Práctica 3 - Lógica intuicionista y clásica - PLP

Philips

1er Cuatrimestre 2025

Extra

Spamear **PBC** a la hora de demostrar en lógica clásica.

Siempre chequear, en todos los pasos de un procedimiento, si lo que estoy probando tiene **sentido lógico**.

Intuición de la disyunción ($\vee$)

A la hora de usar $\vee e$ recordar usar la disyunción con algo que aparezca en el contexto (así puedo probar la primera regla directamente o en pocos pasos), y por sobre todo, algo con lo que pueda construir lo que quiera probar.

Modus Tollens

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, \neg Q, P \vdash P \Rightarrow Q}\,\text{ax} \qquad \dfrac{}{P \Rightarrow Q, \neg Q, P \vdash P}\,\text{ax}}{P \Rightarrow Q, \neg Q, P \vdash Q}\,\Rightarrow\text{e} \qquad \dfrac{}{P \Rightarrow Q, \neg Q, P \vdash \neg Q}\,\text{ax}}{P \Rightarrow Q, \neg Q, P \vdash \bot}}{P \Rightarrow Q, \neg Q \vdash \neg P}\,\neg\text{i}$$

Ejercicio 5

Demostrar en deducción natural que las siguientes fórmulas son teoremas **sin usar principios de razonamiento clásicos** salvo que se indique lo contrario. Recordemos que una fórmula σ es un teorema si y sólo si vale $\vdash \sigma$:

I. Modus ponens relativizado: ?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q \Rightarrow \tau}\,\text{ax} \qquad \dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q}\,\text{ax}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \tau}\,\Rightarrow\text{e}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q \vdash \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\rho \Rightarrow q \Rightarrow \tau \vdash (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\vdash (\rho \Rightarrow q \Rightarrow \tau) \Rightarrow (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}$$

II. Reducción al absurdo:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \bot), p \vdash p \Rightarrow \bot}\,\text{ax} \qquad \dfrac{}{(p \Rightarrow \bot), p \vdash p}\,\text{ax}}{(p \Rightarrow \bot), p \vdash \bot}\,\Rightarrow\text{e}}{(p \Rightarrow \bot) \vdash \neg p}\,\neg\text{i}}{\vdash (p \Rightarrow \bot) \Rightarrow \neg p}\,\Rightarrow\text{i}$$

III. Introducción de la doble negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg\text{e}}{p \vdash \neg\neg p}\,\neg\text{i}}{\vdash p \Rightarrow \neg\neg p}\,\Rightarrow\text{i}$$

IV. Eliminación de la triple negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg\neg p, p \vdash \neg\neg p}\,\text{III} \qquad \dfrac{}{\neg\neg\neg p, p \vdash \neg\neg\neg p}\,\text{ax}}{\neg\neg\neg p, p \vdash \bot}\,\neg\text{e}}{\neg\neg\neg p \vdash \neg p}\,\neg\text{i}}{\vdash \neg\neg\neg p \Rightarrow \neg p}\,\Rightarrow\text{i}$$

V. Contraposición:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \neg\sigma}\,\text{ax} \qquad \dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p}\,\text{ax} \qquad \dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p \Rightarrow \sigma}\,\text{ax}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \sigma}\,\Rightarrow\text{e}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \bot}\,\neg\text{e}}{(p \Rightarrow \sigma), \neg\sigma \vdash \neg p}\,\neg\text{i}}{(p \Rightarrow \sigma) \vdash \neg\sigma \Rightarrow \neg p}\,\Rightarrow\text{i}}{\vdash (p \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg p)}\,\Rightarrow\text{i}$$

VI. Adjunción: $((p \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (p \Rightarrow \sigma \Rightarrow \tau)$

$(\Rightarrow)$ ida

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash p$}
\RightLabel{$\wedge$i}
\BinaryInfC{$((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma) \Rightarrow \tau$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \tau$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$((p \wedge \sigma) \Rightarrow \tau), p \vdash \sigma \Rightarrow \tau$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$((p \wedge \sigma) \Rightarrow \tau) \vdash (p \Rightarrow \sigma \Rightarrow \tau)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash ((p \wedge \sigma) \Rightarrow \tau) \Rightarrow (p \Rightarrow \sigma \Rightarrow \tau)$}
\end{prooftree}
$$

$(\Rightarrow)$ vuelta

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash (p \wedge \sigma)$}
\RightLabel{$\wedge e_2$}
\UnaryInfC{$(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash \sigma$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma \Rightarrow \tau$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash \tau$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$(p \Rightarrow \sigma \Rightarrow \tau) \vdash ((p \wedge \sigma) \Rightarrow \tau)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash (p \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((p \wedge \sigma) \Rightarrow \tau)$}
\end{prooftree}
$$

VII. Ley de Morgan (I): $\neg(p \vee \sigma) \Leftrightarrow (\neg p \wedge \neg \sigma)$

$(\Rightarrow)$ ida

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg(p \vee \sigma), \sigma \vdash \sigma$}
\RightLabel{$\vee i_2$}
\UnaryInfC{$\neg(p \vee \sigma), \sigma \vdash p \vee \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\neg(p \vee \sigma), \sigma \vdash \neg(p \vee \sigma)$}
\RightLabel{$\neg$e}
\BinaryInfC{$\neg(p \vee \sigma), \sigma \vdash \bot$}
\RightLabel{$\neg$i}
\UnaryInfC{$\neg(p \vee \sigma) \vdash \neg \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\vee i_1$}
\UnaryInfC{$\Gamma \vdash p \vee \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \neg(p \vee \sigma)$}
\RightLabel{$\neg$e}
\BinaryInfC{$\Gamma \equiv \neg(p \vee \sigma), p \vdash \bot$}
\RightLabel{$\neg$i}
\UnaryInfC{$\neg(p \vee \sigma) \vdash \neg p$}
\RightLabel{$\wedge$i}
\BinaryInfC{$\neg(p \vee \sigma) \vdash \neg p \wedge \neg \sigma$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash \neg(p \vee \sigma) \Rightarrow (\neg p \wedge \neg \sigma)$}
\end{prooftree}
$$

