

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

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these slides available at : philipsaville.co.uk ·

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

EFFECTFUL SEMANTICS

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

- 1) take in some inputs
- 2) do some work
- 3) return some outputs

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

- 1) take in some inputs
- 2) do some work
- 3) return some outputs

eg,

$x : \text{nat} + x + 1 : \text{nat}$



$$\begin{aligned} N &\longrightarrow N \\ x &\longmapsto x + 1 \end{aligned}$$

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

- 1) take in some inputs
- 2) do some work interacting with the world
- 3) return some outputs

eg,

$x : \text{nat} + x + 1 : \text{nat}$



$N \longrightarrow N$
 $x \mapsto x + 1$

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

- 1) take in some inputs
- 2) do some work interacting with the world
- 3) return some outputs

effects

- print to screen
- memory
- non-determinism
- probability

EFFECTFUL SEMANTICS

modelling programs



inputs



outputs

- 1) take in some inputs
- 2) do some work interacting with the world
- 3) return some outputs

eg // $x : \text{nat} \vdash \text{print "hi"}; x + 1 : \text{nat}$

$\mathbb{N} \longrightarrow \mathcal{L}^* \times \mathbb{N} : n \mapsto (\text{hi}, n+1)$

effects

- print to screen
- memory
- non-determinism
- probability

EFFECTFUL SEMANTICS

modelling programs

$$\begin{array}{c} \text{inputs} \\ (\text{with types}) \end{array} \rightsquigarrow \left\{ \begin{array}{c} \text{effectful program} \\ \Gamma \vdash M : A \end{array} \right. \rightsquigarrow \text{output type}$$

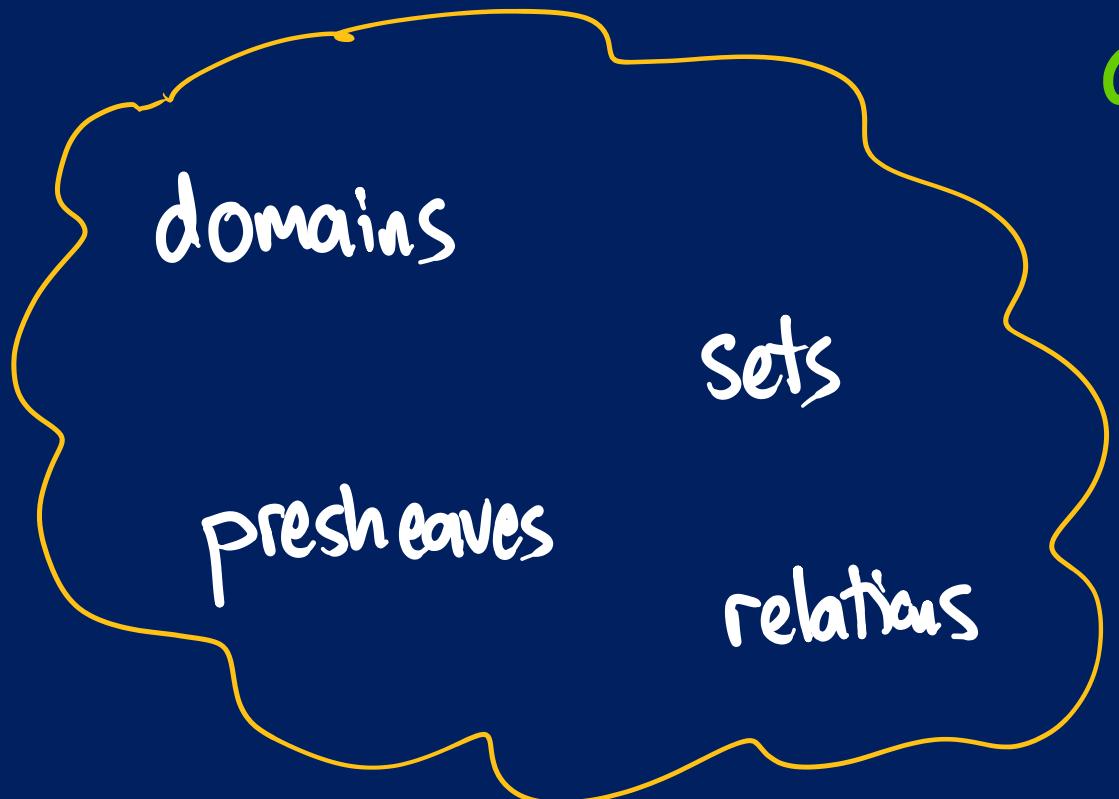
and arrow in some structured category

effects

- print to screen
- memory
- non-determinism
- probability

PREMONOIDAL & FREYD

PREMONOIDAL & FREYD ~~B~~CATEGORIES



a variety of semantic models

{ a unifying framework

strong monads (Moggi)

premonoidal categories

Freyd categories

(Power-
Robinson,
Power)

EFFECTFUL SEMANTICS IN 2-CATEGORIES:

PREMONOIDAL & FREYD BICATEGORIES

2-dimensions = more refined Semantics

↳ intensional information,
rewrites between programs
(often: via a universal property)

