Solutions
Math 291-3
Homework 4
April 29, 2022

Exercise 1 (Colley 5.5.32 and 5.5.36) Determine the value of each of the following integrals, where W is as descirbed.

(a) $\iiint\limits_W \frac{z}{\sqrt{x^2+y^2}}\,dV(x,y,z)$, where W is the solid region bounded below by the plane $z=12$, below by the paraboloid $z=2x^2+2y^2-6$, and lies outside the cylinder $x^2+y^2=1$.

(b) $\iiint\limits_W (x+y+z)\,dV(x,y,z)$, where W is the solid region in the first octant between the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, where $0<a<b$.

(If you set this up directly in terms of spherical coordinates, you'll get an integral which involves a lot of messy computation, and which will require some not-so-obvious trig identity. Perhaps try to find a way to cut down on the amount of computation required.)

(a)

(b)

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Exercise 2 (Colley 5.5.29 and 5.5.37, altered)

(a) Evaluate

$$\int_{-1}^{1} \int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \int_{0}^{4-x^2-4y^2} e^{x^2+4y^2+z}\, dz\, dx\, dy$$

by using cylindrical-like coordinates.

(b) Determine the value of

$$\iiint_W z^2\, dV(x, y, z)$$

where W is the solid region lying above the cone $z = \sqrt{3x^2 + 3y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 6z$. In addition (not in the book), determine the value of this integral if W is the region lying above the elliptic cone $z = \sqrt{3x^2 + 12y^2}$ and inside the ellipsoid $x^2 + 4y^2 + z^2 = 6z$ instead.

(a)

(b)

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Exercise 3 (Colley 5.5.42) Find the volume of the intersection of the three solid cylinders

$$x^2 + y^2 \leq a^2, \quad x^2 + z^2 \leq a^2, \quad y^2 + z^2 \leq a^2.$$

(Hint: First draw a careful sketch, then note that, by symmetry, it suffices to calculate the volume of a portion of the intersection.)

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Exercise 4 For now, take the following fact for granted: if $f : \mathbb{R}^n \to \mathbb{R}$ continuous and $\vec{x}_0 \in \mathbb{R}^n$, then

$$\lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(\vec{x})\, dV_n(\vec{x}) = f(\vec{x}_0).$$

In the solutions I'll prove this result.
Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is C^1, injective, and that $DT(\vec{x})$ is invertible at every $\vec{x} \in \mathbb{R}^n$. Show that for each $\vec{x}_0 \in \mathbb{R}^n$,

$$\lim_{r\to 0^+} \frac{\mathrm{Vol}_n(T(B_r(\vec{x}_0)))}{\mathrm{Vol}_n(B_r(\vec{x}_0))} = |\det(DT(\vec{x}_0))|.$$

(This emphasizes the "infinitesimal expansion factor" interpretation of $|\det(DT(\vec{x}_0)|$, since it says that the ratio between the volume of the image of a ball under T and the volume of that ball itself for very small radii is essentially $|\det(DT(\vec{x}_0)|$.)

Hint: Express the numerator of the limit as an integral and use a change of variables.

pf: Define $f(\vec{u}) = 1$ for all $\vec{u} \in T(B_r(\vec{x}))$. Then, because $DT(\vec{x})$ is invertible, T is injective, and T is C^1. Since f is a constant function, so it is integrable on $T(D)$. Then, by the change of variable theorem,

$$\begin{aligned}
\lim_{r\to 0^+} \frac{\mathrm{Vol}_n(T(B_r(\vec{x}_0)))}{\mathrm{Vol}_n(B_r(\vec{x}_0))} &= \lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{T(B_r(\vec{x}_0))} f(\vec{u})\, dV_n\vec{u} \\
&= \lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(T(\vec{x}))|det(DT(\vec{x}_0))|\, dV_n\vec{x} \\
&= |det(DT(\vec{x}_0))| \lim_{r\to 0^+} \frac{1}{\mathrm{Vol}_n(B_r(\vec{x}_0))} \int_{B_r(\vec{x}_0)} f(T(\vec{x}))\, dV_n\vec{x} \\
&= |det(DT(\vec{x}_0))| f(T(\vec{x}_0)) \\
&= |det(DT(\vec{x}_0))| \times 1 \\
&= |det(DT(\vec{x}_0))|
\end{aligned}$$

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Exercise 5 Let C be a smooth curve in $\mathbb{R}^n$ with parametrization $\vec{x} : [a, b] \to \mathbb{R}^n$. Show that C lies on a hypersphere (i.e. the set of points at a fixed distance away from the origin) if, and only if, $\vec{x}(t)$ and $\vec{x}\,'(t)$ are orthogonal for every $t \in [a, b]$.

Hint: To say that C lies on a hypersphere is to say that $\|\vec{x}(t)\|$ equals the same value for all t, which is the same as saying that $\|\vec{x}(t)\|^2$ is constant in t.

