}}{\partial t}\mathbf{e}_{0123}\\ &= \rho\mathbf{e}_0 + \vec{J} = \mathbf{J}\\ \partial\mathbf{A} &= \nabla\cdot\vec{A} + (\nabla\times\vec{A})\mathbf{e}_{123} + (\nabla V)\mathbf{e}_0 - \mathbf{e}_0\tfrac{\partial V}{\partial t}\mathbf{e}_0 - \mathbf{e}_0\tfrac{\partial\vec{A}}{\partial t}\\ &= -\vec{E}\mathbf{e}_0 + \vec{B}\mathbf{e}_{123} = -\mathbf{F}\end{aligned}$$

where in the last step we used $-\mathbf{e}_0\frac{\partial\vec{A}}{\partial t} = \frac{\partial\vec{A}}{\partial t}\mathbf{e}_0$.

## 13.2 The Maxwell equations in $\mathbb{R}^{1,3}$

We use the following definitions for the potential $\mathbf{A}$, the current $\mathbf{J}$, the differential operator $\partial$, and the electromagnetic field $\mathbf{F}$:

$$\begin{aligned}
&A^\alpha = (\tfrac{1}{c}V, A_x, A_y, A_z), \qquad A_\alpha = (\tfrac{1}{c}V, -A_x, -A_y, -A_z)\\
&J^\alpha = (\rho c, J_x, J_y, J_z), \qquad J_\alpha = (\rho c, -J_x, -J_y, -J_z)\\
&\partial^\alpha = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, -\tfrac{\partial}{\partial x}, -\tfrac{\partial}{\partial y}, -\tfrac{\partial}{\partial z}), \qquad \partial_\alpha = (\tfrac{1}{c}\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z})\\
&(F^{01}, F^{02}, F^{03}) = (-\tfrac{1}{c}E_x, -\tfrac{1}{c}E_y, -\tfrac{1}{c}E_z)\\
&(F^{23}, F^{31}, F^{12}) = (-B_x, -B_y, -B_z).
\end{aligned}$$

This leads to the d'Alembert operator

$$\partial^\alpha \partial_\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

and the equations

$$\partial^\alpha \partial_\alpha A^\beta = \mu J^\beta.$$

The equations

$$\begin{aligned}
\partial^0 A^1 - \partial^1 A^0 &= \tfrac{1}{c}\tfrac{\partial}{\partial t}A_x + \tfrac{\partial}{\partial x}\tfrac{V}{c} = -\tfrac{1}{c}E_x, \ \dots\\
\partial^1 A^2 - \partial^2 A^1 &= -\tfrac{\partial}{\partial x}A_y + \tfrac{\partial}{\partial y}A_x = -B_z, \ \dots
\end{aligned}$$

give

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha,$$

and

$$\begin{aligned}
\partial_\alpha F^{\alpha 0} &= \tfrac{\partial}{\partial x}\tfrac{E_x}{c} + \tfrac{\partial}{\partial y}\tfrac{E_y}{c} + \tfrac{\partial}{\partial z}\tfrac{E_z}{c} = \tfrac{1}{c}\nabla\cdot\vec{E} = \tfrac{\rho}{c\varepsilon} = \mu J^0\\
\partial_\alpha F^{\alpha 1} &= -\tfrac{1}{c}\tfrac{\partial}{\partial t}\tfrac{E_x}{c} + \tfrac{\partial}{\partial y}B_z - \tfrac{\partial}{\partial z}B_y = \mu J^1, \ \dots
\end{aligned}$$

give the Maxwell equations

$$\partial_\alpha F^{\alpha\beta} = \mu J^\beta.$$

In the Clifford algebra $\mathcal{C}\ell_{1,3}$ we have

$$\begin{aligned}
\partial &= \gamma_0 \tfrac{1}{c}\tfrac{\partial}{\partial t} - \gamma_1 \tfrac{\partial}{\partial x} - \gamma_2 \tfrac{\partial}{\partial y} - \gamma_3 \tfrac{\partial}{\partial z}\\
\mathbf{F} &= -\tfrac{1}{c}E_x\gamma_{01} - \tfrac{1}{c}E_y\gamma_{02} - \tfrac{1}{c}E_z\gamma_{03} - B_x\gamma_{23} - B_y\gamma_{31} - B_z\gamma_{12}\\
\mathbf{A} &= \tfrac{1}{c}V\gamma_0 + A_x\gamma_1 + A_y\gamma_2 + A_z\gamma_3\\
\mathbf{J} &= c\rho\gamma_0 + J_x\gamma_1 + J_y\gamma_2 + J_z\gamma_3
\end{aligned}$$

and the equations

$$\begin{aligned}
\partial\mathbf{A} &= \gamma_{01}(\tfrac{1}{c}\tfrac{\partial}{\partial t}A_x + \tfrac{\partial}{\partial x}\tfrac{V}{c}) + \dots + \gamma_{12}(-\tfrac{\partial}{\partial x}A_y + \tfrac{\partial}{\partial y}A_x)\\
\partial\mathbf{F} &= \gamma_0(\tfrac{\partial}{\partial x}\tfrac{E_x}{c} + \tfrac{\partial}{\partial y}\tfrac{E_y}{c} + \tfrac{\partial}{\partial z}\tfrac{E_z}{c}) + \gamma_1(-\tfrac{1}{c^2}\tfrac{\partial}{\partial t}E_x - \tfrac{\partial}{\partial z}B_y + \tfrac{\partial}{\partial y}B_z) + \dots
\end{aligned}$$

which lead to

$$\boxed{\partial \mathbf{A} = \mathbf{F}, \quad \partial \mathbf{F} = \mu \mathbf{J}, \quad \partial^2 \mathbf{A} = \mu \mathbf{J}}$$

