Solving ODEs
Euler Method & RK2/4

http://www.mpia.de/homes/mordasini/UKNUM/UeB_ode.pdf

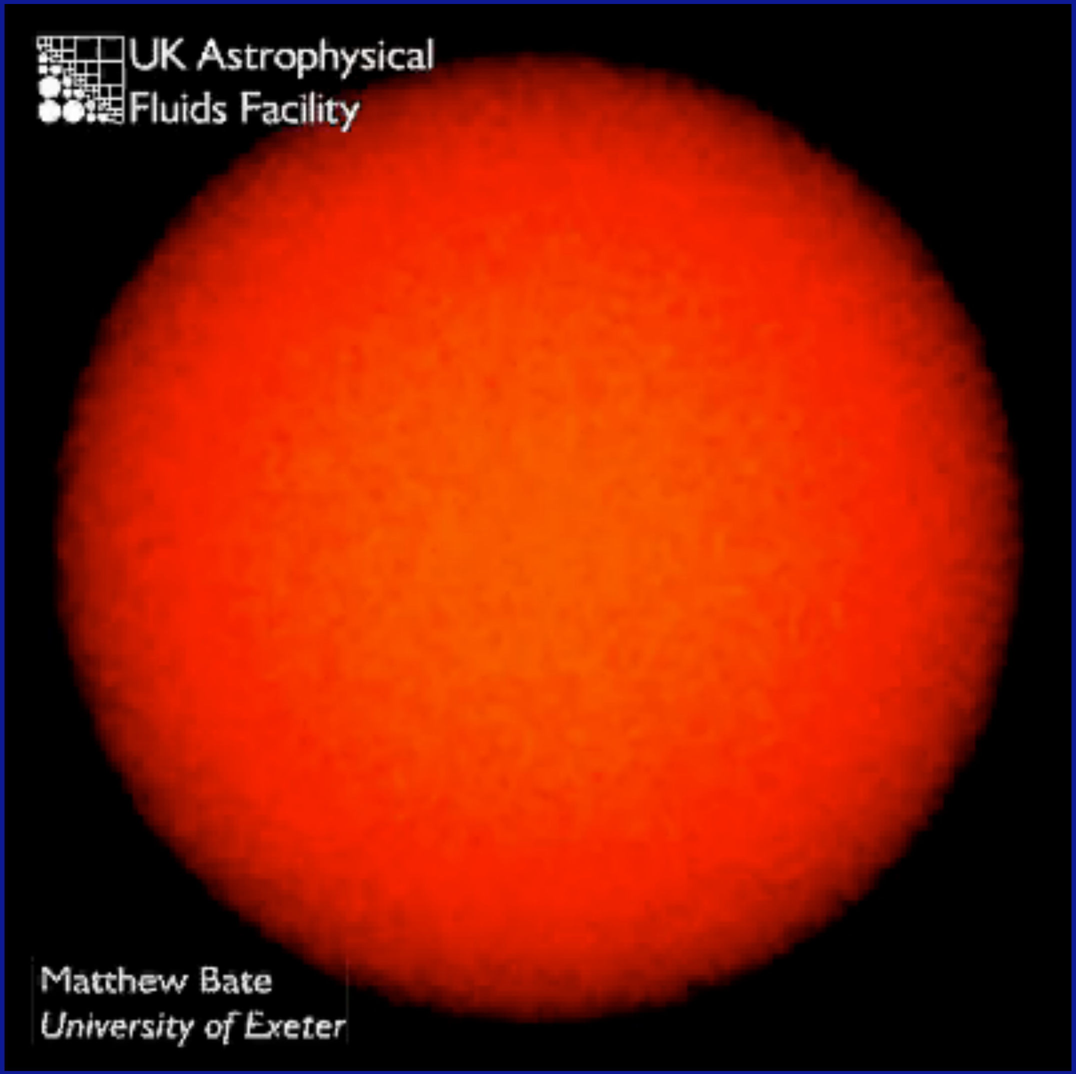
Major: All Engineering Majors

Authors: Autar Kaw, Charlie Barker

http://numericalmethods.eng.usf.edu

Transforming Numerical Methods Education for STEM

Simulation example: Formation of Stars



SPH simulation with gravity and super-sonic turbulence.

Initial conditions:

Uniform density
$1000M_{sun}$
1pc diameter
Temperature 10K

es

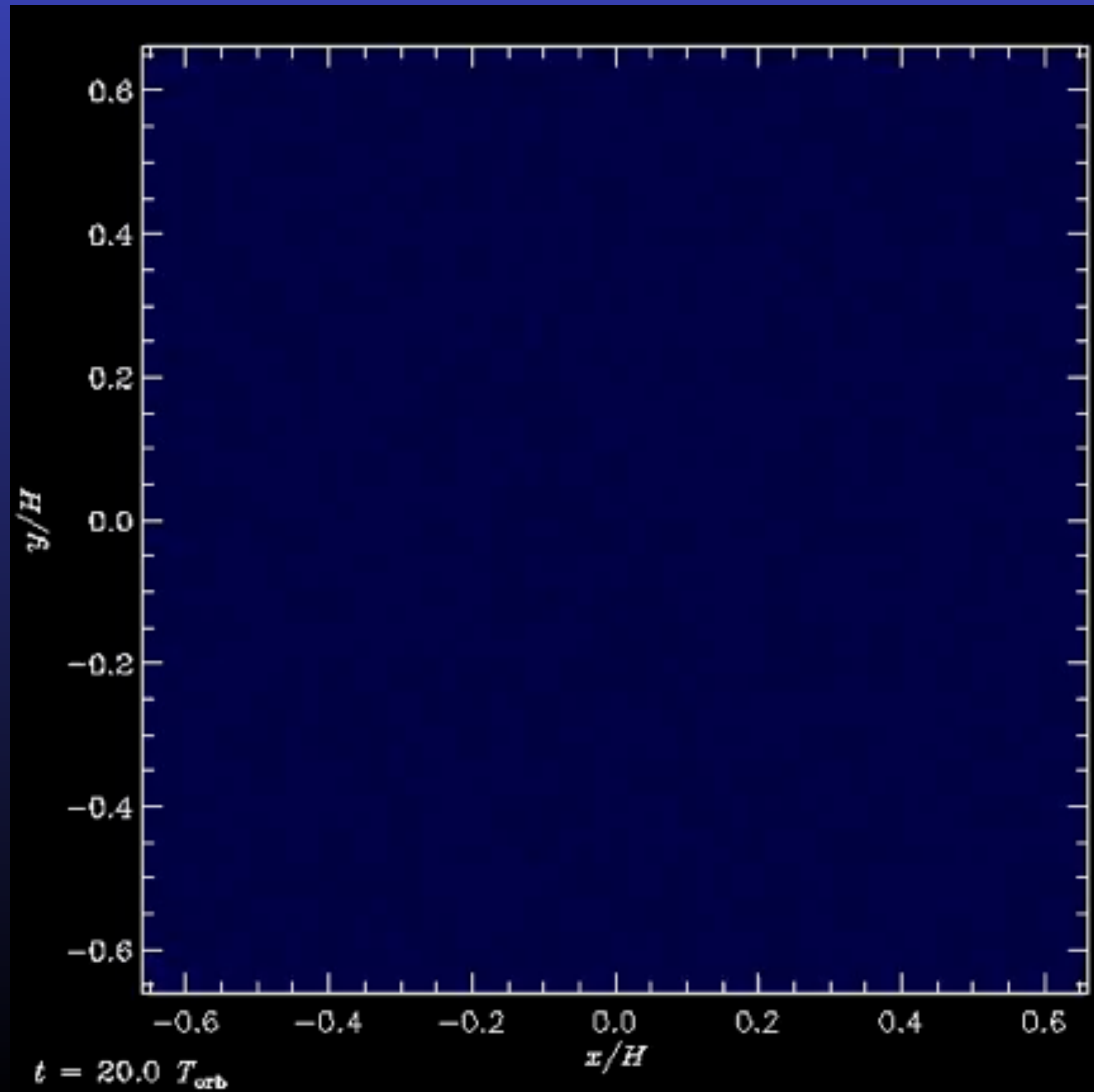
Turbulence and Accretion in 3D Global
MHD Simulations of Stratified Protoplanetary Disk

Gravoturbulent formation of planetesimals - Concentration in Zonal Flows:

