36702 TA Session 3

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Part I. Homework 2 Review

Q1. Standard Deviation of # Defaults

- Part a. Simulation of Correlated Defaults

Exp Deflts	Firm1	Firm2	Firm3	Firm4	Firm5
E(dr)	0.50	0.40	0.30	0.20	0.10

Firm	PD_i	Correlation Matrix $\rho_{i,j}$				
1	0.5	1	0.05	0.1	0.15	0.2
2	0.4	0.05	1	0.25	0.30	0.35
3	0.3	0.10	0.25	1	0.40	0.45
4	0.2	0.15	0.30	0.40	1	0.50
5	0.1	0.20	0.35	0.45	0.50	1

	Portfolio	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
μ	1.4997	0.4999	0.4000	0.2997	0.2002	0.0999
σ	1.2048	0.5000	0.4899	0.4581	0.4001	0.2999

Analytical answers:

- $\mu_i = PD_i$; $\mu_{portfolio} = \Sigma(\mu_i) = 1.5$
- $Std = \sqrt{a \cdot COV \cdot a'} = \mathbf{1.2050}$, where a = [1, 1, 1, 1, 1] is a 1x5 row vector and COV is the 5x5 variance-covariance matrix of the default indicators
- The simulation results are close enough

- Part b. Simulation of Uncorrelated Defaults

Exp Deflts	Firm1	Firm2	Firm3	Firm4	Firm5
E(dr)	0.50	0.40	0.30	0.20	0.10

Firm	PD_i	Correlation Matrix $\rho_{i,j}$				
1	0.5	1	0	0	0	0
2	0.4	0	1	0	0	0
3	0.3	0	0	1	0	0
4	0.2	0	0	0	1	0
5	0.1	0	0	0	0	1

	Portfolio	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
μ	1.4997	0.4999	0.3999	0.2997	0.2002	0.1000
σ	0.9746	0.5000	0.4899	0.4581	0.4001	0.3000

Analytical answers:

- $\mu_i = PD_i$; $\mu_{portfolio} = \Sigma(\mu_i) = 1.5$
- $Std = \sqrt{a \cdot COV \cdot a'} = \mathbf{0.9747}$, where a = [1, 1, 1, 1, 1] is a 1x5 row vector and COV is the 5x5 variance-covariance matrix of the default indicators
- The simulation results are close enough

Q1 Technical Note1: Variance-Covariance Matrix

- The analytical solution follows the standard matrix math. However, note that it's the covariance of default indicators (D_i), not the covariance of the latent variables (Z_i), where i = {1,2,3,4,5}
- D_i = {0,1}, which is a Bernoulli random variable with mean = pd_i, and variance = $pd_i(1-pd_i)$
- Variance-covariance matrix with variances on the principal diagonal:
  - $COV[D_i,D_j]=E[(D_i-PD_i)(D_j-PD_j)] = E[D_i D_j]-PD_i PD_j = PDJ_{ij} - PD_i PD_j = DCorr_{ij} \sqrt{\mathrm{pd_i}(1-\mathrm{pd_i})\mathrm{pd_j}(1-\mathrm{pd_j})}$

pdj	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	pdj_11	pdj_12	pdj_13	pdj_14	pdj_15
Firm2	pdj_21	pdj_22	pdj_23	pdj_24	pdj_25
Firm3	pdj_31	pdj_32	pdj_33	pdj_34	pdj_35
Firm4	pdj_41	pdj_42	pdj_43	pdj_44	pdj_45
Firm5	pdj_51	pdj_52	pdj_53	pdj_54	pdj_55

pdj_{part_a}	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	0.5	0.207709	0.163888	0.116771	0.06394
Firm2	0.207709	0.4	0.154406	0.113669	0.064809
Firm3	0.163888	0.154406	0.3	0.102897	0.061384
Firm4	0.116771	0.113669	0.102897	0.2	0.051497
Firm5	0.06394	0.064809	0.061384	0.051497	0.1

pdj_{part_b}	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	0.5	0.2	0.15	0.1	0.05
Firm2	0.2	0.4	0.12	0.08	0.04
Firm3	0.15	0.12	0.3	0.06	0.03
Firm4	0.1	0.08	0.06	0.2	0.02
Firm5	0.05	0.04	0.03	0.02	0.1

