Algorithm and Software Design

Algorithm Design: Numerical Problems

Week 03

Learning Objectives

- Able to develop systematic algorithm to solve numerical problems.
- Understanding bisection algorithm, finding area under the curve, and Runge-Kutta method

Finding a Root of a Function

How do you find **a root** of a given function **?**

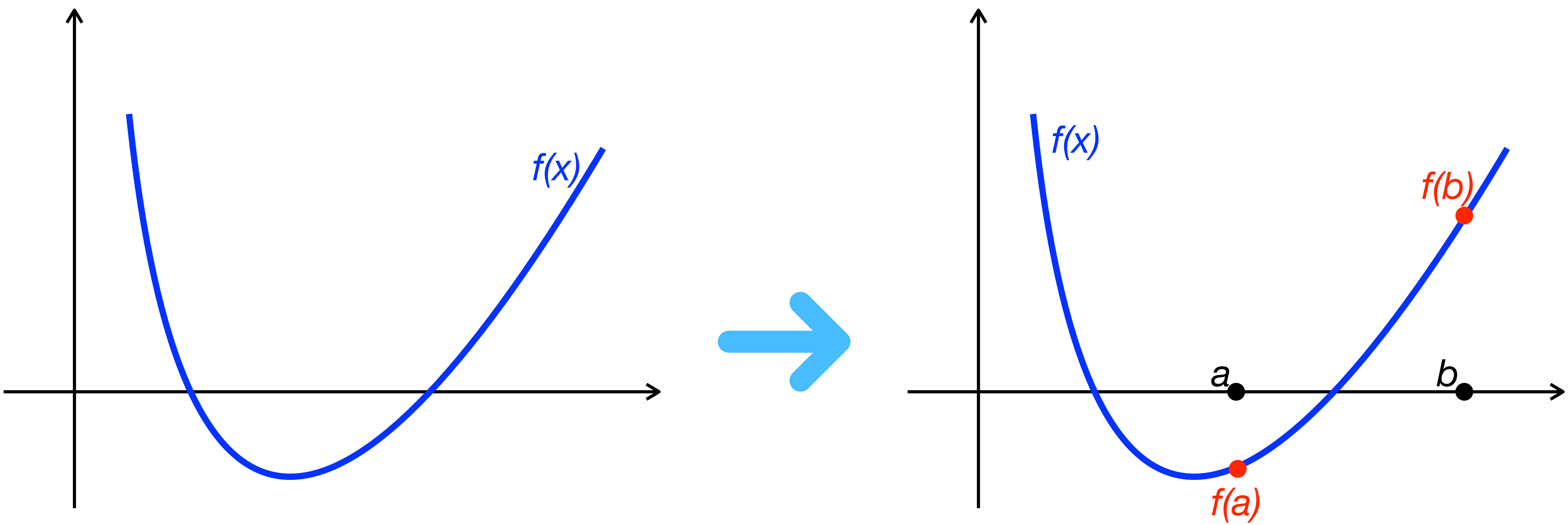
$$f(x) = x^2 - 1$$

Analytically, you may do the following way →

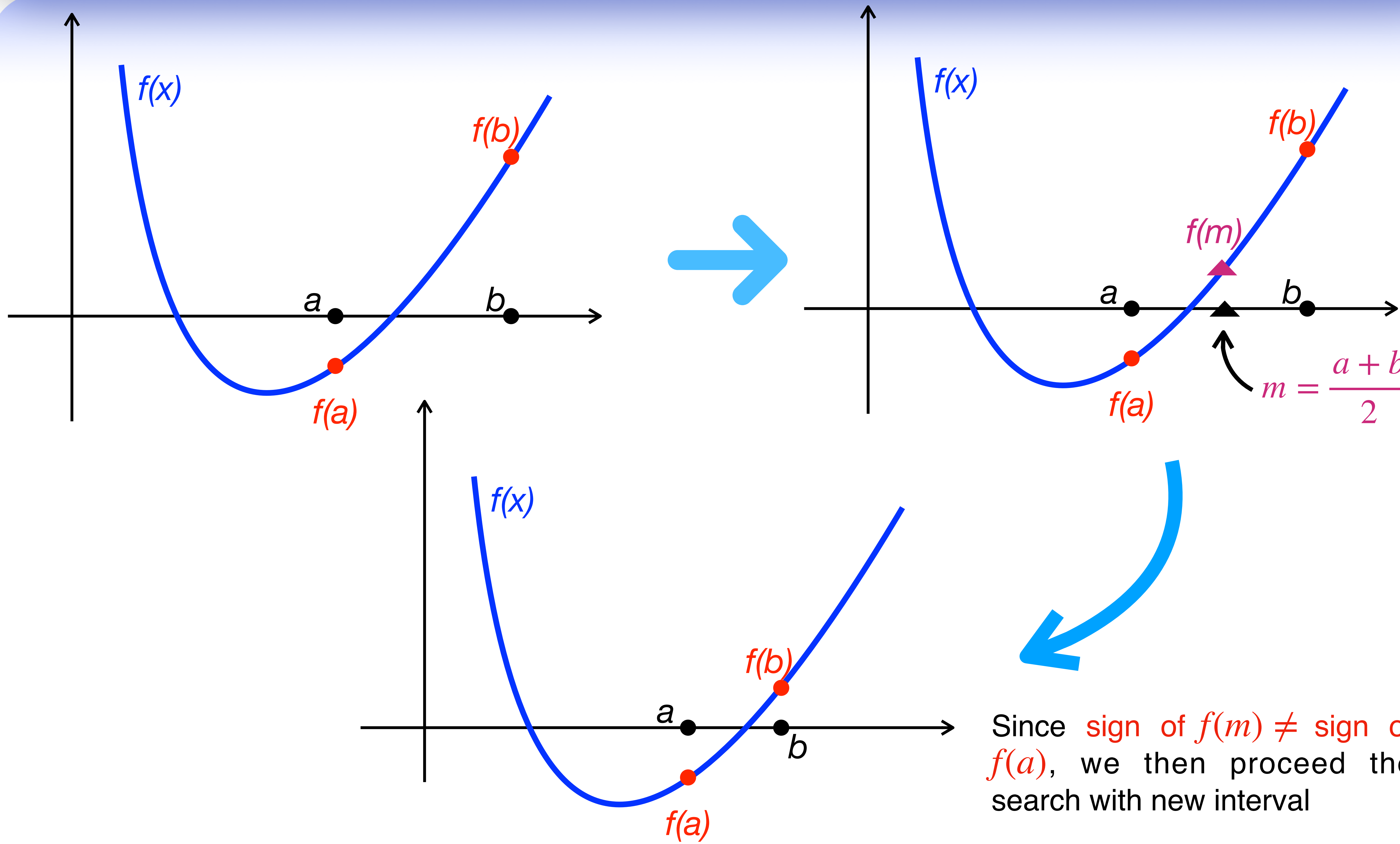
$$x_1; x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Bisection Method

The Bisection Method is a *successive* approximation method that narrows down an interval that contains a root of the function *f(x)*.



