Programming Concepts

Correctness of algorithms

Topics

1. Reasoning about programs

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1. Reasoning about programs in terms of **correctness**

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1. Reasoning about programs in terms of **correctness**
   ⇝ will make use of **assertions**
2. Lots of examples

The problem

Terrible things go wrong with software projects

- American spaceship Mariner sent to Venus in 1960s lost forever - due to software error.
- Mars Climate Orbiter. Wrong units of measurement...
- Software bug in radiation machine in Texas caused numerous deaths due to radiation overdoses
- See The Risks Digest (`http://catless.ncl.ac.uk/Risks/`) for endless examples

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More and more agencies are demanding certified software

Programming errors

1. Language errors:

    e.g. incorrect use of syntax

    Normally spotted by complier or interpreter

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    e.g. incorrect use of syntax

    Normally spotted by complier or interpreter

2. Logical errors, e.g.:

    $\mathrm{J} \leftarrow 1$

    $\textsc{total} \leftarrow 0$

    while $\mathrm{J} \neq 100$ **do**

    $\quad \textsc{total} \leftarrow \textsc{total} + \mathrm{J}$

    $\quad \mathrm{J} \leftarrow \mathrm{J} + 2$

    Normally due to flaws in underlying algorithm

Debugging programs

Logical errors can lie undetected for ages

Test suites can be developed to exercise different execution paths through the program

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Dijkstra:

Debugging

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- only their presence.

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We need methods to develop methods for formally proving that programs are correct:

Debugging programs

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Dijkstra:

Debugging

- cannot be used to demonstrate the *absence* of bugs
- only their presence.

We need methods to develop methods for formally proving that programs are correct: all possible runs of program will produce the correct results

What does correctness mean ?

Recall: Specification of an algorithmic problem:

- a characterisation of all legal inputs
- a description of the required outputs as a function of the inputs

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Partial correctness:

An algorithm is *partially correct* with respect to a specification whenever, for every legal input

- if the algorithm halts, then the output satisfies the required relationship with the original input

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Recall: Specification of an algorithmic problem:

- a characterisation of all legal inputs
- a description of the required outputs as a function of the inputs

Partial correctness:

An algorithm is *partially correct* with respect to a specification whenever, for every legal input

- if the algorithm halts, then the output satisfies the required relationship with the original input

Total correctness:

An algorithm is *totally correct* with respect to a specification whenever, for every legal input

- the algorithm halts
- when the algorithm halts, then the output satisfies the required relationship with the original input

What you need to understand

How to show rigorously that an algorithm is correct for a given problem specification

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How do we do that? **Assertions**

Assertions

What is an assertion?

A mathematical statement about program variables at some specific checkpoint in a program

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Examples:

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Examples:

Input: Array $A[1 \ldots n]$

Output: largest element of A

$\textsc{ptr} \leftarrow 1$

$\textsc{cMax} \leftarrow A[\textsc{ptr}]$

$\Longleftarrow \boxed{\textsc{cMax} = A[1]}$

while $\textsc{ptr} \neq n$ **do**

  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

  if $A[\textsc{ptr}] > \textsc{cMax}$ **then**

    $\textsc{cMax} \leftarrow A[\textsc{ptr}]$

return $\textsc{cMax}$

Assertions

What is an assertion?

A mathematical statement about program variables at some specific checkpoint in a program

Examples:

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What makes an assertion valid?

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Valid assertions

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Which are valid?

Input: Array $A[1 \ldots n]$
Output: largest element of A

$\textsc{ptr} \leftarrow 1$
$\textsc{cMax} \leftarrow A[\textsc{ptr}]$
$\Longleftarrow \boxed{\text{Prop}}$
while $\textsc{ptr} \neq n$ **do**
    $\textsc{ptr} \leftarrow \textsc{ptr} + 1$
    if $A[\textsc{ptr}] > \textsc{cMax}$ **then**
        $\textsc{cMax} \leftarrow A[\textsc{ptr}]$
return $\textsc{cMax}$

When $\boxed{\text{Prop}}$ is

- $\textsc{cMax} = A[1]$?
- $A[\textsc{ptr}] = A[1] + A[n]$?
- $\textsc{cMax} \geq A[\textsc{ptr}]$?

Valid assertions

An assertion is valid if it is true of the program variables every time control passes the checkpoint

Which are valid?

Input: Array $A[1 \ldots n]$
Output: largest ...

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while $\textsc{ptr} \neq n$ **do**
    $\Longleftarrow \boxed{\text{Prop}}$
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    if $A[\textsc{ptr}] > \textsc{cMax}$ **then**
        $\textsc{cMax} \leftarrow A[\textsc{ptr}]$
return $\textsc{cMax}$

When $\boxed{\text{Prop}}$ is

- $\textsc{cMax} = A[\textsc{ptr}]$?
- $\textsc{cMax} \geq A[1]$?
- $A[1] \leq A[\textsc{ptr}]$?

Important assertions

- **Initial assertion:** Captures requirements on legal inputs
- **Final assertion:** Determines properties of data output
- **Loop assertions:** Difficult to assign.
  Must be true every time control goes through the loop.

