Topics in Computer Science: Semantics of programming languages

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- Algorithms and complexity: what can be computed? How fast?

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- Algorithms and complexity: what can be computed? How fast?
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In other words, what is the meaning of a program?

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- A function between two sets, written $A \to B$, is a rule assigning to each element of A an element of B.

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For example, writing $\mathbb{N}$ for the set of natural numbers $0, 1, 2, \ldots$:

1. We get a function $\mathbb{N} \to \mathbb{N}$ as the rule $x \mapsto x + 1$;
2. We get a different function $\mathbb{N} \to \mathbb{N}$ as the rule $x \mapsto 1$.

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Aim: give you some sense of why we want to do this, and how we do it

Aims of the next four lectures

At the end of the next four lectures you should be able to:

- Explain why we do semantics
- Distinguish between operational and denotational approaches
- Be able to prove basic semantic facts about some toy languages
- Outline some directions of where the field goes next

Why do we do semantics?

:(

Your PC ran into a problem that it couldn't handle, and now it needs to restart.

You can search for the error online: HAL_INITIALIZATION_FAILED

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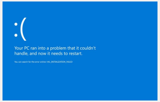
Reliability really matters

In our day-to-day lives software failures are often just annoying:



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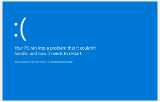

But they can also be life-threatening and catastrophic:

Reliability really matters

Software is at the heart of the modern world.

It's crucial we understand what it does!

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> Higher-order languages. . . encourage the programmer to build abstractions by composing functions. A good compiler must inline many of these calls to recover an efficiently executable program. In principle, inlining is dead simple: just replace the call of a function by an instance of its body. But any compiler-writer will tell you that inlining is a black art, full of delicate compromises that work together to give good performance without unnecessary code bloat.
>
> Simon Peyton-Jones, one of the designers of Haskell

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Reason 3: to design new programming languages and paradigms

(a) by thoroughly understanding the structure of programs,
we can discover useful new constructs
example: effect handlers in Jax and Pyro

(b) we can also make sure new features behave correctly

A cautionary tale

The ML language has a range of powerful features that give strong guarantees on program behaviour

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... but an apparently-innocuous combination of its features broke these guarantees!

Making new features behave correctly

We want programs to do increasingly-fancy things.

Example 1: specify probabilistic models

e.g. for pandemic modelling

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normalize(
  x = sample(bernoulli (2/7));
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Both **probabilistic programs** and **differentiable programs** involve

1. Powerful program features (e.g. loops, recursion, higher-order functions, ...)
2. Sophisticated mathematics (e.g. probability, measure theory, differentiability, ...)

Making sure these combine to do the right thing is both subtle and difficult!

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Do we need semantics?

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Suggestion: define the meaning of a program as the mapping that the compiler defines from the source code to machine code.

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4.7.3.3 Body replacement and execution. Finally the procedure body, modified as above, is inserted in place of the procedure statement and executed. If the procedure is called from a place outside the scope of any non-local quantity of the procedure body, the conflicts between the identifiers inserted through this process of body replacement and the identifiers whose declarations are valid at the place of the procedure statement or function designator will be avoided through suitable systematic changes of the latter identifiers.

From the Revised Report of Algol 60.

The limitations of writing things in words

Side effects = changes to variables resulting from the evaluation of an expression for which the variable is not local.

An Algol 60 program

begin **integer** a;
  integer **procedure** $f(x, y)$; **value** y, x; **integer** y, x;
    $a := f := x + 1$;
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    $a := 0$; $outreal\big(1,\ a\ +\ f(a, g(a))\,/g(a)\big)$ **end**

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See also: the huge on-going effort to verify chip architectures.

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In other words, we want a **mathematical model** of programs.

Aside: syntax vs semantics

Every programming language comes with

1. **Syntax** = the sequences of symbols that are valid in the language
   (e.g. where { and } go in Java)
2. **Semantics** = the meaning of (correctly-written) programs.

Ways of doing semantics

= different ways of modelling programs

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- **Axiomatic**: a program's meaning is given indirectly in terms of the collection of properties it satisfies; these properties are defined via a collection of axioms and rules.

