A type theory for cartesian closed bicategories

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26th June 2019

Bicategories

Categories with axioms 'up to isomorphism'.

Arise where composition is defined by universal property.

Examples throughout mathematics + theoretical computer science:

- Semantics of computation,
- Datatype models,
- Categorical logic,
- Categorical algebra.

Cartesian closed bicategories

Cartesian closed categories ‘up to isomorphism’.

Examples:

- Generalised species and cartesian distributors
  particularly for applications in higher category theory
  (Fiore, Gambino, Hyland, Winskel), (Fiore & Joyal)
- Categorical algebra (operads)
  (Gambino & Joyal)
- Game semantics (concurrent games)
  (Yamada & Abramsky, Winskel *et al.*, Paquet)

Internal monoids

In a category with finite products:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

$\boxed{\text{Unit law}}$

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times M"] \arrow[dr, bend right, "\cong"'] & M \times M \arrow[d, "m"] & M \times 1 \arrow[l, "M \times e"'] \arrow[dl, bend left, "\cong"] \\
& M &
\end{tikzcd}$$

$\boxed{\text{Assoc. law}}$

$$\begin{tikzcd}
(M \times M) \times M \arrow[r, "\cong"] \arrow[d, "m \times M"'] & M \times (M \times M) \arrow[r, "M \times m"] & M \times M \arrow[d, "m"] \\
M \times M \arrow[rr, "m"'] & & M
\end{tikzcd}$$

Internal monoids

In a category with finite products:

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In **Set**: monoids
In **Cat**: strict monoidal categories

Unit law

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Internal pseudomonoids

In **Cat**:

$$1 \xrightarrow{e} M \xleftarrow{m} M \times M$$

$\boxed{\text{Unit 2-cells}}$

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times M"] \arrow[dr, bend right, "\simeq"'] & M \times M \arrow[d, "m"] & M \times 1 \arrow[l, "M \times e"'] \arrow[dl, bend left, "\simeq"] \\
& M &
\arrow[from=1-1, to=2-2, phantom, "\overset{\lambda}{\cong}", near end, shift left=6]
\arrow[from=1-3, to=2-2, phantom, "\overset{\rho}{\cong}", near end, shift right=6]
\end{tikzcd}$$

$\boxed{\text{data}}$

$\boxed{\text{Assoc. 2-cell}}$

$$\begin{tikzcd}
(M \times M) \times M \arrow[r, "\simeq"] \arrow[d, "m \times M"'] & M \times (M \times M) \arrow[r, "M \times m"] \arrow[d, phantom, "\overset{\alpha}{\cong}"] & M \times M \arrow[d, "m"] \\
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$+$ triangle and pentagon laws $\rightsquigarrow$ monoidal category

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...likewise in any fp-bicategory

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\+ triangle and pentagon laws $\rightsquigarrow$ monoidal category

In a CCC every $[X \Rightarrow X]$ becomes a monoid:

$$\left(1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\right)$$

$\boxed{?}$ In a cc-bicategory every $[X \Rightarrow X]$ becomes a pseudomonoid:

$$\left(1 \xrightarrow{\mathrm{Id}_X} [X \Rightarrow X] \xleftarrow{\circ} [X \Rightarrow X] \times [X \Rightarrow X]\right)$$

need to check
coherence laws
(*i.e.* triangle + pentagon)

Coherence

Programme:

1. Construct a type theory $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$ for cartesian closed bicategories
   (this work),
2. Use NBE to prove the type theory is coherent
   bicategorical version of [Fiore2002]
   (my thesis),

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Application:

Algebraic structure definable in every CCC

$\Rightarrow$ algebraic pseudo-structure definable in every cc-bicategory

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   i.e. freeness property for the syntactic model.

