Lecture-04-Trigonometry-and-Differentiation

Warm-up exercises

Remember, if

$$y = ax^n$$

where a is a constant and n is a real number, then

$$\frac{dy}{dx} = anx^{n-1}$$

Hence, given y below, what are the gradient functions, y'?

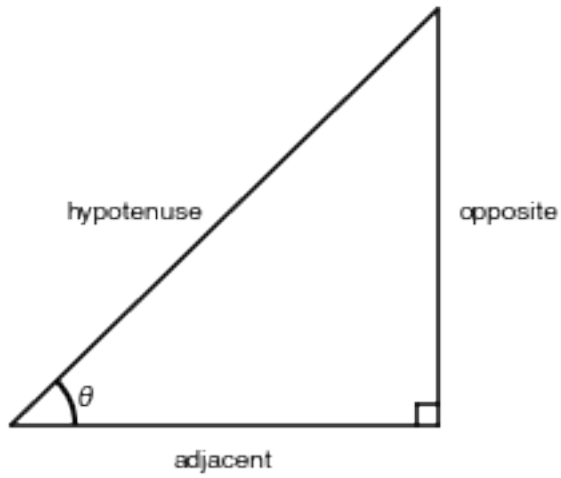
1. $y = 6 \quad y' =$
2. $y = x \quad y' =$
3. $y = 13x - 1 \quad y' =$
4. $y = 6x^2 \quad y' =$
5. $y = 5x^4 - 4 \quad y' =$
6. $y = \frac{x^5}{5} \quad y' =$
7. $y = 1 - x^2 \quad y' =$
8. $y = \sqrt{x^7} \quad y' =$
9. $y = \frac{-2}{x} \quad y' =$
10. $y = \frac{x}{\sqrt{x^3}} \quad y' =$

11. What is the concentration of 2 litres of solution containing 4 moles of solute?

Week 3 challenge: A camel has 300 bananas, and she has to eat 1 per mile as she goes along to fuel her locomotion. However, she can only carry 100 at a time. What is the maximum distance from her starting point in the dessert she can reach? Please send answers to philip.fowler@ndm.ox.ac.uk before the start of week 3.

Trigonmetric functions

You need to know these for e.g. X-ray crystallography.



Definitions

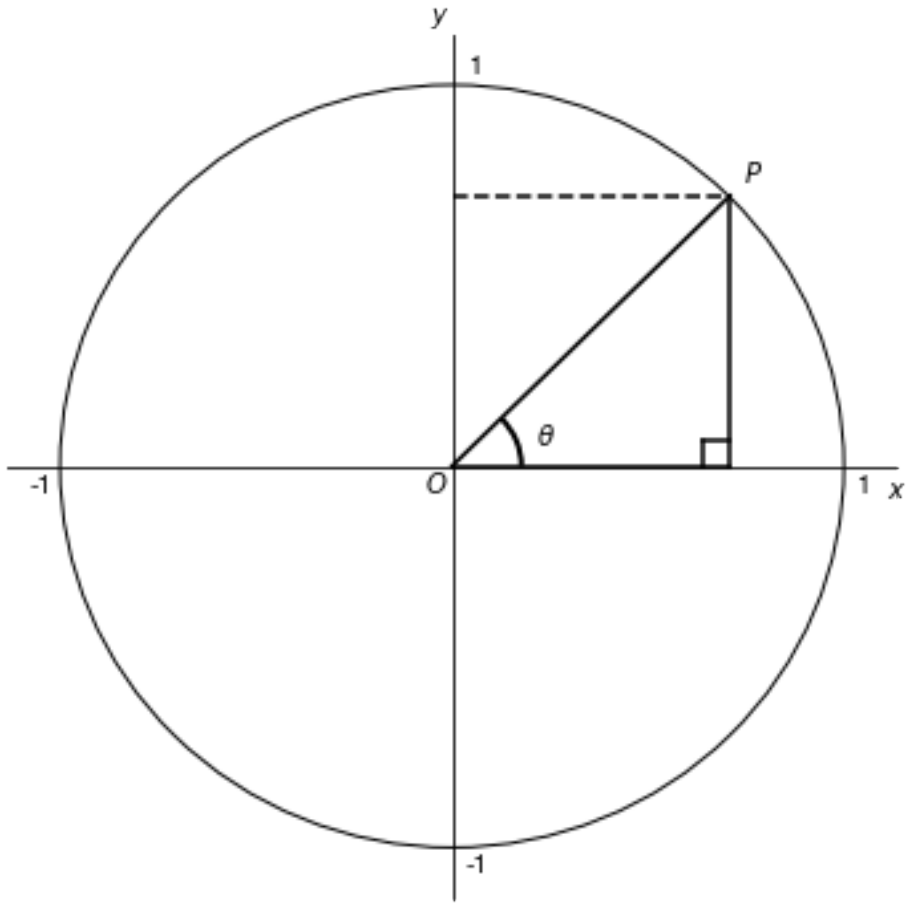
We will only mainly use the basic three functions.

