Portfolio Credit Risk:
Modeling and Estimation

University of Chicago Masters in Financial Mathematics 36702

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Lecture 1

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Credit defaults and connections between defaults

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University Mask Policy

Tonight: Wear a mask while in the classroom.

- **If I am not wearing a mask, it is because I found out that I have the option, and I exercised it.**

Next week: Masks are optional for *<u>everyone</u>*.

- **If you prefer to sit at a distance from those who aren’t wearing masks, send me an Email ASAP.**
  - **If you have suggestions how to handle this, all the better.**

At all times: Circumstances will change, policies will adapt, and we will follow the policies.

The University requires that everyone be fully vaccinated and boosted when eligible.

Portfolio credit risk

A lender has a portfolio of credit exposures like loans.

- **The best case for the lender is that a loan pays on time.**
  - **Upside is limited. The lender is short the borrower's option to default.**

We study the distribution of portfolio credit loss,

- **especially losses great enough to bankrupt the lender.**
  - **When banks fail, the usual reason is credit loss. Such as 2009.**

This point of view is different from most other classes.

- **Option pricing finds the *average* outcome for an *instrument*.**
- **We find the *tail* outcome for a credit-risky *portfolio*.**

Why this might interest you

Your first job might be in credit risk modeling.

- **Banks are required to run large credit risk models.**
- **Trading shops need to control credit risk.**

You might trade credit-risky instruments.

- **The risk is likely measured by the model you'll learn.**

You might create a model using a *small* data set.

- **For example, trading in a new or developing market.**
- **"Wholesale" credit risk *always* has a small data set.**
  - **There are only a few thousand big firms in the world.**
  - **There have been only a handful of stress periods in history.**
  - **Credit loss modeling is a good source of ideas.**

Course agenda

1. Under certain circumstances, the forecasts of models are reliably poor on average.

2. These circumstances appear present when modeling *conditional* credit loss.

- **Example: Use a regression of credit loss on GDP.**

3. Therefore, we use an *unconditional* model.

- **When credit loss goes up and down, we do not attribute this to economic conditions.**

The first four weeks focus on topic 3. The last week you will see *why*.

Tonight's topics

An introduction to credit risk

The saga of RadioShack in 2014-15

Stylizing <u>default</u> and <u>loss given default</u>

The connection between two defaults

The copula connection of many defaults

An introduction to credit risk

Credit risk

Banks face many kinds of *risk*.

- **"Risk" means that the bank might experience a *loss*.**
  - **Credit loss: A borrower fails to repay a loan or other extension of credit.**
  - **Market loss: The bank bets the wrong way or a hedge doesn't work.**
  - **Operational loss: The bank fails to do something it should do, like lose a lawsuit, buy enough insurance, or harbor fraud.**
  - **Reputational loss: The bank loses business when its reputation declines.**
- **Among these, credit risk is most likely to cause bank failure.**

We stylize all sources of credit risk as “loans”.

- **We will speak about “banks” lending to “firms,”**
  - **but credit risk is pervasive in business. For example, a hedge fund has credit risk with respect to its derivative counterparties.**

Loans, covenants, and default

Each loan is part of a legal agreement that places requirements on the borrower:

- **It *must* pay interest and repay the principal on schedule.**
  - **If it doesn't, there is a "money default".**
    - **Default on a loan is usually declared when loan payments are 90 days overdue.**
- **If the firm defaults, it may give power to the lender.**
  - **For example, if the loan is collateralized, the borrower gives up the ownership of those assets identified as collateral.**
- **It *may* also be required also to obey specified "covenants".**
  - **If it doesn't, there is a "covenant default." Examples:**
    - **Maintain earnings above a certain threshold.**
    - **Have cash flow greater than X times the interest on the loan**
    - **Maintain $Y of equity finance for every $1 of bank loan**

If the borrower fails to perform on the agreement, the lender holds the borrower to be in *default*.

Default is expensive

This is intentional! If a firm defaults:

- The loan agreement calls for fees to be paid by the firm.
- The loan agreement gives the bank extra powers.
  - For example, the bank can seize the collateral named in the agreement.
    - Commonly, collateral assets are real estate, inventories, or financial assets.
    - Separate from such “secured” loans, a firm might have “unsecured” loans with no collateral.
- The default might be noted publicly.
  - The firm becomes less able to borrow from other lenders.
- The loan agreement usually has a “cross default” provision.
  - If a firm defaults on one loan, it is held in default on all loans from the bank.
    - Default on an unsecured loan lets the bank seize collateral on the secured loans.

Usually, it is cheaper to pay on time than to default.

- The option to default is usually out of the money.

Next, consider the 2014 default by retailer RadioShack.

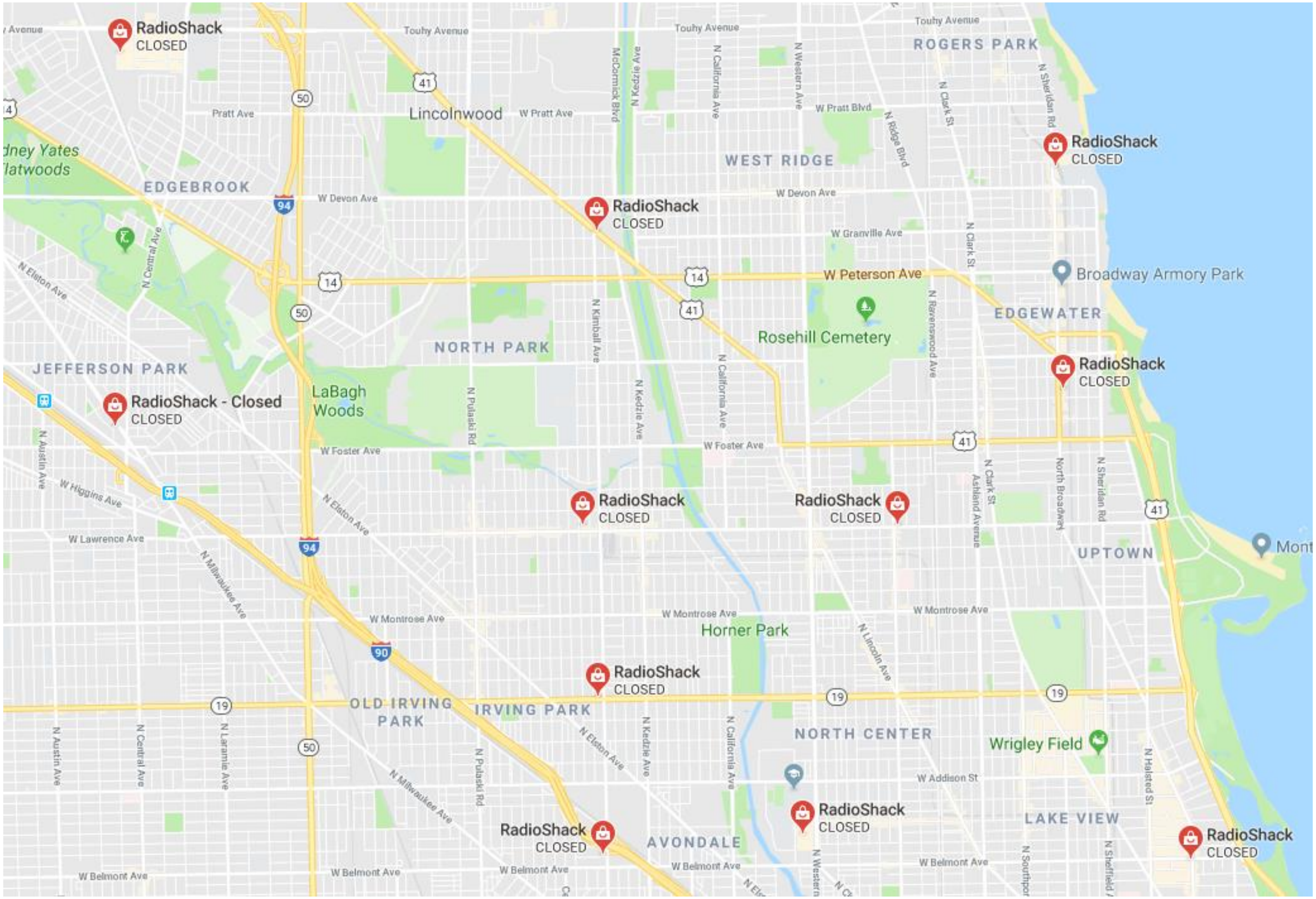
Questions? Comments?

If you have a question, it is likely that others have the same question. Please ask away!

The comments I receive at times like this are more helpful to me than the comments of journal referees. Please make comments!

The saga of RadioShack

RadioShack retail stores, 2016



RadioShack: Orientation

RadioShack was an electronics retailer.