$(\Rightarrow)$ vuelta

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \sigma \vdash \neg p \wedge \neg \sigma$}
\RightLabel{$\wedge e_2$}
\UnaryInfC{$\Gamma, \sigma \vdash \neg \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \sigma \vdash \sigma$}
\RightLabel{$\neg$e}
\BinaryInfC{$\Gamma, \sigma \vdash \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, p \vdash \neg p \wedge \neg \sigma$}
\RightLabel{$\wedge e_1$}
\UnaryInfC{$\Gamma, p \vdash \neg p$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p$}
\RightLabel{$\neg$e}
\BinaryInfC{$\Gamma, p \vdash \bot$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash p \vee \sigma$}
\RightLabel{$\vee$e}
\TrinaryInfC{$\Gamma \equiv (\neg p \wedge \neg \sigma), (p \vee \sigma) \vdash \bot$}
\RightLabel{$\neg$i}
\UnaryInfC{$(\neg p \wedge \neg \sigma) \vdash \neg(p \vee \sigma)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash (\neg p \wedge \neg \sigma) \Rightarrow \neg(p \vee \sigma)$}
\end{prooftree}
$$

VIII. Ley de Morgan (II): $\neg(p \wedge \sigma) \Leftrightarrow (\neg p \vee \neg\sigma)$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(\rho\wedge\sigma),\neg p \vdash \neg\rho}\,\text{ax}}{\neg(\rho\wedge\sigma),\neg p \vdash \neg\rho\vee\neg\sigma}\,\vee i_1 \qquad \dfrac{}{\neg(\rho\wedge\sigma)\vdash p\vee\neg p}\,\text{LEM} \qquad \dfrac{\text{Sigo este caso abajo}}{\Gamma \equiv \neg(\rho\wedge\sigma), p \vdash \neg\rho\vee\neg\sigma}\,(*)}{\neg(\rho\wedge\sigma)\vdash\neg\rho\vee\neg\sigma}\,\vee e}{\vdash \neg(\rho\wedge\sigma)\Rightarrow(\neg\rho\vee\neg\sigma)}\,\Rightarrow i$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\sigma\vdash p}\,\text{ax} \quad \dfrac{}{\Gamma,\sigma\vdash\sigma}\,\text{ax}}{\Gamma,\sigma\vdash\rho\wedge\sigma}\,\wedge i \quad \dfrac{}{\Gamma,\sigma\vdash\neg(\rho\wedge\sigma)}\,\text{ax}}{\Gamma,\sigma\vdash\bot}\,\neg e}{\Gamma,\sigma\vdash\neg\rho\vee\neg\sigma}\,\bot\, e \qquad \dfrac{\dfrac{}{\Gamma,\neg\sigma\vdash\neg\sigma}\,\text{ax}}{\Gamma,\neg\sigma\vdash\neg\rho\vee\neg\sigma}\,\vee i_2 \qquad \dfrac{}{\Gamma\vdash\sigma\vee\neg\sigma}\,\text{LEM}}{\Gamma\vdash\neg\rho\vee\neg\sigma \quad \{*\}}\,\vee e$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\neg\sigma,p\vdash p\wedge\sigma}\,\text{ax}}{\Gamma,\neg\sigma,p\vdash\sigma}\,\wedge e_2 \quad \dfrac{}{\Gamma,\neg\sigma,p\vdash\neg\sigma}\,\text{ax}}{\Gamma,\neg\sigma,p\vdash\bot}\,\neg\text{e}}{\Gamma,\neg\sigma\vdash\neg p}\,\neg\text{i} \quad \dfrac{}{\Gamma,\neg p\vdash\neg p}\,\text{ax} \quad \dfrac{}{\Gamma\vdash\neg p\vee\neg\sigma}\,\text{ax}}{\Gamma\vdash\neg p}\,\vee\text{e} \qquad \dfrac{\dfrac{}{\Gamma\vdash p\wedge\sigma}\,\text{ax}}{\Gamma\vdash p}\,\wedge e_1}{\Gamma\equiv(\neg\rho\vee\neg\sigma),(\rho\wedge\sigma)\vdash\bot}\,\neg\text{e}}{(\neg\rho\vee\neg\sigma)\vdash\neg(\rho\wedge\sigma)}\,\neg\text{i}}{\vdash(\neg\rho\vee\neg\sigma)\Rightarrow\neg(\rho\wedge\sigma)}\,\Rightarrow\text{i}$$

IX. Conmutatividad ($\wedge$):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p\wedge\sigma)\vdash(p\wedge\sigma)}\,\text{ax}}{(p\wedge\sigma)\vdash p}\,\wedge e_1 \qquad \dfrac{\dfrac{}{(p\wedge\sigma)\vdash(p\wedge\sigma)}\,\text{ax}}{(p\wedge\sigma)\vdash\sigma}\,\wedge e_2}{(p\wedge\sigma)\vdash(\sigma\wedge p)}\,\wedge\text{i}}{\vdash(p\wedge\sigma)\Rightarrow(\sigma\wedge p)}\,\Rightarrow\text{i}$$

