

Textbook School Mathematics and the preparation of mathematics teachers*

Stony Brook, October 27, 2014

H. Wu

*A slightly expanded version of the talk on October 27, 2014. I am enormously indebted to Larry Francis for his many suggested improvements.

School mathematics education has been national news for at least two decades.

There have been constant reminders about our students' underachievement in TIMMS, PISA, and NAEP. There have been controversies over the Math Wars. There have been two editions within five years of the report commissioned by Congress on the looming crisis in science and technology, **Rising Above the Gathering Storm**. And in the past two years, the pros and cons of the Common Core Standards have flooded the internet.

Through it all, one conclusion is inescapable: school mathematics education is in crisis.

One can approach this crisis from many angles. Looking into either the administrative incompetence in the education establishment or the *faux pas* in assessment would take up a whole day all by itself. But for people in academia, I think the two most pressing concerns would have to be:

School textbooks are no good.

Teaching in the school classroom is no good.

Mathematicians like to attack problems head-on. To us, the solution is simple:

Write better school textbooks.

Design better teacher preparation.

I can tell you long stories behind these simple statements, but for now I will concentrate on teacher preparation and leave textbooks to the end. Moreover, while pedagogy is important, *the bigger issue by far is teachers' content knowledge deficit*. The rest of this talk will be essentially about the content knowledge deficit.

The kind of content knowledge math teachers need should

- (A) closely parallel what is taught in the *school* classroom,[†] and
- (B) be consistent with the
fundamental principles of mathematics (FPM).

These two requirements pull in opposite directions, and this is what makes math education nontrivial.

[†]This point is stressed in The mis-education of mathematics teachers, and is consonant with the viewpoint of D. K. Cohen & H. C. Hill, *Learning policy: When state education reform works*. Yale University Press, 2001.

Fundamental Principles of Mathematics

- Every concept has a **definition**.
- Every statement is **precise** about what is true and what is not true.
- Every statement is supported by **reasoning**.
- Mathematics is **coherent**: the concepts and skills are logically intertwined to form a whole tapestry.
- Mathematics is **purposeful**: there is a purpose to each skill and concept.

The presence of the FPM in no way implies that we teach teachers only in the definition-theorem-proof format of university mathematics! (Remember (A) above.)

Definitions can be handled in a grade-appropriate manner without sacrificing the FPM. For example, the definition of a fraction can be informal in grades 3 and 4 and, while a formal definition can be introduced in grade 5, it will *not* be an equivalence class of ordered pairs of integers (see, e.g., Section 12.2 of this book).

Reasoning should also be grade-appropriate. For example, the three-line proof of *negative × negative = positive* in abstract algebra can be expanded for the consumption of middle school students (e.g., pp. 405–410 of this book). The location of the max and min of a quadratic function can be done by an elementary method (see Section 11 of this article) without the use of the derivative.

Another example: the proof of the laws of exponents using $\log x$ and $\exp x$ in calculus can be customized for use in high school (see Section 10 of this article). And so on.

The overriding theme throughout this talk will be *the need to provide teachers with content knowledge that informs and parallels school mathematics.*

Teachers must try to teach school mathematics by following the FPM as much as possible because doing so makes mathematics *more teachable* and *more learnable*.

For example, if teachers can give a reason for a skill, there would be no need to browbeat students into memorizing procedures. Or, by explaining to students the *purpose* of the concept or skill they are going to learn, teachers can get more buy-in from them and thereby improve their learning outcomes.

Take the **laws of exponents** for *rational* exponents:

$$a^{\frac{m}{n}} a^{\frac{k}{\ell}} = a^{\frac{m}{n} + \frac{k}{\ell}}$$

$$(a^{\frac{m}{n}})^{\frac{k}{\ell}} = a^{\frac{m}{n} \cdot \frac{k}{\ell}}$$

$$a^{\frac{m}{n}} b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$$

Students are usually taught these laws as mindless procedures for simplifying expressions on standardized tests.

This is understandable because the definition of rational exponents such as $3^{-4/5}$ is usually presented to them as just a *different notation* for $\frac{1}{\sqrt[5]{3^4}}$.

Therefore, $3^{1/2}$ is just a fancy way to rewrite $\sqrt{3}$, and $3^{1/5}$ is just a fancy way to rewrite $\sqrt[5]{3}$. Nothing more.

Why spend all that effort just to learn a new notation?

In the next slide, you will see a textbook introduce fractional exponents by throwing the formula $p = 50f^{0.2}$ at students for no apparent reason. *Why not just write $p = 50\sqrt[5]{f}$ instead?*

Evaluate and rewrite expressions involving rational exponents.

Solve equations involving expressions with rational exponents.

It's important to protect your skin with sunscreen to prevent damage. The sun protection factor (SPF) of a sunscreen indicates how well it protects you. Sunscreen with an SPF of f absorbs about p percent of the UV-B rays, where $p = 50f^{0.2}$.



Rational Exponents You know that an exponent represents the number of times that the base is used as a factor. But how do you evaluate an expression with an exponent that is not an integer like the one above? Let's investigate **rational exponents** by assuming that they behave like integer exponents.

$$\begin{aligned} \left(b^{\frac{1}{2}}\right)^2 &= b^{\frac{1}{2}} \cdot b^{\frac{1}{2}} && \text{Multiplication of exponents} \\ &= b^{\frac{1}{2} + \frac{1}{2}} && \text{Combine like bases.} \\ &= b^1 \text{ or } b && \text{Identity.} \end{aligned}$$

Thus, $b^{\frac{1}{2}}$ is a number with a square equal to b . So $b^{\frac{1}{2}} = \sqrt{b}$.



Concept $b^{\frac{1}{2}}$

Words

For any nonnegative real number b , $b^{\frac{1}{2}} = \sqrt{b}$.