## 13.3 Comparison of $\mathbb{R}^{3,1}$ and $\mathbb{R}^{1,3}$

The equation $\partial \mathbf{A} = \pm \mathbf{F}$ has opposite signs in opposite metrics. This means that the raising part $\partial \wedge \mathbf{A} = \pm \mathbf{F}$ has opposite signs in opposite metrics, whereas the lowering part $\partial \rfloor \mathbf{A} = 0$ is independent of metric. The Maxwell equations $\partial \mathbf{F} = \mu \mathbf{J}$ have the same signs in both metrics. This means that the lowering part $\partial \rfloor \mathbf{F} = \mu \mathbf{J}$ is invariant under the metric swap (and that in both metrics $\partial \wedge \mathbf{F} = 0$). The above unexpected results are consequences of our definition: the differential operator $\partial$ experiences a sign change under the metric swap, that is,

$$\boxed{\operatorname{opp}[\partial] = -\partial}$$

### Spinors and Observables in $\mathbb{R}^{3,1}$

In the rest of this chapter we shall study the Dirac equation, spinors and observables in $\mathbb{R}^{3,1}$. Our special concern will be the behavior of spinors under the transition to the opposite metric.

Since going to the opposite algebra interchanges left and right ideals, we will study real ideal spinors $\phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1-\gamma_{03})$ in conjunction with their *opposite-reverses*

$$\underset{-}{\phi} = \operatorname{opp}[\tilde{\phi}] \in \mathcal{C}\ell_{3,1}\frac{1}{2}(1+\mathbf{e}_{03})$$

(both $\phi$ and the opposite of its reverse $\underset{-}{\phi}$ are in left ideals). Clearly, $\operatorname{opp}[\tilde{\phi}] = \operatorname{opp}[\phi]^{\sim}$.

For instance, the Dirac equation for the real ideal spinors, $\partial\phi\gamma_{21} = e\mathbf{A}\phi + m\phi$, $\phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1-\gamma_{03})$, is transformed by the opposite-reversion to $\partial\hat{\underset{-}{\phi}}\mathbf{e}_{21} = e\mathbf{A}\hat{\underset{-}{\phi}} + m\underset{-}{\phi}$, $\underset{-}{\phi} \in \mathcal{C}\ell_{3,1}\frac{1}{2}(1+\mathbf{e}_{03})$, [2] and further by grade involution to

$$\partial\underset{-}{\phi}\mathbf{e}_{21} = e\mathbf{A}\underset{-}{\phi} - m\hat{\underset{-}{\phi}}.$$

However, this is not a nice formula, because we have to explain the occurrence of the grade involution in the last term. There are even more interpretational difficulties for the opposite-reverses $\underset{-}{\psi} = \operatorname{opp}[\tilde{\psi}]$ [of Dirac spinor $\psi = \psi\frac{1}{2}(1+$

2 Note that $\phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1-\gamma_{03})$ is in a graded minimal left ideal of $\mathcal{C}\ell_{1,3} \simeq \operatorname{Mat}(2,\mathbb{H})$, which is also an ungraded minimal left ideal, while $\underset{-}{\phi} \in \mathcal{C}\ell_{3,1}\frac{1}{2}(1+\mathbf{e}_{03})$ is in a graded minimal left ideal of $\mathcal{C}\ell_{3,1} \simeq \operatorname{Mat}(4,\mathbb{R})$, which is not an ungraded minimal left ideal [the minimal left ideals of $\mathcal{C}\ell_{3,1}$ are not graded].

$\gamma_0)\frac{1}{2}(1+i\gamma_{12})]$ and $\underset{-}{\Phi}=\mathrm{opp}[\tilde{\Phi}]$ [of mother spinor $\Phi=\Phi\frac{1}{2}(1+\gamma_0)]$, since $\underset{-}{\psi}\notin(\mathbb{C}\otimes\mathcal{C}\ell_{3,1})\frac{1}{2}(1\pm i\mathbf{e}_0)$ and $\underset{-}{\Phi}$ is not in any proper left ideal of $\mathcal{C}\ell_{3,1}$. An obvious attempt for a possible solution would be to study $\underset{-}{\eta}=\underset{-}{\psi}\frac{1}{2}(1-i\mathbf{e}_0)$, but then (like evaluating $4\psi\tilde{\psi}^*=Z$ in the case $\mathbb{R}^{1,3}$) for the aggregate of observables

$$4\underset{-}{\eta}\tilde{\underset{-}{\eta}}^*=\frac{1}{2}(\Omega_1-i\mathbf{J}-i\mathbf{S}+\mathbf{K}\mathbf{e}_{0123}+\Omega_2\mathbf{e}_{0123}),\qquad \underset{-}{\eta}\tilde{\underset{-}{\eta}}^*=0,$$

and we would have the inconvenience of an extra factor $\frac{1}{2}$. This shortcoming could be circumvented by multiplying $\underset{-}{\eta}$ by $\sqrt{2}$, but then the relation to the original Dirac spinor $\psi$ would be irrational. Again there is an obvious solution: multiply $\underset{-}{\eta}$ by $1-i$, which has absolute value $\sqrt{2}$, and study instead the *flip* of the opposite-reverse, that is, the *tilted* spinor

$$\underset{\sim}{\psi}=(1-i)\underset{-}{\psi}\frac{1}{2}(1-i\mathbf{e}_0)=\underset{-}{\psi}\frac{1}{2}(1-\mathbf{e}_{012})(1-\mathbf{e}_{12}),\qquad \underset{-}{\psi}=\mathrm{opp}[\tilde{\psi}],$$

for which

$$4\underset{\sim}{\psi}\tilde{\underset{\sim}{\psi}}^*=\Omega_1-i\mathbf{J}-i\mathbf{S}+\mathbf{K}\mathbf{e}_{0123}+\Omega_2\mathbf{e}_{0123},\qquad \underset{\sim}{\psi}\tilde{\underset{\sim}{\psi}}^*=0.$$

