Práctica 3 - Lógica intuicionista y clásica - PLP

Philips

1er Cuatrimestre 2025

Extra

Spamear **PBC** a la hora de demostrar en lógica clásica.

Siempre chequear, en todos los pasos de un procedimiento, si lo que estoy probando tiene **sentido lógico**.

Intuición de la disyunción ($\vee$)
A la hora de usar $\vee e$ recordar usar la disyunción con algo que aparezca en el contexto (así puedo probar la primera regla directamente o en pocos pasos), y por sobre todo, algo con lo que pueda construir lo que quiera probar.

Modus Tollens

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, \neg Q, P \vdash P \Rightarrow Q}\,\text{ax} \quad \dfrac{}{P \Rightarrow Q, \neg Q, P \vdash P}\,\text{ax}}{P \Rightarrow Q, \neg Q, P \vdash Q}\,\Rightarrow\text{e} \quad \dfrac{}{P \Rightarrow Q, \neg Q, P \vdash \neg Q}\,\text{ax}}{P \Rightarrow Q, \neg Q, P \vdash \bot}}{P \Rightarrow Q, \neg Q \vdash \neg P}\,\neg\text{i}$$

Ejercicio 5

Demostrar en deducción natural que las siguientes fórmulas son teoremas **sin usar principios de razonamiento clásicos** salvo que se indique lo contrario. Recordemos que una fórmula σ es un teorema si y sólo si vale $\vdash \sigma$:

I. Modus ponens relativizado: ?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q \Rightarrow \tau}\,\text{ax} \quad \dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q}\,\text{ax}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \tau}\,\Rightarrow\text{e}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q \vdash \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\rho \Rightarrow q \Rightarrow \tau \vdash (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\vdash (\rho \Rightarrow q \Rightarrow \tau) \Rightarrow (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}$$

II. Reducción al absurdo:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \bot), p \vdash p \Rightarrow \bot}\,\text{ax} \quad \dfrac{}{(p \Rightarrow \bot), p \vdash p}\,\text{ax}}{(p \Rightarrow \bot), p \vdash \bot}\,\Rightarrow\text{e}}{(p \Rightarrow \bot) \vdash \neg p}\,\neg\text{i}}{\vdash (p \Rightarrow \bot) \Rightarrow \neg p}\,\Rightarrow\text{i}$$

III. Introducción de la doble negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \quad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg\text{e}}{p \vdash \neg\neg p}\,\neg\text{i}}{\vdash p \Rightarrow \neg\neg p}\,\Rightarrow\text{i}$$

IV. Eliminación de la triple negación:

$$\dfrac{\dfrac{\dfrac{}{\neg\neg\neg p, p \vdash \neg\neg p}\,\text{III} \quad \dfrac{}{\neg\neg\neg p, p \vdash \neg\neg\neg p}\,\text{ax}}{\neg\neg\neg p, p \vdash \bot}\,\neg\text{e}}{\neg\neg\neg p \vdash \neg p}\,\neg\text{i}}{\vdash \neg\neg\neg p \Rightarrow \neg p}\,\Rightarrow\text{i}$$

V. Contraposición:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \neg\sigma}\,\text{ax} \quad \dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p}\,\text{ax} \quad \dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p \Rightarrow \sigma}\,\text{ax}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \sigma}\,\Rightarrow\text{e}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \bot}\,\neg\text{e}}{(p \Rightarrow \sigma), \neg\sigma \vdash \neg p}\,\neg\text{i}}{(p \Rightarrow \sigma) \vdash \neg\sigma \Rightarrow \neg p}\,\Rightarrow\text{i}}{\vdash (p \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg p)}\,\Rightarrow\text{i}$$

VI. Adjunción: $((p \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (p \Rightarrow \sigma \Rightarrow \tau)$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \sigma}\,\text{ax} \quad \dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash p}\,\text{ax}}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma)}\,\wedge\text{i} \quad \dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma) \Rightarrow \tau}\,\text{ax}}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \tau}\,\Rightarrow\text{e}}{((p \wedge \sigma) \Rightarrow \tau), p \vdash \sigma \Rightarrow \tau}\,\Rightarrow\text{i}}{((p \wedge \sigma) \Rightarrow \tau) \vdash (p \Rightarrow \sigma \Rightarrow \tau)}\,\Rightarrow\text{i}}{\vdash ((p \wedge \sigma) \Rightarrow \tau) \Rightarrow (p \Rightarrow \sigma \Rightarrow \tau)}\,\Rightarrow\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash (p \wedge \sigma)}\,\text{ax}}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash \sigma}\,\wedge e_2}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma}\,\Rightarrow\text{i} \quad \dfrac{}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma \Rightarrow \tau}\,\text{ax}}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash \tau}\,\Rightarrow\text{e}}{(p \Rightarrow \sigma \Rightarrow \tau) \vdash ((p \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow\text{i}}{\vdash (p \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((p \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow\text{i}$$

VII. Ley de Morgan (I): $\neg(p \vee \sigma) \Leftrightarrow (\neg p \wedge \neg\sigma)$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \vee \sigma), \sigma \vdash \sigma}\,\text{ax}}{\neg(p \vee \sigma), \sigma \vdash p \vee \sigma}\,\vee i_2 \quad \dfrac{}{\neg(p \vee \sigma), \sigma \vdash \neg(p \vee \sigma)}\,\text{ax}}{\neg(p \vee \sigma), \sigma \vdash \bot}\,\neg\text{e}}{\neg(p \vee \sigma) \vdash \neg\sigma}\,\neg\text{i} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash p \vee \sigma}\,\vee i_1 \quad \dfrac{}{\Gamma \vdash \neg(p \vee \sigma)}\,\text{ax}}{\Gamma \equiv \neg(p \vee \sigma), p \vdash \bot}\,\neg\text{e}}{\neg(p \vee \sigma) \vdash \neg p}\,\neg\text{i}}{\neg(p \vee \sigma) \vdash \neg p \wedge \neg\sigma}\,\wedge\text{i}}{\vdash \neg(p \vee \sigma) \Rightarrow (\neg p \wedge \neg\sigma)}\,\Rightarrow\text{i}$$