Examples

$16^{\frac{1}{2}} = \sqrt{16}$ or 4

$38^{\frac{1}{2}} = \sqrt{38}$

Suppose we tell students, instead, that there is an *interpolation* (a way to “connect the dots”, see the next two slides) of the function

$$n \mapsto 3^n \quad \text{defined on all positive integers } n$$

to a function

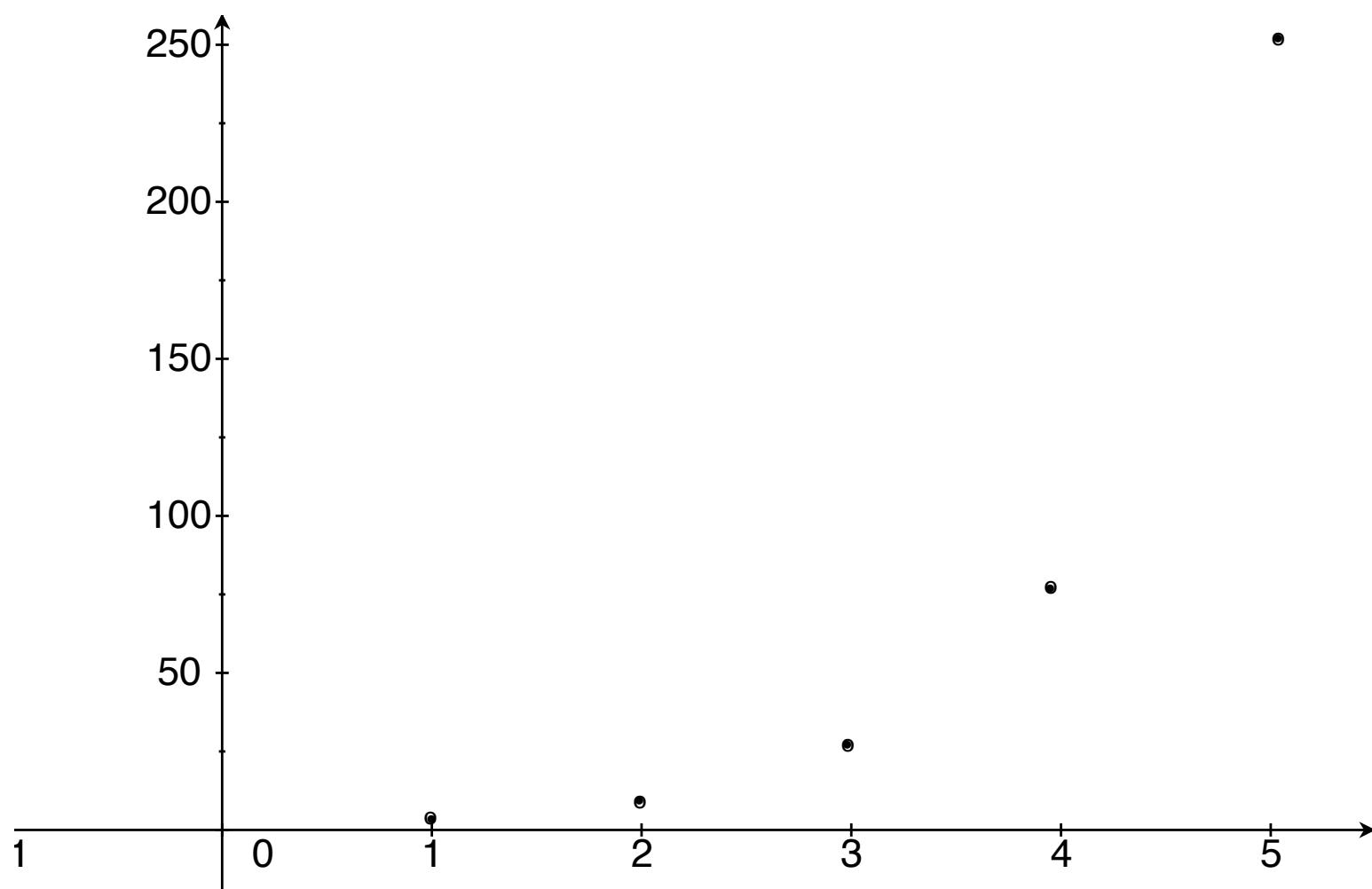
$$x \mapsto 3^x \quad \text{defined for all real numbers } x$$

so that it satisfies the laws of exponents, e.g.,

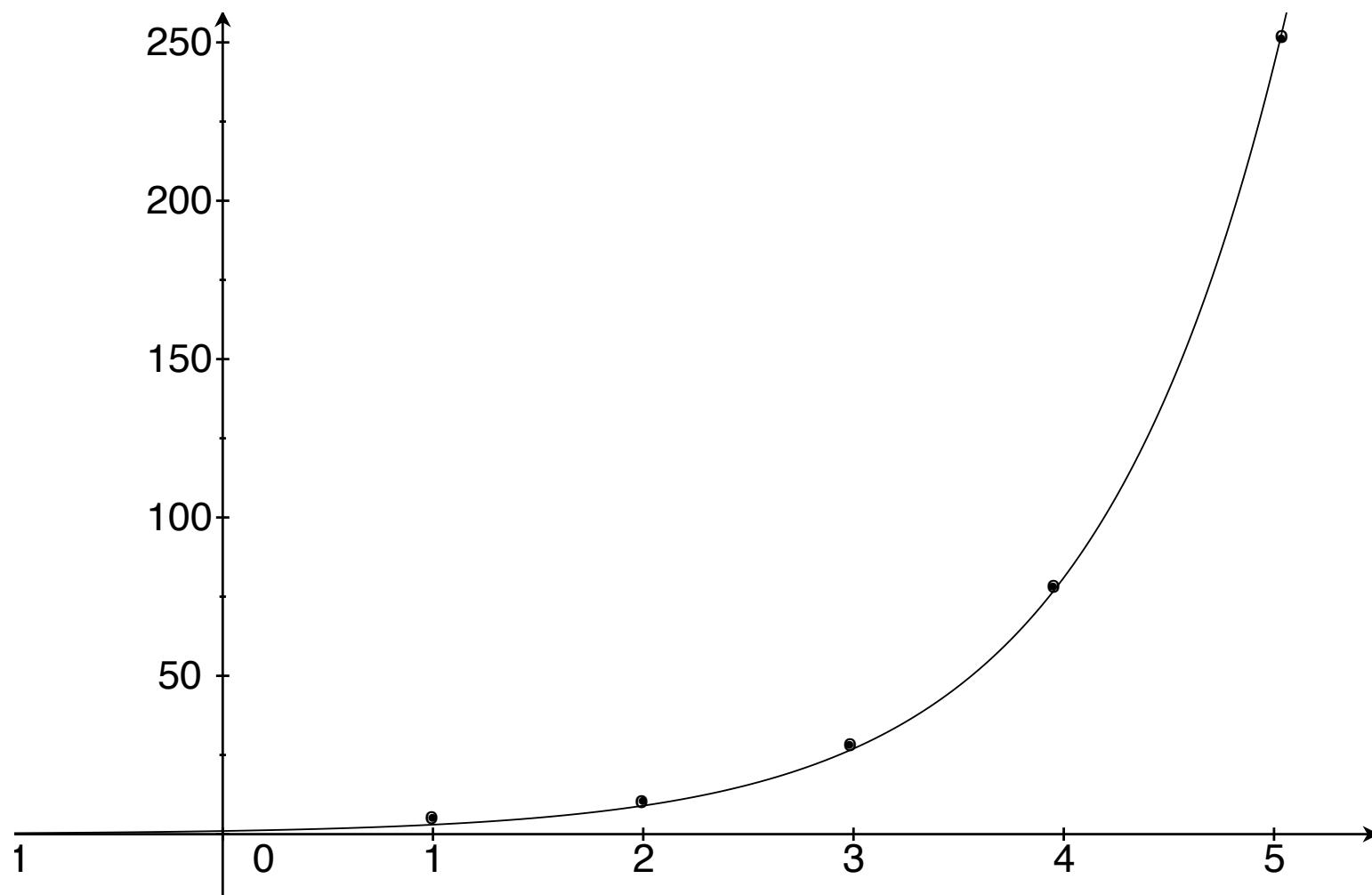
$$3^u \times 3^v = 3^{u+v} \quad \text{and} \quad (3^u)^v = 3^{uv}, \quad \text{etc.},$$

