Portfolio Credit Risk

University of Chicago Masters in Financial Mathematics 36702

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Lecture 4

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Conditional LGD risk

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Week 4 topics

Introduction to LGD

Four ways to model conditional LGD

LGD functions

The Frye-Jacobs LGD function

Testing the Frye-Jacobs LGD function

Motivation

You might get a job where you model conditional LGD.

- **“Stress tests" are required of big banks.**
- **“Current expected credit loss" (CECL) required of all banks.**
- **Credit default swaps or other exposures in changing conditions.**

You might get involved in risks where data is sparce.

- **Risk models are easy to overfit.**
  - **Unneeded model parameters fit the *noise* of the data instead of the *signal*.**
  - **The models make bad forecasts when the noise changes out of sample.**
- **A simpler model is less likely to be overfit.**
  - **Too bad it is harder to simplify than it is to elaborate.**
  - **Even an obvious observation can help simplify.**

Introduction to LGD: topics

Definitions

Measurement of LGD and data averages

Two expectations and three identities

Definitions

Loss *given* default

As seen earlier,

- **A borrower has the option to default, but it is expensive to do so.**
  - **Large corporate borrowers, “firms,” are our main interest.**
- **Firms default if they are unable to make timely payment on debt.**
  - **If the value of assets is greater than liabilities, the firm should be able to sell stock, sell assets, or borrow to get the cash it needs to make timely payments.**
  - **So in a default it is <u>*likely*</u> that at least one lender will not receive full value.**

LGD is the lender’s loss as a fraction of the exposure.

Loans and bonds

Most forms of debt are loans or bonds.

- **Loans are private agreements between a firm and a “bank”,**
  - **or other financial institution, hedge fund, etc.**
  - **Banks recognize a corporate default when the loan is 90 days past due.**
- **Bonds are publicly traded promises to repay on a schedule.**
  - **Bonds are considered in default if a payment is one day late.**

Each debt instrument has a defined *seniority*.

- **In bankruptcy, the seniority of a debt determines the likelihood that the lender will obtain full repayment.**
  - **The most-senior debt gets full repayment if possible.**
  - **If the firm has money is left over, the next-most-senior debt gets paid.**
  - **And so forth down the scale of seniority.**
  - **Net, more-senior debt tends to have lower LGD than junior debt.**

The scale of seniority

Loans are senior to bonds.

A firm probably has multiple bonds outstanding.

- **The names of bonds usually reflect seniority.**
  - **"Senior Debentures" would be more senior.**
  - **"Junior Subordinated Notes" would be among the least senior.**

A bankruptcy judge tends to follow "strict seniority."

- **The banks recover the largest fraction of their exposure.**
- **If there is money left over after the banks are paid, it goes to the holders of the most senior bonds.**
- **And so on down the scale of seniority.**

If there is no bankruptcy, loan recoveries remain private.

- **But the firm must act as it has promised in debt documents.**

Security

Some loans and some bonds have a "second way out":

- **The debt is secured with *collateral*.**
  - **It is much like a consumer auto loan or home loan.**
  - **In the event of default, the debt holder obtains ownership of the identified collateral asset and sells the collateral to obtain partial or full recovery.**
- **If the collateral does not provide a full recovery, the remaining exposure takes its place on the scale of seniority.**

In a given default,

- **LGD is likely to be least for a senior secured bank loan, and**
- **LGD is likely to be greatest for a junior unsecured bond.**

Looking across a large number of defaults,

- **Bank loans usually enjoy substantial recovery.**
- **A "sub" bond might a small fraction of par, on average.**

Strange things happen

A loan secured by a gasoline station defaults.

- The bank gets title to the station and the land under it.
- The gasoline tanks had been leaking.
  - Land under the station must be removed and replaced.
  - The loss to the bank was several times the amount of the loan.
  - LGD is greater than 100%.

A loan secured by a different station defaults.

- The bank itself runs the gas station for months or years.
- Surprisingly, the station becomes highly successful.
  - When the bank finally sells the station, total recovery is greater than the amount of the original loan. LGD is less than zero.

Questions? Comments?

Measurement and averages

LGD data

For a public bond, a measure of LGD is par (100%) minus the post-default price.

- You could also account for the change in the risk-free interest rate over the period from issuance of the bond to default.

Publicly available loan LGD data is scarce.

- In bankruptcies, public documents reveal what banks get.
- A few loans have been publicly rated and traded.
- Otherwise, loans are private. Only the bank knows.

A bank estimates most LGDs by discounting cash flows to the date of the default.

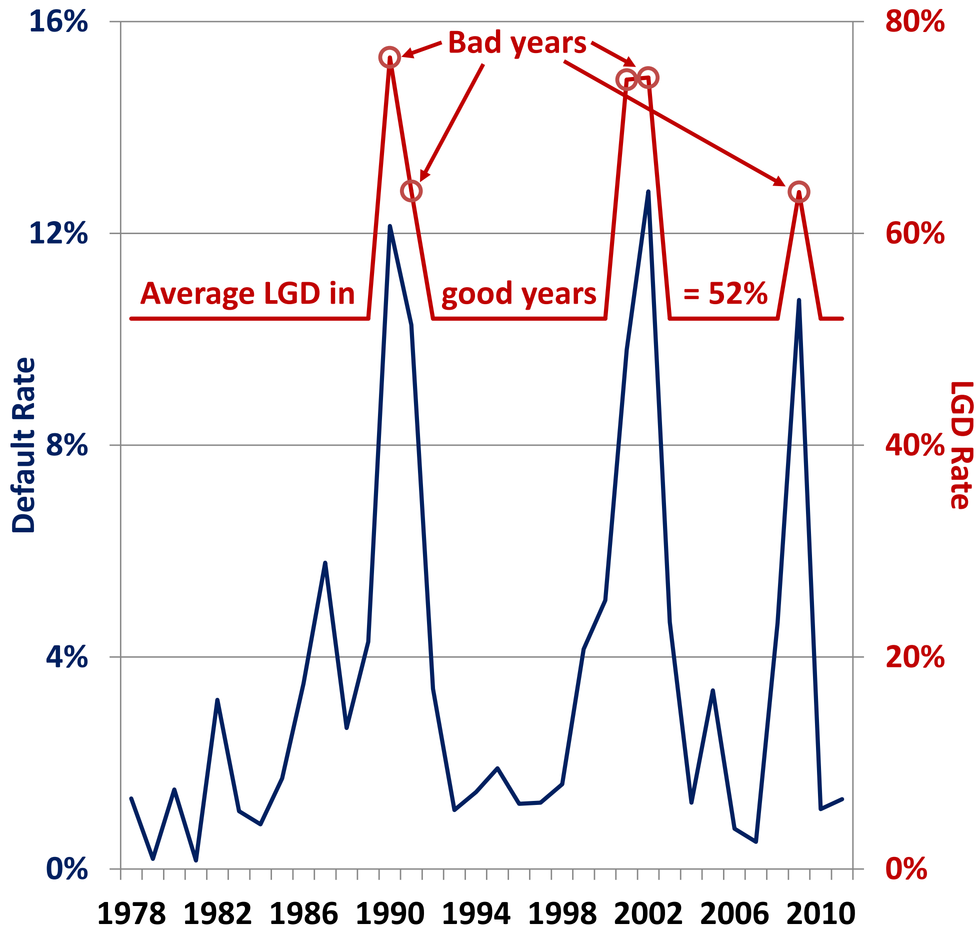
- It can take months or years to fully resolve a default.
  - Worth noting: No observable rate of discount is really appropriate.
  - There is nothing like the risk of a defaulted bank loan.

