Introduction to Differential Geometry Notes

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1 What is a curve?

Def'n 1.1.1
Parameterized curve in $\mathcal{R}^n$ is map $\gamma(t) : (\alpha, \beta) \longrightarrow \mathcal{R}^n$ for some α, β with $-\infty \leq \alpha < \beta < \infty$

Level curves in $\mathcal{R}^n$
i.e. $y^2 - x^2 = 0 \leftarrow$ parabola in $\mathcal{R}^2$
The level curve above is at "level" 0. In general, we could have a level curve at level 'c' $f(x, y) = c$

smooth function $f : (\alpha, \beta) \longrightarrow \mathcal{R}^n$ is said to be smooth if derivative $\frac{d^n f}{dt^n}$ exists $\forall n \geq 1$ and $t \in (\alpha, \beta)$

Def'n 1.1.5
If γ is a parameterized curve, first derivative $\dot{\gamma}$ is called the tangent vector of γ at point $\gamma(t)$

2 Arc Length

Developing intuition for arc length
Suppose we have vector $v = (v_1, ..., v_n)$
$||v|| = \sqrt{v_1^2 + ... + v_n^2}$

Suppose we have curve γ. If δt is very small, then the part of the curve between $\gamma(t)$ and $\gamma(t + \delta t)$ is very small and is neraly a straight line, so its

length is approximately:
$||\gamma(t+\delta t)-\gamma(t)||$

$\frac{(\gamma(t+\delta t)-\gamma(t))}{\delta t} \approx \dot{\gamma}(t)$

$||\gamma(t+\delta t)-\gamma(t)|| \approx ||\dot{\gamma}(t)||\delta t$

Def'n 1.1.1
The arc-length of a surve γ starting at point $\gamma(t_0)$ is the function $s(t)$ given by:
$s(t) = \int_{t_0}^{t} ||\dot{\gamma}(u)||du$

Arc length is differentiable function. Indeed if s is arc-length of curve γ starting at point $\gamma(t_0)$ we have:

$\frac{ds}{dt} = \frac{d}{dt}\int_{t_0}^{t} ||\dot{\gamma}(u)||du = ||\dot{\gamma}(t)||$

Def'n 1.2.3
If $\gamma : (\alpha, \beta) \longrightarrow \mathcal{R}^n$ is parameterized curve, its speed at point $\gamma(t)$ is $||\dot{\gamma}(t)||$ and γ is said to be unit speed curve if $\dot{\gamma}$ is a unit vector $\forall t \in (\alpha, \beta)$.
a unit vector is a vector of speed 1. so $||\dot{\gamma}(t)|| = 1 \forall t \in (\alpha, \beta)$.

Dot product
$a = (a_1, ..., a_n), b = (b_1, ..., b_n)$
$a \cdot b = \sum_{i=1}^{n} a_i b_i$

$\frac{d}{dt}(a \cdot b) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$

Proposition 1.2.4
Let $n(t)$ be unit vector that is smooth function of parameter t. Then dot product

$\dot{n}(t) \cdot n(t) = 0.$

$\forall t$ i.e. $\dot{n}(t)$ is zero or perpendicular to $n(t)$ $\forall t$. In particular, if $\gamma(t)$ is unit speed curve, then $\ddot{\gamma}(t)$ is zero or perpendicular to $\dot{\gamma}$.

Proof: Using the product formula to differentiate both sides of the equation $n \cdot n = 1$ with respect to t gives:
$\dot{n} \cdot n + n \cdot \dot{n} = 0$ so $2\dot{n} \cdot n = 0 \longrightarrow \dot{n} \cdot n = 0$.
For the last part of the proposition, replace $n = \dot{\gamma}$.