- The early 20th century, an amateur radio operator's workplace was known as his "shack," a very cool thing.
  - By the 21st century, neither radio nor shacks were considered cool.

The company filed for bankruptcy in February 2015.

- I reference the Wikipedia summary from January 2015.

The company filed for a second bankruptcy in 2017.

- It could not fulfill the plans that were made in 2015.

RadioShack's Wikipedia Jan 2015

On March 4, 2014

– **RadioShack announced that it planned to close as many as 1,100 lower-performing stores.**

On May 9, 2014

– **the company reported that a conflict with its creditors prevented it from carrying out those closures.**

– **Later Standard & Poor's downgraded to "CCC", warning that the company would have "very small amounts of liquidity early next year, which could lead to a liquidity crisis, default or the company's decision to seek a financial restructuring."**

On September 11, 2014

– **RadioShack admitted it may have to file for bankruptcy.**

Translating to English—1

RadioShack "planned to close stores":

- **RadioShack wanted to close its unprofitable stores.**

It had "conflict with its creditors":

- **The land and buildings of some unprofitable stores had been pledged as collateral on certain loans.**
  - **The lenders prevented RadioShack from closing the stores.**
  - **They believed that it was in their interest that the stores stayed open.**

Later, Standard & Poor's "downgraded" RadioShack:

- **S&P is one of the three major "Nationally Recognized Statistical Rating Organizations (NRSROs)."**
  - **The others are Moody's and Fitch. S&P and Moody's dominate.**
- **Firms can pay an NRSRO to *rate* their bonds or loans.**
  - **A rating gives investors information; the bonds trade with more liquidity.**
  - **In this case, S&P changed its rating lower; this was a *downgrade*.**

Translating to English—2

The new S&P rating was <u>CCC.</u>

- **From best to worst, S&P ratings are:**
  - **AAA, AA, A, and BBB; these are called "Investment grade" ratings.**
  - **BB, B, CCC, CC, and C: these are called "non-investment grade", "high yield", or "junk" ratings.**
  - **SD and D: the firm selectively defaulted or defaulted on all obligations.**
- **Moody's and Fitch have their own scales.**
  - **The scales have the same number of steps.**
  - **Yet each agency says that their ratings do something different.**

S&P said that the company would have "very small amounts of <u>liquidity</u> early next year."

- **That is, RadioShack might not have the <u>cash</u> to repay loans.**

Translating to English—3

"This could lead to default…"

- **This means, maybe RadioShack can't pay the loans it has.**

…or the company might "seek a financial restructuring."

- **That is, maybe RadioShack could arrange for a new set of loans that it has a better chance of repaying.**
  - **The proceeds of the new loans would pay off the old loans.**
  - **There would be a new schedule of repayment and new covenants.**

Then RadioShack admitted it might file for bankruptcy.

- **In bankruptcy, a court *enforces* restructuring or liquidation.**
  - **Not all corporate defaults go to bankruptcy court.**

What else happened

In 10/2014, RSH got a $120 million loan from a hedge fund.

– At the time, RadioShack was losing about $1 million per day.

About 120 days later, RadioShack filed for bankruptcy.

This is odd, don’t you think?

How it turned out for the lenders

Bankruptcy court (nearly) followed "Strict Seniority".

- The company had enough assets to pay 100% of the claims presented by the banks that made Revolving Loans.
- That left enough to pay about 25% of the Term Loan.
- After that, there was approximately zero left to pay the bonds.
  - All lenders know, and agree to, their position on a seniority "ladder.“
  - The hedge fund lender was lumped in this group, I believe.

Debt	Rank	Collateral	Outstanding	Above	Below	Recovery
Revolving Loan	1	All assets	250	0	575	100
Term Loan	2	Non-current assets	250	250	325	25.5
Senior Unsecured Bonds	3	None	325	500	0	0.24

Questions? Comments?

Please ask away!

Stylizing credit loss

Stylizing default and LGD

The standard model simplifies as much as possible.

If the borrower obeys the loan agreement, Loss $\equiv$ 0.

- **The firm pays on time and obeys any covenants.**

If the borrower materially violates the loan agreement, this is an event of *default*.

- **Filing for bankruptcy is one event of default.**

The fraction of the outstanding amount that is lost by the bank is called Loss Given Default (LGD).

We usually assume that loan *exposure* is $1.

- **It is usually trivial to incorporate exposure, and we don't do it.**

The definition of default

In the 20th century, most banks did not define default.

- **They reported to bank supervisors various quantities:**
  - **Non-performing loans, impaired loans, foreclosures, delinquent loans, seriously delinquent loans, and so forth.**

Banks defined "default" when they discovered a model (the one we study) that could help them manage *risk*.

- **The desire to model credit *risk* led to definitions of *default*.**
  - **Different kinds of loans and bonds use different definitions.**

For "wholesale" loans−big loans to big firms−an event of default includes when

- **payments are more than 90 days past due, or**
- **the lender accepts (or is forced to accept) an adverse restructuring, or**
- **there is a bankruptcy filing or certain other events.**

Default-side symbols

D **The (random) default indicator**

= 1 if the firm defaults; = 0 otherwise

(a default on one loan is considered a default on all)

PD **= E [D], the probability that the firm defaults within a planning horizon usually equal to a year.**

PD changes over time depending on events.

N **The number of *<u>firms</u>* having loans in the portfolio.**

DR **= $\sum D_i / N$, the (random) default rate in the portfolio.**

N is the number of firms; $\sum D_i$ is the number that default.

Average default rates

The long-term average US bond default rates:

- **Investment grade (BBB or Baa or better): about 0.20%**
  - **The PD for Apple is probably less than 0.01%.**
- **Non-investment (junk) (BB or Ba or worse): about 3.6%.**
  - **In December 2014, the PD of RadioShack was greater than 50%.**

Estimated values of PD are in the same large range.

Loan PDs are different from bond PDs.

- **Only firms with good credit quality can issue bonds.**
- **Firms can default on bonds and keep paying the banks.**
  - **They have on-going relationships with the banks.**
- **We mostly assume that someone else estimates firm PD.**

Recovery and loss given default

After a default, the bank attempts to ***recover*** as much as it can of the amount that it is owed.

- The bank can gain ownership of identified collateral and sell it.
- Some loans are guaranteed by other firms, and the bank can make claims on those guarantors.
- Sometimes the bank takes over the firm and runs it for a while.
  - The loan agreement states what the lender can do if the borrower defaults.
- A bank's loss depends on its decisions in the recovery phase.

Expressed in dollars, $LGD = $Exposure – $Recovery.

- We usually work in ***ratios***: LGD = 1 – Recovery / Exposure.

Two ways to quantify recovery

Recovery equals the *market price* of the loan or bond.

- **The lender can sell the bond and lose no more.**
  - **This works OK for publicly traded bonds.**
  - **However, very few loans trade in markets.**

Recovery equals the net of all *discounted cash flows*, discounting back to the time of default.

- **Lender decisions after default (accept a bid on the collateral or continue to own it?) affect this "loss given default."**
  - **As a practical matter, it is difficult to keep track and to discount properly, and there is little incentive for doing so.**
  - **Suppose there is a secured loan and an unsecured loan. Of the total recovered, it might be difficult to allocate it to one or the other.**

LGD-side symbols

LGD The (random) fraction of exposure that is lost.
LGD is usually between 0 and 1.
If there is no default, LGD is undefined.

ELGD The mathematical expectation of a loan's LGD.

Loss Loss is usually measured as a fraction of exposure.

For a single loan, $\boldsymbol{Loss = D * LGD}$.

For the k defaulted loans among N loans, $\boldsymbol{Loss = \frac{\sum_{i=1}^{k} D_i LGD_i}{N}}$.

Average LGD rates

The average US junk bond LGD rate is about 60%.

Secured bonds have lower LGDs than unsecured bonds.

- **"Secured" usually means there is collateral.**

Loan LGDs tend to be less than bond LGDs.

- **Loans are more frequently secured by collateral.**
- **Bank loans are almost always *senior* to bonds.**
  - **In bankruptcy, senior debt *must* be paid first.**
  - **Less senior debt holders get whatever is left over.**
- **Loan agreements give banks powers they use to reduce loss.**

But it still comes down to cases.

- **Credit card debt has very high LGD!**

Regarding parameter estimates

We assume for now that PD, ELGD, and the entire distribution of LGD are accurately estimated.

- **Estimating the parameters is specialized work performed by a relatively small number of people.**
  - **The number should be smaller still!**
  - **Too many people think that they can do a good job on this.**

We focus on models of <u>*portfolio*</u> credit risk…

Questions? Comments?

Please ask away!

Portfolios with one or two firms

Modeling a single firm

Draw a random variable having a known distribution.

If the quantile of the RV is less than PD, simulate default.

