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Consistency of Heyting arithmetic in natural deduction

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A proof of the consistency of Heyting arithmetic formulated in natural deduction is given. The proof is a reduction procedure for derivations of falsity and a vector assignment, such that each reduction reduces the vector. By an interpretation of the expressions of the vectors as ordinals each derivation of falsity is assigned an ordinal less than ε_0, thus proving termination of the procedure.

1 Introduction

The limitations of the formal systems of arithmetic, revealed by Gödel's incompleteness theorems in 1931, imply that the consistency of Peano arithmetic can be established only by reducing the problem to certain fundamental principles that are not formalizable within the theory itself. For a proof to be meaningful, these principles should be considered more reliable than the doubtful elements of the theory concerned.

In 1933, Gentzen and Gödel independently established that the consistency of classical Peano arithmetic reduces to the consistency of intuitionistic Heyting arithmetic by a negative translation. Therefore, an inconsistency in the classical calculus cannot stem from the principle of indirect proof that by some was considered a doubtful element. The only remaining principle of arithmetic whose consequences could be doubted was complete induction.

The earliest proofs of the consistency of Peano arithmetic were presented by Gentzen, who worked out a total of four proofs between 1934 and 1939. The first consistency proof was withdrawn from publication due to criticism by Bernays for implicit use of the fan theorem, although this assessment was later retracted [1]. However, a galley proof of the article was preserved and excerpts were published posthumously in English translation [5], as well as unabridged in the German original [6]. The critique led to an alteration of the argument and the published second proof [2] that is appended with an ordinal assignment and relies on a constructive proof of the principle of transfinite induction up to the ordinal ε_0. In both the first and the second proof, the arithmetical system is formalized in natural deduction written in sequent calculus style. The third consistency proof [3] is a revised version, conducted in pure sequent calculus, whereas the fourth proof [4], in contrast, proves consistency through a non-derivability. A thorough account of Gentzen's work by von Plato is found in [10].

The combinatorial methods of Gentzen's reduction procedure described in the third proof can be represented in primitive recursive arithmetic (PRA). PRA is a weaker theory than Peano arithmetic and it is generally included in, and often identified with, finitistic logic, because unbounded quantification over the domain of natural numbers is not allowed. Due to this feature, the primitive recursive operation on derivations corresponds to a quantifier-free formula. Therefore, finitistic reasoning together with the principle of transfinite induction restricted to quantifier-free formulas gives the consistency result. It should be noted that the theory, in which the proof is formalizable, is incomparable to Peano arithmetic. The theory is not stronger than Peano arithmetic, since complete induction cannot be proved. But on the other hand, neither is the theory weaker, since it proves the consistency of Peano arithmetic.

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Since the publication of Gentzen's proof, the conducting of the consistency proof in standard natural deduction has been an open problem. The aim of this paper is to solve this problem by giving a consistency proof in natural deduction for Heyting arithmetic. The result is based on a normalization proof by Howard [7], recommended to the author by Per Martin-Löf.

The present consistency proof is performed in the manner of Gentzen, by giving a reduction procedure for derivations of falsity. The procedure is appended with the assignment of a vector to each derivation and it is shown that the reduction reduces the first component. This component can be interpreted as an ordinal less than ε_0, thus ordering the derivations by complexity and proving termination of the process. To prove consistency it needs to be established that no derivation of the simplest kind exists. An important initial task becomes to examine how natural deduction can be extended into an arithmetical system.

2 Logical calculus

The following rules constitute the standard calculus of intuitionistic natural deduction. Negation is a defined concept with $\neg A \equiv A \supset \bot$.

$$\frac{\bot}{C}\ {\scriptstyle \bot E}$$

$$\frac{A \quad B}{A\&B}\ {\scriptstyle \&I} \qquad \frac{A\&B}{A}\ {\scriptstyle \&E} \qquad \frac{A\&B}{B}\ {\scriptstyle \&E}$$

$$\frac{A}{A \vee B}\ {\scriptstyle \vee I} \qquad \frac{B}{A \vee B}\ {\scriptstyle \vee I} \qquad \frac{A \vee B \quad \begin{matrix}[A]\\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots \\ C\end{matrix}}{C}\ {\scriptstyle \vee E}$$

