

Topics in Computer Science: Semantics of programming languages

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- ▶ Foundations: a mathematical theory of computation
- ▶ Algorithms and complexity: what can be computed? How fast?

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- ▶ Algorithms and complexity: what can be computed? How fast?
- ▶ **Semantics**: what is the mathematical structure of computation?

In other words, what is the [meaning](#) of a program?

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For example, writing \mathbb{N} for the set of **natural numbers** $0, 1, 2, \dots$:

1. We get a function $\mathbb{N} \rightarrow \mathbb{N}$ as the rule $x \mapsto x + 1$;
2. We get a different function $\mathbb{N} \rightarrow \mathbb{N}$ as the rule $x \mapsto 1$.

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Aim: give you some sense of **why** we want to do this, and **how** we do it

Aims of the next four lectures

At the end of the next four lectures you should be able to:

- ▶ Explain why we do semantics
- ▶ Distinguish between **operational** and **denotational** approaches
- ▶ Be able to prove basic semantic facts about some toy languages
- ▶ Outline some directions of where the field goes next

Why do we do semantics?



Your PC ran into a problem that it couldn't handle, and now it needs to restart.

You can search for the error online: `HAL_INITIALIZATION_FAILED`

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Bill Gates, April 18, 2002. Keynote address at WinHec 2002

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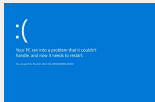
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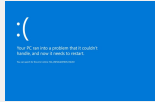
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In our day-to-day lives software failures are often just annoying:



But they can also be life-threatening and catastrophic:



Reliability really matters

Software is at the heart of the modern world.

It's crucial we understand what it does!

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Higher-order languages... encourage the programmer to build abstractions by composing functions. A good compiler must inline many of these calls to recover an efficiently executable program. In principle, inlining is dead simple: just replace the call of a function by an instance of its body. But any compiler-writer will tell you that inlining is a black art, full of delicate compromises that work together to give good performance without unnecessary code bloat.

Simon Peyton-Jones, one of the designers of Haskell

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Reason 3: to design new programming languages and paradigms

- (a) by thoroughly understanding the structure of programs,
we can discover useful new constructs
example: effect handlers in Jax and Pyro
- (b) we can also make sure new features behave correctly

A cautionary tale

The ML language has a range of powerful features that give strong guarantees on program behaviour

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... **but** an apparently-innocuous combination of its features broke these guarantees!

Making new features behave correctly

We want programs to do increasingly-fancy things.