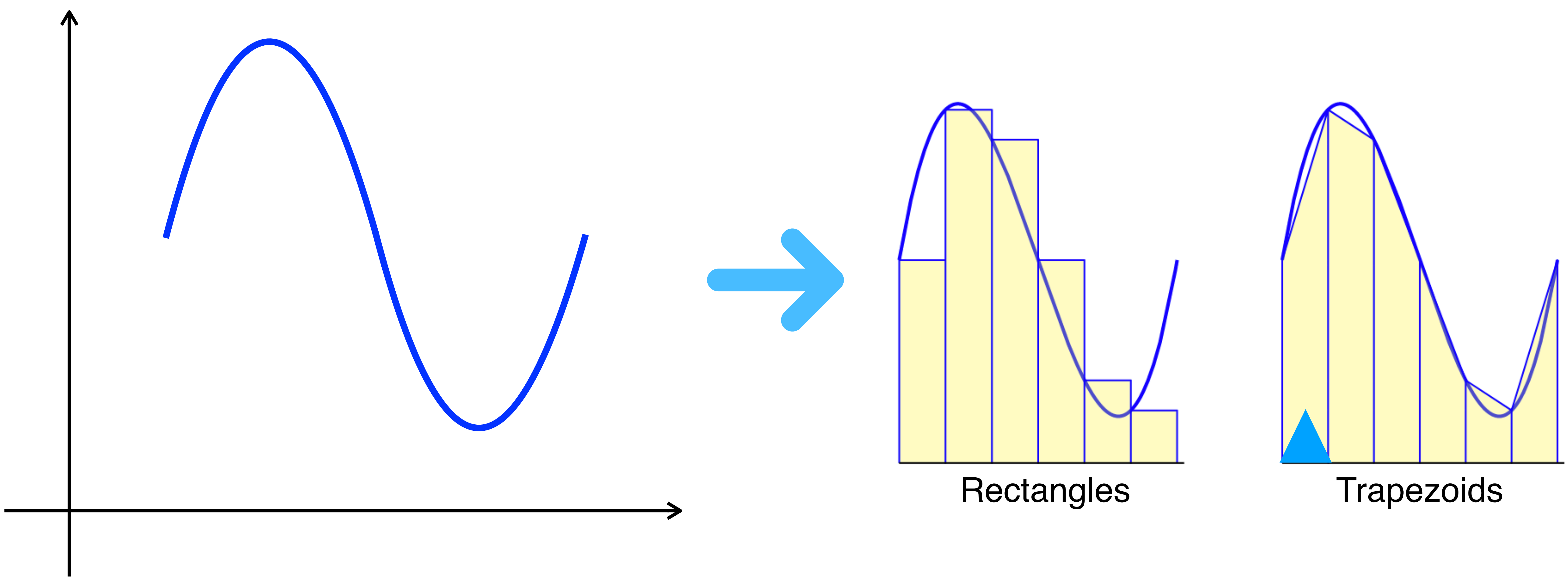
Bisection Method



Since sign of $f(m) \neq$ sign of $f(a)$, we then proceed the search with new interval

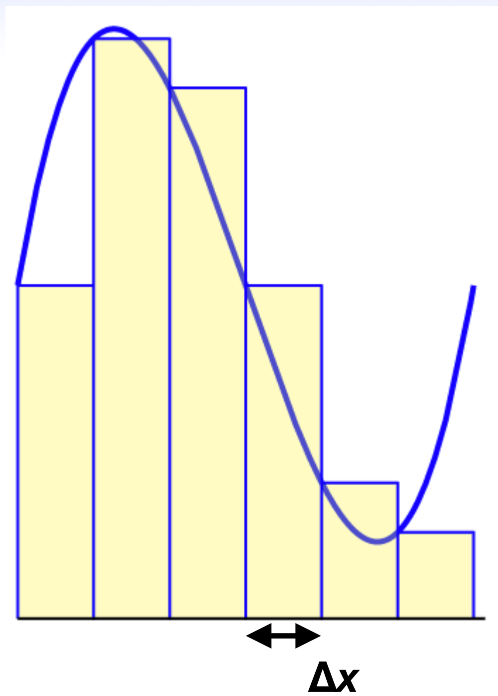
Finding Area Under the Curve

How do you define **an area** under a given curve ?