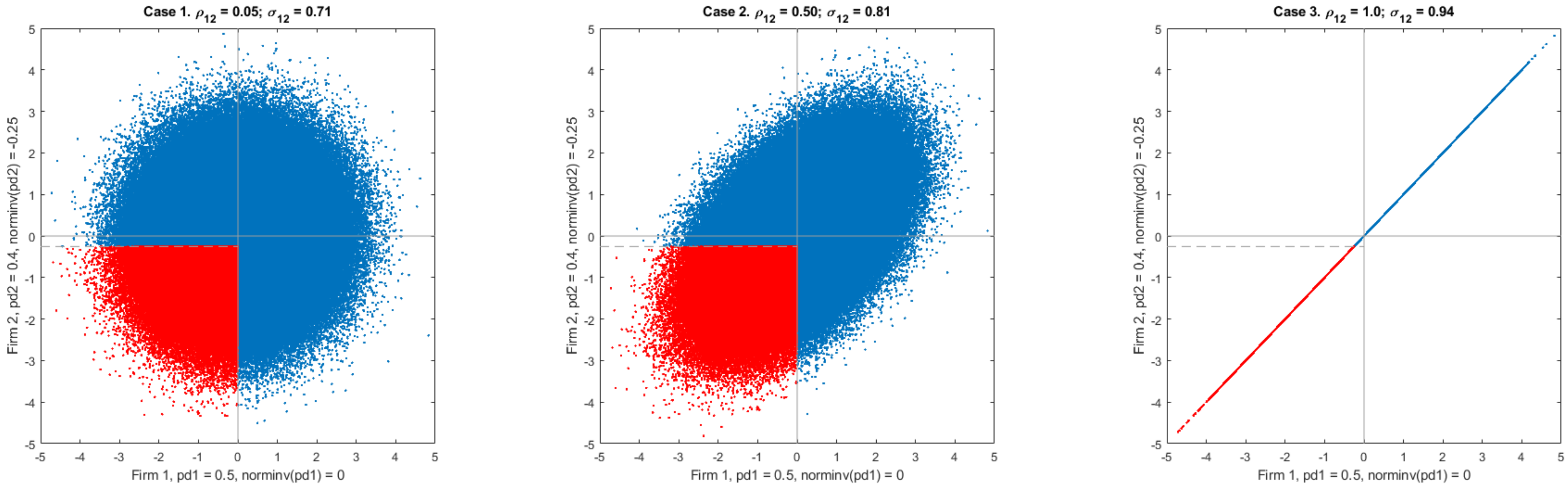
COV	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	Var1	Cov(D1,D2)	Cov(D1,D3)	Cov(D1,D4)	Cov(D1,D5)
Firm2	Cov(D2,D1)	Var2	Cov(D2,D3)	Cov(D2,D4)	Cov(D2,D5)
Firm3	Cov(D3,D1)	Cov(D3,D2)	Var3	Cov(D3,D4)	Cov(D3,D5)
Firm4	Cov(D4,D1)	Cov(D4,D2)	Cov(D4,D3)	Var4	Cov(D4,D5)
Firm5	Cov(D5,D1)	Cov(D5,D2)	Cov(D5,D3)	Cov(D5,D4)	Var5

COV_{part_a}	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	0.25	0.007709	0.013888	0.016771	0.01394
Firm2	0.007709	0.24	0.034406	0.033669	0.024809
Firm3	0.013888	0.034406	0.21	0.042897	0.031384
Firm4	0.016771	0.033669	0.042897	0.16	0.031497
Firm5	0.01394	0.024809	0.031384	0.031497	0.09

COV_{part_b}	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	0.25	0	0	0	0
Firm2	0	0.24	0	0	0
Firm3	0	0	0.21	0	0
Firm4	0	0	0	0.16	0
Firm5	0	0	0	0	0.09

Q1 Caveat. Higher Correlation, Higher Variance

(Plot latent variables, Z_1 and Z_2; then observe σ_{12}, std. of default indicators, as a function of ρ_{12})