(⇒) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \sigma \vdash \neg p \wedge \neg\sigma}\,\text{ax}}{\Gamma, \sigma \vdash \neg\sigma}\,\wedge e_2}{\Gamma, \sigma \vdash \sigma}\,\text{ax}}{\Gamma, \sigma \vdash \bot}\,\neg\text{e} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash \neg p \wedge \neg\sigma}\,\text{ax}}{\Gamma, p \vdash \neg p}\,\wedge e_1 \qquad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma, p \vdash \bot}\,\neg\text{e} \qquad \dfrac{}{\Gamma \vdash p \vee \sigma}\,\text{ax}}{\dfrac{\Gamma \equiv (\neg p \wedge \neg\sigma), (p \vee \sigma) \vdash \bot}{(\neg p \wedge \neg\sigma) \vdash \neg(p \vee \sigma)}\,\neg\text{i}}{\vdash (\neg p \wedge \neg\sigma) \Rightarrow \neg(p \vee \sigma)}\,\Rightarrow\text{i}}\,\vee\text{e}$$

VIII. Ley de Morgan (II): $\neg(p \wedge \sigma) \Leftrightarrow (\neg p \vee \neg\sigma)$

(⇒) ida

$$\dfrac{\dfrac{\dfrac{}{\neg(\rho \wedge \sigma), \neg p \vdash \neg\rho}\,\text{ax}}{\neg(\rho \wedge \sigma), \neg p \vdash \neg\rho \vee \neg\sigma}\,\vee i_1 \qquad \dfrac{}{\neg(\rho \wedge \sigma) \vdash p \vee \neg p}\,\text{LEM} \qquad \dfrac{\text{Sigo este caso abajo}}{\Gamma \equiv \neg(\rho \wedge \sigma), p \vdash \neg\rho \vee \neg\sigma}\,(*)}{\dfrac{\neg(\rho \wedge \sigma) \vdash \neg\rho \vee \neg\sigma}{\vdash \neg(\rho \wedge \sigma) \Rightarrow (\neg\rho \vee \neg\sigma)}\,\Rightarrow i}\,\vee e$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \sigma \vdash p}\,\text{ax} \quad \dfrac{}{\Gamma, \sigma \vdash \sigma}\,\text{ax}}{\Gamma, \sigma \vdash \rho \wedge \sigma}\,\wedge i \qquad \dfrac{}{\Gamma, \sigma \vdash \neg(\rho \wedge \sigma)}\,\text{ax}}{\Gamma, \sigma \vdash \bot}\,\neg e}{\dfrac{\Gamma, \sigma \vdash \neg\rho \vee \neg\sigma}{\Gamma, \neg\sigma \vdash \neg\rho \vee \neg\sigma}\,\bot\, e} \qquad \dfrac{\dfrac{}{\Gamma, \neg\sigma \vdash \neg\sigma}\,\text{ax}}{\Gamma, \neg\sigma \vdash \neg\rho \vee \neg\sigma}\,\vee i_2 \qquad \dfrac{}{\Gamma \vdash \sigma \vee \neg\sigma}\,\text{LEM}}{\Gamma \vdash \neg\rho \vee \neg\sigma \quad \{*\}}\,\vee e$$

(⇒) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg\sigma, p \vdash p \wedge \sigma}\,\text{ax}}{\Gamma, \neg\sigma, p \vdash \sigma}\,\wedge e_2 \qquad \dfrac{}{\Gamma, \neg\sigma, p \vdash \neg\sigma}\,\text{ax}}{\Gamma, \neg\sigma, p \vdash \bot}\,\neg\text{e}}{\Gamma, \neg\sigma \vdash \neg p}\,\neg\text{i} \qquad \dfrac{\dfrac{}{\Gamma, \neg p \vdash \neg p}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash \neg p \vee \neg\sigma}\,\text{ax}}{\Gamma \vdash \neg p}\,\vee\text{e} \qquad \dfrac{\dfrac{}{\Gamma \vdash p \wedge \sigma}\,\text{ax}}{\Gamma \vdash p}\,\wedge e_1}{\dfrac{\Gamma \equiv (\neg\rho \vee \neg\sigma), (\rho \wedge \sigma) \vdash \bot}{(\neg\rho \vee \neg\sigma) \vdash \neg(\rho \wedge \sigma)}\,\neg\text{i}}{\vdash (\neg\rho \vee \neg\sigma) \Rightarrow \neg(\rho \wedge \sigma)}\,\Rightarrow\text{i}}\,\neg\text{e}$$

IX. Conmutatividad ($\wedge$):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \wedge \sigma) \vdash (p \wedge \sigma)}\,\text{ax}}{(p \wedge \sigma) \vdash p}\,\wedge e_1 \qquad \dfrac{\dfrac{}{(p \wedge \sigma) \vdash (p \wedge \sigma)}\,\text{ax}}{(p \wedge \sigma) \vdash \sigma}\,\wedge e_2}{(p \wedge \sigma) \vdash (\sigma \wedge p)}\,\wedge\text{i}}{\vdash (p \wedge \sigma) \Rightarrow (\sigma \wedge p)}\,\Rightarrow\text{i}$$