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Input: Array $A[1 \ldots n]$
Output: largest . . .
$\Longleftarrow$ $\boxed{\text{InitialProp}}$
$\textsc{ptr} \leftarrow 1$
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while $\textsc{ptr} \neq n$ **do**
  $\Longleftarrow$ $\boxed{\text{LoopProp}}$
  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$
  if $A[\textsc{ptr}] > \textsc{cMax}$ **then**
    $\textsc{cMax} \leftarrow A[\textsc{ptr}]$
$\Longleftarrow$ $\boxed{\text{FinalProp}}$
return $\textsc{cMax}$

Assigning valid assertions

- **Initial assertion:** Captures requirements on legal inputs
- **Final assertion:** Determines properties of data output
- **Loop assertions:** Difficult to assign
  Must be true every time control goes through the loop

Input: Array $A[1 \ldots n]$
Output: largest ...
$\Longleftarrow$ $\boxed{n \geq 1}$
$\textsc{ptr} \leftarrow 1$
$\textsc{cMax} \leftarrow A[\textsc{ptr}]$
while $\textsc{ptr} \neq n$ **do**
    $\Longleftarrow$ $\boxed{\text{LoopProp ???}}$
    $\textsc{ptr} \leftarrow \textsc{ptr} + 1$
    if $A[\textsc{ptr}] > \textsc{cMax}$ **then**
        $\textsc{cMax} \leftarrow A[\textsc{ptr}]$
$\Longleftarrow$ $\boxed{\textsc{cMax} \text{ is largest element in } A}$
return $\textsc{cMax}$

Assigning valid assertions

- **Initial assertion:** Captures requirements on legal inputs
- **Final assertion:** Determines properties of data output
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  Must be true every time control goes through the loop

```
Input: Array A[1 . . . n]
Output: largest . . .
⇐ [ n ≥ 1 ]
PTR ← 1
CMAX ← A[PTR]
while PTR ≠ n do
    ⇐ [ LoopProp ??? ]
    PTR ← PTR + 1
    if A[PTR] > CMAX then
        CMAX ← A[PTR]
⇐ [ CMAX is largest element in A ]
return CMAX
```

that is:

- $\textsc{cMax} \geq A[j]$, for j between 1 and n
- $\textsc{cMax}$ occurs in A

How do we assign valid assertions?

We use

- informal mathematical reasoning
- Loop Invariance theorems

$\rightsquigarrow$ best explained via an example

There are semi-automatic software systems

(it is a mathematical fact that you can't do
this completely automatically: see Rice's theorem)

Assigning valid assertions

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    $\textsc{cMax} \leftarrow A[\textsc{ptr}]$

$\Longleftarrow$ $\boxed{\textsc{ptr} = n \text{ and } \textsc{cMax} \text{ is largest element in } A[1 \ldots \textsc{ptr}]}$

return $\textsc{cMax}$

Assigning valid assertions

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$\Longleftarrow$ $\boxed{\textsc{ptr} = n \text{ and } \textsc{cMax} \text{ is largest element in } A[1 \ldots \textsc{ptr}]}$

return $\textsc{cMax}$

$\Longleftarrow$ $\boxed{\text{largest element in A returned}}$

Assigning valid assertions

Input: Array $A[1 \dots n]$

Output: largest element in A

$\Longleftarrow$ $\boxed{n \geq 1}$

$\textsc{ptr} \leftarrow 1$

$\textsc{cMax} \leftarrow A[\textsc{ptr}]$

$\Longleftarrow$ $\boxed{\textsc{cMax} = A[1]}$

while $\textsc{ptr} \neq n$ **do**

$\quad$ $\Longleftarrow$ $\boxed{\textsc{cMax} \text{ is largest element in } A[1 \dots \textsc{ptr}]}$ Where did this come from ?

$\quad$ $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

$\quad$ **if** $A[\textsc{ptr}] > \textsc{cMax}$ **then**

$\quad\quad$ $\textsc{cMax} \leftarrow A[\textsc{ptr}]$

$\Longleftarrow$ $\boxed{\textsc{ptr} = n \text{ and } \textsc{cMax} \text{ is largest element in } A[1 \dots \textsc{ptr}]}$

return $\textsc{cMax}$

$\Longleftarrow$ $\boxed{\text{largest element in A returned}}$

The annotated algorithm as comments

Input: Array $A[1 \dots n]$
Output: largest …

```
PTR ← 1
CMAX ← A[PTR]
// assert: CMAX = A[1]
while PTR ≠ n do
    // assert: CMAX is largest element in A[1...PTR]
    if A[PTR] > CMAX then
        PTR ← PTR + 1
        CMAX ← A[PTR]
// assert:PTR = n and CMAX is largest element in
// A[1...PTR]
return MAX
// assert: largest element in A returned
```

Warning

Termination

- Assertions say nothing about termination
- Final assertion describes properties which are true if the program terminates
- Separate reasoning needs to be done to assure termination
  - Usually involving some quantity which decrease each time round the loop
  - Sometimes involves the manner in which this quantity decreases

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Termination

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- Final assertion describes properties which are true if the program terminates
- Separate reasoning needs to be done to assure termination
  - Usually involving some quantity which decrease each time round the loop
  - Sometimes involves the manner in which this quantity decreases
- Never write a loop, without knowing why it will terminate, for every possible input

An algorithm

What does this do?

Input: number $k \geq 0$
Output: ?????

$\textsc{x} \leftarrow 0$

$\textsc{y} \leftarrow 0$

while $\textsc{x} \neq k$ **do**

$\quad \textsc{y} \leftarrow \textsc{y} + k$

$\quad \textsc{x} \leftarrow \textsc{x} + 1$

return $\textsc{y}$

Input: number $k \geq 0$
Output: $k * k$

$\mathrm{X} \leftarrow 0$

$\mathrm{Y} \leftarrow 0$

while $\mathrm{X} \neq k$ **do**
  $\mathrm{Y} \leftarrow \mathrm{Y} + k$
  $\mathrm{X} \leftarrow \mathrm{X} + 1$

return Y

Assigning valid assertions

Input: number $k \geq 0$

Output: $k * k$

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Assigning valid assertions

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Output: $k * k$

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$\text{X} \leftarrow 0$

$\text{Y} \leftarrow 0$

while $\text{X} \neq k$ **do**

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$\quad \text{X} \leftarrow \text{X} + 1$

return Y

Assigning valid assertions

Input: number $k \geq 0$

Output: $k * k$

$\Longleftarrow$ $\boxed{k \geq 0}$

$\text{X} \leftarrow 0$

$\text{Y} \leftarrow 0$

while $\text{X} \neq k$ **do**

$\Longleftarrow$ $\boxed{y = x * k}$ a loop invariant

$\text{Y} \leftarrow \text{Y} + k$

$\text{X} \leftarrow \text{X} + 1$

return Y

Assigning valid assertions

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Output: $k * k$

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$\mathrm{X} \leftarrow 0$

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$\Longleftarrow$ $\boxed{y = x * k}$

while $\mathrm{X} \neq k$ **do**

$\Longleftarrow$ $\boxed{y = x * k \qquad \text{a loop invariant}}$

$\mathrm{Y} \leftarrow \mathrm{Y} + k$

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return Y

Assigning valid assertions

Input: number $k \geq 0$

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  $\mathrm{Y} \leftarrow \mathrm{Y} + k$

