Strong pseudomonads and premonoidal bicategories

***Abstract*—Strong monads and premonoidal categories play a central role in clarifying the denotational semantics of effectful programming languages. Unfortunately, this theory excludes many modern semantic models in which the associativity and unit laws only hold up to coherent isomorphism: for instance, because composition is defined using a universal property.**

This paper remedies the situation. We define premonoidal bicategories and a notion of strength for pseudomonads, and show that the Kleisli bicategory of a strong pseudomonad is premonoidal.

As often in 2-dimensional category theory, the main difficulty is to find the correct coherence axioms on 2-cells. We therefore justify our definitions with numerous examples and by proving a correspondence theorem between actions and strengths, generalizing a well-known result.

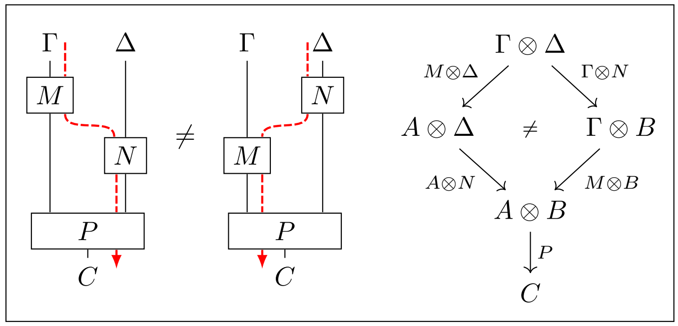


Figure 1: Control flow and data flow in a premonoidal bicategory. Here M and N represent program fragments which cannot be re-ordered—for instance, different print statements—and the dashed red line indicates control flow.

I. Introduction

Moggi famously observed that the category-theoretic notion of *strong monad* encapsulates the structure of effectful programs, and hence one can give a denotational model of an effectful programming language using a strong monad on a cartesian (or more generally, monoidal) category ([55], [56]). The approach was refined by Power & Robinson, whose *premonoidal categories* axiomatize the core structure of Moggi's models [64]. These insights give an abstract framework for studying existing models and constructing new ones, abstracting away from the particular choice of effect.

On the other hand, in recent years a plethora of models have been proposed which are not categories but *bicategories* [2]. These models come with more structure, and typically provide finer-grained or more intensional information that traditional categorical ones. Examples include various kinds of game semantics ([7], [51]), recent models of linear logic based on profunctors ([14], [15], [22]), and models describing the $\beta\eta$-rewrites of the simply-typed λ-calculus ([30], [17]).

This paper is the first step in extending the Moggi–Power–Robinson framework to these new examples. We introduce bicategorical versions of strong monads and premonoidal categories, and validate these definitions through examples and theorems paralleling the categorical theory.

We begin with a brief overview of monadic semantics (Section I-A). Then we introduce and motivate bicategories (Section I-B), and outline our main contributions (Section I-C).

A. Strong monads and premonoidal categories

A *strong monad* on a monoidal category $(\mathbb{C}, \otimes, I)$ is a monad (T, μ, η) equipped with a natural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$ known as a *strength*, compatible with both the monoidal structure of $\mathbb{C}$ and the monadic structure of T ([39], or Section III). In this setting, an effectful program $(\Gamma \vdash M : A)$ is modelled by a Kleisli arrow $\Gamma \to TA$ in $\mathbb{C}$ and substitution into another program $(\Delta, x : A \vdash N : B)$ is modelled using the strength and the Kleisli extension operation:

$$\Delta \otimes \Gamma \xrightarrow{\Delta \otimes M} \Delta \otimes TA \xrightarrow{t} T(\Delta \otimes A) \xrightarrow{\gg\!= N} TB.$$

This approach provides a sound and complete class of semantic models for certain effectful languages [56]. But it also involves some indirect aspects: programs are not interpreted in $\mathbb{C}$ but in the Kleisli category $\mathbb{C}_T$. Premonoidal categories [64] remedy this by directly axiomatising the structure of $\mathbb{C}_T$ required for the semantic interpretation; they more directly reflect the structure of effectful languages in which a monad is not explicit (*e.g.* [55], [20]).

A *premonoidal category* is a category $\mathbb{D}$ equipped with a tensor product $\otimes$ that is only functorial in each argument separately. Effectful programs are interpreted directly as morphisms in $\mathbb{D}$, and the substitution above is interpreted as $\Delta \otimes \Gamma \xrightarrow{\Delta \otimes M} \Delta \otimes A \xrightarrow{N} B$. This approach strictly generalises the previous one: if $(\mathbb{C}, \otimes, I)$ is symmetric monoidal, then any strength for the monad T induces a premonoidal structure on $\mathbb{C}_T$, but there are also examples of premonoidal categories not arising from a monad (*e.g.* [73], [63], [71]).

The lack of functoriality of $\otimes$ reflects the fact that one cannot generally re-order the statements of an effectful program, even if the data flow permits it. So the composition of morphisms in a premonoidal category should be understood as encoding control flow. This is illustrated in Figure 1 using the graphical calculus for premonoidal categories ([34], [67]).

B. The case for bicategories

As indicated above, many recent models form bicategories instead of categories. Roughly speaking, a bicategory is like a category except the unit and associativity laws for the composition of morphisms are replaced by well-behaved isomorphisms. This kind of structure typically arises when composition is defined using a universal construction; for an illustrative example we consider a bicategory of spans, that occurs widely in models of programming languages and computational processes (*e.g.* [51], [13], [26], [1]).

EXAMPLE. *Spans of sets:* Consider a model in which objects are sets and a morphism from A to B consists of a set S and a span of functions $A \longleftarrow S \longrightarrow B$. We can compose pairs of morphisms $A \longleftarrow S \longrightarrow B$ and $B \longleftarrow R \longrightarrow C$ using a pullback in the category of sets:

$$\begin{tikzcd}[column sep=small] & & R \circ S \arrow[dl] \arrow[dr] & & \\ & S \arrow[dl] \arrow[dr] & & R \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

This composition correctly captures a notion of 'plugging together' spans, but is only associative in a weak sense. Indeed, the two ways of taking pullbacks

$$\begin{tikzcd}[column sep=tiny, row sep=small] & & & \bullet \arrow[dl] \arrow[ddrr] & & & \\ & & \bullet \arrow[dl] \arrow[dr] & & & & \\ & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet & & \bullet & & \bullet \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny, row sep=small] & & & \bullet \arrow[ddll] \arrow[dr] & & & \\ & & & & \bullet \arrow[dl] \arrow[dr] & & \\ & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet & & \bullet & & \bullet \end{tikzcd} \tag{1}$$

are generally not equal, but they are canonically isomorphic via the universal property of pullbacks. Similarly, the span $A \xleftarrow{\text{id}} A \xrightarrow{\text{id}} A$ is only a weak identity for composition, because pulling back along id only gives an isomorphic set.

To describe composition in this model, therefore, we require a notion of morphism between spans. If S and S' are spans from A to B, then a map between them is a function $\sigma : S \to S'$ that commutes with the span legs on each side:

$$\begin{tikzcd} & S \arrow[r, dashed, "\sigma"] \arrow[dl] \arrow[drr] & S' \arrow[dll] \arrow[dr] & \\ A & & & B \end{tikzcd}$$

The two iterated composites in (1) are isomorphic as spans, so composition of spans is associative up to isomorphism. Similarly, the identity span is unital up to isomorphism and, because these isomorphisms arise from a universal property, they behave well together. Bicategories axiomatize such situations.

Definition 1 ([2]). *A bicategory $\mathscr{B}$ consists of:*

- *A collection of objects $A, B, \ldots$*
- *For all objects A and B, a collection of morphisms from A to B, themselves related by morphisms: thus we have a* hom-category *$\mathscr{B}(A, B)$ whose objects (typically denoted $f, g : A \to B$) are called* 1-cells*, and whose morphisms (typically denoted $\sigma, \tau : f \Rightarrow g$) are called* 2-cells*. The category structure means we can compose 2-cells between parallel 1-cells.*
- *For all objects A, B, and C, a composition functor*
$$\circ_{A,B,C} : \mathscr{B}(B, C) \times \mathscr{B}(A, B) \longrightarrow \mathscr{B}(A, C)$$
and, for all A, an identity 1-cell $\mathsf{Id}_A \in \mathscr{B}(A, A)$.
- *Coherent structural 2-cells: since the composition of 1-cells is weak, we have a natural family of invertible 2-cells $\mathsf{a}_{f,g,h} : (f \circ g) \circ h \Longrightarrow f \circ (g \circ h)$ instead of the usual associativity equation. Similarly, we have natural families of invertible 2-cells $\mathsf{l}_f : \mathsf{Id}_B \circ f \Longrightarrow f$ and $\mathsf{r}_f : f \circ \mathsf{Id}_A \Longrightarrow f$ instead of the left and right identity laws. These structural 2-cells must satisfy coherence axioms similar to those for a monoidal category.*

The 2-dimensional structure of a bicategory lets one describe relationships between morphisms, which can be used to provide refined semantic information (*e.g.* [30], [17], [74], [57], [37]). To illustrate further, we consider the $\mathbf{Para}$ construction, which is a general way to build models of parametrized processes [6] (see also [11], [68]). In this bicategory, the 2-cells describe how processes can be reparametrized; thus the weakness arises because we are tracking extra information. We will use this bicategory several times, so we spell out the definition in detail.

EXAMPLE*: the* $\mathbf{Para}$ *construction:* Starting from a monoidal category $(\mathbb{C}, \otimes, I)$, we construct a bicategory $\mathbf{Para}(\mathbb{C})$ as follows:

- The objects are those of $\mathbb{C}$.
- A 1-cell from A to B is a parametrized $\mathbb{C}$-morphism, defined as an object $P \in \mathbb{C}$ together with a morphism $f : P \otimes A \to B$ in $\mathbb{C}$. The object P is thought of as a space of parameters.
- A 2-cell from $f : P \otimes A \to B$ to $g : P' \otimes A \to B$ is a reparametrization map, *i.e.* a map $\sigma : P \to P'$ such that $g \circ (\sigma \otimes A) = f$.

Composition of 1-cells is defined using the tensor product of parameters: if $f : P \otimes A \to B$ and $g : Q \otimes B \to C$, then $g \circ f$ is the object $Q \otimes P$ equipped with the map

$$(Q \otimes P) \otimes A \xrightarrow{\cong} Q \otimes (P \otimes A) \xrightarrow{Q \otimes f} Q \otimes B \xrightarrow{g} C$$

where the first map is the associativity of the tensor product.

If we also have $h : R \otimes C \to D$, then the two composites $(h \circ g) \circ f$ and $h \circ (g \circ f)$ have parameter spaces $(R \otimes Q) \otimes P$ and $R \otimes (Q \otimes P)$, respectively. Because the tensor product in a monoidal category is generally only associative up to isomorphism, these 1-cells are only isomorphic in $\mathbf{Para}(\mathbb{C})$. A similar argument applies to the identity laws, so $\mathbf{Para}(\mathbb{C})$ is a bicategory with a, l and r given by $\mathbb{C}$'s monoidal structure. In fact, if $\mathbb{C}$ is symmetric monoidal, this lifts to a symmetric monoidal structure on $\mathbf{Para}(\mathbb{C})$: see Section II-C.