The opposite-reverse of $\psi\gamma_0=\psi$ is $\hat{\underset{-}{\psi}}\mathbf{e}_0=\underset{-}{\psi}$. Therefore, we find

$$(1+i)\underset{\sim}{\psi}=\underset{-}{\psi}(1-i\mathbf{e}_0)=\underset{-}{\psi}+i\hat{\underset{-}{\psi}}=(1+i)\underset{-}{\psi}_0+(1-i)\underset{-}{\psi}_1$$

[$\underset{-}{\psi}_0=\mathrm{even}(\underset{-}{\psi})$, $\underset{-}{\psi}_1=\mathrm{odd}(\underset{-}{\psi})$] which implies $\underset{\sim}{\psi}=\underset{-}{\psi}_0-i\underset{-}{\psi}_1$, or since $i\underset{-}{\psi}=\underset{-}{\psi}\mathbf{e}_{12}$,

$$\underset{\sim}{\psi}=\underset{-}{\psi}_0-\underset{-}{\psi}_1\mathbf{e}_{12}\qquad(\underset{-}{\psi}=\mathrm{opp}[\tilde{\psi}],\quad \psi\in\mathbb{C}\otimes\mathcal{C}\ell_{1,3}).$$

Similarly, we define the *tilted* spinor for the mother of all real spinors $\Phi$ and for the real ideal spinor $\phi$:

$$\underset{\sim}{\Phi}=\underset{-}{\Phi}_0-\underset{-}{\Phi}_1\mathbf{e}_{12}=\underset{-}{\Phi}\,\frac{1}{2}(1-\mathbf{e}_{012})(1-\mathbf{e}_{12}),\qquad \underset{-}{\Phi}=\mathrm{opp}[\tilde{\Phi}],$$
$$\underset{\sim}{\phi}=\underset{-}{\phi}_0-\underset{-}{\phi}_1\mathbf{e}_{12}\neq\underset{-}{\phi}\,\frac{1}{2}(1-\mathbf{e}_{012})(1-\mathbf{e}_{12}),\qquad \underset{-}{\phi}=\mathrm{opp}[\tilde{\phi}].$$

Of course, for the spinor operator $\underset{\sim}{\Psi}=\underset{-}{\Psi}$ $(=\mathrm{opp}[\tilde{\Psi}])$.

The transition back from $\underset{\sim}{\psi}\in(\mathbb{C}\otimes\mathcal{C}\ell_{3,1})\frac{1}{2}(1-i\mathbf{e}_0)\frac{1}{2}(1-i\mathbf{e}_{12})$ to $\psi\in(\mathbb{C}\otimes\mathcal{C}\ell_{1,3})\frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+i\gamma_{12})$ is given by tilting again,

$$\psi=\mathrm{opp}[\tilde{\underset{\sim}{\psi}}_0]-\mathrm{opp}[\tilde{\underset{\sim}{\psi}}_1]\gamma_{12},\qquad \underset{\sim}{\psi}\in\mathbb{C}\otimes\mathcal{C}\ell_{3,1},$$

since

$$\begin{aligned}\mathrm{opp}[\tilde{\underset{\sim}{\psi}}_0]-\mathrm{opp}[\tilde{\underset{\sim}{\psi}}_1]\gamma_{12}&=(\tilde{\psi}_0)\tilde{}-\mathrm{opp}[(-\underset{-}{\psi}_1\mathbf{e}_{12})\tilde{}\,]\gamma_{12}\\&=\psi_0-(-\gamma_{12}\tilde{\psi}_1)\tilde{}\,\gamma_{12}=\psi_0-\psi_1\gamma_{12}\gamma_{12}.\end{aligned}$$

## 13.4 The Dirac equation in $\mathbb{R}^{3,1}$

The opposite-reverse Dirac equation

$$\partial \underline{\phi} \mathbf{e}_{21} = e\mathbf{A}\underline{\phi} - m\hat{\underline{\phi}}, \qquad \underline{\phi} \in \mathcal{C}\ell_{3,1}\,\frac{1}{2}(1+\mathbf{e}_{03}),$$

splits into even and odd parts $[\underline{\phi}_0 = \text{even}(\underline{\phi}),\ \underline{\phi}_1 = \text{odd}(\underline{\phi})]$

$$\partial \underline{\phi}_1 \mathbf{e}_{21} = e\mathbf{A}\underline{\phi}_1 - m\underline{\phi}_0$$
$$\partial \underline{\phi}_0 \mathbf{e}_{21} = e\mathbf{A}\underline{\phi}_0 + m\underline{\phi}_1.$$

Recalling that $\underset{\sim}{\phi}_0 = \underline{\phi}_0$ and $\underset{\sim}{\phi}_1 = \underline{\phi}_1\mathbf{e}_{21}$ we find

$$\partial\, \underset{\sim}{\phi}_0 = e\mathbf{A}\, \underset{\sim}{\phi}_0 \mathbf{e}_{12} - m\, \underset{\sim}{\phi}_1$$
$$\partial\, \underset{\sim}{\phi}_1 = e\mathbf{A}\, \underset{\sim}{\phi}_1 \mathbf{e}_{12} - m\, \underset{\sim}{\phi}_0$$

which added together result in

$$\partial\, \underset{\sim}{\phi} = e\mathbf{A}\, \underset{\sim}{\phi} \mathbf{e}_{12} - m\, \underset{\sim}{\phi}, \qquad \underset{\sim}{\phi} \in \mathcal{C}\ell_{3,1}\,\frac{1}{2}(1+\mathbf{e}_{03}).$$