X. Asociatividad ($\wedge$): $((p\wedge\sigma)\wedge\tau)\Leftrightarrow(p\wedge(\sigma\wedge\tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash\tau}\,\wedge e_2 \quad \dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge\sigma)}\,\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash\sigma}\,\wedge e_2}{((p\wedge\sigma)\wedge\tau)\vdash(\sigma\wedge\tau)}\,\wedge\text{i} \quad \dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge\sigma)}\,\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash p}\,\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge(\sigma\wedge\tau))}\,\wedge\text{i}}{\vdash((p\wedge\sigma)\wedge\tau)\Rightarrow(p\wedge(\sigma\wedge\tau))}\,\Rightarrow\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash(p\wedge(\sigma\wedge\tau))}\,\text{ax}}{\Gamma\vdash(\sigma\wedge\tau)}\,\wedge e_2}{\Gamma\vdash\sigma}\,\wedge e_1 \quad \dfrac{\dfrac{}{\Gamma\vdash(p\wedge(\sigma\wedge\tau))}\,\text{ax}}{\Gamma\vdash p}\,\wedge e_1}{\Gamma\vdash p\wedge\sigma}\,\wedge\text{i} \quad \dfrac{\dfrac{\dfrac{}{\Gamma\vdash p\wedge(\sigma\wedge\tau)}\,\text{ax}}{\Gamma\vdash(\sigma\wedge\tau)}\,\wedge e_2}{\Gamma\vdash\tau}\,\wedge e_2}{\Gamma\equiv(p\wedge(\sigma\wedge\tau))\vdash((p\wedge\sigma)\wedge\tau)}\,\wedge\text{i}}{\vdash(p\wedge(\sigma\wedge\tau))\Rightarrow((p\wedge\sigma)\wedge\tau)}\,\Rightarrow\text{i}$$

XI. Conmutatividad ($\lor$)

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p\lor\sigma),\sigma\vdash\sigma}\,\text{ax}}{(p\lor\sigma),\sigma\vdash(\sigma\lor p)}\,\lor i_1 \qquad \dfrac{\dfrac{}{(p\lor\sigma),p\vdash p}\,\text{ax}}{(p\lor\sigma),p\vdash(\sigma\lor p)}\,\lor i_2 \qquad \dfrac{}{(p\lor\sigma)\vdash(p\lor\sigma)}\,\text{ax}}{(p\lor\sigma)\vdash(\sigma\lor p)}\,\lor\text{e}}{\vdash(p\lor\sigma)\Rightarrow(\sigma\lor p)}\,\Rightarrow\!\text{i}$$

XII. Asociatividad ($\lor$): $((p\lor\sigma)\lor\tau)\Leftrightarrow(p\lor(\sigma\lor\tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta,\sigma\vdash\sigma}\,\text{ax}}{\Delta,\sigma\vdash(\sigma\lor\tau)}\,\lor i_1}{\Delta,\sigma\vdash p\lor(\sigma\lor\tau)}\,\lor i_2 \quad \dfrac{\dfrac{}{\Delta,p\vdash p}\,\text{ax}}{\Delta,p\vdash p\lor(\sigma\lor\tau)}\,\lor i_1 \quad \dfrac{}{\Delta\vdash p\lor\sigma}\,\text{ax}}{\Delta\equiv\Gamma,p\lor\sigma\vdash p\lor(\sigma\lor\tau)}\,\lor\text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma,\tau\vdash\tau}\,\text{ax}}{\Gamma,\tau\vdash\sigma\lor\tau}\,\lor i_2}{\Gamma,\tau\vdash p\lor(\sigma\lor\tau)}\,\lor i_2 \quad \dfrac{}{\Gamma\vdash\Gamma}\,\text{ax}}{\Gamma\equiv((p\lor\sigma)\lor\tau)\vdash(p\lor(\sigma\lor\tau))}\,\lor\text{e}}{\vdash((p\lor\sigma)\lor\tau)\Rightarrow(p\lor(\sigma\lor\tau))}\,\Rightarrow\!\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta,\sigma\vdash\sigma}\,\text{ax}}{\Delta,\sigma\vdash p\lor\sigma}\,\lor i_2}{\Delta,\sigma\vdash(p\lor\sigma)\lor\tau}\,\lor i_1 \quad \dfrac{\dfrac{}{\Delta,\tau\vdash\tau}\,\text{ax}}{\Delta,\tau\vdash(p\lor\sigma)\lor\tau}\,\lor i_2 \quad \dfrac{}{\Delta\vdash\sigma\lor\tau}\,\text{ax}}{\Delta\equiv\Gamma,\sigma\lor\tau\vdash(p\lor\sigma)\lor\tau}\,\lor\text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma,p\vdash p}\,\text{ax}}{\Gamma,p\vdash(p\lor\sigma)}\,\lor i_1}{\Gamma,p\vdash(p\lor\sigma)\lor\tau}\,\lor i_1 \quad \dfrac{}{\Gamma\vdash\Gamma}\,\text{ax}}{\Gamma\equiv(p\lor(\sigma\lor\tau))\vdash((p\lor\sigma)\lor\tau)}\,\lor\text{e}}{\vdash(p\lor(\sigma\lor\tau))\Rightarrow((p\lor\sigma)\lor\tau)}\,\Rightarrow\!\text{i}$$

Ejercicio 6

Demostrar en deducción natural que vale $\vdash \sigma$ para cada una de las siguientes fórmulas. Para estas fórmulas es imprescindible usar **lógica clásica**:

I. Absurdo clásico:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\neg\tau \Rightarrow\bot), \neg\tau \vdash \neg\tau}\text{ax} \qquad \dfrac{}{(\neg\tau \Rightarrow\bot), \neg\tau \vdash (\neg\tau \Rightarrow\bot)}\text{ax}}{(\neg\tau \Rightarrow\bot), \neg\tau \vdash\bot}\Rightarrow\text{e}}{(\neg\tau \Rightarrow\bot) \vdash \tau}\text{PBC}}{\vdash (\neg\tau \Rightarrow\bot) \Rightarrow \tau}\Rightarrow\text{i}$$

II. Ley de Peirce:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \neg\tau}\text{ax} \quad \dfrac{}{\Gamma, \tau \vdash \tau}\text{ax}}{\Gamma, \tau \vdash \bot}\neg\text{e}}{\Gamma, \tau \vdash p}\bot\text{e}}{\Gamma \vdash (\tau \Rightarrow p)}\Rightarrow\text{i} \qquad \dfrac{}{\Gamma \vdash (\tau \Rightarrow p) \Rightarrow \tau}\text{ax}}{\Gamma \vdash \tau}\Rightarrow\text{e} \qquad \dfrac{}{\Gamma \vdash \neg\tau}\text{ax}}{\Gamma \equiv ((\tau \Rightarrow p) \Rightarrow \tau), \neg\tau \vdash \bot}\neg\text{e}}{((\tau \Rightarrow p) \Rightarrow \tau) \vdash \tau}\text{PBC}}{\vdash ((\tau \Rightarrow p) \Rightarrow \tau) \Rightarrow \tau}\Rightarrow\text{i}$$

III. Tercero excluido:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \tau}\text{ax}}{\Gamma, \tau \vdash (\tau \vee \neg\tau)}\vee i_1 \quad \dfrac{}{\Gamma, \tau \vdash \Gamma}\text{ax}}{\Gamma, \tau \vdash \bot}\neg\text{e}}{\Gamma \vdash \neg\tau}\neg\text{i}}{\Gamma \vdash (\tau \vee \neg\tau)}\vee i_2 \qquad \dfrac{}{\Gamma, \tau \vdash \Gamma}\text{ax}}{\Gamma, \tau \vdash \bot}\neg\text{e}}{\Gamma \vdash \neg\tau}\neg\ \text{i}}{\Gamma \vdash (\tau \vee \neg\tau)}\vee i_2 \qquad \dfrac{}{\Gamma \vdash \Gamma}\text{ax}}{\Gamma \equiv \neg(\tau \vee \neg\tau) \vdash \bot}\neg\text{e}}{\vdash \tau \vee \neg\tau}\text{PBC}$$

IV. Consecuencia milagrosa:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \tau}\text{ax} \quad \dfrac{}{\Gamma \vdash \neg\tau \Rightarrow \tau}\text{ax}}{\Gamma \vdash \neg\tau}\Rightarrow\text{e} \qquad \dfrac{}{\Gamma \vdash \neg\tau}\text{ax}}{\Gamma \equiv (\neg\tau \Rightarrow \tau), \neg\tau \vdash \bot}\neg\text{e}}{(\neg\tau \Rightarrow \tau) \vdash \tau}\text{PBC}}{\vdash (\neg\tau \Rightarrow \tau) \Rightarrow \tau}\Rightarrow\text{i}$$

V. Contraposición clásica:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg p}\text{ax} \quad \dfrac{}{\Gamma \vdash \neg p \Rightarrow \neg\tau}\text{ax}}{\Gamma \vdash \neg\tau}\Rightarrow\text{e} \qquad \dfrac{}{\Gamma \vdash \tau}\text{ax}}{\Gamma \equiv (\neg p \Rightarrow \neg\tau), \tau, \neg p \vdash \bot}\neg\text{e}}{(\neg p \Rightarrow \neg\tau), \tau \vdash p}\text{PBC}}{(\neg p \Rightarrow \neg\tau) \vdash (\tau \Rightarrow p)}\Rightarrow\text{i}}{\vdash (\neg p \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow p)}\Rightarrow\text{i}$$

VI. Análisis de casos:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg p}\,\text{ax} \quad \dfrac{}{\Gamma \vdash \tau \Rightarrow p}\,\text{ax}}{\Gamma \vdash \neg \tau}\,\text{MT} \quad \dfrac{}{\Gamma \vdash \neg \tau \Rightarrow p}\,\text{ax}}{\Gamma \vdash p}\,{\Rightarrow}\text{e} \quad \dfrac{}{\Gamma \vdash \neg p}\,\text{ax}}{\Gamma \equiv (\tau \Rightarrow p), (\neg \tau \Rightarrow p), \neg p \vdash \bot}\,\neg\text{e}}{(\tau \Rightarrow p), (\neg \tau \Rightarrow p) \vdash p}\,\text{PBC}}{(\tau \Rightarrow p) \vdash (\neg \tau \Rightarrow p) \Rightarrow p}\,{\Rightarrow}\text{i}}{\vdash (\tau \Rightarrow p) \Rightarrow (\neg \tau \Rightarrow p) \Rightarrow p}\,{\Rightarrow}\text{i}
$$

VII. Implicación vs. disyunción: $(\tau \Rightarrow p) \Leftrightarrow (\neg \tau \lor p)$

($\Rightarrow$) ida

$$
\dfrac{}{\vdash (\tau \Rightarrow p) \Rightarrow (\neg \tau \lor p)}
$$

($\Rightarrow$) vuelta

$$
\dfrac{}{\vdash (\neg \tau \lor p) \Rightarrow (\tau \Rightarrow p)}
$$

Ejercicio 9

Demostrar las siguientes tautologías utilizando deducción natural.