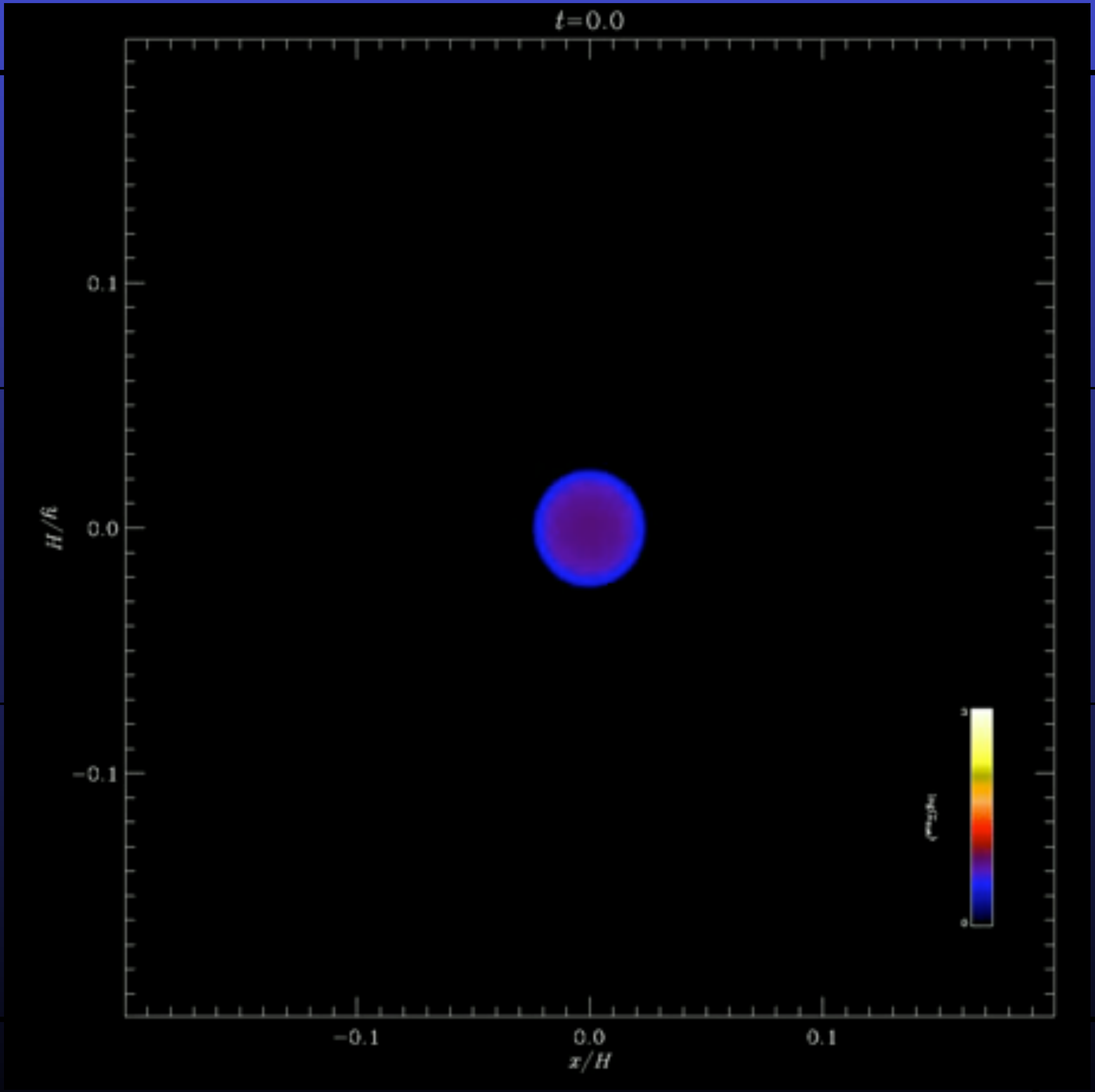
Formation Of Planetesimals From pressure trapped / gravitational Bound heaps of gravel - here magnetic turbulence:

Johansen, Klahr & Henning 2011.
Vortices: Raettig, Klahr & Lyra

512 ^2 simulation
64 Mio particles
Entire project used 15 Mio. CPU hours.



Local collaps 100 x resolution: 0.1 Roche; St: 0.1



Dittrich 2013 PhD thesis

MHD plus self-gravity for the dust, including particle feed back on the gas: Pencil Code: Finite Differenzen / Runge-Kutta 5th order!

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + u_y^{(0)}\frac{\partial \boldsymbol{u}}{\partial y} &= 2\Omega u_y \hat{\boldsymbol{x}} - \frac{1}{2}\Omega u_x \hat{\boldsymbol{y}} - \boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B} \\
&\quad - \frac{1}{\rho}c_{\mathrm{s}}^2\boldsymbol{\nabla}\rho - \frac{\rho_{\mathrm{d}}/\rho}{\tau_{\mathrm{f}}}(\boldsymbol{u}-\boldsymbol{w}) + \boldsymbol{f}_\nu(\boldsymbol{u},\rho)\,, \\
\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho + u_y^{(0)}\frac{\partial \rho}{\partial y} &= -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} + f_{\mathrm{D}}(\rho)\,, \\
\frac{\partial \boldsymbol{A}}{\partial t} + u_y^{(0)}\frac{\partial \boldsymbol{A}}{\partial y} &= \frac{3}{2}\Omega A_y \hat{\boldsymbol{x}} + \boldsymbol{u}\times\boldsymbol{B} + \boldsymbol{f}_\eta(\boldsymbol{A})\,, \\
\nabla^2\Phi &= 4\pi G(\rho+\rho_{\mathrm{d}})\,.
\end{aligned}$$

gas

$$\begin{aligned}
\frac{\partial \boldsymbol{v}^{(i)}}{\partial t} &= 2\Omega v_y^{(i)}\hat{\boldsymbol{x}} - \frac{1}{2}\Omega v_x^{(i)}\hat{\boldsymbol{y}} - \Omega^2 z - \boldsymbol{\nabla}\Phi(\boldsymbol{x}^{(i)}) - \frac{1}{\tau_{\mathrm{f}}}[\boldsymbol{v}^{(i)} - \boldsymbol{u}(\boldsymbol{x}^{(i)})]\,, \\
\frac{\partial \boldsymbol{x}^{(i)}}{\partial t} &= \boldsymbol{v}^{(i)} + u_y^{(0)}\hat{\boldsymbol{y}}\,.
\end{aligned}$$

dust

Poisson equation solved via FFT in parallel mode: up to 256^3 cells

Euler Method

http://numericalmethods.eng.usf.edu

Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope} = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