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Exercise 6 (Colley 3.1.35) Let $\vec{x}(t)$ be a path of class C^1 that does not pass through the origin in $\mathbb{R}^3$. Suppose $\vec{x}(t_0)$ is a point on the image of $\vec{x}$ closest to the origin and $\vec{x}\,'(t_0) \neq \vec{0}$. Show that $\vec{x}(t_0)$ is orthogonal to $\vec{x}\,'(t_0)$.

pf: Define $f(t) = \|\vec{x}(t)\|^2 = \vec{x}(t) \cdot \vec{x}(t)$. Since $\vec{x}(t_0)$ is the point on the image of $\vec{x}$ closest to the origin, $\|\vec{x}(t)\|$ is the smallest when $t = t_0$. Then, $\|\vec{x}(t)\|^2$ is also the smallest when $t = t_0$. Since $f(t)$ is continuous, and thus differentiable, so t_0 is a critical point for f such that $f\,'(t_0) = 0$. Then,

$$0 = f\,'(t_0) = (\vec{x}(t_0) \cdot \vec{x}(t_0))\,' = \vec{x}\,'(t_0) \cdot \vec{x}(t_0) + \vec{x}(t_0) \cdot \vec{x}\,'(t_0) = 2\vec{x}\,'(t_0) \cdot \vec{x}(t_0)$$

Since both $\vec{x}(t_0)$ and $\vec{x}\,'(t_0)$ does not equal to zero, so $\vec{x}(t_0)$ and $\vec{x}\,'(t_0)$ is orthogonal. ∎

Exercise 7 (Colley 3.2.14) Consider the path $\vec{x}(t) = (e^{-t}\cos(t), e^{-t}\sin(t))$.

(a) Argue that the path spirals toward the origin as $t \to +\infty$.

(b) Show that, for any a, the improper integral

$$\int_a^\infty \|\vec{x}'(t)\|\, dt$$

converges.

(c) Interpret what the result in part (b) says about the path $\vec{x}$.

(a)

(b)

(c)

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Exercise 8 (Colley 3.2.15) Suppose that a curve is given in polar coordinates by an equation of the form $r = f(\theta)$, where f is C^1. Derive the formula

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta$$

for the length L of the curve between the points $(f(\alpha), \alpha)$ and $(f(\beta), \beta)$ given in polar coordinates.

pf: Define $\vec{g}(\theta) = (f(\theta)cos(\theta), f(\theta)sin(\theta))$. Then, since f is C^1,

$$\begin{aligned}
L &= \int_\alpha^\beta \|\vec{g}\,'(\theta)\|\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f\,'(\theta)cos(\theta) - f(\theta)sin(\theta))^2 + (f\,'(\theta)sin(\theta) + f(\theta)cos(\theta))^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{(f\,'(\theta)cos(\theta))^2 + (f(\theta)sin(\theta))^2 + (f\,'(\theta)sin(\theta))^2 + (f(\theta)cos(\theta))^2}\, d\theta \\
&= \int_\alpha^\beta \sqrt{f\,'(\theta)^2 + f(\theta)^2}\, d\theta
\end{aligned}$$

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Exercise 9 Suppose C is a smooth C^1 curve in $\mathbb{R}^n$ with two parametrizations

$$\vec{x} : [a, b] \to \mathbb{R}^n \quad \text{and} \quad \vec{y} : [c, d] \to \mathbb{R}^n$$

related by $\vec{y} = \vec{x} \circ \tau$ for some C^1, bijective map $\tau : [c, d] \to [a, b]$ with $\tau'(u) \neq 0$ for every $u \in [c, d]$. Show that the tangent vector at a point along C determined by $\vec{x}$ points in the same direction as the one determined by $\vec{y}$ if, and only if, $\tau'(u)$ is positive for all $u \in [c, d]$.

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Exercise 10 Suppose C is a smooth curve in $\mathbb{R}^n$ with parametrization $\vec{x} : [a, b] \to \mathbb{R}^n$, and that $f : C \to \mathbb{R}$ is a continuous function on C. The **scalar line integral** of f over C is defined to be:

$$\int_C f \, ds \stackrel{def}{=} \int_a^b f(\vec{x}(t)) \|\vec{x}\,'(t)\| \, dt.$$

Show that this definition is independent of parametrization. To be clear, this means that if $\vec{y} : [c, d] \to \mathbb{R}^n$ is another parametrization of C which is related to $\vec{x}$ via

$$\vec{y} = \vec{x} \circ \tau$$

for some C^1, bijective map $\tau : [c, d] \to [a, b]$ with $\tau'(u) \neq 0$ for every $u \in [c, d]$, you want to show that the integral above is the same as the one obtained by using $\vec{y}$ instead of $\vec{x}$.

Suppose $\vec{y} : [c, d] \to \mathbb{R}$ is another parametrization of C which is related to $\vec{x}$ such that $\vec{y}(u) = \vec{x} \circ \tau(u)$ for all $u \in [c, d]$. Since we know that f is integrable, and both $\vec{x}, \vec{y}$ are parametrization for f, and τ is a bijective map from $\vec{y}$ to $\vec{x}$. Then, by the change of variable theorem,

$$\begin{aligned}
\int_c^d f(\vec{y}(u)) |\vec{y}\,'(u)| \, du &= \int_c^d f(\vec{x} \circ \tau(u)) \|\vec{x}\,'(\tau(u))\tau\,'(u)\| \, du \\
&= \int_c^d f(\vec{x}(\tau(u))) \|\vec{x}\,'(\tau(u))\| \|\tau\,'(u)\| \, du \\
&= \int_a^b f(\vec{x}(t)) \|\vec{x}\,'(t)\| \, dt
\end{aligned}$$

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