FINM 32000: Final Exam Practice Solutions

March 2022

1a.

	t=0		t=1		t=2		t=3	
3							3	
						1/3		
2					2			
				1/2		2/3		
1			1				1	
		3/5		1/2		1/2		
0	0.312				0			
		2/5		3/5		1/2		
-1			-0.72				-1	
				2/5		3/5		
-2					-1.8			
						2/5		
-3							-3	

With exercise payoffs displayed in the left-hand margin, and optimized expected values in yellow, and risk-neutral probabilities in white, backward induction finds time-0 value 0.312.

1b. The time-1 optimized value is -0.72, which is greater than the exercise/stopping payoff -1, so do not stop.

2a. $f(x+h) \approx f(x) + hf'(x) + \frac{1}{2}h^2 f''(x)$ and $f(x+2h) \approx f(x) + 2hf'(x) + \frac{1}{2}(2h)^2 f''(x)$

2b.

$$\frac{af(x) + bf(x+h) + cf(x+2h)}{h} \approx \frac{(a+b+c)f(x) + (b+2c)hf'(x) + (\frac{1}{2}b + 2c)h^2 f''(x)}{h}$$

so solve the system of three equations

$$a + b + c = 0, \qquad b + 2c = 1, \qquad \frac{1}{2}b + 2c = 0$$

to find $a = -1.5, b = 2, c = -0.5$

2c. By 2b, the time-0 delta is $(-1.5 \times 1.24 + 2 \times 1.40 - 0.5 \times 1.62)/5 = 0.026$

The time-0 gamma is $(1.24 - 2 \times 1.40 + 1.62)/5^2 = 0.0024$

The time-0.1 gamma is $(1.39 - 2 \times 1.20 + 1.03)/5^2 = 0.0008$

The time-0 charm is $\frac{0.0008 - 0.0024}{0.1} = -0.016$

3a. $\operatorname{Var} Y = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = (1/8) - (1/8)^2 = 7/64$ so the MC estimate has variance $\boxed{7/(64M)}$.

3b. Optimal β by L6.5 is

$$\frac{\operatorname{Cov}(X,Y)}{\operatorname{Var} X} = \frac{\mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y)}{\mathbb{E}X^2 - (\mathbb{E}X)^2} = \frac{1/8 - (1/2)(1/8)}{(1/2) - (1/2)^2} = \frac{1/16}{1/4} = \boxed{\frac{1}{4}}$$

The optimized variance by L6.5 is

$$\frac{7}{64M}\left(1 - \frac{\operatorname{Cov}^2(X,Y)}{\operatorname{Var} X \operatorname{Var} Y}\right) = \frac{7}{64M}\left(1 - \frac{1/256}{(1/4)(7/64)}\right) = \boxed{\frac{3}{32M}}$$

3c. We have $\operatorname{Cov}(Y,\tilde{Y}) = \mathbb{E}(Y\tilde{Y}) - (\mathbb{E}Y)(\mathbb{E}\tilde{Y}) = 0 - (1/8)^2 = -1/64$, so variance of the AV estimate is $\frac{1}{2M}(\operatorname{Var} Y + \operatorname{Cov}(Y,\tilde{Y})) = (7/64 - 1/64)/(2M) = \boxed{3/(64M)}$

Comment: So antithetics do better than controls in this case, mainly because X isn't a very good control variate for Y, with a correlation of only 1/7.

4a. Using the implied volatility function from the homework, $\operatorname{ImpVol}(T_3) = \boxed{0.220}$. Now apply

$$\operatorname{ImpVol}^2(T) = \frac{1}{T}\int_0^T \sigma^2(t)\mathrm{d}t$$

repeatedly. Letting σ_1 and $\sigma_2 = 0.320$ and σ_3 denote the values of $\sigma(t)$ on the three disjoint time intervals ending at T_1, T_2, and T_3 respectively, we find $\sigma_1 = \operatorname{ImpVol}(T_1) = \boxed{0.25}$ and

$$\operatorname{ImpVol}^2(T_2) = \frac{1}{0.25}\left(\sigma_1^2{\times}0.1{+}\sigma_2^2{\times}0.15\right) \text{ and } \operatorname{ImpVol}^2(T_3) = \frac{1}{0.5}\left(\sigma_1^2{\times}0.1{+}\sigma_2^2{\times}0.15{+}\sigma_3^2{\times}0.25\right)$$

which implies $\operatorname{ImpVol}(T_2) = \boxed{0.294}$ and $\sigma_3 = \boxed{0.102}$. Plug $\operatorname{ImpVol}(T_1)$ and $\operatorname{ImpVol}(T_2)$ into Black-Scholes to find $C(T_1) = \boxed{3.15}$ and $C(T_2) = \boxed{5.86}$.

4b. We still have $\sigma_1 = \operatorname{ImpVol}(T_1) = 0.25$. Given $\operatorname{ImpVol}(T_3)$, the values of (σ_2, σ_3) which maximize $\operatorname{ImpVol}(T_2)$ would be the extreme case that $\sigma_3 = \boxed{0}$, making all of the volatility between T_1 and T_3 attributable to σ_2 and none attributable to σ_3. Solving

$$\operatorname{ImpVol}^2(T_3) = \frac{1}{0.5}\left(0.25^2 \times 0.1 + \sigma_2^2 \times 0.15 + 0^2 \times 0.25\right)$$

produces $\sigma_2 = \boxed{0.625}$ and therefore $\operatorname{ImpVol}(T_2) = \sqrt{\frac{1}{0.25}(\sigma_1^2 \times 0.1 + \sigma_2^2 \times 0.15)} = \boxed{0.509}$

5. See `ipynb` file.