theory_airy

September 23, 2020

1 Theory: Airy beam

[1]:
```
# from beams package ./beams/test_airy.py
from test_airy import plot_airy, plot_airy_cubic, airy_numerical,␣
↪plot_pupil_function
```

1.0.1 References:

1. Siviloglou, G. A. & Christodoulides, D. N. Accelerating finite energy Airy beams. Opt. Lett., OL 32, 979–981 (2007).
2. Siviloglou, G. A., Broky, J., Dogariu, A. & Christodoulides, D. N. Observation of Accelerating Airy Beams. Phys. Rev. Lett. 99, 213901 (2007).
3. Niu, L. et al. Generation of One-Dimensional Terahertz Airy Beam by Three-Dimensional Printed Cubic-Phase Plate. IEEE Photonics Journal 9, 1–7 (2017).
4. Vettenburg, T. et al. Light-sheet microscopy using an Airy beam. Nat Meth 11, 541–544 (2014).

1.1 The Airy beam

Folloing the work of Siviloglou et al. [[1], [2]].

The Airy beam is a solution to the paraxial equation of diffraction in 1D, and comes in the form:

$$\Phi(\xi, s) = \mathrm{Ai}(s - (\xi/2)^2) \exp(i(s\xi/2) - i(\xi^3/12))\,,$$

where Ai is the Airy function, $s = x/x_0$, $\xi = z/kx_0^2$, $k = 2\pi n/\lambda_0$, x_0 is a dimensionless coordinate.

Such an Airy beam is 'non-diffracting', however, has infinite energy.

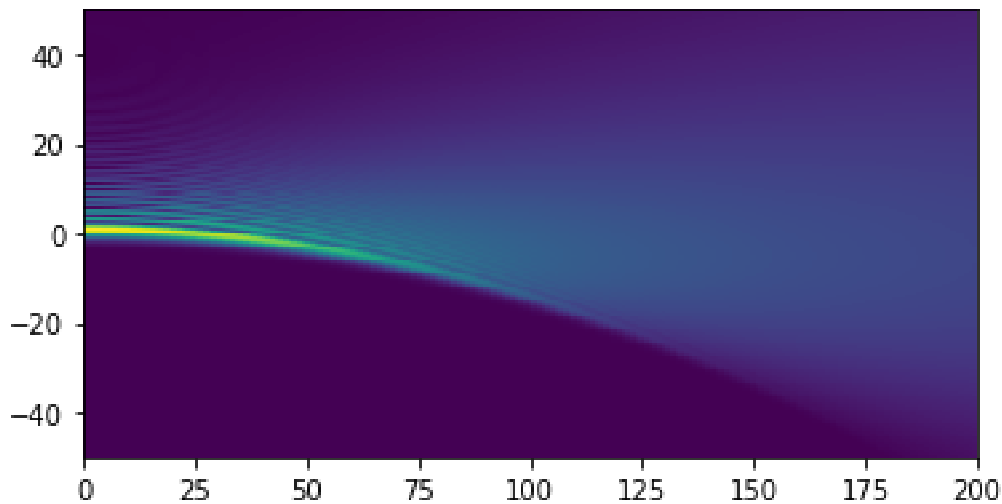
A finite energy version can be realised by constraining the intensity in the cross-section by an exponent $\exp(\alpha s)$, resulting in the following form:

$$\Phi(\xi, s) = \mathrm{Ai}(s - (\xi/2)^2 + i\alpha\xi) \exp(\alpha s - \alpha\xi^2/2 - i\xi^3/12 + i\alpha^2\xi/2 + is\xi/2)\,.$$

note that α here is not the alpha set by the cubic mask.

An example of an Airy beam is given below, where $\alpha = 0.1$, $x0 = 1e - 6$, $\lambda_0 = 488e - 9$

[2]:
```
plot_airy()
```



The question is, how does this relate to the cubic phase mask in the pupil plane and the focussing parameters set by the NA?