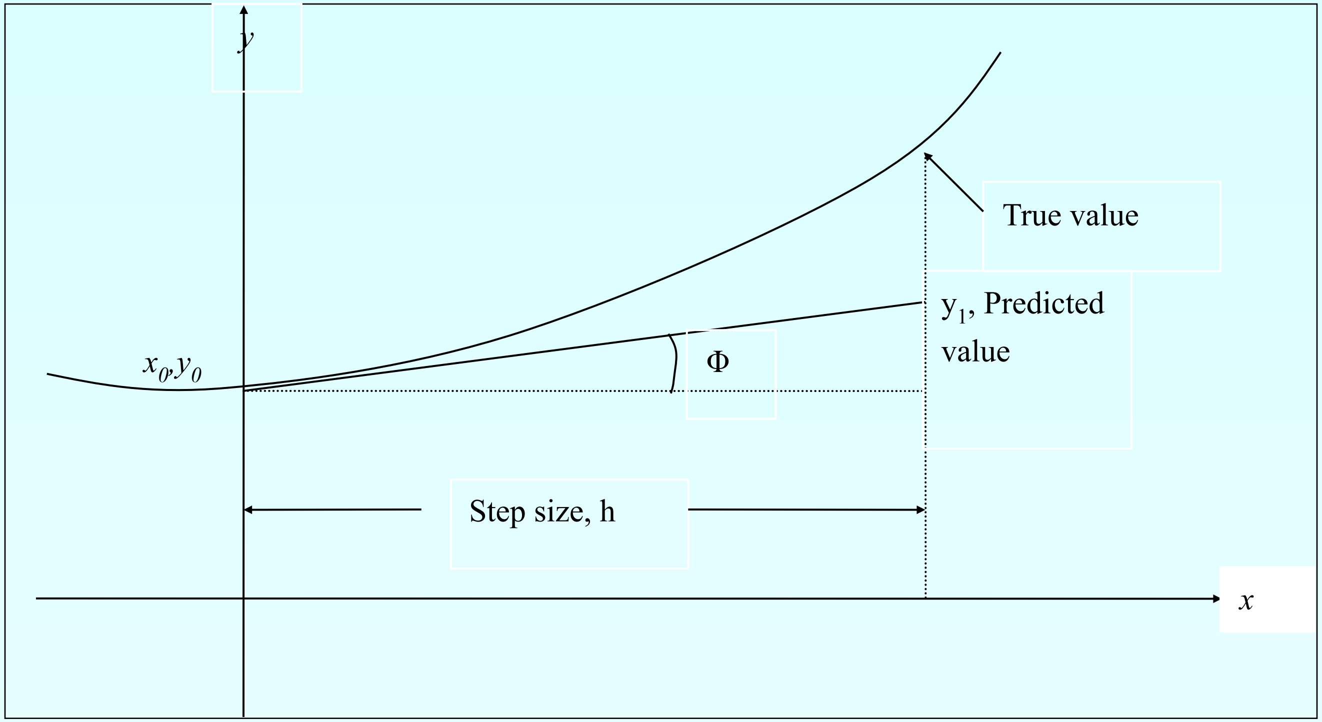


Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$h = x_{i+1} - x_i$$

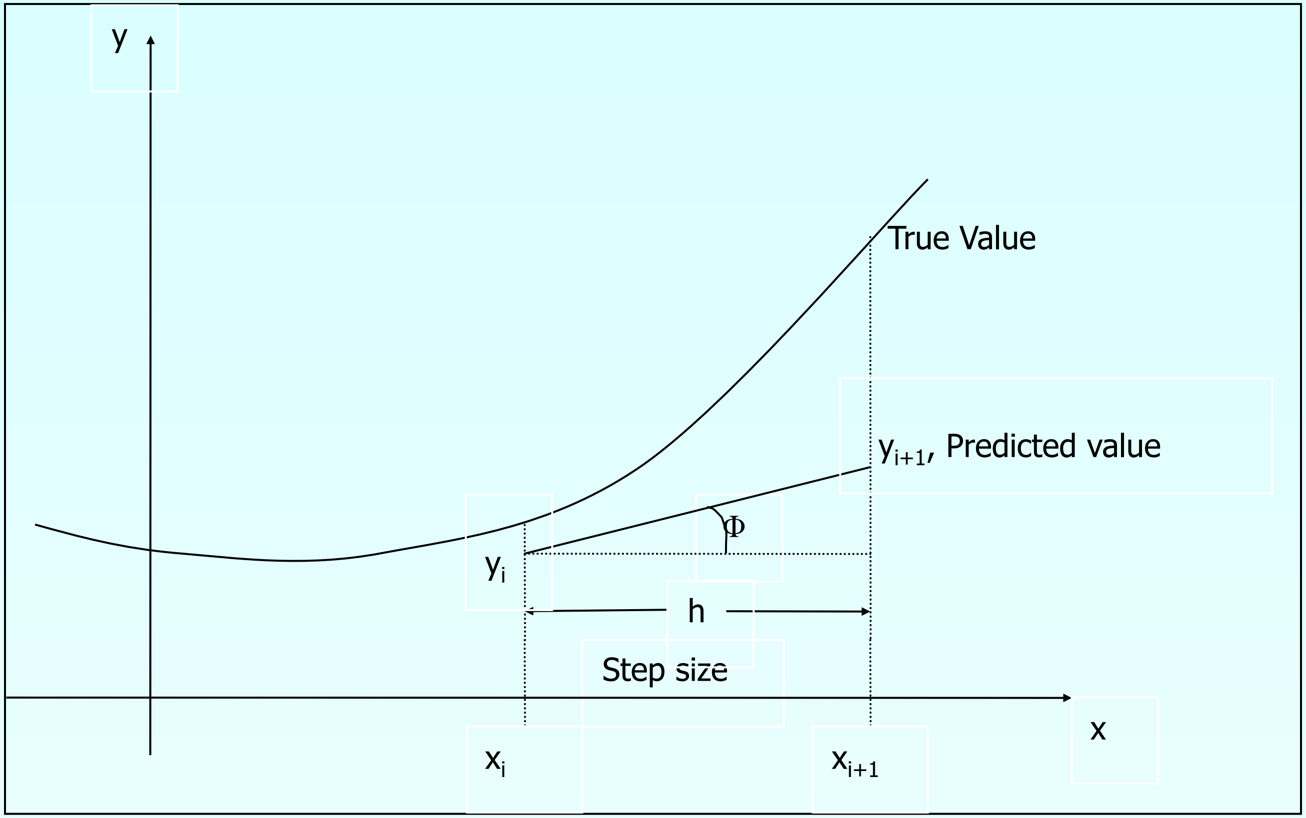


Figure 2. General graphical interpretation of Euler's method

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i,\theta_i)h$$

$$\begin{aligned}\theta_1 &= \theta_0 + f(t_0,\theta_0)h\\ &= 1200 + f(0,1200)240\\ &= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240\\ &= 1200 + (-4.5579)240\\ &= 106.09K\end{aligned}$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

Solution Cont

Step 2: For $i = 1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$\begin{aligned}
\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\
&= 106.09 + f(240, 106.09)240 \\
&= 106.09 + \left(-2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)\right)240 \\
&= 106.09 + (0.017595)240 \\
&= 110.32K
\end{aligned}$$

θ_2 is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}(0.00333\theta)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison of Exact and Numerical Solutions

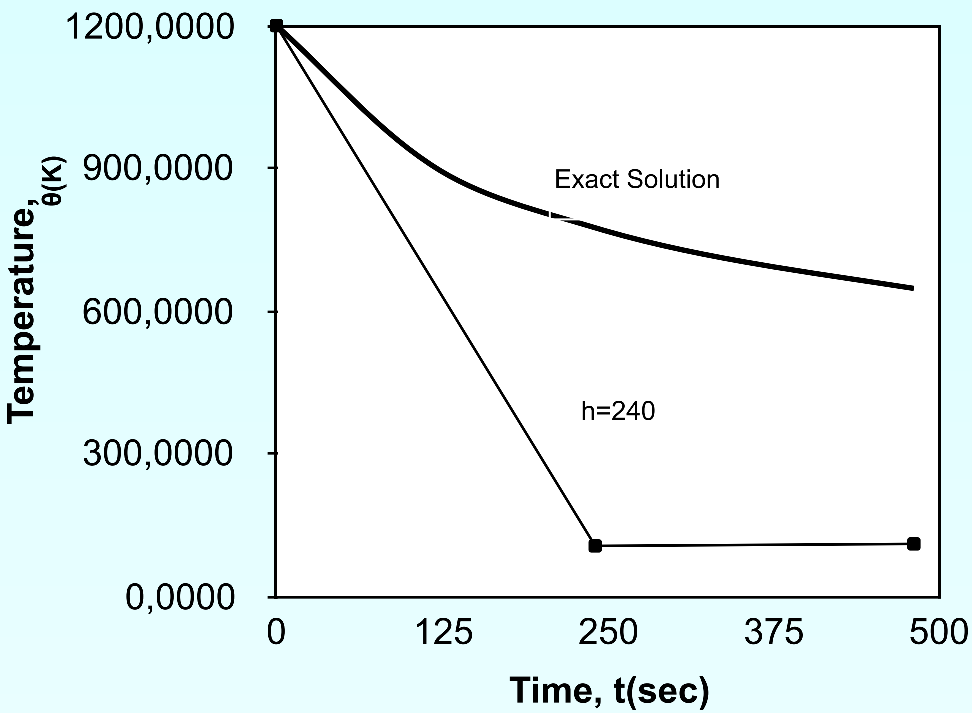


Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	-987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60		32.607	5.0352

$\theta(480) = 647.57K$ (exact)

