Práctica 3 - PLP

Philips

1er Cuatrimestre 2025

Deducción natural

$$\dfrac{\dfrac{}{\Gamma \vdash A}\,\text{Ax} \quad \dfrac{}{\Gamma \vdash A \Rightarrow B}\,\text{Ax}}{\dfrac{\Gamma \vdash B}{\Gamma' \vdash A \Rightarrow B}\,\Rightarrow\text{I}}\,\Rightarrow\text{E}$$

Ejercicio 5

I. Modus ponens relativizado: ?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q \Rightarrow \tau}\,\text{ax} \quad \dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q}\,\text{ax}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \tau}\,\Rightarrow\text{e}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q \vdash \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\rho \Rightarrow q \Rightarrow \tau \vdash (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\vdash (\rho \Rightarrow q \Rightarrow \tau) \Rightarrow (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}$$

II. Reducción al absurdo:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \bot), p \vdash p \Rightarrow \bot}\,\text{ax} \quad \dfrac{}{(p \Rightarrow \bot), p \vdash p}\,\text{ax}}{(p \Rightarrow \bot), p \vdash \bot}\,\Rightarrow\text{e}}{(p \Rightarrow \bot) \vdash \neg p}\,\neg\text{i}}{\vdash (p \Rightarrow \bot) \Rightarrow \neg p}\,\Rightarrow\text{i}$$

III. Introducción de la doble negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \quad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg\text{e}}{p \vdash \neg\neg p}\,\neg\text{i}}{\vdash p \Rightarrow \neg\neg p}\,\Rightarrow\text{i}$$

IV. Eliminación de la triple negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg\neg p, p \vdash \neg\neg p}\,\text{III} \quad \dfrac{}{\neg\neg\neg p, p \vdash \neg\neg\neg p}\,\text{ax}}{\neg\neg\neg p, p \vdash \bot}\,\neg\text{e}}{\neg\neg\neg p \vdash \neg p}\,\neg\text{i}}{\vdash \neg\neg\neg p \Rightarrow \neg p}\,\Rightarrow\text{i}$$

V. Contraposición:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \neg\sigma}\,\text{ax} \quad \dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p}\,\text{ax} \quad \dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p \Rightarrow \sigma}\,\text{ax}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \sigma}\,\Rightarrow\text{e}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \bot}\,\neg\text{e}}{(p \Rightarrow \sigma), \neg\sigma \vdash \neg p}\,\neg\text{i}}{(p \Rightarrow \sigma) \vdash \neg\sigma \Rightarrow \neg p}\,\Rightarrow\text{i}}{\vdash (p \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg p)}\,\Rightarrow\text{i}$$

VI. Adjunción: $((p \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (p \Rightarrow \sigma \Rightarrow \tau)$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \sigma}\,\text{ax} \quad \dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash p}\,\text{ax}}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma)}\,\wedge\text{i} \quad \dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma) \Rightarrow \tau}\,\text{ax}}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \tau}\,\Rightarrow\text{e}}{((p \wedge \sigma) \Rightarrow \tau), p \vdash \sigma \Rightarrow \tau}\,\Rightarrow\text{i}}{((p \wedge \sigma) \Rightarrow \tau) \vdash (p \Rightarrow \sigma \Rightarrow \tau)}\,\Rightarrow\text{i}}{\vdash ((p \wedge \sigma) \Rightarrow \tau) \Rightarrow (p \Rightarrow \sigma \Rightarrow \tau)}\,\Rightarrow\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash (p \wedge \sigma)}\,\text{ax}}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash \sigma}\,\wedge e_2}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma}\,\Rightarrow\text{i} \quad \dfrac{}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma \Rightarrow \tau}\,\text{ax}}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash \tau}\,\Rightarrow\text{e}}{(p \Rightarrow \sigma \Rightarrow \tau) \vdash ((p \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow\text{i}}{\vdash (p \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((p \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow\text{i}$$