Bicategories

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  2-cells
$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}$$

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$$\begin{tikzcd} X \arrow[rr, bend left=60, "f"] \arrow[rr, "f'" description] \arrow[rr, bend right=60, "f''"'] \arrow[rr, phantom, bend left=30, "\Downarrow\alpha"] \arrow[rr, phantom, bend right=30, "\Downarrow\alpha'"] & & Y \end{tikzcd}$$

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- Functors

$$\mathbf{1} \xrightarrow{\mathrm{Id}_X} \mathcal{B}(X, X)$$

$$\mathcal{B}(Y, Z) \times \mathcal{B}(X, Y) \xrightarrow{\circ_{X,Y,Z}} \mathcal{B}(X, Z)$$

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$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & Z \end{tikzcd}$$

Bicategories

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- *Hom-categories* $\big(\mathcal{B}(X, Y), \bullet, \mathrm{id}\big)$:

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  2-cells
$$\begin{tikzcd} X \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] & Y \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

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- Invertible 2-cells

$$(h \circ g) \circ f \xRightarrow{\mathbf{a}_{h,g,f}} h \circ (g \circ f)$$

$$\mathrm{Id}_X \circ f \xRightarrow{\mathbf{l}_f} f$$

$$g \circ \mathrm{Id}_X \xRightarrow{\mathbf{r}_g} g$$

subject to a triangle law and pentagon law.

Cartesian closed bicategories

Bicategories $\mathcal{B}$ equipped with *biuniversal* 1-cells

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$$\text{(fp)} \qquad \pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$$

$$\text{(cc)} \qquad \mathrm{eval} : (A \Rightarrow B) \times A \to B$$

NB: Differ from the 'cartesian bicategories' of Carboni and Walters!

Cartesian closed bicategories

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inducing families of equivalences

$$\mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \quad \simeq \quad \prod_{i=1}^{n} \mathcal{B}(X, A_i)$$

$$\mathcal{B}(X, A \Rightarrow B) \quad \simeq \quad \mathcal{B}(X \times A, B)$$

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inducing families of equivalences

$$\begin{tikzcd} \mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \arrow[r, bend left, "{(\pi_1 \circ -, \ldots, \pi_n \circ -)}"] \arrow[r, phantom, "\perp\simeq"] & \prod_{i=1}^{n} \mathcal{B}(X, A_i) \arrow[l, bend left, "{\langle -, \ldots, = \rangle}"] \end{tikzcd}$$

(tupling)

$$\begin{tikzcd} \mathcal{B}(X, A \Rightarrow B) \arrow[r, bend left, "{\mathrm{eval}_{A,B} \circ (- \times A)}"] \arrow[r, phantom, "\perp\simeq"] & \mathcal{B}(X \times A, B) \arrow[l, bend left, "\lambda"] \end{tikzcd}$$

(currying)

NB: Differ from the 'cartesian bicategories' of Carboni and Walters!

Substitution and composition

In any CCC:

$$[\![x_k[u_1/x_1, \ldots, u_n/x_n]]\!] = [\![u_k]\!] = \pi_k \circ \langle [\![u_1]\!], \ldots, [\![u_n]\!] \rangle$$

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Question: what is bicategorical substitution?

An algebraic theory with substitution:

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- A *substitution* rule

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  such that

$$\begin{aligned} x_k[u_i/x_i] &= u_k \qquad\qquad (1 \leqslant k \leqslant n) \\ t[x_i/x_i] &= t \\ t[u_i/x_i][v_j/y_j] &= t\,[u_i[v_j/y_j]/x_i] \end{aligned}$$

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$\boxed{\textbf{Note: } \text{every clone defines a category}}$

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- Structural isomorphisms

$$\begin{aligned}
\mathsf{p}^{(k)}[u_1, \ldots, u_n] &\xRightarrow{\varrho^{(k)}_{u_\bullet}} u_k \qquad\qquad (1 \leqslant k \leqslant n) \\
t[\mathsf{p}^{(1)}, \ldots, \mathsf{p}^{(n)}] &\xRightarrow{\iota_t} t \\
t\,[u_\bullet]\,[v_\bullet] &\xRightarrow{\mathsf{assoc}_{t;u_\bullet;v_\bullet}} t[v_\bullet[u_\bullet]]
\end{aligned}$$

subject to a triangle law and pentagon law.