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$\Longleftarrow \boxed{y = x * k \text{ and } x = k}$

return Y

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$\Longleftarrow$ $\boxed{y = x * k \text{ and } x = k}$

return Y

$\Longleftarrow$ $\boxed{k * k \text{ returned}}$

Finding the most appropriate loop invariant is the creative step

The annotated algorithm

Input: integer k

Output: $k * k$

```
// assert: k ≥ 0
X ← 0
Y ← 0
// assert: Y = X * k
while X ≠ k do
    // assert: Y = X * k
    Y ← Y + k
    X ← X + 1
// assert: Y = X * k and
    X = k
return Y
// assert: k * k returned
```

The annotated algorithm

Input: integer k

Output: $k * k$

```
// assert: k ≥ 0
X ← 0
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// assert: Y = X * k
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    // assert: Y = X * k
    Y ← Y + k
    X ← X + 1
// assert: Y = X * k and
    X = k
return Y
// assert: k * k returned
```

k is a logical variable

- is unchanged by execution

Reasoning about programs

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How can we propogate valid assertions through an algorithm?

For each block of pseudocode, we think about if something is true beforehand, then what is true after?

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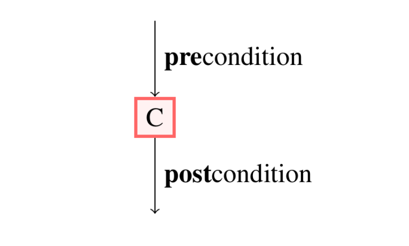

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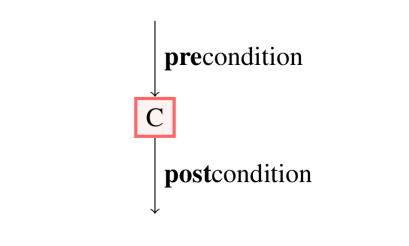

We'll say this is valid if, whenever *precondition* is true before we run C, then *postcondition* is true after we run C.

Reasoning about programs: Floyd-Hoare logic

We write:

$$\{Pre\}\ C\ \{Post\}$$

where

- Pre is a mathematical assertion - the precondition
- Post is a mathematical assertion - the postcondition
- C is some program code

Reasoning about programs: Floyd-Hoare logic

We write:

{Pre} C {Post}

where

- Pre is a mathematical assertion - the precondition
- Post is a mathematical assertion - the postcondition
- C is some program code

{Pre} C {Post} is valid if

- whenever the precondition is true, and the code is executed
- if the code terminates, then the postcondition is true

Which are valid?

{Pre} Code {Post}

Pre	Code	Post
$X = Y + 2$	$Y \leftarrow Y + 1$	$X > Y * 2$
$X = Y + 1$	$Y \leftarrow Y + X$	$X > Y * 2$
$X + Y > k$	$X \leftarrow X + k$ $Y \leftarrow Y - 1$	$Y > k$
$X > Y$	$X \leftarrow X + 1$ $Y \leftarrow Y - 1$	$X - Y > 0$

Rules for applying valid assertions

1. Elementary logic
   - $\rightsquigarrow$ for manipulating/rearranging the assertions
2. Sequential rule
   - $\rightsquigarrow$ for propagating valid assertions through simple actions
3. Loop Invariant theorems
   - $\rightsquigarrow$ for propagating valid assertions through while and for-loops

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All justified using Floyd-Hoare logic

Using logic

Suppose Prop logically implies Newprop:

Then any occurrence of $\boxed{\text{Prop}}$ can be $\boxed{\text{NewProp}}$

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Then any occurrence of $\boxed{\text{Prop}}$ can be $\boxed{\text{NewProp}}$

Example

while $\mathrm{X} \neq k$ **do**

$\Longleftarrow \boxed{y = x * k \qquad \text{a loop invariant}}$

$\mathrm{Y} \leftarrow \mathrm{Y} + k$

$\mathrm{X} \leftarrow \mathrm{X} + 1$

$\Longleftarrow \boxed{y = x * k \text{ and } x = k}$

return Y

Using logic

Suppose Prop logically implies Newprop:

Then any occurrence of $\boxed{\text{Prop}}$ can be $\boxed{\text{NewProp}}$

Example

while $\mathrm{X} \neq k$ **do**

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$\mathrm{Y} \leftarrow \mathrm{Y} + k$

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$\Longleftarrow \boxed{y = x * k \text{ and } x = k}$

return Y

$\Longleftarrow \boxed{k * k \text{ returned}}$

Using logic

Suppose Prop logically implies Newprop:

Then any occurrence of $\boxed{\text{Prop}}$ can be $\boxed{\text{NewProp}}$

Example

while $\mathrm{X} \neq k$ **do**

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$\Longleftarrow \boxed{y = x * k \text{ and } x = k}$

return Y

$\Longleftarrow \boxed{k * k \text{ returned}}$

Because $y = x * k$ and $x = k$ logically implies $y = k * k$

Propogating assertions

Given a valid triple, how do we propogate assertions through our algorithm?

{Pre} C {Post} is valid if

- whenever the precondition is true, and the code is executed
- if the code terminates, then the postcondition is true

Sequential rule

Suppose {Pre} Code {Post} is valid

Then

...