for all real numbers u and v .

We also tell them that these functions $x \mapsto 3^x$ are the ones that describe natural phenomena related to growth and decay.



Graph of $n \mapsto 3^n$



Graph of $x \mapsto 3^x$

Then on the basis of

$$3^u \times 3^v = 3^{u+v} \quad \text{and} \quad (3^u)^v = 3^{uv}$$

for all real numbers u and v , we prove that

$$3^{-4/5} = \frac{1}{\sqrt[5]{3^4}}$$

and, in general, for positive integers m and n , we prove that

$$3^{-m/n} = \frac{1}{\sqrt[n]{3^m}}$$

This explains why the definitions of rational exponents are what they are and why we need them in mathematics.

In particular, the laws of exponents describe some remarkable properties of the **exponential functions** $x \mapsto a^x$.

The *purpose* of studying the laws of exponents is therefore to get to know the basic properties of a special class of functions that show up each time we look at a natural phenomenon related to growth and decay.

In particular, the laws of exponents describe some remarkable properties of the **exponential functions** $x \mapsto a^x$.

The *purpose* of studying the laws of exponents is therefore to get to know the basic properties of a special class of functions that show up each time we look at a natural phenomenon related to growth and decay.

If students see the laws of exponents from this perspective, they just might learn more and better.

At the moment these fundamental principles of mathematics play almost no role in school classrooms.

Teachers' teaching and students' learning are based on **Textbook School Mathematics (TSM)**, the mathematics in school textbooks for at least the past 40 years.

TSM contradicts almost everything in the FPM. Here are the salient characteristics of TSM.

1. Definitions are absent or mangled.

For example:

number

the concept of “less than”

product of fractions

decimal

percent

length, area, volume

variable

congruence

graph of an equation

exponential function

fraction

the concept of “equal”

quotient of fractions

negative number

rate, constant rate

slope

expression

similarity

graph of an inequality

logarithm

Comments:

Anyone not familiar with K–12 will not believe that these fundamental concepts could be left undefined, or less than completely defined, in TSM.

Here are some TSM “definitions”: a **fraction** is “part of a whole” or a “piece of pizza”, but not a number; **percent** is “out of a hundred”; a **variable** is a “quantity that changes or varies”; **congruent** is “same size and same shape”; **similar** is “same shape but not necessarily the same size”, and so on.

Number? negative number? rate? constant rate? length of a curve? area of a region? volume of a solid?

Use your imagination.

2. No precision.

TSM blurs the line between reasoning and heuristics: often it is not clear whether a statement is an assumption, a theorem, or a definition. For example, consider the following claims:

A fraction is a ratio.

A fraction $\frac{a}{b}$ is the division $a \div b$.

$a^0 = 1$ for all positive a .

$a^{-m} = \frac{1}{a^m}$ for all positive a and all integers m .

The graph of a quadratic function is a parabola.

Two nonvertical lines are parallel (or coincide) if they have the same slope, and are perpendicular if the product of their slopes is -1 .

The graph of an inequality $ax + by < c$ is a half-plane.

Rational expressions can be added, subtracted, multiplied, and divided like fractions.

The graph of a quadratic function is a parabola.

Two nonvertical lines are parallel (or coincide) if they have the same slope, and are perpendicular if the product of their slopes is -1 .

The graph of an inequality $ax + by < c$ is a half-plane.

Rational expressions can be added, subtracted, multiplied, and divided like fractions.

Many teachers, including a good many in high school, do not know the difference between a definition and a theorem.

Comments:

A reasonable explanation of any of these statements would require more time than is at my disposal, but the following two pages from an Algebra I textbook on parallel and perpendicular lines will serve to give a glimpse into this aspect of TSM.

Notice that on the first page, the theorem that “*nonvertical lines are parallel if they have the same slope and different y -intercepts*” is given as a *Key Concept*. Naturally, no proof is given.

The same comment applies to the situation with *perpendicular lines* on the second page.

Two distinct lines in a coordinate plane either intersect or are *parallel*. **Parallel lines** are lines in the same plane that never intersect.

Essential Understanding You can determine the relationship between two lines by comparing their slopes and *y*-intercepts.