- Example: Draw Q ~ U [0, 1]; D = 1 if q < PD, otherwise D = 0.
- Example: Draw Z ~ N [0, 1]; D = 1 if z < Φ^{-1}[PD], otherwise D = 0.
  - Upper-case "phi," Φ, symbolizes the standard normal CDF.
- Q and Z are called latent variables responsible for default.

If D = 0, then Loss = 0.

If D = 1, then draw LGD from the loan's LGD distribution.

- The "loss given default" LGD distribution ***anticipates*** default.
- Loss = LGD.

Repeat this process to produce a distribution of default for the firm and a distribution of loss for the loan.

Connecting a second firm

The portfolio has loans to Firm 1 and Firm 2.

- **We assume good estimates of $\boldsymbol{PD_1}$ and $\boldsymbol{PD_2}$.**
- **We also assume a good estimate of $\boldsymbol{PDJ}$.**
  - **This is the probability that both firms default in the next year.**
  - **It can be articulated as $\boldsymbol{PDJ_{1,2}}$.**

If the defaults are independent, then $\boldsymbol{PDJ = PD_1 PD_2}$.

- **This is the definition of "independent."**

But in general, defaults are *<u>dependent</u>*.

- **This makes sense.**
- **And it is in the data.**

It makes sense

Suppose that Apple defaults on its debts. Several other kinds of companies might also default:

- **One that supplies Apple with, say, cell phones.**
- **One with material sales on the Ap Store.**
- **One sensitive to whatever caused the Apple default, such as a shortage of a rare-earth metal.**
- **One that has no apparent connection whatever to Apple other than being in business at the same time.**
- **…**

It is in the data

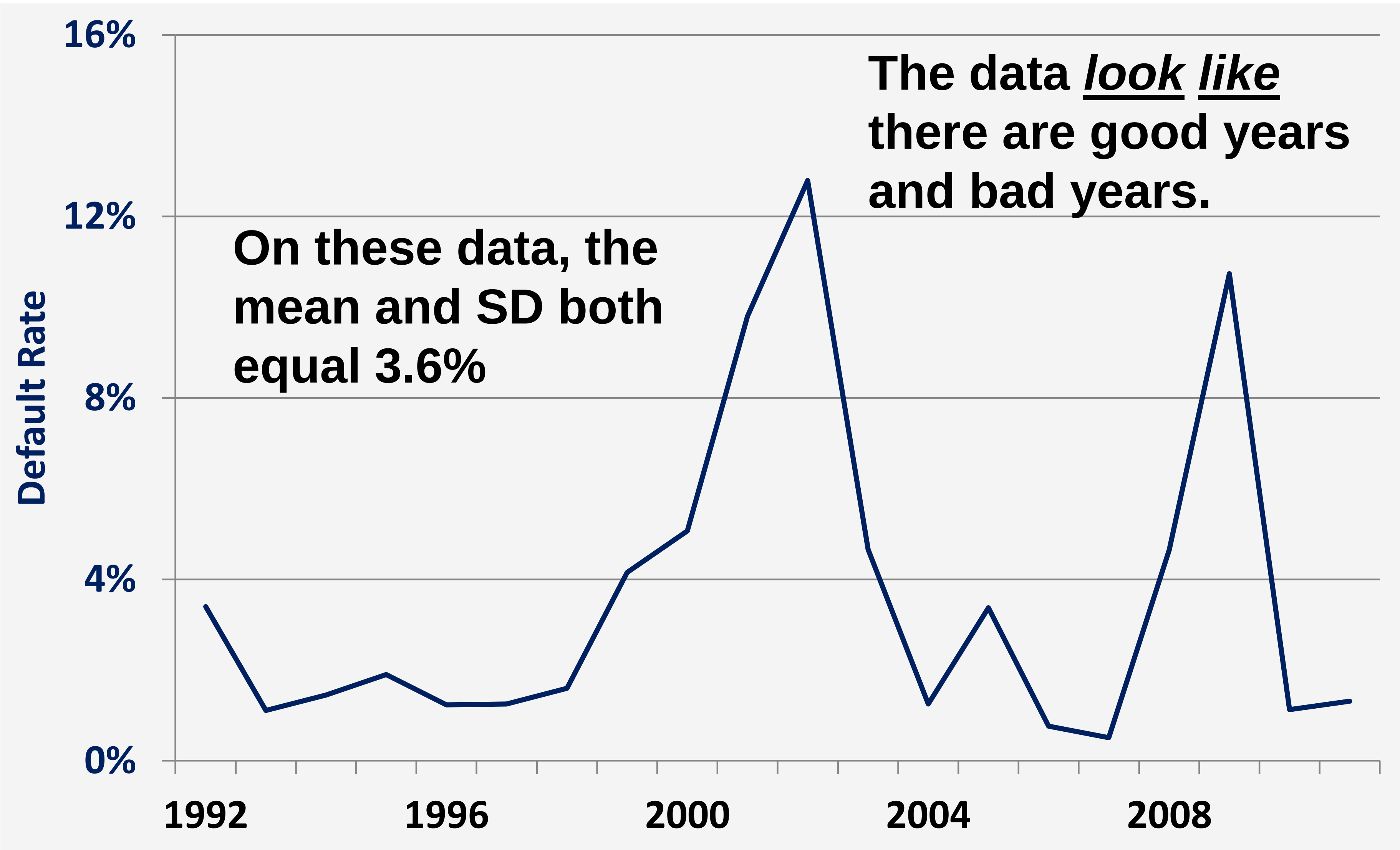
- **Suppose that you have the default rate each year for a portfolio of 1000 statistically identical firms.**
  - **Suppose the average default rate is 3.6%.**
  - **Suppose the standard deviation is 3.6%. (Coincidence.)**

- **If all firms had PD = 0.036 and *<u>were</u>* independent,**
  - **then the SD of the default *<u>rate</u>* is** $\sqrt{\mathbf{0.036\,(1-0.036)/1000}} = \mathbf{0.59\%}$.

- **The SD of the data is six times too big. This is said to be a "significant" difference. The hypothesis that the firms are independent would be rejected.**

- **Real life isn't this tidy, but here's some data…**

Altman junk bond default rates



Firms tend to be connected

The data can be explained by positive correlation.

Usually, correlation is positive: $PDJ > PD_1 PD_2$.

- **Firms are more likely to default when other firms default.**

There might be exceptions.

- **Imagine a law firm specializing in bankruptcy.**

There are three common ways to state the degree of connection between firms:

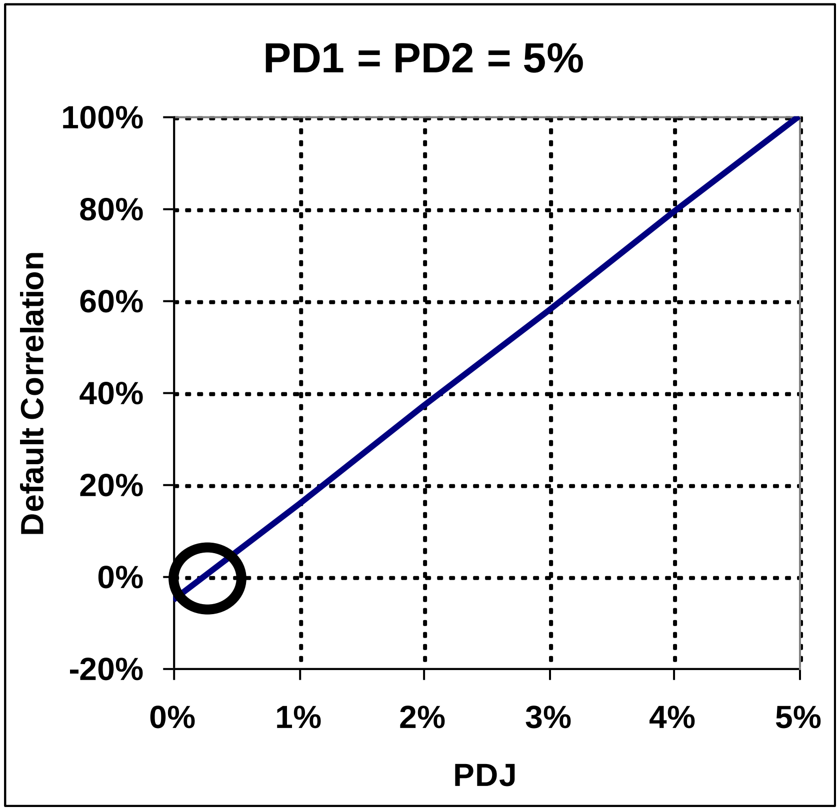
- **The value of PDJ.**
- **The correlation of the default events D_1 and D_2.**
- **The correlation of the latent variables Z_1 and Z_2.**

Default correlation

Default correlation is correlation between D_1 and D_2:

$$Dcorr[D_1, D_2] = \frac{Cov[D_1, D_2]}{\sqrt{Var[D_1]Var[D_2]}}$$

$$= \frac{PDJ - PD_1 PD_2}{\sqrt{PD_1(1-PD_1)PD_2(1-PD_2)}}$$



If PDJ > PD_1 * PD_2, Dcorr > 0.