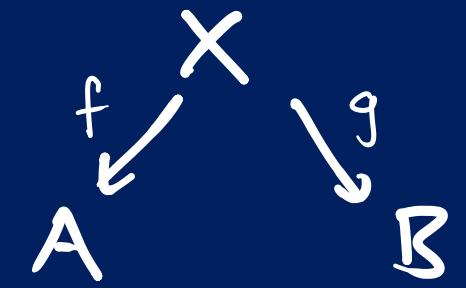
2-dimensions = more refined Semantics

↳ intensional information,
rewrites between programs

eg //

- maps $A \rightsquigarrow B$ as Spans
- composition by pullback

only associative up to TSO?



$$\begin{aligned} a &\underset{\text{via } x}{\underset{\Leftrightarrow}{\sim}} b \\ \Leftrightarrow f(x) &= a \\ g(x) &= b \end{aligned}$$

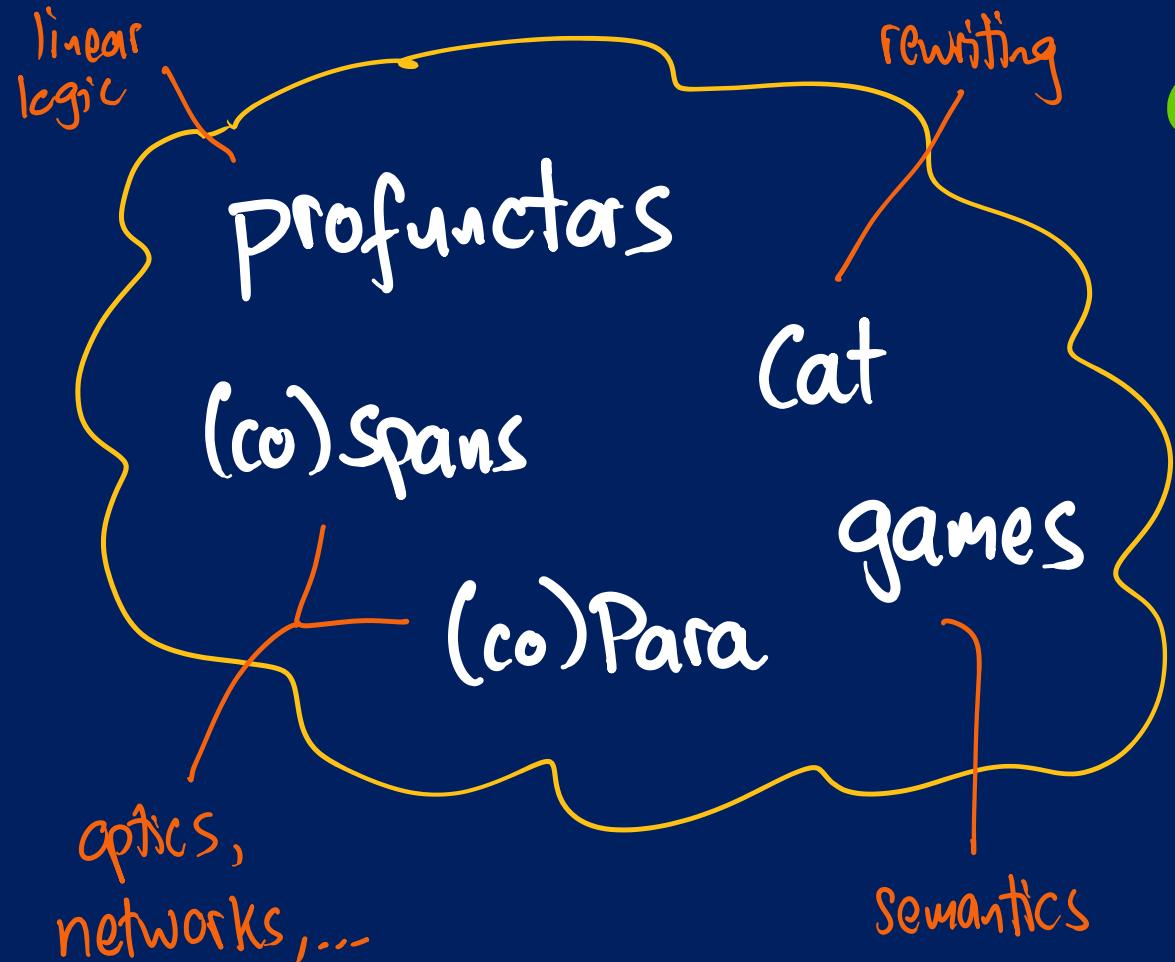
2-dimensions = more refined Semantics

↳ intensional information,
rewrites between programs

inputs
(with types) }
 $\vdash M : A$ effectful program
 } output type

and 1-cell in some structured bicategory

PREMONOIDAL & FREYD BICATEGORIES



a variety of semantic models

{ a unifying framework

~~strong pseudonaturals~~

premonoidal bicategories

Freyd bicategories

THIS
WORK

key point

print "a";
print "b"

≠

print "b";
print "a"

key point

$$\begin{array}{l} \Gamma \vdash P : A \\ \Delta \vdash Q : B \end{array}$$

run P to V ;
run Q to W ;
return (V, W)

run Q to W ;
run P to V ;
return (V, W)