X. Asociatividad ($\wedge$): $((p \wedge \sigma) \wedge \tau) \Leftrightarrow (p \wedge (\sigma \wedge \tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p \wedge \sigma) \wedge \tau) \vdash ((p \wedge \sigma) \wedge \tau)}\,\text{ax}}{((p \wedge \sigma) \wedge \tau) \vdash \tau}\,\wedge e_2 \quad \dfrac{\dfrac{\dfrac{}{((p \wedge \sigma) \wedge \tau) \vdash ((p \wedge \sigma) \wedge \tau)}\,\text{ax}}{((p \wedge \sigma) \wedge \tau) \vdash (p \wedge \sigma)}\,\wedge e_1}{((p \wedge \sigma) \wedge \tau) \vdash \sigma}\,\wedge e_2}{((p \wedge \sigma) \wedge \tau) \vdash (\sigma \wedge \tau)}\,\wedge \text{i} \quad \dfrac{\dfrac{\dfrac{}{((p \wedge \sigma) \wedge \tau) \vdash ((p \wedge \sigma) \wedge \tau)}\,\text{ax}}{((p \wedge \sigma) \wedge \tau) \vdash (p \wedge \sigma)}\,\wedge e_1}{((p \wedge \sigma) \wedge \tau) \vdash p}\,\wedge e_1}{((p \wedge \sigma) \wedge \tau) \vdash (p \wedge (\sigma \wedge \tau))}\,\wedge \text{i}}{\vdash ((p \wedge \sigma) \wedge \tau) \Rightarrow (p \wedge (\sigma \wedge \tau))}\,\Rightarrow \text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \wedge (\sigma \wedge \tau))}\,\text{ax}}{\Gamma \vdash (\sigma \wedge \tau)}\,\wedge e_2}{\Gamma \vdash \sigma}\,\wedge e_1 \quad \dfrac{\dfrac{}{\Gamma \vdash (p \wedge (\sigma \wedge \tau))}\,\text{ax}}{\Gamma \vdash p}\,\wedge e_1}{\Gamma \vdash p \wedge \sigma}\,\wedge \text{i} \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash p \wedge (\sigma \wedge \tau)}\,\text{ax}}{\Gamma \vdash (\sigma \wedge \tau)}\,\wedge e_2}{\Gamma \vdash \tau}\,\wedge e_2}{\Gamma \equiv (p \wedge (\sigma \wedge \tau)) \vdash ((p \wedge \sigma) \wedge \tau)}\,\wedge \text{i}}{\vdash (p \wedge (\sigma \wedge \tau)) \Rightarrow ((p \wedge \sigma) \wedge \tau)}\,\Rightarrow \text{i}$$

XI. Conmutatividad ($\vee$)

$$\dfrac{\dfrac{\dfrac{}{(p \vee \sigma), \sigma \vdash \sigma}\,\text{ax}}{(p \vee \sigma), \sigma \vdash (\sigma \vee p)}\,\vee i_1 \quad \dfrac{\dfrac{}{(p \vee \sigma), p \vdash p}\,\text{ax}}{(p \vee \sigma), p \vdash (\sigma \vee p)}\,\vee i_2 \quad \dfrac{}{(p \vee \sigma) \vdash (p \vee \sigma)}\,\text{ax}}{(p \vee \sigma) \vdash (\sigma \vee p)}\,\vee \text{e}}{\vdash (p \vee \sigma) \Rightarrow (\sigma \vee p)}\,\Rightarrow \text{i}$$

XII. Asociatividad ($\vee$): $((p \vee \sigma) \vee \tau) \Leftrightarrow (p \vee (\sigma \vee \tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, \sigma \vdash \sigma}\,\text{ax}}{\Delta, \sigma \vdash (\sigma \vee \tau)}\,\vee i_1}{\Delta, \sigma \vdash p \vee (\sigma \vee \tau)}\,\vee i_2 \quad \dfrac{\dfrac{}{\Delta, p \vdash p}\,\text{ax}}{\Delta, p \vdash p \vee (\sigma \vee \tau)}\,\vee i_1 \quad \dfrac{}{\Delta \vdash p \vee \sigma}\,\text{ax}}{\Delta \equiv \Gamma, p \vee \sigma \vdash p \vee (\sigma \vee \tau)}\,\vee \text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \tau}\,\text{ax}}{\Gamma, \tau \vdash \sigma \vee \tau}\,\vee i_2}{\Gamma, \tau \vdash p \vee (\sigma \vee \tau)}\,\vee i_2 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \equiv ((p \vee \sigma) \vee \tau) \vdash (p \vee (\sigma \vee \tau))}\,\vee \text{e}}{\vdash ((p \vee \sigma) \vee \tau) \Rightarrow (p \vee (\sigma \vee \tau))}\,\Rightarrow \text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, \sigma \vdash \sigma}\,\text{ax}}{\Delta, \sigma \vdash p \vee \sigma}\,\vee i_2}{\Delta, \sigma \vdash (p \vee \sigma) \vee \tau}\,\vee i_1 \quad \dfrac{\dfrac{}{\Delta, \tau \vdash \tau}\,\text{ax}}{\Delta, \tau \vdash (p \vee \sigma) \vee \tau}\,\vee i_2 \quad \dfrac{}{\Delta \vdash \sigma \vee \tau}\,\text{ax}}{\Delta \equiv \Gamma, \sigma \vee \tau \vdash (p \vee \sigma) \vee \tau}\,\vee \text{e} \quad \dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash p}\,\text{ax}}{\Gamma, p \vdash (p \vee \sigma)}\,\vee i_1}{\Gamma, p \vdash (p \vee \sigma) \vee \tau}\,\vee i_1 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \equiv (p \vee (\sigma \vee \tau)) \vdash ((p \vee \sigma) \vee \tau)}\,\vee \text{e}}{\vdash (p \vee (\sigma \vee \tau)) \Rightarrow ((p \vee \sigma) \vee \tau)}\,\Rightarrow \text{i}$$

Ejercicio 6

Demostrar en deducción natural que vale $\vdash \sigma$ para cada una de las siguientes fórmulas. Para estas fórmulas es imprescindible usar **lógica clásica**:

I. Absurdo clásico:

$$\dfrac{\dfrac{\dfrac{}{(\neg\tau \Rightarrow \bot), \neg\tau \vdash \neg\tau}\,\text{ax} \quad \dfrac{}{(\neg\tau \Rightarrow \bot), \neg\tau \vdash (\neg\tau \Rightarrow \bot)}\,\text{ax}}{(\neg\tau \Rightarrow \bot), \neg\tau \vdash \bot}\,\Rightarrow\text{e}}{(\neg\tau \Rightarrow \bot) \vdash \tau}\,\text{PBC}}{\vdash (\neg\tau \Rightarrow \bot) \Rightarrow \tau}\,\Rightarrow\text{i}$$