1.2 Airy equivalence in the pupil plane

From Fourier optics, we know that for an ideal lens, the pupil plane relates to the image plane via:

$$E(x_2, y_2) = \mathrm{FT}\{E(x_1, y_1)\}, \text{ at } (u = \frac{x_2}{\lambda f}, v = \frac{y_2}{\lambda f})$$

where f is the focal length

At the focal plane, $\xi = 0$, $\Phi(s) = \mathrm{Ai}(s)\exp(\alpha s)$. Taking the fourier transform (matlab symbolic) yeilds [[1]]:

$$E(s_2) = \exp(i/3(s_2 + i\alpha)^3)$$
$$E(s_2) = \exp(\alpha^3/3 - i\alpha^2 s_2 - \alpha s_2^2 + i s_2^3/3)$$
$$E(s_2) = \exp(-\alpha s_2^2)\exp(i s_2^3/3 - i\alpha^2 s_2 + \alpha^3/3)$$

Notably, the first exponent is a Gaussian and the second is dominated by the cubic phase evolution with spatial coordinate. We can approximate this as:

$$E(s_2) = \exp(-\alpha s_2^2)\exp(i s_2^3/3)$$

or [[3]] (however here, we note that this is strictly an approximation)

$$E(x_2) = \exp(-x_2^2/w_0^2)\exp(i\beta x_2^3)$$

where $\alpha = w_0^{-2}(3\beta)^{-2/3}$ and $x_2 = (3\beta)^{-1/3} s_2$

Note, we evaluate at $s_2 = s/(\lambda f)$, so $x_2 = (3\beta)^{-1/3} s/(\lambda f)$ (*This will be useful for numerical solutions*)

In an experiment, we can create an Airy using a cubic phase mask in the pupil, with a phase given as:

$$\phi(u_x, u_y) = \exp(i\beta(u_x^3 + u_y^3))$$

or, in 1D

$$\phi(u_x) = \exp(i\beta u_x^3)$$

Note that in Vettenburgs publication [[4]] this is scaled by λ, i.e. $\phi = \exp(i\beta u_x^3/\lambda)$ and is the 'correct' representation that yeild β values on the order of 5-10. Let's define this as $\beta' = \beta/\lambda$.

Another note: for consistency with theory, this cubic phase factor is defined as β, and α is the exponential apodisation in the Airy equation.

1.3 Verification

We can see a clear equivalence between the cubic mask and the Fourier transform of the Airy beam. To bring this to a conclusion, we can observe the similarities between:

1. the theoretical equations derived above, and summarised below
2. a numerical simulation of refocussing of a cubic phase mask

1.3.1 1

Let the pupil function be defined as a Gaussian envelope and a cubic phase:

$$E(x_2) = \exp(-x_2^2/w_0^2)\exp(i\beta' x_2^3)$$

where $\beta' = \beta/\lambda$. Then, the field at the focus (f) becomes

$$\Phi(x_1, z) = \mathrm{Ai}(s - (\xi/2)^2 + i\alpha\xi)\exp(\alpha s - \alpha\xi^2/2 - i\xi^3/12 + i\alpha^2\xi/2 + is\xi/2)\,.$$

where:

$$\begin{aligned}
\alpha &= w_0^{-2}(3\beta')^{-2/3} \\
x_0 &= \lambda f(3\beta')^{1/3}/(2\pi) \\
s &= x_1/x_0 \\
\xi &= z/(kx_0^2)
\end{aligned}$$

NOTE, the propagation invariance α is dependent only on the relationship of the Gaussian waist and the β parameter, while the f sets the 'scaling factor'. Also note, the z scaling is the square of the lateral scaling as is the case of a Gaussian beam Rayleigh range with respect to beam waist.