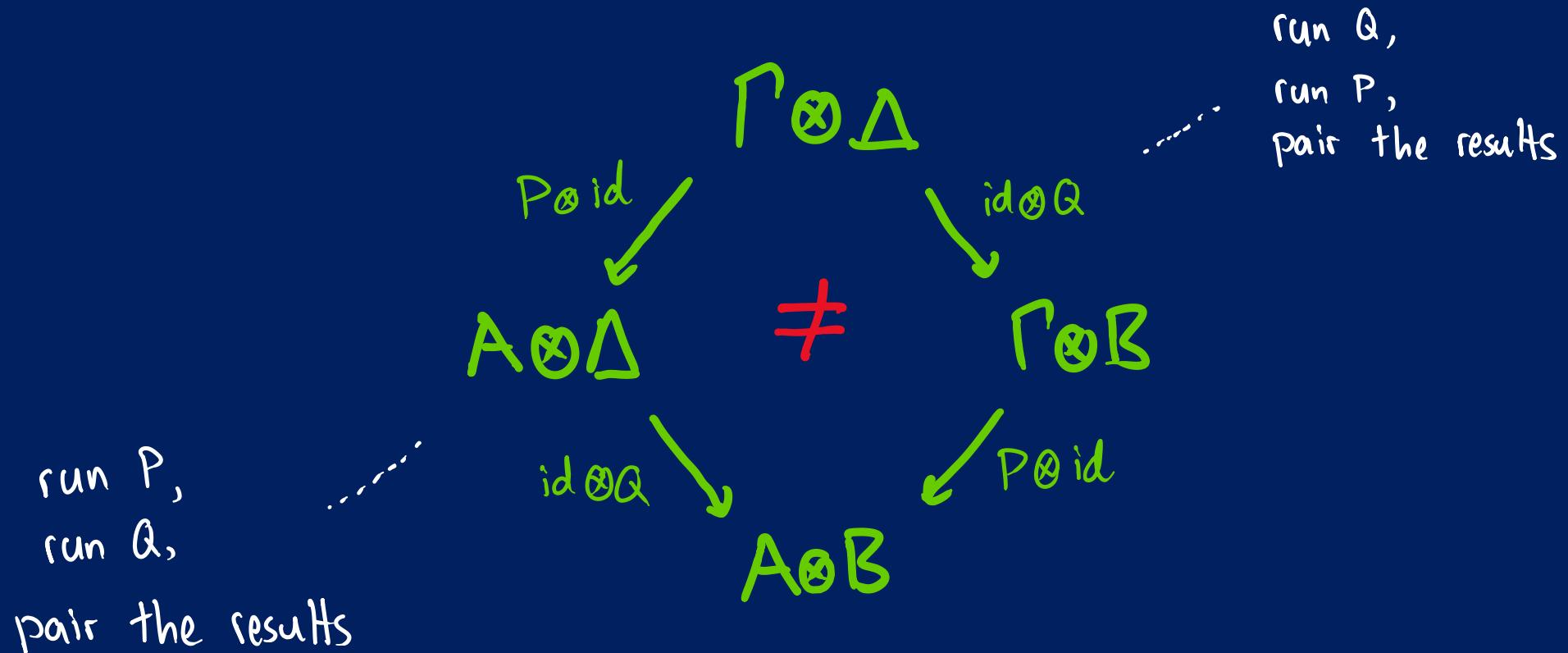
$$\begin{array}{c} \Gamma \otimes \Delta \xrightarrow{\quad} A \otimes \Delta \xrightarrow{\quad} A \otimes B \\ \text{P} \otimes \text{id} \qquad \qquad \text{id} \otimes Q \end{array}$$

$$\begin{array}{c} \Gamma \otimes \Delta \xrightarrow{\quad} \Gamma \otimes B \xrightarrow{\quad} A \otimes B \\ \text{id} \otimes Q \qquad \qquad \text{P} \otimes \text{id} \end{array}$$