Historical bond LGDs

The history of bond LGDs shows three broad behaviors:

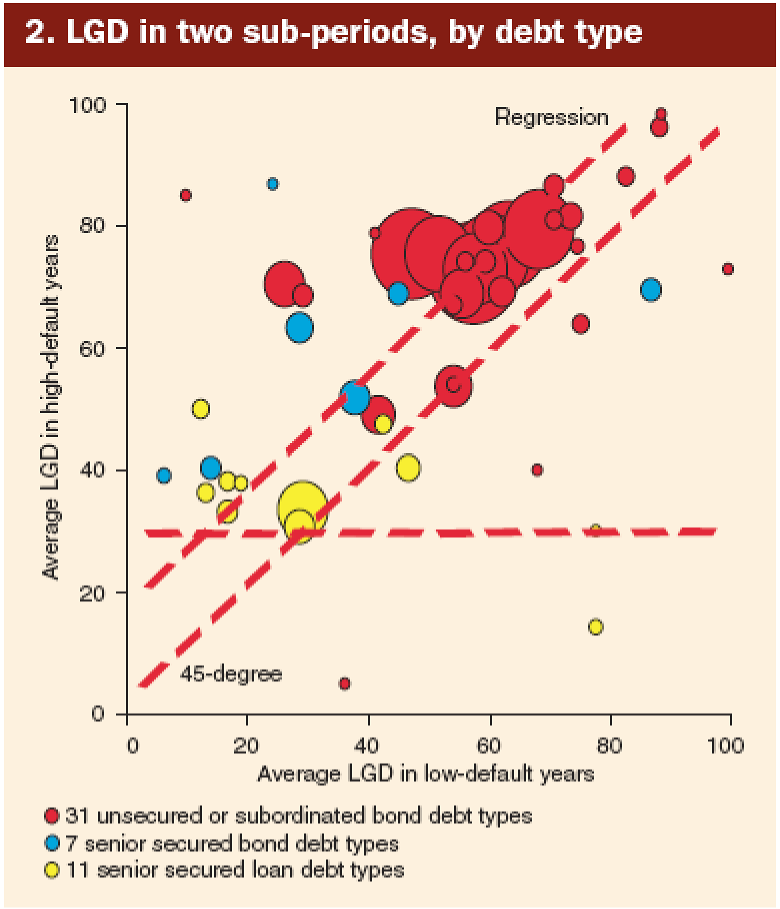
- **Average LGD is elevated when the default rate is elevated.**
- **The elevation is moderate.**
- **The elevation is similar across different seniorities and levels of security.**

Positive, moderate response



Altman's junk bond data by years

Similar response



Moody's loans and bonds

Questions? Comments?

Two expectations and three identities

Roundup of symbols

	Default	Loss given default
Unconditional expectation:	PD	ELGD
Conditional expectation:	cPD	cLGD
Observed variable:	DR	LGD

ELGD: We assume there is a good estimate for each debt.

- With enough data, the estimator could be the long-term average.

cLGD: Our main interest.

- How much should a debt holder expect to lose in a bad year?

LGD: The loss in a specific instance. A random variable.

- The standard deviation around cLGD is large for junior debt.
  - The standard deviation around ELGD is even larger.

Three identities for any loan

These slides use "loan" to signify any debt instrument.

Loss = D * LGD

- **If D = 0, then Loss = 0;   if D = 1, then Loss = LGD.**
  - **Both Loss and LGD are expressed as fractions of the exposure amount.**

EL = PD * ELGD

- **E [Loss] = E [D * LGD]**
- **= E [D] * E [LGD]**
  - **LGD does not depend on default because LGD *assumes* default.**
  - **"A conditional variable is independent of whatever it is conditioned on."**
- **= PD * ELGD**
- **Note, this identity also works for a portfolio.**
  - **It applies in turn to every debt in a portfolio because expectation is linear.**

cLoss = cPD * cLGD

- **Same proof, but now under a particular set of conditions.**

Loan expectations

Suppose a loan has two states:

	Prob	cPD	cLGD	cLoss
State 1	2/3	0.1	0.3	0.03
State 2	1/3	0.4	0.6	0.24

PD = .1 (2/3) + .4 (1/3) = .2

EL = .03 (2/3) + .24 (1/3) = .1

ELGD = EL / PD = .1 / .2 = .5

An equivalent calculation is "default-weighted LGD":

ELGD = .3 (2/3) (.1 / PD) + .6 (1/3) (.4 / PD)

= .02 / PD + .08 / PD = .1 / .2 = .5

These weights reflect probability *and* relative frequency.

Two expectations of LGD

Note: ELDG is *not* expected cLGD.

- **That would average over *conditions*.**

By contrast, ELGD is a property of a *loan*.

- **The loan might default in a bad year or a good year.**
  - **The loan is more likely to default in a bad year, and in a bad year it tends to have an elevated LGD.**
  - **The loan is less likely to default In a good year, and in a good year it tends to have a depressed LGD.**
  - **To find the average LGD for a loan, you must account for the default rate.**
- **When forming expectations about the loan's possible LGD, default in either kind of year must be accounted for.**

The next slide shows that expected cLGD would be an average over *conditions*, and this is something different.

Expected cLGD

Suppose the same loan as before:

	Prob	cPD	cLGD	cLoss
State 1	2/3	0.1	0.3	0.03
State 2	1/3	0.4	0.6	0.24

E [*c*LGD] = .3 (2/3) + .6 (1/3) = .4

- **Basel regulations refer to this as "time-weighted LGD."**
  - **Take average LGD each year, then take the average of averages.**

Note that E [cLGD] < ELGD. When cLGD is elevated, more defaulted loans are produced.

- **Averaging over *conditions* produces E [cLGD].**
- **Averaging over *loans* produces ELGD**
  - **More defaults arise when LGD is elevated.**

Say it in math

Suppose you have a distribution of cPD, $f_{cPD}[r]$.

Suppose that cLGD is a function of cPD, $cLGD = g[cPD]$.

$$E[cLGD] = \int_0^1 g[r]\, f_{cPD}[r]\, dr$$

$$E[LGD] = EL/PD = \frac{1}{PD}\int_0^1 r\, g[r]\, f_{cPD}[r]\, dr$$

Introduction to LGD: Summary

- In a default, a firm can't pay what it owes.
  - Usually, one or more of the lenders experiences loss.
    - Anticipating this, lenders want security and seniority.
- Each loan has specified Seniority (its place on the scale of seniority) and security ("second way out").
  - These influence a loan's expected LGD.
    - Still, the loss on each defaulted loan is highly random.
- Three identities for any given loan:
  - Loss = D * LGD
  - cLoss = cPD * cLGD
  - EL = PD * ELGD

Questions? Comments?

Four ways to model cLGD

Four ways to model cLGD

1. Ignore it.

2. Pretend to not ignore it, then ignore it.

3. Model it naïvely.
   - Look for strong data correlations.
     - The hope is that a strong data correlation reflects a relationship present in the population.

4. Use what you already know.
   - You'll see that every loan has an LGD function.
   - Frye and Jacobs make a particular choice.
   - The Frye-Jacobs LGD function is ***testable***,
     - and it has survived testing so far.

1. Ignore, ignore, ignore

Of the approaches to systematic LGD risk, ignoring it has the longest history and greatest popularity.

This approach makes the simulation model easy:

- **Simulate the defaults as usual.**
- **In each run, Loss = DR * ELGD; ELGD is a fixed number.**
- **Done.**

CreditMetrics© makes this slightly more sophisticated:

- **In it, LGD is random, but the distribution of LGD *does not depend* on conditions in the simulation run.**
  - **The only LGD risk comes from the randomness of a small portfolio.**

2. Pretend to not ignore

To pretend to not ignore systematic LGD risk, do this:

- **Test H_0: LGD does not respond to economic variables.**
- **Assemble data of such poor quality that H_0 is not rejected.**
- **Conclude that *there* *is* *no* systematic LGD risk.**

Note the sequence of steps:

- **H_0 is implausible.**
  - **LGD is an *economic* variable. Why should it be independent of others?**
- **Be sure to use a short, poor-quality data set.**
- **Then, conclude that the implausible hypothesis is *true*.**

As you can imagine, the people who do this have PhDs.

No one believes it anymore

When I wrote Collateral Damage (2000), there was one carefully observed downturn, 1990-91.

- **I found that LGD went up significantly in 1990-91.**
  - **Skeptics still believed that LGD is independent of other variables.**

In the tech recession (2001), LGD went up again.