$\Longleftarrow \boxed{\text{Pre}}$

Code

...

Sequential rule

Suppose {Pre} Code {Post} is valid

Then

. . .

$\Longleftarrow$ $\boxed{\text{Pre}}$

Code

. . .

can be extended to:

. . .

$\Longleftarrow$ $\boxed{\text{Pre}}$

Code

$\Longleftarrow$ $\boxed{\text{Post}}$

. . .

Application of sequential rule

$\dots$

$\Longleftarrow \boxed{\text{SUM} = 0}$

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

Application of sequential rule

$\dots$

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can be extended to:

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$\Longleftarrow \boxed{\text{SUM} = \text{PTR}}$

Application of sequential rule

$\ldots$

$\Longleftarrow \boxed{\text{SUM} = 0}$

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

can be extended to:

$\Longleftarrow \boxed{\text{SUM} = 0}$

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

$\Longleftarrow \boxed{\text{SUM} = \text{PTR}}$

because

$$\{\text{SUM} = 0\} \begin{array}{l} \text{PTR} \leftarrow 1 \\ \text{SUM} \leftarrow \text{SUM} + \text{PTR} \end{array} \{\text{SUM} = \text{PTR}\}$$

is valid

Propogating assertions

What about **while**-loops?

Idea: use invariants.

{Pre} C {Post} is valid if

- whenever the precondition is true, and the code is executed
- if the code terminates, then the postcondition is true

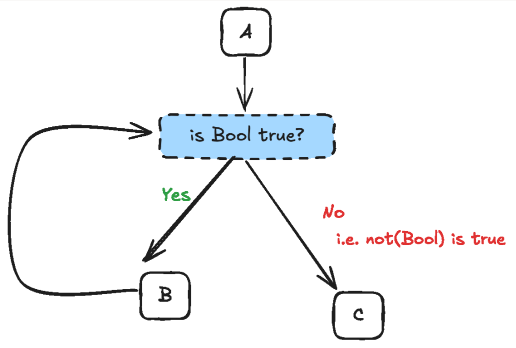
Propogating assertions: while-loops

A

while *condition* **do**

| B

C



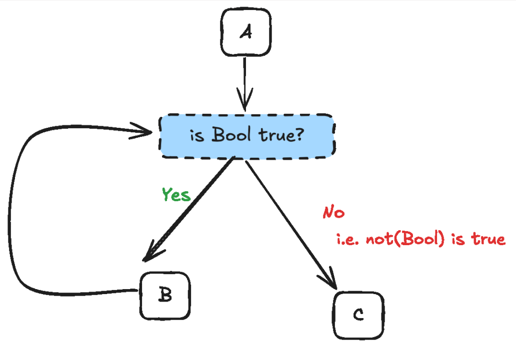
Propogating assertions: while-loops

A

while *condition* **do**

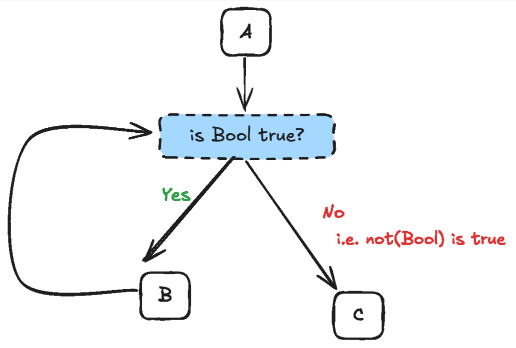
| B

C



An invariant is a property which, if it's true before doing the loop, is also true after doing the loop.

A
while *condition* **do**
| B
C



An invariant is a property which, if it's true before doing the loop, is also true after doing the loop.

⇝ To actually get into the loop, we need *condition* to be true!

Invariants

while *Bool* **do** Body

Invariants for while-loops

The mathematical statement *Inv* is a While-invariant if

- {Bool and Inv} Body {Inv} is valid

Invariants

while *Bool* **do** Body

Invariants for while-loops

The mathematical statement *Inv* is a While-invariant if

- {Bool and Inv} Body {Inv} is valid

in English: *Inv* is preserved each time the body is executed

while *Bool* **do** Body

Invariants for while-loops

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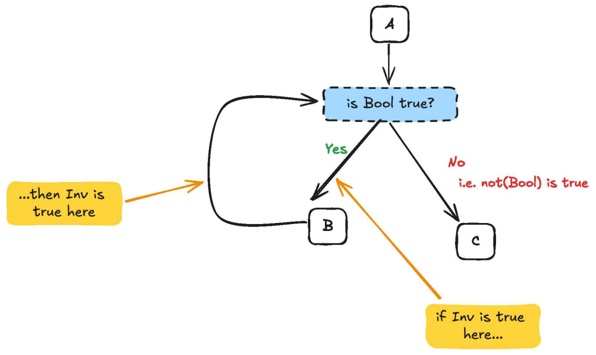
- {Bool and Inv} Body {Inv} is valid



while *Bool* **do** Body

Invariants for while-loops

The mathematical statement *Inv* is a While-invariant if

- {Bool and Inv} Body {Inv} is valid

in English: *Inv* is preserved each time the body is executed

Which are invariants ?