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\ {\scriptstyle \supset I} \qquad \frac{A \supset B \quad A}{B}\ {\scriptstyle \supset E}$$

$$\frac{A(y/x)}{\forall x A}\ {\scriptstyle \forall I} \qquad \frac{\forall x A}{A(t/x)}\ {\scriptstyle \forall E}$$

$$\frac{A(t/x)}{\exists x A(x)}\ {\scriptstyle \exists I} \qquad \frac{\exists x A(x) \qquad \begin{matrix}[A(y/x)]\\ \vdots \\ C\end{matrix}}{C}\ {\scriptstyle \exists E}$$

In the formula denoted by $A(t/x)$ every free occurrence of the variable x in $A(x)$ has been substituted with the term t. The standard variable restriction holds in the rules $\forall I$ and $\exists E$: the eigenvariable y must not be free in the conclusion of the rule, nor in any assumption that the conclusion depends on, except for the discarded assumption $A(y/x)$ in the existential rule. The premise with logical structure, which is eliminated in the conclusion of an elimination rule, is the *major premise* of the rule. The other premises are *minor premises*.

3 Arithmetical rules and induction

The language of Heyting arithmetic consists of the constant 0, the unary functional symbol s, the binary functional symbols $+$ and $\cdot$ and the binary predicate symbol $=$.

Definition 3.1 *Terms* are inductively defined. The constant 0 and variables are terms and if t and t' are terms then $s(t), t + t'$ and $t \cdot t'$ are also terms. Terms are *closed* if they do not contain any variable.

Formal expressions for the natural numbers, *numerals*, are inductively defined: 0 is a numeral and if $\overline{m}$ is a numeral, then $s(\overline{m})$ is also a numeral.

The axioms of Heyting arithmetic can be formulated as rules of natural deduction, expanding the logical calculus. Together with an induction rule the logical and arithmetical rules constitute the system of Heyting arithmetic (HA). Negri and von Plato [8] developed the general method for converting mathematical axioms into rules for the primary purpose of proving cut elimination in systems of sequent calculus. As mentioned in the paper, these rules of sequent calculus can be translated into non-logical introduction and elimination rules of natural deduction. The specific system for arithmetic was first used by von Plato [9] to prove the disjunction and existential properties.

Rules for the equality relation:

$$\frac{}{t = t}\ \mathit{Ref} \qquad \frac{t = t'}{t' = t}\ \mathit{Sym} \qquad \frac{t = t' \quad t' = t''}{t = t''}\ \mathit{Tr}$$

Recursion rules:

$$\frac{}{t + 0 = t}\ {+}\,\mathit{Rec}\,0 \qquad \frac{}{t + s(t') = s(t + t')}\ {+}\,\mathit{Rec}\,s$$

$$\frac{}{t \cdot 0 = 0}\ \cdot\,\mathit{Rec}\,0 \qquad \frac{}{t \cdot s(t') = t \cdot t' + t}\ \cdot\,\mathit{Rec}\,s$$

Replacement rules:

$$\frac{t = t'}{s(t) = s(t')}\ s\,\mathit{Rep}$$

$$\frac{t = t'}{t + t'' = t' + t''}\ {+}\,\mathit{Rep} \qquad \frac{t' = t''}{t + t' = t + t''}\ {+}\,\mathit{Rep}$$

$$\frac{t = t'}{t \cdot t'' = t' \cdot t''}\ \cdot\,\mathit{Rep} \qquad \frac{t' = t''}{t \cdot t' = t \cdot t''}\ \cdot\,\mathit{Rep}$$

Infinity rules:

$$\frac{s(t) = 0}{\bot}\ \mathit{Inf}_1 \qquad \frac{s(t) = s(t')}{t = t'}\ \mathit{Inf}_2$$

Induction rule:

$$\frac{A(0/x) \quad \begin{array}{c}[A(y/x)] \\ \vdots \\ A(s(y)/x)\end{array}}{A(t/x)}\ \mathit{Ind}$$

The eigenvariable y of the induction rule obeys the standard variable restriction and the induction formula A is arbitrary.