Comparison with exact results

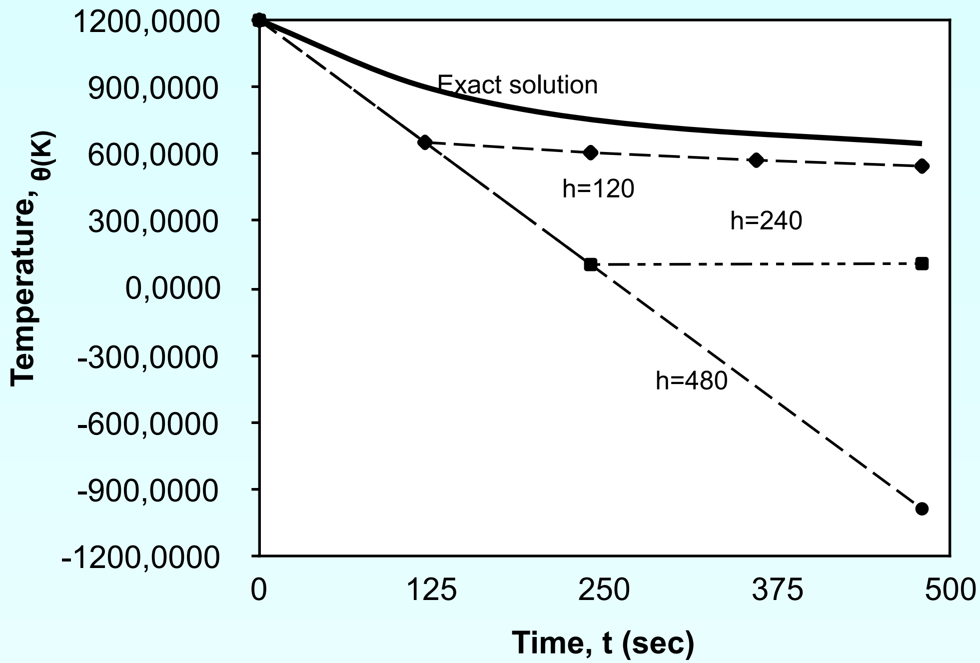


Figure 4. Comparison of Euler's method with exact solution for different step sizes

Effects of step size on Euler's Method

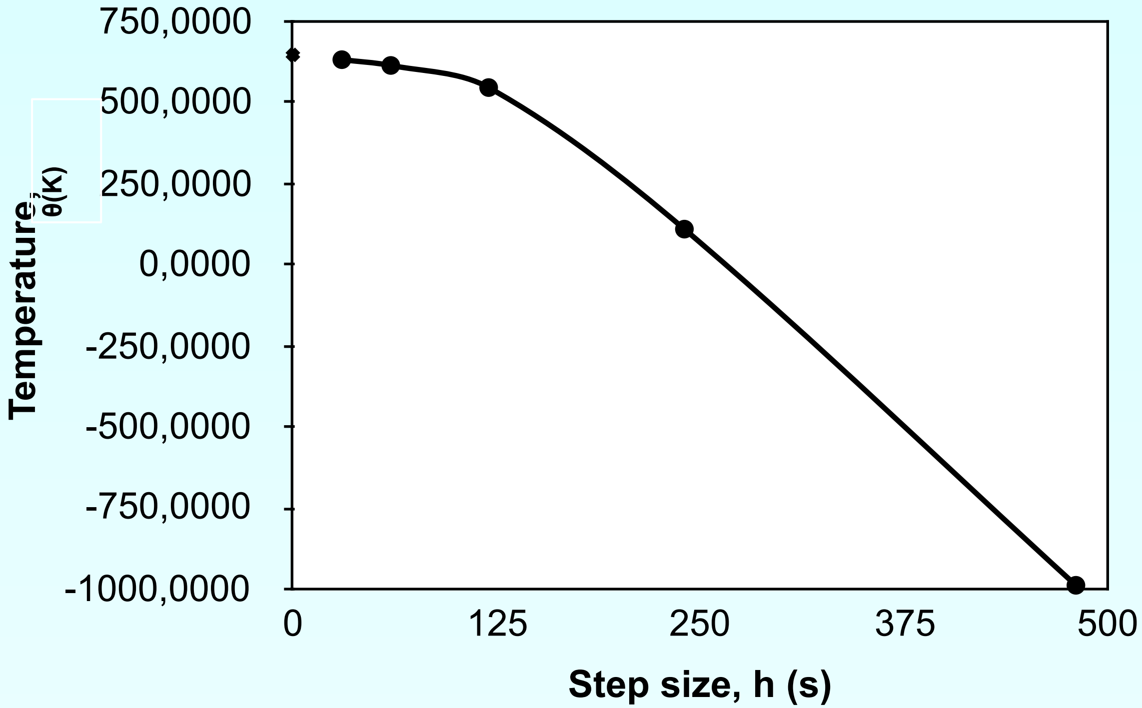


Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \dots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \dots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots \qquad E_t \propto h^2$$

Runge 2^{nd} Order Method

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Runge-Kutta 2nd Order Method

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Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x, y), y(0) = y_0$

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + \left(a_1 k_1 + a_2 k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + p_1 h, y_i + q_{11} k_1 h\right)$$

Heun's Method

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

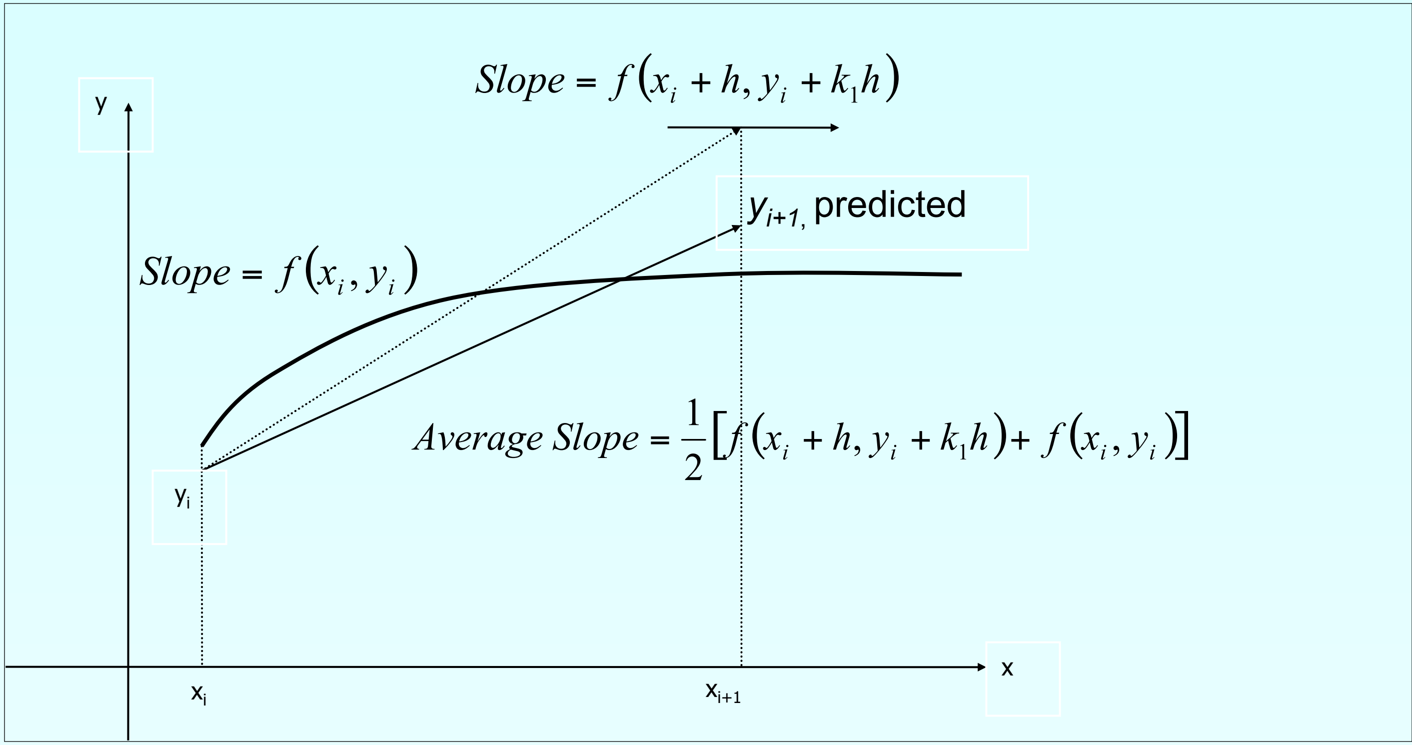

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right),\ \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Heun's method. Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i=0, t_0=0, \theta_0=\theta(0)=1200K$

$$\begin{aligned} k_1 &= f(t_0, \theta_o) \\ &= f(0, 1200) \\ &= -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) \\ &= -4.5579 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_0 + h, \theta_0 + k_1 h) \\ &= f(0+240, 1200+(-4.5579)240) \\ &= f(240, 106.09) \\ &= -2.2067\times10^{-12}\left(106.09^4 - 81\times10^8\right) \\ &= 0.017595 \end{aligned}$$

$$\begin{aligned} \theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240 \\ &= 1200 + (-2.2702)240 \\ &= 655.16K \end{aligned}$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$
\begin{aligned}
k_1 &= f(t_1, \theta_1) \\
&= f(240, 655.16) \\
&= -2.2067\times10^{-12}\left(655.16^4 - 81\times10^8\right) \\
&= -0.38869
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\
&= f(240 + 240, 655.16 + (-0.38869)240) \\
&= f(480, 561.87) \\
&= -2.2067\times10^{-12}\left(561.87^4 - 81\times10^8\right) \\
&= -0.20206
\end{aligned}
$$

$$
\begin{aligned}
\theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\
&= 655.16 + (-0.29538)240 \\
&= 584.27K
\end{aligned}
$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}(0.0033333\theta)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison with exact results