Example 1: specify probabilistic models

e.g. for pandemic modelling

```
1  normalize(  
2    x = sample(bernoulli (2/7));  
3    r = if x then 3 else 10;  
4    observe 4 from poisson(r);  
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6  )
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Both **probabilistic programs** and **differentiable programs** involve

1. Powerful program features (e.g. loops, recursion, higher-order functions, ...)
2. Sophisticated mathematics (e.g. probability, measure theory, differentiability, ...)

Making sure these combine to do the right thing is both subtle and difficult!

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Do we need semantics?

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Suggestion: define the meaning of a program as the mapping that the compiler defines from the source code to machine code.

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4.7.3.3 Body replacement and execution. Finally the procedure body, modified as above, is inserted in place of the procedure statement and executed. If the procedure is called from a place outside the scope of any non-local quantity of the procedure body, the conflicts between the identifiers inserted through this process of body replacement and the identifiers whose declarations are valid at the place of the procedure statement or function designator will be avoided through suitable systematic changes of the latter identifiers.

From the Revised Report of Algol 60.

The limitations of writing things in words

Side effects = changes to variables resulting from the evaluation of an expression for which the variable is not local.

An ALGOL 60 program

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begin integer a;  
  integer procedure  $f(x, y)$ ; value  $y, x$ ; integer  $y, x$ ;  
     $a := f := x + 1$ ;  
  integer procedure  $g(x)$ ; integer  $x$ ;  $x := g := a + 2$ ;  
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See also: the huge on-going effort to verify chip architectures.

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- ▶ *precise* (no ambiguity)
- ▶ *concise* (readable to humans!)

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In other words, we want a **mathematical model** of programs.

Aside: syntax vs semantics

Every programming language comes with

1. **Syntax** = the sequences of symbols that are valid in the language
(e.g. where { and } go in Java)
2. **Semantics** = the meaning of (correctly-written) programs.

Ways of doing semantics

= different ways of modelling programs

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- ▶ **Axiomatic:** a program's meaning is given indirectly in terms of the collection of properties it satisfies; these properties are defined via a collection of axioms and rules.

Ways of doing semantics

Different approaches complement one another:

1. correctness of the proof rules of an axiomatic semantics relies on an underlying denotational or operational semantics,
2. correctness of an implementation with respect to a denotational semantics requires a proof that the operational and denotational semantics agree,
3. in proving facts about an operational semantics it can be of great help to use a denotational semantics, which abstracts away from unimportant, implementation details.

Time to see some semantics!

What we'll see next:

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language of numerical expressions.

This is a short course! The aim is to give a flavour of how things go;
at the end we'll add some features and think how to adapt things for that

The language NumExp

A VERY basic language NumExp

$Num \quad \mathbf{n} ::= 0 \mid 1 \mid 2 \mid \dots$

$Exp \quad \mathbf{e} ::= \mathbf{n} \mid \mathbf{e} \oplus \mathbf{e} \mid \mathbf{e} \otimes \mathbf{e}$

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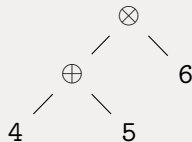
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To be precise, these represent *abstract syntax* trees, like:



Our informal reading of what NumExp programs mean:

1. Every numeral \mathbf{n} is evaluated to the corresponding number n ;
2. To find the value associated with an expression of the form $\mathbf{e}_0 \oplus \mathbf{e}_1$ we evaluate the expressions \mathbf{e}_0 and \mathbf{e}_1 and take the sum of the results;
