Perspectives and Hints for Homework 1

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Q1. Know Thy Correlations

- Given PD's and PDJ's

PD_1	PD_2	PD_3	$PDJ_{1,2}$	$PDJ_{1,3}$	$PDJ_{2,3}$
0.1	0.2	0.3	0.06	0.06	0.06

1. Find the three values of <u>correlation</u>: $\rho_{1,2}$, $\rho_{1,3}$, $\rho_{2,3}$

2. Find the three values of <u>default correlation</u>: DCorr[D_1, D_2], DCorr[D_1, D_3], and DCorr[D_2, D_3]

Q1 Hints: ρ_{ij} versus DCorr[D_i, D_j]

- Note the difference between ρ_{ij} and DCorr[D_i, D_j]
  - Theory check: What are the variables underlying each of the two correlation measures? (See L1.S39 – 43)
  - Bonus: How are the two quantities related, i.e., ρ_{ij} versus DCorr[D_i, D_j]?
  - Bonus on bonus: Lecture 1 describes "three common ways to state the degree of connection between firms". Given one, is it possible to infer the other two? (Hint: L1.S42)
  - We will build more intuitions in later homework questions

Q1. Hints for Solving ρ_{ij}

- Given the PDs and PDJ, solve ρ_{ij} from

$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}]\, dZ_j\, dZ_i \rightarrow \boldsymbol{\rho_{ij}}$$

  - The process calls for
    - numerically implementing the double integral and
    - then inverting the function to solve for ρ_{ij}
  - Theory check: What is the condition for a function to be invertible?

Q2. Joint Probabilities of Default

- Given that each PD = 0.10 and ρ_{ij}'s =

$$\begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \rightarrow$$

PD_1	PD_2	PD_3	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
0.1	0.1	0.1	0.4	0.5	0.6

1. Find the three values of PDJ:

$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi\left[Z_i, Z_j, \rho_{ij}\right] dZ_j \, dZ_i = ?$$

2. State the range of possible values for the probability that all three firms default.

   Hint: try to stylize your solution using the Venn diagrams as examples (L1.S48)

3. State the probability that all three default under the Gauss copula.

$$PDJ_{123} = \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \phi_3\left[\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix}\right] dZ_1 \, dZ_2 dZ_3 = ?$$

Q2. Hints on Implementation

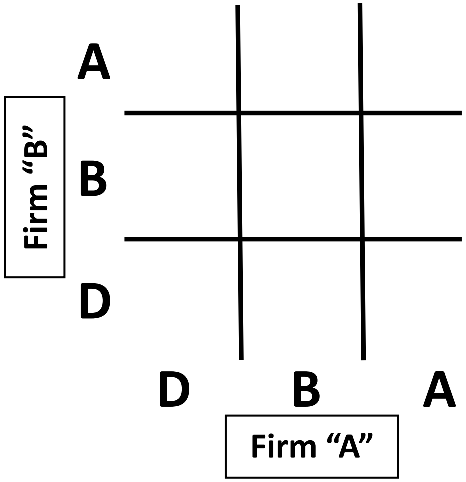
- Implement a numerical function to do a triple integral
  - Hint on hint: This might become a programming challenge. Try to code one piece of the equation at a time and make sure that the parentheses are balanced.
  - You might want to use a simple case to perform sanity check on the implementation, e.g., what should triple integral produce if all three firms are independent?

Q3. Credit Worthiness and Dynamics

Q3. Suppose a firm rated A has correlation 0.4 with a firm rated B. They will obey the transition matrix in the next period.

Transition probabilities			
	A	B	D
A	0.5	0.4	0.1
B	0.3	0.5	0.2

Create a three-by-three grid and fill in the cells with probabilities that sum to 1.00. Two digits of accuracy is sufficient, e.g., 0.66. Assume that all transitions obey a Gauss copula.



Q3 Hints: Rating Transition Matrix

- A firm's credit worthiness is measured by credit ratings.
  - Firm A is rated A and Firm B is rated B today.
  - Rating A is better than rating B, which is better than rating D.
  - D = default.

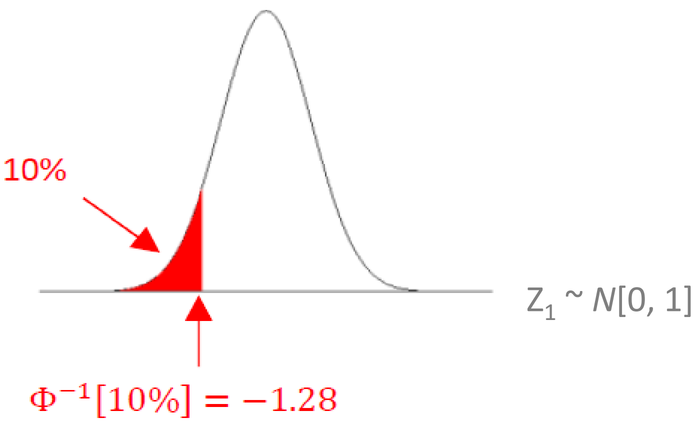
- In the next period (e.g., in 12 months), "things" can change. So would a firm's credit worthiness. The probabilities of a firm's new rating are given in the transition matrix. For example,
  - The probability of Firm A remaining at the rating of A is 50%.
  - The default probabilities are given, PD_1 = 10% and PD_2 = 20%.