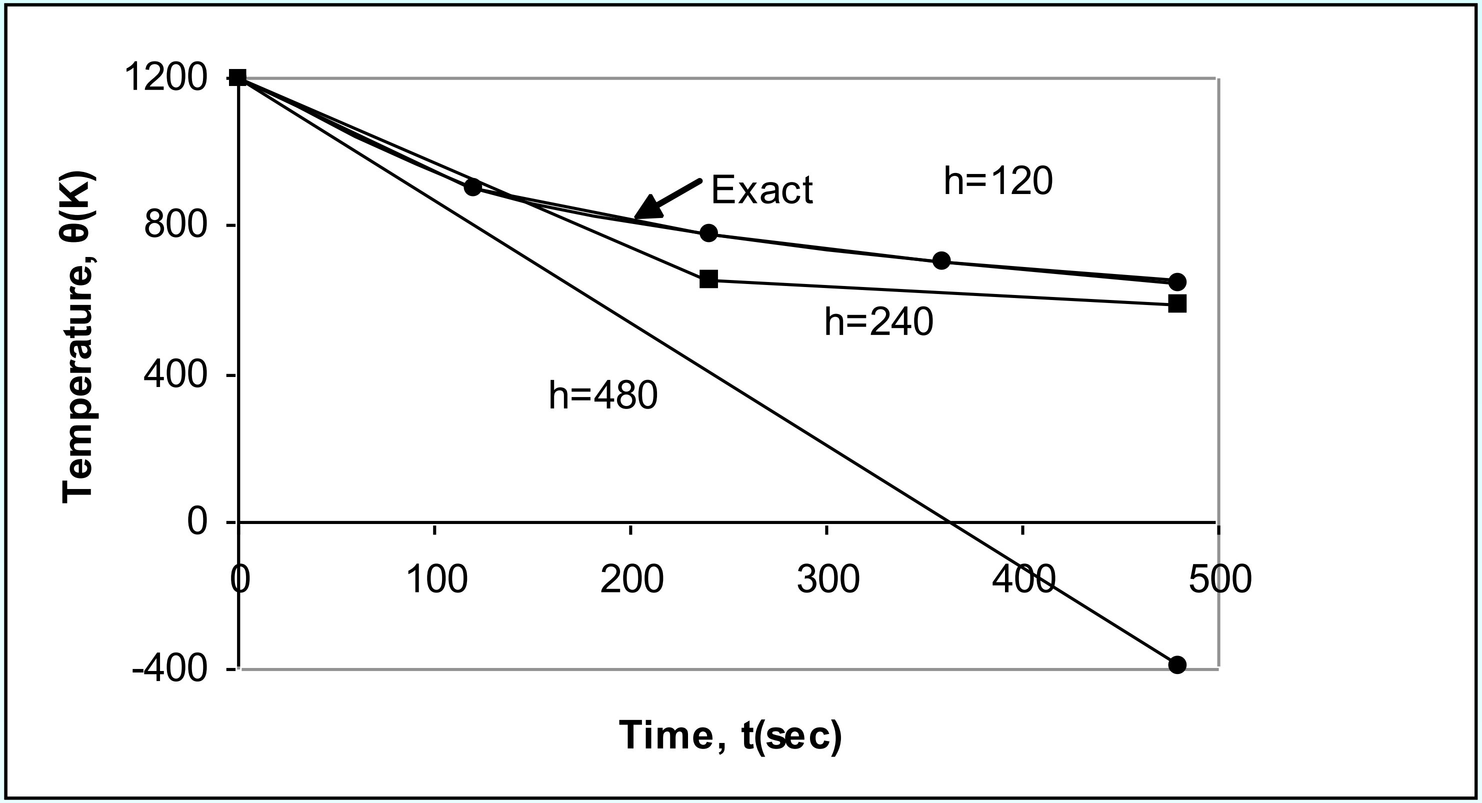

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	−393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	−3.7762	0.58313
60	649.91	−2.3406	0.36145

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Effects of step size on Heun's Method

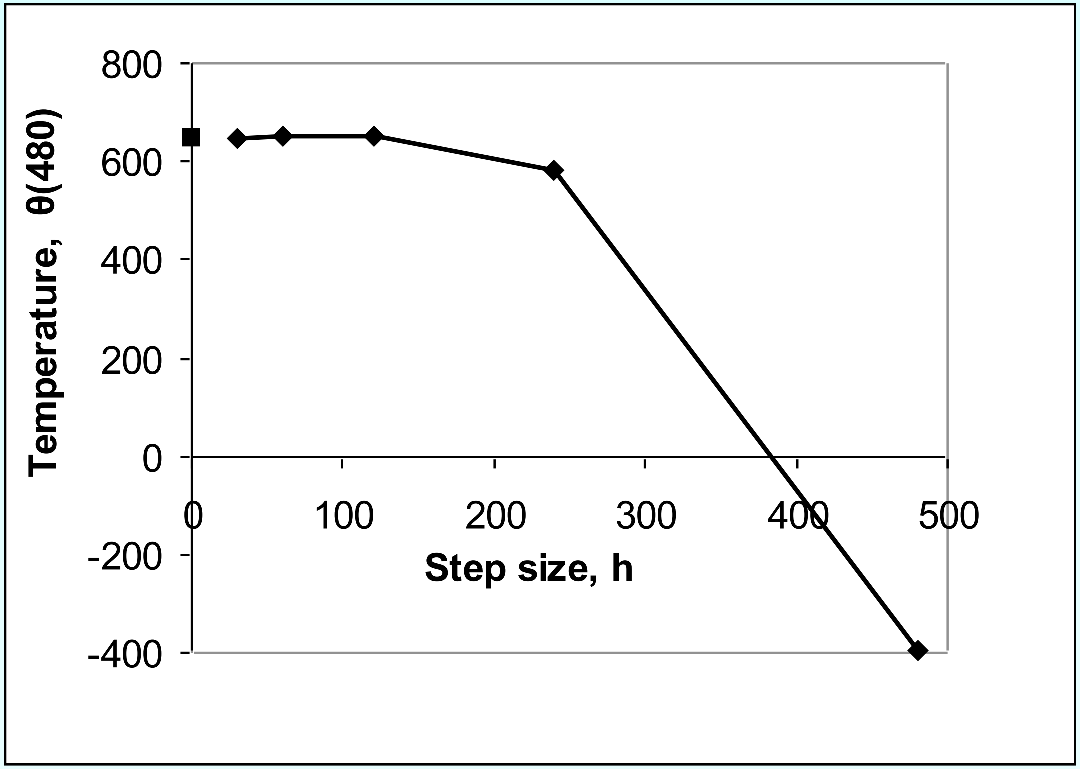

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25

$\theta(480) = 647.57K$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\left\|\in_t\right\|\%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$\theta(480) = 647.57K$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

