Statistics 435/535/711 2 March 2022
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Some ARMA Models

To begin study of the use of ARIMA models to describe time series data, we look at simulations of some autoregressive moving average (ARMA) processes. In this set of notes we focus mainly on some simple pure autoregressive (AR) and pure moving average (MA) processes. The purpose is to gain understanding of the behavior of these processes. To simplify the discussion here, we consider processes with mean zero. A white noise sequence with mean zero and variance σ^2, $\{\varepsilon_t\}$, serves as a building block for each ARMA process.

There are polynomials associated with the autoregressive and moving average parts of ARMA processes. The zeros of the polynomials play a crucial role in determining the properties of the processes. They can be real-valued or complex conjugate in pairs. In all cases the polynomial zeros are required to have values less than 1 in magnitude, so that the models are properly structured and identified.

A. Autoregressive process of order 1, AR(1)

$y_t - \phi_1 y_{t-1} = \varepsilon_t$ The autoregressive polynomial is $z - \phi_1$, and its zero is the value of the autoregressive parameter. A large negative zero (close to −1) produces fast oscillations (high frequency activity) for the time series. A large positive zero (close to 1) leads to time series which fluctuate slowly (low frequency activity).

1. $y_t + 0.9 y_{t-1} = \varepsilon_t \qquad z + 0.9$

2. $y_t - 0.9 y_{t-1} = \varepsilon_t \qquad z - 0.9$

3. $y_t + 0.2 y_{t-1} = \varepsilon_t \qquad z + 0.2$

4. $y_t - 0.2 y_{t-1} = \varepsilon_t \qquad z - 0.2$

The following code produces simulations of these four AR(1) processes.

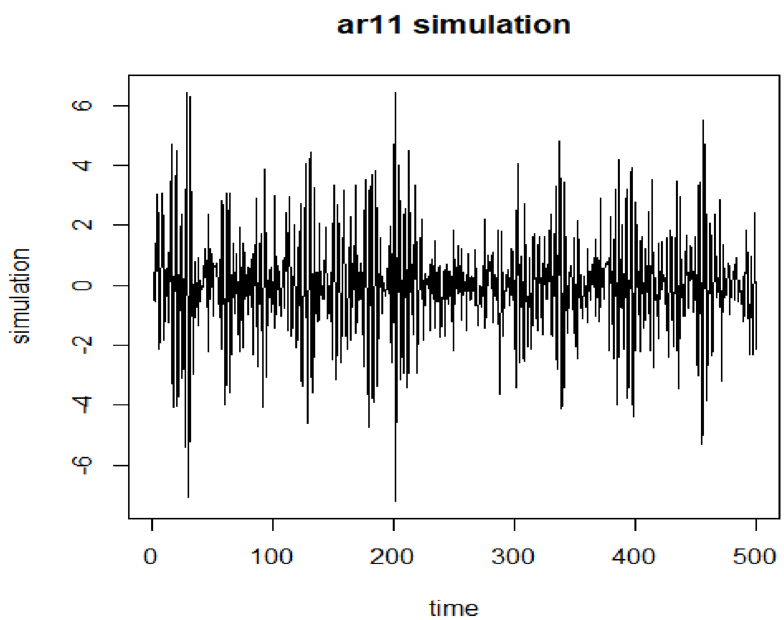
```
> ar1<-matrix(c(rep(0,2400)),ncol=4)
> a<-c(-0.9,0.9,-0.2,0.2)
> #simulate normal data with mean 0, variance 1
> w<-matrix(rnorm(2400),ncol=4)
> for(j in 1:4){
+ ar1[1,j]<-w[1,j]
+ for(i in 2:600){
+ i1<-i-1
+ ar1[i,j]<-a[j]*ar1[i1,j]+w[i,j]
+ }
+ }
> rws<-101:600
> ar11<-ar1[rws,1];ar12<-ar1[rws,2];ar13<-ar1[rws,3];ar14<-ar1[rws,4]
> df<-data.frame(ar11,ar12,ar13,ar14)
> write.table(df,"F:/Stat71122Spring/ar1.txt")
```

Let's input the simulated data and examine the autocorrelations and partial autocorrelations for each series. Partial correlations will be discussed later in these notes. For now we want to examine the characteristics of the AR(1) autocorrelations and partial autocorrelations. We'll also look at the spectral density for the first two series.

```
> ar1<-read.table("F:/Stat71122Spring/ar1.txt")
> attach(ar1)
> head(ar1)
        ar11       ar12        ar13       ar14
1 -0.4909008  0.2911734 -0.04872121 -0.4157873
2  1.3872301  0.8049932 -0.67822475  0.3850513
3 -0.5499547 -0.2819570  0.15682765  0.3609472
4  3.0224429 -1.6286868 -0.79827828 -0.2727354
5 -2.1196836  0.7169710  0.18245128 -1.7752766
6  2.4334682 -0.7120016 -2.96267924  0.7774789
```

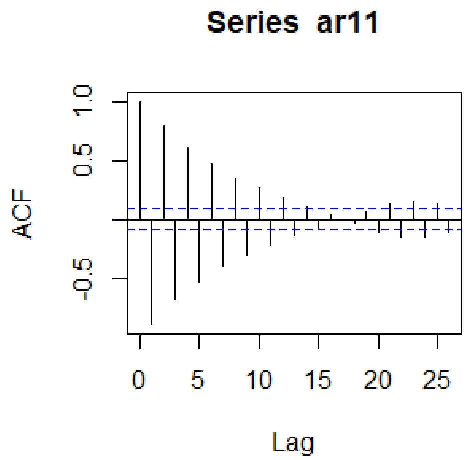
We start with examination of the first simulated AR(1) process.

```
> plot(ts(ar11),xlab="time",ylab="simulation",main="ar11 simulation")
```



The simulation shows a great deal of high frequency movement. This occurs because the autoregressive polynomial has a negative zero which is close to -1.

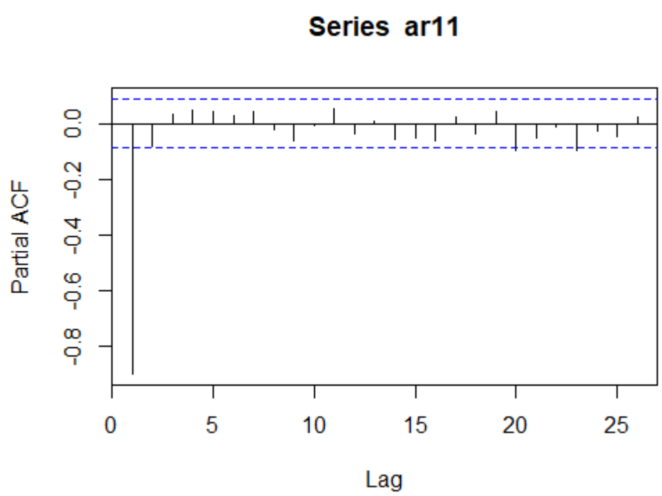
```
> acf(ar11)
```




```
> acf(ar11,lag.max=15,plot=F)

Autocorrelations of series 'ar11', by lag

     0      1      2      3      4      5      6      7      8      9     10 
 1.000 -0.898  0.790 -0.688  0.607 -0.530  0.467 -0.403  0.343 -0.302  0.267 
    11     12     13     14     15 
-0.227  0.183 -0.143  0.098 -0.071 
```

For this AR(1) model the parameter $\phi_1 = -0.9$, and the theoretical correlation at lag k is equal to $\phi_1^k = (-0.9)^k$. As we see, the simulation provides a good approximation to the theoretical values for small lag values. The partial correlations, by contrast, have a very different pattern.

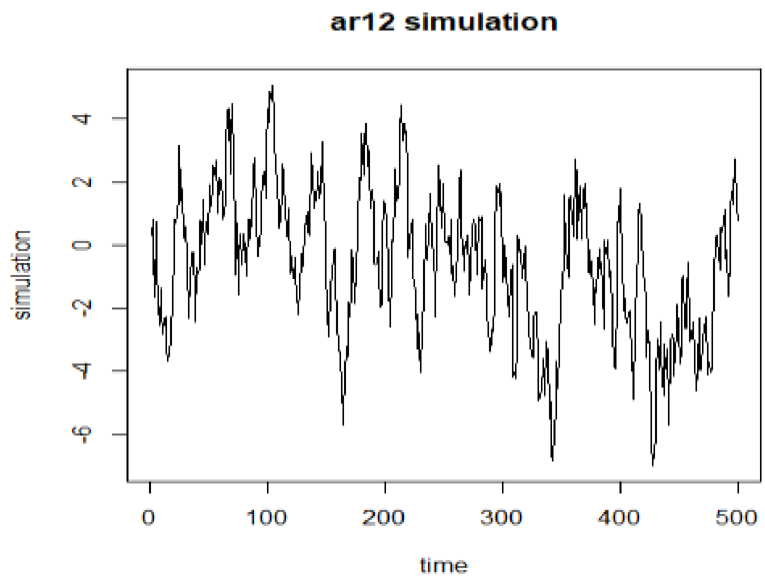
```
pacf(ar11)
```



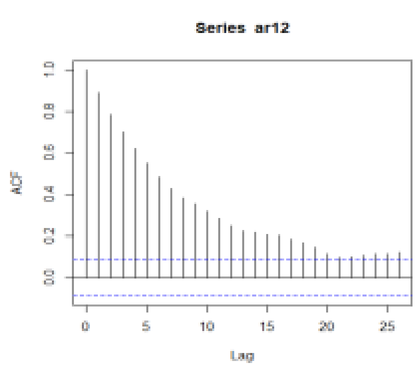
The theoretical partial autocorrelation at lag 1 is equal to −0.9, and the theoretical partial autocorrelations at all higher lags are 0 for this model. That is, the partial autocorrelations cut off at lag 1, which is the case for all AR(1) models. The simulation produces this pattern very well. Note that, by contrast, the autocorrelations for this AR(1) process don't cut off, but tail off.

Let's turn to the second simulated process, which has parameter $\phi_1 = 0.9$. It has a very different structure.

```
> plot(ts(ar12),xlab="time",ylab="simulation",main="ar12 simulation")
```



This plot is much smoother. It is basically devoid of high frequency movement, and this is due to the sign of the autoregressive coefficient—the zero of the autoregressive polynomial is positive and close to 1.



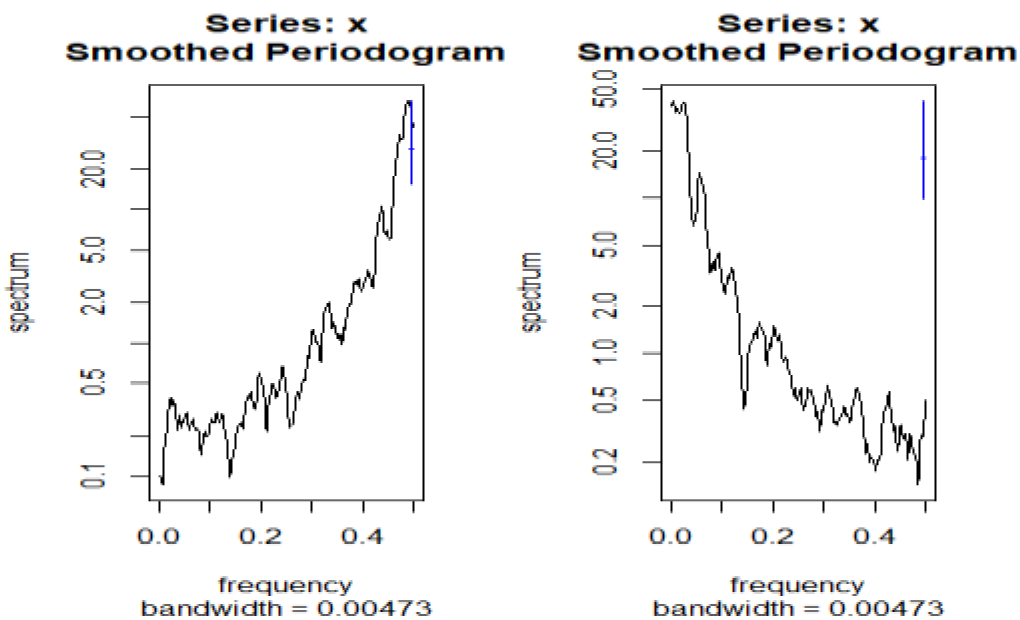

```
Autocorrelations of series 'ar12', by lag

    0     1     2     3     4     5     6     7     8     9    10    11    12
1.000 0.889 0.785 0.699 0.624 0.554 0.486 0.428 0.385 0.357 0.320 0.286 0.251
   13    14    15
0.227 0.220 0.207
```

```
Partial autocorrelations of series 'ar12', by lag

     1      2      3      4      5      6      7      8      9     10     11
 0.889 -0.022  0.027  0.004 -0.012 -0.025  0.008  0.036  0.048 -0.047 -0.001
    12     13     14     15
-0.024  0.034  0.065 -0.019
```

The sample partial autocorrelations clearly show a cutoff at lag 1.