Although the system is intuitionistic it is possible to derive the law of excluded middle for atomic formulas, $\forall x \forall y(x = y \vee \neg x = y)$, by the induction rule. Therefore, the formula $A \vee \neg A$ can be derived for an arbitrary quantifier-free formula A.

Definition 3.2 A *purely arithmetical* derivation is a derivation where only arithmetical rules occur. (Induction is not included among the arithmetical rules.)

4 Properties of arithmetical derivations

The overall aim of Gentzen-style consistency proofs is to reduce complex derivations of a contradiction to simpler derivations. Therefore, a natural starting point is to consider the most elementary kind of derivations. The primary goal is to prove consistency for these purely arithmetical derivations.

Lemma 4.1

(i) *For a closed term* t *there exists a unique numeral* $\overline{m}$*, for which there is a purely arithmetical derivation of* $t = \overline{m}$*.*

(ii) *Let* t *and* t' *be closed terms and assume that there is a purely arithmetical derivation of the formula* $t = t'$*. Then there is a purely arithmetical derivation of the formula* $q(t/x) = q(t'/x)$ *for an arbitrary term* $q(x)$*.*

(iii) *Let* t *and* t' *be closed terms and assume that there is a purely arithmetical derivation of the formula* $t = t'$*. Then for arbitrary terms* $q(x)$ *and* $r(x)$ *there is a purely arithmetical derivation of* $q(t'/x) = r(t'/x)$ *from the open assumption* $q(t/x) = r(t/x)$*.*

(iv) *Let* t *and* t' *be closed terms and assume that there is a purely arithmetical derivation of the formula* $t = t'$*. Then for an arbitrary formula* A *the formula* $A(t/x) \supset A(t'/x)$ *can be derived without Ind.*

Proof. The proof of (i) – (iii) is by induction on the complexity of the term and the proof of (iv) is by induction on the complexity of the formula A. By proving that there is a derivation of $A(t')$ from the open assumption $A(t)$ a proof of (iv) is obtained through a final implication introduction. If A is an atomic formula, then the claim is proved in (iii). For the inductive step it is assumed that formula A is a compound formula. Assuming that A is a conjunction $B\&C$, the following derivation may be constructed:

$$\dfrac{\dfrac{\dfrac{[B(t)\&C(t)]}{B(t)}\,\&E \quad \dfrac{[B(t)\&C(t)]}{C(t)}\,\&E}{\begin{matrix} \vdots & & \vdots \\ B(t') & & C(t') \end{matrix}}}{\dfrac{B(t')\&C(t')}{(B(t)\&C(t)) \supset (B(t')\&C(t'))}\,\supset I}\,\&I$$

The other cases are similar. □

Lemma 4.2 *There is no purely arithmetical derivation of falsity.*

Proof. By the uniqueness of the numeral equal to a term, the premise of rule *Inf*$_1$ cannot be derived without open assumptions. Therefore, it is not possible to derive falsity. □

5 Assignment of vectors to derivations

The normalization proof of Howard [7] provides a unique ordinal assignment up to ε_0 to terms of Gödel's theory T of primitive recursive functionals and proves that restricted reductions of the terms reduce the ordinals. In addition, a non-unique assignment is given for general reductions. By the well-ordering of ε_0, each reduction sequence terminates into a normal form, thereby proving strong normalization.

The unique assignment can be adapted to derivations in natural deduction. If each derivation is assigned an ordinal number as a measure of its complexity, then derivations can be ordered aiming for a proof of termination. The assignment of ordinal numbers to terms in T is indirect through a vector assignment and an interpretation of vectors as ordinals. A detour by vectors is needed, because the length of the vector provides an additional parameter for the calculations. This parameter is used in the definition of two operations (also originating in Howard's paper) that will provide desired properties of vectors. The length of the vector assigned to a formula in a derivation will depend on the complexity of the formula.