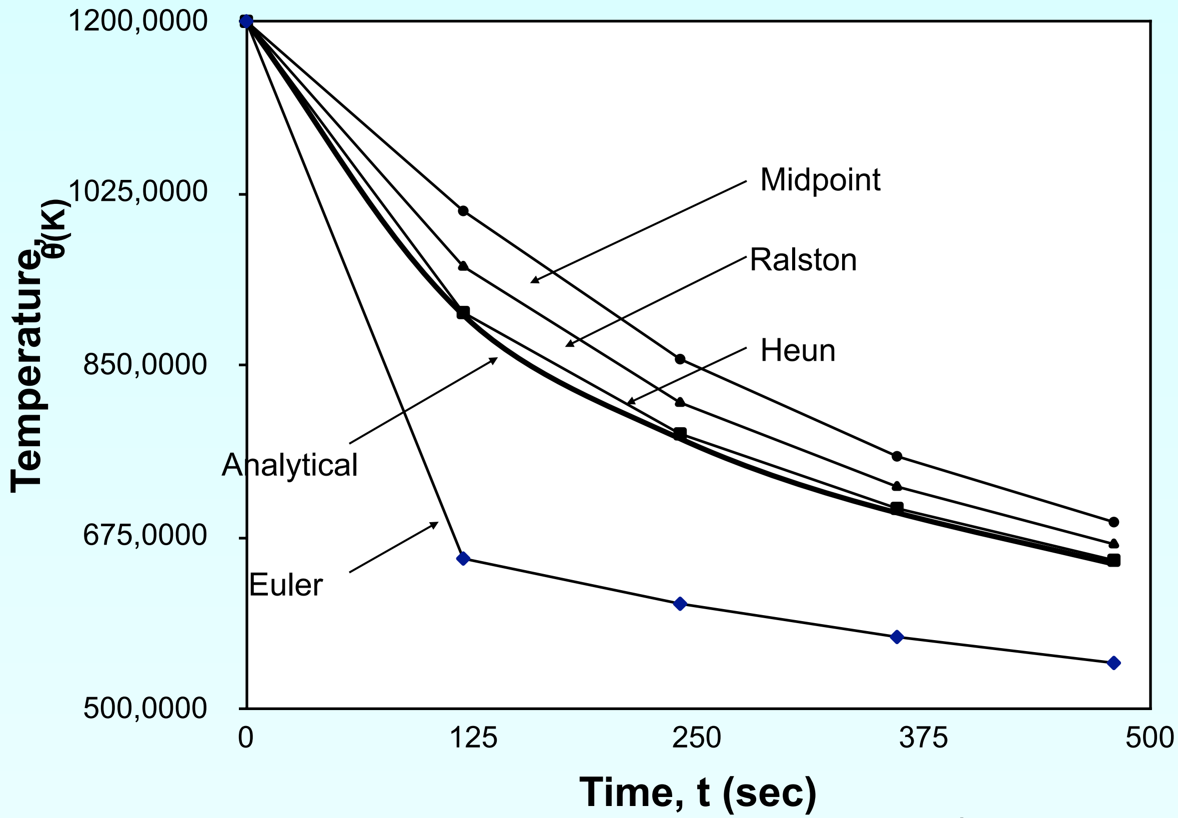

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

Additional Resources

For all resources on this topic such as digital audiovisual lectures, primers, textbook chapters, multiple-choice tests, worksheets in MATLAB, MATHEMATICA, MathCad and MAPLE, blogs, related physical problems, please visit

http://numericalmethods.eng.usf.edu/topics/runge_kutta_2nd_method.html

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Runge-Kutta 4th Order Method

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Runge-Kutta 4th Order Method

For $$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

Runge Kutta 4th order method is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method.

Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$

$$k_1 = f(t_0, \theta_o) = f(0,1200) = -2.2067\times 10^{-12}\left(1200^4 - 81\times 10^8\right) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$

$$= f(120, 653.05) = -2.2067\times 10^{-12}\left(653.05^4 - 81\times 10^8\right) = -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)240\right)$$

$$= f(120, 1154.0) = 2.2067\times 10^{-12}\left(1154.0^4 - 81\times 10^8\right) = -3.8954$$

$$k_4 = f(t_0 + h, \theta_0 + k_3 h) = f\left(0 + (240), 1200 + (-3.984)240\right)$$

$$= f(240, 265.10) = 2.2067\times 10^{-12}\left(265.10^4 - 81\times 10^8\right) = 0.0069750$$

Solution Cont

$$\theta_1 = \theta_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 1200 + \frac{1}{6}\left(-4.5579 + 2\left(-0.38347\right) + 2\left(-3.8954\right) + \left(0.069750\right)\right)240$$

$$= 1200 + \frac{1}{6}\left(-2.1848\right)240$$

$$= 675.65K$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta\left(240\right) \approx \theta_1 = 675.65K$$

Solution Cont

Step 2: $i=1, t_1=240, \theta_1=675.65K$

$$k_1=f(t_1,\theta_1)=f(240,675.65)=-2.2067\times10^{-12}(675.65^4-81\times10^8)=-0.44199$$

$$k_2=f\left(t_1+\frac{1}{2}h,\theta_1+\frac{1}{2}k_1h\right)=f\left(240+\frac{1}{2}(240),675.65+\frac{1}{2}(-0.44199)240\right)$$
$$=f(360,622.61)=-2.2067\times10^{-12}(622.61^4-81\times10^8)=-0.31372$$

$$k_3=f\left(t_1+\frac{1}{2}h,\theta_1+\frac{1}{2}k_2h\right)=f\left(240+\frac{1}{2}(240),675.65+\frac{1}{2}(-0.31372)240\right)$$
$$=f(360,638.00)=2.2067\times10^{-12}(638.00^4-81\times10^8)=-0.34775$$

$$k_4=f(t_1+h,\theta_1+k_3h)=f(240+(240),675.65+(-0.34775)240)$$
$$=f(480,592.19)=2.2067\times10^{-12}(592.19^4-81\times10^8)=-0.25351$$

Solution Cont

$$\theta_2 = \theta_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 675.65 + \frac{1}{6}\left(-0.44199 + 2\left(-0.31372\right) + 2\left(-0.34775\right) + \left(-0.25351\right)\right)240$$

$$= 675.65 + \frac{1}{6}\left(-2.0184\right)240$$

$$= 594.91K$$

θ_2 is the approximate temperature at

$$t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta\left(480\right) \approx \theta_2 = 594.91K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}\left(0.00333\theta\right)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison with exact results

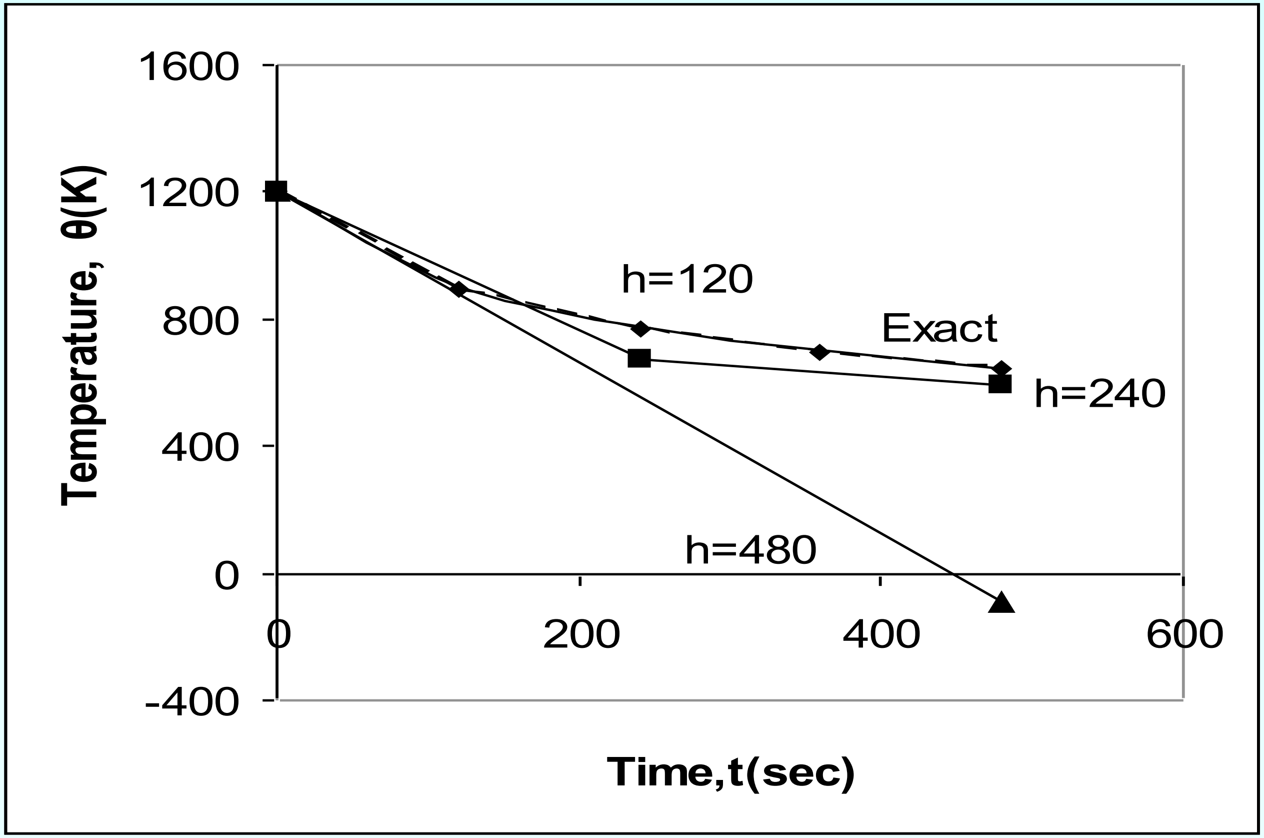

Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	θ (480)	E_t	$\lvert\epsilon_t\rvert\%$
480	−90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