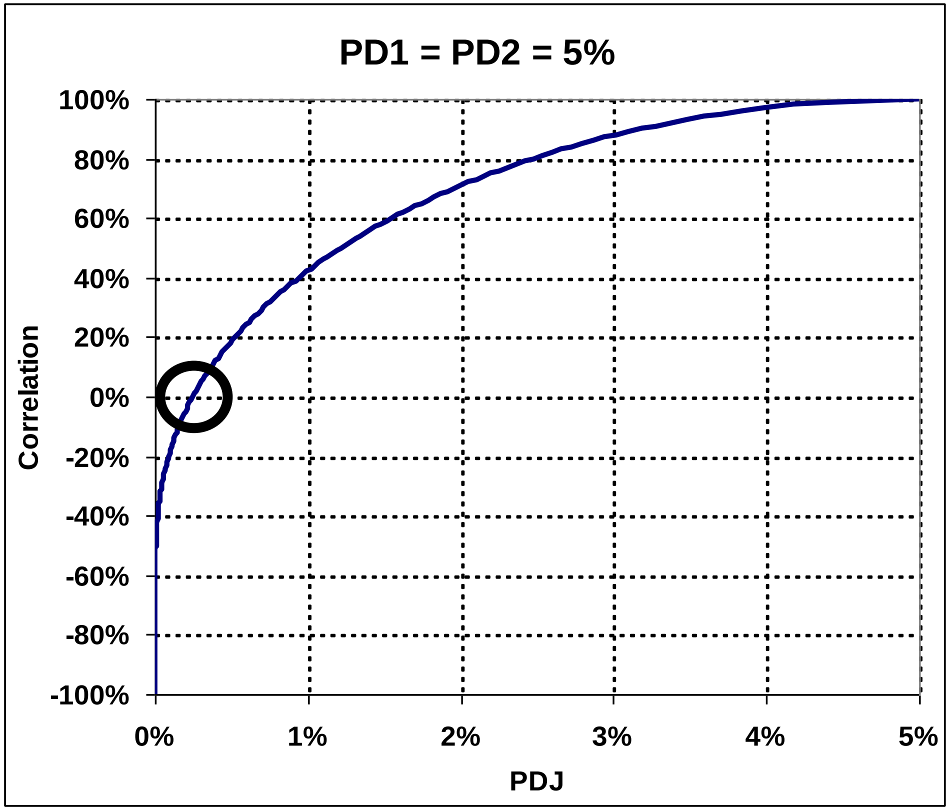
Holding fixed PD_1 and PD_2, greater default correlation ⇒ greater PDJ ⇒ greater risk

Latent variable correlation

Values of ρ between -1 and 1 produce values of PDJ between 0 and Min [PD_1, PD_2].

For $PD_1 = PD_2 = 5\%$,

$$\int_{-\infty}^{\Phi^{-1}[.05]} \int_{-\infty}^{\Phi^{-1}[.05]} \phi[z_1, z_2, \rho] dz_1 dz_2$$



Greater ρ ⇒

greater PDJ ⇒

greater risk

Three equivalent expressions

PDJ, Dcorr, and ρ express dependence between 2 defaults.

- The three expressions are completely equivalent.

Of the three, risk models make the most use of ρ.

- One reason: PDJ and Dcorr are highly sensitive to PD.
  - If a firm's PD changes, its PDJ with every other firm changes.
  - If a firm's PD changes, its Dcorr with every other firm changes.
  - But if a firm's PD changes, its ρ with the other firms may ***not*** change.
    (It does not change enough that analysts can discern the change.)
- Second reason: most pairs of firms have ρ in a narrow range.
  - $5\% < \rho < 15\%$ captures the bulk of the pairs of firms.

We refer to ρ as "correlation" or "credit correlation."

- This is easier to say than "the correlation between the latent variables responsible for default."
- "Correlation" does ***not*** mean Dcorr; "correlation" means ρ.

Modeling two firms

The most common way to model the defaults of two (or more) firms is to use two standard normal latent variables.

Repeatedly draw $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim \mathrm{N}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$**.**

Set D_1 **= 1 if** $Z_1 < \Phi^{-1}[PD_1]$**; set** D_2 **= 1 if** $Z_2 < \Phi^{-1}[PD_2]$**.**

The average rate at which both default estimates PDJ.

There would be many ways to obtain the same results.

- **For example, draw a uniform** Q_1 **and set** D_1 **= 1 if** $Q_1 < PD_1$.
- **Draw independent** Q_2 **and set** D_2 **= 1 if** $Q_2 < PDJ/PD_1$**.**

Questions? Comments?

Please ask away!

Prob$[\boldsymbol{D_1 \cap D_2 \cap D_3}]$

Suppose there is a portfolio having three firms.

Let each PD equal 10% and let each PDJ equal 1%.

- **By definition, each pair of defaults is independent.**
- **The matrix of correlations is** $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

What is the probability that all three firms default?

Trick question!

- There isn't enough information to answer it.
- There are eight primitive events:
  - Two outcomes for each of three firms; $2^3 = 8$.
- We seek the probability that all three firms default.
- We have 7 only facts to work with:
  - We know 3 PDs, 3 PDJs, and we know that the sum of the probabilities of the eight events equals 1.0.
- We need one more fact or assumption.
  - We don't need to assume Prob[$D_1 \cap D_2 \cap D_3$] = 0.001…

Three possible assumptions

Perhaps the three firms never default at the same time.

- **Prob$[D_1 \cap D_2 \cap D_3]$ = 0.**

Perhaps when two default, the third one also defaults.

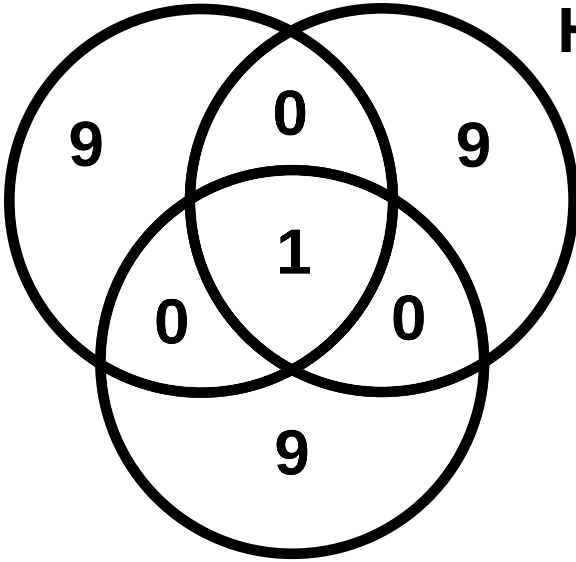
- **Prob$[D_1 \cap D_2 \cap D_3]$ = 0.01.**

Perhaps the three firms are not just *pairwise* independent but also *jointly* independent.

- **Prob$[D_1 \cap D_2 \cap D_3]$ = 0.001**
- **This assumption is automatic under the "Gauss copula."**

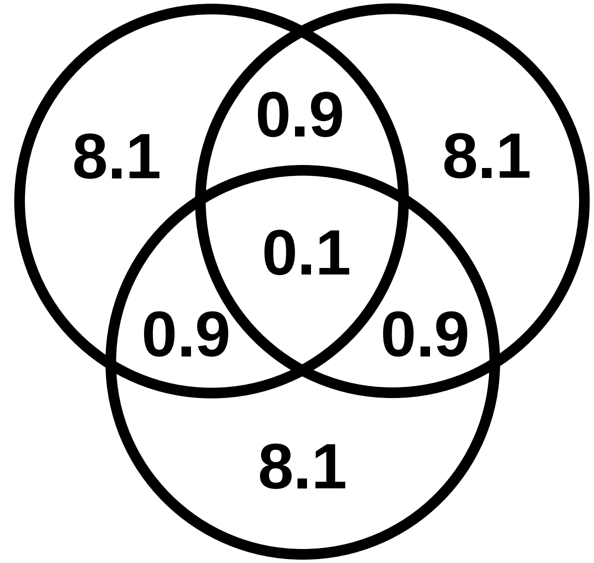
$$\int_{-\infty}^{\boldsymbol{\Phi}^{-1}[.1]} \int_{-\infty}^{\boldsymbol{\Phi}^{-1}[.1]} \int_{-\infty}^{\boldsymbol{\Phi}^{-1}[.1]} \boldsymbol{\phi}_3 \left[\begin{pmatrix} \boldsymbol{z_1} \\ \boldsymbol{z_2} \\ \boldsymbol{z_3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \boldsymbol{dz}_1 \boldsymbol{dz}_2 \boldsymbol{dz}_3 = 0.001$$