Ways of doing semantics

Different approaches complement one another:

1. correctness of the proof rules of an axiomatic semantics relies on an underlying denotational or operational semantics,
2. correctness of an implementation with respect to a denotational semantics requires a proof that the operational and denotational semantics agree,
3. in proving facts about an operational semantics it can be of great help to use a denotational semantics, which abstracts away from unimportant, implementation details.

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What we'll see next:

> operational and denotational semantics for a very basic language of numerical expressions.

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This is a short course! The aim is to give a flavour of how things go;
at the end we'll add some features and think how to adapt things for that

The language `NumExp`

A VERY basic language NumExp

$$
\begin{array}{llll}
\mathit{Num} & \mathtt{n} & ::= & \mathtt{0} \mid \mathtt{1} \mid \mathtt{2} \mid \ldots \\
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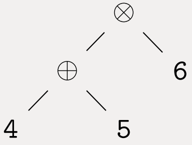
To be precise, these represent *abstract sytax* trees, like:



Our informal reading of what `NumExp` programs mean:

1. Every numeral $\mathtt{n}$ is evaluated to the corresponding number n;
2. To find the value associated with an expression of the form $\mathtt{e}_0 \oplus \mathtt{e}_1$ we evaluate the expressions $\mathtt{e}_0$ and $\mathtt{e}_1$ and take the sum of the results;
3. To find the value associated with an expression of the form $\mathtt{e}_0 \otimes \mathtt{e}_1$ we evaluate the expressions $\mathtt{e}_0$ and $\mathtt{e}_1$ and take the product of the results

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Examples:

- $\texttt{42}$ is evaluated to 42,
- $(\texttt{1} \oplus \texttt{2}) \otimes \texttt{3}$ is evaluated to 9,
- $(\texttt{1} \oplus \texttt{2}) \otimes (\texttt{3} \oplus \texttt{4})$ is evaluated to 21.

Operational semantics for `NumExp`

Giving operational semantics to `NumExp`

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$$\frac{\text{premise}_1 \quad \ldots \quad \text{premise}_n}{\text{conclusion}} \text{ side-condition (rule name)}$$

"if we have all the premises, and the side-condition holds, we get the conclusion"

Examples of logical rules

$$\frac{}{\mathtt{n}_0 \oplus \mathtt{n}_1 \rightarrow \mathtt{n}}\; n = n_0 + n_1 (sum) \qquad \frac{\mathsf{e}_0 \rightarrow \mathsf{e}_0'}{\mathsf{e}_0 \oplus \mathsf{e}_1 \rightarrow \mathsf{e}_0' \oplus \mathsf{e}_1}\;(sumL)$$

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It's sometimes more natural to read from top to bottom: *sumL* says

> To evaluate $\mathtt{e}_0 \oplus \mathtt{e}_1$ one step, first evaluate $\mathtt{e}_0$ one step.

Small-step operational semantics for `NumExp`

Small-step = we describe every step the computation makes,
not just its final result.

$$\frac{}{\mathtt{n}_0 \oplus \mathtt{n}_1 \to \mathtt{n}}\; n = n_0 + n_1 (sum) \qquad \frac{\mathsf{e}_0 \to \mathsf{e}_0'}{\mathsf{e}_0 \oplus \mathsf{e}_1 \to \mathsf{e}_0' \oplus \mathsf{e}_1}\;(sumL) \qquad \frac{\mathsf{e}_1 \to \mathsf{e}_1'}{\mathsf{e}_0 \oplus \mathsf{e}_1 \to \mathsf{e}_0 \oplus \mathsf{e}_1'}\;(sumR)$$

$$\frac{}{\mathtt{n}_0 \otimes \mathtt{n}_1 \to \mathtt{n}}\; n = n_0 \times n_1 (prod) \qquad \frac{\mathsf{e}_0 \to \mathsf{e}_0'}{\mathsf{e}_0 \otimes \mathsf{e}_1 \to \mathsf{e}_0' \otimes \mathsf{e}_1}\;(prodL) \qquad \frac{\mathsf{e}_1 \to \mathsf{e}_1'}{\mathsf{e}_0 \otimes \mathsf{e}_1 \to \mathsf{e}_0 \otimes \mathsf{e}_1'}\;(prodR)$$

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We think of this as saying

> If we start from $(1 \oplus 2) \otimes (3 \oplus 4)$ then $3 \otimes (3 \oplus 4)$ is an intermediate result in the program's execution.