key point

$$\Gamma \vdash P : A$$
$$\Delta \vdash Q : B$$

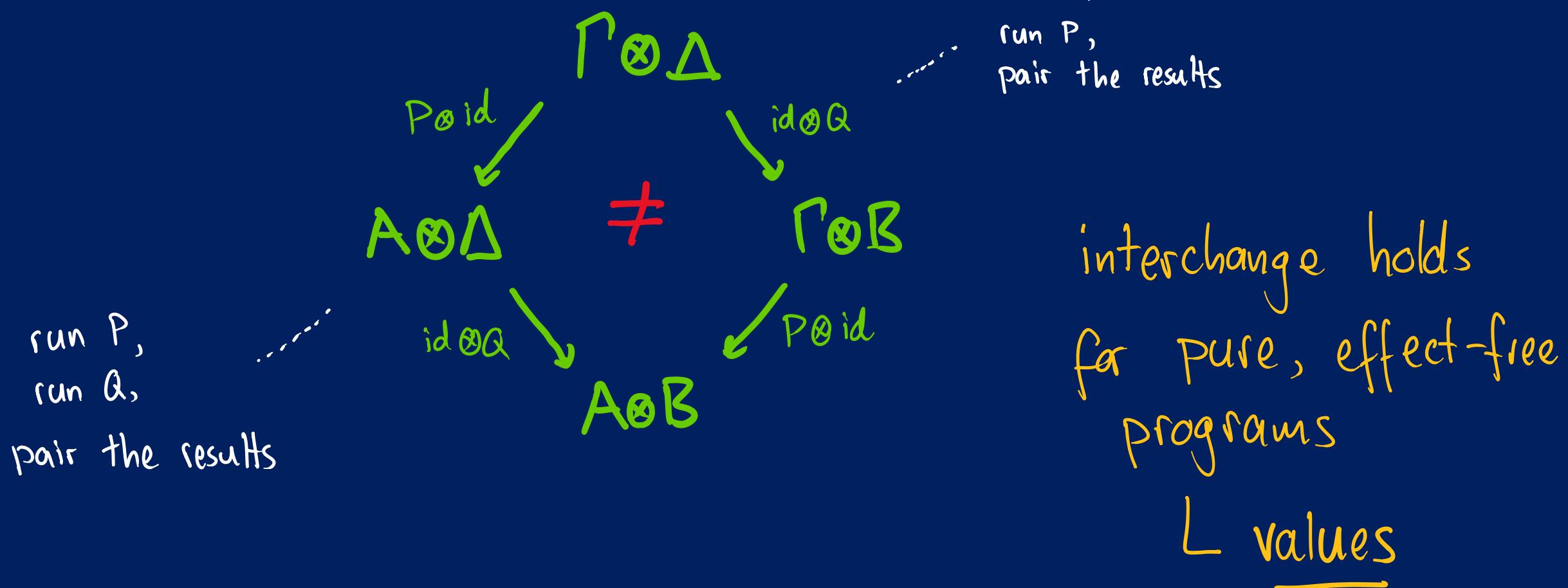
INTERCHANGE FAILS



key point

$$\Gamma \vdash P : A$$
$$\Delta \vdash Q : B$$

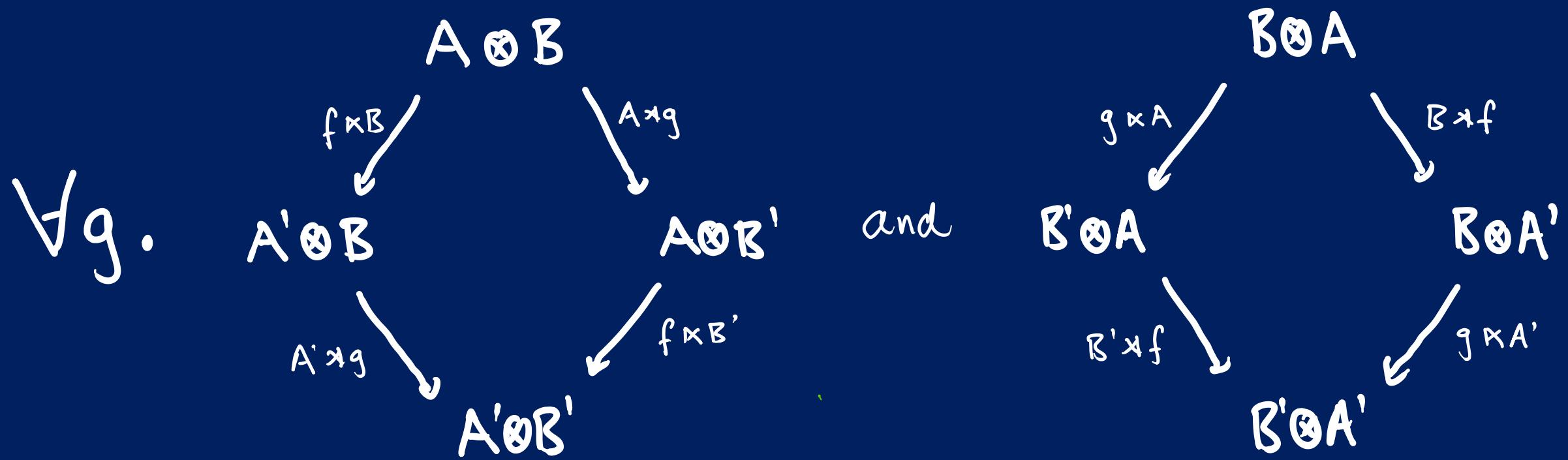
INTERCHANGE FAILS



a binoidal category (\mathcal{C}, \otimes) has

- $\otimes : \text{ob } \mathcal{C} \times \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{C}$
- $I \in \mathcal{C}$
- for every $A, B \in \mathcal{C}$, functors $A \rtimes (-), (-) \ltimes B : \mathcal{C} \rightarrow \mathcal{C}$
s.t. $A \rtimes B = A \otimes B = A \ltimes B$

a map $f: A \rightarrow A'$ is central if



get category $Z(\mathcal{C}) \hookrightarrow \mathcal{C}$

a premonoidal category $(\mathcal{C}, \otimes, I)$ has:

binoidal

- $\otimes : \text{ob } \mathcal{C} \times \text{ob } \mathcal{C} \rightarrow \text{ob } \mathcal{C}$
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s.t. $A \rtimes B = A \otimes B = A \ltimes B$
 - central natural isomorphisms α, λ, ρ
- s.t. $\Delta + \circlearrowleft$ hold
- $\} \text{ so } \mathcal{Z}(\mathcal{C}) \text{ is monoidal}$