Bool	Body	Inv
$X > 0$	$X \leftarrow X + 1$ $Y \leftarrow Y - 1$	$X + Y = k$
$Y > 0$	$Y \leftarrow Y + k$ $X \leftarrow X + 1$	$Y = k * X$

While loop invariance theorem

while *Bool* **do** Body

Suppose Inv

1. is an invariant
2. is true *before* the While statement starts

Then, whenever the While statement terminates, if ever, we know that

1. Inv remains true
2. Bool is false (i.e. *not(Bool)* is true)

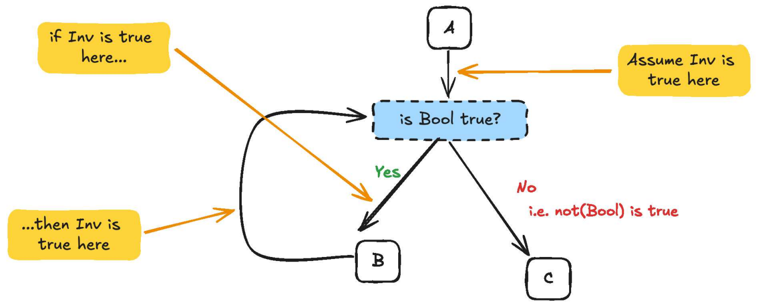
Why is this true? Look at the picture again

Suppose Inv

1. is an invariant
2. is true *before* the While statement starts

Then, whenever the While statement terminates, if ever, we know that

1. Inv remains true
2. Bool is false



Using loop invariance theorem

$\Longleftarrow \boxed{\text{Inv}}$

while *Bool* **do**

  $\Longleftarrow \boxed{\text{Inv}}$

  Body

...

can be extended to

$\Longleftarrow \boxed{\text{Inv}}$

while *Bool* **do**

  $\Longleftarrow \boxed{\text{Inv}}$

  Body

$\Longleftarrow \boxed{\text{Inv and not Bool}}$

Example use

Input: number $k \geq 0$

Output: $k * k$

$\Longleftarrow \boxed{k \geq 0}$

$\mathrm{X} \leftarrow 0$

$\mathrm{Y} \leftarrow 0$

$\Longleftarrow \boxed{y = x * k}$

while $\mathrm{X} \neq k$ **do**

  $\Longleftarrow \boxed{y = x * k \qquad \text{a loop invariant}}$

  $\mathrm{Y} \leftarrow \mathrm{Y} + k$

  $\mathrm{X} \leftarrow \mathrm{X} + 1$

return Y

Example use

Input: number $k \geq 0$

Output: $k * k$

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  $\mathrm{X} \leftarrow \mathrm{X} + 1$

$\Longleftarrow$ $\boxed{y = x * k \text{ and } x = k}$

return Y

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$\mathrm{X} \leftarrow \mathrm{X} + 1$

$\Longleftarrow \boxed{y = x * k \text{ and } x = k}$

return Y

because

- $\mathrm{Y} = \mathrm{X} * k$ before loop is entered
- $\{\mathrm{X} \neq k \text{ and } \mathrm{Y} = \mathrm{X} * k\}\ \begin{matrix} \mathrm{Y} \leftarrow \mathrm{Y} + k \\ \mathrm{X} \leftarrow \mathrm{X} + 1 \end{matrix}\ \{\mathrm{Y} = \mathrm{X} * k\}$ is valid

Examples

Example: summing numbers

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

while $\textsc{ptr} \neq n$ **do**

  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

  $\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

return $\textsc{sum}$

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return $\textsc{sum}$

Goal: establish that the sum of first n positive numbers is returned

Example: summing numbers

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

while $\textsc{ptr} \neq n$ **do**

  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

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return $\textsc{sum}$

$\Longleftarrow$ $\boxed{\text{sum of first } n \text{ positive numbers returned}}$

Example: summing numbers

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow 1$

while $\text{PTR} \neq n$ **do**

  $\Longleftarrow$ $\boxed{\text{loop invariant ???}}$

  $\text{PTR} \leftarrow \text{PTR} + 1$

  $\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

return SUM

$\Longleftarrow$ $\boxed{\text{sum of first } n \text{ positive numbers returned}}$

What is the relevant loop assertion?

Assigning assertions

Input: number $n \geq 1$
Output: sum of first n positive numbers

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

while $\textsc{ptr} \neq n$ **do**

  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

  $\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

return $\textsc{sum}$

Assigning assertions

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

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while $\textsc{ptr} \neq n$ **do**

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return $\textsc{sum}$

Assigning assertions

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

$\Longleftarrow \boxed{\textsc{sum} = \textsc{ptr}}$

while $\textsc{ptr} \neq n$ **do**

  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

  $\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

return $\textsc{sum}$

Assigning assertions

Input: number $n \geq 1$

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because

$$\{n \geq 1\} \begin{array}{l} \textsc{ptr} \leftarrow 1 \\ \textsc{sum} \leftarrow 1 \end{array} \{\textsc{sum} = \textsc{ptr}\}$$

is valid

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Input: number $n \geq 1$

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return $\textsc{sum}$

Assigning assertions

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  $\text{PTR} \leftarrow \text{PTR} + 1$

  $\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

return SUM

because

$$\{\text{PTR} \neq n \text{ and } Inv\} \begin{array}{l} \text{PTR} \leftarrow \\ \text{PTR} + 1 \\ \text{SUM} \leftarrow \\ \text{SUM} + \text{PTR} \end{array} \{Inv\}$$

is valid

Inv: $\text{SUM} = 1 + \cdots + \text{PTR}$

Assigning assertions

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

$\text{PTR} \leftarrow 1$

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  $\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

$\Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR} \text{ and } \text{PTR} = n}$

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Inv: $\text{SUM} = 1 + \cdots + \text{PTR}$

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Input: number $n \geq 1$

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  $\text{PTR} \leftarrow \text{PTR} + 1$

  $\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

$\Longleftarrow$ $\boxed{\text{SUM} = 1 + \cdots + \text{PTR} \text{ and } \text{PTR} = n}$

return SUM

because of Loop Invariance Theorem:

- Inv is before the loop is entered
- Inv is a loop invariant

Inv: $\text{SUM} = 1 + \cdots + \text{PTR}$

Assigning assertions

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

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$\Longleftarrow \boxed{\text{SUM} = \text{PTR}}$

while $\text{PTR} \neq n$ **do**

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$\Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR} \text{ and } \text{PTR} = n}$

return SUM

$\Longleftarrow \boxed{\text{sum of first } n \text{ positive numbers returned}}$

