Solutions
Math 291-3
Homework 8
May 31, 2022

Exercise 1 Compute

$$\int_C (y^2x + x^2 + yx^5)\,dx + (x^2y + x - \sin(y)(y+1)^{y\sin(y)+y^2+3})\,dy$$

where C is the top half of the unit circle oriented clockwise. To be clear, C is NOT closed.

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Exercise 2 Let D be a compact region in $\mathbb{R}^2$ to which Green's Theorem applies. Suppose u is C^2 and **harmonic** on D, meaning that $u_{xx} + u_{yy} = 0$ on D. If $u(x, y) = 0$ for all $(x, y) \in \partial D$, show that $u = 0$ on all of D.

(Thus if a harmonic function is zero on the boundary of a region, then it must be zero throughout the entire region. This implies that the values of a harmonic function throughout a region are fully determined by its values on the boundary alone, which is a key property of harmonic functions.)

Hint: Apply Green's Theorem to the vector field $\vec{F} = -uu_y\vec{i} + uu_x\vec{j}$.

pf: Define $\vec{F} = -uu_y\vec{i} + uu_x\vec{j}$. Since u is C^2 on D, then we can apply Green's Theorem on D to get

$$0 = \oint_{\partial D} \vec{F} \cdot d\vec{s} = \iint\limits_D curl(\vec{F})\, dA(x,t) = \iint\limits_D u_x^2 + uu_{xx} + u_y^2 + uu_{yy}\, dA(x,y) = \iint\limits_D u_x^2 + u_y^2\, dA(x,y)$$

Since $u_x^2, u_y^2 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, so

$$\iint\limits_D u_x^2\, dA(x,y), \iint\limits_D u_y^2\, dA(x,y) \geq 0$$

And since $0 = \iint_D u_x^2 + u_y^2\, dA(x,y)$, So

$$\iint\limits_D u_x^2\, dA(x,y) = \iint\limits_D u_y^2\, dA(x,y) = 0$$

Now we wanna show that $u_x^2 = u_y^2 = 0$ for all $(x, y) \in D$. Without loss of generality, we will show that $u_x^2 = 0$. If $u_x^2 = 0$, then we are done. So suppose there exists $(x_0, y_0) \in D$ such that $u_x^2(x_0, y_0) = c > 0$. Since u_x is continous, then there exists $r > 0$, such that for all $B_r(x_0, y_0) \subseteq D$ and for all $\vec{x} \in B_r(x_0, y_0)$, $u_x^2(\vec{x}) > 0$. Then,

$$\iint\limits_D u_x^2\, dA(x,y) \geq \iint\limits_{B_r(x_0,y_0)} u_x^2\, dA(x,y) > 0$$

Contradiction! Thus $u_x^2(x, y) = 0$ for all $(x, y) \in D$. We can get $u_y^2(x, y) = 0$ for all $(x, y) \in D$ using the exact same argument. Then, $u_x = u_y = 0$. So u is a constant function in D. Since $u(x, y) = 0$ for $(x, y) \in \partial D$, then $u(x, y) = 0$ for $(x, y) \in D$. ∎

Exercise 3 (Colley 7.2.3, 7.2.24) This problem has two unrelated parts.

(a) Find the flux of $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ across the surface S consisting of the triangular portion of the plane $2x - 2y + z = 2$ that is cut out by the coordinate planes. Here assume that S is oriented with upward-pointing normal vectors.

(b) Let $F = 2x\vec{i} + 2y\vec{j} + z^2\vec{k}$. Find $\iint_S \vec{F} \cdot d\vec{S}$, where S is the portion of the cone $x^2 + y^2 = z^2$ between the planes $z = -2$ and $z = 1$, oriented with outward-pointing normal vectors.

(a)

(b)

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Exercise 4 (Colley 7.3.11, 7.3.13b) This problem has two unrelated parts.