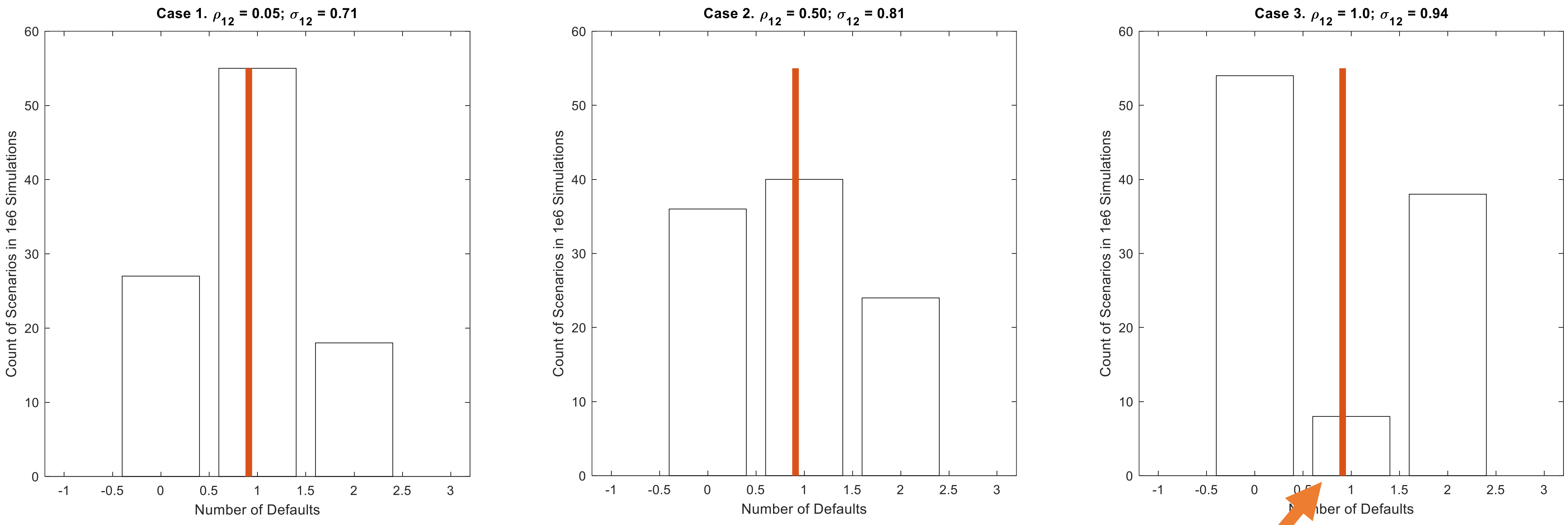


- Red area: $PDJ_{12} = \int_{-\infty}^{\Phi^{-1}[PD_1]} \int_{-\infty}^{\Phi^{-1}[PD_2]} \phi[Z_1, Z_2, \rho_{12}]\, dZ_2\, dZ_1$
- ρ_{12}=.05, pdj_{12}=.21, σ_{12}=.71; ρ_{12}=.50, pdj_{12}=.28, σ_{12}=.81; ρ_{12}=1.0, pdj_{12}=.40, σ_{12}=.94

More Intuition on Why Higher ρ Gives Higher σ

(σ or sigma is the standard deviation of #defaults in the portfolio)

There is an intrinsic reason why correlated defaults have a larger variance than the uncorrelated defaults…