VII. Ley de Morgan (I): $\neg(p \vee \sigma) \Leftrightarrow (\neg p \wedge \neg\sigma)$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \vee \sigma), \sigma \vdash \sigma}\,\text{ax}}{\neg(p \vee \sigma), \sigma \vdash p \vee \sigma}\,\vee i_2 \quad \dfrac{}{\neg(p \vee \sigma), \sigma \vdash \neg(p \vee \sigma)}\,\text{ax}}{\neg(p \vee \sigma), \sigma \vdash \bot}\,\neg\text{e}}{\neg(p \vee \sigma) \vdash \neg\sigma}\,\neg\text{i} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash p \vee \sigma}\,\vee i_1 \quad \dfrac{}{\Gamma \vdash \neg(p \vee \sigma)}\,\text{ax}}{\Gamma \equiv \neg(p \vee \sigma), p \vdash \bot}\,\neg\text{e}}{\neg(p \vee \sigma) \vdash \neg p}\,\neg\text{i}}{\neg(p \vee \sigma) \vdash \neg p \wedge \neg\sigma}\,\wedge\text{i}}{\vdash \neg(p \vee \sigma) \Rightarrow (\neg p \wedge \neg\sigma)}\,\Rightarrow\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \sigma \vdash \neg p \wedge \neg\sigma}\,\text{ax}}{\Gamma, \sigma \vdash \neg\sigma}\,\wedge e_2 \quad \dfrac{\dfrac{}{\Gamma, p \vdash \neg p \wedge \neg\sigma}\,\text{ax}}{\Gamma, p \vdash \neg p}\,\wedge e_1 \quad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma, p \vdash \bot}\,\neg\text{e} \quad \dfrac{}{\Gamma \vdash p \vee \sigma}\,\text{ax}}{\Gamma, \sigma \vdash \sigma}\,\text{ax}}{\Gamma, \sigma \vdash \bot}\,\neg\text{e}}{\Gamma \equiv (\neg p \wedge \neg\sigma), (p \vee \sigma) \vdash \bot}\,\vee\text{e}}{(\neg p \wedge \neg\sigma) \vdash \neg(p \vee \sigma)}\,\neg\text{i}}{\vdash (\neg p \wedge \neg\sigma) \Rightarrow \neg(p \vee \sigma)}\,\Rightarrow\text{i}$$

VIII. Ley de Morgan (II): $\neg(p \wedge \sigma) \Leftrightarrow (\neg p \vee \neg\sigma)$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\rho\wedge\sigma),\neg p\vdash\neg\rho}\,\text{ax}}{\neg(\rho\wedge\sigma),\neg p\vdash\neg\rho\vee\neg\sigma}\vee i_1 \quad \dfrac{}{\neg(\rho\wedge\sigma)\vdash p\vee\neg p}\,\text{LEM} \quad \dfrac{\text{Sigo este caso abajo}}{\Gamma\equiv\neg(\rho\wedge\sigma),p\vdash\neg\rho\vee\neg\sigma}\,(*)}{\neg(\rho\wedge\sigma)\vdash\neg\rho\vee\neg\sigma}\vee e}{\vdash\neg(\rho\wedge\sigma)\Rightarrow(\neg\rho\vee\neg\sigma)}\Rightarrow i$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\sigma\vdash p}\,\text{ax} \quad \dfrac{}{\Gamma,\sigma\vdash\sigma}\,\text{ax}}{\Gamma,\sigma\vdash\rho\wedge\sigma}\wedge i \quad \dfrac{}{\Gamma,\sigma\vdash\neg(\rho\wedge\sigma)}\,\text{ax}}{\Gamma,\sigma\vdash\bot}\neg e}{\Gamma,\sigma\vdash\neg\rho\vee\neg\sigma}\bot e \quad \dfrac{\dfrac{}{\Gamma,\neg\sigma\vdash\neg\sigma}\,\text{ax}}{\Gamma,\neg\sigma\vdash\neg\rho\vee\neg\sigma}\vee i_2 \quad \dfrac{}{\Gamma\vdash\sigma\vee\neg\sigma}\,\text{LEM}}{\Gamma\vdash\neg\rho\vee\neg\sigma\quad\{*\}}\vee e$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\neg\sigma,p\vdash p\wedge\sigma}\,\text{ax}}{\Gamma,\neg\sigma,p\vdash\sigma}\wedge e_2 \quad \dfrac{}{\Gamma,\neg\sigma,p\vdash\neg\sigma}\,\text{ax}}{\Gamma,\neg\sigma,p\vdash\bot}\neg\text{e}}{\Gamma,\neg\sigma\vdash\neg p}\neg\text{i} \quad \dfrac{}{\Gamma,\neg p\vdash\neg p}\,\text{ax} \quad \dfrac{}{\Gamma\vdash\neg p\vee\neg\sigma}\,\text{ax}}{\Gamma\vdash\neg p}\vee\text{e} \quad \dfrac{\dfrac{}{\Gamma\vdash p\wedge\sigma}\,\text{ax}}{\Gamma\vdash p}\wedge e_1}{\Gamma\equiv(\neg\rho\vee\neg\sigma),(\rho\wedge\sigma)\vdash\bot}\neg\text{i}}{(\neg\rho\vee\neg\sigma)\vdash\neg(\rho\wedge\sigma)}\neg\text{i}}{\vdash(\neg\rho\vee\neg\sigma)\Rightarrow\neg(\rho\wedge\sigma)}\Rightarrow\text{i}$$

IX. Conmutatividad ($\wedge$):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p\wedge\sigma)\vdash(p\wedge\sigma)}\,\text{ax}}{(p\wedge\sigma)\vdash p}\wedge e_1 \quad \dfrac{\dfrac{}{(p\wedge\sigma)\vdash(p\wedge\sigma)}\,\text{ax}}{(p\wedge\sigma)\vdash\sigma}\wedge e_2}{(p\wedge\sigma)\vdash(\sigma\wedge p)}\wedge\text{i}}{\vdash(p\wedge\sigma)\Rightarrow(\sigma\wedge p)}\Rightarrow\text{i}$$