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Note: every biclone defines a bicategory

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$$t[\mathsf{p}^{(1)}, \ldots, \mathsf{p}^{(n)}] \xRightarrow{\iota_t} t$$

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subject to a triangle law and pentagon law.

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Judgements:

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Judgements:

- Relating *terms*: $\Gamma \vdash t : B$

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- Relating *rewrites*: $\Gamma \vdash \tau : t \Rightarrow t' : B$

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- Relating *rewrites*: $\Gamma \vdash \tau : t \Rightarrow t' : B$
- Equational theory $\Gamma \vdash \tau \equiv \tau' : t \Rightarrow t' : B$

Vertical composition: $$\frac{\Gamma \vdash \tau' : t' \Rightarrow t'' : B \qquad \Gamma \vdash \tau : t \Rightarrow t' : B}{\Gamma \vdash \tau' \bullet \tau : t \Rightarrow t'' : B}$$

Identities: $$\frac{\Gamma \vdash t : B}{\Gamma \vdash \mathrm{id}_t : t \Rightarrow t : B}$$

A type theory for biclones

A *substitution* functor

$$\mathbb{C}(X_1, \ldots, X_n; Y) \times \prod_{i=1}^{n} \mathbb{C}(\Gamma; X_i) \to \mathbb{C}(\Gamma; Y)$$
$$t, (u_1, \ldots, u_n) \mapsto t[u_1, \ldots, u_n]$$
$$\tau, (\sigma_1, \ldots, \sigma_n) \mapsto \tau[\sigma_1, \ldots, \sigma_n]$$

A type theory for biclones

A *substitution* functor

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Explicit substitution:

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash t : B \qquad (\Delta \vdash u_i : A_i)_{i=1.,n}}{\Delta \vdash t\,\{x_i \mapsto u_i\} : B}$$

$$\frac{x_1 : A_1, \ldots, x_n : A_n \vdash \tau : t \Rightarrow t' : B \qquad (\Delta \vdash \sigma_i : u_i \Rightarrow u_i' : A_i)_{i=1,\ldots,n}}{\Delta \vdash \tau\,\{x_i \mapsto \sigma_i\} : t\,\{x_i \mapsto u_i\} \Rightarrow t'\,\{x_i \mapsto u_i'\} : B}$$

$\rightsquigarrow$ binds the variables $x_1, \ldots, x_n$

A type theory for biclones

Structural isomorphisms $\varrho^{(k)}, \iota, \mathsf{assoc}$

A type theory for biclones

Structural isomorphisms $\varrho^{(k)}, \iota,$ assoc

Distinguished invertible rewrites *e.g.*:

$$\frac{(\Delta \vdash u_i : A_i)_{i=1,\ldots,n}}{x_1 : A_1, \ldots, x_n : A_n \vdash \varrho^{(k)}_{u_\bullet} : x_k \{x_i \mapsto u_i\} \stackrel{\cong}{\Longrightarrow} u_k : A_k} \ (1 \leqslant k \leqslant n)$$

The syntactic model is free

The syntactic model is free

syntactic model

biclone

$$\begin{tikzcd}
\mathrm{Syn}(\mathcal{S}) \arrow[rr, dashed, "\exists!\ \text{strict}\ h^{\#}"] & & \mathcal{C} \\
\mathcal{S} \arrow[u, hook] \arrow[urr, "\forall\, h"'] & &
\end{tikzcd}$$

signature

The syntactic model is free

syntactic model
on unary contexts

bicategory

$$\begin{tikzcd} \mathrm{Syn}(\mathcal{S})\big|_1 \arrow[r, dashed, "\exists!\ \text{strict } h^{\#}"] & \mathcal{B} \\ \mathcal{S}\big|_1 \arrow[u, hook] \arrow[ur, "\forall\ h"'] & \end{tikzcd}$$