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Both of the following judgements are derivable:

$$(\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4}) \to \mathtt{3} \otimes (\mathtt{3} \oplus \mathtt{4}) \qquad (\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4}) \to (\mathtt{1} \oplus \mathtt{2}) \otimes \mathtt{7}$$

and here are their derivations:

$$\dfrac{\dfrac{}{\mathtt{1} \oplus \mathtt{2} \to \mathtt{3}}\ {}^{3=1+2\,(sum)}}{(\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4}) \to \mathtt{3} \otimes (\mathtt{3} \oplus \mathtt{4})}\ (prodL) \qquad \dfrac{\dfrac{}{\mathtt{3} \oplus \mathtt{4} \to \mathtt{7}}\ {}^{7=3+4\,(sum)}}{(\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4}) \to (\mathtt{1} \oplus \mathtt{2}) \otimes \mathtt{7}}\ (prodR)$$

Derivations

Definition

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and here are their derivations:

$$\frac{\overline{1 \oplus 2 \to 3}\ {}^{3=1+2\,(sum)}}{(1 \oplus 2) \otimes (3 \oplus 4) \to 3 \otimes (3 \oplus 4)}\ (prodL) \qquad \frac{\overline{3 \oplus 4 \to 7}\ {}^{7=3+4\,(sum)}}{(1 \oplus 2) \otimes (3 \oplus 4) \to (1 \oplus 2) \otimes 7}\ (prodR)$$

Question: Is $3 \otimes 2 \to 5$ derivable? What about $(1 \oplus 2) \otimes (3 \oplus 4) \to 3 \otimes 7$?

Specifying an evaluation order

Our rules so far specify that the left-most operand always gets evaluated first:

$$\frac{\mathsf{e}_1 \to \mathsf{e}_1'}{\mathsf{e}_0 \oplus \mathsf{e}_1 \to \mathsf{e}_0 \oplus \mathsf{e}_1'}\ (sumR) \quad \text{and} \quad \frac{\mathsf{e}_1 \to \mathsf{e}_1'}{\mathsf{e}_0 \otimes \mathsf{e}_1 \to \mathsf{e}_0 \otimes \mathsf{e}_1'}\ (prodR)$$

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With these rules $(\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4}) \to (\mathtt{1} \oplus \mathtt{2}) \otimes \mathtt{7}$ is no longer derivable.

Exercise: show that the following judgements are now derivable:

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Specifying an evaluation order — these choices can be **very** important!

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Composing evaluation steps

So far we have said what steps our programs can take, one by one.

Now we want to talk about *sequences* of those steps, and their eventual outcomes.

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Definition

We define the *multiple-step evaluation* relation $\rightarrow^\star$ as follows. We write $\mathtt{e} \rightarrow^\star \mathtt{e}'$ if either:

1. $\mathtt{e} = \mathtt{e}'$ or
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The relation $\rightarrow^\star$ is called *the reflexive and transitive closure of* $\rightarrow$.

If $\mathtt{e} \rightarrow^\star \mathtt{n}$ we say that $\mathtt{n}$ is the *final answer* of $\mathtt{e}$.

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Question: is it true that every expression has a final answer? Are there expressions with more than one final answer?

Summarising small-step operational semantics

- We describe individual computation steps using the $\to$ relation;
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This is very fine-grained, but sometimes can be more than we need.

Next we'll look at the big-step approach, where we only describe the final answer of a computation.

Big-step Operational Semantics

Big-step semantics describes the overall result of an execution. We write

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$$\frac{}{\texttt{n} \Downarrow \texttt{n}}\ (num) \qquad \frac{\texttt{e}_0 \Downarrow \texttt{n}_0 \quad \texttt{e}_1 \Downarrow \texttt{n}_1}{\texttt{e}_0 \oplus \texttt{e}_1 \Downarrow \texttt{n}}\ n{=}n_0{+}n_1\ (sum) \qquad \frac{\texttt{e}_0 \Downarrow \texttt{n}_0 \quad \texttt{e}_1 \Downarrow \texttt{n}_1}{\texttt{e}_0 \otimes \texttt{e}_1 \Downarrow \texttt{n}}\ n{=}n_0{\times}n_1\ (prod)$$

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This hides the information about evaluation order, but for our language we can prove it doesn't matter.

A benefit: much fewer rules!