3. To find the value associated with an expression of the form $\mathbf{e}_0 \otimes \mathbf{e}_1$ we evaluate the expressions \mathbf{e}_0 and \mathbf{e}_1 and take the product of the results

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Examples:

- ▶ 42 is evaluated to 42,
- ▶ $(1 \oplus 2) \otimes 3$ is evaluated to 9,
- ▶ $(1 \oplus 2) \otimes (3 \oplus 4)$ is evaluated to 21.

Operational semantics for NumExp

Giving operational semantics to NumExp

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“after evaluating e by one step, the expression e' remains to be evaluated”.

$$\frac{\text{premise}_1 \quad \dots \quad \text{premise}_n}{\text{conclusion}} \text{ side-condition (rule name)}$$

“if we have all the premises, and the side-condition holds, we get the conclusion”

Examples of logical rules

$$\frac{}{\mathbf{n}_0 \oplus \mathbf{n}_1 \rightarrow \mathbf{n}} \quad n = n_0 + n_1 (sum)$$

$$\frac{\mathbf{e}_0 \rightarrow \mathbf{e}'_0}{\mathbf{e}_0 \oplus \mathbf{e}_1 \rightarrow \mathbf{e}'_0 \oplus \mathbf{e}_1} \quad (sumL)$$

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It's sometimes more natural to read from top to bottom: *sumL* says

To evaluate $\mathbf{e}_0 \oplus \mathbf{e}_1$ one step, first evaluate \mathbf{e}_0 one step.

Small-step operational semantics for NumExp

Small-step = we describe every step the computation makes,
not just its final result.

$$\begin{array}{lll} \frac{}{\mathbf{n}_0 \oplus \mathbf{n}_1 \rightarrow \mathbf{n}} \quad n = n_0 + n_1 (sum) & \frac{\mathbf{e}_0 \rightarrow \mathbf{e}'_0}{\mathbf{e}_0 \oplus \mathbf{e}_1 \rightarrow \mathbf{e}'_0 \oplus \mathbf{e}_1} \quad (sumL) & \frac{\mathbf{e}_1 \rightarrow \mathbf{e}'_1}{\mathbf{e}_0 \oplus \mathbf{e}_1 \rightarrow \mathbf{e}_0 \oplus \mathbf{e}'_1} \quad (sumR) \\[1em] \frac{}{\mathbf{n}_0 \otimes \mathbf{n}_1 \rightarrow \mathbf{n}} \quad n = n_0 \times n_1 (prod) & \frac{\mathbf{e}_0 \rightarrow \mathbf{e}'_0}{\mathbf{e}_0 \otimes \mathbf{e}_1 \rightarrow \mathbf{e}'_0 \otimes \mathbf{e}_1} \quad (prodL) & \frac{\mathbf{e}_1 \rightarrow \mathbf{e}'_1}{\mathbf{e}_0 \otimes \mathbf{e}_1 \rightarrow \mathbf{e}_0 \otimes \mathbf{e}'_1} \quad (prodR) \end{array}$$

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We think of this as saying

If we start from $(1 \oplus 2) \otimes (3 \oplus 4)$ then $3 \otimes (3 \oplus 4)$ is an intermediate result in the program's execution.

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Definition

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A judgement $\mathbf{e} \rightarrow \mathbf{e}'$ is *derivable* if there is a derivation whose conclusion is $\mathbf{e} \rightarrow \mathbf{e}'$.

Both of the following judgements are derivable:

$$(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow 3 \otimes (3 \oplus 4) \qquad (1 \oplus 2) \otimes (3 \oplus 4) \rightarrow (1 \oplus 2) \otimes 7$$

and here are their derivations:

$$\frac{\frac{}{1 \oplus 2 \rightarrow 3} \quad 3=1+2 \text{ (sum)}}{(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow 3 \otimes (3 \oplus 4)} \text{ (prodL)}$$

$$\frac{\frac{}{3 \oplus 4 \rightarrow 7} \quad 7=3+4 \text{ (sum)}}{(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow (1 \oplus 2) \otimes 7} \text{ (prodR)}$$

Derivations

Definition

A judgement $e \rightarrow e'$ is *derivable* if there is a derivation whose conclusion is $e \rightarrow e'$.

Both of the following judgements are derivable:

$$(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow 3 \otimes (3 \oplus 4) \qquad (1 \oplus 2) \otimes (3 \oplus 4) \rightarrow (1 \oplus 2) \otimes 7$$

and here are their derivations:

$$\frac{\frac{}{1 \oplus 2 \rightarrow 3} \quad 3=1+2 \text{ (sum)}}{(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow 3 \otimes (3 \oplus 4)} \text{ (prodL)} \qquad \frac{\frac{}{3 \oplus 4 \rightarrow 7} \quad 7=3+4 \text{ (sum)}}{(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow (1 \oplus 2) \otimes 7} \text{ (prodR)}$$

Question: Is $3 \otimes 2 \rightarrow 5$ derivable? What about $(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow 3 \otimes 7$?

Specifying an evaluation order

Our rules so far specify that the left-most operand always gets evaluated first:

$$\frac{\mathbf{e}_1 \rightarrow \mathbf{e}'_1}{\mathbf{e}_0 \oplus \mathbf{e}_1 \rightarrow \mathbf{e}_0 \oplus \mathbf{e}'_1} \text{ (sumR)} \quad \text{and} \quad \frac{\mathbf{e}_1 \rightarrow \mathbf{e}'_1}{\mathbf{e}_0 \otimes \mathbf{e}_1 \rightarrow \mathbf{e}_0 \otimes \mathbf{e}'_1} \text{ (prodR)}$$

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We could instead choose to evaluate the right-most operand first:

$$\frac{e_1 \rightarrow e'_1}{n \oplus e_1 \rightarrow n \oplus e'_1} \text{ (sumR)} \quad \text{and} \quad \frac{e_1 \rightarrow e'_1}{n \otimes e_1 \rightarrow n \otimes e'_1} \text{ (prodR)}$$

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With these rules $(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow (1 \oplus 2) \otimes 7$ is no longer derivable.

Exercise: show that the following judgements are now derivable:

$$(1 \oplus 2) \otimes (3 \oplus 4) \rightarrow 3 \otimes (3 \oplus 4) \quad 3 \otimes (3 \oplus 4) \rightarrow 3 \otimes 7 \quad 3 \otimes 7 \rightarrow 21$$

Specifying an evaluation order — these choices can be **very** important!

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Composing evaluation steps

So far we have said what steps our programs can take, one by one.

Now we want to talk about *sequences* of those steps, and their eventual outcomes.

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Definition

We define the *multiple-step evaluation* relation \rightarrow^* as follows. We write $e \rightarrow^* e'$ if either:

1. $e = e'$ or
2. there is a finite sequence $e \rightarrow e_1 \rightarrow e_2 \rightarrow \cdots \rightarrow e_k \rightarrow e'$.

The relation \rightarrow^* is called *the reflexive and transitive closure of \rightarrow* .

If $e \rightarrow^* n$ we say that n is the *final answer* of e .

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Question: what is the final answer of $(1 \oplus 2) \otimes (3 \oplus 4)$?

Question: is it true that every expression has a final answer? Are there expressions with more than one final answer?

Summarising small-step operational semantics

- ▶ We describe individual computation steps using the \rightarrow relation;
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This is very fine-grained, but sometimes can be more than we need.

Next we'll look at the **big-step** approach, where we only describe the final answer of a computation.

Big-step Operational Semantics

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$$\frac{}{n \Downarrow n} \text{ (num)} \quad \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1}{e_0 \oplus e_1 \Downarrow n} \text{ } n=n_0+n_1 \text{ (sum)} \quad \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1}{e_0 \otimes e_1 \Downarrow n} \text{ } n=n_0 \times n_1 \text{ (prod)}$$

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This hides the information about evaluation order, but for our language we can prove it doesn't matter.

A benefit: much fewer rules!

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$$\frac{\frac{\frac{}{1 \Downarrow 1} \text{ (num)}}{1 \oplus 2 \Downarrow 3} \quad \frac{\frac{}{2 \Downarrow 2} \text{ (num)}}{3=1+2 \text{ (sum)}} \quad \frac{\frac{}{3 \Downarrow 3} \text{ (num)}}{3 \oplus 4 \Downarrow 7} \quad \frac{\frac{}{4 \Downarrow 4} \text{ (num)}}{7=3+4 \text{ (sum)}}}{(1 \oplus 2) \otimes (3 \oplus 4) \Downarrow 21} \quad 21=3 \times 7 \text{ (prod)}$$

Evaluating operational semantics

Pros:

- ▶ Fine-grained description of program behaviour
- ▶ Makes precise our intuition about which steps the program makes when:
very useful when multiple features are involved
- ▶ Quite easy to write down (e.g. you could feed the rules into a proof assistant)

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Next up: a **denotational** semantics for NumExp

Denotational semantics for NumExp

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Remember our original intuitive description of what NumExp-expressions mean:

1. Every numeral n is evaluated to the corresponding number n ;
2. To find the value associated with an expression of the form $e_0 \oplus e_1$ we evaluate the expressions e_0 and e_1 and take the sum of the results;
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What is a NumExp-expression “really” representing? A natural number!

From expressions to numbers

Let's turn our intuitive idea into something precise

Intuition