Key Concept Slopes of Parallel Lines

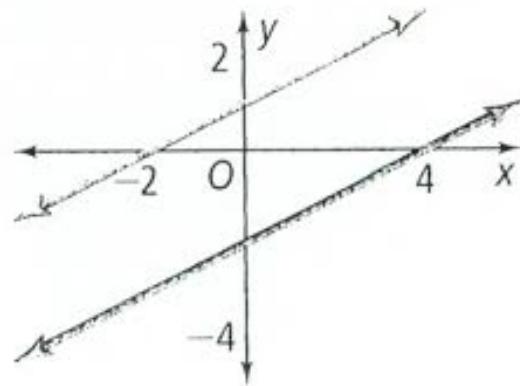
Words

Nonvertical lines are parallel if they have the same slope and different *y*-intercepts. Vertical lines are parallel if they have different *x*-intercepts.

Example

The graphs of $y = \frac{1}{2}x + 1$ and $y = \frac{1}{2}x - 3$ are lines that have the same slope, $\frac{1}{2}$, and different *y*-intercepts. The lines are parallel.

Graph



You can use the fact that the slopes of parallel lines are the same to write the equation of a line parallel to a given line.

You can also use slope to determine whether two lines are *perpendicular*.
Perpendicular lines are lines that intersect to form right angles.

Take note

Key Concept Slopes of Perpendicular Lines

Words

Two nonvertical lines are perpendicular if the product of their slopes is -1 . A vertical line and a horizontal line are also perpendicular.

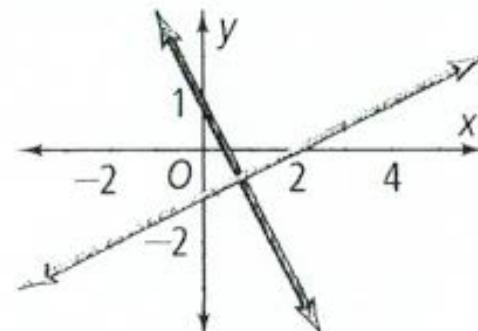
Example

The graph of $y = \frac{1}{2}x + 1$ has a slope of $\frac{1}{2}$.

The graph of $y = -2x - 1$ has a slope of -2 .

Since $\frac{1}{2}(-2) = -1$, the lines are perpendicular.

Graph



Two numbers whose product is -1 are **opposite reciprocals**. So, the slopes of perpendicular lines are opposite reciprocals. To find the opposite reciprocal of $-\frac{3}{4}$, for example, first find the reciprocal, $-\frac{4}{3}$. Then write its opposite, $\frac{4}{3}$. Since $-\frac{3}{4} \cdot \frac{4}{3} = -1$, $\frac{4}{3}$ is the opposite reciprocal of $-\frac{3}{4}$.

3. Grade-appropriate reasoning is absent or flawed.

This is the inevitable consequence of having no definitions or flawed definitions.

Why division of fractions is equal to invert-and-multiply.

Why a fraction can be converted to a decimal by the long division of the numerator by the denominator.

Why $(-a)(-b) = ab$.

Why $\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$.

Why we can solve an equation by manipulating symbols.

Why the graph of $ax + by = c$ is a line.

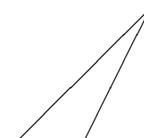
Why simultaneous linear equations can be solved by taking the coordinates of the point of intersection of the graphs.

Why the minimum of $f(x) = ax^2 + bx + c$ lies on the line $x = -\frac{b}{2a}$.

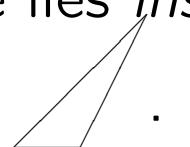
Why c is a root of a polynomial $p(x)$ if $p(c) = 0$.

Why $\sin x = \cos(x - 90)$ and $\cos x = -\sin(x - 90)$ for *all* real numbers x .

Why the area of a triangle is $\frac{1}{2}$ base times height:



Comments:

- The lack of any proof for $(-a)(-b) = ab$ is too well-known for comments.
- Below, we will discuss in some detail how an equation is solved in TSM and the reason that the graph of $ax + by = c$ is a line.
- The transition of sine and cosine from functions of acute angles to functions defined on \mathbf{R} is done with hand-waving, and students end up learning the basic identities such as $\sin x = \cos(x - 90)$ *for all $x \in \mathbf{R}$* by rote.
- The area formula for a triangle is almost always proved *only* for the case where the foot of the altitude lies *inside* the base of the triangle  , but not for the case  .

4. Incoherence. TSM makes a *disjointed* presentation of school mathematics.

No continuity from whole numbers to fractions, to rational numbers, and to rational expressions.

No continuity from arithmetic to algebra, and to *formal* algebra.

No continuity from middle school geometry (rotations, reflections, and translations) to high school geometry.

No continuity from length to area and to volume.

5. The purpose of studying anything is usually well hidden.

Reflections, rotations, and translations are taught in middle school as entertainment about art appreciation. (*The relevance to congruence is never mentioned.*)

Completing the square is taught as a trick to get the quadratic formula. (*No mention is made of the fact that it is the main tool to bring quadratic polynomials to normal form, e.g., make quadratic functions understandable.*)

Rounding whole numbers and decimals is taught as a mindless rote-skill. (*No mention is made of the fact that it is a reasonable—and scientific—method to deal with information that is unreliable or of no interest.*)

Absolute value is taught as a rote-skill unrelated to anything else: it is about how to get rid of the negative sign. (*For example, is it related to making estimations?*)

Sine and cosine are mainly taught as part of right-triangle geometry. (*No emphasis on the fact that they are the basic functions defined on \mathbb{R} that model periodic phenomena, such as your voice on the cell phone.*).

Comments on Points 4 and 5:

The incoherence of the TSM-geometry curriculum from middle school to high school deserves a separate comment.

Point 5 mentions that reflections, rotations, and translations are taught in middle school as entertainment about art appreciation. It is against this background that one begins to understand the comment made in Point 4, to the effect that there is no continuity from middle school geometry to high school geometry.

Let us now follow students' journey in TSM-geometry all through K-12.

In elementary and middle school, students are told that *congruence* means “same size and same shape”. In middle school, they learn about reflections, translations, and rotations and use them to gain an appreciation of the beauty of symmetries in nature and in art. Apparently, these transformations are not relevant *within mathematics*.