unary signature

The syntactic model is free

syntactic model
on unary contexts

bicategory

$$
\begin{tikzcd}
\mathrm{Syn}(\mathcal{S})\big|_1 \arrow[rr, dashed, "\exists!\ \text{strict } h^{\#}"] & & \mathcal{B} \\
\mathcal{S}\big|_1 \arrow[u, hook] \arrow[urr, "\forall\ h"'] & &
\end{tikzcd}
$$

unary signature

$\rightsquigarrow$ An <u>internal language</u> for bicategories.

fp-Bicategories

fp-Bicategories

$\boxed{\text{1-cells}}$

$$\pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$$

$\boxed{\text{Adjoint equivalences}}$

$$\begin{tikzcd} \mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \arrow[rr, bend left, "(\pi_1 \circ -, \ldots, \pi_n \circ -)"] \arrow[rr, phantom, "\perp\simeq"] & & \prod_{i=1}^{n} \mathcal{B}(X, A_i) \arrow[ll, bend left, "\langle -, \ldots, = \rangle"] \end{tikzcd}$$

A type theory for fp-bicategories

1-cells $\pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$

1-cells $\pi_i : \Pi_n(A_1, \ldots, A_n) \to A_i \qquad (1 \leqslant i \leqslant n)$

Projections $\dfrac{}{p : \Pi_n(A_1, \ldots, A_n) \vdash \pi_i(p) : A_i} \; {\scriptstyle (1 \leqslant i \leqslant n)}$

A type theory for fp-bicategories

$$\text{Equivalences} \quad \begin{tikzcd} \mathcal{B}\left(X, \Pi_n(A_1, \ldots, A_n)\right) \arrow[r, bend left, "(\pi_1 \circ -, \ldots, \pi_n \circ -)"] \arrow[r, phantom, "\perp\simeq"] & \prod_{i=1}^{n} \mathcal{B}(X, A_i) \arrow[l, bend left, "\langle -, \ldots, = \rangle"] \end{tikzcd}$$

A type theory for fp-bicategories

$$\text{Equivalences} \quad \begin{tikzcd} \mathcal{B}(X, \Pi_n(A_1, \ldots, A_n)) \arrow[r, bend left, "(\pi_1 \circ -, \ldots, \pi_n \circ -)"] \arrow[r, phantom, "\perp\simeq"] & \prod_{i=1}^{n} \mathcal{B}(X, A_i) \arrow[l, bend left, "\langle -, \ldots, = \rangle"] \end{tikzcd}$$

$$\varpi^{(i)} \bullet (\pi_i \circ (-)) \uparrow \quad \frac{\pi_i \circ u \Rightarrow t_i : A_i \quad (i = 1, \ldots, n)}{u \Rightarrow \langle t_1, \ldots, t_n \rangle : \Pi_n(A_1, \ldots, A_n)} \quad \downarrow \mathsf{p}^{\dagger}(-, \ldots, =)$$

for a counit $(\varpi^{(i)} : \pi_i \circ \langle t_1, \ldots, t_n \rangle \Rightarrow t_i : A_i)_{i=1,\ldots,n}$

A type theory for fp-bicategories

Equivalences

$$\mathcal{B}(X, \Pi_n(A_1, \ldots, A_n)) \underset{\langle -, \ldots, = \rangle}{\overset{(\pi_1 \circ -, \ldots, \pi_n \circ -)}{\rightleftarrows}} \perp\simeq \prod_{i=1}^{n} \mathcal{B}(X, A_i)$$

syntactic sugar

$$\varpi^{(i)} \bullet \pi_i\{(-)\} \quad \frac{\pi_i\{u\} \Rightarrow t_i : A_i \quad (i = 1, \ldots, n)}{u \Rightarrow \mathrm{tup}(t_1, \ldots, t_n) : \Pi_n(A_1, \ldots, A_n)} \quad \mathsf{p}^{\dagger}(-, \ldots, =)$$

for a counit $\quad (\varpi^{(i)} : \pi_i\{\mathrm{tup}(t_1, \ldots, t_n)\} \Rightarrow t_i : \Pi_n(A_1, \ldots, A_n))_{i=1,\ldots,n}$