Definition 5.1 The *level* of a formula A, denoted $l(A)$, is inductively defined.

1. The level of an atomic formula and falsity is 0.
2. The level of a conjunction $A\&B$ is $\max\{l(A), l(B)\}$.
3. The level of a disjunction $A \vee B$ is $\max\{l(A), l(B)\}$.
4. The level of an implication $A \supset B$ is $\max\{l(A) + 1, l(B)\}$.
5. The level of a universally quantified formula $\forall x A$ is $l(A)$.
6. The level of an existentially quantified formula $\exists x A$ is $l(A)$.

5.1 The theory $\mathcal{E}$

The vectors that will be assigned to derivations and formulas of derivations are vectors of expressions. These expressions are defined by introducing an axiomatic theory $\mathcal{E}$ of an order relation $\prec$ on expressions. In Section 8.1 the expressions of this theory will be interpreted as ordinals.

Definition 5.2 *Expressions* are inductively defined.

(i) The constants $0, 1$ and ω are expressions.

(ii) For all formulas A and all i in $\mathbb{N}$, the variable x_i^A is an expression.

(iii) If f and g are expressions, then $f + g$ and (f, g) are also expressions.

Equality between expressions is treated axiomatically and obeys reflexivity and the replacement axiom and the weak order relation $f \preccurlyeq g$ is defined as $f \prec g$ or $f = g$ and the relation $f \succ g$ as $g \prec f$. The axioms of the theory $\mathcal{E}$ are listed below.

1. If $f \prec g$ and $g \prec h$, then $f \prec h$.
2. If $f \prec g$, then $\neg f = g$.
3. $f + g = g + f$ and $(f + g) + h = f + (g + h)$.
4. If $f \prec g$, then $f + h \prec g + h$.
5. $f + g = f$ if and only if $g = 0$.
6. $0 \preccurlyeq f$ and $0 \prec 1 \prec \omega$.
7. If $f \prec \omega$ and $g \prec \omega$, then $f + g \prec \omega$.
8. $(f, g + h) = (f, g) + (f, h)$.
9. If $g \prec c$ and $h \prec c$, then $(g, f) + (h, f) \preccurlyeq (c, f)$.
10. If $f \prec g$, then $(h, f) \prec (h, g)$.
11. If $f \prec g$ and $\neg h = 0$, then $(f, h) \prec (g, h)$.
12. $(0, f) = f$.
13. $(f, (g, h)) = (f + g, h)$

If $f \succ 0$ and $h \succ 0$, then by axioms 12, 11 and 8 the following inequality holds:

(5.1) $$(f, g) + h \prec (f, g + h) \text{ if } f \succ 0 \text{ and } h \succ 0.$$

5.2 Vectors of expressions

Expressions can be divided into classes C_i with the property that each expression in a class contains no variable that has a lower index than the class. A class of vectors $\mathbf{C}$ can be defined by presupposing that each component of the vector belongs to the corresponding class of expressions. The vectors in $\mathbf{C}$ will be the ones assigned to formulas in a derivation and two operations, the box- and the delta-operation acting upon these vectors will shortly be defined.

If f_i is an expression for each $0 \leqslant i \leqslant n$, then the $n + 1$-tuple $\boldsymbol{f} = \langle f_0, \dots, f_n \rangle$ is a vector of length n. For $0 \leqslant i \leqslant n$ the expression f_i, also denoted $(\boldsymbol{f})_i$, is called the ith component of $\boldsymbol{f}$. For $i > \mathrm{length}(\boldsymbol{f})$, the component $(\boldsymbol{f})_i$ is defined to be 0. Addition of vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ is done component by component, i.e. $\boldsymbol{f} + \boldsymbol{g} = \langle f_0 + g_0, \dots, f_n + g_n \rangle$ where $n = \max\{\mathrm{length}(\boldsymbol{f}), \mathrm{length}(\boldsymbol{g})\}$.

Finally, a vector of variables is defined for every formula A with the level $l(A) = n$, $\boldsymbol{x}^A = \langle x_0^A, \dots, x_n^A \rangle$.