II. Ley de Peirce:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \neg\tau}\,\text{ax} \quad \dfrac{}{\Gamma, \tau \vdash \tau}\,\text{ax}}{\Gamma, \tau \vdash \bot}\,\neg\text{e}}{\Gamma, \tau \vdash p}\,\bot\text{e}}{\Gamma \vdash (\tau \Rightarrow p)}\,\Rightarrow\text{i} \quad \dfrac{}{\Gamma \vdash (\tau \Rightarrow p) \Rightarrow \tau}\,\text{ax}}{\Gamma \vdash \tau}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma \vdash \neg\tau}\,\text{ax}}{\Gamma \equiv ((\tau \Rightarrow p) \Rightarrow \tau), \neg\tau \vdash \bot}\,\neg\text{e}}{((\tau \Rightarrow p) \Rightarrow \tau) \vdash \tau}\,\text{PBC}}{\vdash ((\tau \Rightarrow p) \Rightarrow \tau) \Rightarrow \tau}\,\Rightarrow\text{i}$$

III. Tercero excluido:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \tau \vdash \tau}\,\text{ax}}{\Gamma, \tau \vdash (\tau \vee \neg\tau)}\,\vee i_1 \quad \dfrac{}{\Gamma, \tau \vdash \Gamma}\,\text{ax}}{\Gamma, \tau \vdash \bot}\,\neg\text{e}}{\Gamma \vdash \neg\tau}\,\neg\text{i}}{\Gamma \vdash (\tau \vee \neg\tau)}\,\vee i_2 \quad \dfrac{}{\Gamma, \tau \vdash \Gamma}\,\text{ax}}{\Gamma, \tau \vdash \bot}\,\neg\text{e}}{\Gamma \vdash \neg\tau}\,\neg\text{ i}}{\Gamma \vdash (\tau \vee \neg\tau)}\,\vee i_2 \quad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \equiv \neg(\tau \vee \neg\tau) \vdash \bot}\,\neg\text{e}}{\vdash \tau \vee \neg\tau}\,\text{PBC}$$

IV. Consecuencia milagrosa:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \tau}\,\text{ax} \quad \dfrac{}{\Gamma \vdash \neg\tau \Rightarrow \tau}\,\text{ax}}{\Gamma \vdash \neg\tau}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma \vdash \neg\tau}\,\text{ax}}{\Gamma \equiv (\neg\tau \Rightarrow \tau), \neg\tau \vdash \bot}\,\neg\text{e}}{(\neg\tau \Rightarrow \tau) \vdash \tau}\,\text{PBC}}{\vdash (\neg\tau \Rightarrow \tau) \Rightarrow \tau}\,\Rightarrow\text{i}$$

V. Contraposición clásica:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg p}\,\text{ax} \quad \dfrac{}{\Gamma \vdash \neg p \Rightarrow \neg\tau}\,\text{ax}}{\Gamma \vdash \neg\tau}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma \vdash \tau}\,\text{ax}}{\Gamma \equiv (\neg p \Rightarrow \neg\tau), \tau, \neg p \vdash \bot}\,\neg\text{e}}{(\neg p \Rightarrow \neg\tau), \tau \vdash p}\,\text{PBC}}{(\neg p \Rightarrow \neg\tau) \vdash (\tau \Rightarrow p)}\,\Rightarrow\text{i}}{\vdash (\neg p \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow p)}\,\Rightarrow\text{i}$$

VI. Análisis de casos:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg p}\,\text{ax} \quad \dfrac{}{\Gamma \vdash \tau \Rightarrow p}\,\text{ax}}{\Gamma \vdash \neg\tau}\,\text{MT} \quad \dfrac{}{\Gamma \vdash \neg\tau \Rightarrow p}\,\text{ax}}{\Gamma \vdash p}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma \vdash \neg p}\,\text{ax}}{\Gamma \equiv (\tau \Rightarrow p), (\neg\tau \Rightarrow p), \neg p \vdash \bot}\,\neg\text{e}}{(\tau \Rightarrow p), (\neg\tau \Rightarrow p) \vdash p}\,\text{PBC}}{(\tau \Rightarrow p) \vdash (\neg\tau \Rightarrow p) \Rightarrow p}\,\Rightarrow\text{i}}{\vdash (\tau \Rightarrow p) \Rightarrow (\neg\tau \Rightarrow p) \Rightarrow p}\,\Rightarrow\text{i}$$

VII. Implicación vs. disyunción: $(\tau \Rightarrow p) \Leftrightarrow (\neg\tau \vee p)$

($\Rightarrow$) ida

$$\dfrac{}{(\tau \Rightarrow p) \Rightarrow (\neg\tau \vee p)}$$

($\Rightarrow$) vuelta

$$\dfrac{}{(\neg\tau \vee p) \Rightarrow (\tau \Rightarrow p)}$$

Ejercicio 9

Demostrar las siguientes tautologías utilizando deducción natural.

I. $(P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow (P \Rightarrow Q)}\,\text{ax} \quad \dfrac{}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P}\,\text{ax}}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow Q}\,\Rightarrow\text{e} \quad \dfrac{}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P}\,\text{ax}}{(P \Rightarrow (P \Rightarrow Q)), P \vdash Q}\,\Rightarrow\text{e}}{(P \Rightarrow (P \Rightarrow Q)) \vdash (P \Rightarrow Q)}\,\Rightarrow\text{i}}{\vdash (P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)}\,\Rightarrow\text{i}$$

