Optimization - Existence

Let us consider the optimization problem

$$\text{Find} \quad \xi_* = \inf_{\xi \in H^1_\#((0,2\pi),\mathbb{R}^4)} \int_0^{2\pi} G\,\dot{\xi}(t)\cdot\dot{\xi}(t)\,dt$$

under the constraint

$$h_c \sum_{k\in\mathbb{N}_3} \left(\int_J \det(\xi(t)|\dot{\xi}(t)|\tau_{k+1}|\tau_{k+2})\,dt \right) f_k$$

$$+ h_\theta \sum_{k\in\mathbb{N}_3} \left(\int_J \det(\xi(t)|\dot{\xi}(t)|\tau_k|\tau_4)\,dt \right) f_{k+3}$$

$$= \delta p.$$

To simplify notation, let us denote by V the set of functions in $H^1_\#(J,\mathbb{R}^4)$, which satisfies the constraint.

1) T.s.: $V \neq \emptyset$.

Recall that $\sqrt{\det \Delta_g}\,(\Delta_h \Delta_g)^{-1}\,\delta p = \sum_{n\in\mathbb{N}} \frac{v_n \wedge u_n}{n}$,

where $\underline{v} := (v_n)_n$ and $\underline{u} := (u_n)_n$ are the transformed Fourier coefficients of $U^T\xi$. Since any $w \in \bigwedge^2 \mathbb{R}^4$ can be written as the sum of two simple bivectors, it is always possible to find such Fourier coefficient.

2) The energy functional $\mathcal{G}(\xi) := \int_J G\dot{\xi}(t)\cdot\dot{\xi}(t)dt$ is clearly bounded from below by 0 since G is positive definite. Hence,

$$\alpha = \inf_{\xi \in \bar{V}} \int_J G\dot{\xi}(t)\cdot\dot{\xi}(t)\,dt$$

exists, and we can find a sequence $(\xi_n)_n \subset \bar{V}$ such that

$$\mathcal{G}(\xi_n) \downarrow \alpha, \quad n \longrightarrow +\infty.$$

3) Note that $\mathcal{G}$ is a norm on $H^1_\#(J,\mathbb{R}^4)$. Indeed, by Poincaré's inequality, there exists some constant $C>0$ such that

$$\forall \xi \in H^1_\#(J,\mathbb{R}^4) : \|\xi\|_{L^2} \leq C\|\dot{\xi}\|_{L^2}.$$

If we denote K_- the smallest eigenvalue of G, one readily finds that

$$\|\xi\|^2_{H^1_\#} \leq (1+C^2)\|\dot{\xi}\|^2_{L^2} \leq \left(\frac{1+C^2}{K_-}\right)\int_J G\dot{\xi}\cdot\dot{\xi}\,dt.$$

Hence, we have in particular for n sufficiently large that

$$\|\xi\|^2_{H^1_\#} \leq 2\left(\frac{1+C^2}{K_-}\right)\alpha.$$

4) Since $(\xi_n) \subset V$ is bounded, there exists a weakly convergent subsequence

$$\xi_{n_k} \rightharpoonup \xi_\infty \quad \text{in } H^1_\#$$

<u>Q</u>: Why is V closed under weak convergence?

5) Since G is a norm and thus lower semi-continuous with respect to weak convergence, it follows that

$$\alpha \leq G(\xi_\infty) \leq G(\liminf_n \xi_n) = \alpha,$$

therefore ξ_∞ is the desired solution. $\square$.

~~4) Par équivalence des normes $\|\xi\|_{H^1}$ et $\|\dot{\xi}\|_{L^2}$, nous avons $\dot{\xi}_{n_k} \xrightarrow{L^2} \dot{\xi}_\infty$.~~

4) (The equivalence of the H^1-norm and the L^2-norm of the derivative implies that $\dot{\xi}_{n_k} \xrightarrow{L^2} \dot{\xi}_n$.) By the Rellich-Kondrachov Theorem, we have $\xi_{n_k} \xrightarrow{L^2} \xi_\infty$ for at least a subsequence.

So we have a product of a weakly convergent and a L^2 - convergent sequence as integrand, which implies that

$$\delta p = \int_J F(\xi_{n_k})\,\dot{\xi}_{n_k}\,dt \xrightarrow{k\to\infty} \delta p = \int_J F(\xi_\infty)\,\dot{\xi}_\infty\,dt.$$

We need the Theorem:

<u>Thm</u>: $\left.\begin{array}{l} u_n \xrightarrow{L^2} u_\infty \\ v_n \xrightarrow{L^2} v_\infty \end{array}\right\} \Rightarrow \int u_n v_n \xrightarrow{n\to\infty} \int u_\infty v_\infty.$

Proof. $(u_n)_n \subset L^2$ bounded due to weak convergence.

$$u_n v_n - u_\infty v_\infty = u_n(v_n - v_\infty) + (u_n - u_\infty) v_n$$

Hölder

$$\Rightarrow \left|\int (u_n v_n - u_\infty v_\infty)\right| \leq \underbrace{\|u_n\|_{L^2}}_{\text{bounded}} \underbrace{\|v_n - v_\infty\|_{L^2}}_{\to 0}$$

$$+ \underbrace{\left|\int (u_n - u_\infty) v_n\right|}_{\to 0}$$

□.