Big-step operational semantics

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Evaluating operational semantics

Pros:

- Fine-grained description of program behaviour
- Makes precise our intuition about which steps the program makes when: very useful when multiple features are involved
- Quite easy to write down (e.g. you could feed the rules into a proof assistant)

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Next up: a denotational semantics for `NumExp`

Denotational semantics for `NumExp`

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Remember our original intuitive description of what `NumExp`-expressions mean:

1. Every numeral $\mathtt{n}$ is evaluated to the corresponding number n;
2. To find the value associated with an expression of the form $\mathtt{e}_0 \oplus \mathtt{e}_1$ we evaluate the expressions $\mathtt{e}_0$ and $\mathtt{e}_1$ and take the sum of the results;
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What is a `NumExp`-expression "really" representing? A natural number!

From expressions to numbers

Let's turn our intuitive idea into something precise

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1. $[\![\mathtt{n}]\!] := n$;
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Example:

$$\begin{aligned}
[\![(\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4})]\!] &= [\![\mathtt{1} \oplus \mathtt{2}]\!] \times [\![\mathtt{3} \oplus \mathtt{4}]\!] \\
&= ([\![\mathtt{1}]\!] + [\![\mathtt{2}]\!]) \times ([\![\mathtt{3}]\!] + [\![\mathtt{4}]\!]) \\
&= (1 + 2) \times (3 + 4) \\
&= 21.
\end{aligned}$$

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This defines a function $\mathit{Exp} \to \mathbb{N}$ from the set of `NumExp`-expressions to the set of natural numbers.

Here $\mathbb{N}$ is the semantic domain. By changing the semantic domain, or the denotation of expressions, we can study different kinds of properties.

Using the denotational semantics

Intuitively, the behaviour of an expression only using $\oplus$ shouldn't depend on the order of brackets.

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We can make this intuition precise:

Proposition

For all `NumExp`-*expressions* $\mathtt{e}_1$, $\mathtt{e}_2$, $\mathtt{e}_3$, $[\![(\mathtt{e}_1 \oplus \mathtt{e}_2) \oplus \mathtt{e}_3]\!] = [\![\mathtt{e}_1 \oplus (\mathtt{e}_2 \oplus \mathtt{e}_3)]\!]$.

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Proof.

$$\begin{aligned}
[\![(\mathsf{e}_1 \oplus \mathsf{e}_2) \oplus \mathsf{e}_3]\!] &= [\![\mathsf{e}_1 \oplus \mathsf{e}_2]\!] + [\![\mathsf{e}_3]\!] \\
&= ([\![\mathsf{e}_1]\!] + [\![\mathsf{e}_2]\!]) + [\![\mathsf{e}_3]\!] \\
&= [\![\mathsf{e}_1]\!] + ([\![\mathsf{e}_2]\!] + [\![\mathsf{e}_3]\!]) \\
&= [\![\mathsf{e}_1]\!] + [\![\mathsf{e}_2 \oplus \mathsf{e}_3]\!] \\
&= [\![\mathsf{e}_1 \oplus (\mathsf{e}_2 \oplus \mathsf{e}_3)]\!]
\end{aligned}$$

$\square$

Tying things together

We've seen two perspectives on `NumExp`-expressions:

1. Operational: the steps they take in running to a final answer;
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How do these relate?

A correspondence theorem

Theorem

For all expressions $\mathtt{e}$ *and numbers* n, $[\![\mathtt{e}]\!] = n$ *if and only if* $\mathtt{e} \Downarrow \mathtt{n}$.

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Base case: if $\mathtt{e}$ is a numeral, say $\mathtt{n}$, then both $[\![\mathtt{e}]\!] = n$ and $\mathtt{e} \Downarrow \mathtt{n}$.

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Base case: if $\texttt{e}$ is a numeral, say $\texttt{n}$, then both $[\![\texttt{e}]\!] = n$ and $\texttt{e} \Downarrow \texttt{n}$.
Inductive step: we suppose that $\texttt{e} = \texttt{e}_1 \oplus \texttt{e}_2$ and that the theorem holds for both $\texttt{e}_1$ and $\texttt{e}_2$, that is:

1. We assume that for all numbers k_1 and k_2, $[\![\texttt{e}_1]\!] = k_1$ if and only if $\texttt{e}_1 \Downarrow \texttt{k}_1$ and $[\![\texttt{e}_2]\!] = k_2$ if and only if $\texttt{e}_2 \Downarrow \texttt{k}_2$,
2. We prove that for all numbers n, $[\![\texttt{e}_1 \oplus \texttt{e}_2]\!] = n$ if and only if $\texttt{e}_1 \oplus \texttt{e}_2 \Downarrow \texttt{n}$.

