# FINM 32000: Homework 5

Due Friday February 25, 2022 at 11:59pm

## Problem 1

Let $r$ be the constant interest rate. Let $0 < T_1 < T_2$.

(a) Let $F_t$ be the time-$t$ forward price for $T_2$-delivery of some arbitrary underlying $S$, not necessarily tradeable. By definition of *forward price*, a forward contract paying $S_{T_2} - F_t$ at time $T_2$ has time-$t$ value 0.

Let $f_t$ be the time-$t$ value of a $T_2$-forward contract on the same underlying, but with some delivery price $K$ (not necessarily equal to $F_t$).

Express $f_t$ in terms of $K$ and $F_t$ and a discount factor.

Hint: consider a portfolio long one $(K, T_2)$-forward contract, and short one $(F_t, T_2)$-forward contract. The portfolio's time-$t$ value can be expressed in two ways, so the two expressions must be equal.

(b) If $S$ is a *stock* paying no dividends, the forward price must be $F_t = S_t e^{r(T_2-t)}$; otherwise, arbitrage would exist.

> If, say, $F_t > S_t e^{r(T_2-t)}$, then arbitrage would exist: at time $t$, borrow $S_t$ dollars, buy the stock, and short the forward (with delivery price $F_t$ and time-$t$ value 0). At time $T_2$, deliver the stock, and receive $F_t$, which is more than enough to cover your accumulated debt of $S_t e^{r(T_2-t)}$ dollars.

However, if $S$ is the spot price of a barrel of crude oil (so, for all $t$, the time-$t$ price for time-$t$ delivery is $S_t$ per barrel), then this argument fails. Explain briefly (one or two sentences, no math) why *this specific arbitrage* does not apply to crude oil, by specifically pinpointing, in the quote above, why we cannot simply replace "stock" with "crude oil".

Hint: Consider practical complications.

So we need more assumptions to relate $F_t$ and $S_t$ (here and in (c,d,e,f,g), the $S$ denotes spot crude oil, and $F_t$ denotes the time-$t$ forward price for $T_2$-delivery crude oil). One approach is to model the risk-neutral dynamics of $S$. Under risk-neutral measure, assume that $S$ satisfies

$$S_t = \exp(X_t)$$
$$\mathrm{d}X_t = \kappa(\alpha - X_t)\mathrm{d}t + \sigma \mathrm{d}W_t.$$

Then, since $r$ is constant and $\mathbb{E}_t(e^{-r(T_2-t)}(S_{T_2} - F_t))$ must be 0, one can calculate

$$F_t = \mathbb{E}_t(S_{T_2}) = \exp\left[e^{-\kappa(T_2-t)} \log S_t + \left(1 - e^{-\kappa(T_2-t)}\right)\alpha + \frac{\sigma^2}{4\kappa}\left(1 - e^{-2\kappa(T_2-t)}\right)\right],$$

where $\mathbb{E}_t$ is time-$t$ conditional expectation. Suppose $\kappa = 0.472$, $\alpha = 4.4$, $\sigma = 0.368$, $r = 0.05$, and the time-0 spot price is $S_0 = 106.9$.

Let $C$ be the time-0 price of a $K$-strike $T_1$-expiry European call on $F$. So this call pays $(F_{T_1} - K)^+$. Let the call option have strike $K = 103.2$ and expiration $T_1 = 0.5$. Let the forward mature at $T_2 = 0.75$. See the `ipynb` file.

(c) Estimate $C(S_0)$ using Monte Carlo simulation of $S$ with 100 timesteps on $[0, T_1]$. Choose the number of paths large enough that the standard error [the sample standard deviation, divided by the square root of the number of paths] is less than 0.05. Report the standard error. Don't use any variance reduction technique.