C. This paper: semantics for effects in bicategories

The central aim of this paper is to 'bicategorify' the definitions of strong monad and premonoidal category. A definition in category theory typically involves data subject to equations expressed as commutative diagrams. In bicategory theory these equations are replaced by invertible 2-cells filling each diagram, and then we have equations on these 2-cells.

The main challenge, therefore, is knowing what the 2-cell equations should be. This can be difficult and subtle, even for experts. For example, symmetric monoidal structure is apparently innocuous and well-understood, but the history of its bicategorical definition is littered with missteps (see [69] for an overview). We therefore justify our definitions both by providing examples and by showing that our definitions are consistent with other standard bicategorical constructions.

Contributions: First, we define a notion of *strength* for pseudomonads (Section III) and give a range of examples, including a bicategorical version of the fact that every monad on $\mathbf{Set}$ is canonically strong (Section III-C).

Next we show that, just as strengths on a monad correspond to certain actions on its Kleisli category (see *e.g.* [49, Proposition 4.3]), so strengths for a pseudomonad correspond to certain actions on its Kleisli bicategory (Section IV and Theorem 1). This correspondence is well-known in the categorical setting, where actions are sometimes used directly to axiomatise models for effectful languages (see *e.g.* [54]).

In Section V we turn to premonoidal structure. We introduce premonoidal bicategories and show that the Kleisli bicategory of a strong pseudomonad is premonoidal (Theorem 2). We also observe that *graded* monads ([70], [50], [36]) which track quantitative information about side effects, have a natural notion of Kleisli bicategory and that this is premonoidal.

We finish with a summary of the main results and related work, and suggestions for further investigation (Section VI).

II. Background: pseudofunctors, pseudomonads, and monoidal bicategories

Many concepts in category theory have corresponding versions for bicategories. We first summarise the basic definitions of pseudofunctors, pseudonatural transformations, and modifications (Section II-A), then discuss the bicategorical notions of monad (Section II-B) and monoidal structure (Section II-C) we need for this paper. For reasons of space we only give a brief outline and omit the coherence axioms. For a full overview of the basic bicategorical definitions, see [42]; for the definition of (symmetric) monoidal bicategories, including many beautiful diagrams, see [72]. Readable introductions to the wider subject of bicategories include [2], [35]; a more theoretical-computer science perspective is available in [61], [62].

A. Basic notions

Morphisms of bicategories are called pseudofunctors. Just as bicategories are categories 'up to isomorphism', so pseudofunctors are functors 'up to isomorphism'.

Definition 2. *A* pseudofunctor $F : \mathscr{B} \to \mathscr{C}$ *consists of:*

- *A mapping* $F : ob(\mathscr{B}) \to ob(\mathscr{C})$ *on objects;*
- *A functor* $F_{A,B} : \mathscr{B}(A, B) \to \mathscr{C}(FA, FB)$ *for every* $A, B \in \mathscr{B}$*;*
- *A* unitor $\psi_A : \mathsf{Id}_{FA} \xRightarrow{\cong} F(\mathsf{Id}_A)$ *for every* $A \in \mathscr{B}$*;*
- *A* compositor $\phi_{f,g} : F(f) \circ F(g) \xRightarrow{\cong} F(f \circ g)$ *for every composable pair of 1-cells* f *and* g*, natural in* f *and* g*.*

This data is subject to three axioms similar to those for strong monoidal functors (see e.g. *[42]).*

We generally abuse notation by referring to a pseudofunctor (F, ϕ, ψ) simply as F; where there is no risk of confusion, we shall employ similar abuses for structure throughout. A pseudofunctor is called *strict* if ϕ and ψ are both the identity.

Example 1. *Every endofunctor* F *on a monoidal category* $(\mathbb{C}, \otimes, I)$ *with a strength* $t_{A,B} : A \otimes F(B) \to F(A \otimes B)$ *(see* e.g. *[39]) determines a strict endo-pseudofunctor* $\widetilde{F}$ *on* $\mathbf{Para}(\mathbb{C})$*. The action on objects is the same, and on 1-cells* $\widetilde{F}(P \otimes A \xrightarrow{f} B)$ *is the object* P *together with the composite* $\big(P \otimes FA \xrightarrow{t} F(P \otimes A) \xrightarrow{Ff} FB\big)$*.*

Transformations between pseudofunctors are like natural transformations, except one must say in what sense naturality holds for each 1-cell.

Definition 3. *For pseudofunctors* $F, G : \mathscr{B} \to \mathscr{C}$*, a* pseudonatural transformation $\eta : F \Rightarrow G$ *consists of:*

- *A 1-cell* $\eta_A : FA \to GA$ *for every* $A \in \mathscr{B}$*;*
- *For every* $f : A \to B$ *in* $\mathscr{B}$ *an invertible 2-cell*

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\eta_A"'] \arrow[dr, phantom, "\overset{\overline{\eta}_f}{\Leftarrow}"] & FB \arrow[d, "\eta_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd} \tag{2}$$

natural in f *and satisfying identity and composition laws.*

Example 2. *Every natural transformation* $\sigma : F \Rightarrow F'$ *between strong endofunctors* (F, s) *and* (G, t) *which is compatible with the strengths ('strong natural transformation': see* e.g. *[49]) determines a pseudonatural transformation* $\widetilde{\sigma} : \widetilde{F} \Rightarrow \widetilde{G}$ *on* $\mathbf{Para}(\mathbb{C})$*. Each component* $(\widetilde{\sigma})_A$ *is just* $\widetilde{\sigma_A}$*, and for a 1-cell* $f : P \otimes A \to B$ *the 2-cell* $\overline{\widetilde{\sigma}}_f$ *witnessing naturality is the canonical isomorphism* $I \otimes P \xrightarrow{\cong} P \otimes I$ *in* $\mathbb{C}$*.*

Because bicategories have a second layer of structure, there is also a notion of map between pseudonatural transformations.

Definition 4. *A* modification $\boldsymbol{m} : \eta \to \theta$ *between pseudonatural transformations* $F \Rightarrow G : \mathscr{B} \to \mathscr{C}$ *consists of a 2-cell* $\boldsymbol{m}_B : \eta_B \Rightarrow \theta_B$ *for every* $B \in \mathscr{B}$*, subject to an axiom expressing compatibility between* $\boldsymbol{m}$ *and each* $\overline{\eta}_f$ *and* $\overline{\theta}_f$*.*

For any bicategories $\mathscr{B}$ and $\mathscr{C}$ there exists a bicategory $\mathrm{Hom}(\mathscr{B}, \mathscr{C})$ with objects pseudofunctors, 1-cells pseudonatural transformations, and 2-cells modifications. This is also a product bicategory $\mathscr{B} \times \mathscr{C}$ formed component-wise: objects are pairs (B, C), and 1-cells and 2-cells are obtained using the cartesian product of categories: $(\mathscr{B} \times \mathscr{C})((B, C), (B', C')) := \mathscr{B}(B, B') \times \mathscr{C}(C, C')$.

B. Pseudomonads and Kleisli bicategories

The bicategorical correlate of a monad is a *pseudomonad*.

Definition 5 ([46])**.** *A* pseudomonad *on a bicategory* $\mathscr{B}$ *consists of a pseudofunctor* $T : \mathscr{B} \to \mathscr{B}$ *equipped with:*

- Unit *and* multiplication *pseudonatural transformations* $\eta : \mathrm{id} \Rightarrow T$ *and* $\mu : T^2 \Longrightarrow T$*, where* $T^2 = T \circ T$*;*
- *Invertible modifications* $\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{p}$ *with components*

$$\begin{tikzcd} T^3A \arrow[r, "\mu_{TA}"] \arrow[d, "T\mu_A"'] \arrow[dr, phantom, "\overset{\boldsymbol{m}_A}{\Rrightarrow}"] & T^2A \arrow[d, "\mu_A"] \\ T^2A \arrow[r, "\mu_A"'] & TA \end{tikzcd} \qquad \begin{tikzcd} & TA \arrow[dl, "\eta_{TA}"'] \arrow[dr, "T\eta_A"] \arrow[d, equal] & \\ T^2A \arrow[r, "\mu_A"'] \arrow[ur, phantom, "\boldsymbol{n}_A \Rrightarrow"] & TA & T^2A \arrow[l, "\mu_A"] \arrow[ul, phantom, "\Lleftarrow \boldsymbol{p}_A"] \end{tikzcd}$$

replacing the usual monad laws, and satisfying two further coherence axioms (see e.g. *[40]).*

Pseudomonads have a rich theory (*e.g.* [46], [47], [10]). Here we explore their application to denotational semantics but there are many other applications, including universal algebra (in the study of variable binding [31], [16] and operads [25]) and formal category theory (*e.g.* [60]). The dual notion of pseudo*comonad* also arises naturally, for instance in the context of linear logic (*e.g.* [15], [58], [22]).

A simple example is given by the Writer pseudomonad on $\mathbf{Cat}$, the bicategory with objects small categories, 1-cells functors, and 2-cells natural transformations. The structural isomorphisms a, l and r are all the identity (giving a *2-category*).

Example 3. *Let* $(\mathbb{C}, \otimes, I)$ *be a monoidal category. The pseudofunctor* $(-) \times \mathbb{C} : \mathbf{Cat} \to \mathbf{Cat}$ *has a pseudomonad structure with 1-cell components*

$$\eta_{\mathbb{D}} = \mathbb{D} \xrightarrow{\cong} \mathbb{D} \times 1 \xrightarrow{\mathbb{D} \times I} \mathbb{D} \times \mathbb{C}$$
$$\mu_{\mathbb{D}} = (\mathbb{D} \times \mathbb{C}) \times \mathbb{C} \xrightarrow{\cong} \mathbb{D} \times (\mathbb{C} \times \mathbb{C}) \xrightarrow{\mathbb{D} \times \otimes} \mathbb{D} \times \mathbb{C}$$

and 2-cell components $\boldsymbol{m}, \boldsymbol{n}$ *and* $\boldsymbol{p}$ *given by the associator and unitors for the monoidal structure in* $\mathbb{C}$.

Example 4. *Every strong monad* (T, μ, η, t) *on a monoidal category* $(\mathbb{C}, \otimes, I)$ *determines a pseudomonad on* $\mathbf{Para}(\mathbb{C})$*: the underlying pseudofunctor is* $\widetilde{T}$ *and the pseudonatural transformations are* $\widetilde{\mu}$ *and* $\widetilde{\eta}$*. (This remains true if the monoidal structure is replaced by an* action *[59].)*

Just as monads have Kleisli categories, pseudomonads have Kleisli bicategories [8]. If T is a pseudomonad on $\mathscr{B}$, the *Kleisli bicategory* $\mathscr{B}_T$ has the same objects as $\mathscr{B}$ and hom-categories $\mathscr{B}_T(A, B) := \mathscr{B}(A, TB)$. The identity on A is the 1-cell $\eta_A \in \mathscr{B}(A, TA)$ and the composite of $f \in \mathscr{B}(A, TB)$ and $g \in \mathscr{B}(B, TC)$ is defined to be

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC. \tag{3}$$

The structural 2-cells $\mathsf{a}, \mathsf{l}, \mathsf{r}$ in $\mathscr{B}_T$ are constructed using the pseudomonad structure.

In (3) we have drawn the composition of morphisms in the bicategory $\mathscr{B}$ as we would in a category, ignoring issues of weak associativity and identity. This relies on *coherence*: see Section II-D for a discussion.