Next, let's plot estimated spectra for the first two simulations.

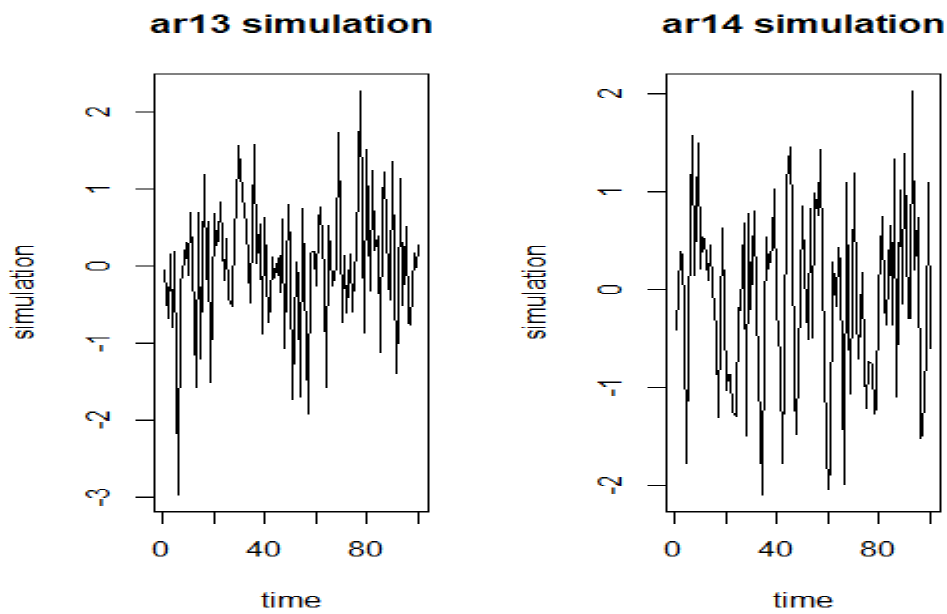
```
> par(mfrow=c(1,2))
> spectrum(ar11,span=8)
> spectrum(ar12,span=8)
```



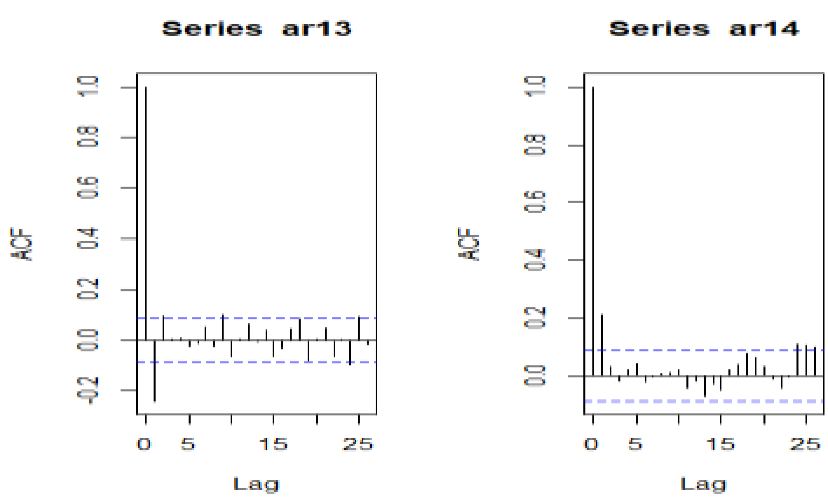
The first plot shows prominent activity at high frequencies, and the second plot indicates prominent low frequency movement for the series.

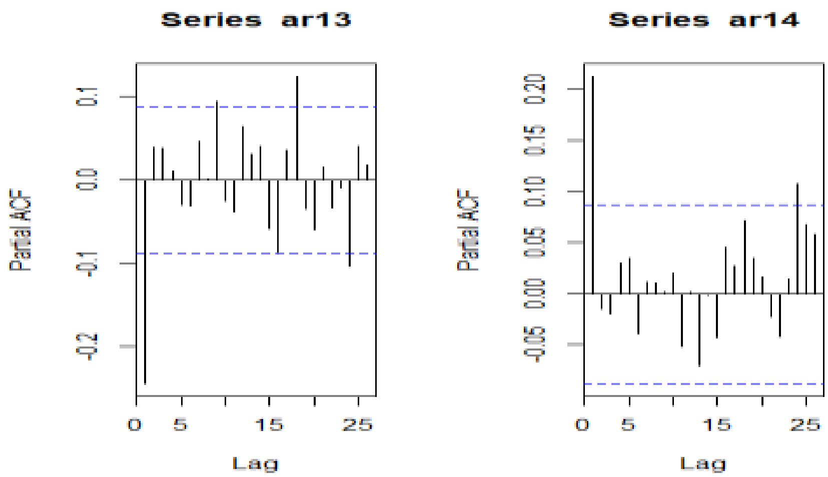
The third and fourth simulated AR(1) series have very weak signals. The plots of these series show just the first 100 points.

```
> par(mfrow=c(1,2))
> plot(ts(ar13[1:100]),xlab="time",ylab="simulation",main="ar13 simulation")
> plot(ts(ar14[1:100]),xlab="time",ylab="simulation",main="ar14 simulation")
```



Relative to the first two series, the range of variation is much more constrained. The small values of the autoregressive parameter produces the weak signals. In fact, the variation is not much more than one would see from the noise component alone, which is normal with standard deviation 1. The acf and pacf plots follow.





The pacf plots should show significance only at lag 1. The few very modest false positives are to be expected in estimates from data.

B. Autoregressive process of order 2, AR(2)

$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} = \varepsilon_t$ The autoregressive polynomial is $z^2 - \phi_1 z - \phi_2$.

1. $y_t - 1.3y_{t-1} + 0.4y_{t-2} = \varepsilon_t$ $\quad (z-0.5)(z-0.8)$

2. $y_t + 1.3y_{t-1} + 0.4y_{t-2} = \varepsilon_t$ $\quad (z+0.5)(z+0.8)$

3. $y_t - 0.9y_{t-1} + 0.81y_{t-2} = \varepsilon_t$ $\quad \left(z-0.9e^{i2\pi/6}\right)\left(z-0.9e^{-i2\pi/6}\right)$

4. $y_t - 1.55885y_{t-1} + 0.81y_{t-2} = \varepsilon_t$ $\quad \left(z-0.9e^{i2\pi/12}\right)\left(z-0.9e^{-i2\pi/12}\right)$

5. $y_t - 0.99y_{t-1} + 0.9801y_{t-2} = \varepsilon_t$ $\quad \left(z-0.99e^{i2\pi/6}\right)\left(z-0.99e^{-i2\pi/6}\right)$

The polynomials associated with the first two models have two real-valued zeros, and the polynomials associated with the last three models have a conjugate pair of complex-valued zeros. Processes with complex-valued zeros exhibit pseudo-cyclical behavior. The larger the amplitudes of the complex-valued zeros are, the more prominent is the pseudo-cyclical behavior. Amplitudes range between 0 and 1.

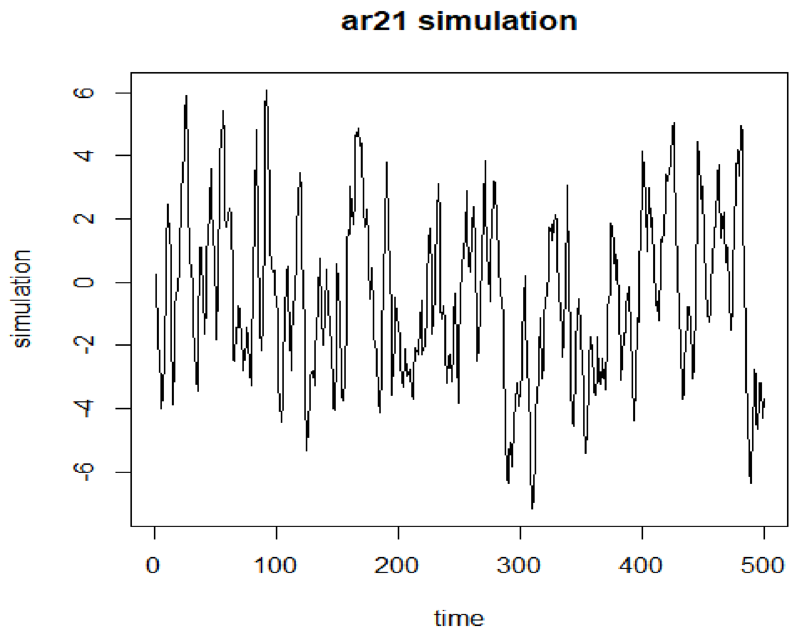
Simulation for the models follows.

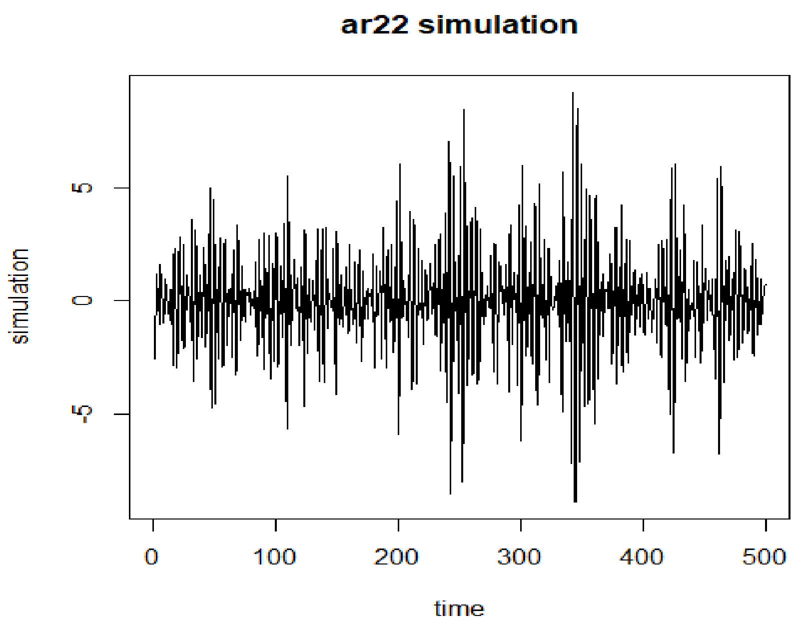
```
> ar2<-matrix(c(rep(0,3000)),ncol=5)
> a<-matrix(c(1.3,-1.3,0.9,1.55885,0.99,-0.4,-0.4,-0.81,-0.81,-
0.9801),ncol=2)
> #simulate normal data with mean 0, variance 1
> w<-matrix(rnorm(3000),ncol=5)
> for(j in 1:5){
+ ar2[1,j]<-w[1,j]
+ ar2[2,j]<-a[j,1]*ar2[1,j]+w[2,j]
+ for(i in 3:600){
+ i1<-i-1;i2<-i-2
+ ar2[i,j]<-a[j,1]*ar2[i1,j]+a[j,2]*ar2[i2,j]+w[i,j]
+ }
+ }
> rws<-101:600
> ar21<-ar2[rws,1];ar22<-ar2[rws,2];ar23<-ar2[rws,3];ar24<-ar2[rws,4];ar25<-
ar2[rws,5]
> df<-data.frame(ar21,ar22,ar23,ar24,ar25)
> write.table(df,"F:/Stat71122Spring/ar2.txt")
```

The first two AR(2) models differ only in the sign attached to one coefficient. However, properties of the two processes differ greatly.