- **where $\boldsymbol{\phi_3}$ is the PDF of 3 jointly standard normal variables.**

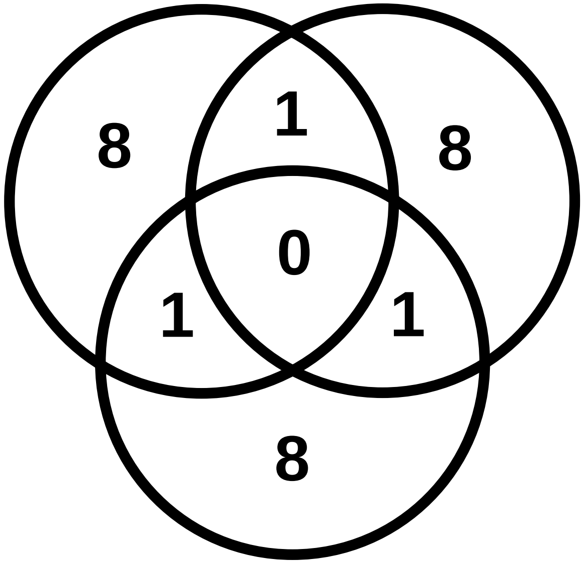


High-risk copula

Numbers in percent



Gauss copula
is more like the low-risk copula
than like the high-risk copula.



Low-risk copula

Ignore the third firm…

Low-risk and high-risk copulas

The Gauss copula says P [all default] equals 0.001.

But here are a Low-risk copula and a High-risk copula:

- **Note that for *__each__* copula:**
  - **$P[D_1] = P[D_2] = P[D_3] = 0.10$.**
  - **$P[D_1,D_2] = P[D_1,D_3] = P[D_2,D_3] = 0.01$.**
- **P[all default] can be anything ≤ 0.01,**
  - **not just what Gauss says it would be.**
  - **Key fact: an independent pair *__need__* *__not__* be independent of the third firm, even though Gauss says that it *__must__* be.**

	Probabilities under copulas		
Event	Gauss	Low Risk	High Risk
000	0.729	0.73	0.72
001	0.081	0.08	0.09
010	0.081	0.08	0.09
011	0.009	0.01	0
100	0.081	0.08	0.09
101	0.009	0.01	0
110	0.009	0.01	0
111	0.001	0.00	0.01

The Gauss copula is more like the low-risk copula than the high-risk copula.

Questions? Comments?

Non-Gauss copulas

With three firms, there is one "missing fact".

- **Each value of the "missing fact" produces a different copula.**
  - **We used three values for the probability that all three firms default:**
  - **0.00 (Low-risk), 0.01 (High-risk), and the value of the triple Gauss integral.**

If there are 300 firms−not a big portfolio−there would be about 2^{300} missing facts.

- **This is larger than the number of electrons in the universe.**
  - **We are going to need help with this.**

The Gauss copula supplies all the facts we need.

- **You might not like valuing 2^{300} 300-dimension integrals.**
  - **Fortunately there are shortcuts.**

You might imagine a large set of other useful copulas.

- **You will not find it!**

A second useful copula

Define multivariate normal variance *mixture* distributions:

$$X = \sqrt{W}\, Z; \quad Z \sim N_d[0, \Sigma], \quad W \text{ is scalar}$$

If W is fixed, then $X \sim N_d[0, W\,\Sigma]$.

If W is random, independent and nice, then

$$E[X] = 0; \quad Var[X] = E[W]\,\Sigma; \quad Corr[X] = Corr[Z]$$

Although X inherits its correlations from Z, the X variables are *not* connected by a Gauss copula.

- **For example, the X variables are uncorrelated, they are *not* independent. The next slide gives a proof…**

$$\boldsymbol{X} = \sqrt{\boldsymbol{W}}\,\boldsymbol{Z};\quad \boldsymbol{Z} \sim \boldsymbol{N}_{\boldsymbol{d}}[\mathbf{0}, \mathbf{I}]$$

Choose two X variables and take their absolute values.

- **If these are independent, then the expectation of their product equals the product of their expectations. It isn't so!**

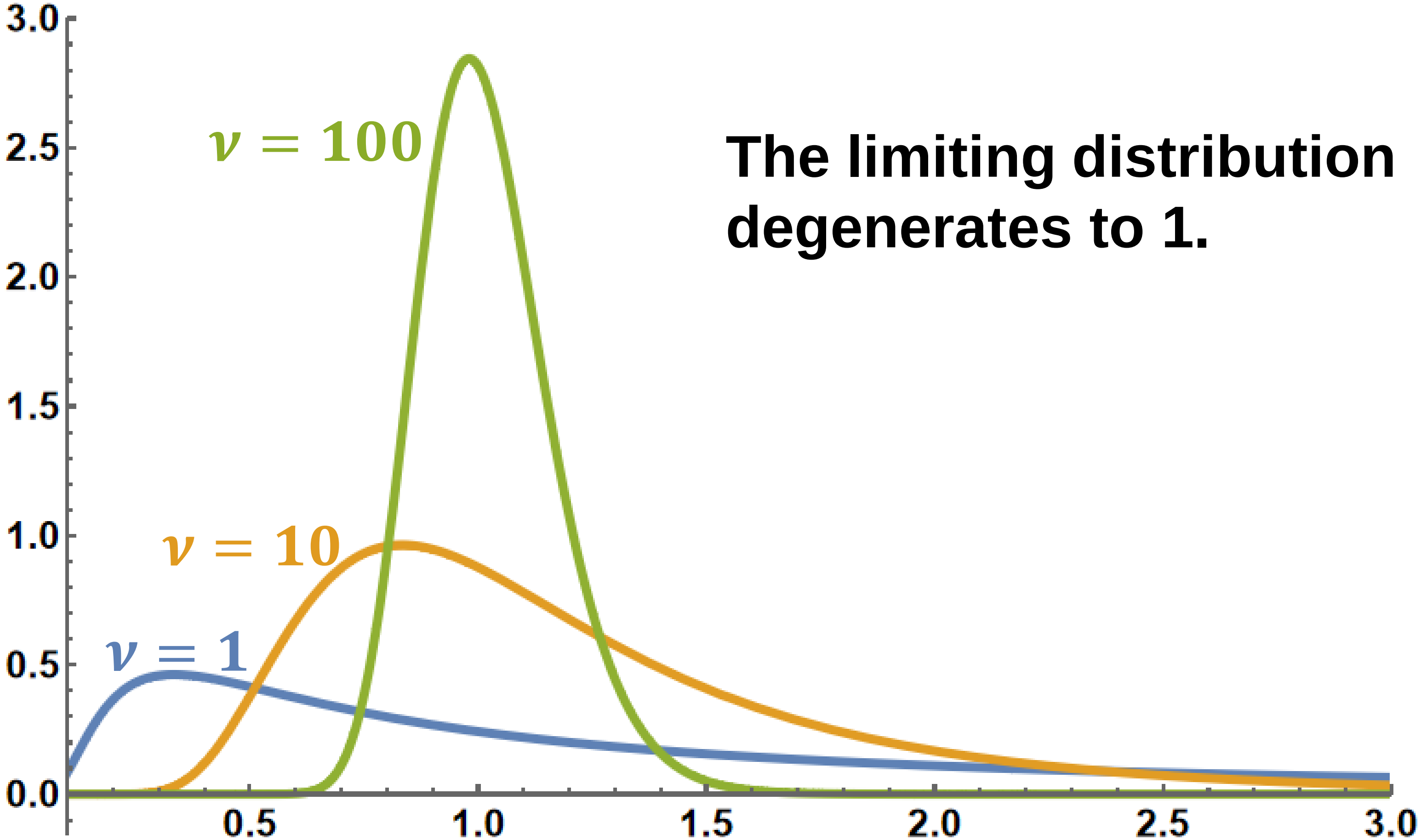
$$\begin{aligned} \boldsymbol{E}[\,|\boldsymbol{X}_1|\,|\boldsymbol{X}_2|\,] &= \boldsymbol{E}[\,\boldsymbol{W}\,|\boldsymbol{Z}_1|\,|\boldsymbol{Z}_2|\,] \\ &= \boldsymbol{E}[\boldsymbol{W}]\;\boldsymbol{E}[\,|\boldsymbol{Z}_1|\,]\;\boldsymbol{E}[\,|\boldsymbol{Z}_2|\,] \\ &> \left(\boldsymbol{E}\left[\sqrt{\boldsymbol{W}}\right]\right)^2 \boldsymbol{E}[\,|\boldsymbol{Z}_1|\,]\;\boldsymbol{E}[\,|\boldsymbol{Z}_2|\,] \\ &= \boldsymbol{E}[\,|\boldsymbol{X}_1|\,]\;\boldsymbol{E}[\,|\boldsymbol{X}_2|\,] \end{aligned}$$

Step 3 logic: Let ω symbolize the random variable $\sqrt{W}$:

$$\begin{aligned} \boldsymbol{Var}[\boldsymbol{\omega}] = \boldsymbol{E}\left[(\boldsymbol{\omega} - \boldsymbol{E}[\boldsymbol{\omega}])^2\right] &= \boldsymbol{E}\left[\boldsymbol{\omega}^2\right] - 2\boldsymbol{\omega}\boldsymbol{E}[\boldsymbol{\omega}] + (\boldsymbol{E}[\boldsymbol{\omega}])^2 \\ &= \boldsymbol{E}\left[\boldsymbol{\omega}^2\right] - (\boldsymbol{E}[\boldsymbol{\omega}])^2 > \mathbf{0}. \end{aligned}$$

A mixing distribution

Let $W \sim Inverse\ Gamma\ Distribution[\frac{\nu}{2}, \frac{\nu}{2}]$, ν an integer.