C. Monoidal bicategories

A monoidal bicategory is a bicategory equipped with a unit object and a tensor product which is only weakly associative and unital. To motivate the various components of the construction, we explain how a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ induces a monoidal structure on $\mathbf{Para}(\mathbb{C})$, with the same action on objects.

The idea is that we can combine the parameters using $\otimes$. For 1-cells $f : P \otimes A \to B$ and $g : P' \otimes A' \to B'$, we set $f \mathbin{\widetilde{\otimes}} g$ to be the object $P \otimes P'$ equipped with

$$(P \otimes P') \otimes (A \otimes A') \xrightarrow{\cong} (P \otimes A) \otimes (P' \otimes A') \xrightarrow{f \otimes g} B \otimes B'$$

where the first map is defined using the symmetry of $\otimes$. On 2-cells, we use the tensor product of maps in $\mathbb{C}$. This construction does not strictly preserve identities and composition, but it does preserve them up to isomorphism. Thus, we get a pseudofunctor $\widetilde{\otimes} : \mathbf{Para}(\mathbb{C}) \times \mathbf{Para}(\mathbb{C}) \longrightarrow \mathbf{Para}(\mathbb{C})$.

We examine the sense in which this tensor is associative and unital, by lifting the structural isomorphisms from $\mathbb{C}$. Note that every map $f : A \to B$ in $\mathbb{C}$ determines a 1-cell $\widetilde{f}$ in $\mathbf{Para}(\mathbb{C})$ given by the object I and the composite $(I \otimes A \xrightarrow{\cong} A \xrightarrow{f} B)$, where $\cong$ is the unit isomorphism. If f has an inverse f^{-1}, the composite $\widetilde{f} \circ \widetilde{f^{-1}}$ has parameter $I \otimes I$ and thus cannot be the identity. But it is isomorphic to the identity: the pair $(\widetilde{f}, \widetilde{f^{-1}})$ is known as an *equivalence* (an 'isomorphism up to isomorphism'). Thus, although the tensor $\otimes$ on $\mathbb{C}$ is associative and unital up to isomorphism, the tensor $\widetilde{\otimes}$ on $\mathbf{Para}(\mathbb{C})$ is only associative and unital up to equivalence. The structural 1-cells are all pseudonatural in a canonical way (Example 2).

Following the general pattern of bicategorification, the triangle and pentagon axioms of a monoidal category now only hold up to isomorphism: one route round the pentagon has three sides and the other has two, so one composite has parameter $I^{\otimes 3}$ and the other has parameter $I^{\otimes 2}$. These are canonically isomorphic, so we get families of invertible 2-cells witnessing the categorical axioms. All the structure we have defined so far has used the canonical isomorphisms of $\mathbb{C}$, so these families are actually modifications on $\mathbf{Para}(\mathbb{C})$. Moreover, by the axioms of a monoidal category, these structural modifications satisfy axioms of their own.

In summary, a monoidal bicategory is a bicategory equipped with an object I, a pseudofunctor $\widetilde{\otimes}$, pseudonatural families of equivalences witnessing the weak associativity and unitality of $\widetilde{\otimes}$, and invertible modifications witnessing the axioms of a monoidal category.

We now make this precise, starting with the definition of equivalences. These generalize equivalences of categories.

Definition 6. *An* equivalence *between objects* A *and* B *in a bicategory* $\mathscr{B}$ *is a pair of 1-cells* $f : A \to B$ *and* $f^{\bullet} : B \to A$ *together with invertible 1-cells* $f \circ f^{\bullet} \Rightarrow \mathsf{Id}_B$ *and* $\mathsf{Id}_A \Rightarrow f^{\bullet} \circ f$.

A *pseudonatural equivalence* is a pseudonatural transformation in which each component has the structure of an equivalence; this induces an equivalence in the hom-bicategory.

The definition is now as advertised. To state it, we introduce some notation for the 2-cell diagrams—known as *pasting diagrams*—that we will use in the rest of the paper.

Notation 1. *To save space and improve readability,*

- *We use juxtaposition for the tensor product, e.g.* $(AB)C$ *means* $(A \otimes B) \otimes C$*;*
- *We omit the subscripts on the components of pseudonatural transformations and modifications,* e.g. $\boldsymbol{m}$ *instead of* $\boldsymbol{m}_A$*;*
- *We use a subscript notation for the action of a pseudofunctor* T*,* e.g. T_{AT_B} *means* $T(A \otimes T(B))$.
- *We write* $\cong$ *for any pseudonaturality 2-cell as in* (2)*, and in equations we omit the arrows showing the directions of 2-cells. These labels can be inferred from the type.*

$$
\begin{tikzcd}
((AB)C)D \arrow[r, "\alpha"] \arrow[d, "\alpha D"'] & (AB)(CD) \arrow[r, "\alpha"] \arrow[d, phantom, "\mathfrak{p}\Uparrow"] & A(B(CD)) \\
(A(BC))D \arrow[rr, "\alpha"'] & & A((BC)D) \arrow[u, "A\alpha"']
\end{tikzcd}
\quad
\begin{tikzcd}
& AB \arrow[d, phantom, "\mathfrak{m}\Uparrow"] & \\
(AI)B \arrow[ur, "\rho B"] \arrow[rr, "\alpha"'] & & A(IB) \arrow[ul, "A\lambda"']
\end{tikzcd}
\quad
\begin{tikzcd}
(IA)B \arrow[r, "\alpha"] \arrow[d, "\lambda B"'] \arrow[dr, phantom, "\mathfrak{l}"] & I(AB) \arrow[dl, bend left, "\lambda"] \\
AB &
\end{tikzcd}
\quad
\begin{tikzcd}
(AB)I \arrow[r, "\alpha"] \arrow[dr, bend right, "\rho"'] \arrow[dr, phantom, "\mathfrak{r}"] & A(BI) \arrow[d, "A\rho"] \\
& AB
\end{tikzcd}
$$

Figure 2: The structural modifications of a monoidal bicategory

Definition 7 (*e.g.* [72])**.** *A* monoidal bicategory *is a bicategory $\mathscr{B}$ equipped with a pseudofunctor $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ and an object $I \in \mathscr{B}$, together with the following data:*

- *Pseudonatural equivalences α, λ and ρ with components $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ (the* associator*), $\lambda_A : I \otimes A \to A$, and $\rho_A : A \otimes I \to A$ (the* unitors*);*
- *Invertible modifications $\mathfrak{p}, \mathfrak{l}, \mathfrak{m}$ and $\mathfrak{r}$ with components shown in Figure 2, subject to coherence axioms.*

A symmetric *monoidal bicategory is a monoidal bicategory equipped with a pseudonatural equivalence β with components $\beta_{A,B} : A \otimes B \to B \otimes A$, called the* braiding*, and invertible modifications governing the possible shufflings of three objects and expressing the symmetry of the braiding, subject to coherence axioms.*

For example (see *e.g.* [72] for full details), the cartesian product on the category $\mathbf{Set}$ induces a monoidal structure on the bicategory $\mathbf{Span}(\mathbf{Set})$ introduced in Section I-B. The pseudofunctor $\otimes$ is defined on objects as $A \otimes A' = A \times A'$, and for spans $A \leftarrow S \to B$ and $A' \leftarrow S' \to B'$ we take the component-wise product to obtain $A \times A' \leftarrow S \times S' \to B \times B'$.

We also record the outcome of our discussion above; this establishes a conjecture made in [6].

Example 5. *If $(\mathbb{C}, \otimes, I)$ is a symmetric monoidal category, this lifts to a symmetric monoidal structure on $\mathbf{Para}(\mathbb{C})$.*

General point: The coherence axioms of a monoidal bicategory can be difficult to verify directly. However, in many cases of interest the monoidal structure is induced from a more fundamental construction, as in $\mathbf{Span}(\mathbf{Set})$ above. This gives a systematic method for constructing (symmetric) monoidal bicategories, described abstractly in [75].

D. Coherence theorems

As we have seen, bicategorical structures involve considerable data and many equations. Much of the difficulty, however, is tamed by various *coherence theorems*. These generally show that any two parallel 2-cells built out of the structural data are equal. Appropriate coherence theorems apply to bicategories [45] pseudofunctors [28], (symmetric) monoidal bicategories ([27], [29]) and pseudomonads [40].

We rely heavily on the coherence of bicategories and pseudofunctors when writing pasting diagrams of 2-cells: in particular we omit all compositors and unitors for pseudofunctors, and ignore the weakness of 1-cell composition. Thus, strictly speaking our diagrams do not type-check, but coherence guarantees the resulting 2-cell is the same no matter how one fills in the structural details. This is standard practice; for a detailed justification see *e.g.* [69, §2.2].

III. STRONG PSEUDOMONADS

We follow the categorical setting by first saying what it means for a pseudo*functor* to be strong, then giving the additional data and axioms to make a pseudo*monad* strong. Section III-C contains many examples.

A. Strong pseudofunctors

For the moment we only consider strengths on the left. In all diagrams below we follow our Notation 1.

Definition 8. *Let $(\mathscr{B}, \otimes, I)$ be a monoidal bicategory. A* left strength *for a pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ is a pseudonatural transformation $t_{A,B} : A \otimes TB \to T(A \otimes B)$, equipped with invertible modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ expressing the compatibility of t with the left unitor and the associator:*

$$
\begin{tikzcd}
T_{IA} \arrow[dr, bend right, "T_\lambda"'] \arrow[dr, phantom, "\boldsymbol{x}"] & IT_A \arrow[l, "t"'] \arrow[d, "\lambda"] \\
& T_A
\end{tikzcd}
\quad
\begin{tikzcd}
(AB)T_C \arrow[rr, "t"] \arrow[d, "\alpha"'] \arrow[drr, phantom, "\boldsymbol{y}"] & & T_{(AB)C} \arrow[d, "T_\alpha"] \\
A(BT_C) \arrow[r, "At"'] & AT_{BC} \arrow[r, "t"'] & T_{A(BC)}
\end{tikzcd}
$$

These modifications must themselves be compatible with the monoidal structure, as per the three axioms of Figure 3.

A left strength for a pseudofunctor T can be used to define a parametrised version of the functorial action: for any map $\Gamma \otimes X \to Y$ we can now define a map $\Gamma \otimes TX \to TY$. This suggests the following (recall Example 1 and Example 2).

Example 6. *If (F, t) is a strong functor on a symmetric monoidal category $(\mathbb{C}, \otimes, I)$, then the induced pseudofunctor $\widetilde{F}$ on $\mathbf{Para}(\mathbb{C})$ is also strong.*

B. Strong pseudomonads

If a strong pseudofunctor $T : \mathscr{B} \to \mathscr{B}$ is also a pseudomonad, then we must ask for additional data to relate the strength and the monad structure, and this data must be compatible with the modifications $\boldsymbol{x}, \boldsymbol{y}$ we already have.

Definition 9. *Let $(\mathscr{B}, \otimes, I)$ be a monoidal bicategory. A* left strength *for a pseudomonad (T, η, μ) consists of a left strength $(t, \boldsymbol{x}, \boldsymbol{y})$ for the underlying pseudofunctor, together with invertible modifications*

$$
\begin{tikzcd}
AT_B^2 \arrow[rr, "A\mu"] \arrow[d, "t"'] \arrow[drr, phantom, "\boldsymbol{w}\Uparrow"] & & AT_B \arrow[d, "t"] \\
T_{AT_B} \arrow[r, "T_t"'] & T^2_{AB} \arrow[r, "\mu"'] & T_{AB}
\end{tikzcd}
\quad
\begin{tikzcd}
AB \arrow[r, "A\eta"] \arrow[dr, bend right, "\eta"'] \arrow[dr, phantom, "\boldsymbol{z}"] & AT_B \arrow[d, "t"] \\
& T_{AB}
\end{tikzcd}
$$

expressing the compatibility of t with the pseudomonad structure. This is subject to the coherence axioms of Figure 4, namely: three axioms expressing compatibility with the monad structure, two axioms laws relating $\boldsymbol{x}$ respectively with $\boldsymbol{z}$ and $\boldsymbol{w}$, and two axioms relating $\boldsymbol{y}$ respectively with $\boldsymbol{z}$ and $\boldsymbol{w}$.