(d) Estimate $\partial C/\partial S$ by using Monte Carlo simulation to calculate $(C(S_0 + 0.01) - C(S_0))/0.01$. For the $C(S_0+0.01)$ calculation, *reuse* the same normal random variables which you generated for the $C(S_0)$ calculation. (Do not re-*generate* random variables to compute $C(S_0 + 0.01)$)

(e) Calculate analytically $\partial f_0/\partial S$, where $f_0$ is the time-0 value of a position long one forward contract on a barrel of crude oil, with maturity $T_2$ and some fixed delivery price $K$.

(f) Suppose you want to hedge a position short one call (so your hedge portfolio should replicate a position long one call), by continuously rebalancing a position in $T_2$-maturity forward contracts. Your hedge portfolio at time 0 should be long how many forward contracts? Your final answer should be a number.

The delivery price $K$ of the forward contracts is irrelevant to the answer here; it would affect only how many units of the bank account to carry in the portfolio (which I am not asking you to compute).

(g) Consider the following "purchase agreement" contract. The holder of this contract receives time-$T_2$ delivery of $\theta$ barrels of crude oil, and pays, at time $T_2$, a delivery price of $K$ dollars per barrel. The $\theta$ is chosen at time $T_1$ by the holder of the purchase agreement, subject to the restriction that $4000 \leq \theta \leq 5000$; in particular, $\theta = 0$ is not a valid choice, because the contract is a commitment to purchase at least 4000 barrels. Using your answer to (c), without running any new simulations, find the time-0 value of this contract.

Here $K, T_1, T_2$ have the same values as on the previous page.

Hint: Assume the holder acts optimally; thus $\theta$ is either 4000 or 5000, depending on $F_{T_1}$.

## Problem 2

Playing no-limit hold 'em (in a cash game, not a tournament), Jamie is all-in on the turn, and Patrik has called. Given the 8 cards that have been revealed (4 pocket, 4 on the board), Patrik is ahead, but Jamie does have 10 outs. There is no chance of a tie. If you are not familiar with poker, you can ignore all of the above, and just start reading at the next sentence:

In other words: there are 44 unrevealed cards, of which 10 would make Jamie the better hand, and the other 34 would make Patrik the better hand.

At this stage, the usual procedure is that one more card will be revealed. If the card is one of the 10 that favor Jamie, then Patrik collects 0% of the money in the pot. If it is one of the 34 that favor Patrik, then Patrik collects 100% of the money in the pot. (The dollar value of the pot has been finalized; it does not matter in this problem, what value it is.)

(a) Find the expectation and standard deviation of the fraction of the pot that Patrik will collect, when the last card (the "river") is dealt in the usual way.

(b) Suppose that, before the river is dealt, Jamie and Patrik agree to "run it three times," with replacement.[1] This means that the last card will be dealt three times, with replacement and reshuffling after each deal of that card. For each of the three iterations of the river card, whoever wins with that card gets one-third of the pot. Therefore Patrik will win either 0%, or 1/3, or 2/3, or 100% of the full pot, depending on whether 0, 1, 2, or 3 of the cards to be dealt turn out to favor Patrik.

Find the expectation and standard deviation of the fraction of the pot that Patrik will collect.

(c) Same question as (b) but *without* replacement after each deal of the river card. Is the standard deviation larger or smaller than the standard deviation in part (b), and does that make sense?

Hint for standard deviation in (c): One approach is to calculate the probability distribution of Patrik's winnings. For instance the probability that Patrik wins exactly 2/3 of the full pot is

$$\frac{\binom{34}{2}\binom{10}{1}}{\binom{44}{3}} \approx 42.36\%$$

and do likewise for the other outcomes. Then use the definition of variance.

Hint for expectation calculation in (b) and (c): $\mathbb{E}(X + Y + Z) = \mathbb{E}X + \mathbb{E}Y + \mathbb{E}Z$.

[1] Replacement is not actually done in practice. Running it twice, or three times, or four times, is conventionally done *without* replacement. So part (c) is more realistic.