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The denotation of an expression \mathbf{e} is given *by recursion*:

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Example:

$$\begin{aligned}\llbracket (1 \oplus 2) \otimes (3 \oplus 4) \rrbracket &= \llbracket 1 \oplus 2 \rrbracket \times \llbracket 3 \oplus 4 \rrbracket \\ &= (\llbracket 1 \rrbracket + \llbracket 2 \rrbracket) \times (\llbracket 3 \rrbracket + \llbracket 4 \rrbracket) \\ &= (1 + 2) \times (3 + 4) \\ &= 21.\end{aligned}$$

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This defines a function $Exp \rightarrow \mathbb{N}$ from the set of NumExp-expressions to the set of natural numbers.

Here \mathbb{N} is the **semantic domain**. By changing the semantic domain, or the denotation of expressions, we can study different kinds of properties.

Using the denotational semantics

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We can make this intuition precise:

Proposition

For all NumExp-expressions e_1, e_2, e_3 , $\llbracket (e_1 \oplus e_2) \oplus e_3 \rrbracket = \llbracket e_1 \oplus (e_2 \oplus e_3) \rrbracket$.

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Proof.

$$\begin{aligned}\llbracket (e_1 \oplus e_2) \oplus e_3 \rrbracket &= \llbracket e_1 \oplus e_2 \rrbracket + \llbracket e_3 \rrbracket \\ &= (\llbracket e_1 \rrbracket + \llbracket e_2 \rrbracket) + \llbracket e_3 \rrbracket \\ &= \llbracket e_1 \rrbracket + (\llbracket e_2 \rrbracket + \llbracket e_3 \rrbracket) \\ &= \llbracket e_1 \rrbracket + \llbracket e_2 \oplus e_3 \rrbracket \\ &= \llbracket e_1 \oplus (e_2 \oplus e_3) \rrbracket\end{aligned}$$



Tying things together

We've seen two perspectives on `NumExp`-expressions:

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We've seen two perspectives on NumExp-expressions:

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How do these relate?

A correspondence theorem

Theorem

For all expressions e and numbers n , $\llbracket e \rrbracket = n$ if and only if $e \Downarrow n$.

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Proof.

By structural induction on e .

Base case: if e is a numeral, say n , then both $\llbracket e \rrbracket = n$ and $e \Downarrow n$.

Inductive step: we suppose that $e = e_1 \oplus e_2$ and that the theorem holds for both e_1 and e_2 , that is:

1. We assume that for all numbers k_1 and k_2 , $\llbracket e_1 \rrbracket = k_1$ if and only if $e_1 \Downarrow k_1$ and $\llbracket e_2 \rrbracket = k_2$ if and only if $e_2 \Downarrow k_2$,
2. We prove that for all numbers n , $\llbracket e_1 \oplus e_2 \rrbracket = n$ if and only if $e_1 \oplus e_2 \Downarrow n$.

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Suppose that $\llbracket \mathbf{e}_1 \oplus \mathbf{e}_2 \rrbracket = n$. By definition, $\llbracket \mathbf{e}_1 \oplus \mathbf{e}_2 \rrbracket = \llbracket \mathbf{e}_1 \rrbracket + \llbracket \mathbf{e}_2 \rrbracket$. Then $\llbracket \mathbf{e}_1 \rrbracket$ and $\llbracket \mathbf{e}_2 \rrbracket$ are two numbers, let's call them k_1 and k_2 , that add up to n . By (1), $\mathbf{e}_1 \Downarrow \mathbf{k}_1$ and $\mathbf{e}_2 \Downarrow \mathbf{k}_2$, which by the rule *sum* of big step operational semantics means that $\mathbf{e}_1 \oplus \mathbf{e}_2 \Downarrow \mathbf{n}$.

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Conversely, suppose that $\mathbf{e}_1 \oplus \mathbf{e}_2 \Downarrow \mathbf{n}$. This can only be proved using the *sum* rule, so there must be numbers k_1 and k_2 such that $n = k_1 + k_2$ and $\mathbf{e}_1 \Downarrow \mathbf{k}_1$ and $\mathbf{e}_2 \Downarrow \mathbf{k}_2$. By (1) then $\llbracket \mathbf{e}_1 \rrbracket = k_1$ and $\llbracket \mathbf{e}_2 \rrbracket = k_2$. Hence

$$\llbracket \mathbf{e}_1 \oplus \mathbf{e}_2 \rrbracket = \llbracket \mathbf{e}_1 \rrbracket + \llbracket \mathbf{e}_2 \rrbracket = k_1 + k_2 = n.$$



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What we've seen

For the language **NumExp**, defined by

$$\begin{array}{lcl} \textit{Num} & \mathbf{n} & ::= 0 \mid 1 \mid 2 \mid \dots \\ \textit{Exp} & \mathbf{e} & ::= \mathbf{n} \mid \mathbf{e} \oplus \mathbf{e} \mid \mathbf{e} \otimes \mathbf{e} \end{array}$$

we've seen two semantic approaches:

1. **Operational:** the steps they take in running to a final answer