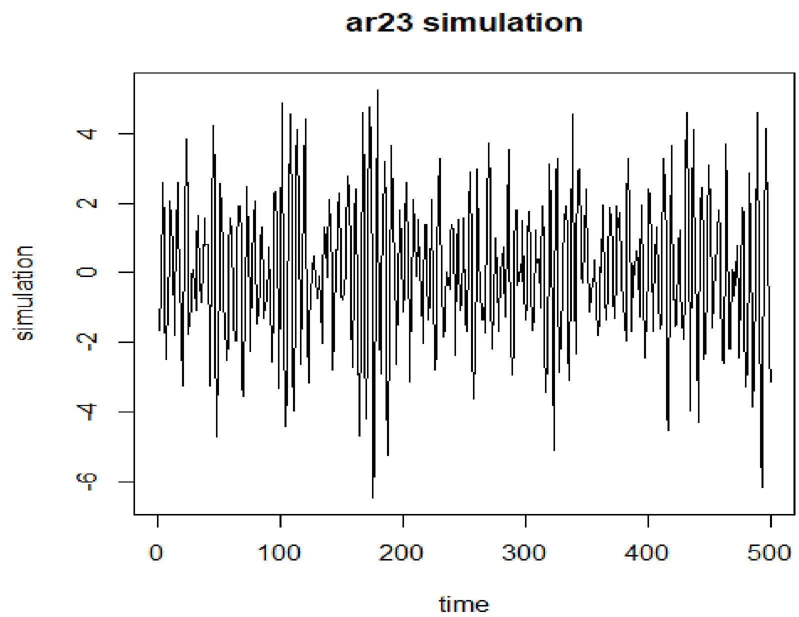
```
> ar2<-read.table("F:/Stat71122Spring/ar2.txt")
> attach(ar2)
> head(ar2)
        ar21       ar22       ar23       ar24      ar25
1  0.2385154 -2.5273103 -1.6426976 -0.5775141 -5.450575
2 -1.8056227  1.1695132 -0.4026677 -0.9009281 -8.077427
3 -1.6116874 -0.3052529 -0.4600749 -0.5866770 -3.436405
4 -2.6593938 -0.4808650  2.5822237 -0.1700035  6.278737
5 -3.0510132  1.6023943  1.2074706  0.9779893  8.944933
6 -4.0024139 -1.0628926 -1.0056045  1.0880485  3.098856
```

Let’s look at time plots of the five simulated series.



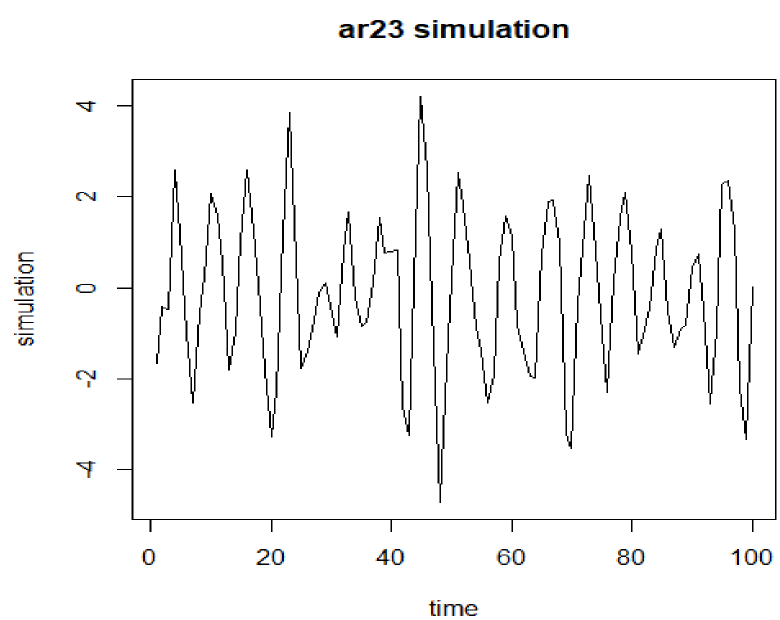


These two series don't exhibit pseudo-cyclical behavior—the polynomial zeros are not complex-valued. The second series has much more intense high frequency activity. And recall that the difference between the two is determined by just one algebraic sign.

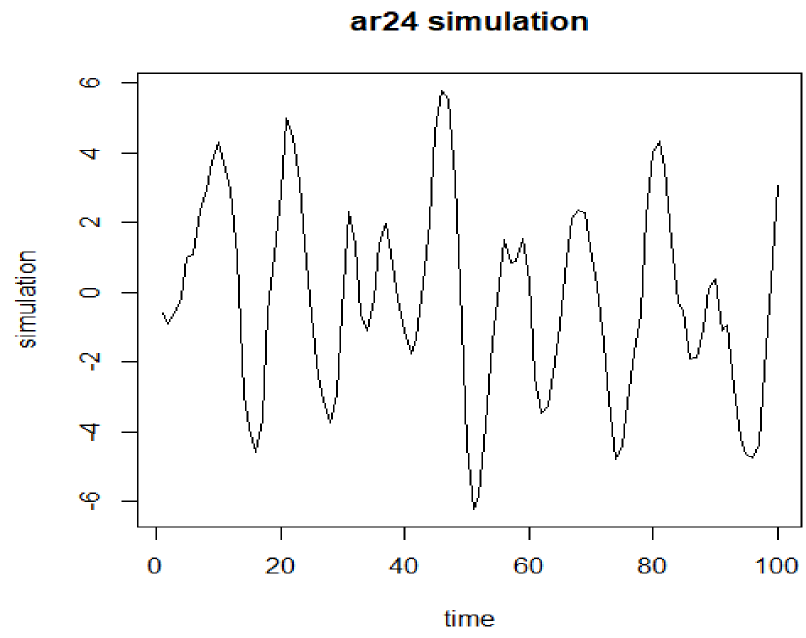


This series show pseudo-cyclical behavior, with a period length of six observations. Let's magnify the plot by looking at just the first 100 observations.

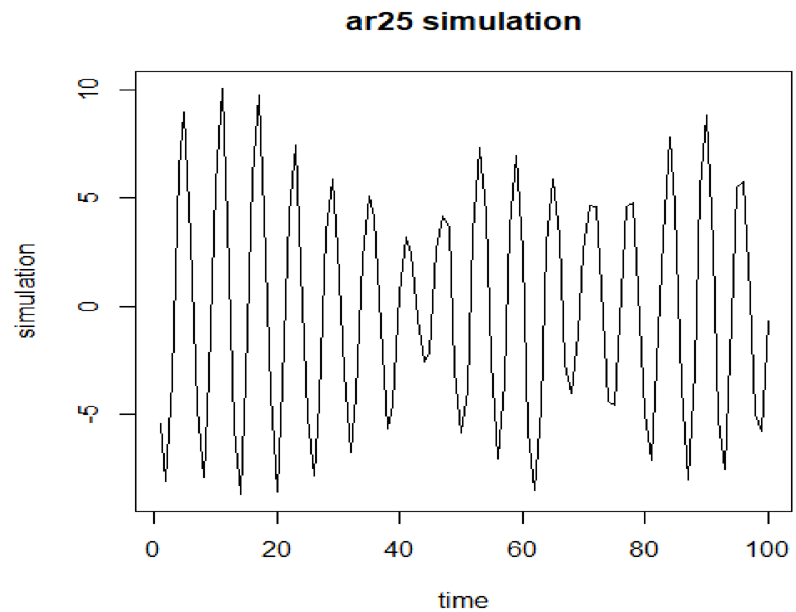
```
> plot(ts(ar23[1:100]),xlab="time",ylab="simulation",main="ar23 simulation")
```



The fourth AR(2) series is pseudo-cyclical with a period length of 12. The plot below shows the first 100 observations.



And the last simulated AR(2) series is pseudo-cyclical with a period length of six. It differs from the third series in that the pseudoperiods are more regular. The plot shows the first 100 observations.

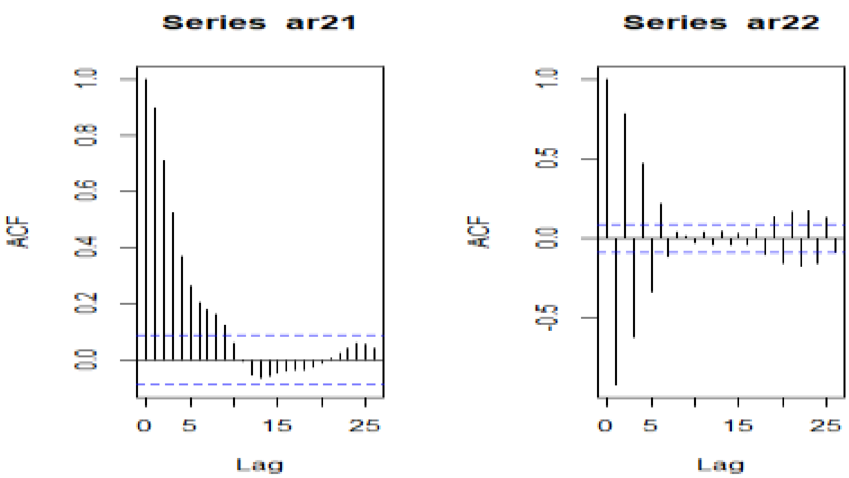


The prominent pseudo-cyclical behavior of the last three series occurs because they have autoregressive polynomials with complex zeros with amplitudes near 1.

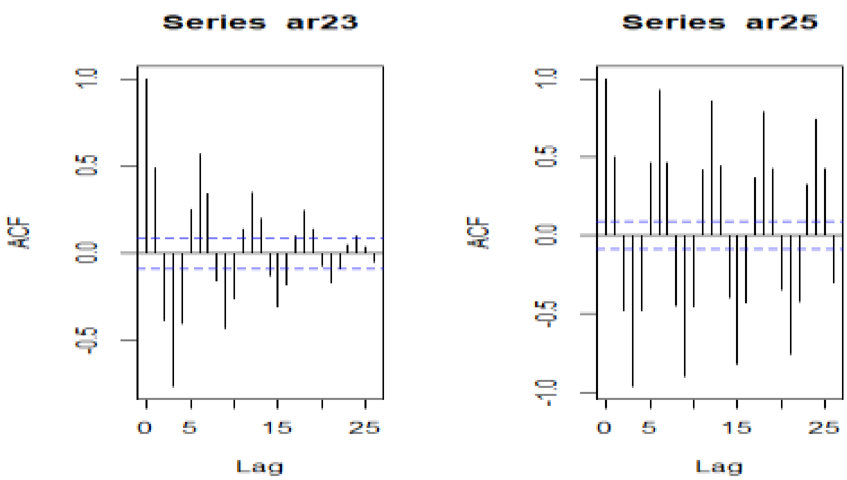
The polynomial zeros for the third model are the conjugate pair $0.9\exp(i2\pi/6)$, $0.9\exp(-i2\pi/6)$. The amplitude for each zero is 0.9, and the arguments for this complex pair are $\pm 2\pi/6$. The period length is determined by the denominator, here 6. The fourth model has complex zero pair with amplitude 0.9 and period length 12. The fifth model yields amplitude 0.99 and period length 6; its greater amplitude, extremely close to 1, produces a very prominent pseudo-cyclical activity.

Next, we look at plots of the estimated autocorrelations and partial autocorrelations for these AR(2) simulations.