Inv: $\text{SUM} = 1 + \cdots + \text{PTR}$

Assigning assertions

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow 1$

$\Longleftarrow \boxed{\text{SUM} = \text{PTR}}$

while $\text{PTR} \neq n$ **do**

$\quad \Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR}}$

$\quad \text{PTR} \leftarrow \text{PTR} + 1$

$\quad \text{SUM} \leftarrow \text{SUM} + \text{PTR}$

$\Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR} \text{ and } \text{PTR} = n}$

return SUM

$\Longleftarrow \boxed{\text{sum of first } n \text{ positive numbers returned}}$

because $\text{SUM} = 1 + \cdots + \text{PTR}$ and $\text{PTR} = n$
logically implies $\text{SUM} = 1 + \cdots + n$

Inv: $\text{SUM} = 1 + \cdots + \text{PTR}$

The annotated algorithm

Input: number $n \geq 1$

Output: sum of first n positive numbers

```
// assert:   n ≥ 0
PTR ← 1
SUM ← 1
// assert:   SUM = PTR
while PTR ≠ n do
    // assert:   SUM = 1 + ··· + PTR
    PTR ← PTR + 1
    SUM ← SUM + PTR

// assert:   SUM = 1 + ··· + PTR and PTR = n
return SUM
// assert:   sum of first n positive numbers
    returned
```

Mathematical interlude

Problem:

Use of dots in $\textsc{sum} = 1 + \cdots + \textsc{ptr}$ too imprecise

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Use of dots in $\textsc{sum} = 1 + \cdots + \textsc{ptr}$ too imprecise

Solution:

Use mathematical notation:

> $\sum_{k=m}^{n} P(k)$ is precise mathematical notation for the sum $P(m) + \cdots + P(n)$

Here k is a *dummy variable*, part of the mechanics

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Use of dots in $\textsc{sum} = 1 + \cdots + \textsc{ptr}$ too imprecise

Solution:

Use mathematical notation:

> $\sum_{k=m}^{n} P(k)$ is precise mathematical notation for the sum $P(m) + \cdots + P(n)$

Here k is a *dummy variable*, part of the mechanics

Variations:

$$\sum_{m \le k \le n} P(k) \text{ means } P(m) + \cdots + P(n)$$

Summing integers with maths notation

Input: number $n \geq 1$

Output: sum of first n positive numbers

```
// assert:  n ≥ 0
PTR ← 1
SUM ← 1
// assert:  SUM = PTR
while PTR ≠ n do
    // assert:  SUM = Σ_{k=1}^{PTR} k
    PTR ← PTR + 1
    SUM ← SUM + PTR

// assert:  SUM = Σ_{k=1}^{PTR} k and PTR = n
return SUM
// assert:  sum of first n positive numbers
   returned
```

Summing integers: a variation

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow 1$

while $\text{PTR} < n$ **do**

  $\text{PTR} \leftarrow \text{PTR} + 1$

  $\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

return SUM

Summing integers: a variation

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

$\Longleftarrow \boxed{\textsc{sum} = \textsc{ptr}}$

while $\textsc{ptr} < n$ **do**

- $\Longleftarrow \boxed{\textsc{sum} = 1 + \cdots + \textsc{ptr}}$
- $\textsc{ptr} \leftarrow \textsc{ptr} + 1$
- $\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

return $\textsc{sum}$

Summing integers: a variation

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

$\text{PTR} \leftarrow 1$

$\text{SUM} \leftarrow 1$

$\Longleftarrow \boxed{\text{SUM} = \text{PTR}}$

while $\text{PTR} < n$ **do**

- $\Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR}}$
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- $\text{SUM} \leftarrow \text{SUM} + \text{PTR}$

$\Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR} \text{ and } \text{PTR} \not< n}$

return SUM

Summing integers: a variation

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

$\Longleftarrow \boxed{\textsc{sum} = \textsc{ptr}}$

while $\textsc{ptr} < n$ **do**

> $\Longleftarrow \boxed{\textsc{sum} = 1 + \cdots + \textsc{ptr}}$
>
> $\textsc{ptr} \leftarrow \textsc{ptr} + 1$
>
> $\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

$\Longleftarrow \boxed{\textsc{sum} = 1 + \cdots + \textsc{ptr} \text{ and } \textsc{ptr} \not< n}$

return $\textsc{sum}$

Problem: $\textsc{sum} = 1 + \cdots + \textsc{ptr}$ and $\textsc{ptr} \not< n$
does not imply $\textsc{sum} = 1 + \cdots + n$

Strengthening the loop invariant

New loop invariant:

$\textsc{sum} = 1 + \cdots + \textsc{ptr}$ and $\textsc{ptr} \leq n$

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Strengthening the loop invariant

New loop invariant:

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Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow \boxed{n \geq 1}$

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while $\text{PTR} < n$ **do**

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$\text{PTR} \leftarrow \text{PTR} + 1$

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$\Longleftarrow \boxed{\text{SUM} = 1 + \cdots + \text{PTR} \text{ and } \text{PTR} \leq n \text{ and } \text{PTR} \not< n}$

return SUM

Strengthening the loop invariant

New loop invariant:

$\textsc{sum} = 1 + \cdots + \textsc{ptr}$ and $\textsc{ptr} \leq n$

Input: number $n \geq 1$

Output: sum of first n positive numbers

$\Longleftarrow$ $\boxed{n \geq 1}$

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

$\Longleftarrow$ $\boxed{\textsc{sum} = \textsc{ptr} \text{ and } \textsc{ptr} \leq n}$

while $\textsc{ptr} < n$ **do**

$\Longleftarrow$ $\boxed{\textsc{sum} = 1 + \cdots + \textsc{ptr} \text{ and } \textsc{ptr} \leq n}$