Similarly, the flip of the opposite-reverse, or simply tilted, Dirac spinor $\underset{\sim}{\psi}$ obeys

$$\partial\, \underset{\sim}{\psi} = ie\mathbf{A}\, \underset{\sim}{\psi} - m\, \underset{\sim}{\psi}, \qquad \underset{\sim}{\psi} \in (\mathbb{C}\otimes\mathcal{C}\ell_{3,1})\frac{1}{2}(1-i\mathbf{e}_0)\frac{1}{2}(1-i\mathbf{e}_{12}),$$

a formula found essentially in Benn & Tucker 1987 p. 284 (and p. 256). So the tilted mother spinor $\underset{\sim}{\Phi} = 4\,\mathrm{Re}(\underset{\sim}{\psi}) = \underset{\sim}{\Psi}(1-\mathbf{e}_{012})$ [3] obeys

$$\partial\, \underset{\sim}{\Phi} = e\mathbf{A}\, \underset{\sim}{\Phi} \mathbf{e}_{12} - m\, \underset{\sim}{\Phi}, \qquad \underset{\sim}{\Phi} \in \mathcal{C}\ell_{3,1}\,\frac{1}{2}(1-\mathbf{e}_{012}),$$

which decomposed into even and odd parts (and recalling that $\underset{\sim}{\Phi}_0 = -\underset{\sim}{\Phi}_1\mathbf{e}_{012}$, $\underset{\sim}{\Phi}_1 = -\underset{\sim}{\Phi}_0\mathbf{e}_{012}$) results in the Dirac-Hestenes equation in the opposite metric,

$$\boxed{\partial\, \underset{\sim}{\Psi} \mathbf{e}_{21} - e\mathbf{A}\, \underset{\sim}{\Psi} = m\, \underset{\sim}{\Psi} \mathbf{e}_0}$$

where $\underset{\sim}{\Psi} : \mathbb{R}^{3,1} \to \mathcal{C}\ell_{3,1}^+$.

## 13.5 Bilinear covariants in $\mathbb{R}^{3,1}$

Recall that for $\underset{\sim}{\psi} \in (\mathbb{C}\otimes\mathcal{C}\ell_{3,1})\frac{1}{2}(1-i\mathbf{e}_0)\frac{1}{2}(1-i\mathbf{e}_{12})$

$$4\,\underset{\sim}{\psi}\,\underset{\sim}{\bar{\psi}}^* = \Omega_1 - i\mathbf{J} - i\mathbf{S} + \mathbf{K}\mathbf{e}_{0123} + \Omega_2\mathbf{e}_{0123}, \qquad \underset{\sim}{\psi}\,\underset{\sim}{\tilde{\psi}}^* = 0.$$

Compute the bilinear covariants in the opposite metric $\mathbb{R}^{3,1}$:

3 Note that $\underset{\sim}{\Phi} = \underset{\sim}{\phi}(1-\mathbf{e}_{123})$ and $\underset{\sim}{\phi} = \underset{\sim}{\Phi}\frac{1}{2}(1+\mathbf{e}_{03})$.

$$\Omega_1 = 4\langle \bar{\underset{\sim}{\psi}}^* \underset{\sim}{\psi}\rangle_0$$
$$J_\mu = 4\langle \bar{\underset{\sim}{\psi}}^* i\mathbf{e}_\mu \underset{\sim}{\psi}\rangle_0 \qquad (J^0 = -J_0)$$
$$S_{\mu\nu} = -4\langle \bar{\underset{\sim}{\psi}}^* i\mathbf{e}_{\mu\nu} \underset{\sim}{\psi}\rangle_0$$
$$K_\mu = -4\langle \bar{\underset{\sim}{\psi}}^* \mathbf{e}_{0123}\mathbf{e}_\mu \underset{\sim}{\psi}\rangle_0$$
$$\Omega_2 = -4\langle \bar{\underset{\sim}{\psi}}^* \mathbf{e}_{0123} \underset{\sim}{\psi}\rangle_0.$$

(Observe that the coordinates $J^\mu$ of $\mathbf{J}$ have the same numerical values in $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{3,1}$. In contrast, the coordinates $J_\mu$ are opposite in $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{3,1}$.)

## 13.6 Fierz identities in $\mathbb{R}^{3,1}$

The Fierz identities in the opposite metric are

$$\mathbf{J}^2 = -(\Omega_1^2 + \Omega_2^2), \qquad \mathbf{K}^2 = -\mathbf{J}^2$$
$$\mathbf{J}\cdot\mathbf{K} = 0, \qquad \mathbf{J}\wedge\mathbf{K} = -(\Omega_2 + \mathbf{e}_{0123}\Omega_1)\mathbf{S}.$$

Note also that $\mathbf{S}^2 = (\Omega_2 - \mathbf{e}_{0123}\Omega_1)^2$.

**Exercise.** Derive the real theory from the mother of all real tilted spinors $4\,\mathrm{Re}(\underset{\sim}{\psi}) = \underset{\approx}{\Phi} = \underset{\approx}{\Phi}\frac{1}{2}(1-\mathbf{e}_{012}) \in \mathcal{C}\ell_{3,1}\,\frac{1}{2}(1-\mathbf{e}_{012})$. ∎

## 13.7 Decomposition of boomerangs in $\mathbb{R}^{3,1}$

Write for $\underset{\sim}{\psi} \in (\mathbb{C}\otimes\mathcal{C}\ell_{3,1})\frac{1}{2}(1-i\mathbf{e}_0)\frac{1}{2}(1-i\mathbf{e}_{12})$

$$Z = 4\underset{\sim}{\psi}\,\bar{\underset{\sim}{\psi}}^*$$
$$K = \Omega_1 + \mathbf{K}\mathbf{e}_{0123} + \Omega_2\mathbf{e}_{0123}, \qquad L = -\mathbf{J} - \mathbf{S}$$
$$\underset{\approx}{S} = \Omega_1 - i\mathbf{S} + \Omega_2\mathbf{e}_{0123}, \qquad \underset{\approx}{\Sigma} = 1 - i\mathbf{e}_{0123}\mathbf{J}\mathbf{K}^{-1}$$
$$P = \Omega_1 - i\mathbf{J} + \Omega_2\mathbf{e}_{0123}, \qquad \Pi = 1 - i\mathbf{J}(\Omega_1 + \Omega_2\mathbf{e}_{0123})^{-1}$$
$$\Gamma = K(\Omega_1 + \Omega_2\mathbf{e}_{0123})^{-1}$$