II. Voy a usar deducción natural clásica: $(P \Rightarrow Q) \Rightarrow ((\neg P \Rightarrow Q) \Rightarrow Q)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg Q, \neg P \vdash \neg P}\,\text{ax} \quad \dfrac{}{\Gamma, \neg Q, \neg P \vdash \neg P \Rightarrow Q}\,\text{ax}}{\Gamma, \neg Q, \neg P \vdash Q}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, \neg Q, \neg P \vdash \neg Q}\,\text{ax}}{\Gamma, \neg Q, \neg P \vdash \bot}\,\neg\text{e}}{\Gamma, \neg Q \vdash P}\,\text{PBC} \quad \dfrac{}{\Gamma, \neg Q \vdash P \Rightarrow Q}\,\text{ax}}{\Gamma, \neg Q \vdash Q}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, \neg Q \vdash \neg Q}\,\text{ax}}{\Gamma, \neg Q \vdash \bot}\,\neg\text{e}}{\Gamma \equiv (P \Rightarrow Q), (\neg P \Rightarrow Q) \vdash Q}\,\text{PBC}}{(P \Rightarrow Q) \vdash (\neg P \Rightarrow Q) \Rightarrow Q}\,\Rightarrow\text{i}}{\vdash (P \Rightarrow Q) \Rightarrow ((\neg P \Rightarrow Q) \Rightarrow Q)}\,\Rightarrow\text{i}$$

Ejercicio 10

Demostrar las siguientes tautologías utilizando deducción natural:

I.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow (P \Rightarrow Q)}\,\text{ax} \quad \dfrac{}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P}\,\text{ax}}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P \Rightarrow Q}\,\Rightarrow\text{e} \quad \dfrac{}{(P \Rightarrow (P \Rightarrow Q)), P \vdash P}\,\text{ax}}{(P \Rightarrow (P \Rightarrow Q)), P \vdash Q}\,\Rightarrow\text{e}}{(P \Rightarrow (P \Rightarrow Q)) \vdash (P \Rightarrow Q)}\,\Rightarrow\text{i}}{\vdash (P \Rightarrow (P \Rightarrow Q)) \Rightarrow (P \Rightarrow Q)}\,\Rightarrow\text{i}$$

II.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash R \wedge Q}\,\text{ax}}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash R}\,\wedge e_1 \quad \dfrac{}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash R \Rightarrow \neg Q}\,\text{ax}}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash \neg Q}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash R \wedge Q}\,\text{ax}}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash Q}\,\wedge e_2}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash \bot}\,\neg\text{e}}{(R \Rightarrow \neg Q), (R \wedge Q) \vdash P}\,\bot\text{e}}{(R \Rightarrow \neg Q) \vdash ((R \wedge Q) \Rightarrow P)}\,\Rightarrow\text{i}}{\vdash (R \Rightarrow \neg Q) \Rightarrow ((R \wedge Q) \Rightarrow P)}\,\Rightarrow\text{i}$$

III.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \vdash P \wedge Q}\,\text{ax}}{\Gamma, P \vdash Q}\,\wedge e_2}{\Gamma \vdash (P \Rightarrow Q)}\,\Rightarrow\text{i} \quad \dfrac{}{\Gamma \vdash (P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)}\,\text{ax}}{\Gamma \vdash R \Rightarrow \neg Q}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma \vdash R \wedge Q}\,\text{ax}}{\Gamma \vdash R}\,\wedge e_1}{\Gamma \vdash \neg Q}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma \vdash R \wedge Q}\,\text{ax}}{\Gamma \vdash Q}\,\wedge e_2}{\Gamma \equiv ((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)), (R \wedge Q) \vdash \bot}\,\bot\text{e}}{((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \vdash \neg(R \wedge Q)}\,\neg\text{i}}{\vdash ((P \Rightarrow Q) \Rightarrow (R \Rightarrow \neg Q)) \Rightarrow \neg(R \wedge Q)}\,\Rightarrow\text{i}$$

Ejercicio 11

Probar que los siguientes secuentes son válidos sin usar principios de razonamiento clásicos:

I.
$$\dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R, S \wedge T \vdash S \wedge T}\,\text{ax}}{(P \wedge Q) \wedge R, S \wedge T \vdash S}\,\wedge e_1 \qquad \dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R, S \wedge T \vdash (P \wedge Q) \wedge R}\,\text{ax}}{(P \wedge Q) \wedge R, S \wedge T \vdash P \wedge Q}\,\wedge e_1}{(P \wedge Q) \wedge R, S \wedge T \vdash Q}\,\wedge e_2}{(P \wedge Q) \wedge R, S \wedge T \vdash Q \wedge S}\,\wedge \text{i}$$

II.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R \vdash (P \wedge Q) \wedge R}\,\text{ax}}{(P \wedge Q) \wedge R \vdash R}\,\wedge e_2 \qquad \dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R \vdash P(\wedge Q) \wedge R}\,\text{ax}}{(P \wedge Q) \wedge R \vdash P(\wedge Q)}\,\wedge e_1}{(P \wedge Q) \wedge R \vdash Q}\,\wedge e_2}{(P \wedge Q) \wedge R \vdash (Q \wedge R)}\,\wedge \text{i} \qquad \dfrac{\dfrac{\dfrac{}{(P \wedge Q) \wedge R \vdash (P \wedge Q) \wedge R}\,\text{ax}}{(P \wedge Q) \wedge R \vdash (P \wedge Q)}\,\wedge e_1}{(P \wedge Q) \wedge R \vdash P}\,\wedge e_1}{(P \wedge Q) \wedge R \vdash P \wedge (Q \wedge R)}\,\wedge \text{i}$$

III.
$$\dfrac{\dfrac{\dfrac{}{P \Rightarrow (P \Rightarrow Q), P \vdash P \Rightarrow (P \Rightarrow Q)}\,\text{ax} \qquad \dfrac{}{P \Rightarrow (P \Rightarrow Q), P \vdash P}\,\text{ax}}{P \Rightarrow (P \Rightarrow Q), P \vdash P \Rightarrow Q}\,\Rightarrow \text{e} \qquad \dfrac{}{P \Rightarrow (P \Rightarrow Q), P \vdash P}\,\text{ax}}{P \Rightarrow (P \Rightarrow Q), P \vdash Q}\,\Rightarrow \text{e}$$