$(\mathcal{C}, \otimes, I)$ symmetric monoidal

$t_{A,B} : A \otimes T(B) \rightarrow T(A \otimes B)$
{}
+ axioms

e.g.: for (T, t) strong, \mathcal{C}_T is premonoidal:

$$A \times g := (A \otimes B \xrightarrow{A \otimes g} A \otimes TB' \xrightarrow[t_{A,B'}]{\text{strength}} T(A \otimes B'))$$

$$f \ltimes B := (A \otimes B \xrightarrow{f \otimes B} T(A') \otimes B \xrightarrow{\quad} T(A' \otimes B))$$

built from t
and symmetry

e.g.: $[\mathcal{C}, \mathcal{C}]_u$ is premonoidal

↳ obj: functors $\mathcal{C} \rightarrow \mathcal{C}$

maps: unnatural transformations

ie. families of arrows

$$\{\delta_c : FC \rightarrow GC\}_{c \in \mathcal{C}}$$

idea

Symmetric
monoidal \mathcal{C}

premonoidal categories axiomatise \mathcal{C}_T — strong

= monoidal categories without interchange

idea

Symmetric
monoidal \mathcal{C}

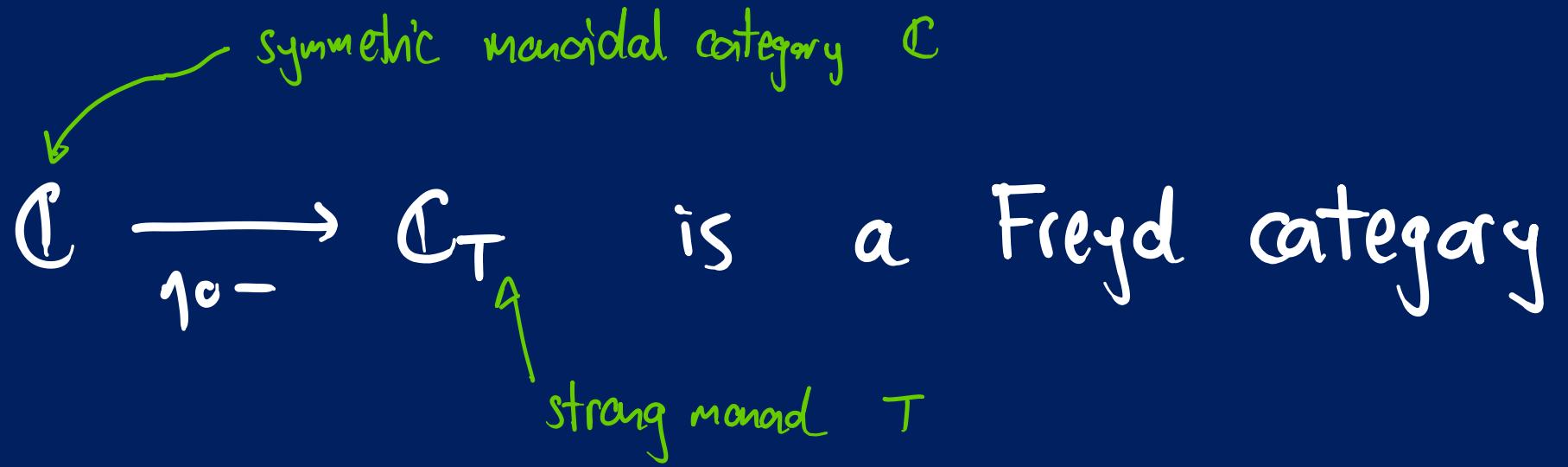
premonoidal categories axiomatise \mathcal{C}_T ← strong
= monoidal categories without interchange

Freyd categories axiomatise $\mathcal{C} \xrightarrow{\eta_0-} \mathcal{C}_T$
= premonoidal categories with a choice of "centre"

a Freyd category has

- $(\mathcal{C}, \otimes, I)$ premonoidal
- (V, \otimes, I) monoidal (often cartesian)
- $J: V \rightarrow \mathcal{C}$ identity on objects,
s.t.
 - ① J lands in $\mathcal{Z}(\mathcal{C})$
 - ② J preserves premonoidal structure
(strictly)

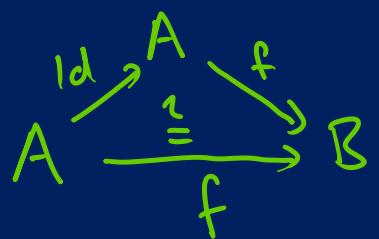
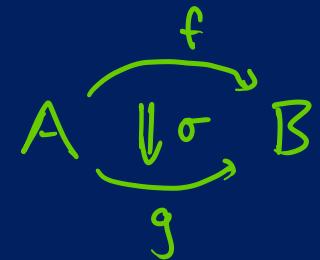
eg :