$$\sin\theta = \frac{o}{h}$$
$$\cos\theta = \frac{a}{h}$$
$$\tan\theta = \frac{o}{a}$$

We can rewrite $\tan\theta$ as

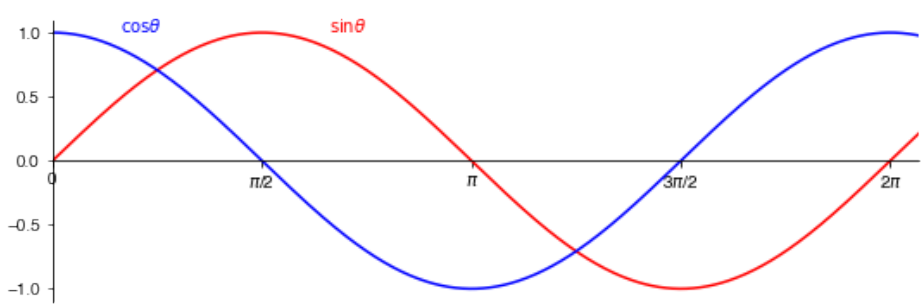
$$\tan\theta = \frac{o}{h}.\frac{h}{a} = \frac{\sin\theta}{\cos\theta}$$

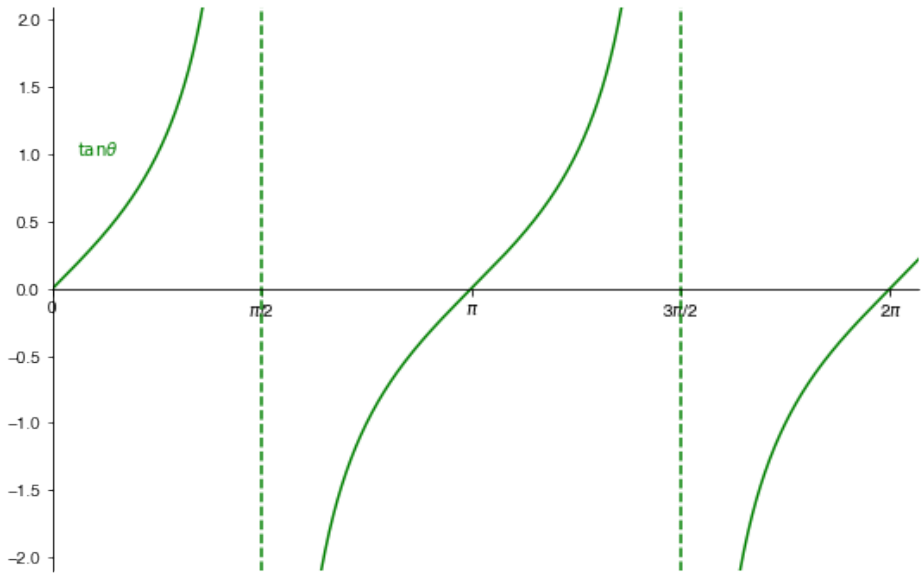
Projection onto Cartesian axes



- $\sin\theta$ is the *projection* of OP on the y-axis.
- $\cos\theta$ is the *projection* of OP on the x-axis.