(a) Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with normals pointing in the positive y-direction. Let

$$\vec{F} = (2xyz + 5z)\,\vec{i} + e^x \cos(yz)\,\vec{j} + x^2 y\,\vec{k}.$$

Determine

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S}.$$

(b) Evaluate

$$\oint_C (y^3 + \cos(x))\,dx + (\sin(y) + z^2)\,dy + x\,dz$$

where C is the smooth closed curve parametrized (and oriented by) the path $\vec{x}(t) = (\cos(t), \sin(t), \sin(2t)), 0 \leq t \leq 2\pi$. Note that this path lies on the surface $z = 2xy$.

(a)

(b)

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Exercise 5 (Colley 7.3.12) Let S be the surface defined as $z = 4 - 4x^2 - y^2$ with $z \geq 0$ and oriented with normal vectors that have a nonnegative $\vec{k}$-component. Let $\vec{F}(x, y, z) = x^3\,\vec{i} + e^{y^2}\,\vec{j} + ze^{xy}\,\vec{k}$. Find $\iint\limits_S \nabla \times \vec{F} \cdot d\vec{S}$.

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Exercise 6 The goal of this problem is to prove a special case of Stokes' Theorem. Suppose S is the portion of the graph of $z = f(x,y)$, where f is C^2, for (x,y) in a compact region D in the xy-plane with boundary consisting of a single smooth curve. Thus S is parametrized by

$$\vec{X}(x,y) = (x, y, f(x,y)),\ (x,y) \in D.$$

Give S the upward orientation and ∂S the induced orientation. Let $\vec{x}(t) = (x(t), y(t)),\ a \le t \le b$ be parametric equations for ∂D and suppose that ∂S is parametrized by

$$(x(t), y(t), f(x(t), y(t))),\ a \le t \le b.$$

Let $\vec{F}$ be a C^1 vector field of the form $\vec{F} = (P, Q, R)$.

(a) Show that

$$\int_{\partial S} (P,Q,R) \cdot d\vec{s} = \int_{\partial D} (P(\vec{X}(x,y)) + R(\vec{X}(x,y)) f_x(x,y)\,,\ Q(\vec{X}(x,y)) + R(\vec{X}(x,y)) f_y(x,y)) \cdot d\vec{s}$$

(b) Use Green's Theorem to replace the right-hand-side of part (a) with an equivalent double integral over D. This will involve the use of the chain rule.

(c) Use the given parametrization for S to show that the double integral over D produced in part (b) is equal to

$$\iint_S (R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot d\vec{S}.$$

The vector field $(R_y - Q_z, P_z - R_x, Q_x - P_y)$ is the curl of (P, Q, R), so we have shown that Stokes' Theorem holds in this special case.

(a) Based on the parametrization of ∂S and ∂D, we have

$$\begin{aligned}
\int_{\partial S} (P,Q,R) \cdot d\vec{s} &= \int_a^b \begin{bmatrix} P(x(t), y(t), f(x(t), y(t))) \\ Q(x(t), y(t), f(x(t), y(t))) \\ R(x(t), y(t), f(x(t), y(t))) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \\ f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t) \end{bmatrix} dt \\
&= \int_a^b (P + R f_x(x(t), y(t))) x'(t) + (Q + R f_y(x(t), y(t))) y'(t)\, dt \\
&= \int_a^b \begin{bmatrix} P + R f_x(x(t), y(t)) \\ Q + R f_y(x(t), y(t)) \end{bmatrix} \cdot \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} dt \\
&= \int_{\partial D} (P(\vec{X}(x,y)) + R(\vec{X}(x,y)) f_x(x,y)\,,\ Q(\vec{X}(x,y)) + R(\vec{X}(x,y)) f_y(x,y)) \cdot d\vec{s}
\end{aligned}$$