$\textsc{ptr} \leftarrow \textsc{ptr} + 1$

$\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

$\Longleftarrow$ $\boxed{\textsc{sum} = 1 + \cdots + \textsc{ptr} \text{ and } \textsc{ptr} \leq n \text{ and } \textsc{ptr} \not< n}$

return $\textsc{sum}$

$\Longleftarrow$ $\boxed{\text{sum of first } n \text{ numbers returned}}$

The annotated variation

Input: number $n \geq 1$

Output: sum of first n positive numbers

```
// assert:  n ≥ 0
PTR ← 1
SUM ← 1
// assert:  SUM = PTR
while PTR < n do
    // assert:  SUM = ∑_{k=1}^{k=PTR} k and PTR ≤ n
    PTR ← PTR + 1
    SUM ← SUM + PTR

// assert:  SUM = ∑_{k=1}^{k=PTR} k and PTR ≤ n and PTR ≮ n

return SUM
// sum of first n positive numbers returned
```

Summing integers: variation

Input: number $n \geq 1$

Output: sum of first n positive numbers

`// assert:` $n \geq 0$

$\textsc{ptr} \leftarrow 1$

$\textsc{sum} \leftarrow 1$

`// assert:` $\textsc{sum} = \textsc{ptr}$

while $\textsc{ptr} \leq n$ **do**

  `// assert:` $\textsc{sum} = \sum_{k=1}^{k=\textsc{ptr}} k$ `and` $\textsc{ptr} \leq n+1$

  $\textsc{ptr} \leftarrow \textsc{ptr} + 1$

  $\textsc{sum} \leftarrow \textsc{sum} + \textsc{ptr}$

`// assert:` $\textsc{sum} = \sum_{k=1}^{k=\textsc{ptr}} k$ `and` $\textsc{ptr} \leq n+1$ $\textsc{ptr} \not\leq n$

return $\textsc{sum}$

`// sum of first` n `positive numbers returned`

What we’ve seen so far

Ways of propogating assertions using valid Floyd-Hoare triples:

1. Basic logic (we’ll come back to this at the end of the module)
2. Use the sequential rule
3. Use the **while**-loop invariant theorem

What we've seen so far

Ways of propogating assertions using valid Floyd-Hoare triples:

1. Basic logic (we'll come back to this at the end of the module)
2. Use the sequential rule
3. Use the **while**-loop invariant theorem

What about **for**-loops?

What do we want?

The theorem for **while**:

$\Longleftarrow$ $\boxed{\text{Inv}}$

while *Bool* **do**

  $\Longleftarrow$ $\boxed{\text{Inv}}$

  Body

...

can be extended to

$\Longleftarrow$ $\boxed{\text{Inv}}$

while *Bool* **do**

  $\Longleftarrow$ $\boxed{\text{Inv}}$

  Body

$\Longleftarrow$ $\boxed{\text{Inv and not Bool}}$

What do we want?

The theorem for **while**:

$\Longleftarrow \boxed{\text{Inv}}$

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| $\Longleftarrow \boxed{\text{Inv}}$

| Body

…

can be extended to

$\Longleftarrow \boxed{\text{Inv}}$

while *Bool* **do**

| $\Longleftarrow \boxed{\text{Inv}}$

| Body

$\Longleftarrow \boxed{\text{Inv and not Bool}}$

What we want:

$\Longleftarrow \boxed{\text{Inv}(0)}$

for $i \leftarrow 1$ **to** n **do**

| ???

…

can be extended to:

$\Longleftarrow \boxed{\text{Inv}(0)}$

for $i \leftarrow 1$ **to** n **do**

| ???

$\Longleftarrow \boxed{\text{Inv}(n)}$

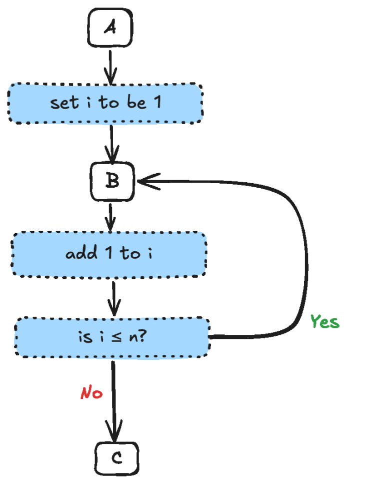
For-loop invariance theorem, in pictures