In a high school geometry course, suddenly students are told that, from now on, every assertion in geometry must be *proved* (although almost nothing else in TSM is ever—or has ever been—proved). They are confronted with *axioms* and *propositions* and *theorems*. They learn about the *congruence of triangles* by measuring angles and sides, and they take ASA, SAS, and SSS on faith. Gone is “same size and same shape”, and gone is the congruence of any figures other than polygons.

At the end of the high school geometry course, sometimes reflections, rotations, and translations make a comeback and are used to “*verify*” that they actually move triangles to congruent triangles; there could even be a hint that, at least for polygons, congruence can be achieved by the use of these transformations.

After so many twists and turns in students’ encounters with “congruence” in K–12, do they know what *congruence* means anymore when they get out of high school? Is it any surprise that many believe geometry is just a sham?

The above list of what is wrong with TSM is by no means exhaustive. The mathematical flaws in TSM are deep and pervasive, and can no way be fully captured in a short list.

Now let us look at the big picture.

TSM is powerful because its presence in the school classroom is *regenerative*. Here is the life-cycle of a math teacher:

- She learns TSM in K–12 as a student,
→ in college she learns either advanced mathematics
or more TSM but not *correct school mathematics*,
→ as a teacher she is forced to regurgitate TSM to
her own students,
→ *some of whom become teachers of the next generation
and in turn inflict TSM on their own students.*

We must help teachers replace their knowledge of TSM with correct school mathematics. This can only be accomplished by sustained hard work: what we are proposing is nothing less than the painstaking reconstruction of teachers' knowledge base.

The only realistic option is to reform pre-service professional development.

But there are obstacles:

- (A) Lack of funding. (*Remember that sustained hard work will be involved. It will take long-term financial commitment.*)
- (B) Lack of awareness of the corrosive effects of TSM.
(This lack occurs in both the education and mathematics communities.)
- (C) Lack of human resources to do the needed professional development. (*This is an inevitable consequence of (B).*)

Obviously, we have to solve problem (B) before worrying about problem (C).

Let us begin with problem (B) and start with university math departments.

Mathematicians tend to believe that the royal road to producing better teachers is to teach more (university) mathematics with redoubled vigor.

Intellectual trickle down theory: School mathematics is thought to be the most trivial and most elementary part of the mathematics that mathematicians do. So once pre-service teachers learn “good” mathematics, they will come to know school mathematics as a matter of course.

Unfortunately,

School mathematics $\not\subset$ **University mathematics**,

in the same way that

Civil engineering $\not\subset$ Newtonian mechanics.

Electrical engineering $\not\subset$ Maxwell's theory of E&M.

Let us illustrate how *school mathematics* is different from *university mathematics*.

Example 1. *What is a fraction?*

University mathematics: A fraction is an equivalence class of ordered pairs of integers $\{(a, b)\}$ so that $b \neq 0$ and $(a, b) \sim (c, d)$ iff $ad = bc$. Then we write $\frac{a}{b}$ for the equivalence class $\{(a, b)\}$.

Can you tell this to ten-year-olds who must begin to learn about fractions? We must abandon the university-level abstraction and devise a grade-appropriate approach for K–12: *use the number line to define and develop fractions.*

Example 2. *What is constant speed?*

University mathematics: The distance function has constant derivative.

Problems of constant speed are a staple of grades 6–11 and, as mentioned above, “constant speed” seems never to be defined in TSM. We must find ways of explaining constant speed to twelve-year-olds without mentioning derivatives.

Example 3. *What is a line in the plane, and what is its slope?*

University mathematics: A **line** is the graph of $ax + by = 0$ where one of a and $b \neq 0$. When $b \neq 0$, the **slope** of the line is, *by definition*, $\frac{-a}{b}$.

Most thirteen-year-olds cannot accept this definition of a line. They are still struggling to grasp what a linear equation of two variables is, and what the *graph of an equation* means. For these students, a line has to be taken in the naive sense of Euclid and its slope has to be defined geometrically (see, e.g., Section 4 of this article).

Example 4. *How to find the extremum of a quadratic function?*

University mathematics: Knee-jerk reaction: differentiate and set the derivative to be equal to 0.

Sixteen-year-olds have to solve optimization problems involving quadratic functions, and they know nothing about calculus. The optimization must be done in a more elementary way, e.g., see Section 12 of this article.

Many, many more examples could be given, but these four should be enough of an indication of the difference between university mathematics and school mathematics.

Moral: Mathematicians cannot help teachers overcome TSM simply by teaching them advanced mathematics.

The advanced courses required of math majors can help pre-service teachers see that there is a *parallel universe* of “good” mathematics where things *probably* make sense.

But does this “good” mathematics have anything to do with the mathematics they will be teaching in schools?

Moral: Mathematicians cannot help teachers overcome TSM simply by teaching them advanced mathematics.

The advanced courses required of math majors can help pre-service teachers see that there is a *parallel universe* of “good” mathematics where things *probably* make sense.

But does this “good” mathematics have anything to do with the mathematics they will be teaching in schools?

If you have been immersed in TSM for thirteen years, you wouldn't believe it. Mathematicians' job is to show teachers, step-by-step, that school mathematics can also be taught in accordance with the FPM.

The romantic notion dies hard, the belief that the exposure to “good” mathematics will be transformative: once teachers experience it, they will never go back to TSM again.

Unfortunately, teachers will not shed the habits acquired over thirteen years of immersion in TSM without a protracted struggle and *without a lot of help*.

The romantic notion dies hard, the belief that the exposure to “good” mathematics will be transformative: once teachers experience it, they will never go back to TSM again.

Unfortunately, teachers will not shed the habits acquired over thirteen years of immersion in TSM without a protracted struggle and *without a lot of help*.

In discussing teacher preparation, *the damage done by TSM to teachers is the stark reality we must face*.

But in mathematics education,

*TSM is the elephant in the room that
everybody wants to ignore.*

It will be instructive to look at three examples.

Example 1. The volume on *The Mathematical Education of Teachers II* (MET2), published by CBMS in 2012, rejects the *intellectual trickle down theory*. So that is good.