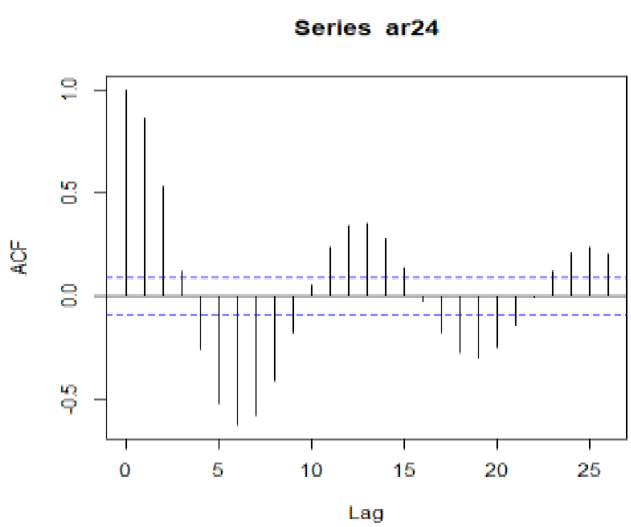
```
> acf(ar21)
> par(mfrow=c(1,2))
> acf(ar21)
> acf(ar22)
```



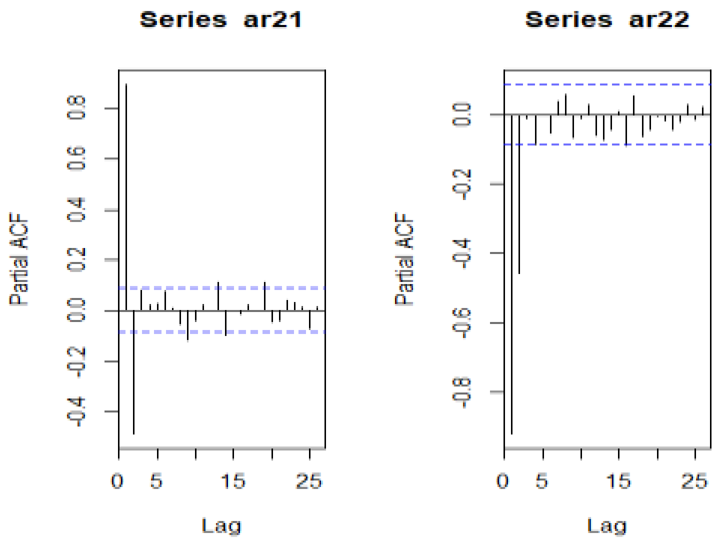
```
> par(mfrow=c(1,2))
> acf(ar23)
> acf(ar25)
```

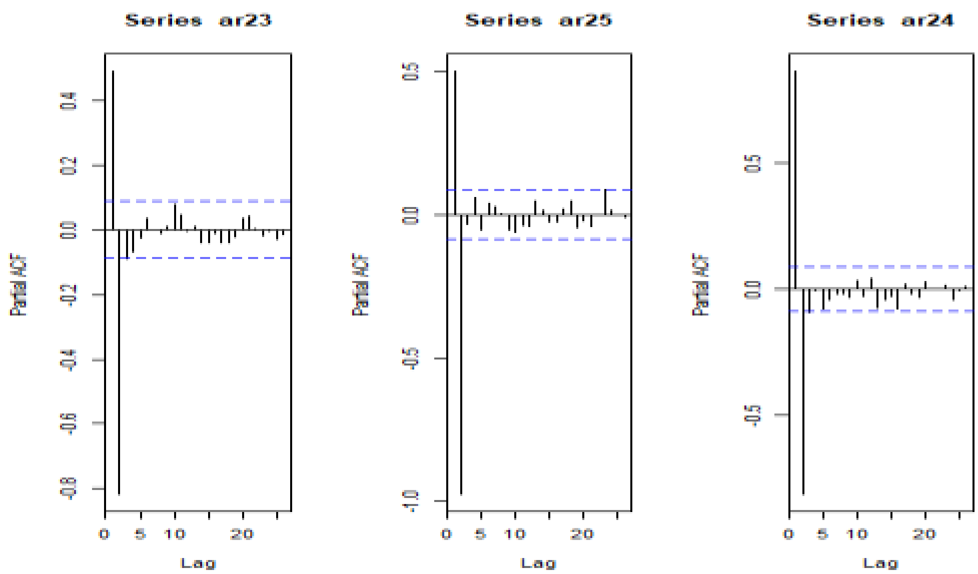



```
> acf(ar24)
```



The last three plots show the period lengths corresponding to the pseudo-cyclical behavior of the series. The partial autocorrelation plots which follow all cut off at lag 2, as expected for AR(2) series.

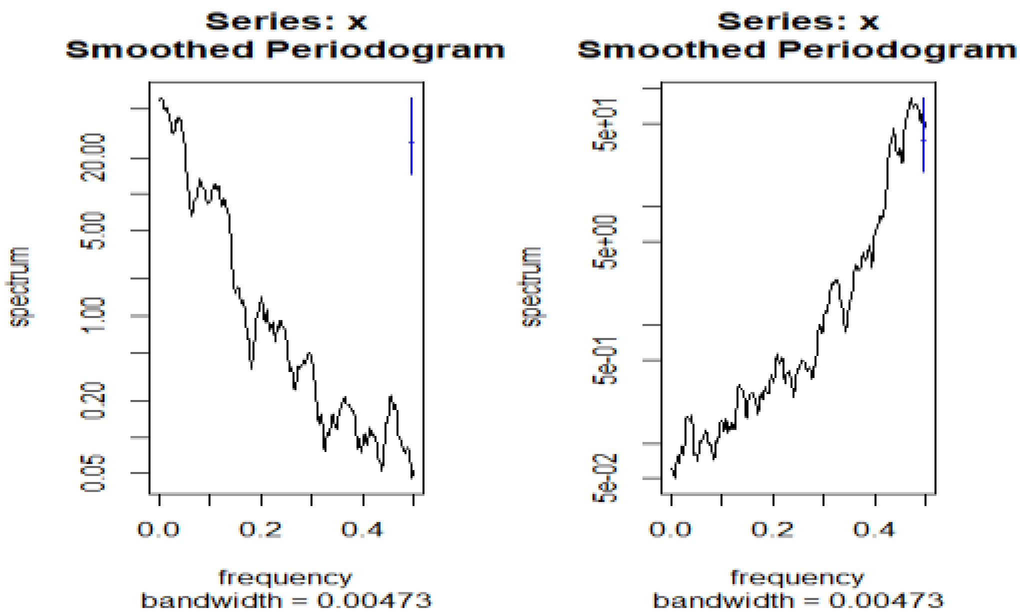




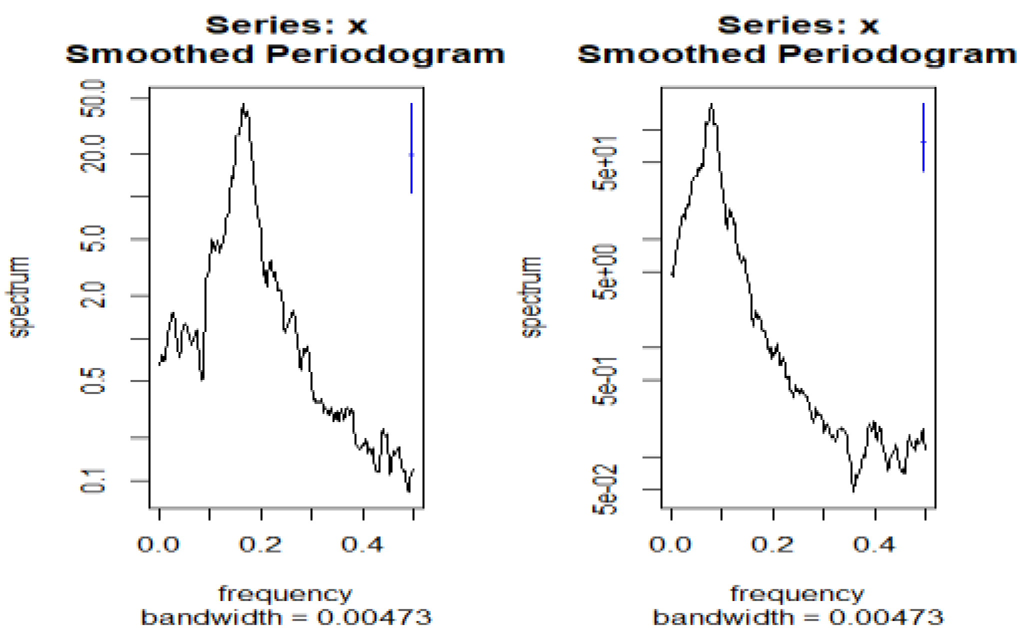
The first plot shows some modestly significant false positives. This is to be expected with estimates from data.

Spectral plots provide useful information.

```
> par(mfrow=c(1,2))
> spectrum(ar21,span=8)
> spectrum(ar22,span=8)
```




```
> par(mfrow=c(1,2))
> spectrum(ar23,span=8)
> spectrum(ar24,span=8)
```



The first series has predominantly low frequency movement, and the second predominantly high frequency fluctuation. For the third and fourth series, the spectral peaks are centered at frequencies 1/6 and 1/12, respectively, indicating the lengths of the pseudo-cycles.

C. Autoregressive process of order 4, AR(4)

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} - \phi_4 y_{t-4} = \varepsilon_t$$

The autoregressive polynomial is $z^4 - \phi_1 z^3 - \phi_2 z^2 - \phi_3 z - \phi_4$.

1. $y_t - 1.6 y_{t-1} + 0.25 y_{t-2} + 0.582 y_{t-3} - 0.216\, y_{t-4} = \varepsilon_t$

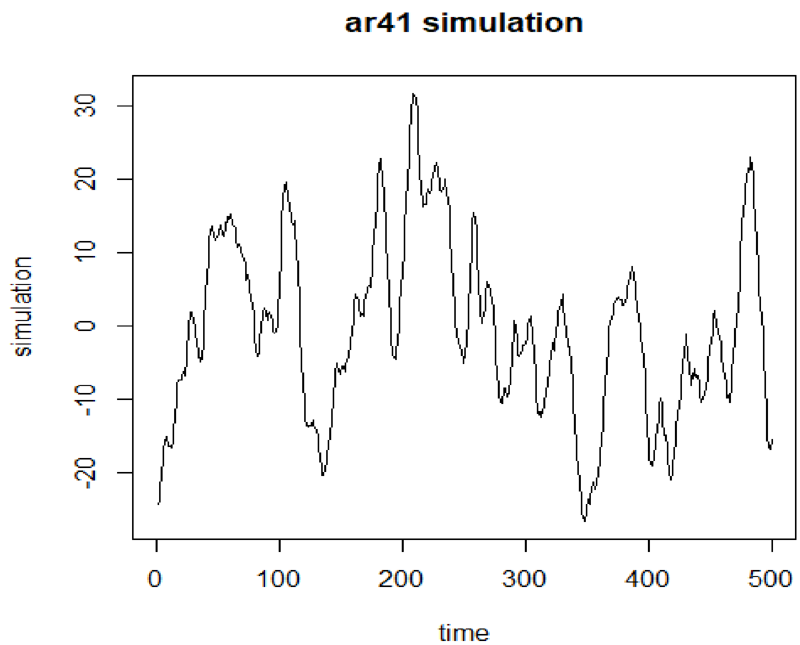
   $$(z-0.5)(z-0.8)(z+0.6)(z-0.9)$$

2. $y_t + 0.05 y_{t-1} + 0.8575 y_{t-2} - 0.04275 y_{t-3} + 0.731025\, y_{t-4} = \varepsilon_t$

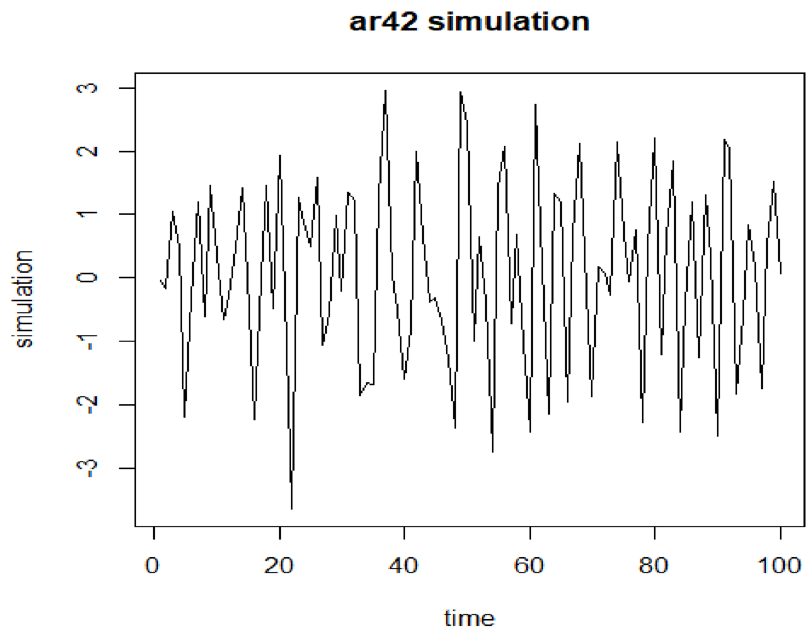
   $$\begin{aligned}&\left(z-0.9e^{i2\pi/6}\right)\left(z-0.9e^{-i2\pi/6}\right)\left(z-0.95e^{i2\pi/3}\right)\left(z-0.95e^{-i2\pi/3}\right)\\ &=\left(z^2-0.9z+0.81\right)\left(z^2+0.95z+0.9025\right)\\ &=z^4+0.05z^3+0.8575z^2-0.04275z+0.731025\end{aligned}$$