1.3.2 2

We verify this numerically by taking the Fourier transform of $E(x_2)$, evaluated at $u = x_1/(\lambda f)$

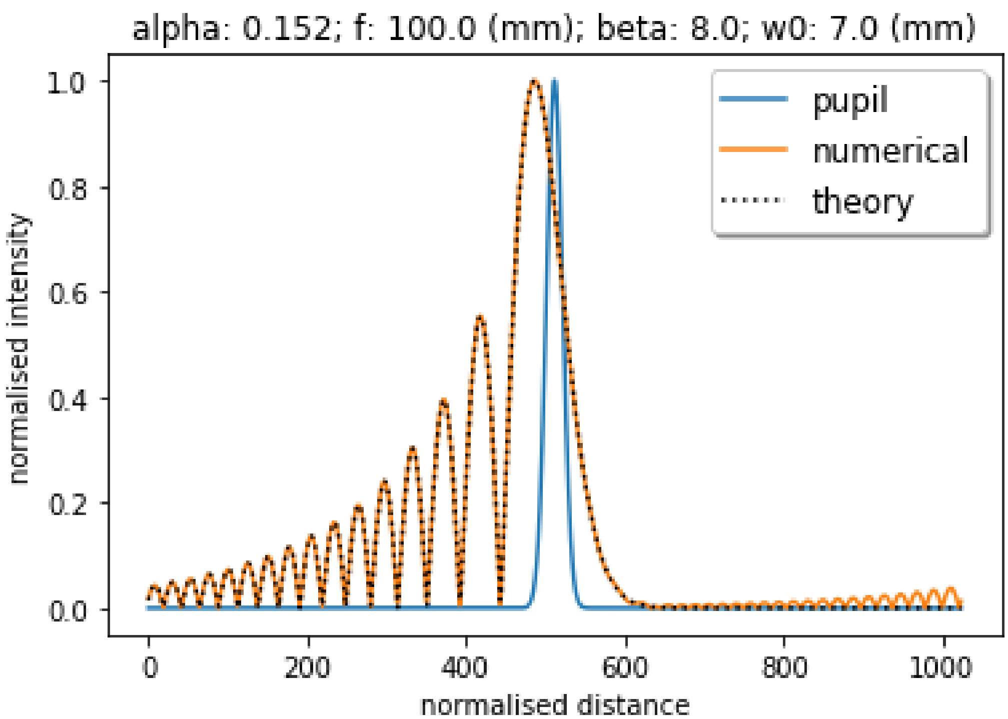
```python
# let x1 be a zero-centred coordinate vector
pixPerM = 1/(x[1]-x[0]) # sampling frequency
x2 = np.linspace(pixPerM/2, -pixPerM/2, x.shape[0])
```

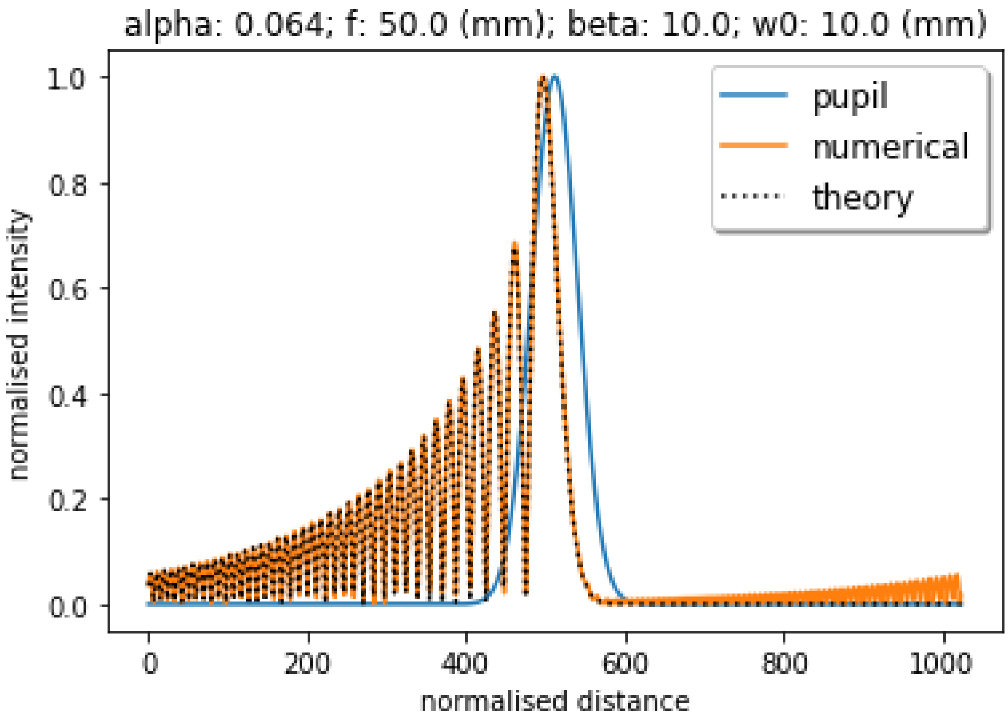
```
x2 = x2*lambda0*f

# if E2 is a function of x2, then E1 is a function of x1
E1 = np.fft.fftshift(np.fft.fft(E2))
```