The limiting distribution degenerates to 1.

Student's *t*

- **The marginal distributions that result are Student's *t*.**
  - **The number of degrees of freedom equals ν.**
- **The connection between the variables is the t-copula.**
  - **Not the same as the Gauss copula.**
    - **Uncorrelated jointly normal variables are independent, for one thing.**
  - **The copula is different for each value of ν.**
    - **As ν grows, the Inverse Gamma degenerates to 1.0, and the t-copula becomes the same as the Gauss copula.**
- **The probability that all three firms default is 0.0343,**
  - **assuming ν = 4. Over three times what Gauss says!**

Gauss　　　t (ν = 4)

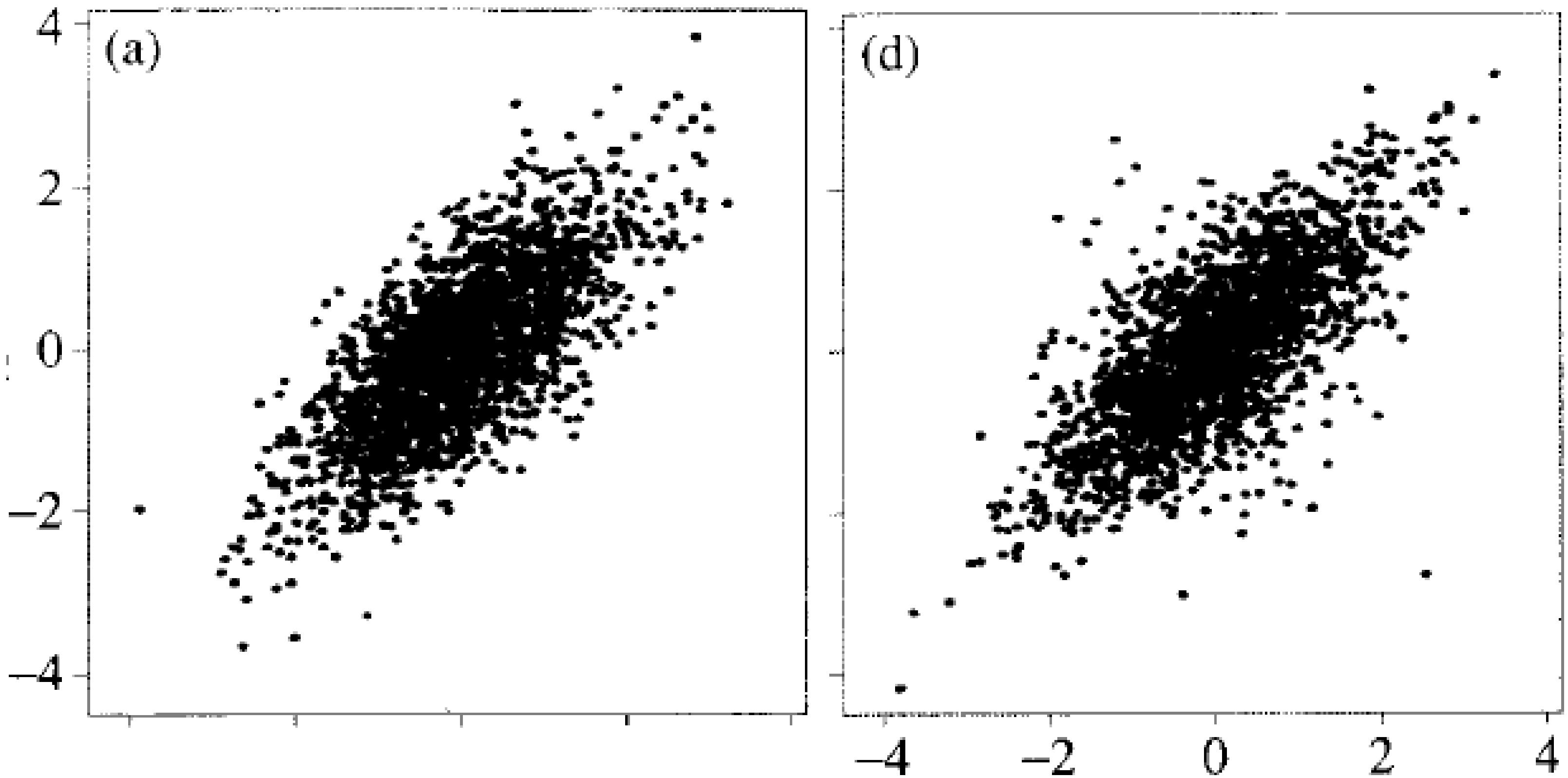


Figure 5.4. Two thousand simulated points from four distributions with standard normal margins, constructed using the copula data from Figure 5.3 ((a) Gaussian, (b) Gumbel, (c) Clayton and (d) t). The Gaussian picture shows points from a standard bivariate normal with correlation 70%; other pictures show distributions with non-Gauss copulas constructed to have a linear correlation of roughly 70%. See Example 5.11 for parameter choices and

Other copulas

We go no further with t-copulas.

- They may be good contenders to the Gauss copula.
- Just not worth the effort now.

Paul Embrechts says there are three uses for copulas:

- Pedagogy, pedagogy, and stress testing.
  - (It is hilarious when he says it.)

I'll show two more copula swarms taken from his book.

- They have explicit mathematical expressions.
  - The Gauss and Student's t copulas are "implicit." One must integrate.
- It would be a miracle if they are helpful in practice.