The 2-dimensional story -

a bicategory \mathcal{B} has

- objects A, B, \dots
- 1-cells $f, g, \dots : A \rightarrow B$
- 2-cells $\sigma, \tau : f \Rightarrow g$
- natural isomorphisms



$$\left. \begin{array}{l} \alpha : (f \circ g) \circ h \xrightarrow{\cong} f \circ (g \circ h) \\ \lambda : \text{Id}_A \circ f \xrightarrow{\cong} f \\ \rho : f \circ \text{Id}_B \xrightarrow{\cong} f \end{array} \right\} \begin{array}{l} \text{subject to} \\ \Delta + \circlearrowleft \text{ laws} \end{array}$$

categories

"bicategorify"

bicategories

equations
of maps



isomorphisms
between maps,
subject to equations

hard part: what equations ??

examples of bicategories

- Cat (any 2-category)
- Prof , generalised species, ... (any Kleisli bicategory)
- $\text{Span}(\mathbb{C})$, Rel , ...
- $\text{Para}(\mathbb{C})$
- bimodules over a ring
-

(Katsurada, Smirnov, Melliès, ...)

e.g.:

a graded monad is a lax monoidal
functor $T : \mathbb{E} \rightarrow [\mathbb{C}, \mathbb{C}]$

grades

$$\mu : T_e \circ T_{e'} \Rightarrow T_{e \cdot e'}$$

$$\eta : id \Rightarrow T_i$$

for (\mathbb{E}, \bullet, i) monoidal

e.g., $\mathbb{E} := (\mathbb{N}_{\leq}, \perp, \cdot)$
 $\mathbb{C} := \text{Set}$
 $T_n := (\text{lists of length } \leq n)$

(cf. Katsunata)

e.g.: a ^{strong} graded monad is a lax monoidal
functor $T : \mathbb{E} \rightarrow [\mathbb{C}, \mathbb{C}]$ ^{strong}
grades

$$\begin{array}{l} \mu : T_e \circ T_{e'} \Rightarrow T_{e \cdot e'} \\ \eta : id \Rightarrow T_i \end{array} \quad \left. \right\} \text{strong}$$

eg :

a ^{strong} \wedge graded monad is a lax monoidal
functor $T : \mathbb{E} \rightarrow [\mathbb{C}, \mathbb{C}]$ ^{strong}

a version
of coPara

get a bicategory Kl_T :

- obj = those of \mathbb{C}
- 1-cells $A \rightarrow B = (e, A \xrightarrow{f} T_e B)$
- 2-cells $\sigma : f \Rightarrow g =$ regadings

$$\begin{array}{ccc} A & \xrightarrow{f} & T_e B \xrightarrow{T_{e,g}} T_e T_{e'} C \\ & & \searrow g \circ f \\ & & T_{e \cdot e'} C \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{f} & T_e B \\ & \downarrow T_{e,B} & \\ & g & T_{e'} B \end{array}$$

PREMONOIDAL AND FREYD BICATEGORIES

Some bicategorical words

$$F(\text{Id}) \cong \text{Id}$$

$$F(f) \circ F(g) \cong F(f \circ g)$$

pseudofunctor = map of bicategories

pseudo natural
transformation = map of pseudofunctors
naturality up to isomorphisms