1.3.3 Results

We can see that the models are equivalent. Note, the numerical model has aliasing and precision errors

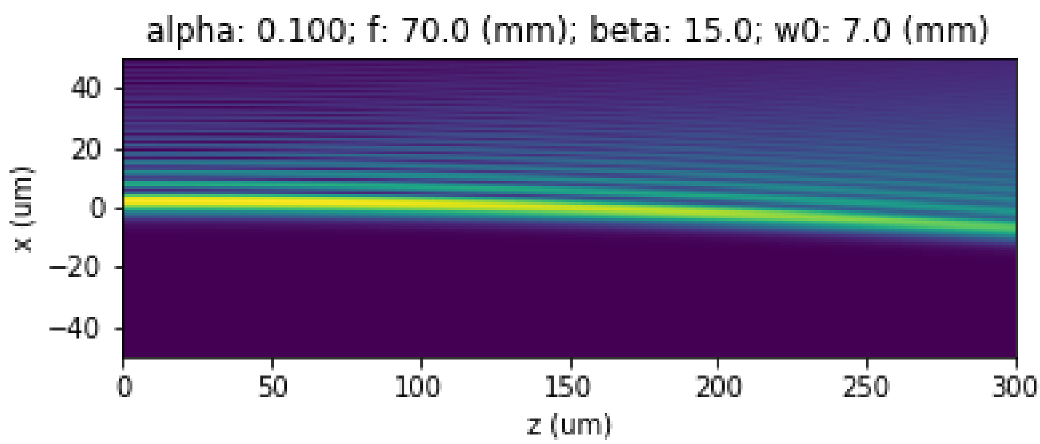
[3]:
```
airy_numerical(beta = 8, f = 100e-3, w0 = 7e-3)
airy_numerical(beta = 10, f = 50e-3, w0 = 10e-3)
```

