

Introduction to Differential Geometry Notes

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1 What is a curve?

Def'n 1.1.1

Parameterized curve in \mathcal{R}^n is map $\gamma(t) : (\alpha, \beta) \longrightarrow \mathcal{R}^n$ for some α, β with $-\infty \leq \alpha < \beta < \infty$

Level curves in \mathcal{R}^n

i.e. $y^2 - x^2 = 0 \leftarrow$ parabola in \mathcal{R}^2

The level curve above is at "level" 0. In general, we could have a level curve at level 'c' $f(x, y) = c$

smooth function $f : (\alpha, \beta) \longrightarrow \mathcal{R}^n$ is said to be smooth if derivative $\frac{d^n f}{dt^n}$ exists $\forall n \geq 1$ and $t \in (\alpha, \beta)$

Def'n 1.1.5

If γ is a parameterized curve, first derivative $\dot{\gamma}$ is called the tangent vector of γ at point $\gamma(t)$

2 Arc Length

Developing intuition for arc length

Suppose we have vector $v = (v_1, \dots, v_n)$

$$\|v\| = \sqrt{v_1^2 + \dots + v_n^2}$$

Suppose we have curve γ . If δt is very small, then the part of the curve between $\gamma(t)$ and $\gamma(t + \delta t)$ is very small and is nearly a straight line, so its

length is approximately:

$$\|\gamma(t + \delta t) - \gamma(t)\|$$

$$\frac{(\gamma(t+\delta t) - \gamma(t))}{\delta t} \approx \dot{\gamma}(t)$$

$$\|\gamma(t + \delta t) - \gamma(t)\| \approx \|\dot{\gamma}(t)\| \delta t$$

Def'n 1.1.1

The arc-length of a curve γ starting at point $\gamma(t_0)$ is the function $s(t)$ given by:

$$s(t) = \int_{t_0}^t \|\dot{\gamma}(u)\| du$$

Arc length is differentiable function. Indeed if s is arc-length of curve γ starting at point $\gamma(t_0)$ we have:

$$\frac{ds}{dt} = \frac{d}{dt} \int_{t_0}^t \|\dot{\gamma}(u)\| du = \|\dot{\gamma}(t)\|$$

Def'n 1.2.3

If $\gamma : (\alpha, \beta) \rightarrow \mathcal{R}^n$ is parameterized curve, its speed at point $\gamma(t)$ is $\|\dot{\gamma}(t)\|$ and γ is said to be unit speed curve if $\dot{\gamma}$ is a unit vector $\forall t \in (\alpha, \beta)$.
a unit vector is a vector of speed 1. so $\|\dot{\gamma}(t)\| = 1 \forall t \in (\alpha, \beta)$.

Dot product

$$a = (a_1, \dots, a_n), b = (b_1, \dots, b_n)$$

$$a \cdot b = \sum_{i=1}^n a_i b_i$$

$$\frac{d}{dt}(a \cdot b) = \frac{da}{dt} \cdot b + a \cdot \frac{db}{dt}$$

Proposition 1.2.4

Let $n(t)$ be unit vector that is smooth function of parameter t . Then dot product

$$\dot{n}(t) \cdot n(t) = 0.$$

$\forall t$ i.e. $\dot{n}(t)$ is zero or perpendicular to $n(t) \forall t$. In particular, if $\gamma(t)$ is unit speed curve, then $\ddot{\gamma}(t)$ is zero or perpendicular to $\dot{\gamma}$.

Proof: Using the product formula to differentiate both sides of the equation $n \cdot n = 1$ with respect to t gives:

$$\dot{n} \cdot n + n \cdot \dot{n} = 0 \text{ so } 2\dot{n} \cdot n = 0 \rightarrow \dot{n} \cdot n = 0.$$

For the last part of the proposition, replace $n = \dot{\gamma}$.