A Guide to *Writing* Proofs

Propositions

Britain is an island

Every island can be circumnavigated

If a set is non-empty it contains an element

All Martians like pepperoni on their pizza

The factorial of 6 is 27

Non-propositions

Could you please pass the salt?

Ready steady, go

Vote for Tom Cruise

Show your work clearly

Good luck to Sunderland

False Reasoning Principles

By superiority: 298743 is a prime number because I say so.

By similarity: This proposition is true because it is very similar to one which I proved yesterday.

By obviousness: Obvious!

By rumour: I read somewhere on the Internet that this proposition was true.

By intimidation: This is so trivial.

By plausibility: It sounds reasonable.

Deductive Argument

Example:

S1: Britain is an island

S2: Every island can be circumnavigated

Therefore
S3: Britain can be circumnavigated

A sequence of propositions, each one of which is either a *premise*, which is taken for granted, or follows logically from the previous ones.

Valid Arguments?

If Abraham Lincoln was Ethiopian, then he was African. Abraham Lincoln was not African. Therefore he was not Ethiopian.

If astrology is a true science, then the economy is improving. The economy is improving. Therefore, astrology is a true science.

If it is cloudy, then it is going to rain. If it is going to rain, then I should take my raincoat with me.
Therefore if it is cloudy, I should take my raincoat with me.

Logical structure of Propositions

Elementary or atomic:

Britain is an island

Can **not** be broken down further into propositions

Decomposition:

Cats and Dogs are here

Can be decomposed into:

Cats are here **and** Dogs are here

The Logical Structure of Propositions

Conjunction, and: Jill is twelve **and** Jack is fourteen

Disjunction, or: I am going to the movies **or** I am going to the pub

Negation, not: I am **not** going to the movies

Implication, implies: x is fourteen **implies** y must be greater than 2

Propositional meta-variables

If Abraham Lincoln was Ethiopian, then he was African. Abraham Lincoln was not African.

Therefore he was not Ethiopian.

Atomic Propositions:

- A: Abraham Lincoln was Ethiopian
- B: Abraham Lincoln was African

Formal Argument:

Given premises

- A **implies** B
- **not** B

Is **not** A a logical consequence?

Notation

$$S_1, S_2, \ldots, S_n \vdash \mathrm{P}$$

Means:

There is a valid logical argument, with which we can derive the proposition P from the finite set of premises $S_1, \ldots S_n$

Question:

How can we develop valid logical arguments?

Establishing Conjunctive Propositions

Rules for Introducing **and**

To establish P **and** Q:

1. Establish P
2. Establish Q
3. Conclude (P **and** Q)

Fancy name: Conjunction Introduction – **AndIntro**

How do use Conjunctions? - and

How do we make use of (P **and** Q) ? :

From	(P **and** Q) we can conclude P
From	(P **and** Q) we can conclude Q

Fancy name: Conjunction Elimination: **AndElim**

All very obvious

An example proof

P **and** Q, R $\vdash$ Q **and** R

A proof:

1. P **and** Q	premise
2. R	premise
3. Q	**AndElim** to 1
4. Q **and** R	**AndIntro** to 2,3

Implicative Propositions

Examples:

if the sun is up it is daytime

n is prime *implies* n is odd

for even integer n, n^2 is also an even integer

B only if A

Each has a *premise* and a *conclusion*

Decomposition:

Premise: the sun is up

Conclusion: it is daytime

Proving Implications

Every proof of $\boxed{P \text{ implies } Q}$ has the form:

1. Assume the proposition P to be true
2. Using this assumption establish Q
3. Conclude: $\boxed{P \text{ implies } Q}$ is true

All the work is in the Part 2.

Fancy name: Implication Introduction - **ImpIntro**

An example proof

If n is even then so is n^2

1. Assume n is an even number

2. So there is some k such that $n = 2k$ (Definition of even)

3. Therefore, using 2, $n^2 = 2(2k^2)$

4. Therefore n^2 is even (Definition of even)

5. Therefore, by **ImpIntro** from 1 and 4,

$$\text{if } n \text{ is even then so is } n^2$$

Using Implications: Modus Ponens

Modus Ponens:

From

- P implies Q

- and P

We can conclude

Q is true

Should be called Implication Elimination - **ImpElim**
but Greeks got there first

An example proof

$\boxed{\mathrm{C}\ \textbf{implies}\ \mathrm{R},\ \ \mathrm{R}\ \textbf{implies}\ \mathrm{S} \vdash \mathrm{C}\ \textbf{implies}\ \mathrm{S}}$

1. C **implies** R	premise
2. R **implies** S	premise
3. Assume C	
4. R	using **Modus Ponens** with 1, 3
5. S	using **Modus Ponens** with 2, 4
6. Therefore C **implies** S	by **ImpIntro** applied to 3 – 5

A Valid Argument

If it is cloudy, then it is going to rain. If it is going to rain, then I should take my raincoat with me.
Therefore if it is cloudy, I should take my raincoat with me.

C: It is cloudy

R: it is going to rain

S: I should take my raincoat

$$\boxed{\mathsf{C}\ \textbf{implies}\ \mathsf{R},\ \ \mathsf{R}\ \textbf{implies}\ \mathsf{S} \vdash \mathsf{C}\ \textbf{implies}\ \mathsf{S}}$$

Handling Negation

How to establish the proposition **not** P:

1. Assume proposition P to be true
2. Derive a contradiction, say **false**
3. Conclude $\boxed{\textbf{not}\ \text{P is true}}$

Fancy rule: Negation Introduction, **NotIntro**

Using Contradictions:

If we have established a contradiction **false**, we can conclude *any* proposition.