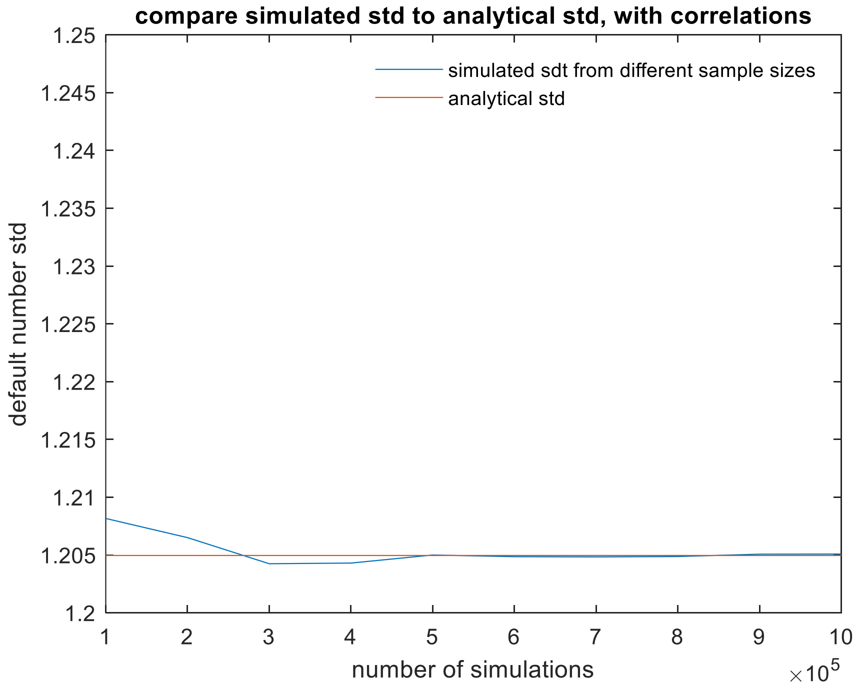


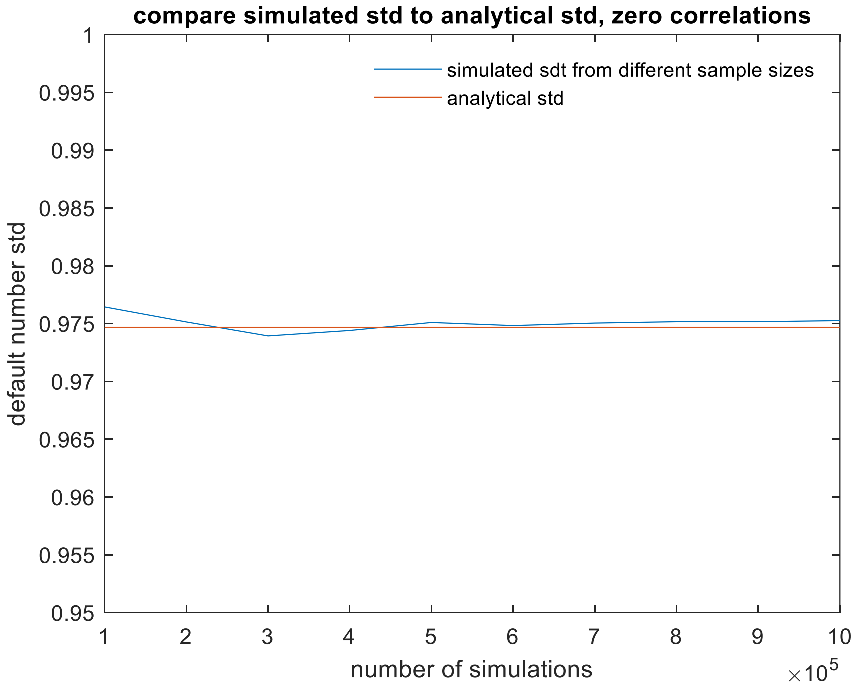
… The reason is that standard deviation measures how dispersed the variables from the mean.

Why is the 1-firm default frequency not zero when ρ_{12} = 1?