$\theta(480) = 647.57K$ (exact)

Effects of step size on Runge-Kutta 4th Order Method

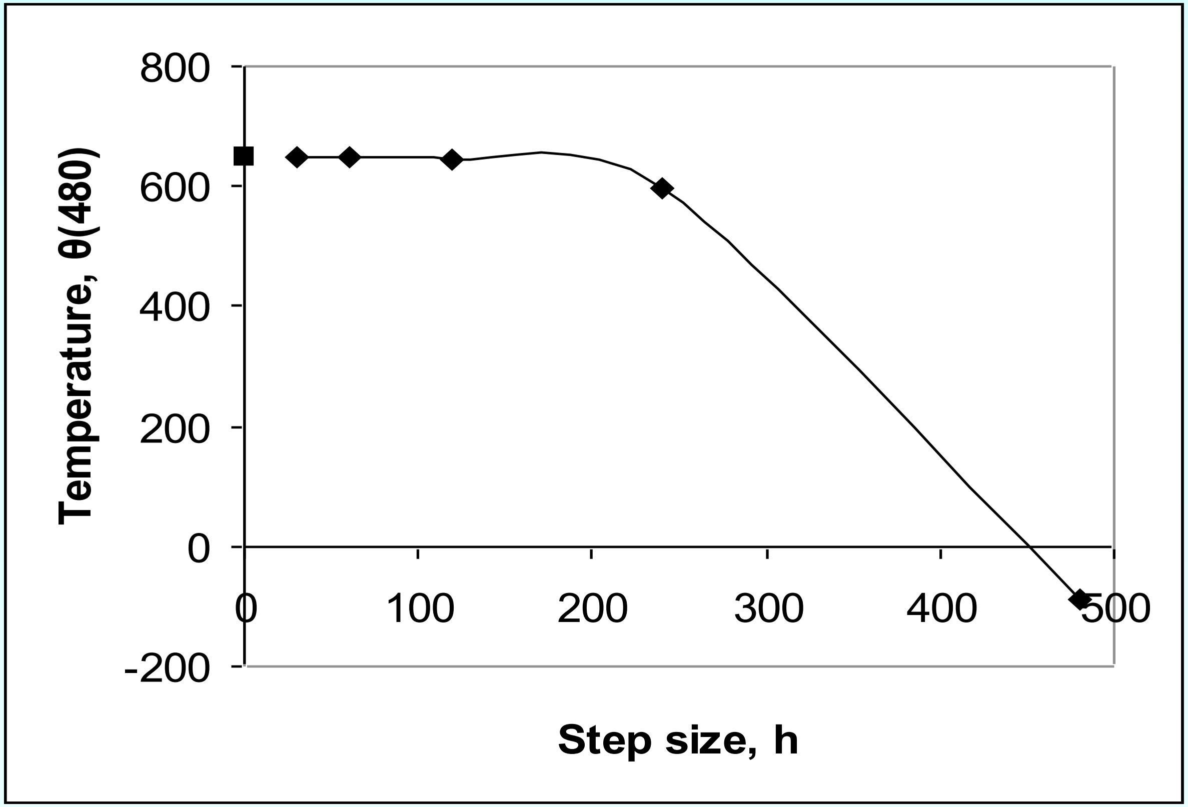

Figure 2. Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

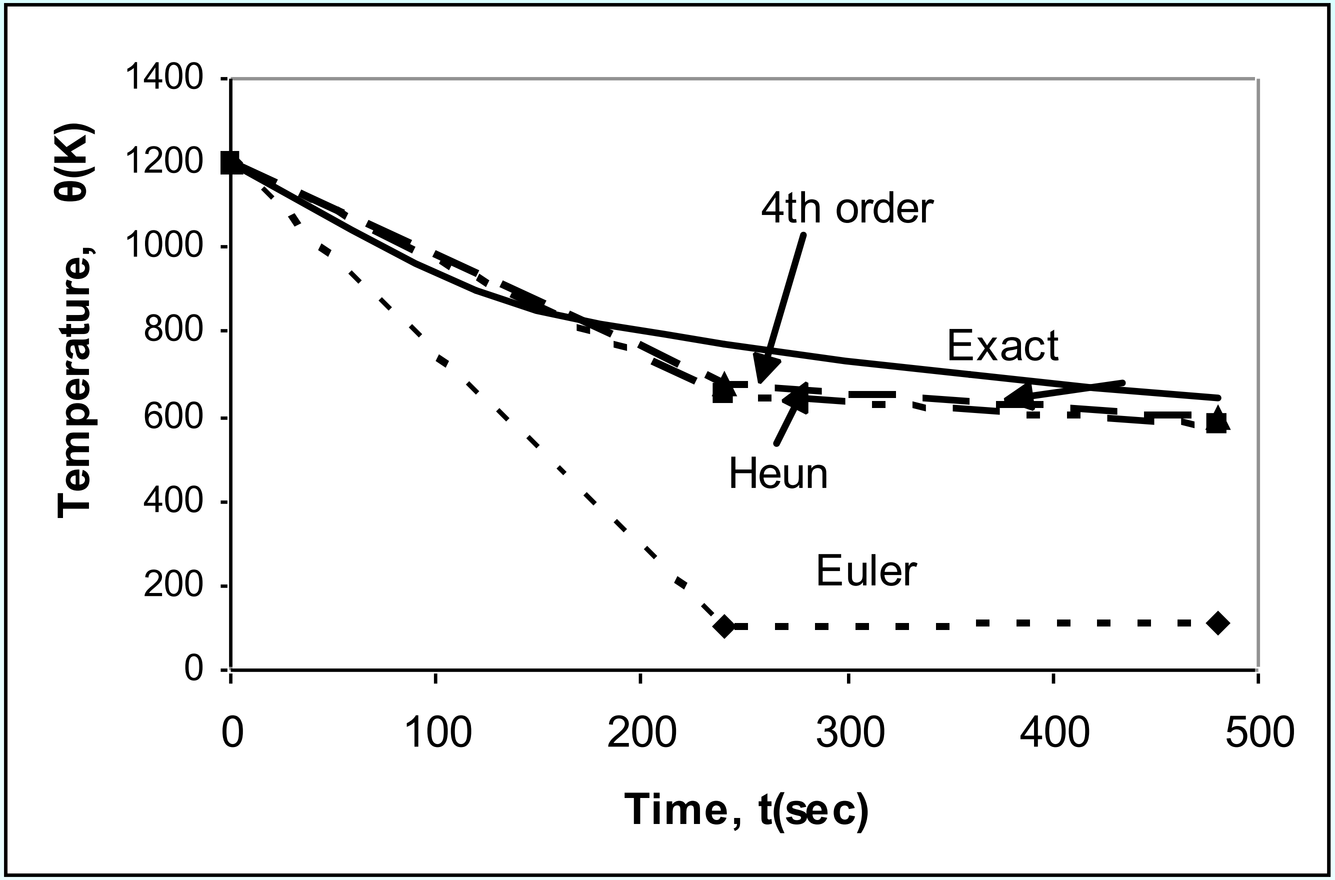

Figure 3. Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.

MHD plus self-gravity for the dust, including particle feed back on the gas: Pencil Code: Finite Differenzen / Runge-Kutta 5th order!