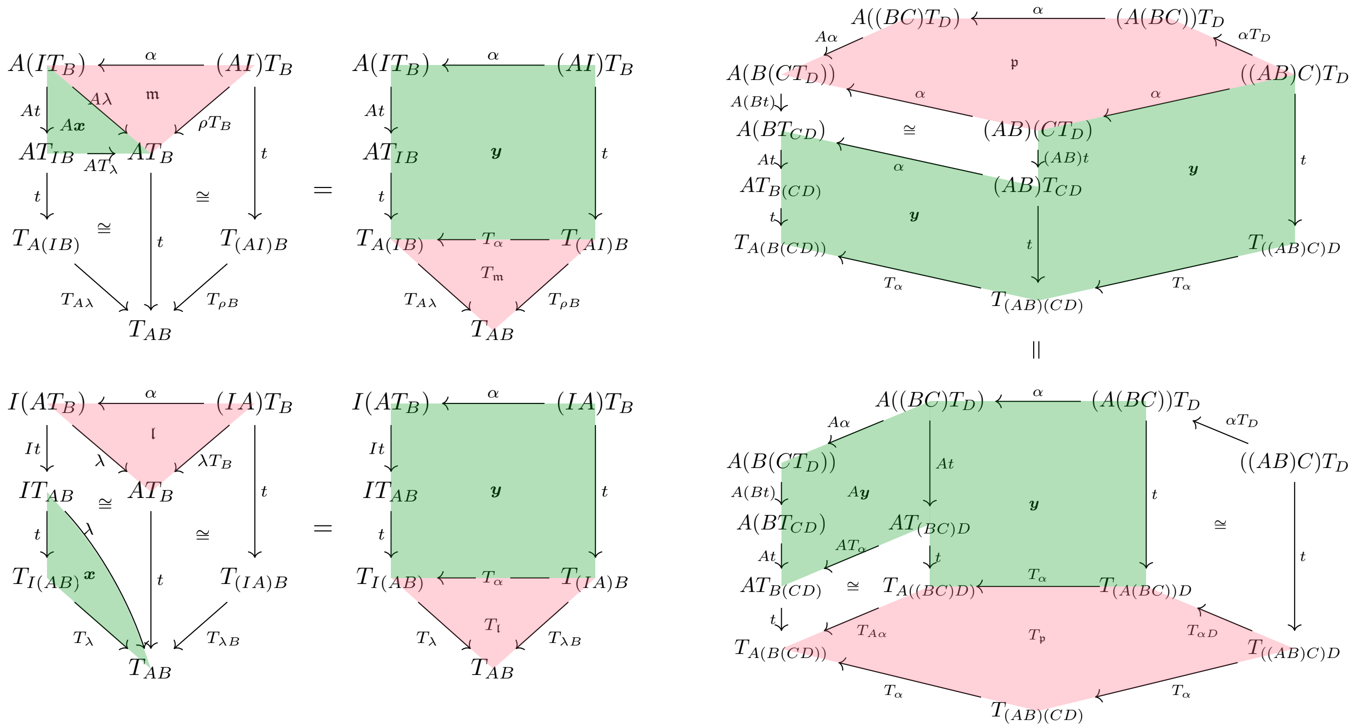


Figure 3: Coherence axioms for a strong pseudofunctor.

Extending Example 4 and Example 6, we obtain:

Example 7. *A strong monad on a symmetric monoidal category* $(\mathbb{C}, \otimes, I)$ *determines a strong pseudomonad on* $\mathbf{Para}(\mathbb{C})$.

C. Examples of strong pseudomonads

In this section we show that several important classes of pseudomonad are strong in the way one would expect from the categorical setting; these help confirm that our definitions are correct. In each case the proof makes heavy use of the relevant coherence theorem for checking the axioms.

First, recall that if (M, m, e) is a monoid in a monoidal category $(\mathbb{C}, \otimes, I)$ then the functor $(-)\otimes M$ becomes a monad with unit and multiplication given via e and m (*c.f.* Example 3). This monad is canonically strong, with strength given by the canonical isomorphism $(A \otimes B) \otimes M \xrightarrow{\cong} A \otimes (B \otimes M)$. Also note that every monad T is strong with respect to the cocartesian structure $(0, +)$, with strength $[T\mathrm{inl} \circ \eta_A, T\mathrm{inr}] : A + TB \to T(A + B)$.

These facts bicategorify. The bicategorical version of a monoid is called a *pseudomonoid* [12], and every pseudomonoid defines a pseudomonad similarly to Example 3.

Lemma 1.
1) *For any pseudomonoid* (M, m, e, a, l, r) *on a monoidal bicategory* $(\mathscr{B}, \otimes, I)$ *the pseudomonad* $(-)\otimes M$ *has a strength given by the associator* α *of* $\otimes$.
2) *Every pseudomonad is canonically strong with respect to the cocartesian monoidal structure* $(+, 0)$.

In particular, a pseudomonoid in $(\mathbf{Cat}, \times, 1)$ is a monoidal category so (1) applies to the Writer pseudomonad.

From this, we can derive a result about pseudomonads on spans. A monad (T, μ, η) is *cartesian* if T preserves pullbacks and every naturality square for μ and η is a pullback square.

Corollary 1. *Any cartesian monad on a lextensive category* $\mathbb{C}$ *(such as* $\mathbf{Set}$*) lifts to a strong pseudomonad on* $\mathbf{Span}(\mathbb{C})$

The next example covers two cases of importance in the semantics of programming languages. The proof follows essentially immediately from the corresponding categorical facts and the particularly strong form of coherence enjoyed by cartesian closed bicategories (see [18, Principle 1.3]).

Lemma 2. *For any cartesian closed bicategory (see* e.g. *[17])* $(\mathscr{B}, \times, 1, \Rightarrow)$ *and objects* $S, R \in \mathscr{B}$*, there exist strong pseudomonads* $S \Rightarrow (S \times -)$ *(the* state *pseudomonad) and* $(- \Rightarrow R) \Rightarrow R$ *(the* continuation *pseudomonad).*

For our final class of examples, recall that every functor F on $\mathbf{Set}$ is canonically strong with respect to the cartesian structure, with $t_{A,B} : A \times FB \to F(A \times B)$ defined by $t_{A,B}(a, w) := F(\lambda b . \langle a, b\rangle)(w)$, and moreover that the same construction makes every monad on $\mathbf{Set}$ strong [56, Proposition 3.4]. A similar fact holds for bicategories.

Proposition 1. *Every pseudofunctor (resp. pseudomonad) on* $(\mathbf{Cat}, \times, 1)$ *has a canonical choice of strength.*

In summary, there are many natural examples of strong pseudomonad. Beyond the general examples we have presented, there are concrete examples arising in denotational semantics, *e.g.* linear continuation pseudomonads in dialogue categories [53], to be studied independently.

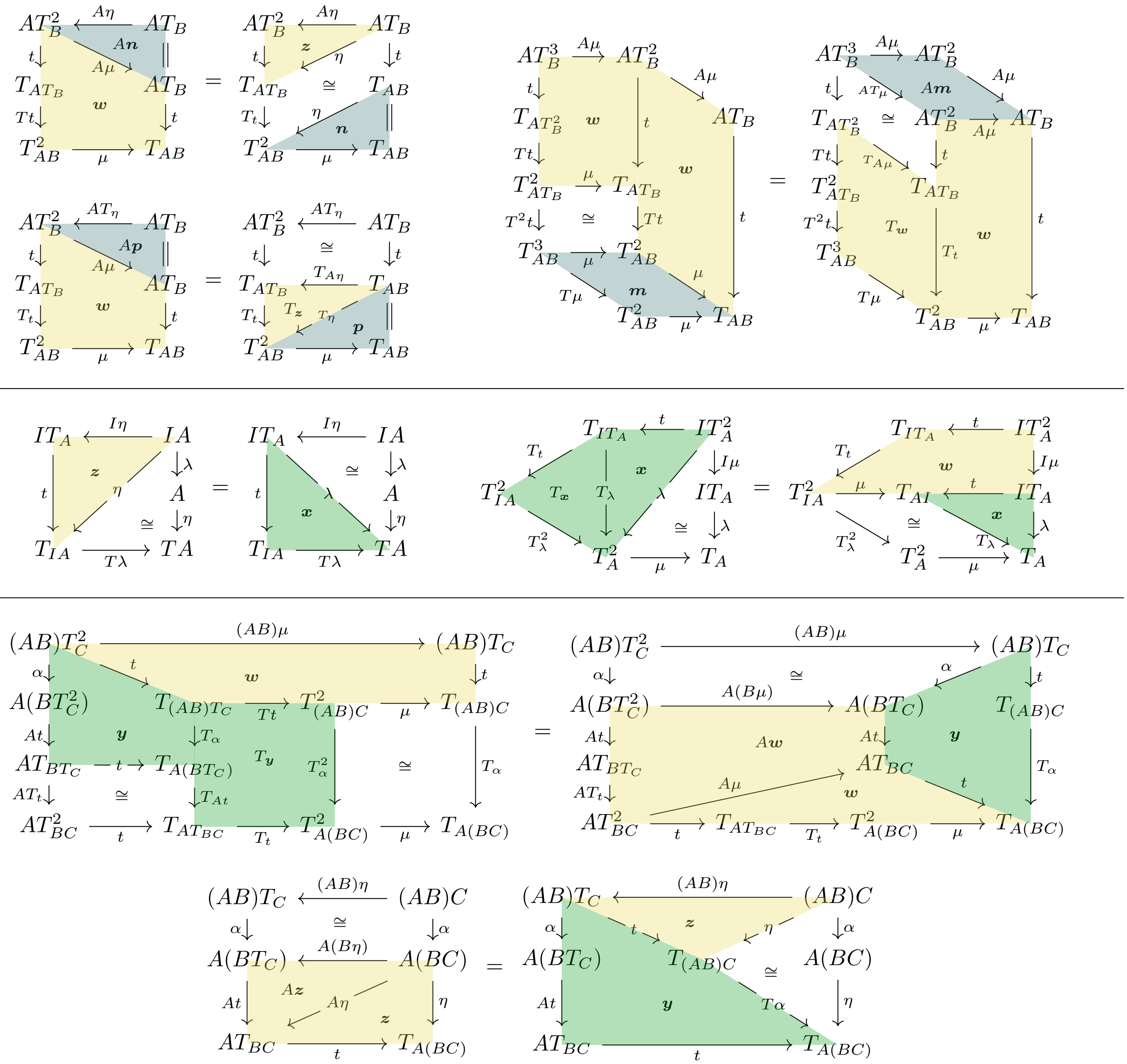


Figure 4: Coherence axioms for a strong pseudomonad.