- **Basel II acknowledged that LGD goes up and down.**
  - **LGD in Basel II became the confusing mess that you saw earlier.**

In the 2008 crisis, LGD went up again.

- **You already saw the LGD chart with the three spikes.**

It is now agreed that LGD goes up in times of stress.

- **I was hoping that Covid 19 would produce a downturn, but no.**

3. Model LGD naively

The naïve approach is to search for data correlations.

- **As you'll see next week, search introduces estimation bias.**
  - **The resulting model can be worse than no model at all.**

A second approach is to tie LGD to risk factor Z.

- **I was the first to do this, and others followed.**
  - **I do not recommend this, because:**

A naïve approach handles LGD and default separately.

- **The bank's *loss* model is the product.**
  - **It contains all the parameters from the default model and all of those from the cLGD model. Complicated! And, worse…**

But wait!

Loss **is a real-money number.**

- **If you lose money, you know it.**

But the definition of "default" is not tied to real money.

- **These days, a bank loan is in default if a payment is 90 days late or if there is a covenant violation like low cash flow.**

The definition of default could *change*.

- **Then, Loss, the *product* of D and LGD, would *change*.**
- **Then a bank might say, "We changed our default *definition*, so now we have less *risk*." Who would take that seriously?**

Bank supervisors did

Basel required banks to estimate loan PDs and LGDs.

- **That's when banks began to define the "default" event.**

At first, banks seemed to define default this way:

- **A historical loan was in "default" if the bank lost money.**
  - **Interns identified defaults using paper documents from long, long ago.**

But then the banks did the Basel calculation.

- **What if "default" included when payment was 90 days late?**
  - **The interns would found more, not less, defaults in the historical data.**
  - **The newly discovered defaults caused no loss to the bank.**
  - **Therefore, average historical LGD would be less…**

This change of definition *reduced* required capital…

$$K = \left[LGD \times N\left(\frac{N^{-1}(PD) + \sqrt{R} \times N^{-1}(0.999)}{\sqrt{1-R}} \right) - (LGD \times PD) \right] \times \left(\frac{1+(M-2.5)\times b}{1-1.5\times b} \right)$$

		Definition 1			Definition 2	
Loan		Default	Loss		Default	Loss
1		0	0		0	0
2		0	0		0	0
3		0	0		0	0
4		0	0		1	0
5		1	0.5		1	0.5

	Definition 1	Definition 2
	EL = .5/5 = 0.1	EL = .5/5 = 0.1
	PD = 1/5 = 0.2	PD = 2/5 = 0.4
	ELGD = .1/.2 = 0.5	ELGD = .1/.4 = 0.25
Basel formula:	$.5 * \Phi\left[\frac{\Phi^{-1}[.2] + \sqrt{.1}\,\Phi^{-1}[.999]}{\sqrt{1-.1}}\right] - .5 * .2$	$.25 * \Phi\left[\frac{\Phi^{-1}[.4] + \sqrt{.1}\,\Phi^{-1}[.999]}{\sqrt{1-.1}}\right] - .25 * .4$
	= 0.28	= 0.19

Two conclusions

1. Many definitions of default would be possible.

- **There is no definition that would be "best".**
- **The dominant definition recognizes relatively many defaults, because Basel rewards low LGD more than low PD.**
- **Supervisors can measure risk any way they want.**

2. It is absurd to imagine that choice of a definition could affect the distribution of Loss.

4. Use what you already know

You know that cLGD = cLoss / cPD.

You have the Vasicek expression for cPD.

Maybe if you had a nice expression for cLoss, you could derive a nice expression for cLGD.

That's a quick preview of the remaining slides.

Questions? Comments?

LGD functions

What would an LGD function do?

Every loan *has* an LGD function

An LGD function has three inputs

PD: The probability of default in the next 12 months

- **PD depends on the firm, its financial condition, etc.**

ELGD: The expected LGD of the loan

- **ELGD depends on seniority, security, guarantees, etc.**

cPD: The *conditional* probability of default.

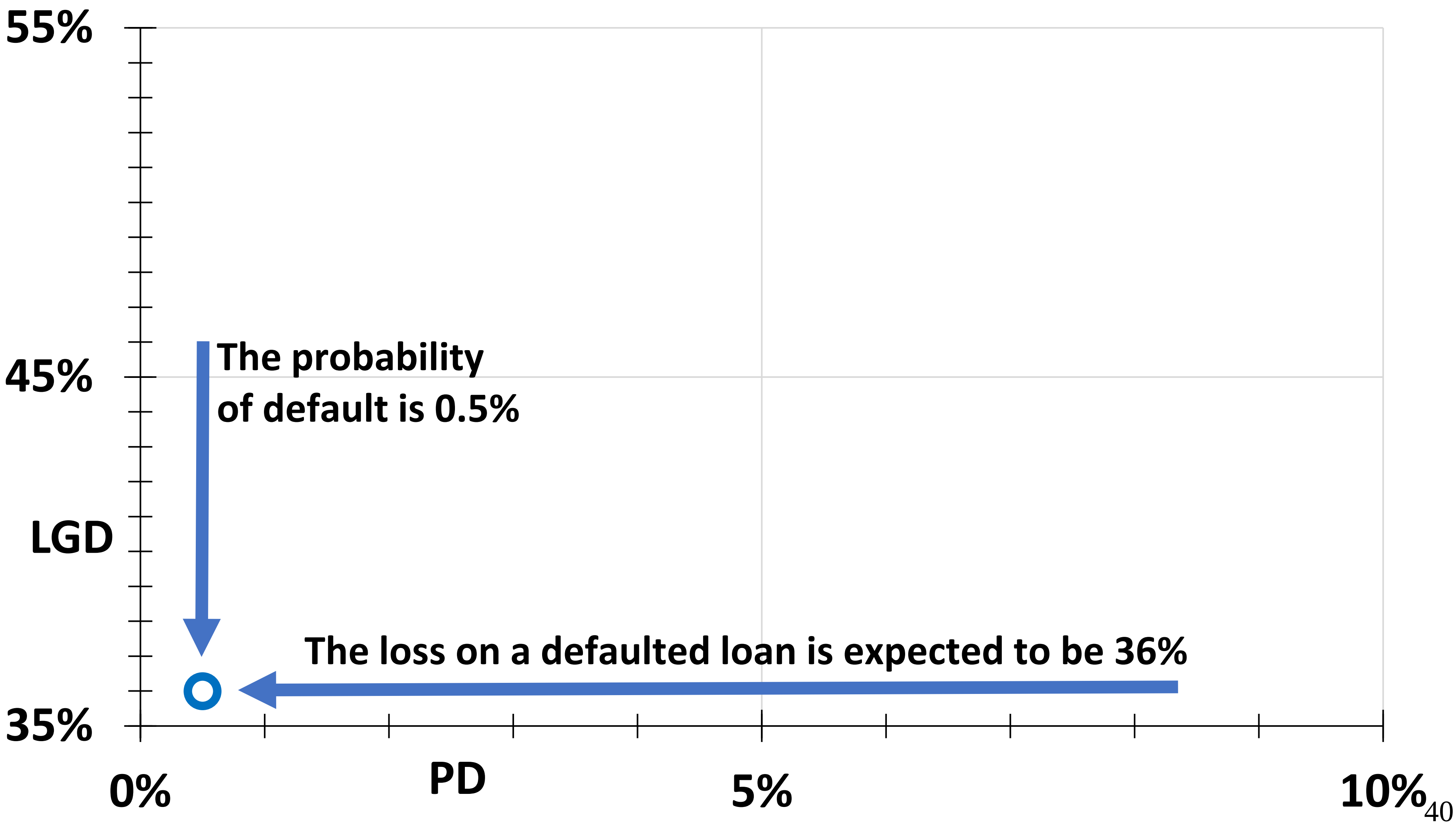
- **The conditions are those present in some scenario.**
  - **In our single-factor models, conditions are determined by Z.**
  - **In a FR stress test, conditions are defined by hypothetical values of several macroeconomic variables.**
    - **Then, cPD would be the "stress default rate."**

There could be more than 3 inputs, but at least 3.

