# Práctica 3 - PLP

Philips

1er Cuatrimestre 2025

## Deducción natural

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A}\,\text{Ax} \qquad \dfrac{}{\Gamma \vdash A \Rightarrow B}\,\text{Ax}}{\Gamma \vdash B}\,\Rightarrow\text{E}}{\Gamma' \vdash A \Rightarrow B}\,\Rightarrow\text{I}$$

## 1 Ejercicio 5

I. Modus ponens relativizado: ?

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q \Rightarrow \tau}\,\text{ax} \qquad \dfrac{}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \rho \Rightarrow q}\,\text{ax}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q,\ \rho \vdash \tau}\,\Rightarrow\text{e}}{\rho \Rightarrow q \Rightarrow \tau,\ \rho \Rightarrow q \vdash \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\rho \Rightarrow q \Rightarrow \tau \vdash (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}}{\vdash (\rho \Rightarrow q \Rightarrow \tau) \Rightarrow (\rho \Rightarrow q) \Rightarrow \rho \Rightarrow \tau}\,\Rightarrow\text{i}$$

II. Reducción al absurdo:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \bot), p \vdash p \Rightarrow \bot}\,\text{ax} \qquad \dfrac{}{(p \Rightarrow \bot), p \vdash p}\,\text{ax}}{(p \Rightarrow \bot), p \vdash \bot}\,\Rightarrow\text{e}}{(p \Rightarrow \bot) \vdash \neg p}\,\neg\text{i}}{\vdash (p \Rightarrow \bot) \Rightarrow \neg p}\,\Rightarrow\text{i}$$

III. Introducción de la doble negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p, \neg p \vdash p}\,\text{ax} \qquad \dfrac{}{p, \neg p \vdash \neg p}\,\text{ax}}{p, \neg p \vdash \bot}\,\neg\text{e}}{p \vdash \neg\neg p}\,\neg\text{i}}{\vdash p \Rightarrow \neg\neg p}\,\Rightarrow\text{i}$$

IV. Eliminación de la triple negación:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg\neg p, p \vdash \neg\neg p}\,\text{III} \qquad \dfrac{}{\neg\neg\neg p, p \vdash \neg\neg\neg p}\,\text{ax}}{\neg\neg\neg p, p \vdash \bot}\,\neg\text{e}}{\neg\neg\neg p \vdash \neg p}\,\neg\text{i}}{\vdash \neg\neg\neg p \Rightarrow \neg p}\,\Rightarrow\text{i}$$

V. Contraposición:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \neg\sigma}\,\text{ax} \qquad \dfrac{\dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p}\,\text{ax} \quad \dfrac{}{(p \Rightarrow \sigma), \neg\sigma, p \vdash p \Rightarrow \sigma}\,\text{ax}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \sigma}\,{\Rightarrow}\text{e}}{(p \Rightarrow \sigma), \neg\sigma, p \vdash \perp}\,\neg\text{e}}{(p \Rightarrow \sigma), \neg\sigma \vdash \neg p}\,\neg\text{i}}{(p \Rightarrow \sigma) \vdash \neg\sigma \Rightarrow \neg p}\,{\Rightarrow}\text{i}}{\vdash (p \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg p)}\,{\Rightarrow}\text{i}
$$

VI. Adjunción: $((p \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (p \Rightarrow \sigma \Rightarrow \tau)$

($\Rightarrow$) ida

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \sigma}\,\text{ax} \quad \dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash p}\,\text{ax}}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma)}\,\wedge\text{i} \qquad \dfrac{}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash (p \wedge \sigma) \Rightarrow \tau}\,\text{ax}}{((p \wedge \sigma) \Rightarrow \tau), p, \sigma \vdash \tau}\,{\Rightarrow}\text{e}}{((p \wedge \sigma) \Rightarrow \tau), p \vdash \sigma \Rightarrow \tau}\,{\Rightarrow}\text{i}}{((p \wedge \sigma) \Rightarrow \tau) \vdash (p \Rightarrow \sigma \Rightarrow \tau)}\,{\Rightarrow}\text{i}}{\vdash ((p \wedge \sigma) \Rightarrow \tau) \Rightarrow (p \Rightarrow \sigma \Rightarrow \tau)}\,{\Rightarrow}\text{i}
$$

($\Rightarrow$) vuelta

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash (p \wedge \sigma)}\,\text{ax}}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma), p \vdash \sigma}\,\wedge e_2}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma}\,{\Rightarrow}\text{i} \qquad \dfrac{}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash p \Rightarrow \sigma \Rightarrow \tau}\,\text{ax}}{(p \Rightarrow \sigma \Rightarrow \tau), (p \wedge \sigma) \vdash \tau}\,{\Rightarrow}\text{e}}{(p \Rightarrow \sigma \Rightarrow \tau) \vdash ((p \wedge \sigma) \Rightarrow \tau)}\,{\Rightarrow}\text{i}}{\vdash (p \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((p \wedge \sigma) \Rightarrow \tau)}\,{\Rightarrow}\text{i}
$$