D. Right-strong pseudomonads

In categorical semantics it is often the case that a monad T supports a strength on both sides. This happens automatically if T has left strength t and the underlying category is symmetric monoidal, because we can define a right strength s using the left strength and the symmetry β as

$$T(A) \otimes B \xrightarrow{\beta} B \otimes TA \xrightarrow{t} T(B \otimes A) \xrightarrow{T\beta} T(A \otimes B). \quad (4)$$

There is a natural notion of right strength for pseudomonads, namely a pseudonatural transformation $T(A) \otimes B \to T(A \otimes B)$ equipped with four modifications analogous to $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ and $\boldsymbol{w}$.

Proposition 2. *Every left-strong pseudomonad on a symmetric monoidal bicategory is right-strong in a canonical way.*

Note on "bistrong" pseudomonads: A monad with a left strength and a right strength is sometimes called *bistrong* when the two strengths satisfy a compatibility axiom (*e.g.* [49]). This condition is automatic when the right strength is induced from the left strength in a symmetric setting. Nevertheless, bistrong pseudomonads are of practical interest: it is easier to work with an abstract right strength rather than the composite (4), since the symmetry increases the complexity of the pasting diagrams. Examples of monoidal bicategories in categorical semantics are generally symmetric, so we do not give the definition of bistrong pseudomonads here; we shall present this elsewhere.

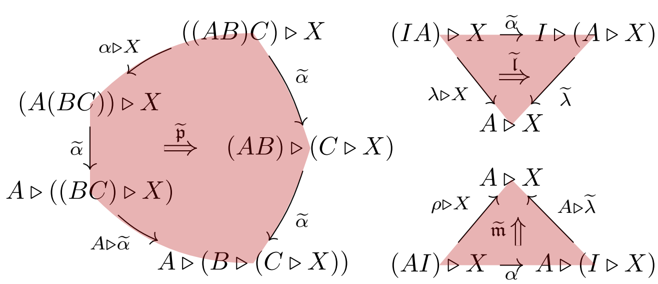
IV. Actions of monoidal bicategories

In this section we justify our definition of strength (Definition 9) by showing a bicategorical version of a well-known correspondence theorem (*e.g.* [49, Proposition 4.3]). We show that to give a left strength for a pseudomonad T on $(\mathscr{B}, \otimes, I)$ is to give a left action of $(\mathscr{B}, \otimes, I)$ on the Kleisli bicategory $\mathscr{B}_T$ that extends the monoidal structure.

We start by defining actions of monoidal bicategories. We are not aware of a definition in the literature, but it is clear what it ought to be. As observed in [33], a left action on a category is equivalently a bicategory with two objects and certain hom-categories taken to be trivial. We therefore define a left action on a bicategory so it is equivalently a *tricategory* (see [27]) with two objects and certain hom-bicategories taken to be trivial. It follows from the coherence of tricategories ([27], [28]) that every diagram of 2-cells constructed using the structural data of an action must commute. More explicitly:

Definition 10. *A* left action *of a monoidal bicategory* $(\mathscr{B}, \otimes, I)$ *on a bicategory* $\mathscr{C}$ *consists of a pseudo-functor* $\triangleright : \mathscr{B} \times \mathscr{C} \to \mathscr{C}$*, together with the following data:*

- *Pseudonatural equivalences* $\widetilde{\lambda}$ *and* $\widetilde{\alpha}$ *with components* $\widetilde{\lambda}_X : I \triangleright X \to X$ *and* $\widetilde{\alpha}_{A,B,X} : (A \otimes B) \triangleright X \to A \triangleright (B \triangleright X)$*;*
- *Invertible modifications as shown below, satisfying the same coherence axioms as* $\mathfrak{p}, \mathfrak{m}$*, and* $\mathfrak{l}$ *in a monoidal bicategory* (e.g. *[72]):*



The tensor product $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ on a monoidal bicategory determines an action of $\mathscr{B}$ on itself, with all the structure given by the monoidal data. We call this the *canonical action* of a monoidal bicategory on itself. Every strong pseudomonad also induces an action.

Proposition 3. *Every strong pseudomonad* (T, t) *on* $(\mathscr{B}, \otimes, I)$ *induces an action of* $\mathscr{B}$ *on the Kleisli bicategory* $\mathscr{B}_T$*, where the pseudofunctor* $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ *is given on objects by* $A \triangleright B = A \otimes B$*, and on morphisms as*

$$f \triangleright g \quad := \quad \left(A \otimes B \xrightarrow{f \otimes g} A' \otimes TB' \xrightarrow{t} T(A' \otimes B')\right)$$

for $f : A \to A'$ *and* $g : B \to TB'$.

The action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ induced by a strong pseudomonad can be seen as an extension of the canonical action $\otimes : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$ given by the monoidal structure. Indeed, we have a pseudonatural transformation

$$\begin{tikzcd} \mathscr{B} \times \mathscr{B}_T \arrow[r, "\triangleright"] & \mathscr{B}_T \\ \mathscr{B} \times \mathscr{B} \arrow[u, "\mathscr{B} \times K"] \arrow[r, "\otimes"'] \arrow[ur, phantom, "\overset{\theta}{\Rightarrow}"] & \mathscr{B} \arrow[u, "K"'] \end{tikzcd} \tag{5}$$

where $K : \mathscr{B} \to \mathscr{B}_T$ is the identity-on-objects pseudofunctor which sends $f : A \to A'$ to $\eta_{A'} \circ f : A \to TA'$. Moreover, the two actions $\triangleright$ and $\otimes$ agree on objects, and the 1-cell components $\theta_{A,B}$ of the transformation are all the identity. Such a transformation is known as an *icon* [41]. The 2-cell components of θ are nontrivial: for each $f : A \to A'$ and $g : B \to B'$ we have an isomorphism

$$\theta_{f,g} : f \triangleright K(g) \overset{\cong}{\Longrightarrow} K(f \otimes g)$$

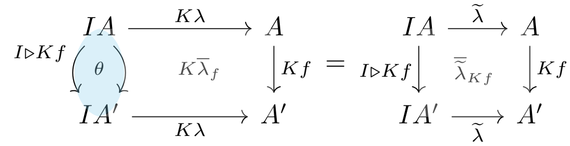
derived from the modification z, satisfying the coherence laws.

A. A correspondence theorem

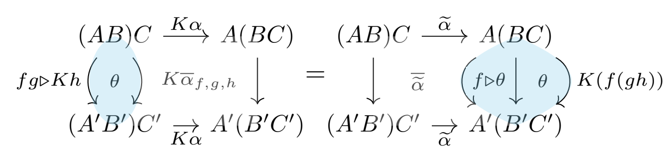
We now prove an equivalence between left strengths and left actions, based on the following definition.

Definition 11. *For a pseudomonad* T *on a monoidal bicategory* $(\mathscr{B}, \otimes, I)$*, an* extension of the canonical action of $\mathscr{B}$ on itself *is a pair* $(\triangleright, \theta)$ *consisting of a left action* $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ *and an icon* θ *as in* (5) *such that the structural data* $\widetilde{\lambda}, \widetilde{\alpha}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{m}}, \widetilde{\mathfrak{p}}$ *for the action extends the corresponding monoidal data via* K *and* θ*. Explicitly, for all objects* A, B, C, D *and morphisms* $f : A \to A'$, $g : B \to B'$*, and* $h : C \to C'$ *in* $\mathscr{B}$*:*

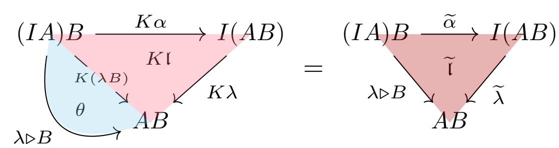
1) $\widetilde{\lambda}_A = K\lambda_A$*, and*



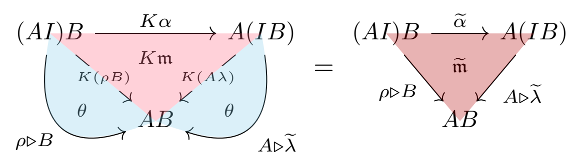
2) $\widetilde{\alpha}_{A,B,C} = K\alpha_{A,B,C}$*, and*



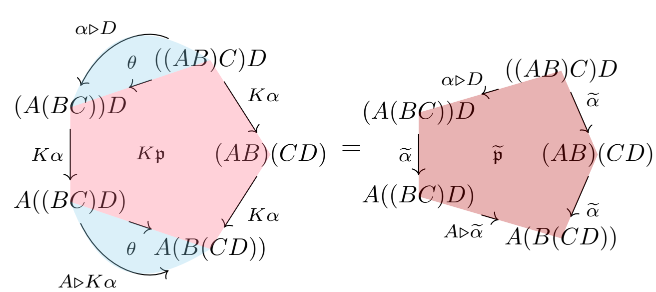
3)



4)



5)



Our correspondence theorem uses the following two categories for a pseudomonad T on $(\mathscr{B}, \otimes, I)$:

- $\mathbf{LeftStr}(T)$, the category whose objects are left strengths for T, and whose morphisms from t to t' are modifications which commute with all the strength data;
- $\mathbf{LeftExt}(T)$, the category whose objects are extensions of the canonical action of $\mathscr{B}$ on itself, and whose morphisms from $(\triangleright, \theta)$ to $(\triangleright', \theta')$ are icons $\triangleright \Rightarrow \triangleright'$ which commute with θ and θ'.

Theorem 1. *For any pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the categories $\mathbf{LeftStr}(T)$ and $\mathbf{LeftExt}(T)$ are equivalent.*

Thus, actions and strengths are related in the bicategorical setting just as they are in the categorical one. This provides further justification for the coherence axioms of Definition 9.

V. Premonoidal bicategories

Finally, we introduce premonoidal bicategories; we then validate this definition by showing the Kleisli bicategory of a bistrong pseudomonad is premonoidal (Section V-E).

A. Premonoidal categories

Premonoidal categories axiomatize the semantics of effectful languages more directly than strong monads, in the sense that a program $(\Gamma \vdash P : A)$ is interpreted as a morphism $\Gamma \to A$. For such a model to be sound, one must deal with the fact that statements in effectful programs cannot generally be re-ordered (recall Figure 1). So if we have another program $(\Delta \vdash Q : B)$, there are two ways of interpreting the pair $\big(\Gamma \otimes \Delta \vdash (P, Q) : A \otimes B\big)$, depending on whether we run P or Q first. As a consequence, $\otimes$ is not a functor of two arguments. In this section we briefly recall the formal definition and the premonoidal structure of a key example, namely the Kleisli category of a bistrong monad.

First, some preliminaries. A *binoidal category* is a category $\mathbb{D}$ equipped with functors

$$A \rtimes (-) : \mathbb{D} \longrightarrow \mathbb{D} \qquad\qquad (-) \ltimes B : \mathbb{D} \longrightarrow \mathbb{D}$$

for every $A, B \in \mathbb{D}$, such that $(A \rtimes -)(B) = (- \ltimes B)(A)$. We write $A \otimes B$, or just AB, for their joint value. (The notation is intended to suggest that $\rtimes$ is a 'complete' functor only on the right.) A map $f : A \to A'$ is *central* if the two diagrams

$$\begin{tikzcd} AB \arrow[r, "A \rtimes g"] \arrow[d, "f \ltimes B"'] & AB' \arrow[d, "f \ltimes B'"] \\ A'B \arrow[r, "A' \rtimes g"'] & A'B' \end{tikzcd} \qquad \begin{tikzcd} BA \arrow[r, "g \ltimes A"] \arrow[d, "B \rtimes f"'] & B'A \arrow[d, "B' \rtimes f"] \\ BA' \arrow[r, "g \ltimes A'"'] & B'A' \end{tikzcd} \tag{6}$$

commute for every $g : B \to B'$. Semantically, f corresponds to a program which may be run at any point without changing the observable result. We can now give the definition.