Graphs





Further definitions

You can also define these trig functions which you may see. Remember you can always rewrite them in terms of the three basic functions.

$$\sec\theta = \frac{1}{\cos\theta}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

We will not use them in this course.

Inverse functions

The *inverse* function 'undoes' the mathematical operation. The *recipricol* of a function $f(x)$ is $\frac{1}{f(x)}$.

The only function for which the *inverse* and the *recipricol* are the same is, unfortunately, x, since $x \times \frac{1}{x} = 1$. Now, we can write $\frac{1}{x} = x^{-1}$.

This often leads to people getting confused about the difference between *inverse* and *reciprocol* and we can very clearly feel that here.

The inverse of $\sin\theta$ is $\sin^{-1}\theta$ (this is sometimes called $\arcsin\theta$). BUT

$$\sin^{-1}\theta \neq \frac{1}{\sin\theta}$$

Instead, the recipricol of $\sin\theta$ is

$$\frac{1}{\sin\theta} = (\sin\theta)^{-1}$$

This also explains why the -1 is placed on the name of the trigonmetric function, as if you placed it after the argument you couldn't tell the difference!

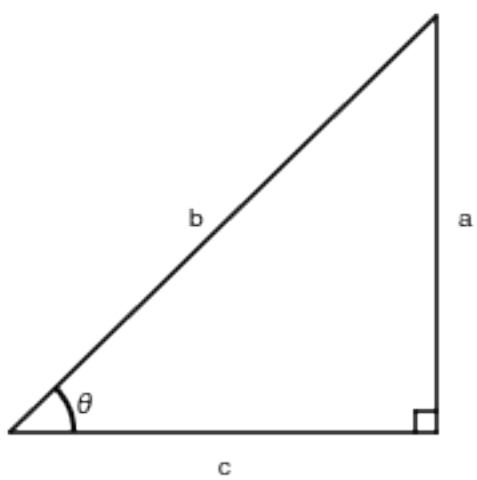
Common misconception:

The inverse of a function 'underdoes' the mathematical operation, whilst the recipricol of a function is $\frac{1}{\text{function}}$. Don't mix them up.

Note that raising a trig function to a power other than -1 doesn't suffer from this confusion, hence

$$\sin^2\theta = (\sin\theta)^2$$

Trigonmetric identities from Pythagoras's theorem



Pythagoras tells us that

$$b^2 = a^2 + c^2$$

by substituting in the definitions of the trigonmetric functions we find that

$$
\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \operatorname{cosec}^2\theta
\end{aligned}
$$

These occasionally come in useful

Radians to degrees

We will measure angles in *radians* not *degrees*. (You'll also find that trig functions in computer programming languages expect angles in radians as well).

$$
\begin{aligned}
360^\circ &= 2\pi \text{ radians} \\
180^\circ &= \pi \text{ radians} \\
90^\circ &= \frac{\pi}{2} \text{ radians} \\
1^\circ &= \frac{\pi}{180} \text{ radians} \\
&= 0.0175 \text{ radians (3s.f.)} \\
1\text{radian} &= \frac{180^\circ}{\pi} \\
&= 57.3^\circ \text{ (3s.f.)}
\end{aligned}
$$