I. $(P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$

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\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow (P \Rightarrow Q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash Q$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)) \vdash (P \Rightarrow Q)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash (P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$}
\end{prooftree}
$$

II. Voy a usar deducción natural clásica: $(P \Rightarrow Q) \Rightarrow ((\neg P \Rightarrow Q) \Rightarrow Q)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg Q, \neg P \vdash \neg P$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg Q, \neg P \vdash \neg P \Rightarrow Q$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$\Gamma, \neg Q, \neg P \vdash Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg Q, \neg P \vdash \neg Q$}
\RightLabel{$\neg$e}
\BinaryInfC{$\Gamma, \neg Q, \neg P \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\Gamma, \neg Q \vdash P$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg Q \vdash P \Rightarrow Q$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$\Gamma, \neg Q \vdash Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, \neg Q \vdash \neg Q$}
\RightLabel{$\neg$e}
\BinaryInfC{$\Gamma, \neg Q \vdash \bot$}
\RightLabel{PBC}
\UnaryInfC{$\Gamma \equiv (P \Rightarrow Q), (\neg P \Rightarrow Q) \vdash Q$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$(P \Rightarrow Q) \vdash (\neg P \Rightarrow Q) \Rightarrow Q$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash (P \Rightarrow Q) \Rightarrow ((\neg P \Rightarrow Q) \Rightarrow Q)$}
\end{prooftree}
$$

Ejercicio 10

Demostrar las siguientes tautologías utilizando deducción natural:

I.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow (P \Rightarrow Q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash P$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$(P \Rightarrow (P \Rightarrow Q)), P \vdash Q$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$(P \Rightarrow (P \Rightarrow Q)) \vdash (P \Rightarrow Q)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash (P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$}
\end{prooftree}
$$

II.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash R \wedge Q$}
\RightLabel{$\wedge e_1$}
\UnaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash R$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash R \Rightarrow \neg Q$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash \neg Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash R \wedge Q$}
\RightLabel{$\wedge e_2$}
\UnaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash Q$}
\RightLabel{$\neg$e}
\BinaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash \bot$}
\RightLabel{$\bot$e}
\UnaryInfC{$(R \Rightarrow \neg Q), (R \wedge Q) \vdash P$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$(R \Rightarrow \neg Q) \vdash ((R \wedge Q) \Rightarrow P)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash (R \Rightarrow \neg Q) \Rightarrow ((R \wedge Q) \Rightarrow P)$}
\end{prooftree}
$$

III.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma, P \vdash P \wedge Q$}
\RightLabel{$\wedge e_2$}
\UnaryInfC{$\Gamma, P \vdash Q$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\Gamma \vdash (P \Rightarrow Q)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$\Gamma \vdash R \Rightarrow \neg Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash R \wedge Q$}
\RightLabel{$\wedge e_1$}
\UnaryInfC{$\Gamma \vdash R$}
\RightLabel{$\Rightarrow$e}
\BinaryInfC{$\Gamma \vdash \neg Q$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash R \wedge Q$}
\RightLabel{$\wedge e_2$}
\UnaryInfC{$\Gamma \vdash Q$}
\RightLabel{$\bot$e}
\BinaryInfC{$\Gamma \equiv ((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)), (R \wedge Q) \vdash \bot$}
\RightLabel{$\neg$i}
\UnaryInfC{$((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \vdash \neg (R \wedge Q)$}
\RightLabel{$\Rightarrow$i}
\UnaryInfC{$\vdash ((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \Rightarrow \neg (R \wedge Q)$}
\end{prooftree}
$$

Ejercicio 11

Probar que los siguientes secuentes son válidos sin usar principios de razonamiento clásicos:

I.

$$\dfrac{\dfrac{\dfrac{}{(P\land Q)\land R, S\land T\vdash S\land T}\,\text{ax}}{(P\land Q)\land R, S\land T\vdash S}\,\land e_1 \qquad \dfrac{\dfrac{\dfrac{}{(P\land Q)\land R, S\land T\vdash (P\land Q)\land R}\,\text{ax}}{(P\land Q)\land R, S\land T\vdash P\land Q}\,\land e_1}{(P\land Q)\land R, S\land T\vdash Q}\,\land e_2}{(P\land Q)\land R, S\land T\vdash Q\land S}\,\land\text{i}$$

II.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(P\land Q)\land R\vdash (P\land Q)\land R}\,\text{ax}}{(P\land Q)\land R\vdash R}\,\land e_2 \qquad \dfrac{\dfrac{\dfrac{}{(P\land Q)\land R\vdash P(\land Q)\land R}\,\text{ax}}{(P\land Q)\land R\vdash P(\land Q)}\,\land e_1}{(P\land Q)\land R\vdash Q}\,\land e_2}{(P\land Q)\land R\vdash (Q\land R)}\,\land\text{i} \qquad \dfrac{\dfrac{\dfrac{}{(P\land Q)\land R\vdash (P\land Q)\land R}\,\text{ax}}{(P\land Q)\land R\vdash (P\land Q)}\,\land e_1}{(P\land Q)\land R\vdash P}\,\land e_1}{(P\land Q)\land R\vdash P\land (Q\land R)}\,\land\text{i}$$

III.

$$\dfrac{\dfrac{\dfrac{}{P\Rightarrow (P\Rightarrow Q), P\vdash P\Rightarrow (P\Rightarrow Q)}\,\text{ax} \qquad \dfrac{}{P\Rightarrow (P\Rightarrow Q), P\vdash P}\,\text{ax}}{P\Rightarrow (P\Rightarrow Q), P\vdash P\Rightarrow Q}\,{\Rightarrow}\text{e} \qquad \dfrac{}{P\Rightarrow (P\Rightarrow Q), P\vdash P}\,\text{ax}}{P\Rightarrow (P\Rightarrow Q), P\vdash Q}\,{\Rightarrow}\text{e}$$