Definition 12 ([64])**.** *A* premonoidal category *is a binoidal category $(\mathbb{D}, \rtimes, \ltimes)$ equipped with a unit object I and central isomorphisms $\rho_A : AI \to A$, $\lambda_A : IA \to A$ and $\alpha_{A,B,C} : (AB)C \to A(BC)$ for every $A, B, C \in \mathbb{D}$, satisfying the triangle and pentagon axioms for a monoidal category and the following naturality conditions:*

$$\begin{tikzcd} IA \arrow[r, "I \rtimes f"] \arrow[d, "\lambda"'] & IA' \arrow[d, "\lambda"] \\ A \arrow[r, "f"'] & A' \end{tikzcd} \quad \begin{tikzcd} AI \arrow[r, "f \ltimes I"] \arrow[d, "\rho"'] & A'I \arrow[d, "\rho"] \\ A \arrow[r, "f"'] & A' \end{tikzcd} \quad \begin{tikzcd} (AB)C \arrow[r, "(f \ltimes B) \ltimes C"] \arrow[d, "\alpha"'] & (A'B)C \arrow[d, "\alpha"] \\ A(BC) \arrow[r, "f \ltimes (BC)"'] & A'(BC) \end{tikzcd}$$

$$\begin{tikzcd} (AB)C \arrow[r, "(A \rtimes g) \ltimes C"] \arrow[d, "\alpha"'] & (AB')C \arrow[d, "\alpha"] \\ A(BC) \arrow[r, "A \rtimes (g \ltimes C)"'] & A(B'C) \end{tikzcd} \quad \begin{tikzcd} (AB)C \arrow[r, "(AB) \rtimes h"] \arrow[d, "\alpha"'] & (AB)C' \arrow[d, "\alpha"] \\ A(BC') \arrow[r, "A \rtimes (B \rtimes h)"'] & A(BC') \end{tikzcd}$$

Notice that α cannot be a natural transformation in all arguments simultaneously, because $\otimes$ is not a functor on $\mathbb{C}$.

One motivating example for premonoidal categories is the Kleisli category of a bistrong monad (T, t, s) on a monoidal category $(\mathbb{C}, \otimes, I)$, where for every A, B, f and g we have

$$\begin{aligned} A \rtimes g \;&:=\; \big(AB \xrightarrow{A \otimes g} AT_{B'} \xrightarrow{t} T_{AB'}\big) \\ f \ltimes B \;&:=\; \big(AB \xrightarrow{f \otimes B} T_{A'}B \xrightarrow{s} T_{A'B}\big). \end{aligned} \tag{7}$$

The structural data is lifted from $\mathbb{C}$. Premonoidal categories are therefore a clean axiomatization of the monadic semantics, which also captures first-order effectful languages whose syntax does not explicitly involve a monad (*e.g.* [55], [65]).

B. Binoidal bicategories and centrality

We now turn to the bicategorical setting. Following the general bicategorical pattern, centrality for 1-cells will be extra data, rather than a property. This raises new subtleties, notably because it affects the type of the structural modifications.

Definition 13. *A* binoidal bicategory *is a bicategory $\mathscr{B}$ equipped with pseudofunctors $A \rtimes (-)$ and $(-) \ltimes B$ for every $A, B \in \mathscr{B}$, such that $A \rtimes B = A \ltimes B$.*

We denote the structure by $(\mathscr{B}, \rtimes, \ltimes)$, and as before we write $A \otimes B$, or just AB, for the joint value on objects.

To define centrality, observe that saying the diagrams in (6) commute for every g is exactly saying that the families of maps $f \ltimes B$ and $B \rtimes f$ are natural in B.

Definition 14 (Central 1-cell)**.** *A* central 1-cell *in a binoidal bicategory $(\mathscr{B}, \rtimes, \ltimes)$ is a 1-cell $f : A \to A'$ with 2-cells*

$$\begin{tikzcd} AB \arrow[r, "A \rtimes g"] \arrow[d, "f \ltimes B"'] \arrow[dr, phantom, "\overset{\mathsf{lc}^f_g}{\Longleftarrow}"] & AB' \arrow[d, "f \ltimes B'"] \\ A'B \arrow[r, "A' \rtimes g"'] & A'B' \end{tikzcd} \qquad \begin{tikzcd} BA \arrow[r, "g \ltimes A"] \arrow[d, "B \rtimes f"'] \arrow[dr, phantom, "\overset{\mathsf{rc}^f_g}{\Longleftarrow}"] & B'A \arrow[d, "B' \rtimes f"] \\ BA' \arrow[r, "g \ltimes A'"'] & B'A' \end{tikzcd} \tag{8}$$

for every $g : B \to B'$, forming pseudonatural transformations $(A \rtimes -) \xRightarrow{f \ltimes -} (A' \rtimes -)$ and $(- \ltimes A) \xRightarrow{- \ltimes f} (- \ltimes A')$.

Definition 15 (Central 2-cell)**.** *A* central 2-cell *σ between central 1-cells $(f, \mathsf{lc}^f, \mathsf{rc}^f)$ and $(f', \mathsf{lc}^{f'}, \mathsf{rc}^{f'})$ is a 2-cell $\sigma : f \Rightarrow f'$ such that the 2-cells $\sigma \ltimes B$ and $B \rtimes \sigma$ (for $B \in \mathscr{B}$) define modifications $\mathsf{lc}^f \Rightarrow \mathsf{lc}^{f'}$ and $\mathsf{rc}^f \Rightarrow \mathsf{rc}^{f'}$, respectively.*

Every monoidal bicategory $(\mathscr{B}, \otimes, I)$ has a canonical binoidal structure, with $A \rtimes (-) := A \otimes (-)$ and $(-) \ltimes B := (-) \otimes B$. Every 1-cell in $\mathscr{B}$ canonically determines a central 1-cell because for any f and g we can set lc_g^f and $(\mathsf{rc}_f^g)^{-1}$ both to be $\mathsf{int}_{f,g}$, defined using the compositor for $\otimes$ as follows:

$$\mathsf{int}_{f,g} \quad := \quad \begin{tikzcd} AB \arrow[r, "A\otimes g"] \arrow[d, "f\otimes B"'] \arrow[dr, "f\otimes g" description] \arrow[dr, phantom, "\phi", shift left=3] & AB' \arrow[d, "f\otimes B'"] \\ A'B \arrow[r, "A'\otimes g"'] \arrow[ur, phantom, "\phi", shift right=3] & A'B' \end{tikzcd} \tag{9}$$

By the functoriality of $\otimes$, every 2-cell becomes central.

For any bistrong pseudomonad (T, s, t) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the Kleisli bicategory $\mathscr{B}_T$ is binoidal: the pseudofunctors $\rtimes$ and $\ltimes$ are given as in (7), and every pure 1-cell $\widetilde{f} := \eta \circ f : A \to TA'$ becomes central with $\mathsf{lc}^{\widetilde{f}}$ and $\mathsf{rc}^{\widetilde{f}}$ constructed using structural data.

C. Premonoidal bicategories

As indicated above, the definition of premonoidal bicategory involves some subtleties not present in the categorical setting, particularly around the 2-cells. We begin by outlining these issues and how we deal with them.

1) Pseudonatural associators and unitors: In Definition 12 the associator α can only be natural in each argument separately because $\otimes$ is not a functor of two arguments. Similarly, we can only ask that the associator is pseudonatural in each argument separately. This entails asking for three pseudonatural transformations: one for each of the three naturality diagrams for α in Definition 12. Precisely, we ask that for every $A, B, C \in \mathscr{B}$ and f, g, h as in that definition, we get 2-cells $\overline{\alpha}_{f,B,C}$, $\overline{\alpha}_{A,g,C}$ and $\overline{\alpha}_{A,B,h}$ giving pseudonatural transformations

$$\begin{aligned} &(\alpha_{-,B,C}, \overline{\alpha}_{-,B,C}) : (- \ltimes B) \ltimes C \Rightarrow (-) \ltimes (B \otimes C) \\ &(\alpha_{A,-,C}, \overline{\alpha}_{A,-,C}) : (A \rtimes -) \ltimes C \Rightarrow A \rtimes (- \ltimes C) \\ &(\alpha_{A,B,-}, \overline{\alpha}_{A,B,-}) : (A \otimes B) \rtimes (-) \Rightarrow A \rtimes (B \rtimes -) \end{aligned} \tag{10}$$

We also need to give unitors ρ and λ, but for these pseudonaturality is straightforward since there is only one argument.

2) Structural modifications in each argument: The main remaining work is in giving the structural modifications. Just as in the case of α above, we cannot ask for modifications $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ and $\mathfrak{r}$ as we do for a monoidal bicategory: instead we must ask for families of 2-cells satisfying the modification condition in each component separately.

This introduces a new issue. In a monoidal bicategory, the fact that the structural modifications are modifications in each argument relies on $\otimes$ being a pseudofunctor of two arguments. For example, the type of $\mathfrak{l}$ (see Figure 2) uses the pseudonatural transformation with components $\lambda_A \otimes B : (IA)B \to AB$. For $g : B \to B'$, the 2-cell witnessing pseudonaturality of this transformation is the interchange isomorphism $\mathsf{int}_{\lambda_A, g}$ defined in (9). This 2-cell does not exist in a premonoidal bicategory, so instead we must use the centrality witness $\mathsf{lc}_g^{\lambda_A}$ for λ_A. Thus, we define $\mathfrak{l}$ to be a family of 2-cells $\mathfrak{l}_{A,B} : (\lambda_A \ltimes B) \Rightarrow \lambda_{A\otimes B} \circ \alpha_{I,A,B}$ that is a modification in each argument, in the sense that for every $A, B \in \mathscr{B}$ we get modifications in $\mathrm{Hom}(\mathscr{B}, \mathscr{B})$ as follows:

$$\begin{tikzcd} (I \rtimes -) \rtimes B \arrow[rr, "\lambda \ltimes B"] \arrow[dr, "\alpha_{I,-,B}"'] & \arrow[d, phantom, "\mathfrak{l}_{-,B}"] & (- \ltimes B) \\ & I \rtimes (- \ltimes B) \arrow[ur, "\lambda_{-\ltimes B}"'] & \end{tikzcd} \qquad \begin{tikzcd} (IA) \rtimes (-) \arrow[rr, "\mathsf{lc}^{\lambda}"] \arrow[dr, "\alpha_{I,A,-}"'] & \arrow[d, phantom, "\mathfrak{l}_{A,-}"] & (A \rtimes -) \\ & I \rtimes (A \ltimes -) \arrow[ur, "\lambda_{A\rtimes -}"'] & \end{tikzcd}$$

Notice that the left-hand diagram is exactly as in the definition of a monoidal bicategory: no adjustments are necessary because each transformation is pseudonatural in the open argument.

