An Average Result for Non-CM Elliptic Curves

Let $E(a, b)$ denote the Elliptic Curve defined by

$$y^2 = x^3 + ax + b$$

An Average Result for Non-CM Elliptic Curves

We want to estimate

$$\pi^f_{E(a,b)}(X) = \#\{p \leq X : 2\sqrt{p} - f(p) < a_{E(a,b),p} < 2\sqrt{p}\}$$

This is the # of primes $p \leq X$ such that a_p is f-almost-extremal.

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Instead we can estimate it on average:

$$\pi^f_{A,B}(X) = \sum_{\substack{|a| \leq A \\ |b| \leq B}} \pi^f_{E(a,b)}(X)$$

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Thm (Birch):
The # of Elliptic Curves $E(a,b)$ with $|a| \leq A$,$|b| \leq B$ such that $a_{E(a,b),p} = r$ is approximately

$$4AB\frac{H(r^2 - 4p)}{2p} + \text{Error}$$

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where L is:

$$L(s,\chi) = \sum_{n \geq 1} \frac{1}{n^s} \left(\frac{\frac{4p-r^2}{f^2}}{n} \right)$$

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Plugging it in:

$$\begin{aligned}
\pi^f_{A,B}(X) &= \sum_{\substack{|a|\le A\\|b|\le B}} \pi^f_{E(a,b)}(X)\\
&= \sum_{\substack{|a|\le A\\|b|\le B}} \#\{p \le X : a_{E(a,b),p} \in (2\sqrt{p} - f(p), 2\sqrt{p})\}\\
&= \sum_{\substack{|a|\le A\\|b|\le B}} \sum_{p\le X} \sum_{\substack{r\in(2\sqrt{p}-f(p),2\sqrt{p})\\a_{E(a,b),p}=r}} 1\\
&= \sum_{p\le X} \sum_{r\in(2\sqrt{p}-f(p),2\sqrt{p})} \sum_{\substack{|a|\le A\\|b|\le B\\a_{E(a,b),p}=r}} 1\\
&= \sum_{p\le X} \sum_{r\in(2\sqrt{p}-f(p),2\sqrt{p})} \frac{H(r^2-4p)}{2p}
\end{aligned}$$

An Average Result for Non-CM Elliptic Curves

Plugging in the definition of H:

$$= \sum_{p \le X} \sum_{r \in (2\sqrt{p} - f(p), 2\sqrt{p})} \frac{H(r^2 - 4p)}{2p}$$

$$= \sum_{\substack{p \le X \\ p \text{ prime}}} \sum_{r \in (2\sqrt{p} - f(p), 2\sqrt{p})} \frac{1}{2p} \sum_{\substack{f^2 \mid r^2 - 4p \\ \frac{r^2-4p}{f^2} = 0,1 \mod 4}} \sqrt{\frac{4p - r^2}{f^2}} L(1, \chi)$$

Where $d = (4p - r^2)/f^2$
Also, remember r refers to a possible value for $a_{E,p}$
All we have to do is estimate this sum!

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Also, remember r refers to a possible value for $a_{E,p}$
All we have to do is estimate this sum!
The plan: move $\sum_p$ to the inside, and use a prime-counting theorem to get rid of it.

Switching the first 2 sums

Let $f(p) = p^\alpha g(p)$ where g is slowly varying
Note that

$$2\sqrt{p} - f(p) < r \leq 2\sqrt{p}$$

if and only if

$$\frac{r^2}{4} \leq p < \frac{r^2}{4} + g(r^2)\left(\frac{r}{2}\right)^{1+2\alpha} + \text{Error stuff}$$

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Let $\ell_r = g(r^2)(r/2)^{1+2\alpha}$, the length of the interval I_r

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Switching the $\sum_p$ and $\sum_r$ sums, and doing some other stuff:

$$\pi_{A,B}^f(X) = \sum_{\substack{p \leq X \\ p \text{ prime}}} \sum_{r \in (2\sqrt{p} - f(p), 2\sqrt{p})} \frac{1}{p} \sum_{\substack{f^2 \mid r^2 - 4p \\ \frac{r^2-4p}{f^2} = 0,1 \mod 4}} \sqrt{\frac{4p - r^2}{f^2}} L(1, \chi)$$

$$= \sum_{r \leq 2\sqrt{X}} \sum_{p \in I_r} \sum_{\substack{f^2 \mid r^2 - 4p \\ \frac{r^2-4p}{f^2} = 0,1 \mod 4}} \frac{1}{p} \frac{\log p}{\log p} \frac{\sqrt{4p - r^2}}{f} L(1, \chi)$$

$$\approx \sum_{r \leq 2\sqrt{X}} \frac{1}{\frac{r^2}{4} \log \frac{r^2}{4}} \sum_{p \in I_r} \sum_{\substack{f^2 \mid r^2 - 4p \\ \frac{r^2-4p}{f^2} = 0,1 \mod 4}} \log p \frac{\sqrt{4p - r^2}}{f} L(1, \chi)$$

So many sigmas!