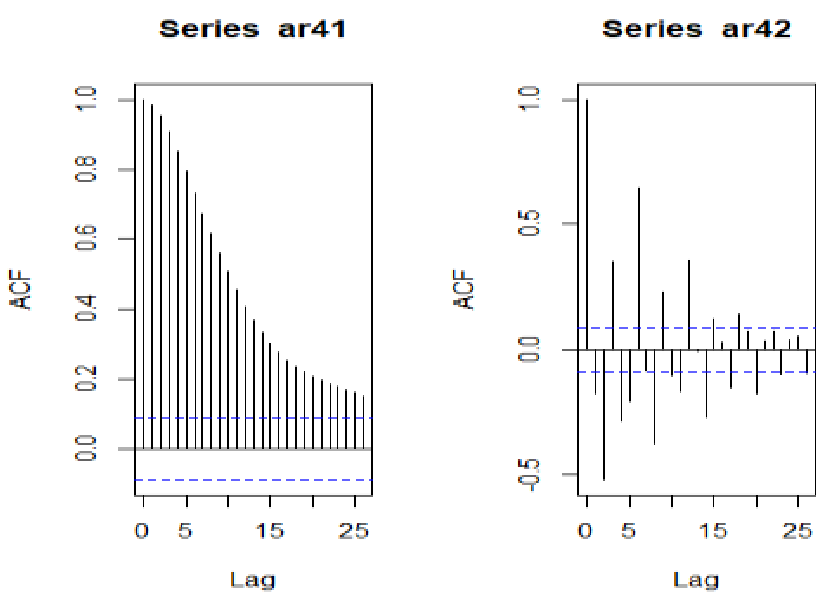
```
> ar4<-matrix(c(rep(0,1200)),ncol=2)
> a<-matrix(c(1.6,-0.05,-0.25,-0.8575,-0.582,0.04275,0.216,-0.731025),ncol=4)
> #simulate normal data with mean 0, variance 1
> w<-matrix(rnorm(1200),ncol=2)
> for(j in 1:2){
+ ar4[1,j]<-w[1,j]
+ ar4[2,j]<-a[j,1]*ar4[1,j]+w[2,j]
+ ar4[3,j]<-a[j,1]*ar4[2,j]+a[j,2]*ar4[1,j]+w[3,j]
+ ar4[4,j]<-a[j,1]*ar4[3,j]+a[j,2]*ar4[2,j]+a[j,3]*ar4[1,j]+w[4,j]
+ for(i in 5:600){
+ i1<-i-1;i2<-i-2;i3<-i-3;i4<-i-4
+ ar4[i,j]<-
a[j,1]*ar4[i1,j]+a[j,2]*ar4[i2,j]+a[j,3]*ar4[i3,j]+a[j,4]*ar4[i4,j]+w[i,j]
+ }
+ }
> rws<-101:600
> ar41<-ar4[rws,1];ar42<-ar4[rws,2]
> df<-data.frame(ar41,ar42)
> write.table(df,"F:/Stat71122Spring/ar4.txt")

> ar4<-read.table("F:/Stat71122Spring/ar4.txt")
> attach(ar4)
> head(ar4)

       ar41        ar42
1 -24.36676 -0.04726313
2 -23.79496 -0.17542237
3 -22.18268  1.05043602
4 -20.36453  0.46851681
5 -18.02847 -2.20085563
6 -16.72893 -0.32347331
```

The first simulated AR(4) series has a time trace which shows trending and suggests irregular long-term cyclical behavior. However, the autoregressive polynomial has only real-valued zeros and there is no cyclical structure in the process.



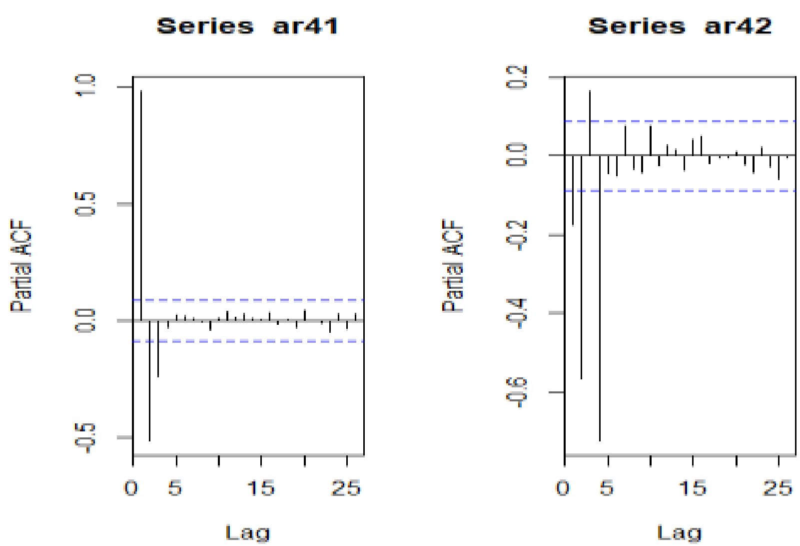
The second simulated series has a very different appearance. It has two pairs of complex conjugate zeros, generating a series with two pseudo-cyclical components having periods of length 3 and 6. The plot shows the first 100 observations.





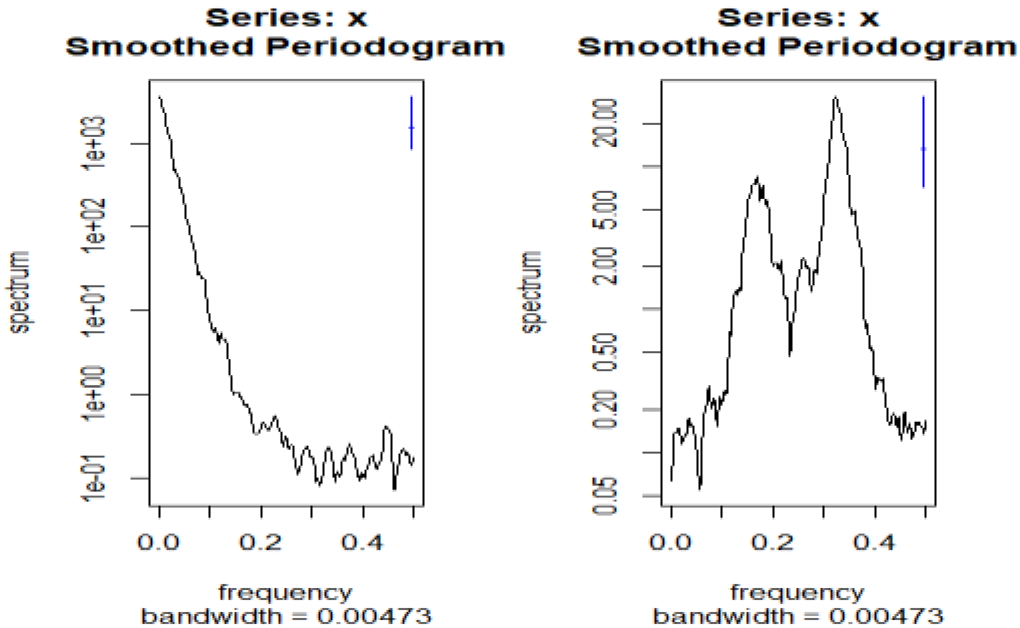
The acf plot on the left attenuates very slowly, akin to what one sees when the time series has a trend. In fact, the first AR(4) series does not have a trend, and the observed behavior is attributable to the values of the autoregressive polynomial zeros. The acf for the second AR(4) series clearly suggests cyclical behavior—there are significant autocorrelations at multiples of lag 3 (as well at some other lags). The pseudo-cyclical structure with period length 6 is not made evident by this plot. Below, we will see that a spectral density plot displays clear evidence of both pseudo-cyclical components.

The estimates of the partial autocorrelations for the two AR(4) series are shown next.



The second series is clearly identified as having AR order 4. The first plot, however, suggests AR(3) structure. This misidentification is not uncommon. The plot gives estimates of the partial autocorrelations, and the estimation is subject to statistical variation. Moreover, there is bias associated with the estimation for some process structures, and this is true for the structure of the first series.

```
> par(mfrow=c(1,2))
> spectrum(ar41,span=8)
> spectrum(ar42,span=8)
```



The first spectrum shows that the series has slow trend-like movement. The second spectrum has peaks centered at frequencies 1/6 and 1/3, indicating the lengths of the pseudo-cycles in the series.

Next, we consider simulations of some MA models. For these, we have chosen the same polynomials used for the AR models above. For MA models, though, we don't need to focus as much on the zeros of the polynomials. Notice that the algebraic sign convention for moving average models differs from that for autoregressive models—there are negative signs attached to the coefficients of the autoregressive models and positive signs attached to the coefficients of the moving average models. This is the convention employed by R. In the literature and in software packages, the sign convention often is different. For example, Tsay's book uses minus signs in front of both the AR and MA parameters.

D. Moving average process of order 1, MA(1)

$y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}$ The moving average polynomial is $z + \theta_1$

1. $y_t = \varepsilon_t + 0.9\varepsilon_{t-1}$ 1 + 0.9z

2. $y_t = \varepsilon_t - 0.9\varepsilon_{t-1}$ 1 − 0.9z

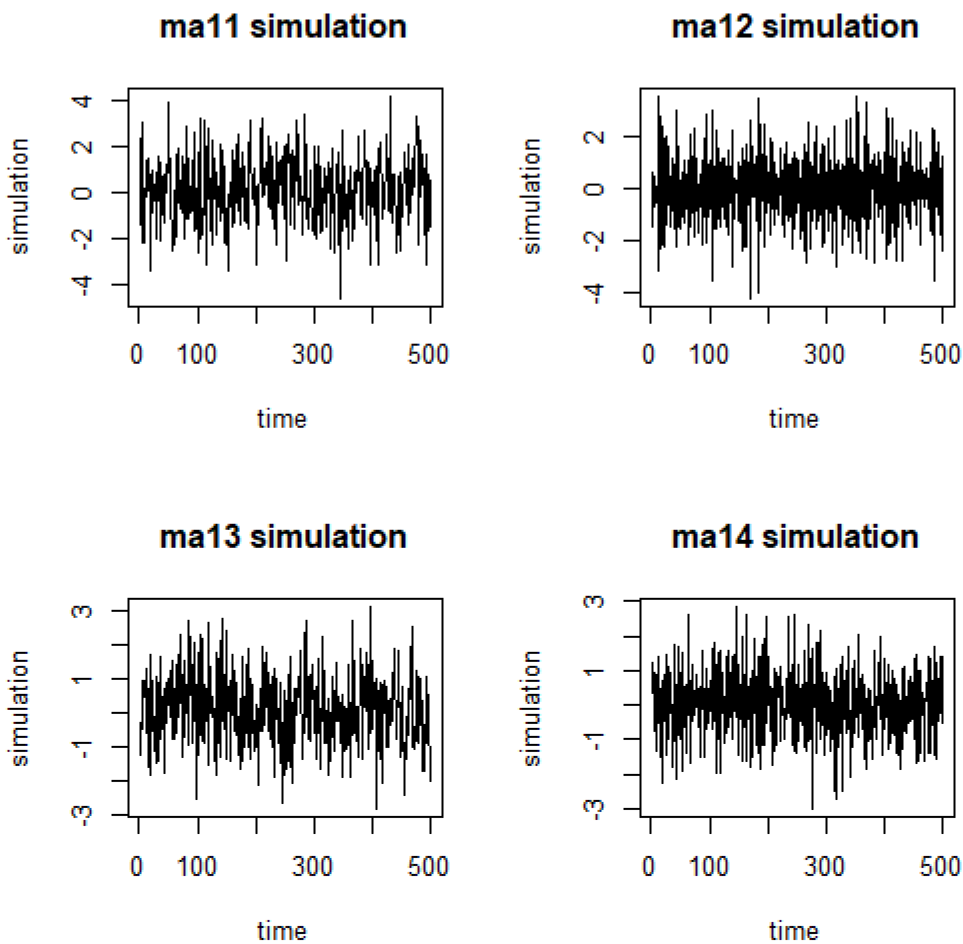
3. $y_t = \varepsilon_t + 0.2\varepsilon_{t-1}$ 1 + 0.2z