Gumbel **Clayton**

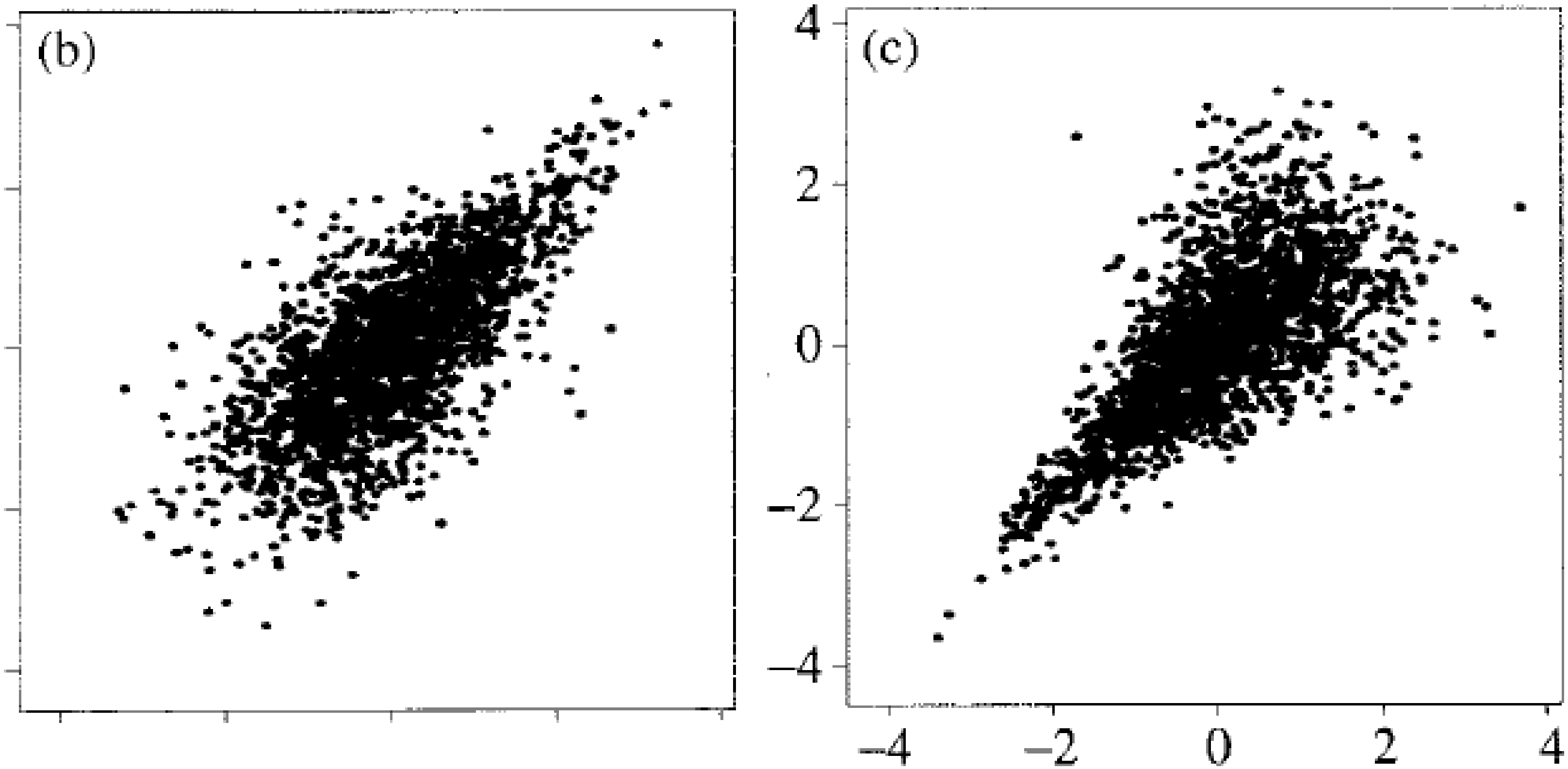


Figure 5.4. Two thousand simulated points from four distributions with standard normal margins, constructed using the copula data from Figure 5.3 ((a) Gaussian, (b) Gumbel, (c) Clayton and (d) t). The Gaussian picture shows points from a standard bivariate normal with correlation 70%; other pictures show distributions with non-Gauss copulas constructed to have a linear correlation of roughly 70%. See Example 5.11 for parameter choices and

Questions? Comments?

Copulas: toward a conclusion

- **I'll present two pedagogic copulas.**
  - **The latent variables are normal but *not* jointly normal.**
    - **The variables are *not* connected by the Gauss copula.**

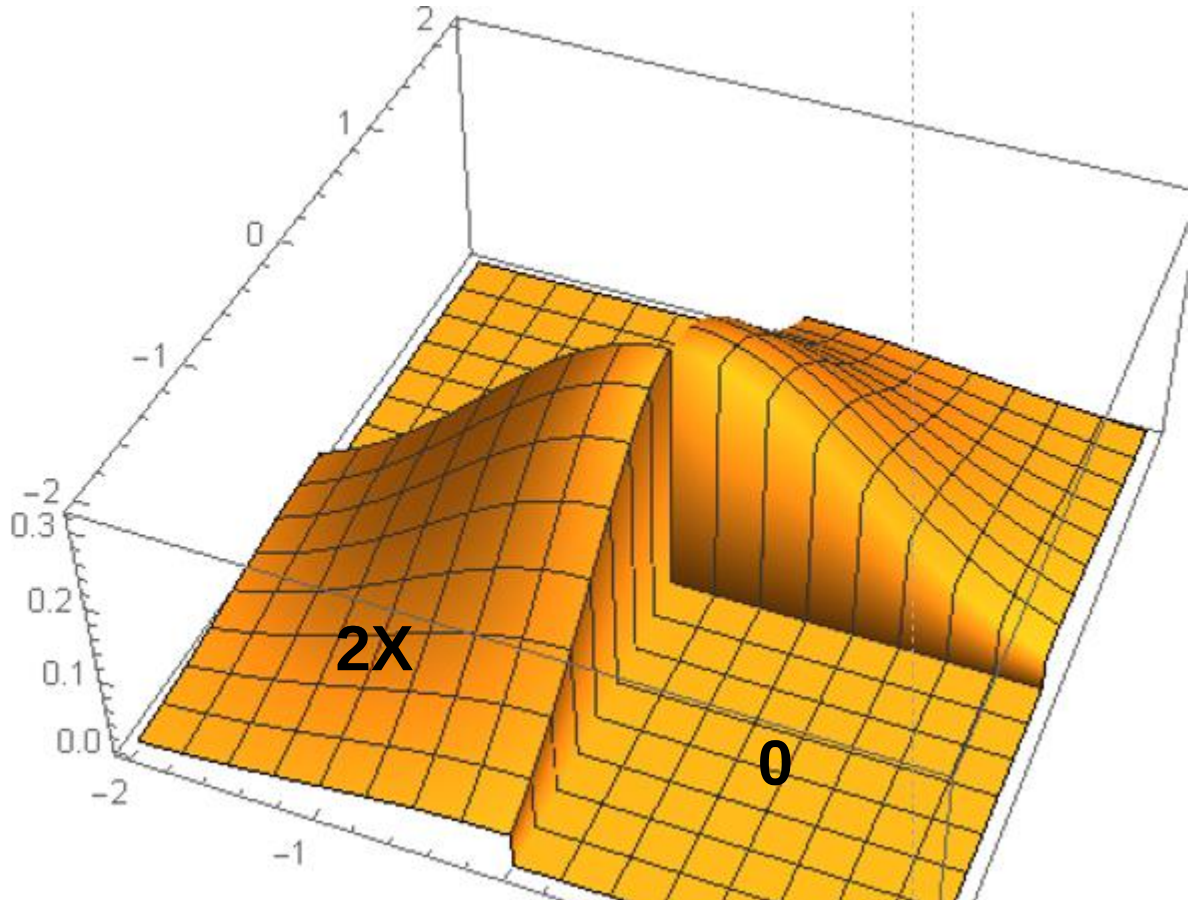
- **Then, the "winner-take-all" credit copula.**
  - **The correlation matrix is not positive definite.**
    - **A Gauss copula is not possible; therefore, it is not present.**

- **At last, I'll define what we're talking about.**
  - **What is a copula?**
  - **What is a Gauss copula?**

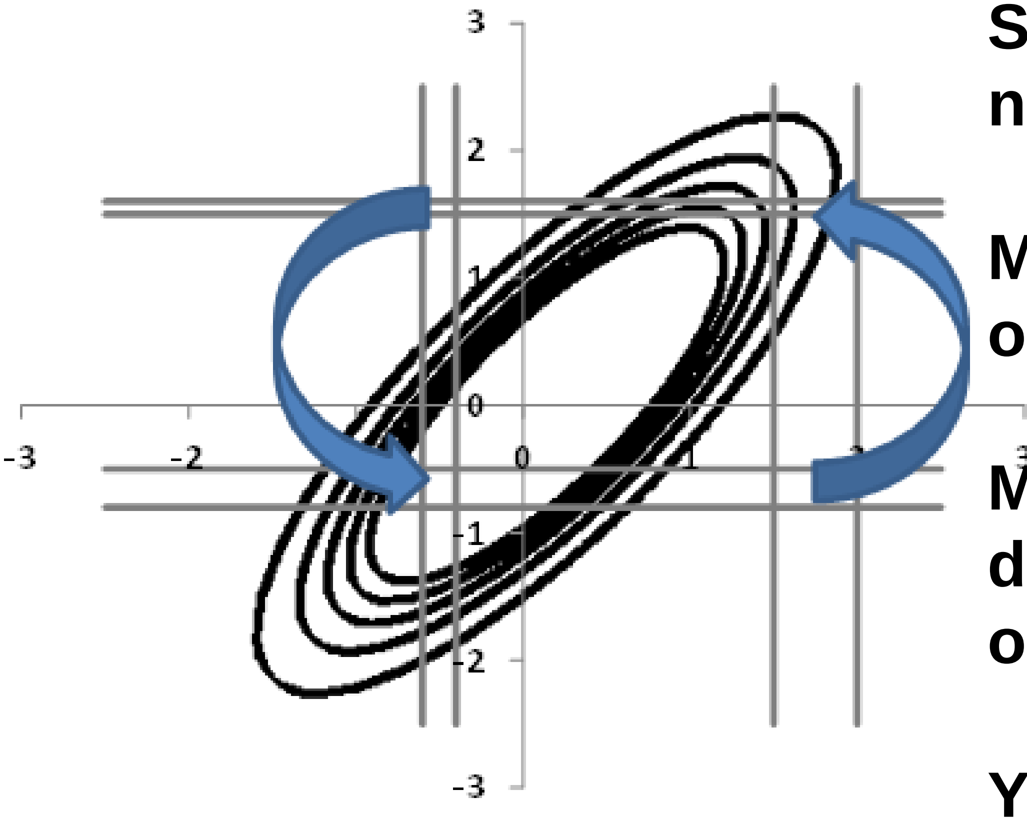
Normal but not jointly normal

Each variable is normal; but they are not *jointly* normal.