Fancy rule: False Elimination, **falseElim**

Handling Negation: Elimination

How do we use proposition **not** P:

Only indirectly, to establish contradictions

The rule **NotElim** :

If we have established

(a) P

(b) **not** P

Then we can conclude **false**

An example: Modus Tollens

$$\boxed{P \ \textbf{implies}\ Q,\ \ \textbf{not}\, Q \vdash \ \textbf{not}\, P}$$

A proof:

1. P **implies** Q	premise
2. **not** Q	premise
3. Suppose P is true	
4. Then Q is true	**MP** to 1
5. Then **false**	**NotElim** to 2,4
6. Therefore **not** P	**NotIntro** to 3–5

Establishing Disjunctive Propositions

Two ways to establish P **or** Q:

1. Establish P	1. Establish Q
2. Conclude (P **or** Q)	2. Conclude (P **or** Q)

It is sufficient to establish *one* of $P,\ Q$

Fancy rule: Or Introduction, **OrIntro**

Using Disjunctive propositions - or

Rules for Eliminating **or**: Case Analysis

To prove R from P **or** Q :

1a. Assume P	1b. Assume Q
2a. Use assumption to prove R	2b. Use assumption to prove R
3. Conclude R	

Two separate cases:

Proof of R, assuming P to be true

Proof of R, assuming Q to be true

Fancy name: **OrElim**

and distributes over or

1.	P **and** (Q **or** R)	premise
2.	P	**AndElim** to 1
3.	Q **or** R	**AndElim** to 1
4.	Assume Q	
5.	P **and** Q	**AndIntro** to 2,4
6.	(P **and** Q) **or** (P **and** R)	**OrIntro** to 5
7.	Assume R	
8.	P **and** R	**AndIntro** to 2,4
9.	(P **and** Q) **or** (P **and** R)	**OrIntro** to 8
10.	(P **and** Q) **or** (P **and** R)	**OrElim** to 3,4–6, 7–9

A Valid Argument?

If the train arrives late and there are no taxis at the station then John is late for his meeting. John is not late for his meeting. The train did arrive late. *Therefore*, there were taxis at the station.

Propositions:

L: the train arrives late

T: there are taxis at the station

J: John arrives late for his meeting

An extra rule required

(L **and not** T) **implies** J, **not** J, L $\vdash$ T

1.	(L **and not** T) **implies** J	premise
2.	**not** J	premise
3.	L	premise
4.	Assume **not** T	
5.	L **and not** T	**AndIntro** to 3,4
6.	J	**MP** to 1,5
7.	**false**	**NotElim** to 2,6
8.	**not** (**not** T)	**NotIntro** to 4–7
9.	Can we now conclude T ?	

Double Negation

Introducing double negations:

$\mathrm{P} \;\vdash\; \textbf{not}\,(\,\textbf{not}\,\mathrm{P})$

Can NOT be derived from our existing rules

Using double negations:

From **not** (**not** P) we can conclude P

Fancy rule: Double negation Elimination, **NotnotElim**

Proof by contradiction

1. Suppose there are only a finite number, say $p_1, \ldots p_n$.
2. Consider the number $(P+1)$ where $P = (p_1 \times p_2 \times \ldots \times p_n)$
3. It is not a prime as it is different from each p_i
4. So $(P+1)$ must be divisible by some p_r
5. So $(P+1) = (p_r \times S) + 1$ where $S = p_1 \times p_{r-1} \times p_{r+1} \times \ldots$
6. Contradiction in 4, 5
7. Therefore from 6, 1 must be false
8. Therefore there are an infinite number of primes.

An unsound rule

A **Silly** rule:

From - (P **implies** Q) - Q 1. Conclude P

Using **Silly**:

We can derive contradictions: If *the moon is in the sky* then *black is white*

the moon is in the sky - obviously true

black is white - obviously false

An unsound deduction

the moon is in the sky $\vdash$ *black is white*

1. *the moon is in the sky* — premise
2. Assume *black is white*
3. *the moon is in the sky* — **falseElim** to 2
4. *black is white* **implies** *the moon is in the sky* — **ImpIntro** to 2,3
5. *black is white* — **Silly** to 4,1

Formal versus Informal Proofs

In an informal proof:

- many steps are omitted

- co-operation of reader is required

- some (obvious) justifications omitted

Reader can construct formal proof if necessary

Informal proof contains sufficient material to construct formal proof.

Proof of $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Each line

- boring but has clear justification

- can (if necessary) be justified *formally*

Hidden formal rules:

Generalisation: Lines 1,10 (from predicate logic)

ImpIntro: Lines 1,2,9

OrElim (Case Analysis): Lines 5,8

Very Confusing: iff

Examples:

The sun is up *if and only if* it is day time

$(A \cup B) \cap A = A$

$P \vee (Q \wedge P)$ *logically equivalent* to P

These are **abbreviations** for two independent propositions, which need two independent proofs.

Proving *iff* Propositions

There are **NO** shortcuts

To prove P *if and only if* Q:

1. Prove implication: $\boxed{P \text{ implies } Q}$
2. Prove implication: $\boxed{Q \text{ implies } P}$

To prove $(A \cup B) \cap A = A$:

1. Prove $\boxed{x \in (A \cup B) \cap A \text{ implies } x \in A}$
2. Prove $\boxed{x \in A \text{ implies } x \in (A \cup B) \cap A}$

Goal-Oriented Reasoning

- Set up a $\boxed{\text{Goal}}$

  $\boxed{\text{assumptions}}$

- Understand both

- Ransack $\boxed{\text{assumptions}}$ for relevant information

- Reduce $\boxed{\text{Goal}}$ to simpler $\boxed{\text{subGoals}}$

- Arrive at Trivial $\boxed{\text{Goals}}$

- Write up proof using $\boxed{\text{Deductive Reasoning Rules}}$