Transition probabilities			
Firm/New Rating	**A**	**B**	**D**
Firm A	0.5	0.4	0.1
Firm B	0.3	0.5	0.2

Q3 Hints: The Underlying Dynamics

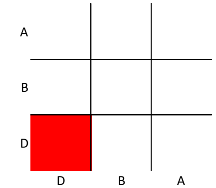
- **Assume Gaussian copula** and build from the previous homework questions.
  - Let Z_1 denote the latent variable that drives the ratings of Firm A, and Z_2 for Firm B.
  - The innovations of the latent variable cause a firm's rating to transition in the next period. For example,

$$Z_1 \sim N[0, 1] \text{ or } P[D_1 = 1] = \int_{-\infty}^{\Phi^{-1}[PD_1]} \phi[z_1]dz_1 = 10\%$$



  - The underlying dynamics also drive the joint default behavior of the two firms:

$$PDJ_{AB} = \int_{-\infty}^{\Phi^{-1}[.1]} \int_{-\infty}^{\Phi^{-1}[.2]} \phi[Z_1, Z_2, 0.4]\, dZ_2 dZ_1$$



Q4. Beyond Gaussian Copula

- Suppose that four firms have PDs equal to 1%, 2%, 3%, and 4%, and the probability that any given pair defaults equals 0.1%.
  - Part 1. What is the matrix of correlations?
  - Part 2. Explain why the defaults of the four firms can or cannot be connected by a Gauss copula.

Q4 Hints: Validity of a Correlation Matrix

- One way to interpret the question: Assume PDs and PDJs can be observed from data,
  - Can we claim that the underlying copula is Gaussian?
  - Even if we can use a Gaussian copula to explain the data (meaning the correlation matrix inferred from data is valid), can we assume that no other copulas are possible?
- Hint: Assume Gaussian copula for Part 1 and see if you can find a valid correlation matrix from the given PDs and PDJs (what makes a correlation matrix valid?! If so, the first half of Part 2 is answered.)

Applying Math in Modeling

- Focus on capturing the behaviors of drivers and dynamics, e.g., what causes default?
  - Using latent variable, Z, as proxy driver
  - Layered dynamics: how Z translates to default event; correlated Z's to represent herd behaviors
- Set up a collection of machinery
  - Analytical approach and asymptotic approach
  - Always helpful to build intuitions and do sanity checks with simulations and plotting
- Always be curious and always seek discipline

Appendix. Notations and Greek Letters

- Notations for normal distribution in this course
  - Unless otherwise specified, we denote a variable having the standard normal distribution as Z ~ N[0,1]
  - Denote normal distribution PDF (probability density function): ϕ, pronounced as /fee/
  - Denote normal distribution CDF (cumulative distribution function): Φ, also pronounced /fee/

Greek letters														
Name	**TeX**	**HTML**	**Name**	**TeX**	**HTML**	**Name**	**TeX**	**HTML**	**Name**	**TeX**	**HTML**	**Name**	**TeX**	**HTML**
Alpha	$\mathrm{A}\ \alpha$	Α α	Digamma	$\digamma\ \digamma$	Ϝ ϝ	Kappa	$\mathrm{K}\ \kappa\ \varkappa$	Κ κ ϰ	Omicron	$\mathrm{O}\ o$	Ο ο	Upsilon	$\Upsilon\ \upsilon$	Υ υ
Beta	$\mathrm{B}\ \beta$	Β β	Zeta	$\mathrm{Z}\ \zeta$	Ζ ζ	Lambda	$\Lambda\ \lambda$	Λ λ	Pi	$\Pi\ \pi\ \varpi$	Π π ϖ	Phi	$\Phi\ \phi\ \varphi$	Φ ϕ φ
Gamma	$\Gamma\ \gamma$	Γ γ	Eta	$\mathrm{H}\ \eta$	Η η	Mu	$\mathrm{M}\ \mu$	Μ μ	Rho	$\mathrm{P}\ \rho\ \varrho$	Ρ ρ ϱ	Chi	$\mathrm{X}\ \chi$	Χ χ
Delta	$\Delta\ \delta$	Δ δ	Theta	$\Theta\ \theta\ \vartheta$	Θ θ ϑ	Nu	$\mathrm{N}\ \nu$	Ν ν	Sigma	$\Sigma\ \sigma\ \varsigma$	Σ σ ς	Psi	$\Psi\ \psi$	Ψ ψ
Epsilon	$\mathrm{E}\ \epsilon\ \varepsilon$	Ε ϵ ε	Iota	$\mathrm{I}\ \iota$	Ι ι	Xi	$\Xi\ \xi$	Ξ ξ	Tau	$\mathrm{T}\ \tau$	Τ τ	Omega	$\Omega\ \omega$	Ω ω

(taken from Wikipedia.com)