[$Z$, $K$, $P$, $\Pi$, $\Gamma$ are not the same as those in the case of $\mathbb{R}^{1,3}$ but instead as a sample $Z$ in $\mathbb{C}\otimes\mathcal{C}\ell_{3,1}$ is obtained by sending $Z$ in $\mathbb{C}\otimes\mathcal{C}\ell_{1,3}$ to $\mathrm{even}(\tilde{Z}) - i\,\mathrm{odd}(\tilde{Z})$]. Then

$$\begin{aligned} Z &= K + iL = K\underset{\approx}{\Sigma} = \underset{\approx}{\Sigma}K = \Pi K = K\bar{\Pi}^* \quad (\neq K\Pi \text{ unless } \Omega_2 = 0) \\ &= P\underset{\approx}{\Sigma} = \underset{\approx}{\Sigma}P = \Pi\underset{\approx}{S} = \underset{\approx}{S}\bar{\Pi}^* \\ &= \Gamma P = P\bar{\Gamma} \qquad \text{(no complex conjugation)} \\ &= K\Big(1 + i\frac{1}{2\Omega_1}L\Big), \end{aligned}$$

$$K^2 = 2\Omega_1 K = -L^2, \qquad KL = LK = 2\Omega_1 L$$
$$\Pi = P(\Omega_1 + \Omega_2 \mathbf{e}_{0123})^{-1}, \qquad \underset{\sim}{\Sigma} = \underset{\sim}{S}(\Omega_1 + \Omega_2 \mathbf{e}_{0123})^{-1}.$$

## 13.8 Multiplication by $i = \sqrt{-1}$ in $\mathbb{C} \otimes \mathcal{C}\ell_{3,1}$

Write

$$\underset{\sim}{j} = (\Omega_1 - \Omega_2 \mathbf{e}_{0123})^{-1}\mathbf{J} = -\mathbf{e}_{0123}\mathbf{S}\mathbf{K}^{-1} = \underset{\sim}{\Psi}\mathbf{e}_0 \underset{\sim}{\Psi}^{-1}$$
$$\underset{\sim}{s} = \mathbf{e}_{0123}\mathbf{J}\mathbf{K}^{-1} = (\Omega_1 + \Omega_2 \mathbf{e}_{0123})^{-1}\mathbf{S} = \underset{\sim}{\Psi}\mathbf{e}_{12} \underset{\sim}{\Psi}^{-1}.$$

Then $K = \underset{\sim}{j}L = \underset{\sim}{s}L$, $\Pi = 1 - i\underset{\sim}{j}$ and $\underset{\sim}{\Sigma} = 1 - i\underset{\sim}{s}$. Also

$$\begin{array}{ll} i\underset{\sim}{\psi} = \underset{\sim}{j}\,\underset{\sim}{\psi} = \underset{\sim}{s}\,\underset{\sim}{\psi} = \underset{\sim}{\psi}\mathbf{e}_{12} = \underset{\sim}{\psi}\mathbf{e}_0 \neq \underset{\sim}{\psi}\mathbf{e}_{0123} & \text{tilted Dirac} \\ \underset{\sim}{j}\,\underset{\sim}{\Phi} = \underset{\sim}{s}\,\underset{\sim}{\Phi} = \underset{\sim}{\Phi}\mathbf{e}_{12} = \underset{\sim}{\Phi}\mathbf{e}_0 & \text{tilted mother} \\ \underset{\sim}{j}\,\underset{\sim}{\phi} = \underset{\sim}{s}\,\underset{\sim}{\phi} = \underset{\sim}{\phi}\mathbf{e}_{12} = \underset{\sim}{\phi}\mathbf{e}_{0123} \neq \underset{\sim}{\phi}\mathbf{e}_0 & \text{tilted ideal} \\ \underset{\sim}{s}\,\underset{\sim}{\Psi} = \underset{\sim}{\Psi}\mathbf{e}_{12} \quad (\text{but } \; \underset{\sim}{j}\,\underset{\sim}{\Psi}\mathbf{e}_0 = -\underset{\sim}{\Psi}) & \text{operator.} \end{array}$$

## 13.9 Some differences between $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{3,1}$

Compute as a sample in $\mathcal{C}\ell_{3,1}$

$$\underset{\sim}{\Phi}\,\tilde{\underset{\sim}{\Phi}} = 0, \qquad \frac{1}{2}\underset{\sim}{\Phi}\,\bar{\underset{\sim}{\Phi}} = K, \qquad \frac{1}{2}\underset{\sim}{\Phi}\mathbf{e}_0\,\bar{\underset{\sim}{\Phi}} = -L$$
$$\frac{1}{2}\underset{\sim}{\Phi}\mathbf{e}_3\,\bar{\underset{\sim}{\Phi}} = 0, \qquad \frac{1}{2}\underset{\sim}{\Phi}\mathbf{e}_3\,\tilde{\underset{\sim}{\Phi}} = -K\mathbf{e}_{0123}.$$

Note that

$$\langle P\gamma_0\tilde{P}\gamma_0\rangle_0 = \langle Q\gamma_0\tilde{Q}\gamma_0\rangle_0 > 0 \quad \text{for non-zero} \quad \psi \in \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$$

while

$$\langle K\mathbf{e}_0\tilde{K}\mathbf{e}_0\rangle_0 = \langle L\mathbf{e}_0\tilde{L}\mathbf{e}_0\rangle_0 \gtrless 0 \quad \text{for non-zero} \quad \underset{\sim}{\psi} \in \mathbb{C}\otimes\mathcal{C}\ell_{3,1}$$
$$\langle K\mathbf{e}_0\bar{K}\mathbf{e}_0^{-1}\rangle_0 \neq \langle L\mathbf{e}_0\bar{L}\mathbf{e}_0^{-1}\rangle_0 > 0$$

so that it is possible that $K = 0$ while $L \neq 0$ (this happens in the case $\Omega_1 = \Omega_2 = 0$, $\mathbf{K} = 0$ of Majorana spinors).

