FINM 32000: Practice Final Exam Questions

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Problem 1

You have a deck of 5 cards randomly shuffled with all permutations of the 5 cards equally likely.

You know that 3 of the cards are white and 2 are black.

You sucessively reveal cards, without replacement of white cards, but with replacement of black cards. For every white card that you reveal, you get 1 dollar, and the white card is burned (not replaced in the deck). For every black card that you reveal, you pay 1 dollar, and the black card is replaced in a uniformly random location in the deck.

So, for example, if you choose to see the first card, your profit at time 1 will be either $+1$ or -1. That card will be returned to the deck, or not, depending on whether it is black or white. Then, if you choose to see a second card, your total profit at time 2 will be $+2$ or 0 or -2.

You may stop playing the game at any stopping time that you choose, that is earlier than or equal to time 3 (which is immediately after the 3rd card is revealed). This game does not allow you to stop later than time 3; therefore at least 2 cards will remain in the deck when you stop.

(a) What is the (time-0, before any cards are revealed) expectation of your final total profit, using the strategy that optimizes expected total profit?

(b) Suppose that after playing one turn, the first card is revealed to be black. In that case, what is the (time-1, after the first card is revealed) expectation of your final total profit? In that case, is it optimal to stop at time 1?

Problem 2

Let f be a smooth function mapping the underlying x to the price $f(x)$ of some contract.

(a) For general $x \in \mathbb{R}$ and $h \to 0$, find the second-order (so it includes zeroth, first, and second order terms) Taylor approximation of $f(x+h)$ in terms of $f(x)$, $f'(x)$, $f''(x)$, and h. Find the second order Taylor appoximation of $f(x+2h)$ in terms of $f(x)$, $f'(x)$, $f''(x)$, and h.

(b) Find the linear combination of $f(x)$, $f(x+h)$, and $f(x+2h)$ that produces a second-order accurate approximation of $f'(x)$. Specifically, find a, b, c such that

$$\frac{af(x) + bf(x+h) + cf(x+2h)}{h} \approx f'(x) \qquad (*)$$

where the error is (you don't need to prove this:) bounded by h^2 times a constant as $h \to 0$.

Hint:

$$f'(x) = 0 \times f(x) + 1 \times f'(x) + 0 \times f''(x)$$

Now compare coefficients.

(c) Let $C(X,t)$ be the time-t value of some contract, given time-t underlying price X. Here is a table of values of $C(X,t)$.

X=20	1.62	1.39
X=15	1.40	1.20
X=10	1.24	1.03
	t=0	t=0.1

Given underlying $X_0 = 10$, find the time-0 delta of the contract, using a second-order accurate finite difference approximation.

Given underlying $X_0 = 15$, find the time-0 gamma of the contract, the time-0 theta of the contract, and the time-0 charm of the contract, where

$$\text{Charm} = \frac{\partial^3 C}{\partial t \partial x^2} = \frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial}{\partial x}C$$

(Your t-derivative calculations do not need to be second-order accurate.)

Problem 3

Let $Y = W \times X \times Z$ where W and X and Z are independently and identically distributed random variables, each one having 50% probability of being 0, and 50% probability of being 1.

(This is a model of the Presidential election contracts[1] from last quarter, but the price data from last quarter are not consistent with the probability assumptions here, so I advise you to treat this as a separate problem.)

Suppose that someone (who fails to see that $\mathbb{E}Y$ can be directly calculated) decides to calculate $\mathbb{E}Y$ by Monte Carlo, by simulating (W, X, Z) and multiplying them together to simulate Y. Suppose he runs M simulations of Y generated in this way, in all parts (a,b,c).

(a) What is the variance of the ordinary Monte Carlo estimate?

(b) Suppose that he uses X as a control variate. What is the optimal coefficient β to multiply the control variate? What is the variance of the resulting control variate estimate?

(c) Instead of using a control variate, suppose that you use antithetic variates to estimate $\mathbb{E}Y$, by simulating pairs $(Y_m, \tilde{Y}_m)$ for $m = 1, \ldots, M$, where

$$Y_m := W_m X_m Z_m$$

and the antithetic variate is defined by

$$\tilde{Y}_m := (1 - W_m)(1 - X_m)(1 - Z_m)$$

What is the variance of your antithetic variate estimate of $\mathbb{E}Y$?

[1] where Y = `Trump.US`, and (W, X, Z) = (`Trump.GA`, `Trump.PA`, `Trump.AZ`)

Problem 4

Let $0 < T_1 < T_2 < T_3$. Assume zero interest rates.

Suppose that a non-dividend-paying stock has dynamics

$$dS_t = \sigma(t) S_t dW_t, \qquad S_0 = 100 \tag{1}$$

where W is Brownian motion under risk-neutral probabilities, and where the time-dependent but *non-random* instantaneous or local volatility function $\sigma : [0, T_3] \to \mathbb{R}$ is a step function, constant within each interval $(0, T_1]$, $(T_1, T_2]$, and $(T_2, T_3]$.

For any $T \in [0, T_3]$, let $C(T)$ be the time-0 price, and $\mathrm{ImpVol}(T)$ be the time-0 Black-Scholes implied volatility, of a European call option on S with strike 100 and expiration T.

You may, but are not required to, use Python to solve these problems. In any case, please show your calculations, and report all results to at least three significant digits (three digits, not counting leading zeros).

(a) Let $T_1 = 0.10$, let $T_2 = 0.25$, let $T_3 = 0.50$. Fill in the 6 blank spaces of the following table.

	ImpVol(T)	σ(t)	C(T)
t in $(0,T_1]$			
T = T_1	0.250		
t in $(T_1,T_2]$		0.320	
T = T_2			
t in $(T_2,T_3]$			
T = T_3			6.20

(b) Again let $T_1 = 0.10$, let $T_2 = 0.25$, let $T_3 = 0.50$.

In this part, there is not a unique way to fill in the 3 blank spaces, so let us specify which one of the many solutions to choose:

Fill in the 3 blank spaces of the following table such that ImpVol(T_2) is *as large as possible*, given ImpVol(T_1) and ImpVol(T_3).

	ImpVol(T)	σ(t)
$T = T_1$	0.250	
t in $(T_1,T_2]$		
$T = T_2$		
t in $(T_2,T_3]$		
$T = T_3$	0.360	

Problem 5

Suppose that a non-dividend-paying stock S follows Black-Scholes dynamics, with interest rate r and volatility σ.

Let $S_0 = 400$ and $T = 1/12$ and $\sigma = 0.3$ and $r = 0.01$. Calculate the strike of a T-expiry call option with delta 0.25, by completing the code in the provided `ipynb` notebook.

Unlike the homework, the where you used the inverse of the normal CDF, here you will use a root-finder (`bisect`) to solve for the strike where the delta is at the desired level.

(Why may we want to use a root finder, rather than the inverse CDF? Because if the implied volatility σ is not flat, but rather a function of strike, then the homework formula is no longer a valid expression for the strike in terms of the delta, while the root-finder approach still works.)

Submit the `ipynb` notebook that includes your code and calculation.