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2. We prove that for all numbers n, $[\![\mathtt{e}_1 \oplus \mathtt{e}_2]\!] = n$ if and only if $\mathtt{e}_1 \oplus \mathtt{e}_2 \Downarrow \mathtt{n}$.

Suppose that $[\![\mathtt{e}_1 \oplus \mathtt{e}_2]\!] = n$. By definition, $[\![\mathtt{e}_1 \oplus \mathtt{e}_2]\!] = [\![\mathtt{e}_1]\!] + [\![\mathtt{e}_2]\!]$. Then $[\![\mathtt{e}_1]\!]$ and $[\![\mathtt{e}_2]\!]$ are two numbers, let's call them k_1 and k_2, that add up to n. By (1), $\mathtt{e}_1 \Downarrow \mathtt{k}_1$ and $\mathtt{e}_2 \Downarrow \mathtt{k}_2$, which by the rule *sum* of big step operational semantics means that $\mathtt{e}_1 \oplus \mathtt{e}_2 \Downarrow \mathtt{n}$.

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Proof.

1. We assume that for all numbers k_1 and k_2, $[\![\texttt{e}_1]\!] = k_1$ if and only if $\texttt{e}_1 \Downarrow \texttt{k}_1$ and $[\![\texttt{e}_2]\!] = k_2$ if and only if $\texttt{e}_2 \Downarrow \texttt{k}_2$,
2. We prove that for all numbers n, $[\![\texttt{e}_1 \oplus \texttt{e}_2]\!] = n$ if and only if $\texttt{e}_1 \oplus \texttt{e}_2 \Downarrow \texttt{n}$.

Suppose that $[\![\texttt{e}_1 \oplus \texttt{e}_2]\!] = n$. By definition, $[\![\texttt{e}_1 \oplus \texttt{e}_2]\!] = [\![\texttt{e}_1]\!] + [\![\texttt{e}_2]\!]$. Then $[\![\texttt{e}_1]\!]$ and $[\![\texttt{e}_2]\!]$ are two numbers, let's call them k_1 and k_2, that add up to n. By (1), $\texttt{e}_1 \Downarrow \texttt{k}_1$ and $\texttt{e}_2 \Downarrow \texttt{k}_2$, which by the rule *sum* of big step operational semantics means that $\texttt{e}_1 \oplus \texttt{e}_2 \Downarrow \texttt{n}$.

Conversely, suppose that $\texttt{e}_1 \oplus \texttt{e}_2 \Downarrow \texttt{n}$. This can only be proved using the *sum* rule, so there must be numbers k_1 and k_2 such that $n = k_1 + k_2$ and $\texttt{e}_1 \Downarrow \texttt{k}_1$ and $\texttt{e}_2 \Downarrow \texttt{k}_2$. By (1) then $[\![\texttt{e}_1]\!] = k_1$ and $[\![\texttt{e}_2]\!] = k_2$. Hence

$$[\![\texttt{e}_1 \oplus \texttt{e}_2]\!] = [\![\texttt{e}_1]\!] + [\![\texttt{e}_2]\!] = k_1 + k_2 = n. \qquad \square$$

Theorem

For all $\texttt{NumExp}$*-expressions* $\texttt{e}$ *and numbers* n*,* $[\![\texttt{e}]\!] = n$ *if and only if* $\texttt{e} \Downarrow \texttt{n}$*.*

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Corollary

For all $\texttt{NumExp}$*-expressions* $\texttt{e}_1$, $\texttt{e}_2$, $\texttt{e}_3$,

$$(\texttt{e}_1 \oplus \texttt{e}_2) \oplus \texttt{e}_3 \Downarrow \texttt{n} \quad \textit{if and only if} \quad \texttt{e}_1 \oplus (\texttt{e}_2 \oplus \texttt{e}_3) \Downarrow \texttt{n}$$

Theorem

For all $\texttt{NumExp}$*-expressions* $\mathtt{e}$ *and numbers* n, $[\![\mathtt{e}]\!] = n$ *if and only if* $\mathtt{e} \Downarrow \mathtt{n}$.