Modification = map of pseudonat. trans.
families of 2-cells + axiom

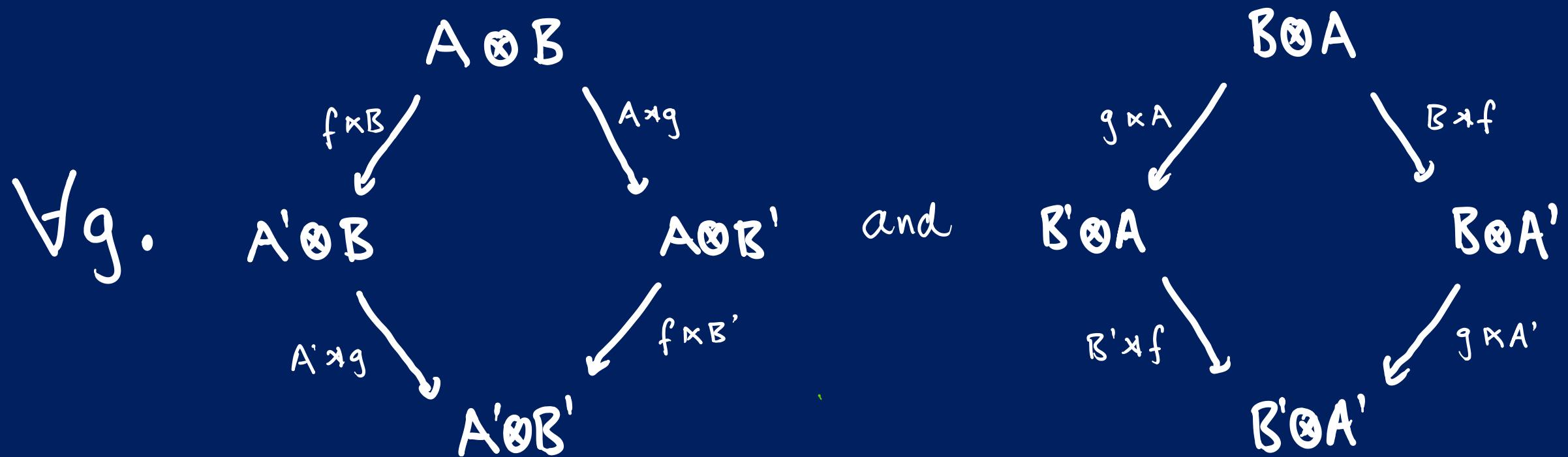
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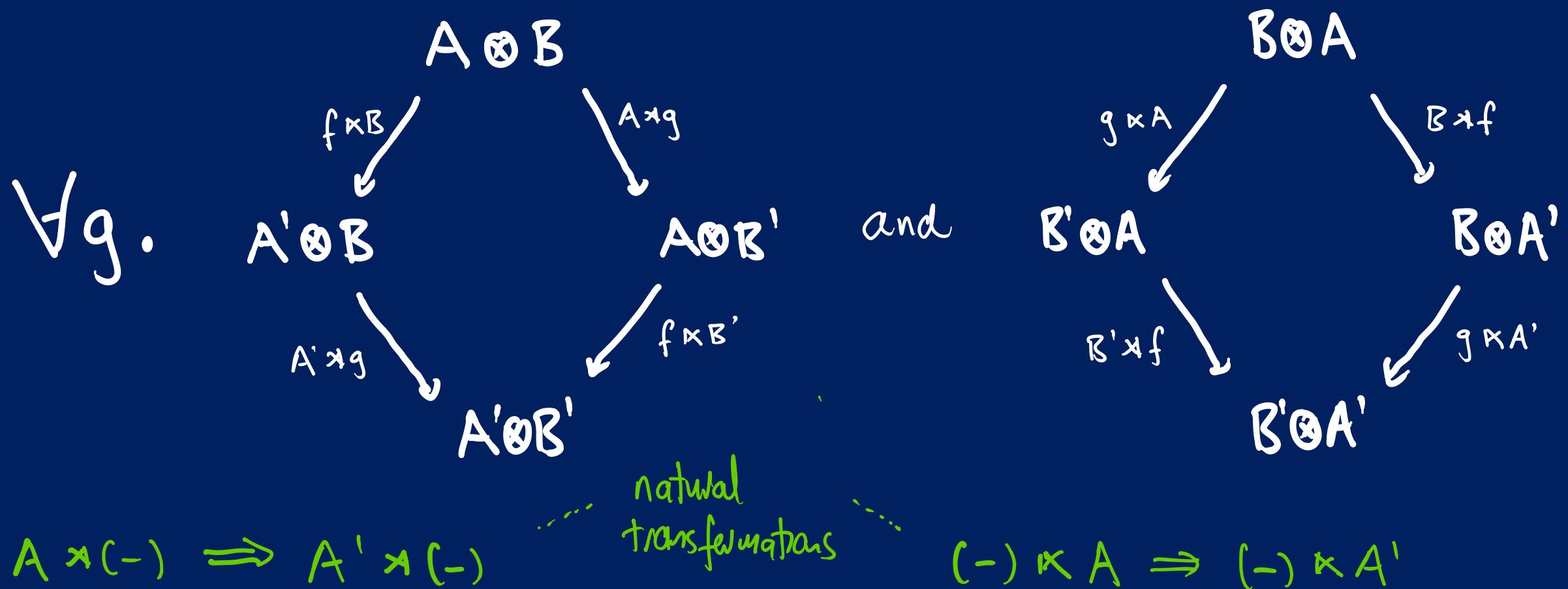
a binoidal bicategory (B, \otimes) has

- $\otimes : \text{ob } B \times \text{ob } B \rightarrow \text{ob } B$
- $I \in B$
- for every $A, B \in B$, functors $A \rtimes (-), (-) \ltimes B : B \rightarrow B$
s.t. $A \rtimes B = A \otimes B = A \ltimes B$
Pseudo

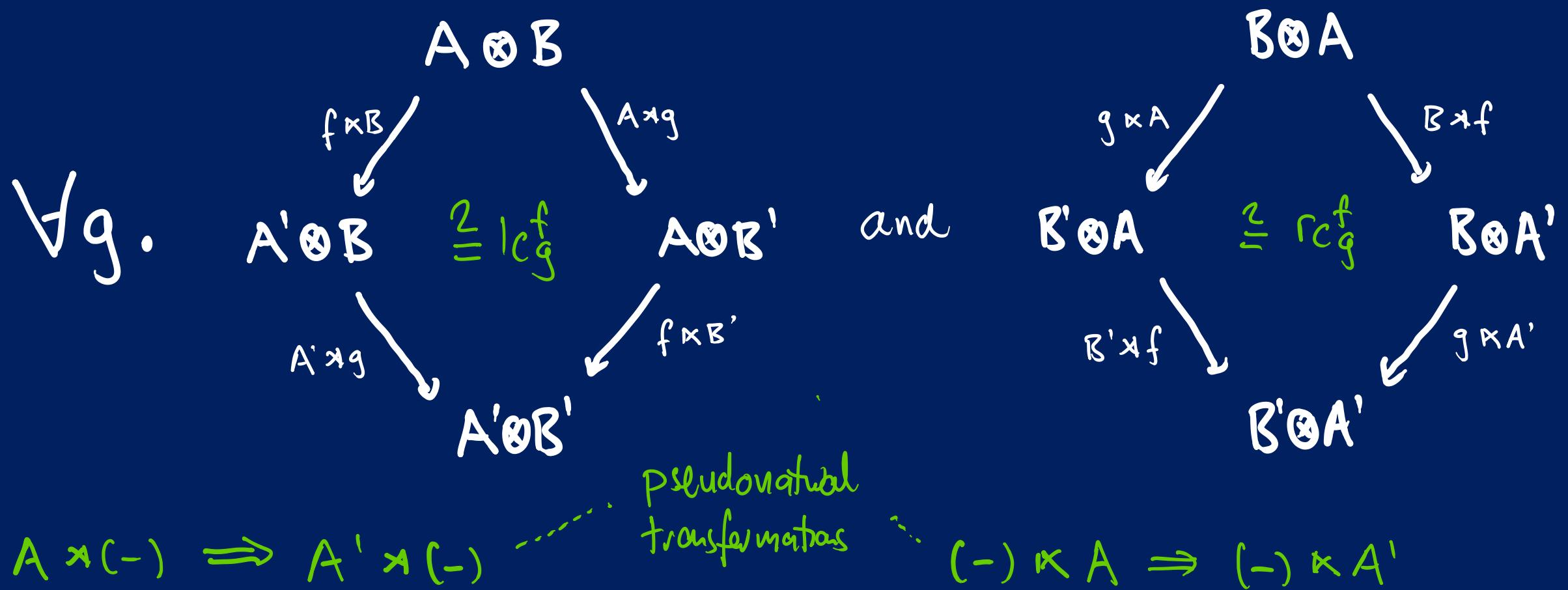
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a map $f: A \rightarrow A'$ is central if