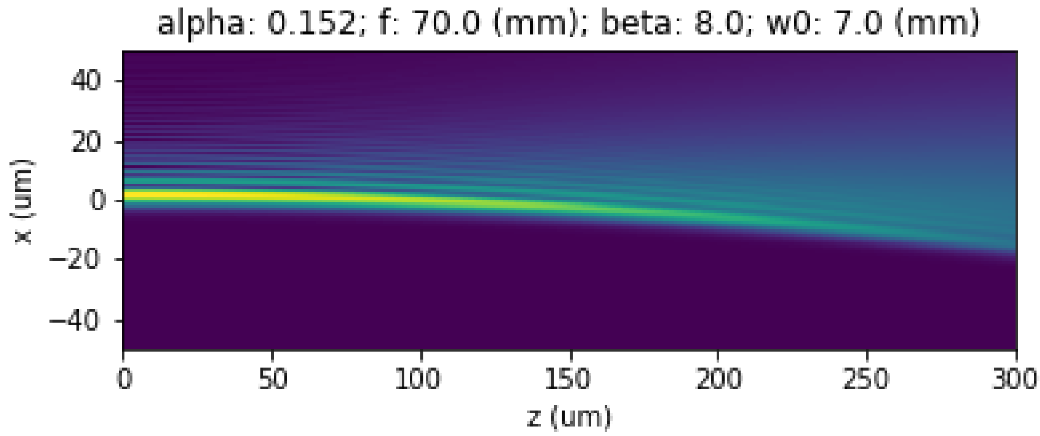


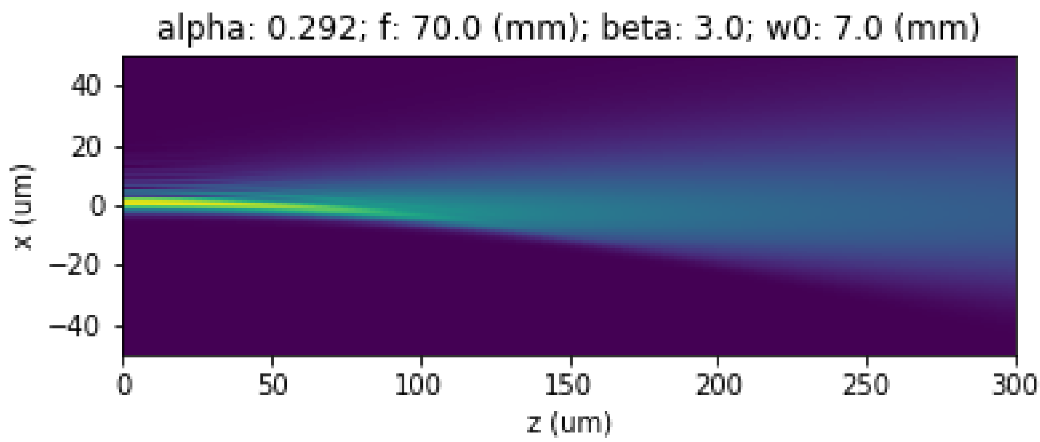


We can also look at the propagation of the Airy beam at focus with respect to the pupil function

[4]:
```
plot_airy_cubic(beta = 15, f = 70e-3, w0 = 7e-3)
plot_airy_cubic(beta = 8, f = 70e-3, w0 = 7e-3)
plot_airy_cubic(beta = 3, f = 70e-3, w0 = 7e-3)
```







1.4 Scale invariant representation

Let's consider a new representation of the pupil plane, where β is scaled by the ω_0 parameter, to provide a more intuitive and scale invariant form.

Let

$$E(x_2) = \exp(-x_2^2/\omega_0^2)\exp(i(2\pi\gamma/\omega_0^3)x_2^3)$$

where γ indicates the number of phase wraps at ω_0.

Note, in different implementations this may be referenced to the 1/e^2, FWHM, radius, etc; thus, this process has to be repeated for each case. This isn't a simple scaling, since we are referencing to a cubed value.

Thus

$$\beta' = 2\pi\gamma/\omega_0^3$$

and

$$\alpha = 1/(6\pi\gamma)^{2/3}$$

Note, this form makes α agnostic of ω_0. Similarly

$$x_0 = \lambda f(6\pi\gamma)^{1/3}/(2\pi\omega_0)$$

1.5 Scale invariant representation (HWHM)

Let's create a representation referenced to the HWHM. Here, $\omega_0 = \sqrt{2}\sigma$. Thus, $r_0 = \omega_0\sqrt{\ln 2}$

Let

$$E(x_2) = \exp(-x_2^2/\omega_0^2)\exp(i(2\pi\gamma/r_0^3)x_2^3)$$

where γ indicates the number of phase wraps at r_0.

Thus

$$\beta' = 2\pi\gamma/(\omega_0\sqrt{\ln 2})^3$$

and

$$\alpha = \ln 2/(6\pi\gamma)^{2/3}$$

Note, this form makes α agnostic of ω_0. Similarly

$$x_0 = \lambda f(6\pi\gamma)^{1/3}/(2\pi\omega_0\sqrt{\ln 2})$$