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Proof.

$$\begin{aligned} (\mathtt{e}_1 \oplus \mathtt{e}_2) \oplus \mathtt{e}_3 \Downarrow n \quad & \text{if and only if} \quad [\![(\mathtt{e}_1 \oplus \mathtt{e}_2) \oplus \mathtt{e}_3]\!] = n \\ & \text{if and only if} \quad [\![\mathtt{e}_1 \oplus (\mathtt{e}_2 \oplus \mathtt{e}_3)]\!] = n \\ & \text{if and only if} \quad \mathtt{e}_1 \oplus (\mathtt{e}_2 \oplus \mathtt{e}_3) \Downarrow \mathtt{n} \end{aligned}$$

□

What we've seen

For the language `NumExp`, defined by

$$\begin{array}{llll} \textit{Num} & \mathtt{n} & ::= & \mathtt{0} \mid \mathtt{1} \mid \mathtt{2} \mid \ldots \\ \textit{Exp} & \mathtt{e} & ::= & \mathtt{n} \mid \mathtt{e} \oplus \mathtt{e} \mid \mathtt{e} \otimes \mathtt{e} \end{array}$$

we've seen two semantic approaches:

1. **Operational:** the steps they take in running to a final answer (both small-step and big-step);
2. **Denotational:** the meaning of the expression, expressed as a natural number.

And we've proven these agree. This can be read in two ways:

1. Our operational semantics correctly captures the 'meaning' of programs;
2. Our denotational semantics correctly captures the way programs run;

`NumExp` is a very limited language!

What other features might we want?

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Possible examples:

- variables
- other forms of data (arrays / lists, boolean values, . . .)
- other control structures (iteration / loops, if-statements, handlers, `call-cc`, . . .)
- procedures / methods / user-defined functions
- interaction with the world: state, I/O, printing, exceptions, probability. . .
- abstract data types (e.g. to define trees, lists, . . .)
- higher-order functions
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- polymorphism

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Challenge: think about how these things work *together*!

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What other features might we want?

In what follows, we'll think about operational and denotational perspectives on some of the simpler cases.

Semantics for variables

As it stands, we can only write things like $(\mathtt{1} \oplus \mathtt{2}) \otimes (\mathtt{3} \oplus \mathtt{4})$.

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- The associativity rule we proved above
- The meaning of $a \oplus b$ should be the same as that of $b \oplus a$ whatever a and b are

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But we know some things are true no matter what numbers we put in, such as:

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- The meaning of $a \oplus b$ should be the same as that of $b \oplus a$ whatever a and b are

To understand these things properly we need variables

What is a variable anyway?

We will use variables in our language as a way to stand for any possible number
(like in high-school algebra)

This is a *functional* perspective. It might not be what you're used to!

Contrast to an *imperative* language, where a variable is a pointer to a memory cell

Adding variables to NumExp

The language $\texttt{NumExp} + \texttt{Var}$ is defined by

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\begin{array}{llcl}
\mathit{Num} & \texttt{n} & ::= & \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \ldots \\
\mathit{Exp} & \texttt{e} & ::= & \texttt{x} \mid \texttt{n} \mid \texttt{e} \oplus \texttt{e} \mid \texttt{e} \otimes \texttt{e}
\end{array}
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We assume $\texttt{x}$ stands for anything in a fixed stock of variables $\texttt{x}, \texttt{y}, \texttt{z}, \ldots$.

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Now we can write things like

$$\texttt{x} \oplus \texttt{2} \qquad \texttt{y} \oplus \texttt{2} \qquad (\texttt{7} \oplus \texttt{x}) \otimes \texttt{y}$$

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$$\texttt{x} \oplus \texttt{2} \qquad \texttt{y} \oplus \texttt{2} \qquad (\texttt{7} \oplus \texttt{x}) \otimes \texttt{y}$$

An expression $\texttt{e}$ is open if it contains a variable (e.g. $\texttt{x}$, $\texttt{2} \oplus \texttt{y}$, $\ldots$)

If it contains no variables, it is closed (e.g. $\texttt{7} \oplus \texttt{3}$)

Operational semantics for variables

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If we know the value of $\mathtt{x}$, we know what to do

But if $\mathtt{x}$ can be anything, the final result can also be lots of things

(but not anything: why not?)