(both small-step and big-step);
2. **Denotational:** the meaning of the expression, expressed as a natural number.

And we've proven these agree. This can be read in two ways:

1. Our operational semantics correctly captures the 'meaning' of programs;
2. Our denotational semantics correctly captures the way programs run;

NumExp is a very limited language!

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Possible examples:

- ▶ variables
- ▶ other forms of data (arrays / lists, boolean values, ...)
- ▶ other control structures (iteration / loops, if-statements, handlers, `call-cc`, ...)
- ▶ procedures / methods / user-defined functions
- ▶ interaction with the world: state, I/O, printing, exceptions, probability...
- ▶ abstract data types (e.g. to define trees, lists, ...)
- ▶ higher-order functions
- ▶ types
- ▶ polymorphism

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Challenge: think about how these things work *together*!

NumExp is a very limited language!

What other features might we want?

In what follows, we'll think about **operational** and **denotational** perspectives on some of the simpler cases.

Semantics for variables

As it stands, we can only write things like $(1 \oplus 2) \otimes (3 \oplus 4)$.

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But we know some things are true no matter what numbers we put in, such as:

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To understand these things properly we need **variables**

What is a variable anyway?

We will use **variables** in our language as a way to stand for any possible number
(like in high-school algebra)

This is a *functional* perspective. It might not be what you're used to!

Contrast to an *imperative* language, where a variable is a pointer to a memory cell

Adding variables to NumExp

The language $\text{NumExp} + \text{Var}$ is defined by

$$\begin{array}{lll} \text{Num} & \mathbf{n} & ::= 0 \mid 1 \mid 2 \mid \dots \\ \text{Exp} & \mathbf{e} & ::= \mathbf{x} \mid \mathbf{n} \mid \mathbf{e} \oplus \mathbf{e} \mid \mathbf{e} \otimes \mathbf{e} \end{array}$$

We assume \mathbf{x} stands for anything in a fixed stock of variables $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$

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An expression e is **open** if it contains a variable (e.g. $x, 2 \oplus y, \dots$)

If it contains no variables, it is **closed** (e.g. $7 \oplus 3$)

Operational semantics for variables

We already know how **closed** terms run. What about **open** ones?

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But if \mathbf{x} can be anything, the final result can also be lots of things
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So \mathbf{x} is *incomplete*: we don't know how it runs till you tell me what \mathbf{x} is

The small-step rules for **NumExp** + **Var** are therefore exactly those of **NumExp**:
we give no rules for running variables

Small-step rules for NumExp + Var

$$\begin{array}{lll} \frac{}{\mathbf{n}_0 \oplus \mathbf{n}_1 \rightarrow \mathbf{n}} \quad n = n_0 + n_1 (sum) & \frac{\mathbf{e}_0 \rightarrow \mathbf{e}'_0}{\mathbf{e}_0 \oplus \mathbf{e}_1 \rightarrow \mathbf{e}'_0 \oplus \mathbf{e}_1} \quad (sumL) & \frac{\mathbf{e}_1 \rightarrow \mathbf{e}'_1}{\mathbf{e}_0 \oplus \mathbf{e}_1 \rightarrow \mathbf{e}_0 \oplus \mathbf{e}'_1} \quad (sumR) \\[1em] \frac{}{\mathbf{n}_0 \otimes \mathbf{n}_1 \rightarrow \mathbf{n}} \quad n = n_0 \times n_1 (prod) & \frac{\mathbf{e}_0 \rightarrow \mathbf{e}'_0}{\mathbf{e}_0 \otimes \mathbf{e}_1 \rightarrow \mathbf{e}'_0 \otimes \mathbf{e}_1} \quad (prodL) & \frac{\mathbf{e}_1 \rightarrow \mathbf{e}'_1}{\mathbf{e}_0 \otimes \mathbf{e}_1 \rightarrow \mathbf{e}_0 \otimes \mathbf{e}'_1} \quad (prodR) \end{array}$$

Examples of valid NumExp + Var-derivations:

$$(1 \oplus 2) \otimes x \rightarrow 3 \otimes x$$

$$(1 \oplus y) \otimes (3 \oplus 4) \rightarrow (1 \oplus y) \otimes 7$$

with their derivations:

$$\frac{\overline{1 \oplus 2 \rightarrow 3} \quad 3=1+2 \text{ (sum)}}{(1 \oplus 2) \otimes x \rightarrow 3 \otimes x} \text{ (prodL)}$$

$$\frac{\overline{3 \oplus 4 \rightarrow 7} \quad 7=3+4 \text{ (sum)}}{(1 \oplus y) \otimes (3 \oplus 4) \rightarrow (1 \oplus y) \otimes 7} \text{ (prodR)}$$

Big-step rules for NumExp + Var

For the big-step rules, we want to say what the eventual output of an expression is.