VII. Ley de Morgan (I): $\neg(p \vee \sigma) \Leftrightarrow (\neg p \wedge \neg\sigma)$

($\Rightarrow$) ida

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg(p \vee \sigma), \sigma \vdash \sigma}\,\text{ax}}{\neg(p \vee \sigma), \sigma \vdash p \vee \sigma}\,\vee i_2 \quad \dfrac{}{\neg(p \vee \sigma), \sigma \vdash \neg(p \vee \sigma)}\,\text{ax}}{\neg(p \vee \sigma), \sigma \vdash \perp}\,\neg\text{e}}{\neg(p \vee \sigma) \vdash \neg\sigma}\,\neg\text{i} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma \vdash p \vee \sigma}\,\vee i_1 \quad \dfrac{}{\Gamma \vdash \neg(p \vee \sigma)}\,\text{ax}}{\Gamma \equiv \neg(p \vee \sigma), p \vdash \perp}\,\neg\text{e}}{\neg(p \vee \sigma) \vdash \neg p}\,\neg\text{i}}{\neg(p \vee \sigma) \vdash \neg p \wedge \neg\sigma}\,\wedge\text{i}}{\vdash \neg(p \vee \sigma) \Rightarrow (\neg p \wedge \neg\sigma)}\,{\Rightarrow}\text{i}
$$

($\Rightarrow$) vuelta

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \sigma \vdash \neg p \wedge \neg\sigma}\,\text{ax}}{\Gamma, \sigma \vdash \neg\sigma}\,\wedge e_2 \quad \dfrac{}{\Gamma, \sigma \vdash \sigma}\,\text{ax}}{\Gamma, \sigma \vdash \perp}\,\neg\text{e} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, p \vdash \neg p \wedge \neg\sigma}\,\text{ax}}{\Gamma, p \vdash \neg p}\,\wedge e_1 \quad \dfrac{}{\Gamma \vdash p}\,\text{ax}}{\Gamma, p \vdash \perp}\,\neg\text{e} \qquad \dfrac{}{\Gamma \vdash p \vee \sigma}\,\text{ax}}{\Gamma \equiv (\neg p \wedge \neg\sigma), (p \vee \sigma) \vdash \perp}\,\vee\text{e}}{(\neg p \wedge \neg\sigma) \vdash \neg(p \vee \sigma)}\,\neg\text{i}}{\vdash (\neg p \wedge \neg\sigma) \Rightarrow \neg(p \vee \sigma)}\,{\Rightarrow}\text{i}
$$

VIII. Ley de Morgan (II): $\neg(p \wedge \sigma) \Leftrightarrow (\neg p \vee \neg \sigma)$

(⇒) ida

$$\dfrac{\dfrac{\dfrac{}{\neg(\rho\wedge\sigma),\neg p\vdash\neg\rho}\,\text{ax}}{\neg(\rho\wedge\sigma),\neg p\vdash\neg\rho\vee\neg\sigma}\,\vee i_1 \qquad \dfrac{}{\neg(\rho\wedge\sigma)\vdash p\vee\neg p}\,\text{LEM} \qquad \dfrac{\text{Sigo este caso abajo}}{\neg(\rho\wedge\sigma),p\vdash\neg\rho\vee\neg\sigma}\,(*)}{\dfrac{\neg(\rho\wedge\sigma)\vdash\neg\rho\vee\neg\sigma}{\vdash\neg(\rho\wedge\sigma)\Rightarrow(\neg\rho\vee\neg\sigma)}\,\Rightarrow i}\,\vee e$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\sigma\vdash p}\,\text{ax} \quad \dfrac{}{\Gamma,\sigma\vdash\sigma}\,\text{ax}}{\Gamma,\sigma\vdash\rho\wedge\sigma}\,\wedge i \quad \dfrac{}{\Gamma,\sigma\vdash\neg(\rho\wedge\sigma)}\,\text{ax}}{\Gamma,\sigma\vdash\bot}\,\neg e}{\Gamma,\sigma\vdash\neg\rho\vee\neg\sigma}\,\bot\ e \qquad \dfrac{\dfrac{}{\Gamma,\neg\sigma\vdash\neg\sigma}\,\text{ax}}{\Gamma,\neg\sigma\vdash\neg\rho\vee\neg\sigma}\,\vee i_2 \qquad \dfrac{}{\Gamma\vdash\sigma\vee\neg\sigma}\,\text{LEM}}{\Gamma\vdash\neg\rho\vee\neg\sigma \quad \{*\}}\,\vee e$$