Definition 5.3 The *classes* C_i of expressions are defined by four clauses.

1. If the expression h contains no variables, then h is in C_i.
2. For every formula A, the variable x_i^A is in C_i.
3. If the expressions f and g are in C_i, then so is $f + g$.
4. If the expression f is in C_{i+1} and the expression g is in C_i, then (f, g) is in C_i.

The class $\mathbf{C}$ consists of all vectors $\boldsymbol{h}$, such that h_i is in C_i for $0 \leqslant i \leqslant \mathrm{length}(\boldsymbol{h})$.

Definition 5.4 The *box-operation* of two vectors $\boldsymbol{f}\Box\boldsymbol{g}$ is defined to be the vector $\boldsymbol{h} = \langle h_0, \dots, h_n\rangle$, where $n = \max\{\text{length}(\boldsymbol{f}), \text{length}(\boldsymbol{g})\}$, such that

$$h_n = f_n + g_n \text{ and}$$

$$h_i = (h_{i+1}, f_i + g_i) \text{ for } 0 \leqslant i < n. \tag{5.2}$$

Note that equation (5.2) in fact holds for all $0 \leqslant i \leqslant n$, because recalling that $h_i = 0$ holds for all $i > \max\{\text{length}(\boldsymbol{f}), \text{length}(\boldsymbol{g})\}$, and relying on axiom 12 the component h_n can be written as a pair and a simple calculation gives $h_i = f_n + g_n = (0, f_n + g_n) = (h_{n+1}, f_n + g_n)$. Another noticeable fact is the commutativity of the box-operation that follows from axiom 3, which states commutativity of addition on the expressions.

Definition 5.5 The *delta-operation* on a formula A of an expression h in $\bigcup C_i$, denoted $\delta^A h$, is a vector in $\mathbf{C}$ of length $l(A) + 1$ that does not contain any component of the vector $\boldsymbol{x}^A$. The vector is defined when the C_i, to which h belongs, is specified.

1. If h is in C_i and contains no component of $\boldsymbol{x}^A$, then $\delta^A h$ is the vector of length $l(A) + 1$, defined by $(\delta^A h)_i = h + 1$ and $(\delta^A h)_j = 1$, when $j \neq i$ and $0 \leqslant j \leqslant l(A) + 1$.
2. If h is x_i^A, then $(\delta^A h)_j = 1$ for $0 \leqslant j \leqslant l(A) + 1$.
3. If h contains a component of $\boldsymbol{x}^A$ and $h = f + g$, where f and g are in C_i, then $\delta^A h = \delta^A f + \delta^A g$.
4. If h contains a component of $\boldsymbol{x}^A$ and $h = (f, g)$, where f is in C_{i+1} and g is in C_i, then

$$(\delta^A h)_j = (\delta^A f)_j + (\delta^A g)_j \text{ if } 0 \leqslant j \leqslant l(A) \quad \text{and} \quad (\delta^A h)_j = 2(\delta^A f)_j + 2(\delta^A g)_j + 1 \text{ if } j = l(A)+1.$$

The delta-operation is also defined for vectors $\boldsymbol{h} = \langle h_0, \dots, h_n\rangle$ in $\mathbf{C}$.

$$(\delta^A \boldsymbol{h})_j = (\delta^A h_0)_j + \cdots + (\delta^A h_n)_j \text{ if } 0 \leqslant j \leqslant l(A) + 1$$

and if $n > l(A) + 1$, then we define

$$(\delta^A \boldsymbol{h})_j = h_j + 1 \text{ for } l(A) + 1 < j \leqslant n.$$

The vector $\delta^A \boldsymbol{h}$ has the length $\max\{l(A) + 1, n\}$.

By the definitions of the operations, both are well-defined operations on vectors in $\mathbf{C}$.