IV.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash Q\Rightarrow (P\Rightarrow R)}\,\text{ax} \qquad \dfrac{}{\Gamma\vdash Q}\,\text{ax}}{\Gamma\vdash P\Rightarrow R}\,{\Rightarrow}\text{e} \qquad \dfrac{}{\Gamma\vdash P}\,\text{ax}}{\Gamma\vdash R}\,{\Rightarrow}\text{e} \qquad \dfrac{}{\Gamma\vdash\neg R}\,\text{ax}}{\Gamma\equiv Q\Rightarrow (P\Rightarrow R), \neg R, Q, P\vdash\bot}\,\neg\text{e}}{Q\Rightarrow (P\Rightarrow R), \neg R, Q\vdash\neg P}\,\neg\text{i}$$

V.

$$\dfrac{\dfrac{\dfrac{}{(P\land Q)\vdash P\land Q}\,\text{ax}}{(P\land Q)\vdash P}\,\land e_1}{\vdash (P\land Q)\Rightarrow P}\,{\Rightarrow}\text{i}$$

VI.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P\Rightarrow\neg Q, Q, P\vdash P\Rightarrow\neg Q}\,\text{ax} \qquad \dfrac{}{P\Rightarrow\neg Q, Q, P\vdash P}\,\text{ax}}{P\Rightarrow\neg Q, Q, P\vdash\neg Q}\,{\Rightarrow}\text{e} \qquad \dfrac{}{P\Rightarrow\neg Q, Q, P\vdash Q}\,\text{ax}}{P\Rightarrow\neg Q, Q, P\vdash\bot}\,\neg\text{e}}{P\Rightarrow\neg Q, Q\vdash\neg P}\,\neg\text{i}$$

VII.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P\Rightarrow Q, (P\land R)\vdash P\Rightarrow Q}\,\text{ax} \qquad \dfrac{\dfrac{}{P\Rightarrow Q, (P\land R)\vdash P\land R}\,\text{ax}}{P\Rightarrow Q, (P\land R)\vdash P}\,\land e_1}{P\Rightarrow Q, (P\land R)\vdash Q}\,{\Rightarrow}\text{e} \qquad \dfrac{\dfrac{}{P\Rightarrow Q, (P\land R)\vdash P\land R}\,\text{ax}}{P\Rightarrow Q, (P\land R)\vdash R}\,\land e_2}{P\Rightarrow Q, (P\land R)\vdash (Q\land R)}\,\land\text{i}}{P\Rightarrow Q\vdash (P\land R)\Rightarrow (Q\land R)}\,{\Rightarrow}\text{i}$$

VIII.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, Q\vdash Q}\,\text{ax} \qquad \dfrac{}{\Gamma, Q\vdash Q\Rightarrow R}\,\text{ax}}{\Gamma, Q\vdash R}\,{\Rightarrow}\text{e}}{\Gamma, Q\vdash (P\lor R)}\,\lor i_2 \qquad \dfrac{\dfrac{}{\Gamma, P\vdash P}\,\text{ax}}{\Gamma, P\vdash (P\lor R)}\,\lor i_1 \qquad \dfrac{}{\Gamma\vdash (P\lor Q)}\,\text{ax}}{\Gamma\equiv Q\Rightarrow R, (P\lor Q)\vdash (P\lor R)}\,\lor\text{e}}{Q\Rightarrow R\vdash (P\lor Q)\Rightarrow (P\lor R)}\,{\Rightarrow}\text{i}$$

IX.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, R\vdash R}\,\text{ax}}{\Gamma, R\vdash (Q\lor R)}\,\lor i_2}{\Gamma, R\vdash P\lor (Q\lor R)}\,\lor i_2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, Q\vdash Q}\,\text{ax}}{\Delta, Q\vdash (Q\lor R)}\,\lor i_1}{\Delta, Q\vdash P\lor (Q\lor R)}\,\lor i_2 \qquad \dfrac{\dfrac{}{\Delta, P\vdash P}\,\text{ax}}{\Delta, P\vdash P\lor (Q\lor R)}\,\lor i_1 \qquad \dfrac{}{\Delta\vdash P\lor Q}\,\text{ax}}{\Delta\equiv\Gamma, P\lor Q\vdash P\lor (Q\lor R)} \qquad \dfrac{}{\Gamma\vdash\Gamma}\,\text{ax}}{\Gamma\equiv (P\lor Q)\lor R\vdash P\lor (Q\lor R)}\,\lor\text{e}$$

X.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, Q\vdash P\land (Q\lor R)}\,\text{ax}}{\Gamma, Q\vdash P}\,\land e_1 \qquad \dfrac{}{\Gamma, Q\vdash Q}\,\text{ax}}{\Gamma, Q\vdash P\land Q}\,\land\text{i}}{\Gamma, Q\vdash (P\land Q)\lor (P\land R)}\,\lor i_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, R\vdash P\land (Q\lor R)}\,\text{ax}}{\Gamma, R\vdash P}\,\land e_1 \qquad \dfrac{}{\Gamma, R\vdash R}\,\text{ax}}{\Gamma, R\vdash P\land R}\,\land\text{i}}{\Gamma, R\vdash (P\land Q)\lor (P\land R)}\,\lor i_2 \qquad \dfrac{\dfrac{}{\Gamma\vdash P\land (Q\lor R)}\,\text{ax}}{\Gamma\vdash Q\lor R}\,\land e_2}{\Gamma\equiv P\land (Q\lor R)\vdash (P\land Q)\lor (P\land R)}\,\lor\text{e}$$