IV.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash Q \Rightarrow (P \Rightarrow R)}\,\text{ax} \qquad \dfrac{}{\Gamma \vdash Q}\,\text{ax}}{\Gamma \vdash P \Rightarrow R}\,\Rightarrow \text{e} \qquad \dfrac{}{\Gamma \vdash P}\,\text{ax}}{\Gamma \vdash R}\,\Rightarrow \text{e} \qquad \dfrac{}{\Gamma \vdash \neg R}\,\text{ax}}{\Gamma \equiv Q \Rightarrow (P \Rightarrow R), \neg R, Q, P \vdash \bot}\,\neg \text{e}}{Q \Rightarrow (P \Rightarrow R), \neg R, Q \vdash \neg P}\,\neg \text{i}$$

V.
$$\dfrac{\dfrac{\dfrac{}{(P \wedge Q) \vdash P \wedge Q}\,\text{ax}}{(P \wedge Q) \vdash P}\,\wedge e_1}{\vdash (P \wedge Q) \Rightarrow P}\,\Rightarrow \text{i}$$

VI.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow \neg Q, Q, P \vdash P \Rightarrow \neg Q}\,\text{ax} \qquad \dfrac{}{P \Rightarrow \neg Q, Q, P \vdash P}\,\text{ax}}{P \Rightarrow \neg Q, Q, P \vdash \neg Q}\,\Rightarrow \text{e} \qquad \dfrac{}{P \Rightarrow \neg Q, Q, P \vdash Q}\,\text{ax}}{P \Rightarrow \neg Q, Q, P \vdash \bot}\,\neg \text{e}}{P \Rightarrow \neg Q, Q \vdash \neg P}\,\neg \text{i}$$

VII.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, (P \wedge R) \vdash P \Rightarrow Q}\,\text{ax} \qquad \dfrac{\dfrac{}{P \Rightarrow Q, (P \wedge R) \vdash P \wedge R}\,\text{ax}}{P \Rightarrow Q, (P \wedge R) \vdash P}\,\wedge e_1}{P \Rightarrow Q, (P \wedge R) \vdash Q}\,\Rightarrow \text{e} \qquad \dfrac{\dfrac{}{P \Rightarrow Q, (P \wedge R) \vdash P \wedge R}\,\text{ax}}{P \Rightarrow Q, (P \wedge R) \vdash R}\,\wedge e_2}{P \Rightarrow Q, (P \wedge R) \vdash (Q \wedge R)}\,\wedge \text{i}}{P \Rightarrow Q \vdash (P \wedge R) \Rightarrow (Q \wedge R)}\,\Rightarrow \text{i}$$

VIII.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, Q \vdash Q}\,\text{ax} \qquad \dfrac{}{\Gamma, Q \vdash Q \Rightarrow R}\,\text{ax}}{\Gamma, Q \vdash R}\,\Rightarrow \text{e}}{\Gamma, Q \vdash (P \vee R)}\,\vee i_2 \qquad \dfrac{\dfrac{}{\Gamma, P \vdash P}\,\text{ax}}{\Gamma, P \vdash (P \vee R)}\,\vee i_1 \qquad \dfrac{}{\Gamma \vdash (P \vee Q)}\,\text{ax}}{\Gamma \equiv Q \Rightarrow R, (P \vee Q) \vdash (P \vee R)}\,\vee \text{e}}{Q \Rightarrow R \vdash (P \vee Q) \Rightarrow (P \vee R)}\,\Rightarrow \text{i}$$

IX.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, R \vdash R}\,\text{ax}}{\Gamma, R \vdash (Q \vee R)}\,\vee i_2}{\Gamma, R \vdash P \vee (Q \vee R)}\,\vee i_2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Delta, Q \vdash Q}\,\text{ax}}{\Delta, Q \vdash (Q \vee R)}\,\vee i_1}{\Delta, Q \vdash P \vee (Q \vee R)}\,\vee i_2 \qquad \dfrac{\dfrac{}{\Delta, P \vdash P}\,\text{ax}}{\Delta, P \vdash P \vee (Q \vee R)}\,\vee i_1 \qquad \dfrac{}{\Delta \vdash P \vee Q}\,\text{ax}}{\Delta \equiv \Gamma, P \vee Q \vdash P \vee (Q \vee R)}\,\vee \text{e} \qquad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \equiv (P \vee Q) \vee R \vdash P \vee (Q \vee R)}\,\vee \text{e}$$

X.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, Q \vdash P \wedge (Q \vee R)}\,\text{ax}}{\Gamma, Q \vdash P}\,\wedge e_1 \qquad \dfrac{}{\Gamma, Q \vdash Q}\,\text{ax}}{\Gamma, Q \vdash P \wedge Q}\,\wedge \text{i}}{\Gamma, Q \vdash (P \wedge Q) \vee (P \wedge R)}\,\vee i_1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, R \vdash P \wedge (Q \vee R)}\,\text{ax}}{\Gamma, R \vdash P}\,\wedge e_1 \qquad \dfrac{}{\Gamma, R \vdash R}\,\text{ax}}{\Gamma, R \vdash P \wedge R}\,\wedge \text{i}}{\Gamma, R \vdash (P \wedge Q) \vee (P \wedge R)}\,\vee i_2 \qquad \dfrac{\dfrac{}{\Gamma \vdash P \wedge (Q \vee R)}\,\text{ax}}{\Gamma \vdash Q \vee R}\,\wedge e_2}{\Gamma \equiv P \wedge (Q \vee R) \vdash (P \wedge Q) \vee (P \wedge R)}\,\vee \text{e}$$