X. Asociatividad ($\wedge$): $((p\wedge\sigma)\wedge\tau)\Leftrightarrow(p\wedge(\sigma\wedge\tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash\tau}\wedge e_2 \quad \dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge\sigma)}\wedge e_2}{((p\wedge\sigma)\wedge\tau)\vdash\sigma}\wedge\text{i} \quad \dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge\sigma)}\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash(\sigma\wedge\tau)}\wedge\text{i} \quad \dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash p}\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge(\sigma\wedge\tau))}\wedge\text{i}}{\vdash((p\wedge\sigma)\wedge\tau)\Rightarrow(p\wedge(\sigma\wedge\tau))}\Rightarrow\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash(p\wedge(\sigma\wedge\tau))}\,\text{ax}}{\Gamma\vdash(\sigma\wedge\tau)}\wedge e_2}{\Gamma\vdash\sigma}\wedge e_1 \quad \dfrac{\dfrac{}{\Gamma\vdash(p\wedge(\sigma\wedge\tau))}\,\text{ax}}{\Gamma\vdash p}\wedge e_1}{\Gamma\vdash p\wedge\sigma}\wedge\text{i} \quad \dfrac{\dfrac{}{\Gamma\vdash p\wedge(\sigma\wedge\tau)}\,\text{ax}}{\Gamma\vdash(\sigma\wedge\tau)}\wedge e_2}{\Gamma\vdash\tau}\wedge e_2}{\Gamma\equiv(p\wedge(\sigma\wedge\tau))\vdash((p\wedge\sigma)\wedge\tau)}\wedge\text{i}}{\vdash(p\wedge(\sigma\wedge\tau))\Rightarrow((p\wedge\sigma)\wedge\tau)}\Rightarrow\text{i}$$

XI. Conmutatividad ($\vee$)

$$\dfrac{\dfrac{\dfrac{}{(p \vee \sigma), \sigma \vdash \sigma}\,\text{ax}}{(p \vee \sigma), \sigma \vdash (\sigma \vee p)} \vee i_1 \quad \dfrac{\dfrac{}{(p \vee \sigma), p \vdash p}\,\text{ax}}{(p \vee \sigma), p \vdash (\sigma \vee p)} \vee i_2 \quad \dfrac{}{(p \vee \sigma) \vdash (p \vee \sigma)}\,\text{ax}}{\dfrac{(p \vee \sigma) \vdash (\sigma \vee p)}{\vdash (p \vee \sigma) \Rightarrow (\sigma \vee p)} \Rightarrow \text{i}} \vee \text{e}$$

XII. Asociatividad ($\vee$): $((p \vee \sigma) \vee \tau) \Leftrightarrow (p \vee (\sigma \vee \tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, \sigma \vdash \sigma}\,\text{ax}}{\Delta, \sigma \vdash (\sigma \vee \tau)} \vee i_1}{\Delta, \sigma \vdash p \vee (\sigma \vee \tau)} \vee i_2 \quad \dfrac{\dfrac{}{\Delta, p \vdash p}\,\text{ax}}{\Delta, p \vdash p \vee (\sigma \vee \tau)} \vee i_1 \quad \dfrac{}{\Delta \vdash p \vee \sigma}\,\text{ax}}{\Delta \equiv \Gamma, p \vee \sigma \vdash p \vee (\sigma \vee \tau)} \vee \text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \tau}\,\text{ax}}{\Gamma, \tau \vdash \sigma \vee \tau} \vee i_2}{\Gamma, \tau \vdash p \vee (\sigma \vee \tau)} \vee i_2 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\dfrac{\Gamma \equiv ((p \vee \sigma) \vee \tau) \vdash (p \vee (\sigma \vee \tau))}{\vdash ((p \vee \sigma) \vee \tau) \Rightarrow (p \vee (\sigma \vee \tau))} \Rightarrow \text{i}} \vee \text{e}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, \sigma \vdash \sigma}\,\text{ax}}{\Delta, \sigma \vdash p \vee \sigma} \vee i_2}{\Delta, \sigma \vdash (p \vee \sigma) \vee \tau} \vee i_1 \quad \dfrac{\dfrac{}{\Delta, \tau \vdash \tau}\,\text{ax}}{\Delta, \tau \vdash (p \vee \sigma) \vee \tau} \vee i_2 \quad \dfrac{}{\Delta \vdash \sigma \vee \tau}\,\text{ax}}{\Delta \equiv \Gamma, \sigma \vee \tau \vdash (p \vee \sigma) \vee \tau} \vee \text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{\Gamma, p \vdash (p \vee \sigma)} \vee i_1}{\Gamma, p \vdash (p \vee \sigma) \vee \tau} \vee i_1 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\dfrac{\Gamma \equiv (p \vee (\sigma \vee \tau)) \vdash ((p \vee \sigma) \vee \tau)}{\vdash (p \vee (\sigma \vee \tau)) \Rightarrow ((p \vee \sigma) \vee \tau)} \Rightarrow \text{i}} \vee \text{e}$$