- **The variables are dependent but have a non-Gauss copula.**



Degaussing a Gauss copula



Start with correlated joint normals.

Move a little density from one box to the other.

Move the same amount of density the other way on the other side.

You have two standard normals connected by a non-Gauss copula.

The winner-take-all copula

Three firms compete in the development of the next gizmo.

The one that develops it first will become successful, and the other two firms will fail and default on their debts.

- **At present, the firms are equal contenders.**
- **Like the low-risk copula, Prob[$D_1 \cap D_2 \cap D_3$] = 0.**

All the PDs are 2/3 and all the PDJs are 1/3.

- **A firm defaults if either of its rivals wins the competition: 2/3.**
- **A pair of firms defaults if the third firm wins the competition: 1/3.**
    - **All correlations are *negative*: The PDJs are less than the product of the PD's.**
    - **Convince yourself that each correlation equals -1…**

Why this is ***not*** a Gauss copula

The normal latent variable correlation matrix is

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

The Eigenvalues of this matrix are {2, 2, -1}.

The matrix is *not* positive definite or semi-definite.

It *cannot* reflect jointly normal variables.

- **Still, it is a realistic copula and super easy to simulate.**
  - **Draw U, a variable that is uniformly distributed on [0,1].**
  - **If $U < \frac{1}{3}$, Firms 1 and 2 default. If $\frac{1}{3} < U < \frac{2}{3}$, Firms 1 and 3 default.**
  - **If $\frac{2}{3} < U$, Firms 2 and 3 default.**

Questions? Comments?

A copula is a multivariate CDF…

A copula is a multivariate CDF having marginal distributions that are Uniform[0,1].

- **A copula is said to connect the set of uniform variables.**
- **It is also said to connect any collection of random variables that each take a copula marginal variable as its quantile.**
  - **The i^{th} such variable would be $Y_i = CDF_i^{-1}[U_i]$, where U_i is the i^{th} uniform marginal variate of the copula and CDF_i is distribution of the i^{th} of the collection of random variables.**

To simulate portfolio defaults with a Gauss copula, we produce a set of correlated jointly normal variables and compare the i^{th} one to $\Phi^{-1}[PD_i]$.

- **But what, exactly, is a Gauss copula?...**

The Gauss copula is defined by the vector central limit theorem

Scalar central limit theorem: Let $X_i \sim$ IID [μ, σ^2], i = 1, …, n

- **Define $\bar{X} = \frac{1}{n} \sum X_i$**
- **Law of large numbers: As n $\rightarrow \infty$, $\bar{X} \rightarrow \mu$**
- **Central limit theorem: As n $\rightarrow \infty$, $\sqrt{n}\,(\bar{X} - \mu) \overset{d}{\rightarrow}$ something.**
  - **The proof makes clear that if z = $\sqrt{n}\,(\bar{x} - \mu)/\sigma$, then $f_z[z] = Exp\left[-\frac{z^2}{2}\right]/\sqrt{2\,\pi}$.**
  - **This *<u>defines</u>* the standard normal distribution.**

Vector central limit theorem: Let $X_i \sim$ IID [μ_k, Σ_k], i = 1, …

- **LLN: As n $\rightarrow \infty$, $\bar{X} \rightarrow \mu_k$**
- **CLT: As n $\rightarrow \infty$, $\sqrt{n}\,(\bar{X} - \mu_k) \overset{d}{\rightarrow}$ N [0_k, Σ_k]**
- **This *<u>defines</u>* the multivariate normal.**
  - **The Gauss copula is the copula of the vector of normal variables, X.**

A definition and two handy facts

Iff a collection of normal variables is connected by a Gauss copula, the variables are said to be jointly normal.

Independent normal variables are jointly normal.

Variables that are linear functions of jointly normal variables are jointly normal.

Summary of copulas

The set of copulas is vast and trackless.

- The copula has all the information of the multivariate distribution except for the marginal distributions.

The Gauss copula is supported by the vector CLT.

- This is about as helpful as a plank in a shipwreck.
  - If you are ever in a shipwreck, one would be extremely helpful.
  - Other copulas don’t even have this going for them.

t-copulas have “tail dependence”.

- If one variable takes an extreme value, others are more likely to take extreme values than predicted by the Gauss copula.
- Too bad small data sets can’t distinguish between copulas.
  - The scatter plots have 2000 dots, and we have about 100 quarters.
  - A hundred dots could look like just about any kind of copula.

Questions? Comments?

Course mechanics

Here is a quick summary

of what you would find in the Syllabus.

Class personnel

Lecturer: Jon Frye, JonFrye@UChicago.edu

- **PhD Economics, Northwestern University**
- **Federal Reserve, derivatives desks, consulting, brokerages**

Teaching Assistant:

- **Lisheng Su, MS Financial Math, Lisheng@UChicago.edu**

Grader:

- **Davide Negri, MS Financial Math, DavideNegri@uchicago.edu**

Get in touch

For homeworks, please Email Davide and *copy* Lisheng.

For most course content, please start with Lisheng.

I usually answer Emails early the next morning.

- **Sometimes I answer by sending a clarifying Announcement.**
- **I can meet by appointment before class in MS Room 302.**

After the exam, contact me any time about anything.

https://canvas.uchicago.edu/courses/41650

Canvas has a module for each week. Each contains:

- The slide deck for the lecture
- Optional readings
  - This week's reading is the CreditMetrics manual of 1997. This was the first comprehensive articulation of the model that you study this week.
- A homework associated to the lecture.
  - This becomes available after the lecture. It is due before the next lecture.

On-line session notes

- There's more about Lisheng's on-line sessions later.

Syllabus and Announcements

- You are responsible to act on the announcements on Canvas.

Homeworks

Follow carefully the Submission Instructions on Canvas.

Homework is to be done by individuals, not by groups.

- Don’t just copy.

The homeworks prepare you to perform well on the exam.

- This preparation is worth more than the points you get.
  - Homework questions help you think through certain issues in advance.
  - Lisheng's online sessions make the connection more direct.

Each homework set is due at 6pm the next Wednesday.

- Late submissions are graded like others, but the score is penalized 50% the first 24 hours and 100% afterwards.
  - Submit early to be safe!

Five online sessions

Lisheng will discuss the homeworks in online sessions.

- **These take place Sundays at 6:00pm Chicago time.**
  - **The dates are April 3, 10, 17, 24, and May 1.**
- **Have you received an invitation to each Zoom session?**
  - **And, did you get invitations to lecture Zoom sessions?**

Online session topics:

- **Questions and guidance regarding course content**
  - **Please send your questions to Lisheng *in advance* if you can.**
- **In-depth discussion of previous homework questions**
- **Tactics and hints for the current homework**

A paper-and-pencil final exam

6:00 to 7:30 Wednesday May 4

I expect each student to take the exam in person.
- **If you require other arrangements, Email me ASAP.**

You must bring your U of C photo ID to the exam.
- **You cannot use books or written notes of any kind.**
- **You cannot use electronic gear.**
  - **This includes everything: No laptops, tablets, cell phones, or calculators**
- **Any violations will be treated as serious.**

How to prepare for the exam

- **The best preparation is to think about homeworks.**
  - **Exam questions tend to go a bit deeper than homeworks.**
    - **If you haven’t thought about the homework questions, you are unlikely to do well on the exam.**

- **The range of scores tends to be very large.**
  - **From the mathematical expectation up to almost perfect.**

- **The duration of the exam is 75 minutes.**
  - **Most students feel time pressure.**
  - **Plan to manage time wisely.**
    - **Recheck your answers on questions you find easier.**
    - **Do not waste time on questions you find harder.**

Grading

Subject to discretion and (usually upward) adjustment,

- **About 84% of your grade stems from the exam.**
- **About 16% of your grade stems from homeworks.**

If you want to take this course Pass / Fail:

- **Contact Meredith Hajinazarian, memuir@uchicago.edu.**
  - **Or use her student Services Page on Canvas. *There is a deadline*.**
  - **She will check that you have unused options remaining.**
- **Then if you pass the exam, your grade will be "P".**

If you want a grade like A, B, or C (or D or F), do nothing.

- **The average final grade in this course is about equal to the average GPA of recent graduates.**

Weekly events

Wednesday class sessions

- Before class starts:
  - Homeworks are due.
- After class ends:
  - I post the next homework question set.
  - The Zoom session is posted to Canvas.

Sunday TA sessions

- Before the session: Email any questions to Lisheng.
- After the session is over:
  - Lisheng posts a recording of the session to Canvas with solutions to the previous homework and hints about the current one.

Any questions or comments?

Don’t forget

Homework Set 1 is due by 6PM Wednesday April 6.

(Follow carefully the Submission Instructions on Canvas.)

Lisheng’s TA session will be 6PM Sunday April 3.