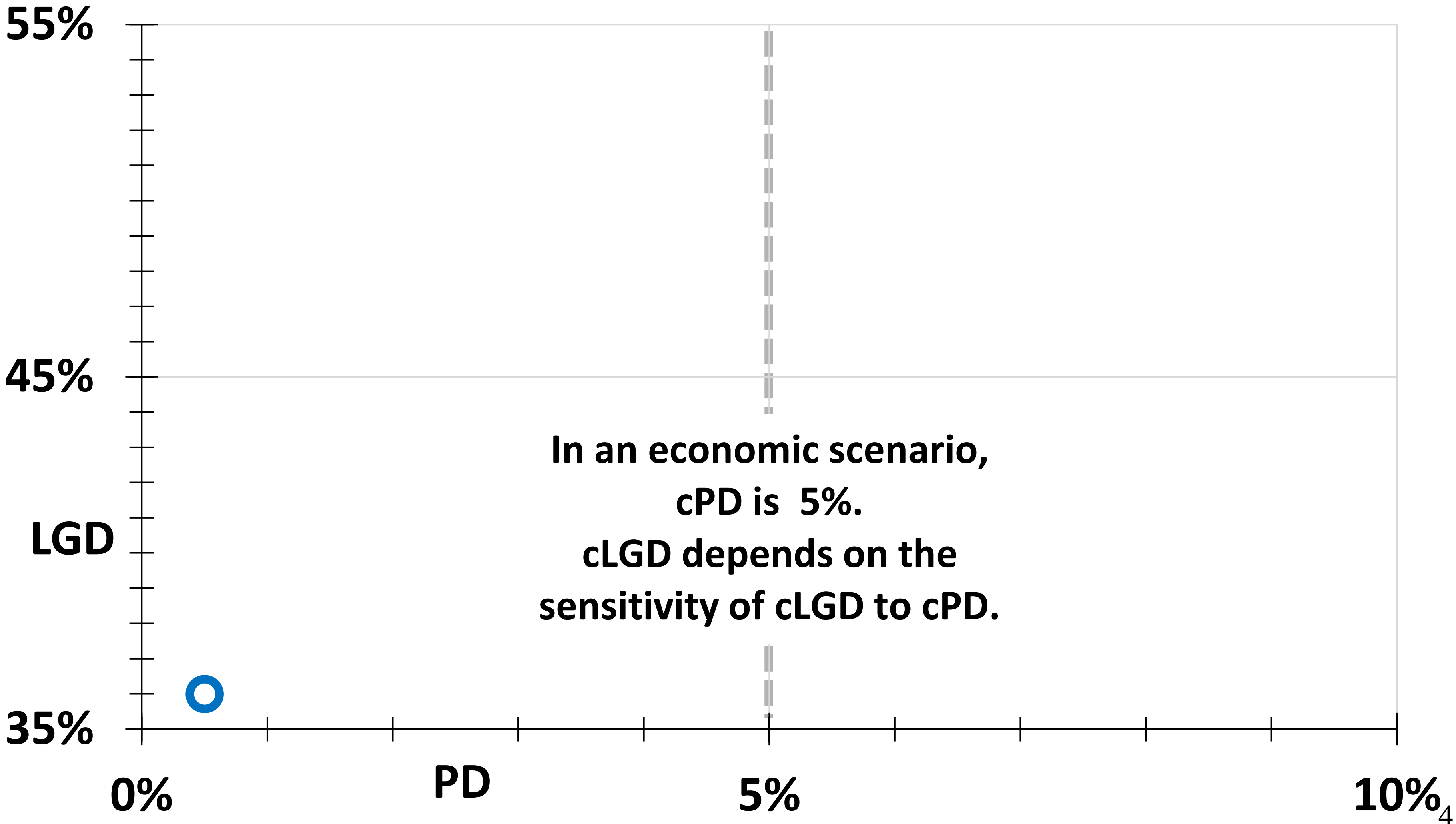
The output is *conditional* LGD, cLGD.

- **cLGD is the LGD to be expected in the conditions that produce cPD; sometimes referred to as "stress LGD."**