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If we know the value of `x`, we know what to do

But if `x` can be anything, the final result can also be lots of things
(but not anything: why not?)

So `x` is *incomplete*: we don't know how it runs till you tell me what `x` is

The small-step rules for `NumExp` + `Var` are therefore exactly those of `NumExp`:
we give no rules for running variables

Small-step rules for $\texttt{NumExp} + \texttt{Var}$

$$\frac{}{\mathtt{n}_0 \oplus \mathtt{n}_1 \to \mathtt{n}}\ n = n_0 + n_1 (sum) \qquad \frac{\mathsf{e}_0 \to \mathsf{e}_0'}{\mathsf{e}_0 \oplus \mathsf{e}_1 \to \mathsf{e}_0' \oplus \mathsf{e}_1}\ (sumL) \qquad \frac{\mathsf{e}_1 \to \mathsf{e}_1'}{\mathsf{e}_0 \oplus \mathsf{e}_1 \to \mathsf{e}_0 \oplus \mathsf{e}_1'}\ (sumR)$$

$$\frac{}{\mathtt{n}_0 \otimes \mathtt{n}_1 \to \mathtt{n}}\ n = n_0 \times n_1 (prod) \qquad \frac{\mathsf{e}_0 \to \mathsf{e}_0'}{\mathsf{e}_0 \otimes \mathsf{e}_1 \to \mathsf{e}_0' \otimes \mathsf{e}_1}\ (prodL) \qquad \frac{\mathsf{e}_1 \to \mathsf{e}_1'}{\mathsf{e}_0 \otimes \mathsf{e}_1 \to \mathsf{e}_0 \otimes \mathsf{e}_1'}\ (prodR)$$

Examples of valid $\texttt{NumExp} + \texttt{Var}$-derivations:

$$(\texttt{1} \oplus \texttt{2}) \otimes \texttt{x} \to \texttt{3} \otimes \texttt{x} \qquad\qquad (\texttt{1} \oplus \texttt{y}) \otimes (\texttt{3} \oplus \texttt{4}) \to (\texttt{1} \oplus \texttt{y}) \otimes \texttt{7}$$

with their derivations:

$$\dfrac{\dfrac{}{\texttt{1} \oplus \texttt{2} \to \texttt{3}}\ {\scriptstyle 3=1+2\,(sum)}}{(\texttt{1} \oplus \texttt{2}) \otimes \texttt{x} \to \texttt{3} \otimes \texttt{x}}\ (prodL) \qquad \dfrac{\dfrac{}{\texttt{3} \oplus \texttt{4} \to \texttt{7}}\ {\scriptstyle 7=3+4\,(sum)}}{(\texttt{1} \oplus \texttt{y}) \otimes (\texttt{3} \oplus \texttt{4}) \to (\texttt{1} \oplus \texttt{y}) \otimes \texttt{7}}\ (prodR)$$

Big-step rules for `NumExp` + `Var`

For the big-step rules, we want to say what the eventual ouput of an expression is.

For an open term we don't know what that is. So we restrict our big-step rules to closed terms.

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Big-step rules (just as for $\texttt{NumExp}$)

$$\frac{}{\mathtt{n} \Downarrow \mathtt{n}}\ (\mathit{num}) \qquad \frac{\mathtt{e}_0 \Downarrow \mathtt{n}_0 \quad \mathtt{e}_1 \Downarrow \mathtt{n}_1}{\mathtt{e}_0 \oplus \mathtt{e}_1 \Downarrow \mathtt{n}}\ {\scriptstyle n = n_0 + n_1}\ (\mathit{sum}) \qquad \frac{\mathtt{e}_0 \Downarrow \mathtt{n}_0 \quad \mathtt{e}_1 \Downarrow \mathtt{n}_1}{\mathtt{e}_0 \otimes \mathtt{e}_1 \Downarrow \mathtt{n}}\ {\scriptstyle n = n_0 \times n_1}\ (\mathit{prod})$$

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An alternative approach would be to say variables don't run:

$$\frac{}{\mathtt{x} \Downarrow \mathtt{x}}\ (\mathit{var})$$

Denotational semantics for variables

The meaning of a closed term is a number:

$$[\![\mathsf{7} \oplus (\mathsf{3} \otimes \mathsf{1})]\!] = [\![\mathsf{7}]\!] + [\![\mathsf{3} \otimes \mathsf{1}]\!] = [\![\mathsf{7}]\!] + ([\![\mathsf{3}]\!] \times [\![\mathsf{1}]\!]) = 21$$

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What is the meaning of an open term?