We repeat this process for each of the four structural modifications to obtain our definition of premonoidal bicategory: Figure 5 gives the diagrams that are not exactly as in the definition of a monoidal bicategory. (To save space we suppress the $\rtimes$ and $\ltimes$: these can be inferred from the bracketing.) Although we have changed the conditions for $\mathfrak{p}, \mathfrak{m}, \mathfrak{l}$ and $\mathfrak{r}$ to be modifications, the equations of a monoidal bicategory are still well-typed. We also require only that the 1-cell data is central, because the centrality of 2-cells such as $\overline{\alpha}_{f,B,C}$ can be derived in the cases of interest: we discuss this briefly in Section V-D.

3) Premonoidal bicategories: We give the main definition. Despite the subtleties outlined above, the passage from premonoidal categories is very natural.

Definition 16. *A* premonoidal bicategory *is a binoidal bicategory $(\mathscr{B}, \rtimes, \ltimes)$ with a* unit $I \in \mathscr{B}$ *and the following data:*

1) *Pseudonatural equivalences $\lambda : I \rtimes (-) \Rightarrow \mathrm{id}$ and $\rho : (-) \ltimes I \Rightarrow \mathrm{id}$ with each 1-cell component central;*
2) *An equivalence $\alpha_{A,B,C}$ for every $A, B, C \in \mathscr{B}$, with pseudonatural data as in* (10)*.*
3) *For each $A, B, C, D \in \mathscr{B}$ 2-cells $\mathfrak{p}_{A,B,C,D}, \mathfrak{m}_{A,B}, \mathfrak{l}_{A,B}$ and $\mathfrak{r}_{A,B}$, such that these form modifications in each argument as in Figure 5 or, if not shown there, in Figure 2.*

This data is subject to the same equations between 2-cells as in a monoidal bicategory (see e.g. *[72]).*

D. Aside: bicategories of central maps

In a premonoidal category $\mathbb{D}$, the central morphisms form a wide subcategory $\mathcal{Z}(\mathbb{D})$ called the *centre*. This defines a right adjoint to the inclusion of monoidal categories into premonoidal categories. In particular, the centre is a monoidal category, and the inclusion $\mathcal{Z}(\mathbb{D}) \hookrightarrow \mathbb{D}$ strictly preserves premonoidal structure.

We briefly discuss bicategories of central 1-cells and 2-cells for a premonoidal bicategory; here the 2-dimensional structure presents new challenges. A first definition is as follows:

Definition 17. *For a premonoidal category $(\mathscr{B}, \rtimes, \ltimes, I)$, we call $\mathcal{Z}(\mathscr{B})$ the bicategory with the same objects, whose 1-cells and 2-cells are the central 1-cells and central 2-cells in $\mathscr{B}$. Composition is defined using the composition of pseudonatural transformations; the structural 2-cells a, l and r are as in $\mathscr{B}$.*

Note that this is *not* a sub-bicategory of $\mathscr{B}$, since a 1-cell may be central in many ways. Instead we have a forgetful pseudofunctor $\mathcal{Z}(\mathscr{B}) \to \mathscr{B}$.

$$
\begin{tikzcd}[column sep=small]
& (-B)(CD) \arrow[dr, "\alpha_{-,B,CD}"] & \\
((-B)C)D \arrow[ur, "\alpha_{-B,C,D}"] \arrow[d, "\alpha_{-,B,C}D"'] & \arrow[d, phantom, "\mathfrak{p}_{-,B,C,D}\Uparrow"] & (-)(B(CD)) \\
(-(BC))D \arrow[rr, "\alpha_{-,BC,D}"'] & & (-)((BC)D) \arrow[u, "\mathsf{rc}^{\alpha}"']
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& (AB)(C-) \arrow[dr, "\alpha_{A,B,C-}"] & \\
((AB)C)(-) \arrow[ur, "\alpha_{AB,C,-}"] \arrow[d, "\mathsf{lc}^{\alpha}"'] & \arrow[d, phantom, "\mathfrak{p}_{A,B,C,-}\Uparrow"] & A(B(C-)) \\
(A(BC))(-) \arrow[rr, "\alpha_{A,BC,-}"'] & & A((BC)(-)) \arrow[u, "A\alpha_{B,C,-}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& (-)B & \\
(-I)B \arrow[ur, bend left, "\rho B"] \arrow[rr, "\alpha_{-,I,B}"'] & \arrow[u, phantom, "\mathfrak{m}_{-,B}\Uparrow"] & (-)(IB) \arrow[ul, bend right, "\mathsf{rc}^{\lambda}"']
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& A(-) & \\
(AI)(-) \arrow[ur, bend left, "\mathsf{lc}^{\rho}"] \arrow[rr, "\alpha_{A,I,-}"'] & \arrow[u, phantom, "\mathfrak{m}_{A,-}\Uparrow"] & A(I-) \arrow[ul, bend right, "A\lambda"']
\end{tikzcd}
\quad
\begin{tikzcd}
(IA)(-) \arrow[r, "\alpha_{I,A,-}"] \arrow[d, "\mathsf{lc}^{\lambda}"'] \arrow[dr, phantom, "\Rrightarrow \mathfrak{l}_{A,-}"] & I(A-) \arrow[dl, bend left, "\lambda"] \\
A(-) &
\end{tikzcd}
\quad
\begin{tikzcd}
(-B)I \arrow[r, "\alpha_{-,B,I}"] \arrow[dr, bend right, "\rho_{-B}"'] \arrow[dr, phantom, "\mathfrak{r}_{-,B}\Rrightarrow"] & (-)(BI) \arrow[d, "\mathsf{rc}^{\rho}"] \\
& AB
\end{tikzcd}
$$

Figure 5: Modification axioms for the structural 2-cells of a premonoidal bicategory, where they differ from those of a monoidal bicategory.

Proposition 4. *For every object A in a premonoidal bicategory $(\mathscr{B}, \rtimes, \ltimes, I)$, the operations $A \rtimes (-)$ and $(-) \ltimes A$ induce pseudofunctors on $\mathcal{Z}(\mathscr{B})$.*

Unlike in the categorical setting, this does not straightforwardly extend to a monoidal structure. The difficulty is due to the fact that, for central 1-cells $(f, \mathsf{lc}^f, \mathsf{rc}^f)$ and $(g, \mathsf{lc}^g, \mathsf{rc}^g)$, it is not generally the case that $\mathsf{lc}^f_g = (\mathsf{rc}^g_f)^{-1}$, as would be the case for a monoidal interchange law (recall (9)). However, this does not seem to be a problem in practice. In models of effectful languages, there is often a bicategory of pure maps for which the problematic equation automatically holds (see Lemma 3 and Lemma 4).

An interesting direction is to axiomatize 'bicategories of pure maps' in which this equation holds; one then recovers many properties of the categorical centre. For example, the structural 2-cells of the premonoidal structure are automatically 2-cells in such bicategories—which is why we do not require them to be central in Definition 16—and the premonoidal structure extends to a monoidal structure. This leads towards a notion of *Freyd bicategory* (*c.f.* [65], [66]). We leave this for future work: see the discussion in Section VI.

E. Premonoidal Kleisli bicategories

We now arrive at our main theorem.

Theorem 2. *For any strong pseudomonad (T, t) on a symmetric monoidal bicategory $(\mathscr{B}, \otimes, I)$, the Kleisli bicategory $\mathscr{B}_T$ is premonoidal, with binoidal structure given as in (7).*

Proof summary. We sketched the binoidal structure above. The structural 1-cells are lifted from $\mathscr{B}$ via the canonical pseudofunctor $\mathscr{B} \to \mathscr{B}_T$, and their pseudonaturality is established by considering the corresponding actions. One exception is the pseudonaturality of α in its middle argument, which uses the compatibility of a left and right strength. □

In constructing this premonoidal structure we make essential use of the following result. For f a morphism in $\mathscr{B}$, write $\tilde{f}$ for the image of f under the pseudofunctor $\mathscr{B} \to \mathscr{B}_T$.

Lemma 3. *For $f \in \mathscr{B}(A, A')$, $\tilde{f} \in \mathscr{B}_T(A, A')$ is canonically central. Moreover, for every $g \in \mathscr{B}(B, B')$, $\mathsf{lc}^{\tilde{f}}_{\tilde{g}} = (\mathsf{rc}^{\tilde{g}}_{\tilde{f}})^{-1}$.*

Further examples: graded monads and **Para***:* Another source of examples, which we were not expecting when we began this work, comes from *graded monads* ([70], [50], [36]). A graded monad goes beyond just tracking side-effects to also tracking quantitative information. Formally, a graded monad on $\mathbb{C}$ consists of a monoidal category $(\mathbb{E}, \bullet, I)$ of *grades* and a lax monoidal functor $T : \mathbb{E} \to [\mathbb{C}, \mathbb{C}]$. Thus, one has a functor $T_e : \mathbb{C} \to \mathbb{C}$ for every $e \in \mathbb{E}$, natural in e, together with natural transformations $\phi_{e,e'} : T'_e \circ T_e \Rightarrow T_{e' \bullet e}$ (acting like a monadic multiplication) and $\psi : \mathrm{id} \Rightarrow T_I$ (acting like a monadic unit) satisfying unit and associativity laws.

Instead of restricting to the symmetric case as we do in Theorem 2, we consider abstract bistrong graded monads. An endofunctor $T : \mathbb{C} \to \mathbb{C}$ equipped with two strengths $t_{A,B} : A \otimes TB \to T(A \otimes B)$ and $s_{A,B} : TA \otimes B \to T(A \otimes B)$ is bistrong if

$$
\begin{tikzcd}
(AT_B)C \arrow[r, "\alpha"] \arrow[d, "tC"'] & A(T_BC) \arrow[r, "As"] & AT_{BC} \arrow[d, "t"] \\
T_{AB}C \arrow[r, "s"'] & T_{(AB)C} \arrow[r, "T_\alpha"'] & T_{A(BC)}
\end{tikzcd}
$$

commutes. Then, a small adjustment to Katsumata's definition of strong graded monads gives the following.

Definition 18 (*c.f.* [36, Definition 2.5])**.** *A* bistrong graded monad *on a monoidal category $(\mathbb{C}, \otimes, I)$ consists of a monoidal category $(\mathbb{E}, \bullet, I)$ of grades and a lax monoidal functor $T : \mathbb{E} \to [\mathbb{C}, \mathbb{C}]_{bistrong}$, where $[\mathbb{C}, \mathbb{C}]_{bistrong}$ is the category of bistrong endofunctors and natural transformations that commute with both strengths (see* e.g. *[49]).*

Previous work (*e.g.* [48]) has used presheaf enrichment to define Kleisli constructions for graded monads. However, there is also a natural bicategorical construction:

Definition 19. *Let T be a bistrong graded monad on $(\mathbb{C}, \otimes, I)$ with grades $(\mathbb{E}, \bullet, I)$. Define a bicategory $\mathcal{K}_T$ with objects those of $\mathbb{C}$ as follows:*

- *1-cells from A to B are pairs (e, f) consisting of a grade e and a map $f : A \to T_eB$;*
- *2-cells $\gamma : (e, f) \Rightarrow (e', f')$ are maps $\gamma : e \to e'$ in $\mathbb{E}$ such that $T_\gamma(B) \circ f = f'$, with composition as in $\mathbb{E}$;*

- *Composition is defined similarly to Kleisli composition: the composite of $f : A \to T_e B$ and $g : B \to T_{e'} C$ is*

$$A \xrightarrow{f} T_e B \xrightarrow{T_e g} T_e(T_{e'} C) \xrightarrow{\phi} T_{e \bullet e'} C$$

The identity on A is $A \xrightarrow{\psi_A} T_I A$, and the structural isomorphisms are the structural isomorphisms in $\mathbb{E}$.