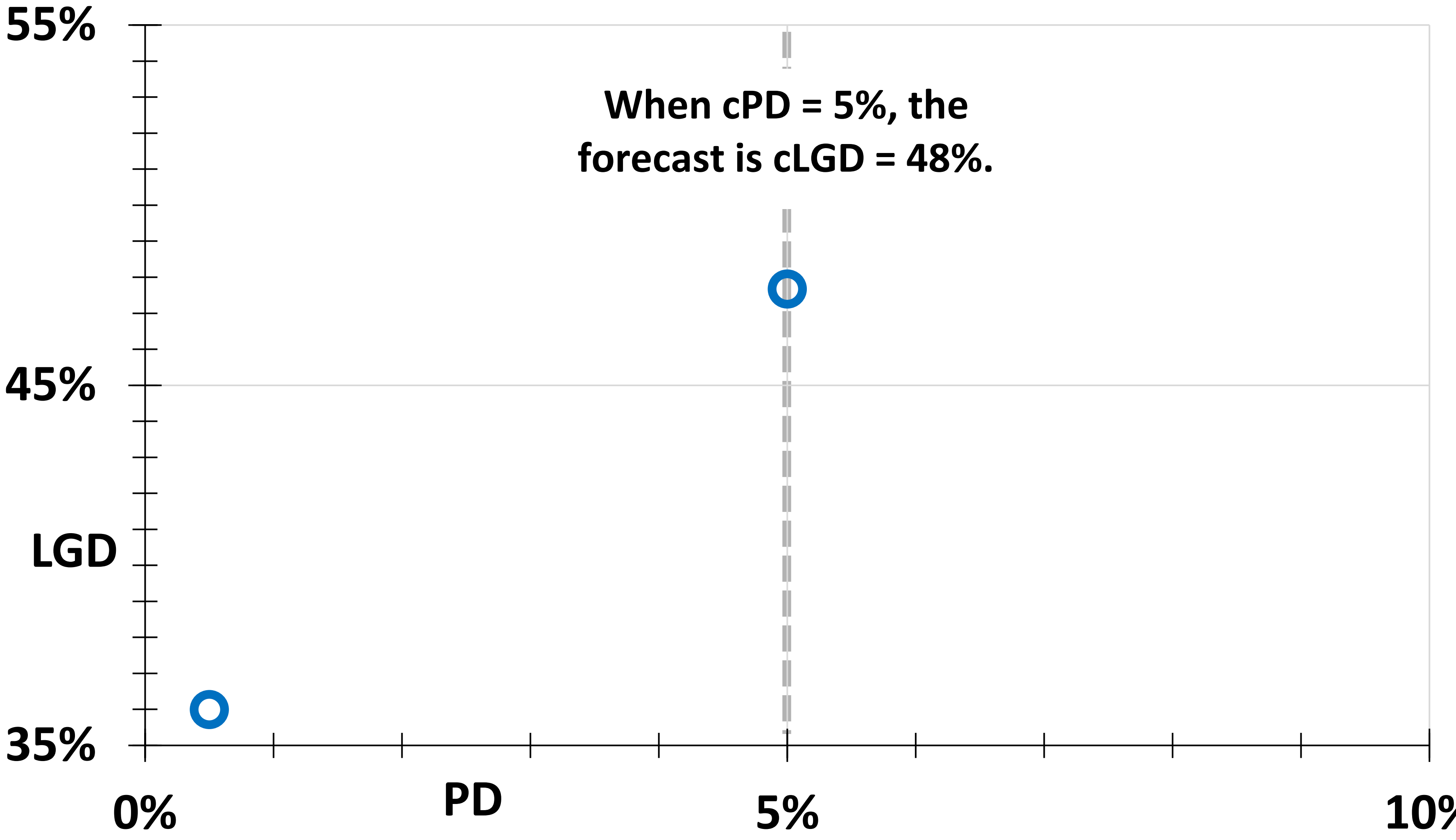
The first two inputs



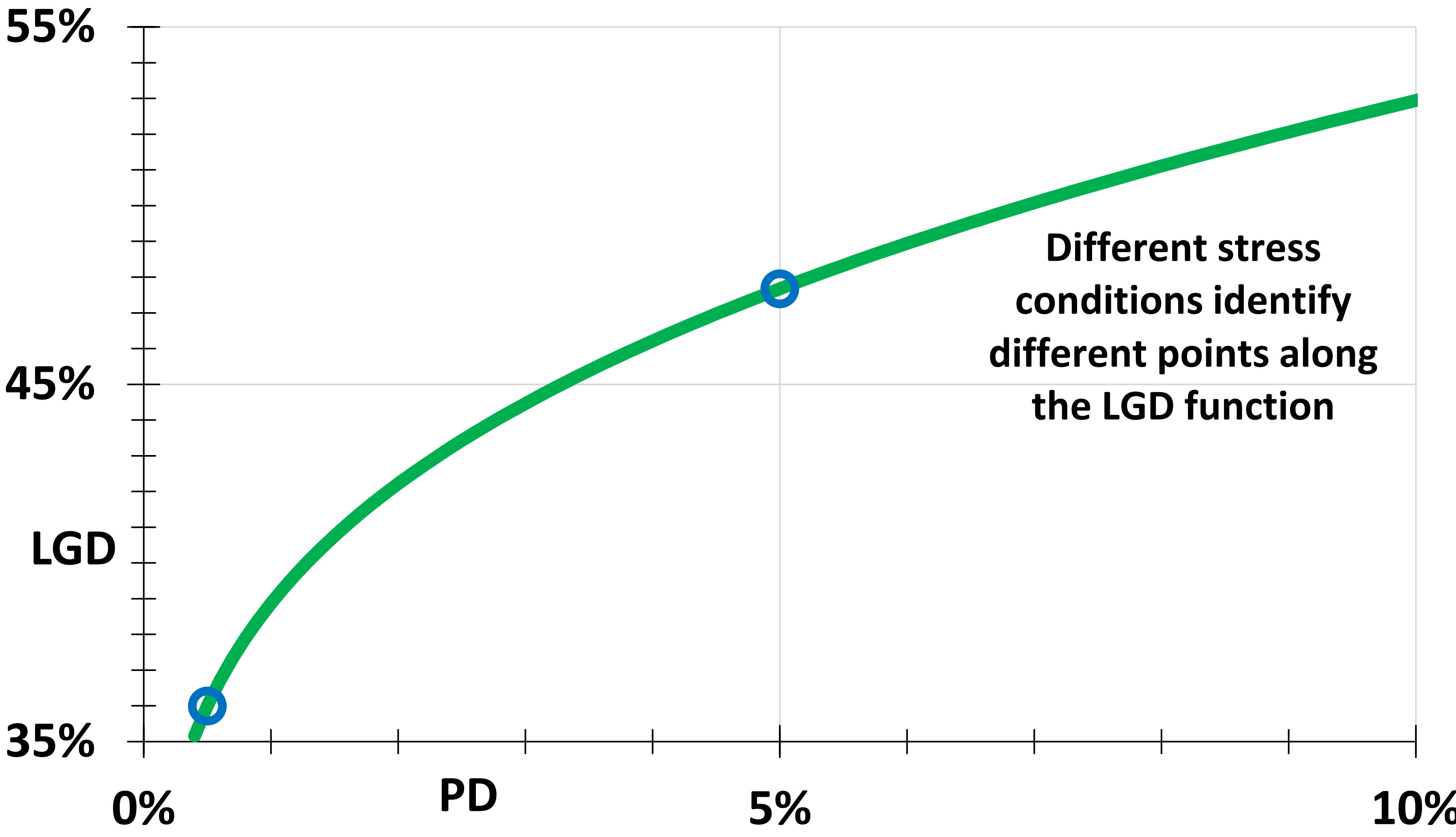
The third input



The value of the LGD function



An LGD function



Discussion?

An LGD function forecasts cLGD for a loan,
given cPD and the loan-specific inputs PD, ELGD.

The simplest LGD function would have no other inputs or parameters.

Every loan has an LGD function

The key assumption

Of two sets of economic conditions, the one having greater conditional default has greater conditional loss.

If this assumption is violated, the following can occur:

	cPD	cLoss
Scenario A	10%	4%
Scenario B	<10%	> 4%

It is hard to imagine conditions that would be expected to bring about both less default and more loss.

cPD and cLoss are assumed to go up and down together.
They are said to be *comonotonic*.

An LGD function

Comonotonicity generalizes perfect correlation.

- **If X is at its q^th quantile, then Y is at its q^th quantile.**
  - **Only difference: the locus of points can be curved.**

To calculate cLGD as a function of cPD:

- **Begin with a value of cPD. Find its quantile.**
- **Find the value of cLoss at the *same* quantile.**
  - **OK to do, because comonotonic variables.**
- **cLGD = cLoss / cPD.**

An LGD function therefore has the form:

$$cLGD\,[\,cPD\,] = CDF^{-1}_{cLoss}\,[\,CDF_{cPD}\,[\,cPD\,]\,]\ /\ cPD$$

This logic can apply to any loan.

Therefore, every loan has an LGD function.

$$\boldsymbol{cLGD[cPD] = CDF_{cLoss}^{-1}\big[CDF_{cPD}[cPD]\big]/cPD}$$

An LGD function takes a single argument, cPD.

- **The effect of a macro variable on cLGD is fully accounted for by its effect on cPD.**
- **cPD contains *all* the information relevant to cLGD.**

When we model cLGD:

- **cLGD is the ratio of cLoss to cPD**
- **cPD and cLoss are comonotonic.**
- **Therefore, cLGD is a function of one argument.**

We need not naïvely search for correlations in data.

- **Such ad hoc specification search degrades forecasts.**
  - **You will see this next week.**
- **The formula up top guarantees that cPD is sufficient.**

Questions? Comments?

The Frye-Jacobs LGD function

Derivation

Alternative hypotheses

Finite portfolios

The tests

The Frye-Jacobs LGD function

- Every loan has an LGD function, and its form depends entirely on two CDFs.

- F-J assume that cPD has a Vasicek distribution.

- F-J assume that cLoss also has a Vasicek distribution, and that the value of ρ is ***identical***.
  – The authors knew that this would reduce the number of parameters, but there was a big payoff: it worked.

- Part of the payoff is that the function is tidier.
  – If the two values of ρ are anything but identical, the function is messier…

Frye-Jacobs derivation

cPD ~ Vasicek [PD, ρ]; $F_{cPD}[cPD] = \Phi\left[\frac{\sqrt{1-\rho}\ \Phi^{-1}[cPD] - \Phi^{-1}[PD]}{\sqrt{\rho}}\right]$,

cLoss ~ Vasicek [EL, ρ]; $F^{-1}_{cLoss}[q] = \Phi\left[\frac{\Phi^{-1}[EL] + \sqrt{\rho}\ \Phi^{-1}[q]}{\sqrt{1-\rho}}\right]$

$$cLGD\ [\ cPD\ ] = F^{-1}_{cLoss}\ [\ F_{cPD}\ [\ cPD\ ]\ ]\ /\ cPD$$

$$= \Phi\left[\frac{\Phi^{-1}[EL] + \sqrt{\rho}\ \Phi^{-1}\left[\Phi\left[\frac{\sqrt{1-\rho}\ \Phi^{-1}[cPD] - \Phi^{-1}[PD]}{\sqrt{\rho}}\right]\right]}{\sqrt{1-\rho}}\right] / \ cPD$$

$$\underbrace{= \Phi\left[\Phi^{-1}[cPD] - k\right]/cPD}_{\textbf{LGD function}};\quad \underbrace{k = (\Phi^{-1}[PD] - \Phi^{-1}[EL])\ /\ \sqrt{1-\rho}}_{\textbf{Definition of } k}$$

LGD function ; Definition of k

LGD function properties

$$\mathbf{cLGD} = \boldsymbol{\Phi}\left[\boldsymbol{\Phi}^{-1}[\boldsymbol{cPD}] - \boldsymbol{k}\right]/\boldsymbol{cPD}$$

This function is monotonic in its argument cPD.