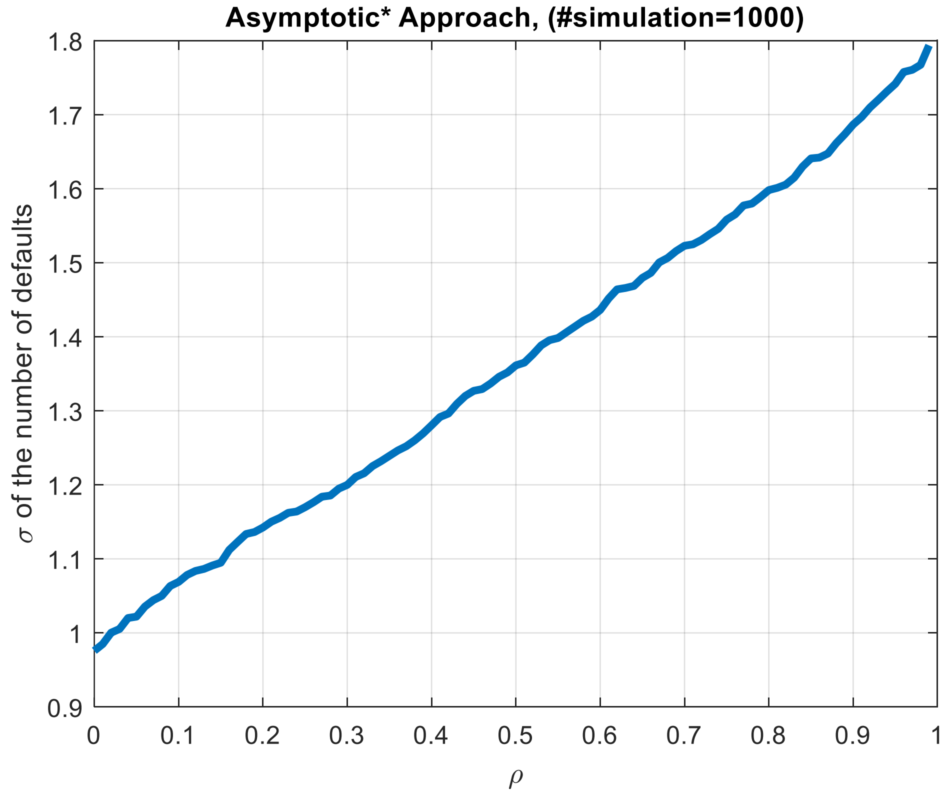
Q1 Technical Note2: Convergence in Simulations

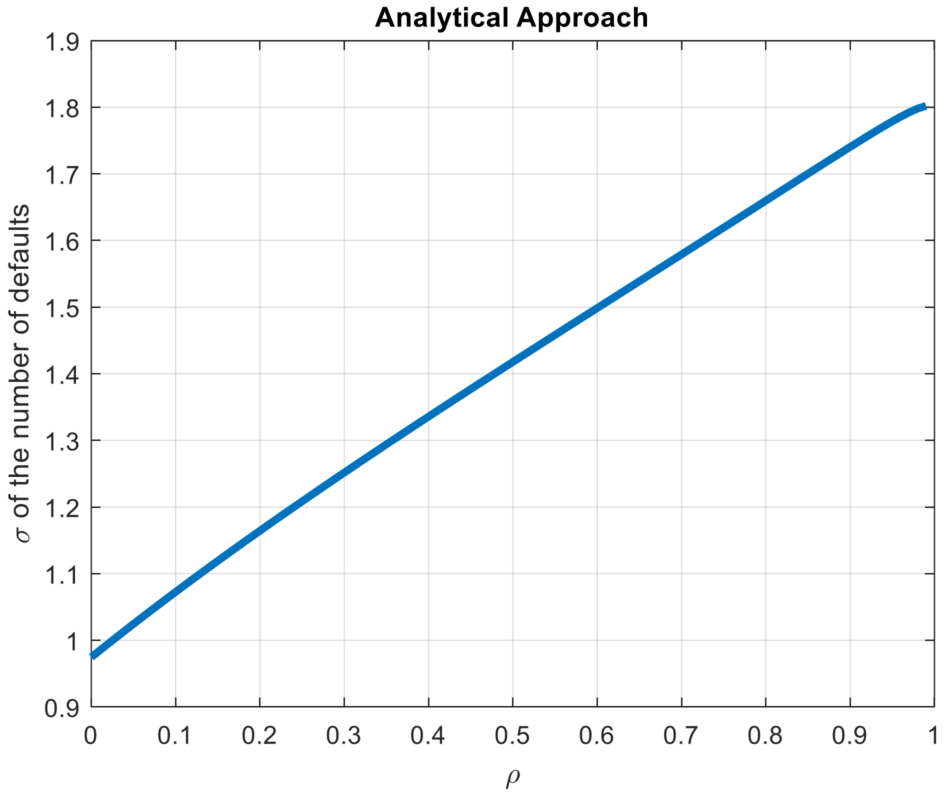
- The “error margin”, the difference between the simulated answer and the analytical solution, looks big in the chart when N = 1e5 (N being the number of simulations)
- Accuracy improves when N is larger, i.e. the simulation results “converge” to the analytical answer





Q2. Visualize σ as a Function of ρ





Q3. Analytical solutions

- Part 1. $\Pr[D_4 = 1, D_5 = 1] = PDJ_{45} = \int_{-\infty}^{\Phi^{-1}[.4]} \int_{-\infty}^{\Phi^{-1}[.5]} \phi[Z_4, Z_5, .45]\, dZ_5\, dZ_4$
- Part 2. $\Pr[D_4 = 1, D_5 = 1 | D_3 = 1] = \frac{PDJ_{45}}{PD_3}$
- Part 3.