For an open term we don't know what that is. So we restrict our big-step rules to closed terms.

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Big-step rules (just as for NumExp)

$$\frac{}{n \Downarrow n} \text{ (num)} \quad \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1}{e_0 \oplus e_1 \Downarrow n} \text{ } n=n_0+n_1 \text{ (sum)} \quad \frac{e_0 \Downarrow n_0 \quad e_1 \Downarrow n_1}{e_0 \otimes e_1 \Downarrow n} \text{ } n=n_0 \times n_1 \text{ (prod)}$$

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An alternative approach would be to say variables don't run:

$$\frac{}{\mathbf{x} \Downarrow \mathbf{x}} \text{ (var)}$$

Denotational semantics for variables

The meaning of a closed term is a number:

$$\llbracket 7 \oplus (3 \otimes 1) \rrbracket = \llbracket 7 \rrbracket + \llbracket 3 \otimes 1 \rrbracket = \llbracket 7 \rrbracket + (\llbracket 3 \rrbracket \times \llbracket 1 \rrbracket) = 21$$

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What is the meaning of an open term?

It depends on the values of the variables: on the **environment**

Definition

An **environment** ρ for a **NumExp** + **Var**-expression \mathbf{e} is a function assigning to each variable in \mathbf{e} a natural number. Write $Env(\mathbf{e})$ for the set of environments of \mathbf{e} .

Formally: ρ is a partial function $Vars \rightarrow \mathbb{N}$ which is defined on all the variables in \mathbf{e} .

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So the meaning of a variable x depends on which environment you have: for each environment ρ we get a value $\rho(x)$.

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So $\llbracket x \rrbracket$ is a *function* $Env(x) \rightarrow \mathbb{N}$

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More generally: what is the denotation of $(3 \otimes \mathbf{x}) \otimes (\mathbf{x} \oplus \mathbf{y})$?

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The denotation of a NumExp + Var-expression \mathbf{e} is a function $Env(\mathbf{e}) \rightarrow \mathbb{N}$. For an environment ρ in $Env(\mathbf{e})$, the denotation of \mathbf{e} is given by recursion:

1. $\llbracket \mathbf{x} \rrbracket(\rho) := \rho(\mathbf{x})$
2. $\llbracket \mathbf{n} \rrbracket(\rho) := n;$
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The definition is still compositional

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What about $\mathbf{x} \otimes \mathbf{y}$?

Tying things together

Proposition

For any closed expression e , we have $e \Downarrow n$ if and only if $\llbracket e \rrbracket = n$.

Proven in the same way as for NumExp

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Something more interesting:

Proposition

For any expression e containing the variables x_1, \dots, x_k , and any environment ρ assigning $\rho(x_1) = m_1, \dots, \rho(x_k) = m_k$:

$$\llbracket e \rrbracket(\rho) = n \quad \text{if and only if} \quad e' \Downarrow n$$

where e' is obtained from e by replacing each x_i by m_i .

Denotational semantics for NumExp + Var

When you have variables, the meaning of programs depends on the values of the variables.

Open expressions don't run, but we can think about their operational semantics by thinking about how we fill in the variables.

For the denotational semantics, we think of programs as depending on a choice of **environment**. So programs are now interpreted as functions.

If we want to study different properties or features, we need different models. A lot of the work in denotational semantics is coming up with these models: makes use of powerful tools from mathematics (algebra, topology, logic, category theory, ...)

Semantics for printing

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Examples of effects:

- ▶ Exceptions
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- ▶ Probability
- ▶ User input
- ▶ Memory / state
(either *global* or *local*)
- ▶ Printing
- ▶ Control (e.g. jumps or **GOTO**)
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Part of the development from 1990 onwards has been theories for programs independent of the choice of effect

Here we study one in particular: printing (simple, but enlightening!)

The language `Print`

We add the ability to print words: (e.g. to screen, or as output)