a central 1-cell $(f, \text{lc}^f, \text{rc}^f)$ has



a premonoidal bicategory (B, \otimes, I) has:

binojidal

- $\otimes : \text{ob } B \times \text{ob } B \rightarrow \text{ob } B$
- $I \in B$
- for every $A, B \in B$, functors $A \rtimes (-), (-) \ltimes B : B \rightarrow B$ ^{pseudo}
s.t. $A \rtimes B = A \otimes B = A \ltimes B$
- central natural equivalences α, λ, ρ ^{pseudo}
- isomorphisms witnessing $\triangleright, \triangleleft, \dots$
subject to equations



eg //

- $[B, B]_u$ is premonoidal

↳ pseudofunctors, unnatural transformations,
families of 2-cells

- Kl_T is premonoidal

↳ for $T : \mathbb{E} \rightarrow [C, \mathbb{I}]_{\text{strong}}$
a strong graded monad

a Freyd bicategory has

- $(\mathcal{B}, \otimes, I)$ premonoidal
- $(\mathcal{V}, \otimes, I)$ monoidal
- plus compatibility equations

- $J: \mathcal{V} \rightarrow \mathcal{B}$ identity on objects,

s.t.

① J factors through $\mathcal{Z}(\mathcal{B})$

② J preserves premonoidal structure up to an icon



how strict? Use examples!

eg //

- for $E \xrightarrow{\tau} [\mathcal{C}, \mathcal{C}]_{\text{strong}}$ a graded monad
 - discrete monoidal 2-category $\mathcal{C} \longrightarrow \mathbf{Kl}_T$ is a Freyd bicategory
- [Bonus] $B \longrightarrow B_T$ is a Freyd bicategory
 - strong pseudomonad

Why believe the definition?

(Levy)

(Freyd category)
 $\mathcal{V} \xrightarrow{\exists} \mathcal{C}$

$$\equiv \left\{ \begin{array}{l} \text{actions } \Delta : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C} \\ \triangleleft : \mathcal{C} \times \mathcal{V} \rightarrow \mathcal{C} \\ \text{s.t. } \exists \text{ is a strict map} \\ \text{of actions:} \\ \mathcal{V} \times \mathcal{C} \xrightarrow{\triangleleft} \mathcal{C} \xleftarrow{\Delta} \mathcal{C} \times \mathcal{V} \\ \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V} \xleftarrow{\otimes} \mathcal{V} \times \mathcal{V} \end{array} \right\}$$

$\mathcal{V} \times \mathcal{C} \xrightarrow{\triangleleft} \mathcal{C} \xleftarrow{\Delta} \mathcal{C} \times \mathcal{V}$
 $\mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V} \xleftarrow{\otimes} \mathcal{V} \times \mathcal{V}$

Why believe the definition?

A THEOREM

(Freyd bicategory) \simeq

all very
canonical

actions $\Delta : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$
 $\square : \mathcal{C} \times \mathcal{V} \rightarrow \mathcal{C}$
s.t. J is a strict map
of actions:

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} & \xleftarrow{\square} & \mathcal{C} \times \mathcal{V} \\ \uparrow \mathcal{V} \times J & \cong & \uparrow J & \cong & \uparrow J \times \mathcal{V} \\ \mathcal{V} \times \mathcal{V} & \xrightarrow{\otimes} & \mathcal{V} & \xleftarrow{\otimes} & \mathcal{V} \times \mathcal{V} \end{array}$$

strict = commutes with all the data

SUMMARY

- 2-dimensional models refine 1-dimensional ones
- bicategorical premonoidal + Freyd structure
is (part of) a framework for these
- definitions backed by examples and lifting
of 1-dimensional correspondence

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FUTURE WORK : closure, internal language, relation to strengths,
coherence, relation to Gray structure, centres, ...