Let us consider its recommendation for the preparation of high school teachers, for example. It is to complete the equivalent of a math major *and*

three courses with a primary focus on high school mathematics from an advanced standpoint.

Any hope for real change now resides with these three courses.

Here are the suggested organizing principles for these three courses (MET2, p. 62):

Emphasize the inherent coherence of the mathematics of high school.

Develop a particular mathematical terrain in depth.

Develop mathematics that is useful in teachers' professional lives.

Here are the suggested organizing principles for these three courses (MET2, p. 62):

Emphasize the inherent coherence of the mathematics of high school.

Develop a particular mathematical terrain in depth.

Develop mathematics that is useful in teachers' professional lives.

Observe that no mention is made of the need to undo *thirteen years of mis-education* (e.g., TSM).

The reality is that, unless these courses directly address—and show how to overcome—TSM, future teachers will lose focus (because everything will seem like “more of the same”) and may never become aware of their urgent need *to undo all the misinformation that TSM has fed them for 13 years.*

Will teachers wake up to the importance of definitions in the teaching and learning of mathematics? Will they understand “coherence” if they have only experienced a jumbled and chaotic presentation of mathematics for the first thirteen years of their schooling? Will they accept that every assertion they make in front of their students should be given a reason? And so on.

Consider, for example, how teachers deal with the issue of teaching definitions in the school classroom.

In their years as K-12 students, prospective teachers learned from TSM that *definitions are not important in mathematics*. A definition, they were told, is nothing more than “one more thing to memorize”.

They went through thirteen years of mathematics education that were essentially *definition-free*. But now we ask them to make definitions the foundation of reasoning when it is their turn to teach.

We ask them not only to make a sea change in the way they think about mathematics, but also to *teach K-12 mathematics as “good” mathematics* in accordance with this sea change.

In their years as K-12 students, prospective teachers learned from TSM that *definitions are not important in mathematics*. A definition, they were told, is nothing more than “one more thing to memorize”.

They went through thirteen years of mathematics education that were essentially *definition-free*. But now we ask them to make definitions the foundation of reasoning when it is their turn to teach.

We ask them not only to make a sea change in the way they think about mathematics, but also to *teach K-12 mathematics as “good” mathematics* in accordance with this sea change.

How likely is that?

We want teachers to present all these topics,

fractions, rational numbers, constant rate, slope of a line in the coordinate plane, the graphs of linear equations, rational expressions, congruence, similarity, the graph of a system of linear inequalities, etc.,

by allowing each to unfold logically from a precise definition.

Remember: *TSM has never shown them a precise definition of any of these concepts.* Even if they learn “good” mathematics in college, they still have not seen it in action in the classroom. Not piecemeal, but systematically, lesson by lesson.

We want teachers to teach mathematics by making essential use of definitions—*not in the parallel universe* of “good” mathematics—but *in the real-world setting of a K–12 classroom.*

By ignoring the long-lasting effects of TSM on teachers, those three courses are not likely to be effective in improving teachers’ mathematical content knowledge for K–12 teaching.

I would suggest using these three semesters to do something much more down-to-earth.

Let us show these pre-service teachers how school mathematics can be taught as “good” mathematics by giving

a *systematic exposition* (see the Appendix of this article for an example) of the high school mathematics curriculum that is as close to the way it is taught in the school classroom as possible, but in a way that also respects the FPM.

Such an exposition will at least have a fighting chance of showing teachers, *explicitly*, why definitions are essential in *school mathematics*. The same goes for all the other fundamental principles of mathematics that are routinely abused in TSM.

If we want a sea change in teachers' conception of mathematics, we won't get it done by waving our hands and feeding them generalities of the type suggested for those three courses.

Let us show them how to do it right, from the ground up.

Until we get this done, the more esoteric recommendations from MET2—such as *research experience* for high school teachers—can wait.

Example 2. The CCSSM (2010) have made substantial inroads in steering the *mathematical* presentations of many K–12 topics away from TSM.

But the CCSSM have also prefaced the content standards with eight **Mathematical Practice Standards (MPS)**.

The MPS have come to be widely regarded as “the best thing about the CCSSM” (see p. 16 of the Spring 2013 issue of the NCSM Newsletter). The consensus is more or less that,

Moving forward with the Standards for Mathematical Practice will significantly change student learning in mathematics by dramatically changing how teachers teach.

Nothing about moving forward by freeing teachers from the shackles of TSM. Just “moving forward”.

Here are the eight MPS for students:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

It is perhaps a sign of the times that many have come to equate the CCSSM with the MPS.

The idea has taken root: in order to implement the CCSSM, one can simply concentrate on the MPS. Memorize the MPS, put them up on the walls of classrooms, quiz students on them, and success will come.

In this vein, let us try to evaluate students' *reasoning* (MP2 & MP3) for writing down the equation of a line passing through two given points $(1, 2)$ and $(3, 4)$ in the plane.

TSM only teaches how to do this by rote, because **slope** is defined in TSM as the difference quotient of the coordinates of *two given points* on the line. The education literature follows suit and concentrates on finding great pedagogical strategies to teach slope *according to this misleading definition*.

The CCSSM want slope to be defined correctly so that *any* two points on the line can be used to compute slope. Therefore both of the following give the equation of the line joining (1, 2) and (3, 4): let (x, y) be an arbitrary point on this line, then

$$\frac{y - 2}{x - 1} = \frac{4 - 2}{3 - 1} \quad \text{and} \quad \frac{y - 4}{x - 3} = \frac{4 - 2}{3 - 1}$$

The reasoning is self-evident.

Now MP2 says students should explain how this equation of the line joining $(1, 2)$ and $(3, 4)$ comes about, and MP3 says students should critique each other's reasoning.

If teachers have never seen the slope of a line defined correctly, would they even know what MP2 and MP3 are talking about? How then will their students learn according to MP2 and MP3?

Educators have to wake up to the fact that the Practice Standards won't mean anything to students (or teachers) until they learn non-TSM school mathematics. *[It is not realistic to expect that students \(or teachers\) can unlearn TSM simply by reading the Practice Standards.](#)*

By the same token, as long as the slope of a line is not defined correctly, TSM does not provide the reasoning for explaining why the graph of a linear equation $ax + by = c$ is a line.