XI.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \wedge R \vdash P \wedge R}\,\text{ax}}{\Gamma, P \wedge R \vdash R}\,\wedge e_2}{\Gamma, P \wedge R \vdash Q \vee R}\,\vee i_2 \quad \dfrac{\dfrac{\dfrac{}{\Gamma, P \wedge Q \vdash P \wedge Q}\,\text{ax}}{\Gamma, P \wedge Q \vdash Q}\,\wedge e_2}{\Gamma, P \wedge Q \vdash Q \vee R}\,\vee i_1 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \vdash (Q \vee R)}\,\vee\text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, P \wedge R \vdash P \wedge R}\,\text{ax}}{\Gamma, P \wedge R \vdash P}\,\wedge e_1 \quad \dfrac{\dfrac{}{\Gamma, P \wedge Q \vdash P \wedge Q}\,\text{ax}}{\Gamma, P \wedge Q \vdash P}\,\wedge e_1 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \vdash P}\,\vee\text{e}$$

$$\dfrac{\Gamma \vdash (Q \vee R) \qquad \Gamma \vdash P}{\Gamma \equiv (P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)}\,\wedge\text{i}$$

XII.

$$\dfrac{\dfrac{\dfrac{}{\neg P \vee Q, P, Q \vdash Q}\,\text{ax} \quad \dfrac{\dfrac{\dfrac{}{\neg P \vee Q, P, \neg P \vdash \neg P}\,\text{ax} \quad \dfrac{}{\neg P \vee Q, P, \neg P \vdash P}\,\text{ax}}{\neg P \vee Q, P, \neg P \vdash \perp}\,\neg\text{e}}{\neg P \vee Q, P, \neg P \vdash Q}\,\perp\text{e} \quad \dfrac{}{\neg P \vee Q, P \vdash \neg P \vee Q}\,\text{ax}}{\neg P \vee Q, P \vdash Q}\,\vee\text{e}}{\neg P \vee Q \vdash P \Rightarrow Q}\,\Rightarrow\text{i}$$

XIII.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \vdash P \Rightarrow \neg Q}\,\text{ax} \quad \dfrac{}{\Gamma, P \vdash P}\,\text{ax}}{\Gamma, P \vdash \neg Q}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma, P \vdash P \Rightarrow Q}\,\text{ax} \quad \dfrac{}{\Gamma, P \vdash P}\,\text{ax}}{\Gamma, P \vdash Q}\,\Rightarrow\text{e}}{\Gamma \equiv P \Rightarrow Q, P \Rightarrow \neg Q, P \vdash \perp}\,\neg\text{e}}{P \Rightarrow Q, P \Rightarrow \neg Q \vdash \neg P}\,\neg\text{i}$$

XIV.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, Q \vdash P \Rightarrow (Q \Rightarrow R)}\,\text{ax} \quad \dfrac{}{\Gamma, Q \vdash P}\,\text{ax}}{\Gamma, Q \vdash Q \Rightarrow R}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, Q \vdash Q}\,\text{ax}}{\Gamma, Q \vdash R}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, Q \vdash \neg R}\,\text{ax}}{\Gamma, Q \vdash \perp}\,\neg\text{e}}{\Gamma \equiv P \Rightarrow (Q \Rightarrow R), P, \neg R \vdash \neg Q}\,\neg\text{i}$$

Ejercicio 12

Probar que los siguientes secuentes son válidos:

I.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg Q \vdash \neg Q}\text{ax}}{\Gamma, \neg Q \vdash (P \wedge \neg Q)}\wedge e_2 \quad \dfrac{}{\Gamma, \neg Q \vdash (P \wedge \neg Q) \Rightarrow R}\text{ax}}{\Gamma, \neg Q \vdash R}\Rightarrow\text{e} \quad \dfrac{}{\Gamma, \neg Q \vdash \neg R}\text{ax}}{\Gamma, \neg Q \vdash \bot}\neg\text{e}}{\Gamma \equiv (P \wedge \neg Q) \Rightarrow R, \neg R, P \vdash Q}\text{PBC}$$

II.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg P \vdash \neg P}\text{ax} \quad \dfrac{}{\Gamma, \neg P \vdash \neg P \Rightarrow Q}\text{ax}}{\Gamma, \neg P \vdash Q}\Rightarrow\text{e} \quad \dfrac{}{\Gamma, \neg P \vdash \neg Q}\text{ax}}{\Gamma, \neg P \vdash \bot}\neg\text{e}}{\Gamma \equiv \neg P \Rightarrow Q, \neg Q \vdash P}\text{PBC}}{\neg P \Rightarrow Q \vdash \neg Q \Rightarrow P}\Rightarrow\text{i}$$

III.

$$\dfrac{\dfrac{\dfrac{}{P \vee Q, R \vdash R}\text{ax} \quad \dfrac{}{P \vee Q, R \vdash (P \vee Q)}\text{ax}}{P \vee Q, R \vdash (P \vee Q) \wedge R}\wedge i}{P \vee Q \vdash R \Rightarrow (P \vee Q) \wedge R}\Rightarrow\text{i}$$

IV.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, (Q \Rightarrow P) \vdash (P \vee (Q \Rightarrow P)) \wedge Q}\text{ax}}{\Gamma, (Q \Rightarrow P) \vdash Q}\wedge e_2 \quad \dfrac{}{\Gamma, (Q \Rightarrow P) \vdash Q \Rightarrow P}\text{ax}}{\Gamma, (Q \Rightarrow P) \vdash P}\Rightarrow\text{e} \quad \dfrac{}{\Gamma, P \vdash P}\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (P \vee (Q \Rightarrow P)) \wedge Q}\text{ax}}{\Gamma \vdash (P \vee (Q \Rightarrow P))}\wedge e_1}{\Gamma \equiv (P \vee (Q \Rightarrow P)) \wedge Q \vdash P}\vee\text{e}$$