Differentiation

We are not going to prove these results; they are

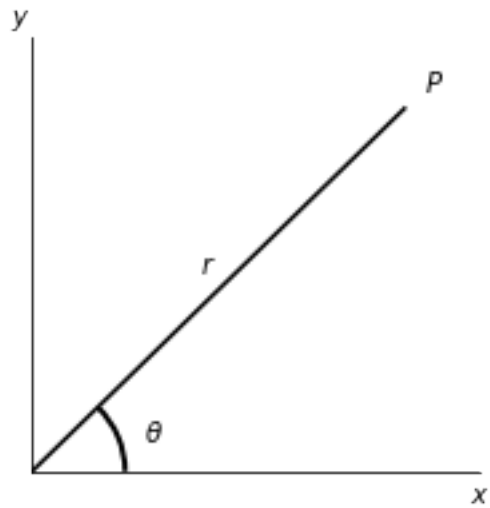
$$
\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x \\
\frac{d}{dx}(\cos x) &= -\sin x \\
\frac{d}{dx}(\tan x) &= \sec^2 x \\
&= \frac{1}{\cos^2 x} \\
&= \frac{1}{(\cos x)^2} \\
&= \frac{1}{\cos x \times \cos x}
\end{aligned}
$$

If you want to convince yourself, take a look at the graph of each trig function and think how the gradient varies.

Polar coordinates

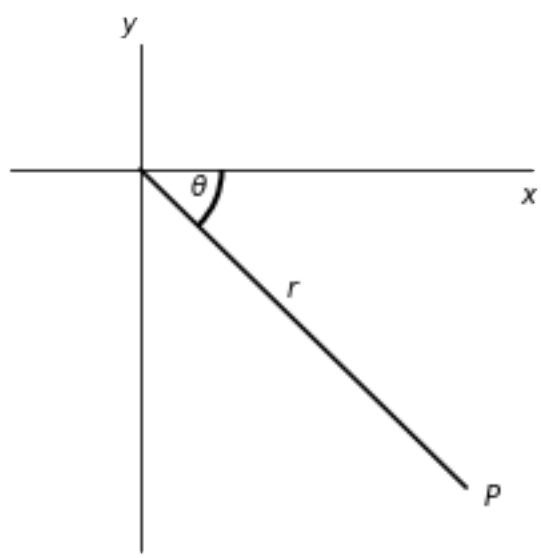
We have plotted and sketched all graphs using *Cartesian* coordinates e.g. (y, x). This isn't the only way of describing a curve; polar coordinates is another option and in some situations this makes the mathematics easier.

A point, P, is described by how far it is from the origin (r) and the angle (θ) the resulting line makes with the horizontal axis (which would be x if we were using Cartesians).



Note that θ conventionally lies between $-\pi$ and π and for positive angles is measured anti-clockwise, so in the above example $\theta = \frac{\pi}{4}$ radians.

If P lies below the x-axis, as below, then $\theta = -\frac{\pi}{4}$ radians.



Since we now know about trig functions, we can transform back to Cartesian coordinates via

$$x = r\cos\theta$$
$$y = r\sin\theta$$

or to go the other way

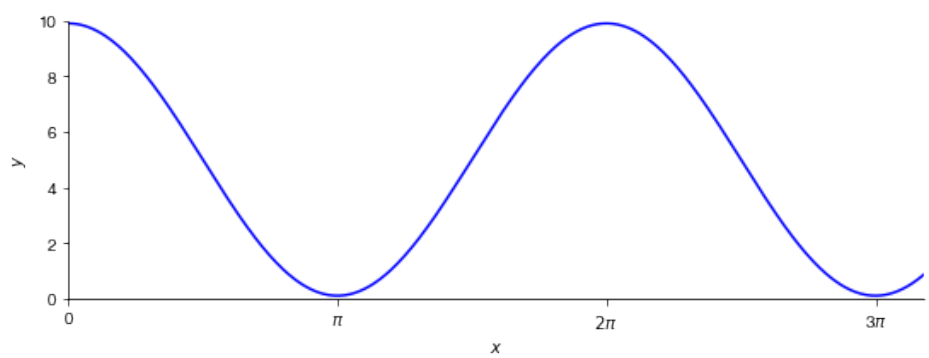
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\frac{y}{x}$$