(b) By Green's Theorem, and Clairaut's Theorem, we have

$$\begin{aligned}
RHS &= \int_{\partial D} (P(\vec{X}(x,y)) + R(\vec{X}(x,y))f_x(x,y)\,,\ Q(\vec{X}(x,y)) + R(\vec{X}(x,y))f_y(x,y)) \cdot d\vec{s} \\
&= \iint\limits_D (Q(\vec{X}(x,y)) + R(\vec{X}(x,y))f_y(x,y))_x - (P(\vec{X}(x,y)) + R(\vec{X}(x,y))f_x(x,y))_y \, dA(x,t) \\
&= \iint\limits_D Q_x(\vec{X}(x,y)) + Q_z(\vec{X}(x,y))f_x(x,y) + R_x(\vec{X}(x,y))f_y(x,y) + R(\vec{X}(x,y))f_{yx}(x,y) \\
&- P_y(\vec{X}(x,y)) - P_z(\vec{X}(x,y))f_y(x,y) - R_y(\vec{X}(x,y))f_x(x,y) - R(\vec{X}(x,y))f_{xy}(x,y)\, dA(x,y) \\
&= \iint\limits_D Q_x(\vec{X}(x,y)) + Q_z(\vec{X}(x,y))f_x(x,y) + R_x(\vec{X}(x,y))f_y(x,y) \\
&- P_y(\vec{X}(x,y)) - P_z(\vec{X}(x,y))f_y(x,y) - R_y(\vec{X}(x,y))f_x(x,y)\, dA(x,y)
\end{aligned}$$

(c) Based on the parametrization of S, we can first determine that

$$N_{\vec{X}}(x,y) = \begin{bmatrix} 1 \\ 0 \\ f_x(x,y) \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} -f_x(x,y) \\ -f_y(x,y) \\ 1 \end{bmatrix}$$

Then,

$$\begin{aligned}
\iint\limits_S (R_y - Q_z, P_z - R_x, Q_x - P_y) \cdot d\vec{S} &= \iint\limits_D \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} \cdot \begin{bmatrix} -f_x(x,y) \\ -f_y(x,y) \\ 1 \end{bmatrix} \, dA(x,y) \\
&= \iint\limits_D -R_y(\vec{X}(x,y))f_x(x,y) + Q_z(\vec{X}(x,y))f_x(x,y) - P_z(\vec{X}(x,y))f_y(x,y) \\
&+ R_x(\vec{X}(x,y))f_y(x,y) + Q_x(\vec{X}(x,y)) - P_y(\vec{X}(x,y))\, dA(x,y)
\end{aligned}$$

This is the same as the double integral we get from part(b).

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Exercise 7 (Colley 7.3.26) Let $\vec{n}(x, y, z)$ be a unit normal vector to a smooth surface S. The directional derivative of a differentiable function $f(x, y, z)$ in the direction of $\vec{n}$ is called a **normal derivative** of f, denoted $\frac{\partial f}{\partial n}$. In particular, from our results on directional derivatives we have

$$\frac{\partial f}{\partial n} = \nabla f \cdot \vec{n}.$$

Suppose that $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function such that for any closed, oriented smooth surface S,

$$\iint_S \frac{\partial f}{\partial n}\, dS = 0.$$

Prove that f is **harmonic**, in the sense that $f_{xx} + f_{yy} + f_{zz} = 0$ throughout $\mathbb{R}^3$.

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Exercise 8 (Colley 7.3.20) Use Gauss's theorem to evaluate

$$\iint_S \vec{F} \cdot d\vec{S}$$

where $\vec{F} = ze^{x^2}\,\vec{i} + 3y\,\vec{j} + (2 - yz^7)\,\vec{k}$ and S is the union of the five "upper" faces of the unit cube $[0,1] \times [0,1] \times [0,1]$, each oriented with normal vectors that point "away" from center of the cube $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Note that the $z = 0$ face is *not* part of S.