A type theory for fp-bicategories

$$\varpi^{(i)} \bullet \pi_i\{(-)\} \left(\frac{\pi_i\{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \mathrm{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)} \right) \mathsf{p}^\dagger(-, \dots, =)$$

$$\varpi^{(i)} \bullet \pi_i\,\{(-)\} \left(\begin{array}{c} \pi_i\,\{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n) \\ \hline\hline u \Rightarrow \mathsf{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n) \end{array} \right) \mathsf{p}^{\dagger}(-, \dots, =)$$

Tupling map
$$\frac{(\Gamma \vdash t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \mathsf{tup}(t_1 \dots, t_n) : \Pi_n(A_1, \dots, A_n)}$$

A type theory for fp-bicategories

$$\varpi^{(i)} \bullet \pi_i\{(-)\} \quad \begin{array}{c} \pi_i\{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n) \\ \hline\hline u \Rightarrow \mathrm{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n) \end{array} \quad \mathsf{p}^{\dagger}(-, \dots, =)$$

Tupling map $$\frac{(\Gamma \vdash t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \mathrm{tup}(t_1 \dots, t_n) : \Pi_n(A_1, \dots, A_n)}$$

Counit (β-law) $$\frac{(\Gamma \vdash t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \varpi^{(k)}_{t_\bullet} : \pi_k\{\mathrm{tup}(t_1 \dots, t_n)\} \overset{\cong}{\Longrightarrow} t_k : A_k} \ (1 \leqslant k \leqslant n)$$

A type theory for fp-bicategories

$$\varpi^{(i)} \bullet \pi_i\{(-)\} \left(\frac{\pi_i\{u\} \Rightarrow t_i : A_i \quad (i = 1, \dots, n)}{u \Rightarrow \mathrm{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)} \right) \mathrm{p}^\dagger(-, \dots, =)$$

Tupling map $$\frac{(\Gamma \vdash t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \mathrm{tup}(t_1 \dots, t_n) : \Pi_n(A_1, \dots, A_n)}$$

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Mediating 2-cell
$$\frac{(\Gamma \vdash \alpha_i : \pi_i\{u\} \Rightarrow t_i : A_i)_{i=1,\dots,n}}{\Gamma \vdash \mathrm{p}^\dagger(\alpha_1, \dots, \alpha_n) : u \Rightarrow \mathrm{tup}(t_1, \dots, t_n) : \Pi_n(A_1, \dots, A_n)}$$

+ three equational rules. $\rightsquigarrow$ η-law is derivable

The syntactic model is free

syntactic model

fp-biclone

$$\begin{tikzcd}
(\mathrm{Syn}(\mathcal{S}), \Pi_n(-)) \arrow[rr, dashed, "\exists!\ \text{strict}\ h^{\#}"] & & (\mathcal{C}, \Pi_n(-)) \\
\mathcal{S} \arrow[u, hook] \arrow[urr, "\forall\ h"'] & &
\end{tikzcd}$$

signature

(Require $\Pi_{\mathbf{1}}(X) = X$)

The syntactic model is free

syntactic model
on unary contexts

fp-bicategory

$$
\begin{tikzcd}
\left(\mathrm{Syn}(\mathcal{S})\big|_1, \Pi_n(-)\right) \arrow[r, dashed, "\exists!\ \text{strict } h^{\#}"] & (\mathcal{B}, \Pi_n(-)) \\
\mathcal{S}\big|_1 \arrow[u, hook] \arrow[ur, "\forall\ h"']
\end{tikzcd}
$$

unary signature

(Require $\Pi_{\mathbf{1}}(X) = X$)