(One could also present this bicategory as $\mathbf{Para}(\mathbb{C}, \mathbb{E}, *)^{\mathrm{op}}$ where maps are parametrized by the left action of $\mathbb{E}$ on $\mathbb{C}$ with $e * A := T_e(A)$.)

If the graded monad T is bistrong, one obtains strict pseudofunctors $A \rtimes (-), (-) \ltimes B : \mathcal{K}_T \to \mathcal{K}_T$ for every $A, B \in \mathcal{K}_T$, defined similarly to (7). Moreover, just as every map in $\mathbb{C}$ determines a 1-cell in $\mathbf{Para}(\mathbb{C})$, every map $f \in \mathbb{C}(A, A')$ determines a 1-cell with grade I in $\mathcal{K}_T$, as $\widetilde{f} := \big(A \xrightarrow{f} A' \xrightarrow{\psi'_A} T_I A'\big)$. This 1-cell canonically determines a central 1-cell, with $\mathsf{lc}^{\widetilde{f}}$ and $\mathsf{rc}^{\widetilde{f}}$ given by the canonical isomorphism in $\mathbb{C}$. Hence, just as for a Kleisli bicategory, the pure maps in $\mathcal{K}_T$ satisfy the 'problematic equation' discussed in Section V-D.

Lemma 4. *For $f \in \mathbb{C}(A, A')$, $\tilde{f} \in \mathcal{K}_T(A, A')$ is canonically central. Moreover, for every $g \in \mathbb{C}(B, B')$, $\mathsf{lc}^{\tilde{f}}_{\tilde{g}} = (\mathsf{rc}^{\tilde{g}}_{\tilde{f}})^{-1}$.*

Arguing similarly to Section II-C, we get the following.

Proposition 5. *For any bistrong graded monad (T, ϕ, ψ) the bicategory $\mathcal{K}_T$ has a canonical choice of premonoidal structure.*

VI. Related work and perspectives

We have introduced strong pseudomonads and premonoidal bicategories, and validated these definitions with examples and theorems paralleling the categorical setting:

- Certain pseudomonads are canonically strong: for instance, any pseudomonad on $(\mathbf{Cat}, \times)$, any pseudomonad with respect to a cocartesian structure $(0, +)$, or any pseudomonad of the form $(-) \otimes M$ for M a pseudomonoid (Section III-C);
- A left strength for a pseudomonad is equivalent to a left action on the Kleisli bicategory extending the monoidal structure (Theorem 1);
- A left strong monad on a symmetric monoidal bicategory is also right strong (Proposition 2);
- The Kleisli bicategory for a strong pseudomonad is premonoidal (Theorem 2).

Our contributions can be understood in two ways: as a practical tool towards new semantic models for programming languages, or from the theoretical perspective of categorical algebra and higher category theory.

A. Related work and perspectives in semantics

An important inspiration for this work was the theory of *dialogue categories* [53], [52], which explains models of linear logic and game semantics in terms of Moggi-style computational effects. A dialogue category is a symmetric monoidal category $\mathbb{C}$ equipped with a negation functor $\neg : \mathbb{C} \longrightarrow \mathbb{C}^{\mathrm{op}}$ such that the double negation functor $\neg\neg : \mathbb{C} \longrightarrow \mathbb{C}$ extends to a strong monad. With our definition of strong pseudomonads, we can incorporate recent bicategorical models (*e.g.* [7], [24], [22], [51], [14]) into this theory. There are also connections to graded monads: see [50], [21].

A natural step would be to bicategorify the definition of *Freyd categories*, which model effectful languages with an explicit collection of pure *values* ([65], [44]). As Section V-D indicates this would involve some subtleties, including an axiomatization of some version of Lemma 3. Such a definition should therefore be validated by a appropriate correspondence theorem, in the style of our Theorem 1, lifting the correspondence between actions and Freyd categories (*e.g.* [43, Appendix B]) to the bicategorical setting. In a similar vein, models of higher-order call-by-value languages can be presented as *closed* Freyd categories [66]. These are equivalent to a strong monad on a monoidal category with enough closed structure ([44], [54]). One would therefore expect corresponding results relating closed Freyd bicategories to strong pseudomonads. Further work is also needed to connect the Kleisli bicategory for a graded monad presented here with existing Kleisli constructions such as that in [48].

In the spirit of Moggi's programme ([55], [56]) it would be of interest to extract a 2-dimensional monadic metalanguage as an internal language for our bicategorical models. Existing work already covers the simply-typed λ-calculus [17].

Finally we note that, although higher-categorical definitions can sometimes be intimidating or intractable, it is often possible (and generally preferable) to identify universal properties that make the axioms hold automatically. It would be of use, therefore, to extend the framework of [75] for constructing symmetric monoidal bicategories to the structures introduced here: the similarities in the construction of the strong pseudomonads on $\mathbf{Para}(\mathbb{C})$ (Example 6) and $\mathbf{Span}(\mathbb{C})$ (Corollary 1) suggest this is possible.

B. Related work in category theory and universal algebra

There are many accounts of strong monads in category theory, for example in terms of enriched structure ([38], [39]). In this paper we have given a direct and explicit definition, but it would be interesting to reconstruct the full theory.

More generally, this work contributes to recent research on Kleisli bicategories and the theory of substitution ([31], [15], [16], [23]). It is also part of a lively line of research studying various kinds of weak monoidal structure in higher dimensions ([4], [5], [19], [23]). From this perspective, the complexities with the 2-cells in the definition of premonoidal bicategories and their centres are perhaps unsurprising, since the interaction between the "funny" tensor product (used to define premonoidal categories [64]), higher-dimensional structure, and interchange laws is subtle, *e.g.* the funny tensor product does not extend to a tensor on the 2-category $\mathbf{Cat}$ (also see *e.g.* [9], [3], [5]).

Finally, we note that Hyland & Power have suggested a notion of pseudo-commutativity for strict bistrong 2-monads [32], which should generalize smoothly to our setting. We will describe bistrong and commutative pseudomonads in future work.

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APPENDIX

Proof of Lemma 1

Lemma. *1) For any pseudomonoid (M, m, e, a, l, r) on a monoidal bicategory $(\mathscr{B}, \otimes, I)$ the pseudomonad $(-)\otimes M$ has a strength given by the associator α of $\otimes$.*

2) Every pseudomonad is canonically strong with respect to the cocartesian monoidal structure $(+, 0)$.

Proof notes. For both claims, one constructs the data by following the corresponding 1-categorical argument and filling the commuting diagrams with the appropriate 2-cells; the equations hold by coherence. For (1, for instance, the structural modifications $\boldsymbol{x}$ and $\boldsymbol{y}$ are given using $\mathfrak{l}$ and $\mathfrak{p}$, respectively, while $\boldsymbol{w}$ and $\boldsymbol{z}$ are given using $\mathfrak{r}$ and $\mathfrak{p}$, respectively, together with the pseudonaturality of $\alpha^{\bullet}$. The axioms hold by the coherence of pseudomonoids [40]. Similarly for (2): the strength has components $[T\mathrm{inl} \circ \eta_A, T\mathrm{inr}] : A + TB \to T(A+B)$ and the equations hold by coherence for cartesian monoidal bicategories and the fact all the structural 2-cells are invertible. □

Proof of Proposition 1

Proposition. *Every pseudofunctor (resp. pseudomonad) on $(\mathbf{Cat}, \times, 1)$ has a canonical choice of strength.*

Proof notes. Similarly to the categorical proof, for any pseudofunctor $F : \mathbf{Cat} \to \mathbf{Cat}$ and $a \in \mathbb{A}$ one has $F(\lambda b\,.\langle a, b\rangle) : F(\mathbb{B}) \to F(\mathbb{A} \times \mathbb{B})$. However, since F is now a pseudofunctor we also have a natural transformation $F(\lambda b\,.\langle f, b\rangle)$ for each $f : a \to a'$ in $\mathbb{A}$, with components $F(\lambda b\,.\langle f, b\rangle)_w : F(\lambda b\,.\langle a, b\rangle) \to F(\lambda b\,.\langle a', b\rangle)$ in $F(\mathbb{A}\times\mathbb{B})$. We may therefore define a functor $t_{\mathbb{A},\mathbb{B}} : \mathbb{A} \times F\mathbb{B} \to F(\mathbb{A}\times\mathbb{B})$ sending a pair of objects (a, w) to $F(\lambda b\,.\langle a, b\rangle)$ and a pair of morphisms $(a \xrightarrow{f} a', w \xrightarrow{g} w')$ to the composite $F(\lambda b\,.\langle a', b\rangle)(g) \circ F(\lambda b\,.\langle f, b\rangle)_w$. This is functorial because F is functorial on natural transformations and $F(\lambda b\,.\langle a, b\rangle)$ is a functor, and pseudonatural via the compositor for F.

Then $\boldsymbol{x}$ and $\boldsymbol{y}$ are defined using the compositor and unitor for F, and the coherence of pseudofunctors [28] ensures the axioms hold. Finally, if T is a pseudomonad then one defines $\boldsymbol{w}$ and $\boldsymbol{z}$ using the pseudonaturality of η and μ: this is similar to the proof in the categorical setting, where one uses the naturality of the unit and multiplication to show the two compatibility laws hold. Again, the axioms follow from coherence [40]. □

Proof of Theorem 1

Theorem. *For any pseudomonad T on a monoidal bicategory $(\mathscr{B}, \otimes, I)$, the categories $\mathbf{LeftStr}(T)$ and $\mathbf{LeftExt}(T)$ are equivalent.*

Proof notes. The proof follows the categorical construction (see [49, Prop. 4.3]). For every left strength t for T, the induced action $\triangleright : \mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$ extends the canonical action of $\mathscr{B}$ on itself, by construction. Conversely, any extension $(\triangleright, \theta)$ induces a strength $t_{A,B} = \mathrm{id}_A \triangleright \mathrm{id}_{TB}$, where id_{TB} is regarded as an element of $\mathscr{B}_T(TB, B)$. These constructions are inverses, up to isomorphism. In each direction, there is a tight match-up between the equations of the given structure and the equations for the required structure; for an outline, see Table I. □

strength	action
strong pseudofunctor axioms (Fig. 3)	axioms for an action (Def. 10)
top row of axioms in Fig. 4	pseudofunctor axioms for $\mathscr{B} \times \mathscr{B}_T \to \mathscr{B}_T$
middle row of axioms in Fig. 4	pseudonaturality axioms for $\widetilde{\lambda}$
bottom row of axioms in Fig. 4	pseudonaturality axioms for $\widetilde{\alpha}$

TABLE I: Relating the data and equations for Theorem 1

Proof of Theorem 2

Theorem. *For any strong pseudomonad (T, t) on a symmetric monoidal bicategory $(\mathscr{B}, \otimes, I)$, the Kleisli bicategory $\mathscr{B}_T$ is premonoidal, with binoidal structure given as in (7).*

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from the

Proof notes. The binoidal structure is as in (7), with s defined as in (4). The structural 1-cells are lifted from $\mathscr{B}$ via the canonical pseudofunctor $\eta \circ (-) : \mathscr{B} \to \mathscr{B}_T$, and their pseudonaturality is established by considering the corresponding actions. One exception is the pseudonaturality of α in its middle argument, which uses the the fact the left and right strengths are defined in terms of one another, hence compatible. $\square$