4. $y_t = \varepsilon_t - 0.2\varepsilon_{t-1}$ 1 − 0.2z

```
> ma1<-matrix(c(rep(0,2400)),ncol=4)
> a<-c(0.9,-0.9,0.2,-0.2)
> #simulate normal data with mean 0, variance 1
> w<-matrix(rnorm(2400),ncol=4)
> for(j in 1:4){
+ ma1[1,j]<-w[1,j]
+ for(i in 2:600){
+ i1<-i-1
+ ma1[i,j]<-w[i,j]+a[j]*w[i1,j]
+ }
+ }
> rws<-101:600
> ma11<-ma1[rws,1];ma12<-ma1[rws,2];ma13<-ma1[rws,3];ma14<-ma1[rws,4]
> df<-data.frame(ma11,ma12,ma13,ma14)
> write.table(df,"F:/Stat71122Spring/ma1.txt")

> ma1<-read.table("F:/Stat71122Spring/ma1.txt")
> attach(ma1)
> head(ma1)

        ma11       ma12       ma13       ma14
1 -1.3556456 -1.4692460 -1.1091863  0.9348938
2 -0.6269314  0.5686331 -0.2874499  0.6246713
3  1.6220255  0.3939217 -1.2574026  1.2109026
4  3.0440840  0.3217352  0.6317336 -0.5399005
5  1.2217002  0.4454690 -0.3269447  0.3354953
6 -0.6537679  0.1464899 -0.4569714 -0.7394737
```

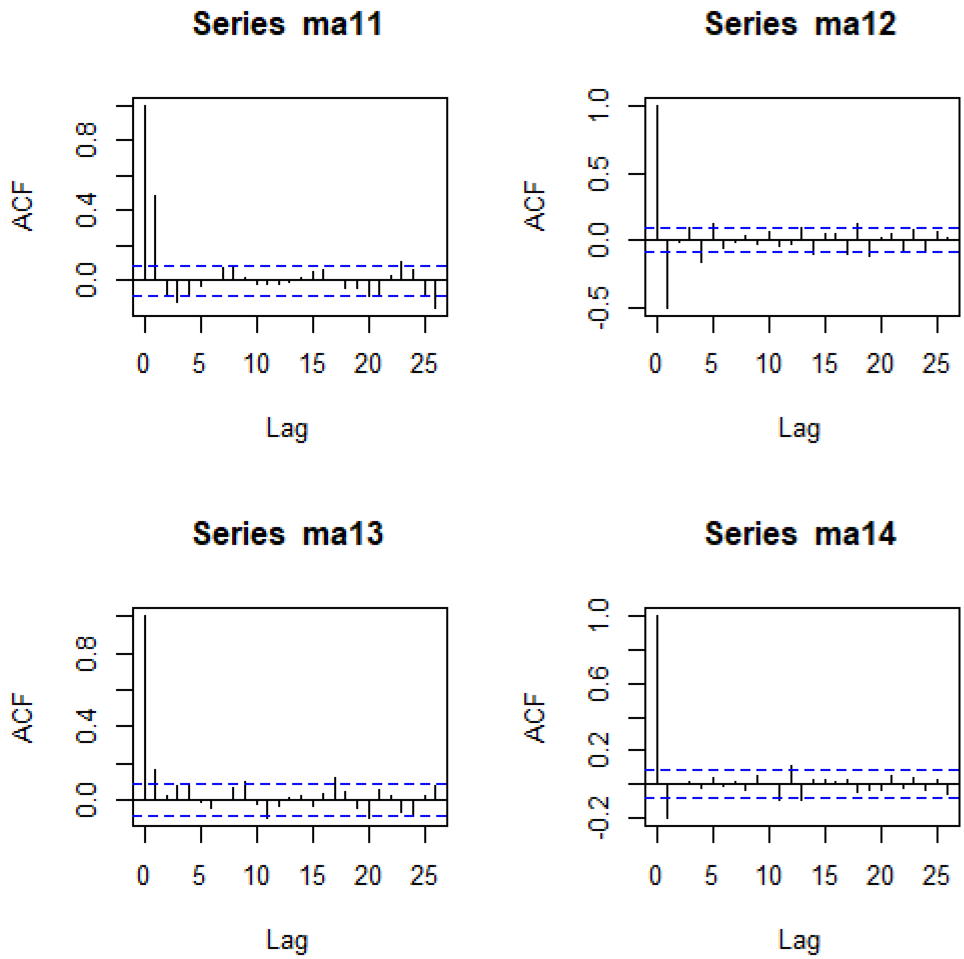
Let’s look at the time plots of all four of the simulated MA(1) series simultaneously.

```
> par(mfrow=c(2,2))
> plot(ts(ma11),xlab="time",ylab="simulation",main="ma11 simulation")
> plot(ts(ma12),xlab="time",ylab="simulation",main="ma12 simulation")
> plot(ts(ma13),xlab="time",ylab="simulation",main="ma13 simulation")
> plot(ts(ma14),xlab="time",ylab="simulation",main="ma14 simulation")
```

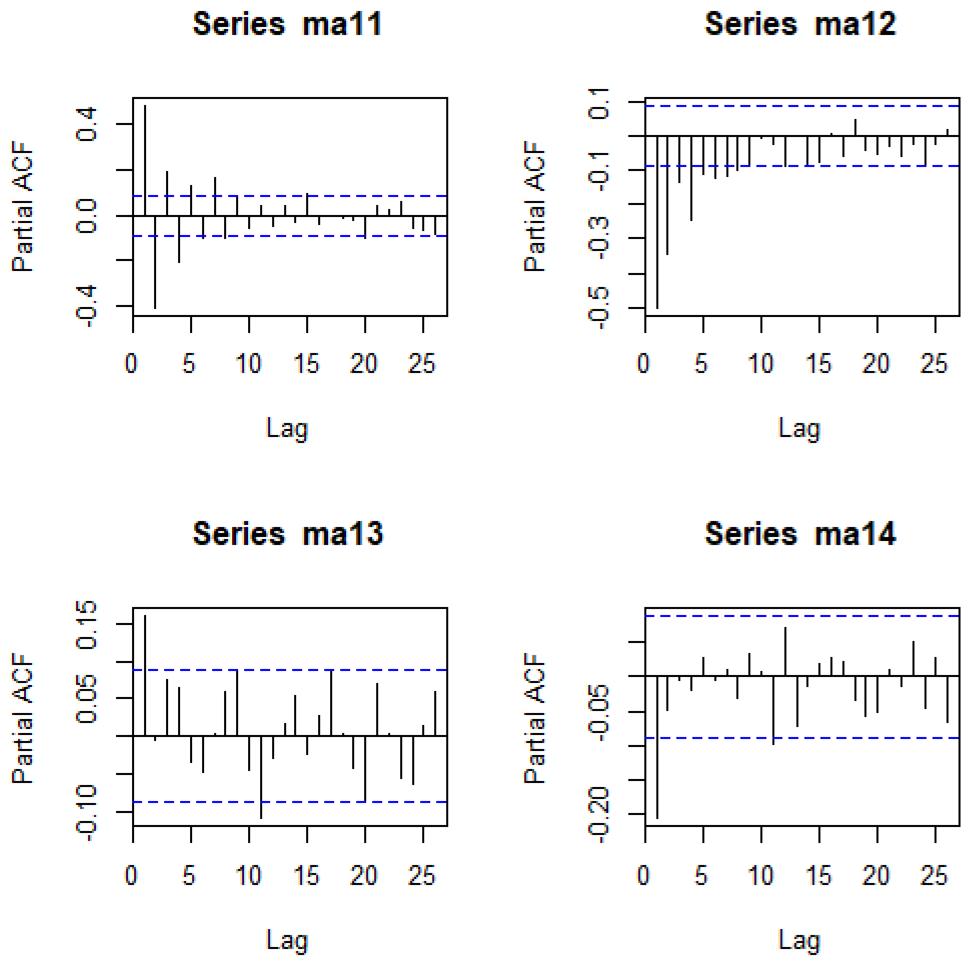


Note that the first two time traces have different appearance. The second MA(1) model exhibits high frequency activity. The last two simulated series are for models with weak signals, but we can still see that the fourth model also has enhanced high frequency movement.

Next, we look at acf and pacf plots.



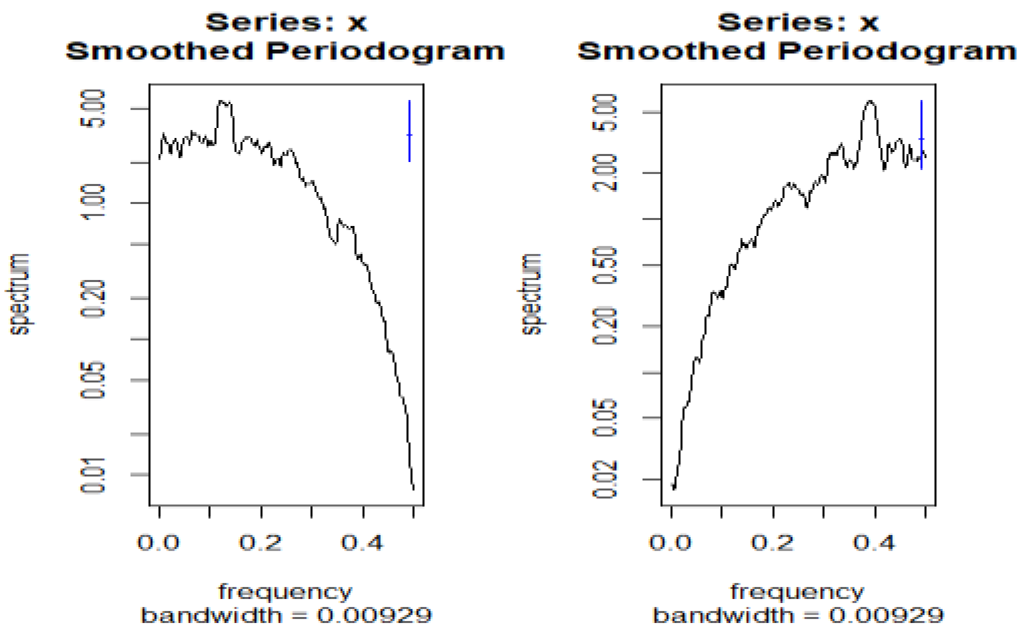
Although there are some modest false positives, it is evident that all these acf plots cut off at lag 1, the order of the moving average models. For all MA(1) models the acf cuts off at lag 1.



The top two plots, which are for MA(1) models with strong signals, show that the partial autocorrelations tail off rather than cut off. The bottom two plots are for MA(1) models with weak signals, and in theory their partial autocorrelations also do not cut off. The patterns observed are attributable to the weak signals and given sample size (a much larger sample size would clearly show behavior which tails off for the bottom two plots).

Spectral estimates for the first two series follow.

```
> par(mfrow=c(1,2))
> spectrum(ma11,span=16)
> spectrum(ma12,span=16)
```



The small peaks in each plot do not describe the structure of these MA(1) models—some false positives. The important takeaway is that the first series has relatively slow fluctuations, and the second series relatively fast fluctuations.