It therefore has come to pass that most teachers only know by rote that the graph of $ax + by = c$ is a line. In the absence of adequate textbooks, the students do likewise. There goes MP2—“*Reason abstractly and quantitatively*”—by the wayside.

Teachers know TSM, and almost all school textbooks as of 2014 continue to promote TSM. By now, teachers also know the MPS backwards and forwards, and perhaps so do the students. Yet none of this can provide students with any correct reasoning for writing down the equation of the line passing through two given points or why the graph of $ax + by = c$ is a line.

Teachers know TSM, and almost all school textbooks as of 2014 continue to promote TSM. By now, teachers also know the MPS backwards and forwards, and perhaps so do the students. Yet none of this can provide students with any correct reasoning for writing down the equation of the line passing through two given points or why the graph of $ax + by = c$ is a line.

How is this as a recipe for implementing the CCSSM?

Shouldn't we first try to help teachers replace their knowledge of TSM with correct mathematics?

Example 3. A recent volume, *Principles to Actions* (NCTM 2014), sets itself the goal of describing “the conditions, structures, and policies that must exist for all students to learn”, among other things. We will refer to this volume as **P-to-A**.

P-to-A makes no mention of TSM or the need to help teachers overcome the damage done by TSM.

Nevertheless, it asks teachers to use “purposeful questions” to “help students make important mathematical connections, and support students in posing their own questions”, (P-to-A, pp. 35, 36)

What will teachers say when students ask them for the purpose of learning the laws of exponents?

Nevertheless, it asks teachers to use “purposeful questions” to “help students make important mathematical connections, and support students in posing their own questions”, (P-to-A, pp. 35, 36)

What will teachers say when students ask them for the purpose of learning the laws of exponents?

Remembering what they themselves learned from TSM as students, will these teachers reply, *“To help you ace standardized tests”*?

P-to-A also wants students to “represent, discuss, and make connections among mathematical ideas in multiple forms”.
(P-to-A, p. 24)

In TSM, concepts are rarely given correct definitions. When students don't even know what a concept is (i.e., without a definition), how can they talk about “multiple forms” of something they only “sort of know” ?

For example, should teachers just follow TSM and the standard education research literature to tell students about the “multiple personalities” of a fraction (a part-to-whole comparison, a decimal, a ratio, an indicated division, an operator, and a measure of continuous and discrete quantities), or should they begin by using the number line to define what a fraction is, emphasizing that a fraction is a *number*?

(It is heartening to note that educators seem to be coming around to this point of view: use the number line to define what a fraction is before doing any fractions arithmetic.)

P-to-A says “effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding” (P-to-A, p. 42).

Most students manage to achieve fluency with the *procedure* of solving linear equations. Let us see what *conceptual understanding* they can achieve about this procedure in TSM.

TSM solves an equation such as $4x - 3 = 2x$ as follows:

Step 1: $-2x + (4x - 3) = -2x + 2x.$

Step 2: $2x - 3 = 0$

Step 3: $(2x - 3) + 3 = 0 + 3$

Step 4: $2x = 3$

Step 5: $x = \frac{3}{2}$

How does Step 1 (adding $-2x$ to both sides), for example, make any sense if TSM does not say what x represents?

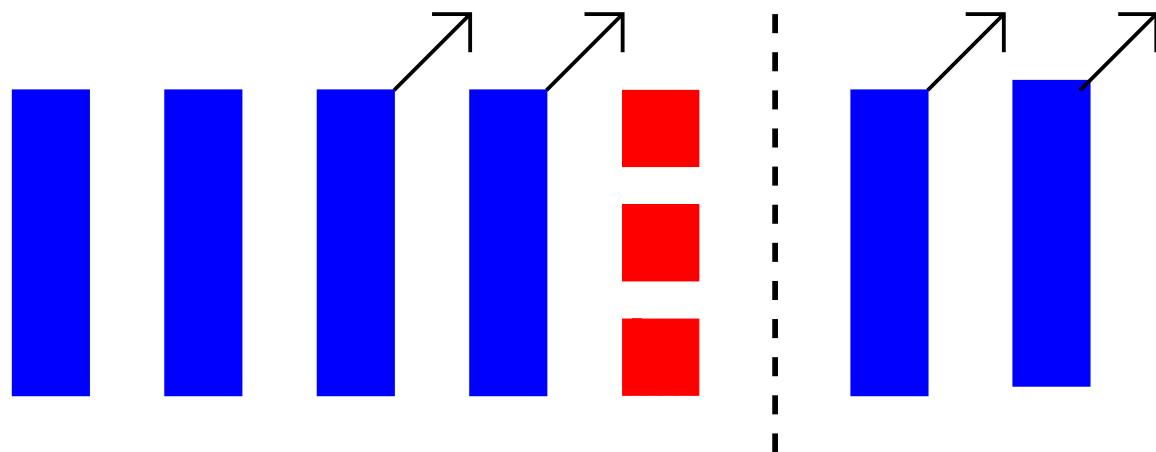
Typically TSM considers x to be an undefinable thing called a “variable”. But $4x - 3$ and $2x$ cannot be equal for a variable number x (both sides are not equal, for example, if $x = 1$), nor can they be equal as elements of the polynomial ring $\mathbf{R}[x]$.

TSM puts forth three different ways to “explain” why adding the $-2x$ to both sides preserves the equality.