It depends on the values of the variables: on the environment

Definition

An environment ρ for a $\mathtt{NumExp} + \mathtt{Var}$-expression $\mathtt{e}$ is a function assigning to each variable in $\mathtt{e}$ a natural number. Write $\mathit{Env}(\mathtt{e})$ for the set of environments of $\mathtt{e}$.

Formally: ρ is a partial function $\mathit{Vars} \rightharpoonup \mathbb{N}$ which is defined on all the variables in $\mathtt{e}$.

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So $[\![\texttt{x}]\!]$ is a *function* $\mathit{Env}(\texttt{x}) \to \mathbb{N}$

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This also depends on ρ!

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Example: given an environment ρ setting $\texttt{x} \mapsto 3$ and $\texttt{y} \mapsto 0$,
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More generally: what is the denotation of $(\texttt{3} \otimes \texttt{x}) \otimes (\texttt{x} \oplus \texttt{y})$?

Denotational semantics for `NumExp` + `Var`

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Denotational semantics for NumExp + Var

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The denotation of a $\texttt{NumExp} + \texttt{Var}$-expression $\texttt{e}$ is a function $\mathit{Env}(\texttt{e}) \to \mathbb{N}$.
For an environment ρ in $\mathit{Env}(\texttt{e})$, the denotation of $\texttt{e}$ is given by recursion:

1. $[\![\texttt{x}]\!](\rho) := \rho(\texttt{x})$
2. $[\![\texttt{n}]\!](\rho) := n$;
3. $[\![\texttt{e}_0 \oplus \texttt{e}_1]\!](\rho) := [\![\texttt{e}_0]\!](\rho) + [\![\texttt{e}_1]\!](\rho)$;
4. $[\![\texttt{e}_0 \otimes \texttt{e}_1]\!](\rho) := [\![\texttt{e}_0]\!](\rho) \times [\![\texttt{e}_1]\!](\rho)$

Denotational semantics for $\texttt{NumExp} + \texttt{Var}$

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The definition is still compositional

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$$[\![\texttt{x} \oplus \texttt{y}]\!](\rho) = [\![\texttt{x}]\!](\rho) + [\![\texttt{y}]\!](\rho) = \rho(\texttt{x}) + \rho(\texttt{y})$$

What about $\texttt{x} \otimes \texttt{y}$?

Tying things together

Proposition

For any closed expression $\texttt{e}$*, we have* $\texttt{e} \Downarrow n$ *if and only if* $[\![\texttt{e}]\!] = \texttt{n}$.

Proven in the same way as for `NumExp`

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For any closed expression $\texttt{e}$, *we have* $\texttt{e} \Downarrow n$ *if and only if* $[\![\texttt{e}]\!] = \texttt{n}$.

Proven in the same way as for `NumExp`

Something more interesting:

Proposition

For any expression $\texttt{e}$ *containing the variables* $\texttt{x}_1, \ldots, \texttt{x}_k$, *and any environment* ρ *assigning* $\rho(\texttt{x}_1) = m_1, \ldots, \rho(\texttt{x}_k) = m_k$:

$$[\![\texttt{e}]\!](\rho) = \texttt{n} \quad \textit{if and only if} \quad \texttt{e}' \Downarrow \texttt{n}$$

where $\texttt{e}'$ *is obtained from* $\texttt{e}$ *by replacing each* $\texttt{x}_i$ *by* $\texttt{m}_\texttt{i}$.

Denotational semantics for `NumExp` + `Var`

When you have variables, the meaning of programs depends
on the values of the variables.

Open expressions don't run, but we can think about their operational semantics
by thinking about how we fill in the variables.

For the denotational semantics, we think of programs as depending on a choice of
environment. So programs are now interpreted as functions.

If we want to study different properties or features, we need different models. A lot
of the work in denotational semantics is coming up with these models: makes use of
powerful tools from mathematics (algebra, topology, logic, category theory, ...)