E. Moving average process of order 2, MA(2)

$y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$ The moving average polynomial is $z^2 + \theta_1 z + \theta_2$

1. $y_t = \varepsilon_t - 1.3\varepsilon_{t-1} + 0.4\varepsilon_{t-2}$ $(z - 0.5)(z - 0.8)$

2. $y_t = \varepsilon_t + 1.3\varepsilon_{t-1} + 0.4\varepsilon_{t-2}$ $(z + 0.5)(z + 0.8)$

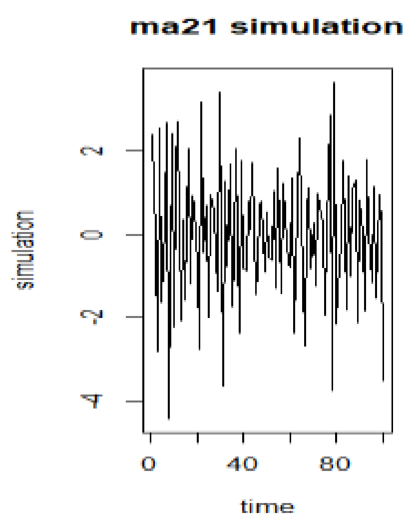
3. $y_t = \varepsilon_t - 0.9\varepsilon_{t-1} + 0.81\varepsilon_{t-2}$ $(z - 0.9e^{i2\pi/6})(z - 0.9e^{-i2\pi/6})$

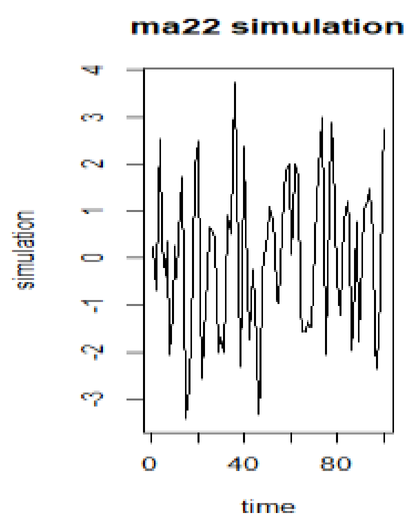
4. $y_t = \varepsilon_t - 1.55885\varepsilon_{t-1} + 0.81\varepsilon_{t-2}$ $(z - 0.9e^{i2\pi/12})(z - 0.9e^{-i2\pi/12})$

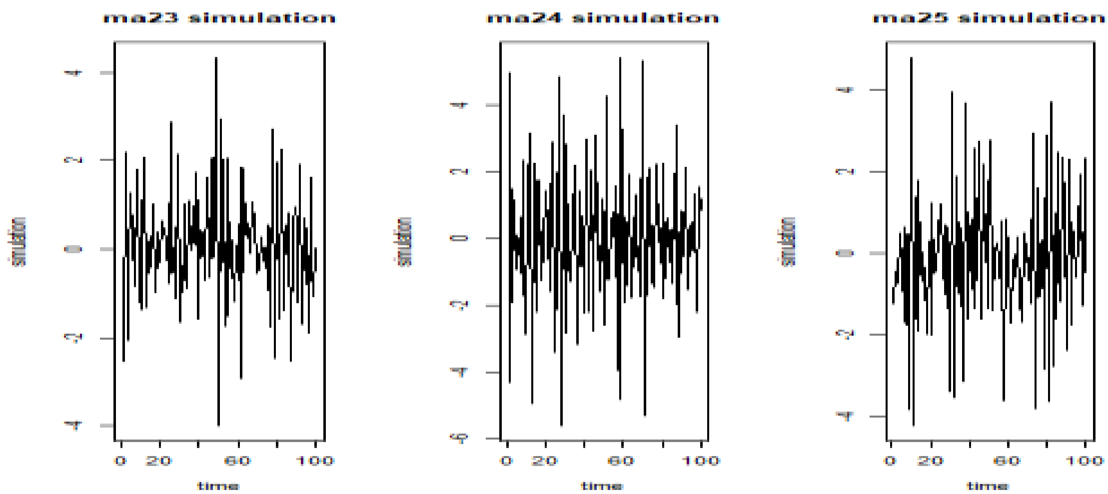
5. $y_t = \varepsilon_t - 0.99\varepsilon_{t-1} + 0.9801\varepsilon_{t-2}$ $(z - 0.99e^{i2\pi/6})(z - 0.99e^{-i2\pi/6})$

```
> ma2<-matrix(c(rep(0,3000)),ncol=5)
> a<-matrix(c(-1.3,1.3,-0.9,-1.55885,-0.99,0.4,0.4,0.81,0.81,0.9801),ncol=2)
> #simulate normal data with mean 0, variance 1
> w<-matrix(rnorm(3000),ncol=5)
> for(j in 1:5){
+ ma2[1,j]<-w[1,j]
+ ma2[2,j]<-w[2,j]+a[j,1]*w[1,j]
+ for(i in 3:600){
+ i1<-i-1;i2<-i-2
+ ma2[i,j]<-w[i,j]+a[j,1]*w[i1,j]+a[j,2]*w[i2,j]
+ }
+ }
> rws<-101:600
> ma21<-ma2[rws,1];ma22<-ma2[rws,2];ma23<-ma2[rws,3];ma24<-ma2[rws,4];ma25<-
ma2[rws,5]
> df<-data.frame(ma21,ma22,ma23,ma24,ma25)
> write.table(df,"F:/Stat71122Spring/ma2.txt")

> ma2<-read.table("F:/Stat71122Spring/ma2.txt")
> attach(ma2)

> head(ma2)
        ma21       ma22       ma23       ma24       ma25
1  2.4014477  0.2511824 -2.5259267 -4.2930961 -1.2333868
2 -0.1403834 -0.6895066  2.1861977  4.9706708 -0.4365336
3 -2.8079826  1.3254046 -2.0439537 -1.9129234 -0.7360099
4  2.5421019  2.5239765 -0.3347117  1.1492171  0.5002535
5 -1.6408666  0.2849773  1.2767551 -0.9236704 -0.2145085
6  0.2801233 -0.3882460  0.3195595  0.0209632 -1.6836227
```

The time plots of the simulations which follow show the first 100 observations for each series.

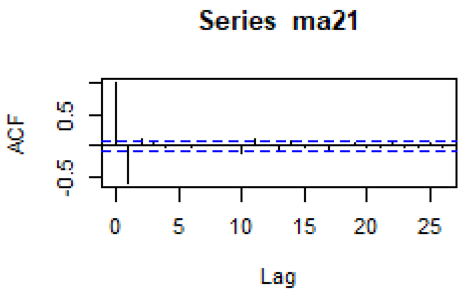


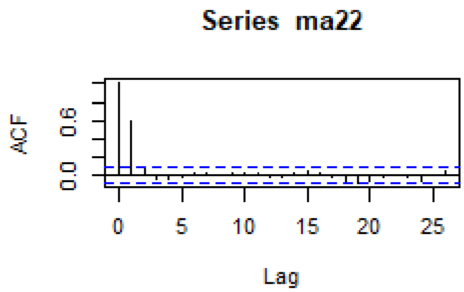


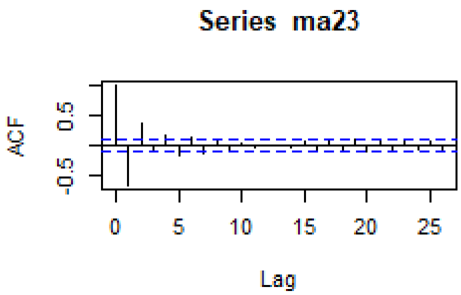


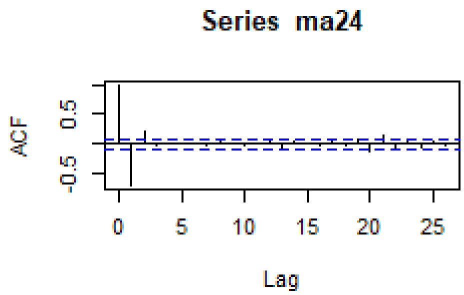
The first plot exhibits high frequency activity, and the second does not. The last three plots are for MA(2) models where the moving average polynomial has complex-valued zeros. None of these models shows pseudo-cyclical behavior, unlike the autoregressive models with the same polynomials.

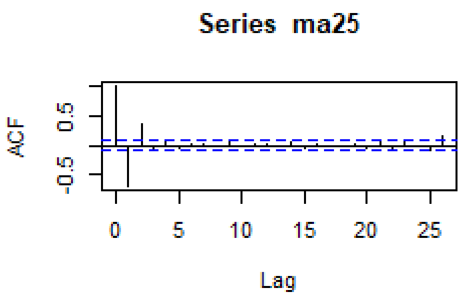
Acf, pacf, and spectral plots follow. Except for some minor sampling error, the acf plots cut off at lag 2, and the pacf plots tail off. The last three acf plots show no evidence of pseudo-cyclical behavior.