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Exercise 9 Let $\vec{F}$ be the vector field

$$\vec{F} = \frac{x\,\vec{i} + y\,\vec{j} + z\,\vec{k}}{(x^2+y^2+z^2)^{3/2}}.$$

Show that the surface integral of $\vec{F}$ over any closed, outward-oriented, smooth C^1 surface in $\mathbb{R}^3$ which encloses the origin is 4π.

(Hint: Show that the integral of $\vec{F}$ over any such surface is the same as the integral of $\vec{F}$ over a small-enough outward-oriented sphere centered at the origin.)

pf: Let S be such surface. And let S_1 be the unit sphere centered at the origin with orientation pointing inward (so pointing out of the region enclosed by S_1 and S). Let the region enclosed by S_1 and S be D. Then, by Gauss's theorem

$$\iint\limits_{S} \vec{F}\cdot d\vec{S} = \iint\limits_{S\cup S_1} \vec{F}\cdot d\vec{S} - \iint\limits_{S_1} \vec{F}\cdot d\vec{S} = \iiint\limits_{D} div(\vec{F})dV - \iint\limits_{S_1} \vec{F}\cdot d\vec{S}$$

We will first handle the first part.

$$\begin{aligned}\iiint\limits_{D} div(\vec{F})dV &= 3\times\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3y^2}{(x^2+y^2+z^2)^{5/2}} - \frac{3z^2}{(x^2+y^2+z^2)^{5/2}}\\ &= \frac{3}{(x^2+y^2+z^2)^{3/2}} - \frac{3}{(x^2+y^2+z^2)^{3/2}}\\ &= 0\end{aligned}$$

Now, for S_1, we will parametrize the upper half of S_1 (with positive z coordinate) and multiply that multiply that by 2 to get the whole sphere. by $\vec{X}(\phi,\theta) = (cos(\theta)sin(\phi), sin(\theta)sin(\phi), cos(\phi)), 0 \le \phi \le \pi, 0 \le \theta \le 2\pi$. Then,

$$N_{\vec{X}}(\phi,\theta) = \begin{bmatrix} cos(\theta)cos(\phi)\\ sin(\theta)cos(\phi)\\ -sin(\phi)\end{bmatrix} \times \begin{bmatrix} -sin(\theta)sin(\phi)\\ cos(\theta)sin(\phi)\\ 0\end{bmatrix} = \begin{bmatrix} cos(\theta)sin^2(\phi)\\ sin(\theta)sin^2(\phi)\\ sin(\phi)cos(\phi)\end{bmatrix}$$

Then,

$$\begin{aligned}\iint\limits_{S_1} \vec{F}\cdot d\vec{S} &= \int_0^{\pi}\int_0^{2\pi} \begin{bmatrix} cos(\theta)sin(\phi)\\ sin(\theta)sin(\phi)\\ cos(\phi)\end{bmatrix} \cdot \begin{bmatrix} cos(\theta)sin^2(\phi)\\ sin(\theta)sin^2(\phi)\\ sin(\phi)cos(\phi)\end{bmatrix} d\theta d\phi\\ &= \int_0^{\pi}\int_0^{2\pi} sin(\phi)\, d\theta d\phi\\ &= \int_0^{\pi} 2\pi sin(\phi)\, d\phi\\ &= [-2\pi cos(\phi)]_0^{\pi}\\ &= 4\pi\end{aligned}$$

Thus we have,

$$\iint\limits_{S} \vec{F}\cdot d\vec{S} = \iiint\limits_{D} div(\vec{F})dV - \iint\limits_{S_1} \vec{F}\cdot d\vec{S} = 0 - 4\pi = -4\pi$$

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Exercise 10 Prove Gauss's Theorem in the special case where E is bounded by the surfaces $x = h_2(y,z)$ on the front and $x = h_1(y,z)$ on the back where $(y,z) \in D$ is the shadow of E in the yz-plane, and $\vec{F}$ has the form $\vec{F} = P(x,y,z)\,\vec{i}$.

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