A

for $i \leftarrow 1$ **to** n **do**

| B

C



For-loop invariance theorem, in pictures

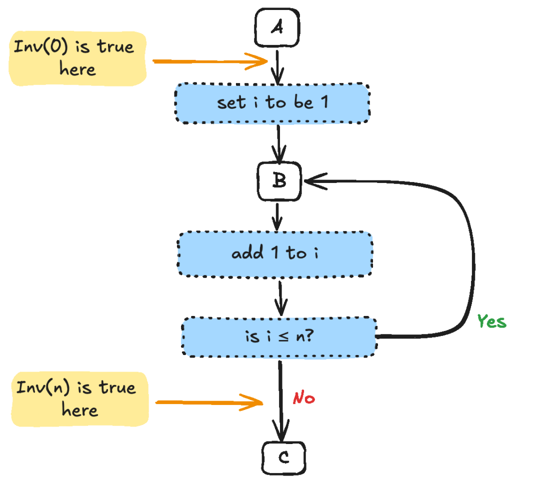
A

for $i \leftarrow 1$ **to** n **do**

  B

C

What we want



For-loop invariance theorem, in pictures

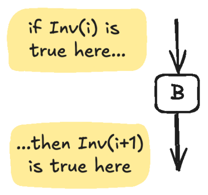
A

for $i \leftarrow 1$ **to** n **do**

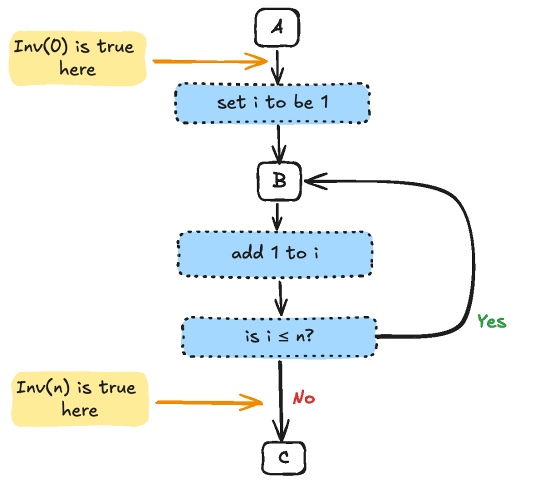
| B

C

How we get it



What we want



For-loop invariance theorem, in pictures

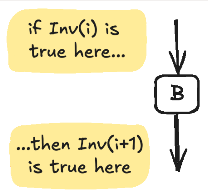
A

for $i \leftarrow 1$ **to** n **do**

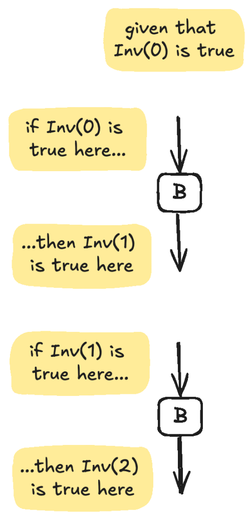
B

C

How we get it



Why this works



For-loop invariance theorem

for $i \leftarrow 1$ **to** n **do** Body

Suppose Inv(i) is a mathematical statement about i. If:

1. Inv(0) is true before the For-statement starts; and
2. {Inv(i)} Body {Inv($i + 1$)}

   In English: Inv(i) is preserved by the Body

Then:

when the for-loop terminates the property Inv(n) is true.

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Example Inv(i): $\textsc{sum} = \sum_{k=1}^{i} A[k]$

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Example placement of valid assertions

Input: Array $A[1 \dots n]$

Output: sum of elements in A

$\textsc{sum} \leftarrow 0$

for $i \leftarrow 1$ **to** n **do**

  $\textsc{sum} \leftarrow \textsc{sum} + A[i]$

return $\textsc{sum}$

Example placement of valid assertions

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  $\Longleftarrow \boxed{\textsc{sum} = A[1] + \cdots + A[i]}$

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$\Longleftarrow \boxed{\textsc{sum} = A[1] + \cdots + A[n]}$

return $\textsc{sum}$

$\Longleftarrow$ $\boxed{\text{sum of elements in } A \text{ returned}}$

The annotated algorithm

Input: Array $A[1 \ldots n]$

Output: sum of elements in A

$\textsc{sum} \leftarrow 0$

// **assert:** $\textsc{sum} = 0$

for $i \leftarrow 1$ **to** n **do**

  $\textsc{sum} \leftarrow \textsc{sum} + A[i]$

  // **assert:** $\textsc{sum} = A[1] + \cdots + A[i]$

// **assert:** $\textsc{sum} = A[1] + \cdots + A[n]$

return $\textsc{sum}$

// **assert:** sum of elements in A returned

Using for-loop invariance theorem, in general

$\Longleftarrow \boxed{\text{Inv}(0)}$

for $i \leftarrow 1$ **to** n **do**

    Body

    $\Longleftarrow \boxed{\text{Inv}(i)}$

...

can be extended to:

$\Longleftarrow \boxed{\text{Inv}(0)}$

for $i \leftarrow 1$ **to** n **do**

    Body

    $\Longleftarrow \boxed{\text{Inv}(i)}$

$\Longleftarrow \boxed{\text{Inv}(n)}$

Summing up

Never write down a looping construct unless

- you have a convincing argument to show it terminates
- you have established a relevant invariant
- you document your program with this information

Summing up

Never write down a looping construct unless

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- you have established a relevant invariant
- you document your program with this information

- The arguments do not have to be very formal
- But they have to be convincing
- Design algorithms and invariant simultaneously
- A priori justification is better than a posteriori justification

A couple more examples

What does this algorithm do ?

Input: Array $A[1 \dots n]$
Output: ????

for $i \leftarrow 2$ **to** n **do**
  $\text{KEY} \leftarrow A[i]$
  $j \leftarrow i - 1$
  while $j > 0$ *and* $A[j] > \text{KEY}$ **do**
    $A[j+1] \leftarrow A[j]$
    $j \leftarrow j - 1$
  $A[j+1] \leftarrow \text{KEY}$
return A

Difficult to know unless you have designed the program

Easy to know if you have designed the program

Insertion sort:

```
Input: Array A[1...n]
Output: Array A sorted
for i ← 2 to n do
    // maintains A[1...i] sorted
    KEY ← A[i]
    j ← i − 1
    while j > 0 and A[j] > KEY do
        // inserts A[i] correctly into A[1...(i − 1)]
        A[j + 1] ← A[j]
        j ← j − 1
    A[j + 1] ← KEY
return A
```

Another example

Input: A number $k \geq 0$

Output: 3(k+1)

$\Longleftarrow \boxed{k \geq 0}$

$\mathrm{X} \leftarrow 0$

$\mathrm{Y} \leftarrow 0$

$\Longleftarrow \boxed{\mathrm{Y} = 3\mathrm{X} \text{ and } \mathrm{X} \leq k + 1}$

while $\mathrm{X} \leq k$ **do**

> $\Longleftarrow \boxed{\mathrm{Y} = 3\mathrm{X} \text{ and } \mathrm{X} \leq k + 1}$
>
> $\mathrm{X} \leftarrow \mathrm{X} + 1$
>
> $\mathrm{Y} \leftarrow \mathrm{Y} + 3$

$\Longleftarrow \boxed{\mathrm{Y} = 3\mathrm{X} \text{ and } \mathrm{X} \leq k + 1 \text{ and } \mathrm{X} > k}$

return Y

$\Longleftarrow \boxed{\mathrm{Y} = 3(k + 1)}$