```
print hello ; e
```

means print the string of symbols h-e-l-l-o, then run `e`.

The language `Print`

$$\begin{array}{lll} \textit{Num} & n & ::= 0 \mid 1 \mid 2 \mid \dots \\ \textit{NumExp} & l & ::= x \mid n \mid l \oplus l \mid l \otimes l \\ \textit{Exp} & e & ::= l \mid e \oplus e \mid e \otimes e \mid \texttt{print } w ; l \end{array}$$

Here w refers to a string over a fixed *alphabet* of symbols.

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Examples:

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$$\mathbf{e} \Downarrow \langle \mathbf{n}, \mathbf{w} \rangle$$

means

running \mathbf{e} has final result
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Example

1. $\text{print hello} ; (3 \oplus 4) \Downarrow \langle 7, \text{hello} \rangle$
2. $(\text{print hello} ; 3) \oplus 4 \Downarrow \langle 7, \text{hello} \rangle$
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Rule for printing:

$$\frac{e \Downarrow \langle n, w \rangle}{\text{print } w' ; e \Downarrow \langle n, w w' \rangle}$$

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What about $(\text{print hello} ; 7) \oplus (\text{print world} ; 3)$?

We have to choose an **evaluation order**. For left to right:

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Big-step semantics for Print

$$\frac{}{n \Downarrow \langle n, \varepsilon \rangle} \text{ (num)} \qquad \frac{e \Downarrow \langle n, w \rangle}{\text{print } w'; e \Downarrow \langle n, w w' \rangle} \text{ (print)}$$

$$\frac{e_0 \Downarrow \langle n_0, w_0 \rangle \quad e_1 \Downarrow \langle n_1, w_1 \rangle}{e_0 \oplus e_1 \Downarrow \langle n, w_0 w_1 \rangle} \quad n = n_0 + n_1 \text{ (sum)}$$

$$\frac{e_0 \Downarrow \langle n_0, w_0 \rangle \quad e_1 \Downarrow \langle n_1, w_1 \rangle}{e_0 \otimes e_1 \Downarrow \langle n, w_0 w_1 \rangle} \quad n = n_0 \times n_1 \text{ (prod)}$$

Example

$$(\text{print hello}; 3) \oplus (\text{print world}; 4) \Downarrow \langle 7, \text{hello world} \rangle$$

What about the denotational semantics?

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A `NumExp` + `Var`-program e was interpreted as a function $Env(e) \rightarrow \mathbb{N}$

What is the meaning of a `Print`-program?

Example:

1. $\llbracket 7 \rrbracket = (7, \varepsilon)$
2. $\llbracket \text{print hello} ; 3 \rrbracket = (3, \text{hello})$
3. $\llbracket \text{print hello} ; 3 \oplus \text{print world} ; 5 \rrbracket = (8, \text{hello world})$
4. $\llbracket x \rrbracket(\rho) = \rho(x)$

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What is the meaning of a **Print**-program?

Definition

For a set S with elements s_1, s_2, \dots we write S^* for the set of strings over S .

The elements are $\varepsilon, s_1, s_1s_2, s_1s_1, \dots$

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For an environment ρ in $Env(\mathbf{e})$, the denotation of \mathbf{e} is given by recursion:

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What is the meaning of a **Print**-program? A function $Env(\mathbf{e}) \rightarrow \mathbb{N} \times Letters^*$

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Something new!

The order we run programs matters!

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This is where the semantics of programs gets [subtle](#) and [interesting](#)

Tying things together

Proposition

For any closed expression e , we have $e \Downarrow \langle n, s \rangle$ if and only if $\llbracket e \rrbracket = (n, s)$.

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Something more interesting:

Proposition

For any expression \mathbf{e} containing the variables $\mathbf{x}_1, \dots, \mathbf{x}_k$, and any environment ρ assigning $\rho(\mathbf{x}_1) = m_1, \dots, \rho(\mathbf{x}_k) = m_k$:

$$\llbracket \mathbf{e} \rrbracket(\rho) = (n, s) \quad \text{if and only if} \quad \mathbf{e}' \Downarrow \langle \mathbf{n}, s \rangle$$

where \mathbf{e}' is obtained from \mathbf{e} by replacing each \mathbf{x}_i by \mathbf{m}_i .

Denotational semantics for Print

When you have an effect, the meaning of programs depends on both the effect, and the order of evaluation.

Open expressions don't run, but we can think about their operational semantics by thinking about how we fill in the variables.

For the denotational semantics, we think of programs as depending on a choice of **environment**. So programs are now interpreted as functions **plus** extra structure to model the effect.

If we want to study different properties or features, we need different models. A lot of the work in denotational semantics is coming up with these models: makes use of powerful tools from mathematics (algebra, topology, logic, category theory, ...)