XI.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \wedge R \vdash P \wedge R}\,\text{ax}}{\Gamma, P \wedge R \vdash R}\,\wedge e_2}{\Gamma, P \wedge R \vdash Q \vee R}\,\vee i_2 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, P \wedge Q \vdash P \wedge Q}\,\text{ax}}{\Gamma, P \wedge Q \vdash Q}\,\wedge e_2}{\Gamma, P \wedge Q \vdash Q \vee R}\,\vee i_1 \qquad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \vdash (Q \vee R)}\,\vee\text{e} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, P \wedge R \vdash P \wedge R}\,\text{ax}}{\Gamma, P \wedge R \vdash P}\,\wedge e_1 \qquad \dfrac{\dfrac{}{\Gamma, P \wedge Q \vdash P \wedge Q}\,\text{ax}}{\Gamma, P \wedge Q \vdash P}\,\wedge e_1 \qquad \dfrac{}{\Gamma \vdash \Gamma}\,\text{ax}}{\Gamma \vdash P}\,\vee\text{e}}{\Gamma \equiv (P \wedge Q) \vee (P \wedge R) \vdash P \wedge (Q \vee R)}\,\wedge\text{i}$$

XII.
$$\dfrac{\dfrac{\dfrac{}{\neg P \vee Q, P, Q \vdash Q}\,\text{ax} \qquad \dfrac{\dfrac{\dfrac{}{\neg P \vee Q, P, \neg P \vdash \neg P}\,\text{ax} \qquad \dfrac{}{\neg P \vee Q, P, \neg P \vdash P}\,\text{ax}}{\neg P \vee Q, P, \neg P \vdash \bot}\,\neg\text{e}}{\neg P \vee Q, P, \neg P \vdash Q}\,\bot\text{e} \qquad \dfrac{}{\neg P \vee Q, P \vdash \neg P \vee Q}\,\text{ax}}{\neg P \vee Q, P \vdash Q}\,\vee\text{e}}{\neg P \vee Q \vdash P \Rightarrow Q}\,\Rightarrow\text{i}$$

XIII.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, P \vdash P \Rightarrow \neg Q}\,\text{ax} \qquad \dfrac{}{\Gamma, P \vdash P}\,\text{ax}}{\Gamma, P \vdash \neg Q}\,\Rightarrow\text{e} \qquad \dfrac{\dfrac{}{\Gamma, P \vdash P \Rightarrow Q}\,\text{ax} \qquad \dfrac{}{\Gamma, P \vdash P}\,\text{ax}}{\Gamma, P \vdash Q}\,\Rightarrow\text{e}}{\Gamma \equiv P \Rightarrow Q, P \Rightarrow \neg Q, P \vdash \bot}\,\neg\text{e}}{P \Rightarrow Q, P \Rightarrow \neg Q \vdash \neg P}\,\neg\text{i}$$

XIV.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, Q \vdash P \Rightarrow (Q \Rightarrow R)}\,\text{ax} \qquad \dfrac{}{\Gamma, Q \vdash P}\,\text{ax}}{\Gamma, Q \vdash Q \Rightarrow R}\,\Rightarrow\text{e} \qquad \dfrac{}{\Gamma, Q \vdash Q}\,\text{ax}}{\Gamma, Q \vdash R}\,\Rightarrow\text{e} \qquad \dfrac{}{\Gamma, Q \vdash \neg R}\,\text{ax}}{\Gamma, Q \vdash \bot}\,\neg\text{e}}{\Gamma \equiv P \Rightarrow (Q \Rightarrow R), P, \neg R \vdash \neg Q}\,\neg\text{i}$$

Ejercicio 12

Probar que los siguientes secuentes son válidos:

I.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg Q \vdash \neg Q}\,\text{ax}}{\Gamma, \neg Q \vdash (P \wedge \neg Q)}\,\wedge e_2 \quad \dfrac{}{\Gamma, \neg Q \vdash (P \wedge \neg Q) \Rightarrow R}\,\text{ax}}{\Gamma, \neg Q \vdash R}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, \neg Q \vdash \neg R}\,\text{ax}}{\dfrac{\Gamma, \neg Q \vdash \bot}{\Gamma \equiv (P \wedge \neg Q) \Rightarrow R, \neg R, P \vdash Q}\,\text{PBC}}\,\neg\text{e}$$

II.
$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg P \vdash \neg P}\,\text{ax} \quad \dfrac{}{\Gamma, \neg P \vdash \neg P \Rightarrow Q}\,\text{ax}}{\Gamma, \neg P \vdash Q}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, \neg P \vdash \neg Q}\,\text{ax}}{\Gamma, \neg P \vdash \bot}\,\neg\text{e}}{\dfrac{\Gamma \equiv \neg P \Rightarrow Q, \neg Q \vdash P}{\neg P \Rightarrow Q \vdash \neg Q \Rightarrow P}\,\Rightarrow\text{i}}\,\text{PBC}$$

III.
$$\dfrac{\dfrac{\dfrac{}{P \vee Q, R \vdash R}\,\text{ax} \quad \dfrac{}{P \vee Q, R \vdash (P \vee Q)}\,\text{ax}}{P \vee Q, R \vdash (P \vee Q) \wedge R}\,\wedge i}{P \vee Q \vdash R \Rightarrow (P \vee Q) \wedge R}\,\Rightarrow\text{i}$$

IV.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, (Q \Rightarrow P) \vdash (P \vee (Q \Rightarrow P)) \wedge Q}\,\text{ax}}{\Gamma, (Q \Rightarrow P) \vdash Q}\,\wedge e_2 \quad \dfrac{}{\Gamma, (Q \Rightarrow P) \vdash Q \Rightarrow P}\,\text{ax}}{\Gamma, (Q \Rightarrow P) \vdash P}\,\Rightarrow\text{e} \quad \dfrac{}{\Gamma, P \vdash P}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (P \vee (Q \Rightarrow P)) \wedge Q}\,\text{ax}}{\Gamma \vdash (P \vee (Q \Rightarrow P))}\,\wedge e_1}{\Gamma \equiv (P \vee (Q \Rightarrow P)) \wedge Q \vdash P}\,\vee\text{e}$$