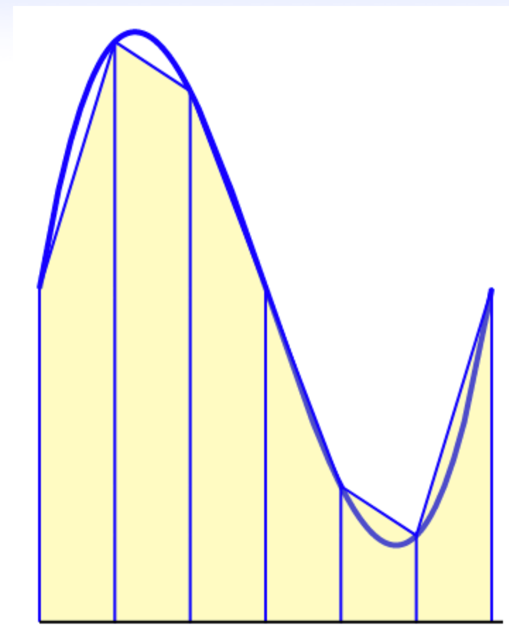


Numerical Integration

Rectangles



Trapezoids



$$\text{Area} = \frac{f(x_0)+f(x_1)}{2}\Delta x + \frac{f(x_1)+f(x_2)}{2}\Delta x + \ldots. + \frac{f(x_{n-1})+f(x_n)}{2}\Delta x$$

$$= \left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \ldots. + f(x_{n-1}) + \frac{f(x_n)}{2}\right)\Delta x$$

Runge-Kutta 4th Order Method (RK4)

How do you get a solution of a given ordinary differential equation ?

$$\frac{dy}{dx} = f(x, y) \ ; \ \boxed{y(x_0) = y_0}$$

Initial condition

Runge-Kutta 4th Order Method (RK4)

In general, the RK4 is written as:

$$y_{n+1} = y_n + h \sum_{i=1}^{4} b_i k_i$$

$$= y_n + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4) \quad \text{where} \quad h = x_{\mathrm{i}+1} - x_{\mathrm{i}}$$

Let's take a look into the first five terms of Taylor expansion:

$$\boxed{\begin{aligned} y_{i+1} = y_i + \frac{dy}{dx}\bigg|_{x_i, y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\bigg|_{x_i, y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\bigg|_{x_i, y_i}(x_{i+1} - x_i)^3 \\ + \frac{1}{4!}\frac{d^4y}{dx^4}\bigg|_{x_i, y_i}(x_{i+1} - x_i)^4 \end{aligned}}$$

Runge-Kutta 4th Order Method (RK4)

$$y_{i+1} = y_i + \frac{dy}{dx}\Big|_{x_i,y_i}(x_{i+1} - x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}\Big|_{x_i,y_i}(x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}\Big|_{x_i,y_i}(x_{i+1} - x_i)^3 + \frac{1}{4!}\frac{d^4y}{dx^4}\Big|_{x_i,y_i}(x_{i+1} - x_i)^4$$

$$\rightarrow \quad \frac{dy}{dx} = f(x, y)\ ;\ y(x_0) = y_0$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!}f'(x_i, y_i)h^2 + \frac{1}{3!}f''(x_i, y_i)h^3 + \frac{1}{4!}f'''(x_i, y_i)h^4$$

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f\left(x_i + \frac{h}{2}, y_i + h\frac{k_1}{2}\right)$$

$$k_3 = f\left(x_i + \frac{h}{2}, y_i + h\frac{k_2}{2}\right)$$

$$k_4 = f\left(x_i + h, y_i + hk_3\right)$$

Runge-Kutta 4th Order Method (RK4)

Based on the following differential equation:

$$\frac{dy}{dx} = \frac{1+y^2}{2}\ ;\ y(0) = 0$$

Define $y(0.40)$ by using 4th order Runge-Kutta!
(Use $h = 0.1$)

Runge-Kutta 4th Order Method (RK4)

$$\frac{dy}{dx} = \frac{1+y^2}{2}\ ;\ y(0) = 0 \quad \Rightarrow \quad f(x,y) = \frac{1+y^2}{2}$$