(⇒) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\neg\sigma,p\vdash p\wedge\sigma}\,\text{ax}}{\Gamma,\neg\sigma,p\vdash\sigma}\,\wedge e_2 \quad \dfrac{}{\Gamma,\neg\sigma,p\vdash\neg\sigma}\,\text{ax}}{\dfrac{\Gamma,\neg\sigma,p\vdash\bot}{\Gamma,\neg\sigma\vdash\neg p}\,\neg\text{i}} \quad \dfrac{}{\Gamma,\neg p\vdash\neg p}\,\text{ax} \quad \dfrac{}{\Gamma\vdash\neg p\vee\neg\sigma}\,\text{ax}}{\Gamma\vdash\neg p} \qquad \dfrac{\dfrac{}{\Gamma\vdash p\wedge\sigma}\,\text{ax}}{\Gamma\vdash p}\,\wedge e_1}{\Gamma\equiv(\neg\rho\vee\neg\sigma),(\rho\wedge\sigma)\vdash\bot}\,\neg\text{e}}{\dfrac{(\neg\rho\vee\neg\sigma)\vdash\neg(\rho\wedge\sigma)}{\vdash(\neg\rho\vee\neg\sigma)\Rightarrow\neg(\rho\wedge\sigma)}\,\Rightarrow\text{i}}\,\neg\text{i}$$

IX. Conmutatividad ($\wedge$):

$$\dfrac{\dfrac{\dfrac{}{(p\wedge\sigma)\vdash(p\wedge\sigma)}\,\text{ax}}{(p\wedge\sigma)\vdash p}\,\wedge e_1 \qquad \dfrac{\dfrac{}{(p\wedge\sigma)\vdash(p\wedge\sigma)}\,\text{ax}}{(p\wedge\sigma)\vdash\sigma}\,\wedge e_2}{\dfrac{(p\wedge\sigma)\vdash(\sigma\wedge p)}{\vdash(p\wedge\sigma)\Rightarrow(\sigma\wedge p)}\,\Rightarrow\text{i}}\,\wedge\text{i}$$

X. Asociatividad ($\wedge$): $((p\wedge\sigma)\wedge\tau) \Leftrightarrow (p\wedge(\sigma\wedge\tau))$

(⇒) ida

$$\dfrac{\dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash\tau}\,\wedge e_2 \quad \dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge\sigma)}\,\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash\sigma}\,\wedge e_2}{((p\wedge\sigma)\wedge\tau)\vdash(\sigma\wedge\tau)} \qquad \dfrac{\dfrac{\dfrac{}{((p\wedge\sigma)\wedge\tau)\vdash((p\wedge\sigma)\wedge\tau)}\,\text{ax}}{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge\sigma)}\,\wedge e_1}{((p\wedge\sigma)\wedge\tau)\vdash p}\,\wedge e_1}{\dfrac{((p\wedge\sigma)\wedge\tau)\vdash(p\wedge(\sigma\wedge\tau))}{\vdash((p\wedge\sigma)\wedge\tau)\Rightarrow(p\wedge(\sigma\wedge\tau))}\,\Rightarrow\text{i}}\,\wedge\text{i}$$

(⇒) vuelta

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash(p\wedge(\sigma\wedge\tau))}\,\text{ax}}{\Gamma\vdash(\sigma\wedge\tau)}\,\wedge e_2}{\Gamma\vdash\sigma}\,\wedge e_1 \quad \dfrac{\dfrac{}{\Gamma\vdash(p\wedge(\sigma\wedge\tau))}\,\text{ax}}{\Gamma\vdash p}\,\wedge e_1}{\Gamma\vdash p\wedge\sigma} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma\vdash p\wedge(\sigma\wedge\tau)}\,\text{ax}}{\Gamma\vdash(\sigma\wedge\tau)}\,\wedge e_2}{\Gamma\vdash\tau}\,\wedge e_2}{\dfrac{\Gamma\equiv(p\wedge(\sigma\wedge\tau))\vdash((p\wedge\sigma)\wedge\tau)}{\vdash(p\wedge(\sigma\wedge\tau))\Rightarrow((p\wedge\sigma)\wedge\tau)}\,\Rightarrow\text{i}}\,\wedge\text{i}$$

XI. Conmutatividad ($\lor$)

$$\dfrac{\dfrac{\dfrac{}{(p\lor\sigma),\sigma\vdash\sigma}\,\text{ax}}{(p\lor\sigma),\sigma\vdash(\sigma\lor p)}\,\lor i_1 \qquad \dfrac{\dfrac{}{(p\lor\sigma),p\vdash p}\,\text{ax}}{(p\lor\sigma),p\vdash(\sigma\lor p)}\,\lor i_2 \qquad \dfrac{}{(p\lor\sigma)\vdash(p\lor\sigma)}\,\text{ax}}{\dfrac{(p\lor\sigma)\vdash(\sigma\lor p)}{\vdash(p\lor\sigma)\Rightarrow(\sigma\lor p)}\,\Rightarrow\text{i}}\,\lor\text{e}$$

XII. Asociatividad ($\lor$): $((p\lor\sigma)\lor\tau)\Leftrightarrow(p\lor(\sigma\lor\tau))$

($\Rightarrow$) ida

$$\dfrac{\dfrac{}{((p\lor\sigma)\lor\tau)\vdash(p\lor(\sigma\lor\tau))}}{\vdash((p\lor\sigma)\lor\tau)\Rightarrow(p\lor(\sigma\lor\tau))}\,\Rightarrow\text{i}$$

($\Rightarrow$) vuelta

$$\dfrac{}{\vdash(p\lor(\sigma\lor\tau))\Rightarrow((p\lor\sigma)\lor\tau)}$$