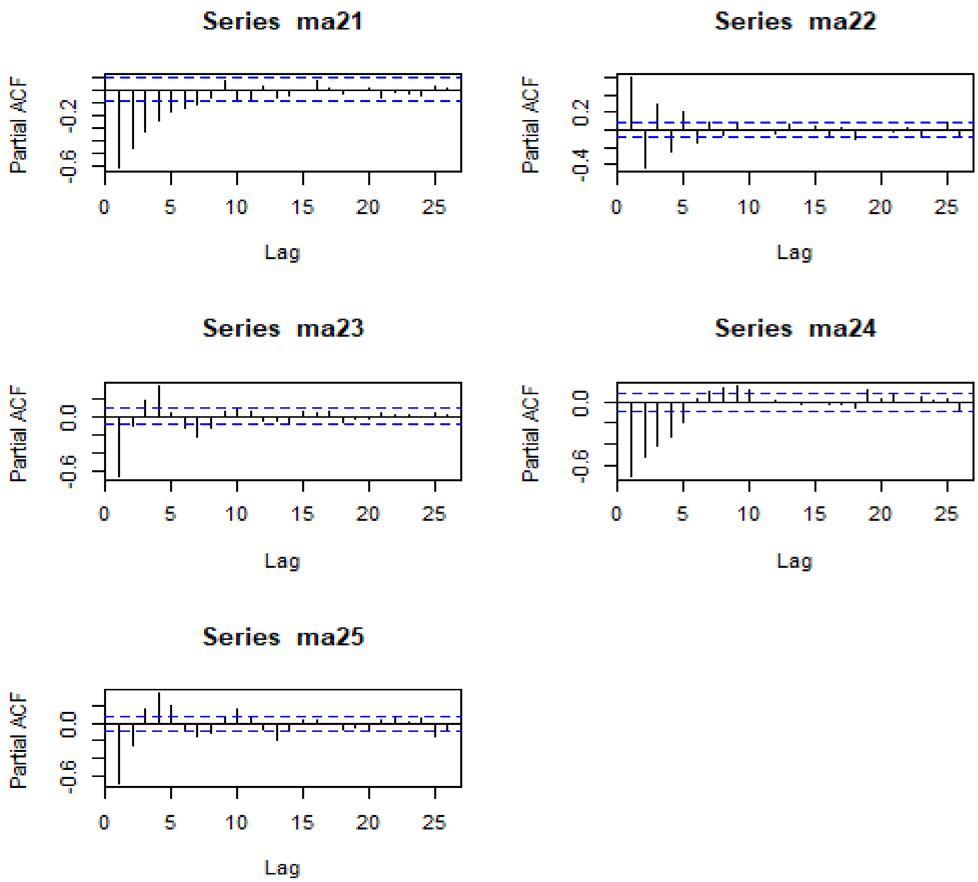

Series ma21
ACF
-0.5
0.5
0
5
10
15
20
25
Lag


Series ma22
ACF
0.0
0.6
0
5
10
15
20
25
Lag

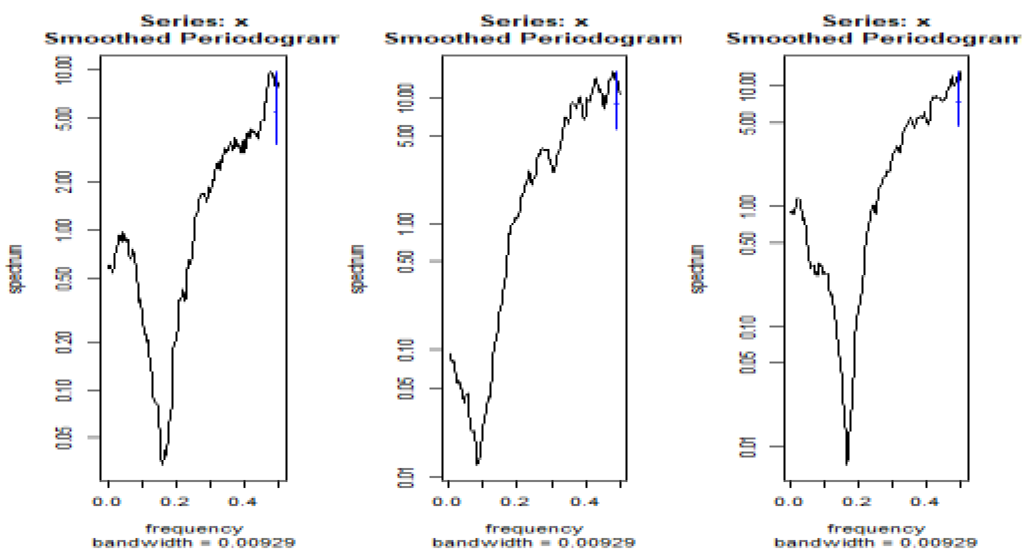

Series ma23
ACF
-0.5
0.5
0
5
10
15
20
25
Lag


Series ma24
ACF
-0.5
0.5
0
5
10
15
20
25
Lag


Series ma25
ACF
-0.5
0.5
0
5
10
15
20
25
Lag




```
> par(mfrow=c(1,3))
> spectrum(ma23,span=16)
> spectrum(ma24,span=16)
> spectrum(ma25,span=16)
```



As we have noted, moving average models for which the polynomials have complex zeros do not have pseudo-cyclical behavior. Thus, spectral peaks do not appear at frequencies corresponding to the arguments of the complex zeros. Here, the last three moving average models have the following zeros:

model ma23 $0.9e^{i2\pi/6}$, $0.9e^{-i2\pi/6}$

model ma24 $0.9e^{i2\pi/12}$, $0.9e^{-i2\pi/12}$

model ma25 $0.99e^{i2\pi/6}$, $0.99e^{-i2\pi/6}$

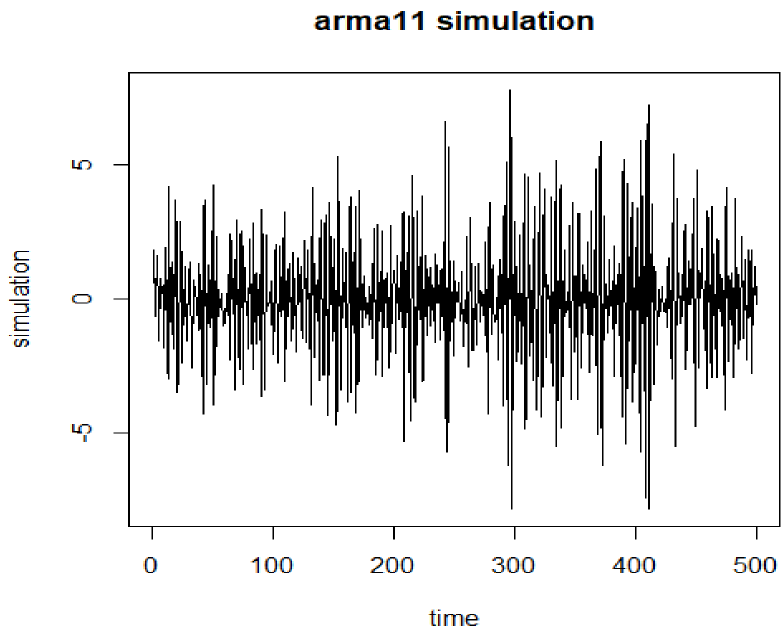
The three spectral estimates have valleys, rather than peaks, at frequencies 1/6, 1/12, and 1/6. The valleys do not indicate the generation of notable features for the time series, as peaks do.

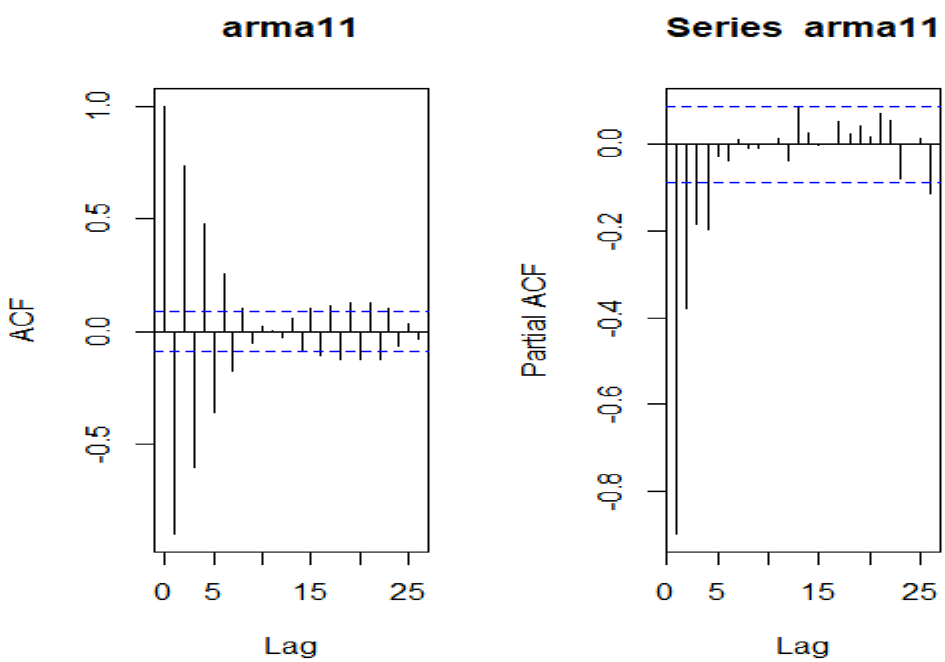
F. Autoregressive moving average process of order (1,1), ARMA(1,1)

$y_t - \phi_1 y_{t-1} = \varepsilon_t + \theta_1 \varepsilon_{t-1}$ The polynomials are $z - \phi_1$ and $z + \theta_1$.

$y_t + 0.8 y_{t-1} = \varepsilon_t - 0.5\varepsilon_{t-1}$ $z + 0.8z$, $z - 0.5$

```
> arma11<-c(rep(0,600))
> a<-c(-0.8,-0.5)
> #simulate normal data with mean 0, variance 1
> w<-rnorm(600)
> arma11[1]<-w[1]
> for(i in 2:600){
+ i1<-i-1
+ arma11[i]<-a[1]*arma11[i1]+w[i]+a[2]*w[i1]
+ }
> rws<-101:600
> arma11<-arma11[rws]
> df<-data.frame(arma11)
> write.table(df,"F:/Stat71122Spring/arma11.txt")

> arma11<-read.table("F:/Stat71122Spring/arma11.txt")
> attach(arma11)

> head(arma11)
       arma11
1  1.83975142
2 -0.65018534
3 -0.08154918
4  1.59695856
5 -0.60272122
6 -1.56841475
```





The acf and pacf plot both tail off, rather than cut off (the apparent cutoff at lag 4 in the pacf plot is not relevant). ARMA models which are not pure autoregressive or pure moving average exhibit this behavior, that both tail off.

Summary and additional remarks

1. Autoregressive moving average (ARMA) models are commonly employed in analysis of time series data. They provide relatively simple structures characterized by a small number of parameters. Special cases are pure autoregressive (AR) models and pure moving average (MA) models.

2. A white noise structure is the basic building block for an ARMA model. For an MA model, the observed time series is a filtering of white noise, to produce colored noise. For an AR model, a filtering of the observed time series produces white noise.

3. The MA and AR structures both have associated polynomials. The zeros of these polynomials occur as real values and complex conjugate pairs. The complex zeros are conveniently described by their amplitudes and phases. The values of the polynomial zeros in the ARMA models determine the structure and behavior of the models. In particular, for AR models the presence of complex zeros produces pseudo-cyclical behavior.

4. AR models are very useful in describing time series data, because the models relate the present value of a time series to its immediate past values.

5. In examining the behavior of ARMA models, we look at their autocorrelation functions, partial autocorrelation functions, and spectral density functions. Pure autoregressive models have an acf which tails off and a pacf which cuts off at the order of the process. Pure moving average models have an acf which cuts off at the order of the process and a pacf which tails off. For mixed autoregressive moving average models, both the acf and the pacf tail off. For all of the models spectral peaks give information about the time series fluctuations.

Autocorrelation Function
and Partial Autocorrelation Function Estimation

Suppose a time series $y_1, \ldots, y_n$ has been observed. The sample mean is

$$\bar{y} = \frac{1}{n}(y_1 + \cdots + y_n).$$

The sample autocorrelation at lag k is

$$r_k = \frac{\sum_{t=1}^{n-k}(y_t - \bar{y})(y_{t+k} - \bar{y})}{\sum_{t=1}^{n}(y_t - \bar{y})^2}, \qquad k = 1,2,\ldots,$$

and $r_{-k} = r_k$. This is an estimate of ρ_k, the population correlation at lag k.

We judge the precision of the estimate r_k by its standard error. In plots and tables of the autocorrelation function R takes the standard error to be $1/\sqrt{n}$ under the null hypothesis that an autocorrelation is 0. Some statistical packages use a slightly different protocol for the standard error.

The sample partial autocorrelation at lag k can be estimated as the negative value of the coefficient of y_{t-k} in the least squares regression of y_t on $y_{t-1}, \ldots, y_{t-k}$, that is, as the coefficient attached to the highest lagged regressor. The standard error of the estimate under the null hypothesis that the partial autocorrelation is 0 is $1/\sqrt{n}$.

Example. Consider expressions for the theoretical acf and pacf of an MA(1) process,

$$y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad -1 < \theta_1 < 1.$$

The acf is

$$\rho_{-1} = \rho_1 = \frac{\theta_1}{1 + \theta_1^2},$$

and $\rho_k = 0$ for $k = \pm 2, \pm 3, \ldots$.

The pacf is more complicated. It is

$$\phi_{kk} = -\frac{\left(-\theta_1\right)^k (1-\theta_1^2)}{1-\theta_1^{2k+2}}, \qquad k = 1, 2, \ldots.$$

The MA(1) process with $\theta_1 = -0.9$ has $\rho_1 = -0.9/1.81 = -0.4972$ and

$$\phi_{kk} = -\frac{0.9^k\, 0.19}{1-(0.81)^{k+1}}, \qquad k = 1, 2, \ldots.$$

The second MA(1) simulation above has this structure. Here are tables of the acf and pacf estimates for the model.

```
> acf(ma12,plot=F)

Autocorrelations of series 'ma12', by lag

     0      1      2      3      4      5      6      7      8      9     10
 1.000 -0.499 -0.011  0.093 -0.156  0.125 -0.056 -0.013  0.036 -0.031  0.062
    11     12     13     14     15     16     17     18     19     20     21
-0.048 -0.034  0.086 -0.103  0.052  0.046 -0.097  0.122 -0.110  0.019  0.047
    22     23     24     25     26
-0.070  0.070 -0.068  0.056  0.023

> pacf(ma12,plot=F)

Partial autocorrelations of series 'ma12', by lag

     1      2      3      4      5      6      7      8      9     10     11
-0.499 -0.346 -0.131 -0.246 -0.110 -0.120 -0.117 -0.099 -0.090 -0.005 -0.026
    12     13     14     15     16     17     18     19     20     21     22
-0.088  0.002 -0.084 -0.073  0.008 -0.061  0.044 -0.042 -0.055 -0.028 -0.056
    23     24     25     26
-0.024 -0.082 -0.021  0.016
```