- **I found this difficult to prove.**
  - **Give it a shot and let me know!**
- **If the ρ*'s* are different, then the LGD function is non-monotonic or has other unwanted characteristics.**

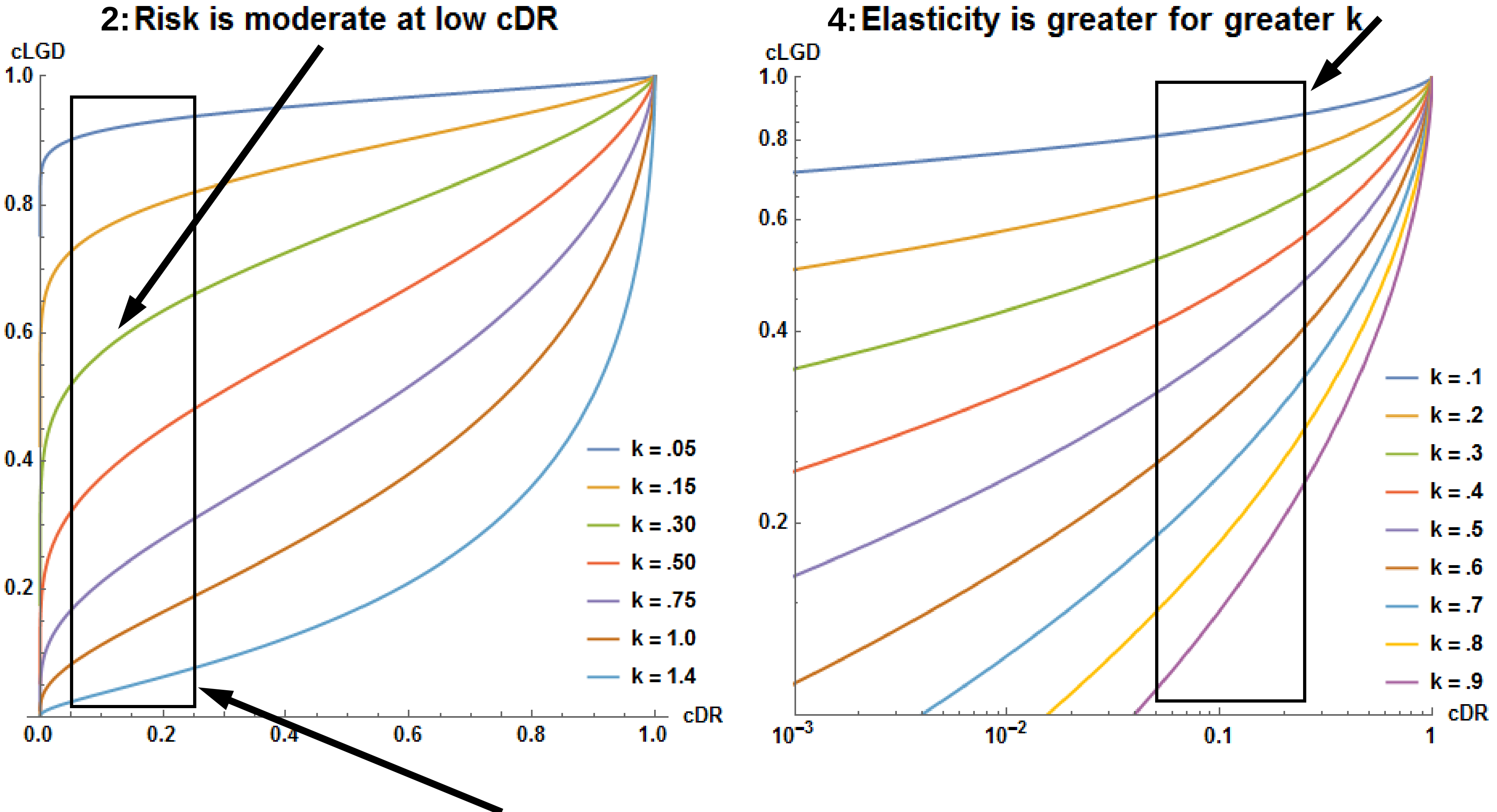
k summarizes the effects of PD, EL, and ρ:

$$\boldsymbol{k} = \frac{\boldsymbol{\Phi}^{-1}[\boldsymbol{PD}] - \boldsymbol{\Phi}^{-1}[\boldsymbol{EL}]}{\sqrt{\mathbf{1} - \boldsymbol{\rho}}}$$

- **It is worth noting that ρ has *little effect*.**
  - **E.g., if $\boldsymbol{\rho = 0.19}$, then the denominator is 0.90.**

The properties are broadly consistent with observation…

1: Bounded on unit square and monotonic



3. Slope is *similar* for all loans on 5% < cPD < 25%.

Questions? Comments?

Toward testing the LGD function

Most of the Frye-Jacobs paper is devoted to an attempt to reject the Frye-Jacobs LGD function in a hypothesis test.

- **We couldn't do it, and no one else has tried that we know.**

To perform the test requires:

- ***alternative*** **hypotheses that use**
- ***finite*** **portfolios that contain**
- **firms in *diverse* rating grades and loans with *diverse* seniority and security.**

We go through these points in order.

Alternative hypotheses

Alternative Hypothesis

The Alternative LGD function has an additional parameter. That parameter has three nice properties.

- **It controls the sensitivity of cLGD to cPD.**
  - **Sensitivity is the only thing that matters to the function, as you saw.**
- **It controls *only* the sensitivity of cLGD to cPD.**
  - **Neither expected loss (EL) nor the distribution of cPD are affected by the value of the additional parameter. Just the LGD function.**
- **At some value of the parameter, the Alternative equals F-J.**

Spoiler: When the extra parameter is fit by MLE, its value does not differ significantly from zero.

- **The simpler Null hypothesis is not rejected.**
- **That's why the F-J LGD function is considered useful.**

Alternative: Steps 0 and 1

Step 0: Let r symbolize cPD. Let $cLGD[\cdot]$ be the Frye-Jacobs LGD function. The mathematical expectation of cLoss is EL:

$$EL = \int_0^1 r\, cLGD[r]\, pdf_{cPD}[r, PD, \rho] dr$$

$$= \int_0^1 \Phi\left[\Phi^{-1}[r] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL]}{\sqrt{1-\rho}}\right] pdf_{cPD}[r, PD, \rho] dr$$

Step 1: This equation holds for any value of EL, such as ψ:

$$\psi = \int_0^1 \Phi\left[\Phi^{-1}[r] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[\psi]}{\sqrt{1-\rho}}\right] pdf_{cPD}[r, PD, \rho] dr$$

Alternative: Steps 2-4

Step 2: Multiply by ELGD^{a}, where *a* is a real number:

$$\psi\, ELGD^{a} = \int_{0}^{1} ELGD^{a}\, \Phi\left[\Phi^{-1}[r] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[\psi]}{\sqrt{1-\rho}}\right] pdf_{cPD}[r, PD, \rho]\, dr$$

Step 3: Set ψ = EL / ELGD^{a}; the left side is now EL:

$$EL = \int_{0}^{1} \underbrace{ELGD^{a}\, \Phi\left[\Phi^{-1}[r] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL/ELGD^{a}]}{\sqrt{1-\rho}}\right]} pdf_{cPD}[r, PD, \rho]\, dr$$

Step 4: The expectation of this is EL! This must be cLoss!

$$cLGD[cPD, a] = ELGD^{a}\, \Phi\left[\Phi^{-1}[cPD] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL/ELGD^{a}]}{\sqrt{1-\rho}}\right] / cPD$$

$$\mathbf{c}LGD = ELGD^{a}\,\Phi\left[\Phi^{-1}[cPD] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL/ELGD^{a}]}{\sqrt{1-\rho}}\right]/cPD$$

We just showed that E [this expression * cPD] = EL.

Therefore, this expression is an LGD function.