## 13.10 Charge conjugate in $\mathbb{R}^{3,1}$

The charge conjugate of the tilted Dirac spinor is obtained as follows:

$$\begin{aligned}
&\psi_C = \hat{\psi}^*\gamma_1 \quad \text{take opposite-reverse} \quad \underline{\psi}_C = \underline{\psi}^*\mathbf{e}_1 \quad \text{and tilt} \\
&\underset{\sim}{\psi}_C = (1-i)\underline{\psi}^*\mathbf{e}_1\tfrac{1}{2}(1-i\mathbf{e}_0) = (1-i)\underline{\psi}^*\tfrac{1}{2}(1+i\mathbf{e}_0)\mathbf{e}_1 \\
&\quad = [(1+i)\underline{\psi}\tfrac{1}{2}(1-i\mathbf{e}_0)]^*\mathbf{e}_1 = -i[(1-i)\underline{\psi}\tfrac{1}{2}(1-i\mathbf{e}_0)]^*\mathbf{e}_1 \\
&\quad = -i\,\underset{\sim}{\psi}^*\mathbf{e}_1 = -\underset{\sim}{\psi}^*\mathbf{e}_1\mathbf{e}_{12} = -\underset{\sim}{\psi}^*\mathbf{e}_2 \qquad \text{or} \\
&\quad = -\underset{\sim}{\psi}^*\mathbf{e}_1\mathbf{e}_0 = \underset{\sim}{\psi}^*\mathbf{e}_{01} \in (\mathbb{C}\otimes\mathcal{C}\ell_{3,1})\tfrac{1}{2}(1-i\mathbf{e}_0)\tfrac{1}{2}(1-i\mathbf{e}_{12}).
\end{aligned}$$

NUMERICAL EXAMPLE

Start from $\mathcal{C}\ell_{1,3}$. Take a column spinor

$$\psi = \begin{pmatrix} 4+3i \\ 1+6i \\ 5+2i \\ 2+i \end{pmatrix} \in \mathbb{C}^4.$$

Then $Z = 4\psi\psi^\dagger\gamma_0$

$$\begin{aligned}
&= 4\begin{pmatrix} 25 & 22-21i & -26-7i & -11-2i \\ 22+21i & 37 & -17-28i & -8-11i \\ 26-7i & 17-28i & -29 & -12+i \\ 11-2i & 8-11i & -12-i & -5 \end{pmatrix} \\
&= \Omega_1 + \mathbf{J} + i\mathbf{S} + i\mathbf{K}\gamma_{0123} + \Omega_2\gamma_{0123}
\end{aligned}$$

where

$$\begin{aligned}
&\Omega_1 = 28 \\
&\mathbf{J} = 96\gamma_0 + 56\gamma_1 + 52\gamma_2 + 36\gamma_3 \\
&\mathbf{S} = 60\gamma_{01} - 12\gamma_{02} - 8\gamma_{03} - 36\gamma_{12} - 40\gamma_{13} + 20\gamma_{23} \\
&\mathbf{K} = 68\gamma_0 + 68\gamma_1 + 44\gamma_2 + 12\gamma_3 \\
&\Omega_2 = -36.
\end{aligned}$$

In the opposite algebra $\mathcal{C}\ell_{3,1}$ we must first fix the matrix representation, for instance, using the Pauli spin matrices $\sigma_k$,

$$\mathbf{e}_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \mathbf{e}_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}$$

corresponding to the tilted primitive idempotent

$$\underset{\sim}{f} = \frac{1}{2}(1 - i\mathbf{e}_0)\,\frac{1}{2}(1 - i\mathbf{e}_{12})$$

and the tilted spinor basis

$$\begin{aligned}
\underset{\sim}{f}_1 &= \frac{1}{4}(1 - i\mathbf{e}_0 - i\mathbf{e}_{12} - \mathbf{e}_{012}) &&= \underset{\sim}{f}\\
\underset{\sim}{f}_2 &= \frac{1}{4}(\mathbf{e}_{13} - i\mathbf{e}_{23} - i\mathbf{e}_{013} - \mathbf{e}_{023}) &&= \mathbf{e}_{13}\underset{\sim}{f}\\
\underset{\sim}{f}_3 &= \frac{1}{4}(\mathbf{e}_3 + i\mathbf{e}_{03} - i\mathbf{e}_{123} + \mathbf{e}_{0123}) &&= \mathbf{e}_3\underset{\sim}{f}\\
\underset{\sim}{f}_4 &= \frac{1}{4}(\mathbf{e}_1 - i\mathbf{e}_2 + i\mathbf{e}_{01} + \mathbf{e}_{02}) &&= \mathbf{e}_1\underset{\sim}{f}.
\end{aligned}$$

Then the tilted spinor is

$$\underset{\sim}{\psi} = \begin{pmatrix} 4+3i \\ 1+6i \\ 2-5i \\ 1-2i \end{pmatrix}$$

and the boomerang $Z = 4\underset{\sim}{\psi}\,\underset{\sim}{\psi}^\dagger(-i\mathbf{e}_0) = 4\underset{\sim}{\psi}\,\underset{\sim}{\psi}^\dagger i\mathbf{e}^0$

$$\begin{aligned}
&= 4\begin{pmatrix} 25 & 22-21i & 7-26i & 2-11i \\ 22+21i & 37 & 28-17i & 11-8i \\ -7-26i & -28-17i & -29 & -12+i \\ -2-11i & -11-8i & -12-i & -5 \end{pmatrix}\\
&= \Omega_1 - i\mathbf{J} - i\mathbf{S} + \mathbf{K}\mathbf{e}_{0123} + \Omega_2\mathbf{e}_{0123}
\end{aligned}$$

where

$$\begin{aligned}
\Omega_1 &= 28\\
\mathbf{J} &= 96\mathbf{e}_0 + 56\mathbf{e}_1 + 52\mathbf{e}_2 + 36\mathbf{e}_3\\
\mathbf{S} &= 60\mathbf{e}_{01} - 12\mathbf{e}_{02} - 8\mathbf{e}_{03} - 36\mathbf{e}_{12} - 40\mathbf{e}_{13} + 20\mathbf{e}_{23}\\
\mathbf{K} &= 68\mathbf{e}_0 + 68\mathbf{e}_1 + 44\mathbf{e}_2 + 12\mathbf{e}_3\\
\Omega_2 &= -36.
\end{aligned}$$

Note that for $u \in \mathbb{C}\otimes \mathcal{C}\ell_{3,1}$, $\bar{u}^* = \mathbf{e}_0 u^\dagger \mathbf{e}_0^{-1}$.

