FINM 32000: Homework 3

Due Friday February 11, 2021 at 11:59pm

Problem 1

Assume that the short rate (the instantaneous spot rate of interest) follows the process

$$\mathrm{d}r_t = \alpha(r_t, t)\mathrm{d}t + \beta(r_t, t)\mathrm{d}W_t$$

where W_t is Brownian motion under risk-neutral probabilities. This framework includes models such as the Vasicek and CIR models, which correspond to particular choices of the functions (α, β), but for part (a), let's leave the α and β as unspecified functions.

(a) Consider an interest rate derivative whose time-T payout has value given by some function $F(r_T)$, and whose time-t price C_t satisfies $C_t = C(r_t, t)$ for some pricing function C.

Apply Ito's rule to find the risk-neutral dynamics of C. Then set its drift equal to rC, to derive a PDE for $C(r, t)$.

Now let's take a specific choice for the functions α and β. Suppose that the risk-neutral dynamics of r are given by

$$\mathrm{d}r_t = \kappa(\theta - r_t)\mathrm{d}t + \sigma\mathrm{d}W_t,$$

with parameters $\kappa = 3$, $\theta = 0.05$, $\sigma = 0.03$. Consider a $T = 5$-year discount bond (a zero-coupon bond which pays 1 at maturity T).

(b) Write code to find the time-0 price of bond by applying a standard central-difference explicit finite difference scheme to the PDE in (a). (Therefore C_n^j will be determined by C_{n+1}^{j+1}, C_{n+1}^j, and C_{n+1}^{j-1}.)

Complete the code in the file `hw3.ipynb`.

(c) Also write code to price the bond using an explicit *upwind* approximation to $\frac{\partial C}{\partial r}$ instead of the usual central difference. Specifically, for those r_j such that $\kappa(\theta - r_j) \geqslant 0$, approximate $\frac{\partial C}{\partial r}(r_j, t_{n+1})$ using the points C_{n+1}^{j+1} and C_{n+1}^j. For those r_j such that $\kappa(\theta - r_j) < 0$, approximate $\frac{\partial C}{\partial r}(r_j, t_{n+1})$ using the points C_{n+1}^j and C_{n+1}^{j-1}. (For $\frac{\partial^2 C}{\partial r^2}$, use the usual approximation).

In (b) and (c), to approximate the PDE's rC term, use the values of r and C at node (n, j). (As we said in class, node $(n+1, j)$ would also be a natural choice, but let's choose n instead of $n+1$). At the grid's upper and lower boundaries r_{max} and r_{min}, impose for all $t < T$ the "linearity" boundary conditions

$$C(r_{max}, t) = 2C(r_{max} - \Delta r, t) - C(r_{max} - 2\Delta r, t)$$
$$C(r_{min}, t) = 2C(r_{min} + \Delta r, t) - C(r_{min} + 2\Delta r, t)$$

(This technique can help in some situations where it is not obvious what boundary conditions to use.) Thus, in each column of the grid, first solve for C in the interior nodes; then deal with the top and bottom nodes.

Now let us do some comparison of the central-difference and upwind schemes.

(d) Suppose $f : \mathbb{R} \to \mathbb{R}$ is smooth in some open neighborhood of x. Show that as $h \to 0$,
$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| = O(h) \quad \text{and} \quad \left| \frac{f(x+h) - f(x-h)}{2h} - f'(x) \right| = O(h^2).$$

using Taylor's theorem. The $O(h)$ means "some function bounded by a constant times h, near $h = 0$." Likewise, $O(h^2)$ means "some function bounded by a constant times h^2, near $h = 0$." Different instances of "O" may mean different functions. The "constants" may depend on x but not h.

(e) For all part (e) calculations: Use the grid spacings $\Delta r = 0.01$ and $\Delta t = 0.01$. Use $r_{max} = 0.35$ and $r_{min} = -0.25$ for the upper and lower boundaries of the grid, respectively.

Run a central-difference calculation and an upwind calculation of the bond price for $r_0 = 0.10$. Which is more accurate? The more accurate of the two solutions should agree, to three significant digits, with the true bond price in this model: 0.7661. The less accurate of the two solutions will be *very* inaccurate.

(f) Based on your answers to (d) and (e), insert either "greater" or "less" in each blank space in the following rule-of-thumb. No explanation necessary.

Ignoring stability issues and considering only consistency (i.e. "truncation error," also known as "local discretization error"), the upwind explicit scheme, which uses one-sided spatial differences, discretizes the PDE with _____ accuracy than the standard explicit scheme, which uses central spatial differences.

However, to actually guarantee convergence, the grid spacing must satisfy certain stability constraints. In a PDE exhibiting strong drift, we have just seen that these constraints may allow the upwind scheme _____ freedom in choosing grid spacing, compared to the standard scheme.

(g) The *continuously-compounded yield-to-maturity* of a zero-coupon bond with time-t price P_t and nonrandom face value P_T to be paid at maturity date T is defined to be
$$\frac{\log(P_T/P_t)}{T - t}$$

where log, as always for us, denotes natural log, and where $P_T = 1$ according to this problem's assumptions. One way to think of the time-t yield to maturity T is as the time-t expectation of the average of the instantaneous spot rates from time t to time T.

Find the yield-to-maturity of a 5-year discount bond, in the case that $r_0 = 0.12$, and in the case that $r_0 = 0.02$. (The "good" results from part (e) may be used here. The "bad" results should not be used, unless you want to fix them by modifying the grid spacings).

Why, intuitively, is the yield for $r_0 = 0.12$ smaller than 0.12, whereas the yield for $r_0 = 0.02$ is greater than 0.02?

Comment: Under these short-rate dynamics, there exist analytic pricing formulas for bonds. So we do not need finite difference methods to value the simple payoff that we have here. But the finite difference scheme can be adjusted to handle contracts for which exact pricing formulas do not exist.

Problem 2

Suppose that a non-dividend-paying stock has dynamics

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t, \tag{1}$$

where W is Brownian motion under risk-neutral probabilities, and where the time-dependent but *non-random* volatility function $\sigma : [0, T] \to \mathbb{R}$ is piecewise continuous and sufficiently integrable. L2.13 relates this local volatility function σ to the Black-Scholes implied volatility σ_{imp}.

(a) Are the dynamics (1) capable of generating a non-constant (with respect to T) term-structure of implied volatility? Are they capable of generating an implied volatility skew (non-constant with respect to K)? Explain briefly.

(b) Let $S_0 = 100$ and $r = 0.05$. At time 0, you observe the prices of at-the-money (this means $K = 100$) European calls at 0.1-year, 0.2-year, and 0.5-year expiries to be 5.25, 7.25, and 9.5, respectively. First find the Black-Scholes *implied* volatilities of the three options. Then find (calibrate) a time-varying *local* volatility function $\sigma : [0, 0.5] \to \mathbb{R}$ consistent with these option prices. A step function suffices (but other answers are also acceptable).

(c) Consistently with your local volatility function σ from part (d), find the time-0 price of an at-the-money European call with expiry 0.4. Do not use a tree or finite difference calculation.