Example

Sketch

$$y = 5 + 4.9\cos x$$

and find any turning points



To find the location of the turning points we must differentiate y with respect to x

$$\frac{dy}{dx} = -4.9\sin x$$
$$0 = -4.9\sin x$$

Because the trig functions repeat (they are *periodic*) there are an infinite number of values of x for which this is true, so typically we give the values of x where there are turning points in the first

repeating unit (first wavelength) of the function. To make this clear we can state the range of values of x in which the turning points are found.

$$\implies x = 0, \pi \quad (0 \leq x < 2\pi)$$

The linear approximation

This is a minor diversion but we'll need it shortly.

Suppose we have a function, say

$$y = 4x^2 + 5x + 6$$

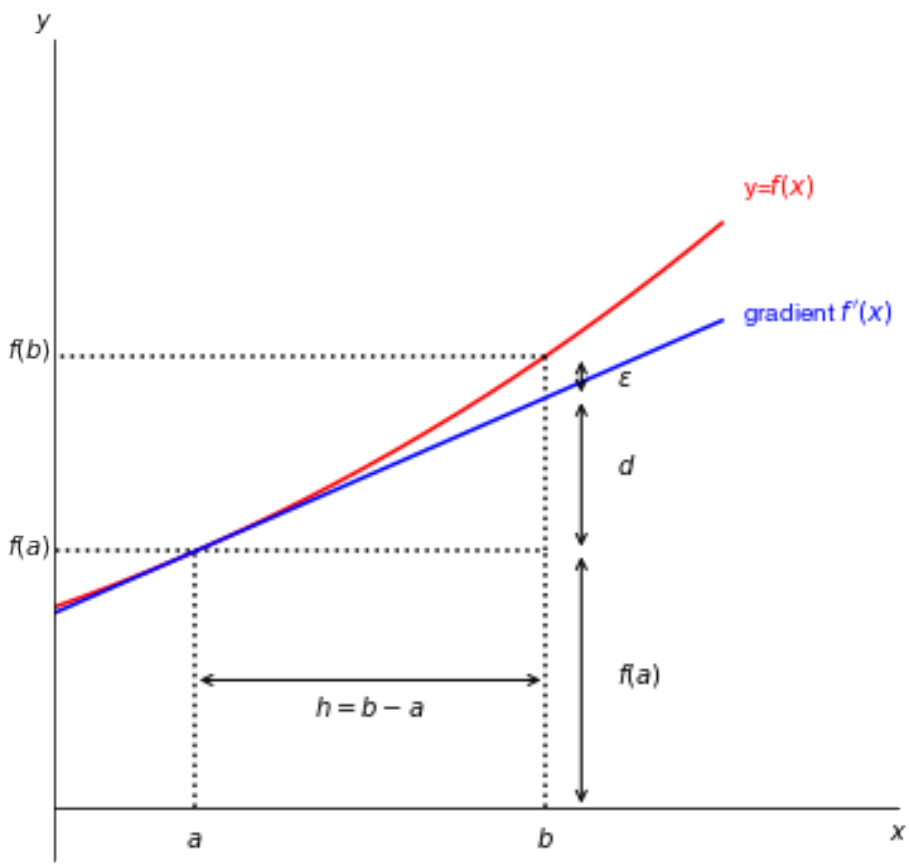
We can evaluate y at a given point e.g. $x = 2$

$$\begin{aligned} y &= 4(2)^2 + 5(2) + 6 \\ &= 32 \end{aligned}$$

Now imagine we want to evalulate y at a point close by, say $x = 2.1$. We can do this one of two ways

1. Directly evaluate the function at $x = 2.1$.
2. Make a good approximation to the answer by using our knowledge of graphs and gradients.

Although it is easy here, it isn't always possible to do option 1. So, let's see how you might do option 2. by drawing a graph.