The next slide shows that the parameter "a" controls the sensitivity of cLGD to cPD…

$$cLGD = ELGD^a\, \Phi\left[\Phi^{-1}[cPD] - \frac{\Phi^{-1}[PD] - \Phi^{-1}[EL/ELGD^a]}{\sqrt{1-\rho}}\right]/cPD$$

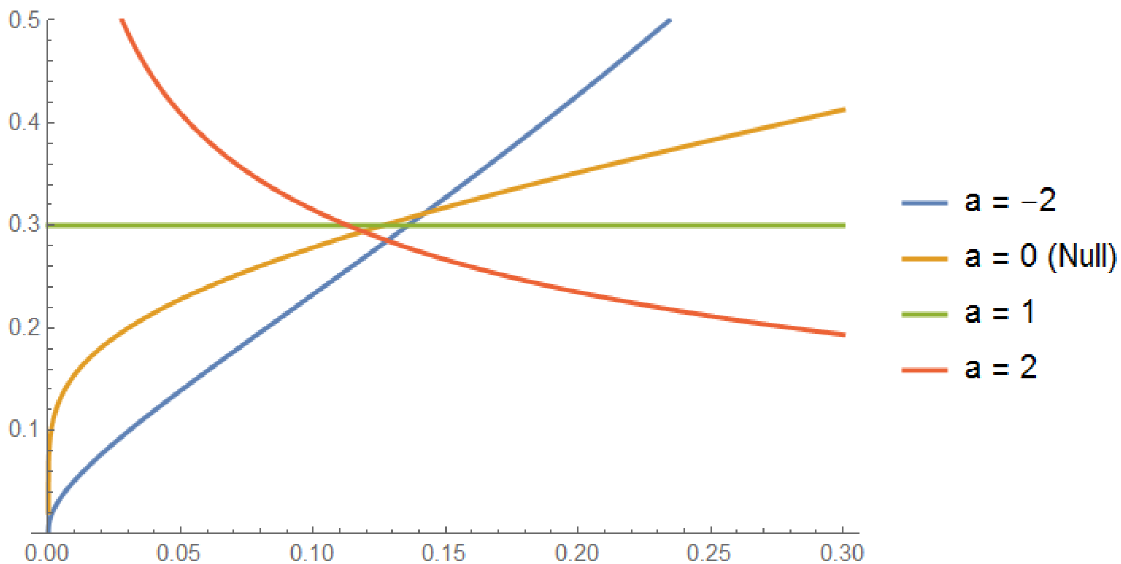
If a = 0, this is the Frye-Jacobs LGD function.

- **The Null Hypothesis nests with the Alternative.**

If a = 1, this is cLGD = ELGD.

- **cLGD can be a constant function and independent from cPD.**

Other values give a monotonic-looking function. →→→→



Summary: Alternative A

We have an alternative LGD function with the new parameter, a, which controls the sensitivity of cLGD.

- a has no effect on EL, so the function can be readily calibrated to a set of loans with a particular values of EL and PD.

We can test whether $a = 0$.

- If is significantly different from 0, we reject Frye-Jacobs.

We can test whether $a = 1$.

- If $a \neq 1$, we reject that cLGD does not depend on conditions.
    - One can't assume that an implausible hypothesis is true, simply because the available data don't allow rejection in a particular model framework.
      I can't reject the null hypothesis that vaccines are worthless, but it hardly proves that they are.
    - The data confidently reject H_0: cLGD does not depend on conditions.

Questions?

Finite portfolios

Finite portfolio

A finite portfolio introduces randomness into the default rate and into average LGD.

We assume that the finite portfolio is *uniform*.

- **All loans have equal PD, all pairs have equal ρ, copula = Gauss.**

Given cPD, the number of defaults is *Binomial*.

  - **Same as when we derived the PMF of the number of defaults on Week 1.**

We assume each LGD is normally distributed around cLGD:

- $LGD_i \sim N\left[\, cLGD[cPD, a], \sigma^2 \right]$**; note the Alternative LGD function.**
- **We assume σ = 20%.**
- **Normality is convenient because we take averages.**
  - **The average of two normal variables is a normal variable.**
  - **Among useful distributions, only the normal has this property.**

Symbols

We are deriving the distribution of portfolio credit loss. Given the name of this course, it is about time.

We define these symbols:

- **N: The number of firms in the portfolio; one loan per firm.**
- **D: The number of defaults among the N firms.**
- **LGD: The *<u>average</u>* LGD rate among the D defaults.**
- **Loss: The portfolio loss rate.**
  - **Loss = LGD * D / N.**
- **cLGD and cPD are conditional expectations as always.**

If there is no default, there is no loss…

Point mass at zero loss

The probability of zero defaults among N loans is:

$$\int_0^1 (1 - cPD)^N \, pdf_{cPD}[cPD] \, dcPD$$

where $pdf_{cPD}[cPD]$ is the Vasicek PDF.

Example calculation: If N = 10, PD = 0.1, ρ = 0.15, then the probability of zero defaults is 0.431.

– **Try it and see!**

Loss when D > 0

We seek the distribution of Loss for a portfolio of N loans that has D > 0 defaults. We assume that the LGD of each defaulted loan is normal:

$$\boldsymbol{LGD_i \sim \mathrm{IID}\ N\,[\,cLGD[\,cPD, a\,]\,, \sigma^2]}$$

Then, portfolio average LGD is also normal:

$$\boldsymbol{LGD \sim N\,[\,cLGD[\,cPD, a\,]\,, \sigma^2/D]}$$

This is the distribution of portfolio average LGD,

- ***<u>conditioned</u>*** **on the random number of defaults, D, and**
- ***<u>conditioned</u>*** **on random cPD.**

Conditioned on D and cPD…

Infer the distribution of Loss from the distribution of LGD:

$$LGD|cPD, D \sim N\,[\,cLGD[\,cDR, a\,]\,, \sigma^2/D]$$

$$LGD = cLGD[cPD, a] + \frac{\sigma}{\sqrt{D}} Y, \quad Y \sim N[0, 1]$$

$$Loss = \frac{D}{N} LGD = \frac{D\, cLGD[cPD, a] + \sqrt{D}\,\sigma\, Y}{N}$$

Invert: $$Y = \frac{N\, Loss - D\, cLGD[cPD, a]}{\sigma\sqrt{D}}\, ;\, \frac{\partial\, Y}{\partial\ Loss} = \frac{N}{\sigma\sqrt{D}}$$

$$pdf_{Loss|D,cPD}[Loss] = \frac{N}{\sigma\sqrt{D}}\,\phi\left[\frac{N\, Loss - D\, cLGD[cPD, a]}{\sigma\sqrt{D}}\right]$$