	(1)	(2)	(3)	(4) = (2)*(3)	(5) = (1)*(4)
Firm	PD	ELGD	EAD	ELGD*EAD	EL
Firm 1	0.1	0.1	$700	$70	$7
Firm 2	0.2	0.2	$600	$120	$24
Firm 3	0.3	0.3	$500	$150	$45
Firm 4	0.4	0.4	$400	$160	$64
Firm 5	0.5	0.5	$300	$150	$75
Firm 4	0.4	0.6	$200	$120	$48
Firm 5	0.5	0.7	$100	$70	$35
			$2800		$298
				298/2800=	10.64%

- Part 4. DCorr_{34} from Lecture 1

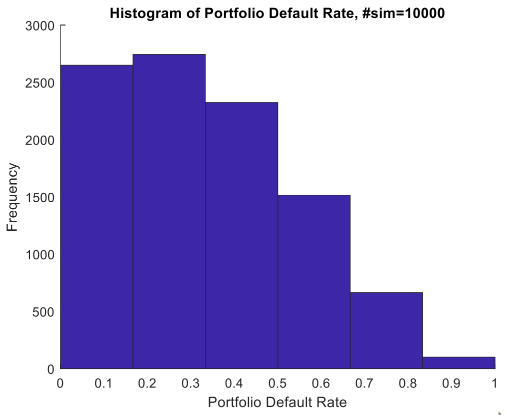
- Q3 Answers:

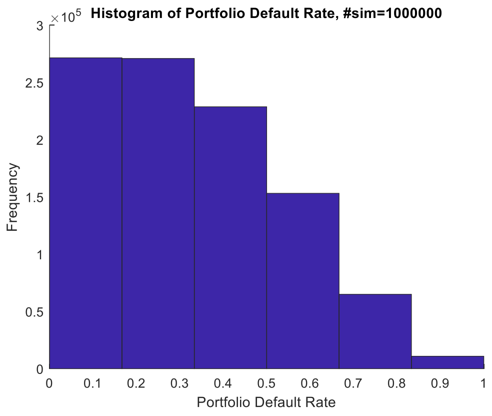
	Prob[D4 = 1 and D5 = 1]	Prob[D_4 = 1 and D_5 = 1 | D_3 =1]	Portfolio Expected Loss Rate	Correlation between D_3 and D_4
Asymptotic*	0.2708	0.4424	0.1065	0.2222
Analytical	0.2718	0.4433	0.1064	0.2177