Note that $Z$ for $\mathbb{R}^{1,3}$ and $Z$ for $\mathbb{R}^{3,1}$ are related via a similarity transformation by $\frac{1}{\sqrt{2}}(I + i\gamma_0)$.

**Summary**

To realize the transition to the opposite metric we use the rules

$$\mathrm{opp}[\partial] = -\partial \quad \text{and} \quad \mathrm{opp}[ab] = b_0a_0 + b_0a_1 + b_1a_0 - b_1a_1$$

and apply reversion to get the tilted spinors also in left ideals. Thus, the two sides of $\partial\mathbf{A} = \mathbf{F}$ in $\mathbb{R}^{1,3}$ are transformed as

$$\begin{aligned}\mathrm{opp}[\partial\mathbf{A}]^{\sim} &= -(-\partial)\tilde{\mathbf{A}} = \partial\mathbf{A}\\ \mathrm{opp}[\mathbf{F}]^{\sim} &= \tilde{\mathbf{F}} = -\mathbf{F}\end{aligned}$$

and so we have $\partial\mathbf{A} = -\mathbf{F}$ in $\mathbb{R}^{3,1}$. The two sides of $\partial\mathbf{F} = \mathbf{J}$ in $\mathbb{R}^{1,3}$ are transformed as

$$\begin{aligned}\mathrm{opp}[\partial\mathbf{F}]^{\sim} &= (-\partial)\tilde{\mathbf{F}} = \partial\mathbf{F}\\ \mathrm{opp}[\mathbf{J}]^{\sim} &= \tilde{\mathbf{J}} = \mathbf{J}\end{aligned}$$

and so we have $\partial\mathbf{F} = \mathbf{J}$ in $\mathbb{R}^{3,1}$. The terms of $\partial\Psi\gamma_{21} - e\mathbf{A}\Psi = m\Psi\gamma_0$ in $\mathbb{R}^{1,3}$ are transformed as

$$\begin{aligned}\mathrm{opp}[\partial\Psi\gamma_{21}]^{\sim} &= (-\partial)\tilde{\Psi}\tilde{\mathbf{e}}_{21} = \partial\tilde{\Psi}\mathbf{e}_{21}\\ \mathrm{opp}[\mathbf{A}\Psi]^{\sim} &= \tilde{\mathbf{A}}\tilde{\Psi} = \mathbf{A}\tilde{\Psi}\\ \mathrm{opp}[\Psi\gamma_0]^{\sim} &= \tilde{\Psi}\tilde{\mathbf{e}}_0 = \tilde{\Psi}\mathbf{e}_0\end{aligned}$$

and so we have $\partial\tilde{\Psi}\mathbf{e}_{21} - e\mathbf{A}\tilde{\Psi} = m\tilde{\Psi}\mathbf{e}_0$ in $\mathbb{R}^{3,1}$. [Earlier in this chapter we wrote $\underset{\sim}{\Psi}$ for $\tilde{\Psi}$.] Note in particular that the Dirac-Hestenes equation has the same form in both metrics, only the spinor operators are reversed. For complex ideal spinors the situation is more complicated, an extra flip is needed to complete the metric tilt.

In our differential operator

$$\partial = \mathbf{e}^0\frac{\partial}{\partial x^0} + \mathbf{e}^1\frac{\partial}{\partial x^1} + \mathbf{e}^2\frac{\partial}{\partial x^2} + \mathbf{e}^3\frac{\partial}{\partial x^3}$$

we have used an orthonormal basis, but this formula gives the same $\partial$ for any basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^{3,1}$ when $\mathbf{e}^\mu \cdot \mathbf{e}_\nu = \delta^\mu_\nu$, that is, when $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$ are reciprocal. In this sense our differential operator is not only Lorentz covariant but also invariant under all of $GL(4, \mathbb{R})$.

Note that the raising differential $\partial \wedge \mathbf{f}$ is metric dependent and therefore it is *not* related to the exterior differential $d \wedge f$ [in a metric inpendent way]. In general, in dimension $n$, not necessarily $n = 4$, the lowering differential $\partial \,\lrcorner\, \mathbf{f}$ is metric independent and related to the exterior differential by

$$\partial \,\lrcorner\, \mathbf{f} = \pm[d \wedge (\mathbf{f} \,\lrcorner\, w^*)] \,\lrcorner\, \mathbf{w}$$

for an $n$-volume $\mathbf{w} \in \bigwedge^n V$ such that $w^* \lrcorner \mathbf{w} = 1$ for $w^* \in \bigwedge^n V^*$. The relation $(\partial \,\lrcorner\, \mathbf{f}) \,\lrcorner\, \mathbf{w} = d \wedge (\mathbf{f} \,\lrcorner\, w^*)$ requires identification of multivector-valued functions with differential forms, which is possible only in flat spaces, while multivectors cannot be differentiated on curved spaces. Such an identification can be carried out by lowering the coordinate-indices of a multivector by means of the metric tensor $g_{\mu\nu}$ (or raising the indices of a differential form by the inverse $g^{\mu\nu}$).