V.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge R}\text{ax}}{\Gamma \vdash R}\wedge e_2 \quad \dfrac{}{\Gamma \vdash R \Rightarrow S}\text{ax}}{\Gamma \vdash S}\Rightarrow\text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge R}\text{ax}}{\Gamma \vdash P}\wedge e_1 \quad \dfrac{}{\Gamma \vdash P \Rightarrow Q}\text{ax}}{\Gamma \vdash Q}\Rightarrow\text{e}}{\Gamma \equiv P \Rightarrow Q,\ R \Rightarrow S, (P \wedge R) \vdash (Q \wedge S)}\wedge i}{P \Rightarrow Q,\ R \Rightarrow S \vdash (P \wedge R) \Rightarrow (Q \wedge S)}\Rightarrow\text{i}$$

VI.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta \vdash P}\text{ax} \quad \dfrac{}{\Delta \vdash P \Rightarrow Q}\text{ax}}{\Delta \vdash Q}\Rightarrow\text{e} \quad \dfrac{}{\Delta \vdash P}\text{ax}}{\Delta \equiv P \Rightarrow Q, P \vdash (P \wedge Q)}\wedge\text{i}}{P \Rightarrow Q \vdash P \Rightarrow (P \wedge Q)}\Rightarrow\text{i} \quad \dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, (P \wedge Q), \vdash P \wedge Q}\text{ax}}{P \Rightarrow Q, (P \wedge Q), \vdash P}\wedge e_1}{P \Rightarrow Q \vdash (P \wedge Q) \Rightarrow P}\Rightarrow\text{i}}{P \Rightarrow Q \vdash ((P \wedge Q) \Rightarrow P) \wedge (P \Rightarrow (P \wedge Q))}\wedge\text{i}$$

VII.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P}\text{ax} \quad \dfrac{}{\Gamma \vdash P \Rightarrow (Q \wedge R)}\text{ax}}{\Gamma \equiv P \Rightarrow (Q \wedge R), P \vdash Q \wedge R}\Rightarrow\text{e}}{P \Rightarrow (Q \wedge R), P \vdash R}\wedge e_2}{P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow R)}\Rightarrow\text{i} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow (Q \wedge R), P \vdash P}\text{ax} \quad \dfrac{}{P \Rightarrow (Q \wedge R), P \vdash P \Rightarrow (Q \wedge R)}\text{ax}}{P \Rightarrow (Q \wedge R), P \vdash (Q \wedge R)}\Rightarrow\text{e}}{P \Rightarrow (Q \wedge R), P \vdash Q}\wedge e_1}{P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow Q)}\Rightarrow\text{i}}{P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)}\wedge\text{i}$$

VIII.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P}\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)}\text{ax}}{\Gamma \vdash (P \Rightarrow R)}\wedge e_2}{\Gamma \vdash R}\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma \vdash P}\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)}\text{ax}}{\Gamma \vdash P \Rightarrow Q}\wedge e_1}{\Gamma \vdash Q}\Rightarrow\text{e}}{\Gamma \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R), P \vdash (Q \wedge R)}\wedge\text{i}}{(P \Rightarrow Q) \wedge (P \Rightarrow R) \vdash P \Rightarrow (Q \wedge R)}\Rightarrow\text{i}$$

IX.

$$\dfrac{\dfrac{}{P \vdash P}\text{ax}}{P \vee (P \wedge Q) \vdash P}\vee i_1$$

X.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, R \vdash R}\text{ax} \quad \dfrac{}{\Gamma, R \vdash R \Rightarrow S}\text{ax}}{\Gamma, R \vdash S}\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma, Q \vdash Q}\text{ax} \quad \dfrac{}{\Gamma, Q \vdash Q \Rightarrow S}\text{ax}}{\Gamma, Q \vdash S}\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma \vdash P}\text{ax} \quad \dfrac{}{\Gamma \vdash P \Rightarrow (Q \vee R)}\text{ax}}{\Gamma \vdash Q \vee R}\Rightarrow\text{e}}{\Gamma \equiv P \Rightarrow (Q \vee R),\ Q \Rightarrow S,\ R \Rightarrow S, P \vdash S}\vee\text{e}}{P \Rightarrow (Q \vee R),\ Q \Rightarrow S,\ R \Rightarrow S \vdash P \Rightarrow S}\Rightarrow\text{i}$$

Ejercicio 13

Probar que los siguientes secuentes son válidos:

I. $$\overline{\neg P \Rightarrow \neg Q \vdash Q \Rightarrow P}$$

II. $$\overline{\neg P \vee \neg Q \vdash \neg(P \wedge Q)}$$

III. $$\overline{\neg P,\ P \vee Q \vdash Q}$$

IV. $$\overline{P \vee Q,\ \neg Q \vee R \vdash P \vee R}$$

V. $$\overline{P \wedge \neg P \vdash \neg(R \Rightarrow Q) \wedge (R \Rightarrow Q)}$$

VI. $$\overline{\neg(\neg P \vee Q) \vdash P}$$

VII. $$\overline{\vdash \neg P \Rightarrow (P \Rightarrow (P \Rightarrow Q))}$$

VIII. $$\overline{P \wedge Q \vdash \neg(\neg P \vee \neg Q)}$$

IX. $$\overline{\vdash (P \Rightarrow Q) \vee (Q \Rightarrow R)}$$