We can see how $f(b)$ is made up of three distances:

$$f(b) = f(a) + d + \epsilon$$

but we know that

$$\begin{aligned} \text{gradient} &= f'(a) \\ &= \frac{d}{h} \\ \implies d &= hf'(a) \end{aligned}$$

$$\therefore f(b) = f(a) + hf'(a) + \epsilon$$

If h is very small then ϵ will also become small and we can neglect it. We then have

$$f(b) = f(a) + hf'(a)$$

Or more generally

$$\boxed{f(x+h) \approx f(x) + f'(x).h}$$

We can see graphically how we are assuming that if we only take tiny steps in x (a bit like zooming in) away from a known point on a function we can assume it looks like a straight line. How true this is depends of course on how rapidly the function changes with x!

This is called a linear approximation as we are assuming the gradient isn't changing.

Example

Without using a calculator what is $\sqrt{5218}$?

We know that $70^2 = 4900$ - this is our $f(a)$. First let's identify the function and then differentiate it so we can use the linear approximation above.

$$\begin{aligned} f(x) &= \sqrt{x} \\ &= x^{1/2} \\ f'(x) &= \frac{1}{2x^{1/2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Now we know that $5218 = 4900 + 318$ and hence $h = 318$. Using the result

$$f(x+h) \approx f(x) + f'(x).h$$

we can write (note where the equals sign changes to an approximately equals!)

$$\begin{aligned} \sqrt{5218} &= \sqrt{4900 + 318} \\ &\approx \sqrt{4900} + \frac{1}{2\sqrt{4900}} \times 318 \\ &\approx 70 + \frac{318}{2 \times 70} \\ &\approx 70 + \frac{318}{140} \\ &\approx 70 + 2.3 \\ &\approx 72.3 \end{aligned}$$

The exact values is 72.24! If we picked another point further away from 4900 (e.g. 5500) then we'd expect the approximation to fare worse.

Thought:

This method will over- or under-estimate depending on the local curvature of the function

For you:

Without using your calculator, estimate 5.3^3. How close are you to the true value?

Calculus (cont.)

Back to calculus: in the last lecture we had worked out how to differentiate the general function x^n and we've just seen how to differentiate the basic trignometric functions. But what about when we start to *combine* functions together?

Differentiating sums, differences and scalar multiples

The differential of a sum of functions is simply the sum of the differentials

$$(f(x) + g(x))' = f'(x) + g'(x)$$

e.g.

$$\begin{aligned} y &= x^2 + \frac{1}{x^2} \\ \frac{dy}{dx} &= 2x - \frac{2}{x^3} \end{aligned}$$

We can can see that differences follow the same pattern by substituting $-h(x) = g(x)$

If a function is multiplied by a scalar then the differential is simply the scalar multiplied by the differential of the function

$$(a.f(x))' = a.f'(x)$$

e.g.

$$\begin{aligned} y &= 6x^3 \\ \frac{dy}{dx} &= 6 \times 3 \times x^2 \\ &= 18x^2 \end{aligned}$$

Differentiating two functions multiplied together

Sometimes we may wish to differentiate two functions multipled together i.e.

$$\begin{aligned} y &= f(x)g(x) \\ \frac{dy}{dx} &= \ ? \end{aligned}$$

This is a little more complex, however sometimes we can simply rewrite the product of the functions as a series (sum) of other functions

e.g.

$$\begin{aligned} y &= (6x^3 - 1)(3x^2 + 4) && \text{multiply out} \\ y &= 18x^5 + 24x^3 - 3x^2 - 4 \\ \frac{dy}{dx} &= 90x^4 + 72x^2 - 6x \end{aligned}$$

But there is another way!

Recall that

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

But this time $y = f(x)g(x)$, hence

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \tag{2}$$

This is where we need the linear approximation result from our diversion. If we let

$$b = x + h$$

and h is small (but we are going to take the limits and let h tend to zero in a bit anyway so that is fine)

$$f(x+h) \approx f(x) + hf'(x)$$

substituting

Warning: This is the first time we've encountered heavy algebra; remember to layout the working clearly with the = signs below one another.