**Exercises**

Show that

1. $\underset{\sim}{\Psi} \mathbf{e}_0 \bar{\underset{\sim}{\Psi}} = \mathbf{J}$. 2. $\underset{\sim}{\Psi} \mathbf{e}_{12} \bar{\underset{\sim}{\Psi}} = \mathbf{S}$.
3. $\underset{\sim}{\Psi}(1 - i\mathbf{e}_0)(1 - i\mathbf{e}_{12}) \bar{\underset{\sim}{\Psi}} = K + iL$. 4. $\underset{\sim}{\phi} \mathbf{e}_0 \bar{\underset{\sim}{\phi}} = -L$.
5. $\underset{\sim}{\psi}_0 = -\underset{\sim}{\psi}_1 \mathbf{e}_{012}$, $\underset{\sim}{\psi}_1 = -\underset{\sim}{\psi}_0 \mathbf{e}_{012}$.
6. $\underline{\psi}_0 + \underline{\psi}_1 \mathbf{e}_{12} \in (\mathbb{C} \otimes \mathcal{C}\ell_{3,1}) \frac{1}{2}(1 + i\mathbf{e}_0)$.

Write $\text{flip}(u) = u_0 - u_1 \mathbf{e}_{12}$ and recall that the opposite-reverse of $\mathbf{A}\psi$ is $\mathbf{A}\underline{\hat{\psi}}$ (for a vector $\mathbf{A}$). Show that

7. $\text{flip}(\mathbf{A}\underline{\hat{\psi}}) = -\mathbf{A}\,\underset{\sim}{\psi} \mathbf{e}_{12}$ $[\Rightarrow \partial\,\underset{\sim}{\psi} = e\mathbf{A}\,\underset{\sim}{\psi} \mathbf{e}_{12} - m\,\underset{\sim}{\psi}]$.
8. $\mathbf{A}\psi$ and $\mathbf{A}\Phi = 4\,\text{Re}(\mathbf{A}\psi)$ correspond to $\mathbf{A}\Psi\gamma_0 = \text{even}(\mathbf{A}\Phi)$, and in the opposite algebra to $-\mathbf{A}\,\underset{\sim}{\Psi}\mathbf{e}_0$.
9. $4\,\underset{\sim}{\psi}_C\,\bar{\underset{\sim}{\psi}}{}_C^* = -4(\underset{\sim}{\psi}\,\bar{\underset{\sim}{\psi}}{}^*)^*$. [This means that $\mathbf{J}$, $\mathbf{S}$ are preserved under charge conjugation while $\Omega_1$, $\mathbf{K}$, $\Omega_2$ swap their signs – as in $\mathbb{R}^{1,3}$. This should be contrasted with Crumeyrolle (1990 p. 135, l. -9), who considers charge conjugation in conjunction with a scalar product of spinors induced by the reversion (composed with complex conjugation), in which case $\mathbf{S}$, $\mathbf{K}$ are preserved and $\Omega_1$, $\mathbf{J}$, $\Omega_2$ swap signs. Crumeyrolle's numerical results are not directly related to the Bjorken & Drell formulation of the Dirac theory, as he induces spinor spaces by totally isotropic subspaces of $\mathbb{C} \otimes \mathbb{R}^{3,1}$. To relate the results one must permute the primitive idempotents by an algebra automorphism of the Clifford algebra in such a way that dimension grades are mixed. The next exercise gives a hint on how the scalar product of spinors induced by the reversion (composed with complex conjugation) can be used to find the bilinear covariants.]
10. $K_\mu = 4\langle \tilde{\underset{\sim}{\psi}}{}^* \mathbf{e}_\mu \underset{\sim}{\psi} \rangle_1 \cdot \mathbf{e}_3$ (find similar formulas for $J_\mu$, $S_{\mu\nu}$).

In the next exercise we have a scalar product of spinors induced by the reversion alone without composing it with complex conjugation.

11. Take a Majorana spinor $\underset{\sim}{\psi} = \underset{\sim}{\psi}_C$ with bilinear covariants $\mathbf{J}$, $\mathbf{S} = \mathbf{J} \wedge \mathbf{s}$. Then the Weyl spinor $u = \frac{1}{2}(1 + i\mathbf{e}_{0123})\,\underset{\sim}{\psi}$ has charge conjugate $u_C = \frac{1}{2}(1 - i\mathbf{e}_{0123})\,\underset{\sim}{\psi}$ so that $\underset{\sim}{\psi} = u + u_C$. Show that

$$\mathbf{J}(u + u_C) = 0 \qquad (3.1.21)$$

$$\mathbf{s}(u + u_C) = -(u + u_C) \qquad (3.1.25)$$

$$4\langle iu\mathbf{e}_{13}\tilde{u}_C \rangle_1 = \tfrac{1}{2}\,\mathbf{J} \qquad (3.1.28)$$

$$\text{Re}(4iu\mathbf{e}_{13}\tilde{u}) = -\tfrac{1}{2}\,\mathbf{S} \qquad (3.1.29)$$

$$4\,\underset{\sim}{\psi}\mathbf{e}_{13}\,\tilde{\underset{\sim}{\psi}}\mathbf{e}_{0123} = \mathbf{J} + \mathbf{S} \qquad (3.1.30/31)$$

The numbering on the right refers to Benn & Tucker 1987 pp. 113-116. Try to work out a translation to their notation, and discuss the physical relevance of the connection between the Majorana and Weyl spinors. Hint: $u_C = \hat{u}^* \mathbf{e}_{13}$ while $u = \frac{1}{2}(1 + i\mathbf{e}_{0123}) \underset{\sim}{\psi}$. Benn & Tucker use the scalar product of spinors

$$(\underset{\sim}{\psi}, \underset{\sim}{\varphi}) \to \mathbf{e}_{13} \tilde{\underset{\sim}{\psi}} \underset{\sim}{\varphi} \in \underset{\sim}{f}(\mathbb{C} \otimes \mathcal{C}\ell_{3,1}) \underset{\sim}{f} \simeq \mathbb{C}.$$

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This material grew out of discussions with Bill Pezzaglia at a meeting in Banff, 1995.

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