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + u_y^{(0)}\frac{\partial \boldsymbol{u}}{\partial y} &= 2\Omega u_y \hat{\boldsymbol{x}} - \frac{1}{2}\Omega u_x \hat{\boldsymbol{y}} - \boldsymbol{\nabla}\Phi + \frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B} \\
&\quad - \frac{1}{\rho}c_s^2\boldsymbol{\nabla}\rho - \frac{\rho_d/\rho}{\tau_f}(\boldsymbol{u}-\boldsymbol{w}) + \boldsymbol{f}_\nu(\boldsymbol{u},\rho)\,, \\
\frac{\partial \rho}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho + u_y^{(0)}\frac{\partial \rho}{\partial y} &= -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} + f_D(\rho)\,, \\
\frac{\partial \boldsymbol{A}}{\partial t} + u_y^{(0)}\frac{\partial \boldsymbol{A}}{\partial y} &= \frac{3}{2}\Omega A_y \hat{\boldsymbol{x}} + \boldsymbol{u}\times\boldsymbol{B} + \boldsymbol{f}_\eta(\boldsymbol{A})\,, \\
\nabla^2\Phi &= 4\pi G(\rho+\rho_d)\,.
\end{aligned}$$

gas

$$\begin{aligned}
\frac{\partial \boldsymbol{v}^{(i)}}{\partial t} &= 2\Omega v_y^{(i)}\hat{\boldsymbol{x}} - \frac{1}{2}\Omega v_x^{(i)}\hat{\boldsymbol{y}} - \Omega^2 z - \boldsymbol{\nabla}\Phi(\boldsymbol{x}^{(i)}) - \frac{1}{\tau_f}[\boldsymbol{v}^{(i)} - \boldsymbol{u}(\boldsymbol{x}^{(i)})]\,, \\
\frac{\partial \boldsymbol{x}^{(i)}}{\partial t} &= \boldsymbol{v}^{(i)} + u_y^{(0)}\hat{\boldsymbol{y}}\,.
\end{aligned}$$

dust

Poisson equation solved via FFT in parallel mode: up to 256^3 cells

Numerical Practical Training, UKNum

WS 2011/2012 (Block Course Feb. 20 - Mar. 02, 2012)

Free Training

- Write a program code for solving a system of 1st order ordinary differential equations of the type $\mathbf{y}' \equiv d\mathbf{y}/dx = f(\mathbf{y}, x)$ using the

  1. Euler forward method
  2. Runge-Kutta 2nd order (RK2)

  with a constant integration step h, as presented in the lecture.

- **Exercise 4.1, 8 points**: Accuracy and precision.
Solve the differential equation $y' = \exp(x)$ using both the Euler and the RK2 scheme. Start with the initial values $x = 0$ and $y(0) = 1$ and integrate up to $x = 1$ with constant integration steps $h = 1, 1/2, 1/3 \ldots 1/10$. Plot y_i as a function of x_i and compare the results directly with the analytical solution. Plot the absolute true error for $x = 1$ as a function of h in a double logarithmic plot. Confirm the order of the integration scheme.

- **Exercise 4.2, 6 points**: Harmonic oscillator.
Solve the 2nd order ordinary differential equation (ODE) of a harmonic oscillator:

$$\ddot{x} \equiv \frac{d^2 x}{d\tau^2} = -kx, \tag{0.1}$$

with $k = 16$. For this, first transform the 2nd order ODE to a set of two 1st order ODE's as shown in the lecture. Start with the initial values $\tau = 0$, $x(0) = 0.8$ and $\dot{x}(0) = 0$ and integrate the system for several periods using the RK2. Find an integration step that conserves the energy $E = \frac{1}{2}kA^2$, where A is the amplitude of the oscillation, with a relative true error of less than 0.01 at the end of the integration. Plot the results of $x(\tau)$ and $\dot{x}(\tau)$. Using the same integration step, how does the solution change when switching to the Euler scheme?

- **Exercise 4.3, 6 points**: Volterra-Lotka System
  Solve with your RK2 program the dimensonless coupled differential equation of the Volterra-Lotka system (see lecture):

$$
\begin{aligned}
\dot{u}_1 &= u_1(1-u_2) \\
\dot{u}_2 &= \alpha\, u_2(u_1-1)
\end{aligned}
$$

  with $\alpha = 0.5$, $u_1(0) = 1$ and $u_2(0) = 2, 3, 4$ and 5 from $\tau = 0$ to (at least) $\tau = 50$. Plot the solution of $u_1(\tau)$ and $u_2(\tau)$, e.g., for $u_2(0) = 3$. Find an appropriate timestep h for the integration. Is there a conserved quantity of the dynamical system? Also plot $u_2(\tau)$ versus $u_1(\tau)$.

4.3 Interacting Populations

Mutual interaction of reproduction rates. Any two populations can be in different relation with each other.

- predator – prey
- competition
- symbiosis

• **Volterra-Lotka System** Lets start with simplest case: 2 interacting populations, e.g., in predator – prey relation

$$\begin{aligned} \frac{\mathrm{d}N_1}{\mathrm{d}t} &= N_1(a - bN_2) \\ \frac{\mathrm{d}N_2}{\mathrm{d}t} &= N_2(cN_1 - d) \end{aligned} \tag{4.9}$$

$$
\begin{aligned}
\frac{\mathrm{d}N_1}{\mathrm{d}t} &= N_1(a - bN_2) \\
\frac{\mathrm{d}N_2}{\mathrm{d}t} &= N_2(cN_1 - d)
\end{aligned}
\tag{4.9}
$$

All parameters a, b, c, d positive. Meaning: Malthusian growth for N_1 without N_2, extinction of N_2 (predator) without prey (N_1) to feed on. Approximately describes, e.g., hare-lynx populationen in the woods of Canada. Other example: Sardines/predator fish. **Rem.**: Economic interest can be in the predator (lynx skins) or in the prey population (sardines); of interest: prediction of population sizes, influence of hunting/fishing. Dimensionless formulation from

$\Rightarrow u_1 = \frac{c}{d}N_1$, $u_2 = \frac{b}{a}N_2$ $\tau = at$, and $\alpha = \frac{d}{a}$ are suitable dimensionless quantities, with dynamics

$$\begin{aligned} \frac{\mathrm{d}u_1}{\mathrm{d}\tau} &= u_1(1-u_2) \\ \frac{\mathrm{d}u_2}{\mathrm{d}\tau} &= \alpha u_2(u_1-1) \end{aligned} \tag{4.10}$$

$\Rightarrow$ Instead of 4 just 1 (!) free parameter in the dynamics. Stationary points:(i) $u_1^* = u_2^* = 0$, (ii) $u_1^* = u_2^* = 1$.

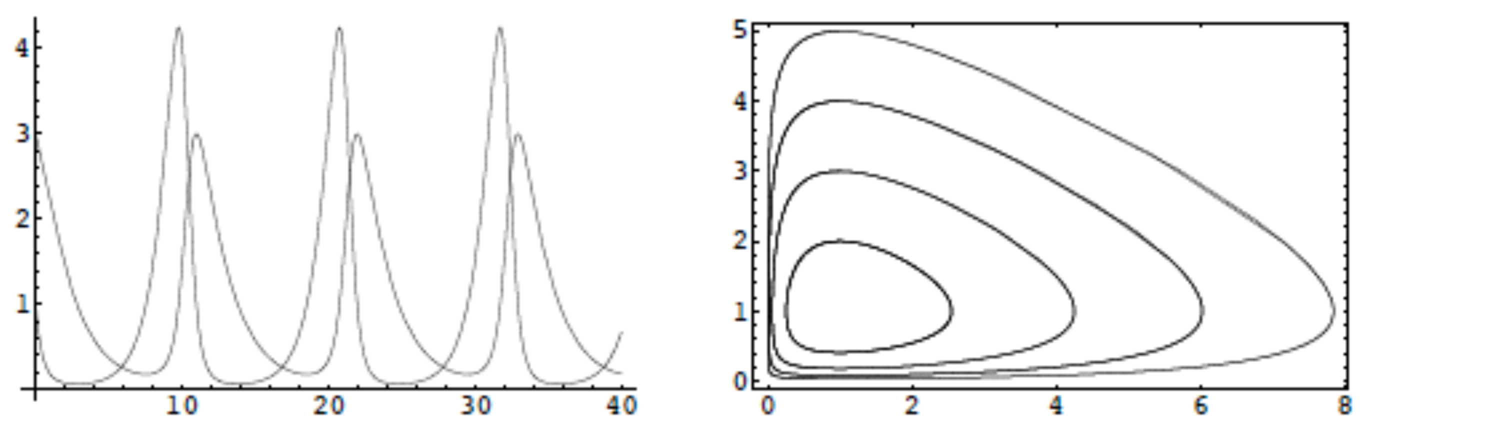

Figure 3: **(a)** Solution of Volterra-Lotka System for $\alpha = 0.5$ and initial condition $u_1(0) = 1$,$u_2(0) = 3$. **(b)** Trajectories for various initial conditions: $u_1(0) = 1$ and $u_2(0) = 2, 3, 4$ and 5.