V.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P \wedge R}\,\text{ax}}{\Gamma \vdash R}\,\wedge e_2 \quad \dfrac{}{\Gamma \vdash R \Rightarrow S}\,\text{ax}}{\Gamma \vdash S}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma \vdash P \wedge R}\,\text{ax}}{\Gamma \vdash P}\,\wedge e_1 \quad \dfrac{}{\Gamma \vdash P \Rightarrow Q}\,\text{ax}}{\Gamma \vdash Q}\,\Rightarrow\text{e}}{\dfrac{\Gamma \equiv P \Rightarrow Q,\ R \Rightarrow S, (P \wedge R) \vdash (Q \wedge S)}{P \Rightarrow Q,\ R \Rightarrow S \vdash (P \wedge R) \Rightarrow (Q \wedge S)}\,\Rightarrow\text{i}}\,\wedge i$$

VI.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Delta \vdash P}\,\text{ax} \quad \dfrac{}{\Delta \vdash P \Rightarrow Q}\,\text{ax}}{\Delta \vdash Q}\,\Rightarrow\text{e} \quad \dfrac{}{\Delta \vdash P}\,\text{ax}}{\Delta \equiv P \Rightarrow Q, P \vdash (P \wedge Q)}\,\wedge\text{i}}{P \Rightarrow Q \vdash P \Rightarrow (P \wedge Q)}\,\Rightarrow\text{i} \quad \dfrac{\dfrac{\dfrac{}{P \Rightarrow Q, (P \wedge Q), \vdash P \wedge Q}\,\text{ax}}{P \Rightarrow Q, (P \wedge Q), \vdash P}\,\wedge e_1}{P \Rightarrow Q \vdash (P \wedge Q) \Rightarrow P}\,\Rightarrow\text{i}}{P \Rightarrow Q \vdash ((P \wedge Q) \Rightarrow P) \wedge (P \Rightarrow (P \wedge Q))}\,\wedge\text{i}$$

VII.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P}\,\text{ax} \quad \dfrac{}{\Gamma \vdash P \Rightarrow (Q \wedge R)}\,\text{ax}}{\Gamma \equiv P \Rightarrow (Q \wedge R), P \vdash Q \wedge R}\,\Rightarrow\text{e}}{P \Rightarrow (Q \wedge R), P \vdash R}\,\wedge e_2}{P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow R)}\,\Rightarrow\text{i} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{P \Rightarrow (Q \wedge R), P \vdash P}\,\text{ax} \quad \dfrac{}{P \Rightarrow (Q \wedge R), P \vdash P \Rightarrow (Q \wedge R)}\,\text{ax}}{P \Rightarrow (Q \wedge R), P \vdash (Q \wedge R)}\,\Rightarrow\text{e}}{P \Rightarrow (Q \wedge R), P \vdash Q}\,\wedge e_1}{P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow Q)}\,\Rightarrow\text{i}}{P \Rightarrow (Q \wedge R) \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)}\,\wedge\text{i}$$

VIII.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)}\,\text{ax}}{\Gamma \vdash (P \Rightarrow R)}\,\wedge e_2}{\Gamma \vdash R}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma \vdash P}\,\text{ax} \quad \dfrac{\dfrac{}{\Gamma \vdash (P \Rightarrow Q) \wedge (P \Rightarrow R)}\,\text{ax}}{\Gamma \vdash P \Rightarrow Q}\,\wedge e_1}{\Gamma \vdash Q}\,\Rightarrow\text{e}}{\Gamma \equiv (P \Rightarrow Q) \wedge (P \Rightarrow R), P \vdash (Q \wedge R)}\,\wedge\text{i}}{(P \Rightarrow Q) \wedge (P \Rightarrow R) \vdash P \Rightarrow (Q \wedge R)}\,\Rightarrow\text{i}$$

IX.
$$\dfrac{\dfrac{}{P \vdash P}\,\text{ax}}{P \vee (P \wedge Q) \vdash P}\,\vee i_1$$

X.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, R \vdash R}\,\text{ax} \quad \dfrac{}{\Gamma, R \vdash R \Rightarrow S}\,\text{ax}}{\Gamma, R \vdash S}\,\Rightarrow\text{e} \quad \dfrac{\dfrac{}{\Gamma, Q \vdash Q}\,\text{ax} \quad \dfrac{}{\Gamma, Q \vdash Q \Rightarrow S}\,\text{ax}}{\Gamma, Q \vdash S}\,\Rightarrow\text{e}}{\Gamma \equiv P \Rightarrow (Q \vee R),\ Q \Rightarrow S,\ R \Rightarrow S, P \vdash S}\,\Rightarrow\text{i}}{P \Rightarrow (Q \vee R),\ Q \Rightarrow S,\ R \Rightarrow S \vdash P \Rightarrow S}\,\Rightarrow\text{i} \quad \dfrac{\dfrac{}{\Gamma \vdash P}\,\text{ax} \quad \dfrac{}{\Gamma \vdash P \Rightarrow (Q \vee R)}\,\text{ax}}{\Gamma \vdash Q \vee R}\,\Rightarrow\text{e}}{\Gamma \equiv P \Rightarrow (Q \vee R),\ Q \Rightarrow S,\ R \Rightarrow S, P \vdash S}\,\vee\text{e}$$

Ejercicio 13

Probar que los siguientes secuentes son válidos:

I. $\overline{\neg P \Rightarrow \neg Q \vdash Q \Rightarrow P}$

II. $\overline{\neg P \vee \neg Q \vdash \neg(P \wedge Q)}$

III. $\overline{\neg P,\ P \vee Q \vdash Q}$

IV. $\overline{P \vee Q,\ \neg Q \vee R \vdash P \vee R}$

V. $\overline{P \wedge \neg P \vdash \neg(R \Rightarrow Q) \wedge (R \Rightarrow Q)}$

VI. $\overline{\neg(\neg P \vee Q) \vdash P}$

VII. $\overline{\vdash \neg P \Rightarrow (P \Rightarrow (P \Rightarrow Q))}$

VIII. $\overline{P \wedge Q \vdash \neg(\neg P \vee \neg Q)}$

IX. $\overline{\vdash (P \Rightarrow Q) \vee (Q \Rightarrow R)}$