The syntactic model is free

syntactic model
on unary contexts

fp-bicategory

$$
\begin{tikzcd}
\left(\mathrm{Syn}(\mathcal{S})\big|_1, \Pi_n(-)\right) \arrow[rr, dashed, "\exists!\ \text{strict}\ h^{\#}"] & & (\mathcal{B}, \Pi_n(-)) \\
\mathcal{S}\big|_1 \arrow[u, hook] \arrow[urr, "\forall\ h"'] & &
\end{tikzcd}
$$

unary signature

(Require $\Pi_{\mathbf{1}}(X) = X$)

$\rightsquigarrow$ An <u>internal language</u> for fp-bicategories.

derived from definition of biadjoint

The syntactic model is free

syntactic model

cc-biclone

$$
\begin{tikzcd}
(\mathrm{Syn}(\mathcal{S}), \Pi_n(-), \Rightarrow) \arrow[rr, dashed, "\exists!\ \text{strict } h^{\#}"] & & (\mathcal{C}, \Pi_n(-), \Rightarrow) \\
\mathcal{S} \arrow[u, hook] \arrow[urr, "\forall\ h"'] & &
\end{tikzcd}
$$

signature

(Require $\Pi_1(X) = X$)

The syntactic model is free

syntactic model
on unary contexts

cc-bicategory with
strict products

$$
\begin{tikzcd}
\left(\mathrm{Syn}(\mathcal{S})\big|_1, \Pi_n(-), \Rightarrow\right) \arrow[rr, dashed, "\exists!\ \text{strict}\ h^{\#}"] & & (\mathcal{B}, \Pi_n(-), \Rightarrow) \\
\mathcal{S}\big|_1 \arrow[u, hook] \arrow[urr, "\forall\ h"'] & &
\end{tikzcd}
$$

unary signature

(Require $\Pi_{\mathbf{1}}(X) = X$)

The syntactic model is free

syntactic model
on unary contexts

unique up to equivalence

cc-bicategory

$$
\begin{tikzcd}
\left(\mathrm{Syn}(\mathcal{S})\big|_1, \Pi_n(-), \Rightarrow\right) \arrow[r, dashed] & (\mathcal{B}, \Pi_n(-), \Rightarrow) \\
\mathcal{S}\big|_1 \arrow[u, hook] \arrow[ur, "\forall\, h"'] &
\end{tikzcd}
$$

unary signature

(Require $\Pi_{\mathbf{1}}(X) = X$)

The syntactic model is free

syntactic model
on unary contexts

unique up to equivalence

cc-bicategory

$$\begin{tikzcd}
\left(\mathrm{Syn}(\mathcal{S})\big|_1, \Pi_n(-), \Rightarrow\right) \arrow[r, dashed] & \left(\mathcal{B}, \Pi_n(-), \Rightarrow\right) \\
\mathcal{S}\big|_1 \arrow[u, hook] \arrow[ur, "\forall\, h"'] &
\end{tikzcd}$$

(Require $\Pi_{\mathbf{1}}(X) = X$)

unary signature

$\rightsquigarrow$ An <u>internal language</u> for cartesian closed bicategories.

STLC up to isomorphism

Embedding of STLC-terms to $\Lambda_{\mathrm{ps}}^{\times,\to}$-terms:

$$
\begin{aligned}
x_k &\mapsto x_k \\
\pi_k(t) &\mapsto \pi_k\,\{\llparenthesis\, t \,\rrparenthesis\} \\
\langle t_1, \ldots, t_n \rangle &\mapsto \mathsf{tup}(\llparenthesis\, t_1 \,\rrparenthesis, \ldots, \llparenthesis\, t_n \,\rrparenthesis) \\
\mathsf{app}(t, u) &\mapsto \mathsf{eval}\,\{\llparenthesis\, t \,\rrparenthesis, \llparenthesis\, u \,\rrparenthesis\} \\
\lambda x.t &\mapsto \lambda x.\llparenthesis\, t \,\rrparenthesis
\end{aligned}
$$