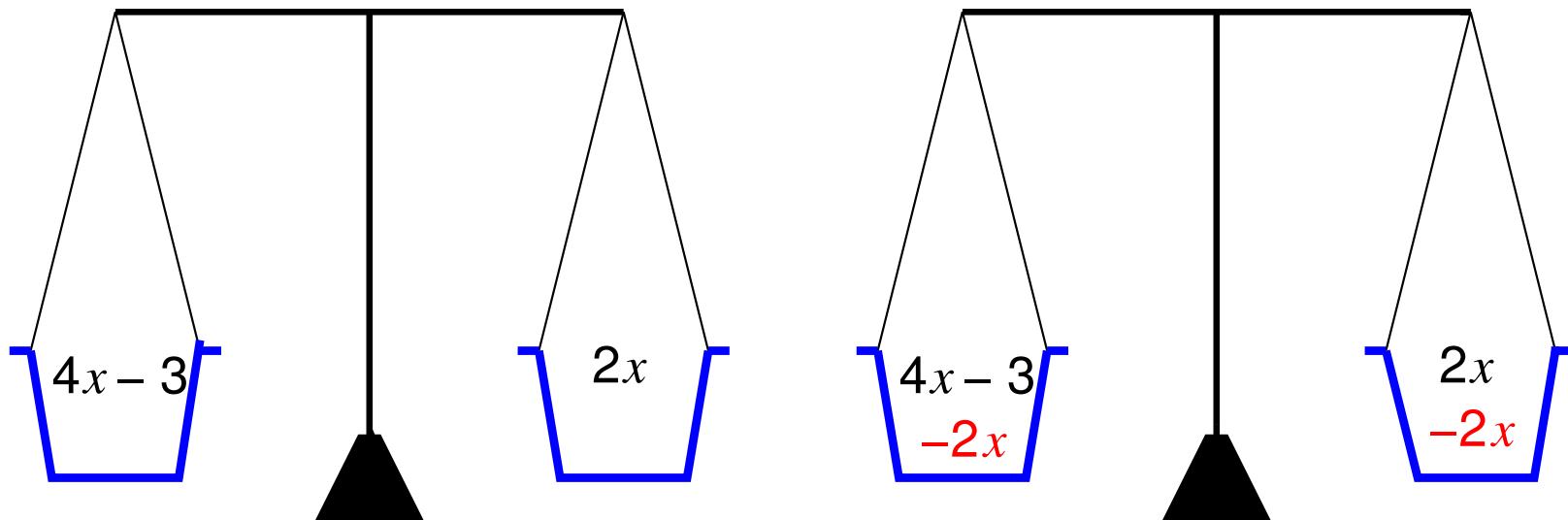
First: Invoke the principle of Euclid that “[Equals added to equals remain equal](#)”.

But equal as *what*? Are $4x - 3$ and $2x$ two numbers or “many” numbers since x is a variable? In what sense are they “equal”? $4x - 3$ is certainly not equal to $2x$ when $x = 1$.

Second: Use algebra tiles to “model” this solution of $4x - 3 = 2x$. Let a blue tile model x and a red square model -1 . It seems “natural” that if we remove two blue tiles on the left, we should also remove two blue tiles on the right.



Third: Use a balance scale to “model” this solution of $4x - 3 = 2x$. It seems “obvious” that if we remove $2x$ (whatever it is) from both sides, the scale will stay in balance.



The second and third “explanations” are just metaphors. They have nothing to do with *mathematical* reasoning. See Section 3 of this article.

As long as teachers’ knowledge of TSM is left undisturbed, this kind of pseudo-reasoning will be the high point of their students’ *conceptual understanding* of solving equations.

When P-to-A enthusiastically recommends actions to realize these and other learning goals, it conveniently ignores the fact that our teachers cannot help with these goals when they are saddled with the *damaged knowledge base* of TSM.

The recommended actions in P-to-A are all phrased in facile generalities. P-to-A says teachers should be provided with all the necessary “resources and support that are essential to enacting the [Mathematical Teaching Practices](#)[‡] for effective teaching and learning” (Page 110). There is nothing said about TSM.

The “support” that districts will provide for teachers’ learning is unlikely to be anything other than the business-as-usual professional development that simply recycles TSM (because the professional developers were themselves brought up in TSM). How is that going to help?

[‡]General statements similar to the MPS.

TSM is the elephant in the room that everybody wants to ignore.

This cannot go on.

Finally, textbooks. Why not just concentrate on writing better textbooks to get rid of **Textbook School Mathematics**? Two reasons:

- *The vicious circle syndrome*: Staff writers for major publishers are themselves products of TSM.
- *The bottom-line mentality*: In order to maximize the sales of their books, publishers do not publish anything that teachers (whose knowledge base at the moment is TSM) don't feel comfortable reading and using with their students.[§]

[§]During my talk on October 27 (2014), Alan Tucker made the comment that he had also observed an analogous phenomenon in the arena of assessment.

Annie Keeghan, who has worked in the educational publishing industry for over 20 years, has written an exposé: “Afraid of your child’s Math Textbooks? You should be.” on the industry’s fixation on the bottom line at the expense of quality.

Annie Keeghan, who has worked in the educational publishing industry for over 20 years, has written an exposé: “Afraid of your child’s Math Textbooks? You should be.” on the industry’s fixation on the bottom line at the expense of quality.

For now, the only hope left of getting better school textbooks is for teachers to dump all the TSM-infested textbooks and shout out of their windows, **“I’m as mad as hell, and I’m not going to take this anymore!”** (Paddy Chayefsky, 1976.[¶])

[¶]See his screenplay for the movie, *Network*.

Then, and only then, will the major publishers listen.

Then, and only then, will the major publishers listen.

But this won't happen until teachers reject TSM.

Helping teachers eradicate TSM is therefore not only imperative for improving their content knowledge, but

*it may also be the only way to get
better school math textbooks written.*

For decades, we have made major mistakes in how we prepare teachers. We have consistently ignored their needs and have made it impossible for them to do their job.

Can we not finally do the right thing by doing the hard work of teaching them correct *school mathematics*?

My lecture today is a plea to the mathematics community to fulfill this particular social obligation. This is an enormous task, but

if we mathematicians don't do it, who will?

If most teachers knew school mathematics that respects the fundamental principles of mathematics rather than just TSM, my lecture would have been about how to enhance teachers' content knowledge or about how to give teachers the research experience that MET2 talks about.

When teachers have an adequate knowledge base, much of what is currently recommended for teacher preparation will become relevant.

If most teachers knew school mathematics that respects the fundamental principles of mathematics rather than just TSM, my lecture would have been about how to enhance teachers' content knowledge or about how to give teachers the research experience that MET2 talks about.

When teachers have an adequate knowledge base, much of what is currently recommended for teacher preparation will become relevant.

If *most teachers knew school mathematics that respects the fundamental principles of mathematics . . .*