Untuk $i = 0$, $x_0 = 0$ dan $y_0 = 0$:

$$k_1 = f(x_i, y_i) = f(0,0) = \frac{1}{2}$$

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + h\frac{k_1}{2}\right) = f(0.05,0.025) = 0.5003125$$

$$k_3 = f\left(x_i + \frac{h}{2}, y_i + h\frac{k_2}{2}\right) = f(0.05,0.025015625) = 0.50031289074707$$

$$k_4 = f\left(x_i + h, y_i + hk_3\right) = f(0.1,0.050031289) = 0.50125156494323$$

Runge-Kutta 4th Order Method (RK4)

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{0.1}{6} \times 3.00250234643737 8$$

$$= 0.050041705773956$$

Runge-Kutta 4th Order Method (RK4)

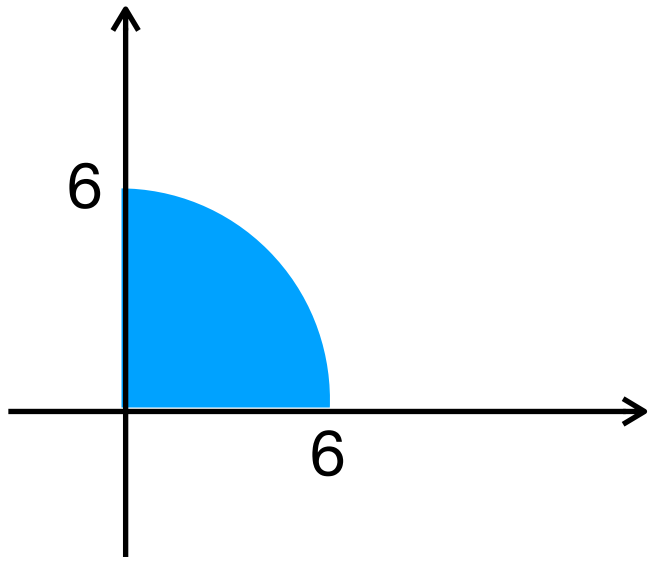
Untuk $i = 1$, $x_1 = 0.1$ dan
$y_1 = 0.050041705773956$:

Assignment

1. By using **bisection method**, find the root of the following function(s):

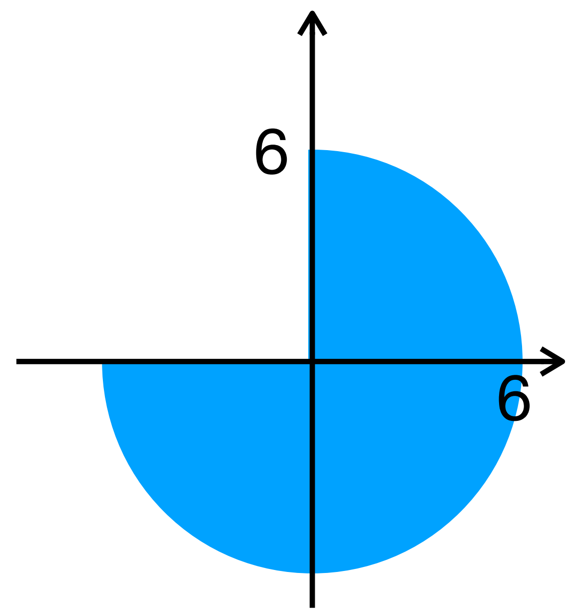
- $f(x) = x^2 - 3x - 2$
- $f(x) = x^3 + x^2 - 3x - 2$

2. By using **numerical integration**, find the area under following curve(s):



Weekly Assignment

1. By using **numerical integration**, find the area under following curve(s):



Weekly Assignment

2. A ball at 1200 K is allowed to cool down in air at an ambient temperature of 300 K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by:

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\,(\theta - 8.1 \times 10^{9}); \quad \theta(0) = 1200\text{K}$$

where θ is in K and t in seconds. Find the temperature at t = 480 s using RungeKutta 4th order method. Assume a step size of h are 30 s, 60 s, 120 s, dan 240 s.