* Number of simulations = 1e5

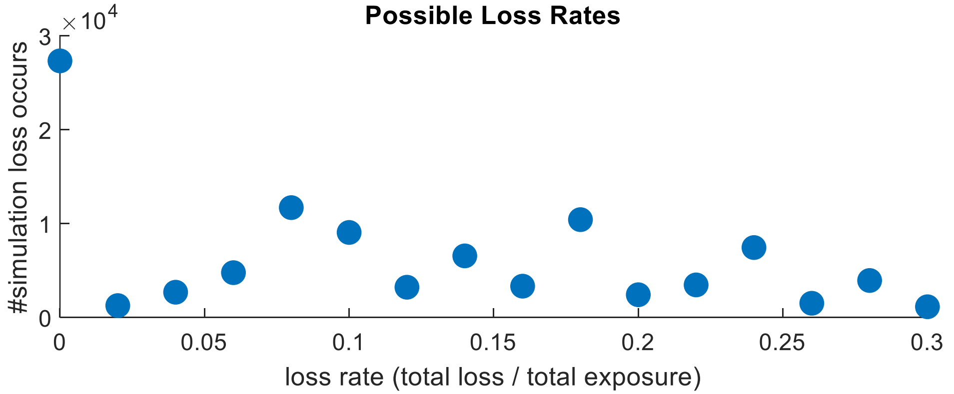
Q4. Plotting the Defaults

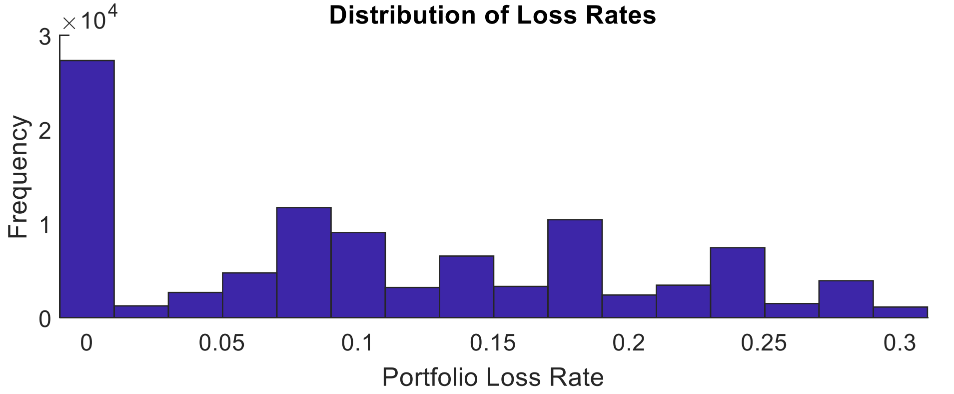
- Part 1. portfolio default rate: ΣD_i / 5, where i = 1… #simulation





- Part 2. portfolio loss rate





Part II. Perspectives and Hints for Homework 3

Q1. Function Derivation & Plotting

- Suppose that the default rate of a portfolio has the triangular distribution: $pdf_{dr}[dr] = 2 - 2dr$. Suppose that in this portfolio lgd is a function of dr: $lgd[dr] = dr^{1/2}$.
  - Derive and state the function $pdf_{lgd}[lgd]$.
  - Produce a single diagram containing plots of three functions $pdf_{dr}[dr], lgd[dr], and\ pdf_{lgd}[lgd]$ for arguments in the range between 0 and 1

Q1. Change of Variable (Chain Rule)

- Hint: S2.L38-39

- It is very useful to learn to derive the PDFs for stochastic distributions using the change of variable (or the chain rule) technique.

- Make sure that you submit both the math derivations AND the plot.

Q2. Function Derivation & Plotting

- Making the same assumptions as in Question 1.
  - Derive and state the function $pdf_{loss}[loss]$. (Hint: loss = dr * lgd)
  - Create a diagram containing the two plots from Question 1 along with the plot of $pdf_{loss}[loss]$ for variables in the range between 0 and 1; limit the vertical axis to the range from zero to 3.
  - State the values of
    - Expected loss, EL
    - Expected LGD, ELGD
    - "Time-weighted" LGD (Hint: L4.S21-22)

Q3. Std. of a Vasicek distribution and Plotting

- Part 1. Express the standard deviation of a Vasicek distribution as an integral that involves the Vasicek PDF. (Hint: This is a math derivation)

- Part 2. For distributions with PD = 0.10, numerically integrate and plot the standard deviation for $0.05 < \rho < 0.95$. (Hint: Would the standard deviation of a Vasicek distribution be similar to the standard deviation of the number of defaults in a portfolio, i.e., an increasing function of ρ?)

- Part 3. On a separate diagram, plot two Vasicek distributions: PD = 0.10, ρ = 0.05 and PD = 0.10, ρ = 0.95, limiting the vertical axis to {0, 0.12}

Q4. Std. of a Vasicek distribution and Plotting

- 4. Suppose two loans have Vasicek distributions. One loan has PD = 0.06, ρ = 0.06, the second loan has PD = 0.03, ρ = 0.20, and both loans respond to the same systematic risk factor.
  - Plot on a single diagram the two inverse CDFs. (Hint: L2.S36)
  - At the lower quantiles, the first loan has greater cPD than the second. The situation is reversed at very high quantiles. Estimate the quantile at which both loans have the same value of cPD.