STLC up to isomorphism

Embedding of STLC-terms to $\Lambda_{\mathrm{ps}}^{\times,\to}$-terms:

$$\begin{aligned} x_k &\mapsto x_k \\ \pi_k(t) &\mapsto \pi_k\,\{(\!|\, t \,|\!)\} \\ \langle t_1, \ldots, t_n \rangle &\mapsto \mathsf{tup}((\!|\, t_1 \,|\!), \ldots, (\!|\, t_n \,|\!)) \\ \mathsf{app}(t, u) &\mapsto \mathsf{eval}\,\{(\!|\, t \,|\!), (\!|\, u \,|\!)\} \\ \lambda x.t &\mapsto \lambda x.(\!|\, t \,|\!) \end{aligned}$$

$$\left(\text{STLC terms } \Gamma \vdash t : B\right)\big/_{\beta\eta} \cong \left(\Lambda_{\mathrm{ps}}^{\times,\to}\text{-terms } \Gamma \vdash t : B\right)\big/_{\cong_B^\Gamma}$$

$$t \cong_B^\Gamma t' \;\Leftrightarrow\; \Gamma \vdash \tau : t \overset{\cong}{\Longrightarrow} t' : B$$
for some invertible τ

Contributions

Key properties of $\Lambda_{\mathrm{ps}}^{\times,\to}$:

Contributions

Key properties of $\Lambda_{\mathrm{ps}}^{\times,\rightarrow}$:

1. Principled development $\Rightarrow$ few rules,
2. An internal language for cc-bicategories,
3. STLC up-to-isomorphism.

$\boxed{\text{1-cells}}$

$$\mathrm{eval}_{A,B} : (A \Rightarrow B) \times A \to B$$

$\boxed{\text{Adjoint equivalences}}$

$$\begin{tikzcd} \mathcal{B}(X, A \Rightarrow B) \arrow[r, bend left, "\mathrm{eval}_{A,B} \circ (- \times A)"] \arrow[r, phantom, "\perp \simeq"] & \mathcal{B}(X \times A, B) \arrow[l, bend left, "\lambda"] \end{tikzcd}$$

Rules for exponentials

explicit weakening by x free variable in context

$$\epsilon \bullet \mathsf{eval}\left\{(-)\left\{\mathrm{inc}_x\right\}, x\right\} \quad \dfrac{\begin{array}{c}(x : A)\\ \mathsf{eval}\left\{u\left\{\mathrm{inc}_x\right\}, x\right\} \Rightarrow t : B\end{array}}{u \Rightarrow \lambda x.t : A \Rightarrow B} \quad \mathsf{e}^\dagger(x\,.\,-)$$

$$\dfrac{}{f : A \Rightarrow B, x : A \vdash \mathsf{eval}(f, x) : B}\ \mathsf{eval} \qquad \dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \Rightarrow B}\ \mathsf{lam}$$

$$\dfrac{\Gamma, x : A \vdash t : B}{\Gamma, x : A \vdash \epsilon_t : \mathsf{eval}\left\{(\lambda x.t)\left\{\mathrm{inc}_x\right\}, x\right\} \Rightarrow t : B}\ \epsilon\text{-intro } (\beta\text{-rule})$$

$$\dfrac{\begin{array}{c}\Gamma, x : A \vdash t : B \qquad \Gamma \vdash u : A \Rightarrow B\\ \Gamma, x : A \vdash \alpha : \mathsf{eval}\left\{u\left\{\mathrm{inc}_x\right\}, x\right\} \Rightarrow t : B\end{array}}{\Gamma \vdash \mathsf{e}^\dagger(x\,.\,\alpha) : u \Rightarrow \lambda x.t : A \Rightarrow B}\ \mathsf{e}^\dagger(x\,.\,\alpha)\text{-intro}$$

+ three equational rules $\rightsquigarrow$ η-rule derivable