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$$\pi^f_{A,B}(X)$$

$$= \sum_{r \le 2\sqrt{X}} \frac{1}{\frac{r^2}{4}\log\frac{r^2}{4}} \sum_{p \in I_r} \sum_{\substack{f^2 \mid r^2-4p \\ \frac{r^2-4p}{f^2} = 0,1 \mod 4}} \log p \frac{\sqrt{4p-r^2}}{f} L(1,\chi)$$

$$= \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j < r \le 2^{j+1}} \frac{1}{\frac{r^2}{4}\log\frac{r^2}{4}} \sum_{p \in I_r} \sum_{\substack{f^2 \mid r^2-4p \\ \frac{r^2-4p}{f^2} = 0,1 \mod 4}} \log p \frac{\sqrt{4p-r^2}}{f} L(1,\chi)$$

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Reason: want to approximate $\log r$ as $j \log 2 \pm \log 2$, want to bound $\frac{1}{r} \le 2^{-j}$ etc

Counting Primes in an Interval

$$\pi(y, h; q, b) = \#\{y \le p \le y + h : y \mod q = b\}$$

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$h/(\phi(q) \log y)$ is how many primes we expect on average, E is the difference from our expectations.

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For all $A > 0$, $0 \leq h \leq X$, $1 \leq Q^2 \leq h/X^{1/6+\epsilon}$

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This is the error bound that lets us do this paper.
But I'm ignoring error terms in this Beamer.

Lemma: Counting primes with weights

$$\sum_{\substack{y \le p \le y+h \\ p \text{ prime} \\ p = b \mod q}} 1 = \frac{h}{\phi(q) \log y}$$

$$\sum_{\substack{y \le p \le y+h \\ p \text{ prime} \\ p = b \mod q}} \log p = \frac{h}{\phi(q)} \text{ Since } p \approx y$$

$$\sum_{\substack{y \le p \le y+h \\ p \text{ prime} \\ p = b \mod q}} \sqrt{p-y} \log p = \frac{h^{3/2}}{\phi(q)} + \text{Error stuff}$$

$$\pi^f_{A,B}(X)$$

$$= \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j \le r < 2^{j+1}} \frac{1}{\frac{r^2}{4}\log\frac{r^2}{4}} \sum_f \frac{1}{f} \sum_{p \in S_f(I_r)} \sqrt{4p - r^2} \cdot \log p \cdot L(1,\chi)$$

$$= \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j \le r < 2^{j+1}} \frac{1}{\frac{r^2}{4}\log\frac{r^2}{4}} \sum_f \frac{1}{f} \sum_{p \in S_f(I_r)} \sqrt{4p - r^2} \cdot \log p \cdot \sum_n \frac{1}{n}\left(\frac{d}{n}\right)$$

$$= \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j \le r < 2^{j+1}} \frac{1}{\frac{r^2}{4}\log\frac{r^2}{4}} \sum_{n,f} \sum_{\substack{a \in \mathbb{Z}/4n\mathbb{Z} \\ a \equiv 0,1\,(4) \\ (r^2 - af^2, 4nf^2) = 4}} \frac{1}{nf}\left(\frac{a}{n}\right) \sum_{\substack{p \in I_r \\ p \equiv \frac{r^2 - af^2}{4}\,(nf^2)}} \begin{array}{c}\sqrt{4p-r^2} \\ \times \\ \log p\end{array}$$

$$= \sum_{2^j \le r < 2^{j+1}} \frac{1}{\frac{r^2}{4}\log\frac{r^2}{4}} \sum_{n,f} \sum_{\substack{a \in \mathbb{Z}/4n\mathbb{Z} \\ a \equiv 0,1\,(4) \\ (r^2 - af^2, 4nf^2) = 4}} \frac{1}{nf}\left(\frac{a}{n}\right) \frac{\ell_r^{3/2}}{\phi(nf^2)}$$

Magic Constant

$$\sum_{n,f} \frac{1}{nf\phi(nf^2)} \sum_{\substack{a \in \mathbb{Z}/4n\mathbb{Z} \\ a \equiv 0,1\,(4) \\ (r^2 - af^2, 4nf^2) = 4}} \left(\frac{a}{n}\right) = C \cdot C(r)$$

Where C is a constant and $C(r)$ is some function of r.

$$\pi^f_{A,B}(X)$$

$$= \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j \le r < 2^{j+1}} \frac{1}{\frac{r^2}{4} \log \frac{r^2}{4}} \sum_{n,f} \sum_{\substack{a \in \mathbb{Z}/4n\mathbb{Z} \\ a \equiv 0,1\,(4) \\ (r^2 - af^2, 4nf^2) = 4}} \frac{1}{nf} \left(\frac{a}{n}\right) \frac{\ell_r^{3/2}}{\phi(nf^2)}$$

$$= C \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j \le r < 2^{j+1}} \frac{\ell_r^{3/2}}{\frac{r^2}{4} \log \frac{r^2}{4}} C(r)$$

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It just so happens that $\sum_r C(r)$ cancels out to C^{-1}

$$= \sum_{j=1}^{\log_2 2\sqrt{X}} \sum_{2^j \leq r < 2^{j+1}} \frac{\ell_r^{3/2}}{\frac{r^2}{4}\log\frac{r^2}{4}} \approx \frac{g(X)^{3/2} X^{\frac{3}{2}\alpha + \frac{1}{4}}}{\log X}$$