$$pdf[Loss]$$

$$= \int pdf[Loss, cPD] \; dcPD$$

$$= \int pdf[cPD] \; pdf[Loss|cPD] \; dcPD$$

$$= \int pdf[cPD] \sum_{D=1}^{N} pdf[Loss, D|cPD] \;\; dcPD$$

$$= \int pdf[cPD] \sum_{D=1}^{N} pdf[Loss|D, cPD] \; pdf[D|cPD] \;\; dcPD$$

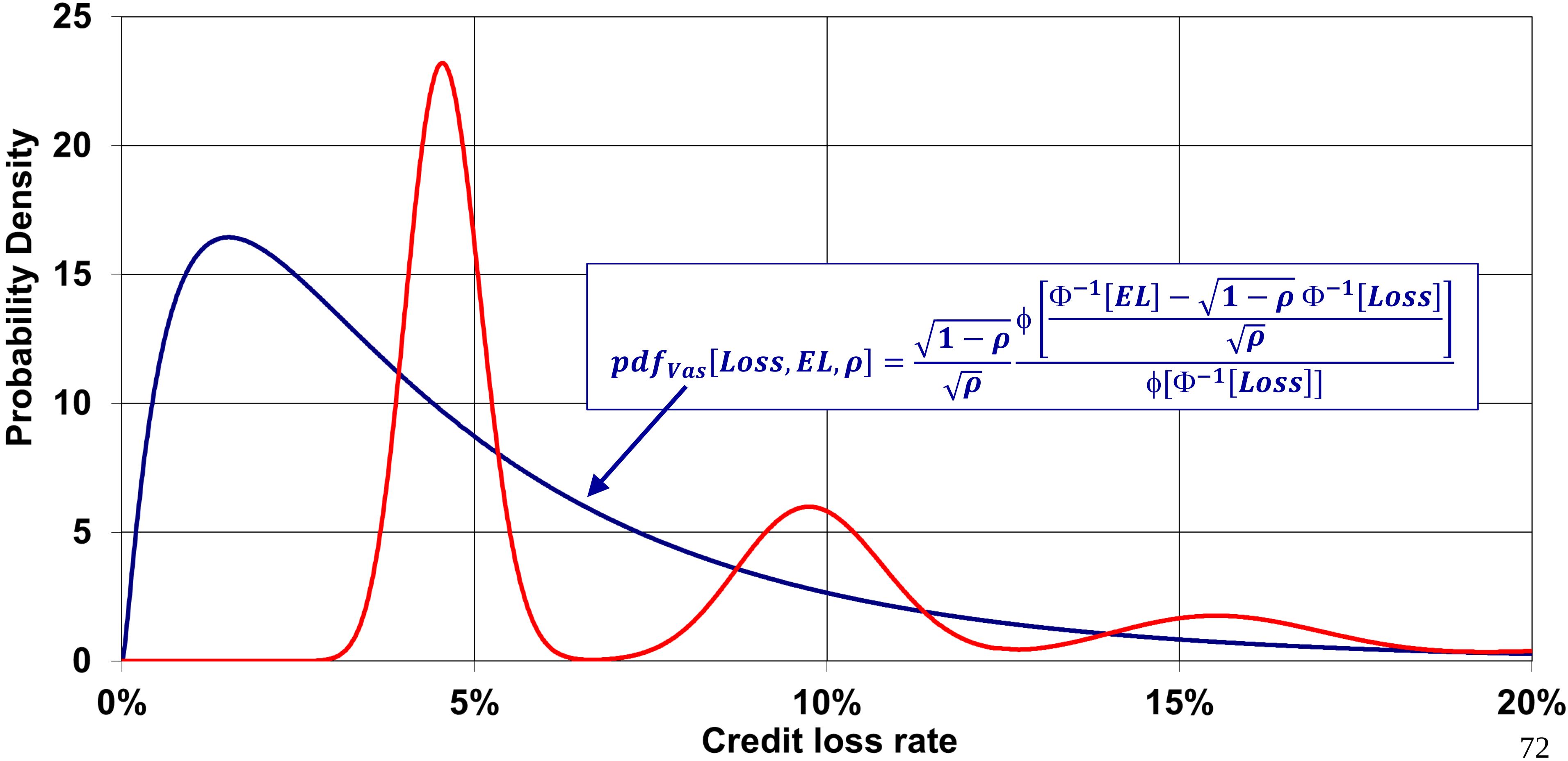
$$= \int pdf_{Vas}[cPD] \sum_{D=1}^{N} \frac{N}{\sigma\sqrt{D}} \;\; \phi\left[\frac{N\ Loss - D\ cLGD[cPD, a]}{\sigma\sqrt{D}}\right] \binom{N}{D} cPD^{D} \, (1 - cPD)^{N-D} \;\; dcPD$$

This the distribution of loss each period of one year.

Distribution of loss in a finite portfolio

N=10, PD=10%, EL=5%, ρ=15%, σ=1%, a=0; Pr[D=0]=0.431

$$\int pdf_{Vas}[cPD] \sum_{D=1}^{N} \frac{N}{\sigma\sqrt{D}} \, \phi\left[\frac{N\,Loss - D\,cLGD[cPD,a]}{\sigma\sqrt{D}}\right] \binom{N}{D} cPD^{D} \, (1-cPD)^{N-D} \, dcPD$$



$$pdf_{Vas}[Loss, EL, \rho] = \frac{\sqrt{1-\rho}}{\sqrt{\rho}} \frac{\phi\left[\frac{\Phi^{-1}[EL] - \sqrt{1-\rho}\,\Phi^{-1}[Loss]}{\sqrt{\rho}}\right]}{\phi[\Phi^{-1}[Loss]]}$$

Questions?

The tests and summary

Multiple grades and classes

There are 5 rating grades and 5 seniority classes.

We assume a single risk factor, Z.

- Then, the losses within different grades and classes are ***conditionally*** ***independent***.
  - We perform change-of-variable to get all losses as functions of Z.
  - We integrate over Z rather than cPD.
  - Because Z is the only tie between sub-portfolios, we can multiply the PDFs of the sub-portfolios to produce the integrand.
    - Once you take care of conditions Z = z, you have is conditional independence.
  - This is messy to write down on a slide, but you get the idea.

Doing this lets us analyze all loans together, all bonds together, or all instruments together.

One test

For both the Null and the Alternative:

- **PD and ρ are set to MLE's based on *default* data.**
- **ELGD equals average LGD in each grade – class combo.**
- **σ is set to 20%**
  - **This is on the low side. I am *not* packing the model with false noise.**

H_0: $a = 0$

H_1: a is set to its MLE, based on the loss data.

- **ELGD reflects the long-term average LGD.**
- **a reflects the relationship between the annual values of LGD and D.**
  - **What is the sensitivity of cLGD to cPD?**

Result: MLE [a] = 0.01

MLE [a] = 0.01 for all loans taken together.

- a = 0.01 is not significantly different from a = 0.
  - The Frye-Jacobs LGD function is not rejected.
- a = 0.01 is significantly different from a = 1.
  - The idea that LGD is fixed is rejected
  - LGD varies with the default rate.

All other tests also show no significance:

- All bonds together; all loans and bonds together
- 23 of 25 combinations of rating and seniority
  - Once F-J is too steep and once F-J is too shallow.
  - 23.75 would be expected not significant if F-J is correct.
- I showed you only Alternative A. Results were similar for three other alternative hypotheses.

We conclude that F-J is consistent with Moody's.

Summary: ***<u>An</u>*** LGD function

Assumption: If a set of conditions are expected to make the default rate go up, they should be expected to make the Loss rate go up.

- **cPD and cLoss are comonotonic.**

Implication: Every loan has an LGD function that maps its cPD to its cLGD.

- **The function depends on two distributions.**
- **There is no limit to the complexity of either one.**

Summary: ***<u>The</u>*** LGD function

Frye and Jacobs assume that $cPD \sim Vas[PD, \rho]$.

- **The firms in a portfolio have a uniform value of PD.**
- **Defaults are connected by a Gauss copula with uniform value of correlation, ρ.**

Frye and Jacobs assume that $cLoss \sim Vas[EL, \rho]$.

The resulting LGD function is strictly monotonic on the unit square for all values of PD, EL, and ρ.

- **No other LGD functions are known with this property.**

Summary: Testing

An Alternative LGD function contains a parameter that controls the sensitivity of cLGD to cPD.

- **It is placed into the PDF of Loss in a finite portfolio.**
- **The PDF is calibrated to Moody's data by MLE.**

The sensitivity parameter is *<u>not</u>* significantly different from zero.

- **The Frye-Jacobs LGD function survives testing.**

The Fed uses F-J in its own models:
https://www.federalreserve.gov/publications/files/2022-march-supervisory-stress-test-methodology.pdf

Questions?

References

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Frye, Modest Means, *Risk*, January 2010.

- A two-parameter credit loss distribution is adequate, given Altman's data. The LGD function is later inferred from this idea.

Frye and Jacobs, Credit loss and systematic loss given default, *Journal of Credit Risk*, Spring 2012

- The sensitivity of cLGD to cPD calibrated to Moody's data is nearly equal to the sensitivity built into the LGD function.

Frye, The link from default to LGD, *Risk*, March 2014

- Tail LGD is better predicted by the LGD function than by linear regression using simulated data from a linear model.

Don’t forget

Homework 4 is due next week at 6PM.

Lisheng’s TA session will be Sunday at 6PM.