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{[f(x)+hf'(x)][g(x)+hg'(x)]-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x)g(x)+hf'(x)g(x)+hf(x)g'(x)+h^2f'(x)g'(x)]-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[hf'(x)g(x)+hf(x)g'(x)+h^2f'(x)g'(x)]}{h} \\
&= \lim_{h\to 0} f'(x)g(x)+f(x)g'(x)+hf'(x)g'(x) \\
&= f'(x)g(x)+f(x)g'(x)
\end{aligned}
$$

This is an important result and is called the PRODUCT RULE. Let's write in a slightly easier form

$$
\boxed{\frac{d}{dx}(fg) = f'g + fg'}
$$

Example 1

Let's try it on our earlier example since we know what the answer should be since we multiplied out the brackets and then differentiated

$$
\begin{aligned}
y &= (6x^3-1)(3x^2+4) \\
\frac{dy}{dx} &= 18x^2(3x^2+4)+6x(6x^3-1) \\
&= 54x^4+72x^2+36x^4-6x \\
&= 90x^4+72x^2-6x
\end{aligned}
$$

It works!

Now you might be thinking "well, I don't have to understand this since I can always multiply the functions out" but you can encounter functions for which that will either be hard or impossible ($y = x\sin x$?)

Tip:

In the example above (and in all below), notice how I keep the order of f and g in my working so that it matches our result above. Approaching problems the same way each time helps build 'maths memory' and helps reduce the number of mistakes you make. In very complicated questions, I'll separate off some of the working (e.g. calculating $f'(x)$ and $g'(x)$).

Example 2

$$y = (x^2 + 1)(x - 2)$$

Since this is a bit more complicated, I'm going to do some working out first before substituting back in. If I was doing this on paper, I'd do it off on the side to make it clear it was separate.

$$\begin{aligned} f(x) &= x^2 + 1 & f'(x) &= 2x \\ g(x) &= (x - 2) & g'(x) &= 1 \end{aligned}$$

Now I've got everything I need and I can use the product rule

$$\begin{aligned} \frac{dy}{dx} &= 2x(x - 2) + 1(x^2 + 1) \\ &= 2x^2 - 4x + x^2 + 1 \\ &= 3x^2 - 4x + 1 \end{aligned}$$

For you:

Multiply out the function and differentiate the terms individually.
Do you get the same answer?

Example 3

Let's make it a bit more complex, but still use a function we can multiple out and differentiate "by brute force" so we can check it works

$$y = x^{-1/2}(x^2 + 6)$$

identifying $f(x)$ and $g(x)$ and differentiating each:

$$\begin{aligned} f(x) &= x^{-1/2} & f'(x) &= -\frac{1}{2}x^{-3/2} \\ g(x) &= (x^2 + 6) & g'(x) &= 2x \end{aligned}$$

now applying the product rule:

$$\begin{aligned} y' &= -\frac{1}{2}x^{-3/2}.(x^2 + 6) + x^{-1/2}.2x \\ &= -\frac{1}{2}x^{1/2} - 3x^{-3/2} + 2x^{1/2} \\ &= \frac{3}{2}x^{1/2} - 3x^{-3/2} \end{aligned}$$

For you:

Expand the original function and differentiate and check you get the same result.

Example 4

Here is a function we *cannot* multiple out so *have* to use the product rule to differentiate.

$$y = x \sin x$$

Since this is the first time we've done a question like this, let's explicitly identify the terms and separately differentiate them to make our life easier

$$f(x) = x \qquad f'(x) = 1$$
$$g(x) = \sin x \qquad g'(x) = \cos x$$

To apply the product rule we substitute for the above terms

$$\begin{aligned}\frac{dy}{dx} &= 1.\sin x + x\cos x \\ &= \sin x + x\cos x\end{aligned}$$

Without the product rule this would have been IMPOSSIBLE.