Lemma 5.6 *If $\boldsymbol{f}$ and $\boldsymbol{g}$ are in $\mathbf{C}$, then $\boldsymbol{f}\Box\boldsymbol{g}$ and $\delta^A\boldsymbol{f}$ are in $\mathbf{C}$.*

5.3 The vector assignment

After all preparations it is now possible to assign a vector $\boldsymbol{f}$ to each formula A in a derivation such that $\text{length}(\boldsymbol{f}) = l(A)$ and $\boldsymbol{f}$ is in $\mathbf{C}$. To increase readability the following notation is introduced: if $\boldsymbol{f} = \langle f_0, \dots, f_n\rangle$, and $\boldsymbol{g} = \langle f_0, \dots, f_m\rangle$ where $m \leqslant n$, then $\boldsymbol{g} = (\boldsymbol{f})\restriction_m$ is the restricted vector.

Definition 5.7 The *vector assigned to a formula* in a derivation is inductively defined as follows:

1. An assumption A is assigned the vector $\boldsymbol{x}^A = \langle x_0^A, \dots, x_n^A\rangle$, where $n = l(A)$.
2. The conclusion of an arithmetical rule without premises is assigned the vector $\langle 0\rangle$.
3. The conclusion of a one-premise arithmetical rule has the same vector as the premise.
4. If the premises of an instance of Tr or $\& I$ are assigned the vectors $\boldsymbol{f}$ and $\boldsymbol{g}$, then the conclusion of the rule is assigned the vector $\boldsymbol{f} + \boldsymbol{g}$.
5. If the premise of an instance of $\& E$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule is assigned a vector $\boldsymbol{g}$ such that $g_i = f_i + 1$ for $0 \leqslant i \leqslant n$, where n is the level of the formula in the conclusion.
6. If the premise of $\vee I$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule is assigned the vector $\boldsymbol{g}$ such that $g_i = f_i + 1$ for $0 \leqslant i \leqslant n$, where n is the level of the formula in the conclusion.
7. If the premises of $\vee E$ are assigned the vectors $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, then the conclusion of the rule is assigned the vector $\boldsymbol{e}$ such that $e_i = (\boldsymbol{f}\Box(\delta^A\boldsymbol{g} + \delta^B\boldsymbol{h}))_i$ for $0 \leqslant i \leqslant n$, where n is the level of the formula in the conclusion and A and B are the discarded assumptions of the rule.

8. If the premise of $\supset I$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule is assigned the vector $\delta^A \boldsymbol{f}$, where A is the discarded assumption of the rule.

9. If the premises of $\supset E$ are assigned the vectors $\boldsymbol{f}$ and $\boldsymbol{g}$, then the conclusion of the rule is assigned the vector $\boldsymbol{h}$ such that $h_i = (\boldsymbol{f} \Box \boldsymbol{g})_i$ for $0 \leqslant i \leqslant n$, where n is the level of the formula in the conclusion.

10. If the premise of $\forall I$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule has the same vector.

11. If the premise of $\forall E$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule is assigned the vector $\boldsymbol{g}$ such that $g_i = f_i + 1$ for $0 \leqslant i \leqslant \mathrm{length}(\boldsymbol{f})$.

12. If the premise of $\exists I$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule has the same vector.

13. If the premises of $\exists E$ are assigned the vectors $\boldsymbol{f}$ and $\boldsymbol{g}$, then the conclusion of the rule is assigned the vector $\boldsymbol{h}$ such that $h_i = (\boldsymbol{f} \Box \delta^{A(x)} \boldsymbol{g})_i$ for $0 \leqslant i \leqslant n$, where n is the level of the formula in the conclusion and $A(x)$ is the discarded assumption of the rule.

14. If the premise of $\bot E$ is assigned the vector $\boldsymbol{f}$, then the conclusion of the rule is assigned the vector $\boldsymbol{g}$ such that $g_i = f_i + 1$ for $0 \leqslant i \leqslant n$, where n is the level of the formula in the conclusion.

15. If the formula concluded by an instance of *Ind* is $A(t)$, then the vector assigned to this formula depends on the term t. Let $\boldsymbol{f} = \langle f_0, \dots, f_{n+1} \rangle$, where $n = l(A)$, be the vector assigned to the derivation of $A(\overline{m'}) \supset A(t')$ described in Lemma 4.1(iv) for some closed term t' for which $t' = \overline{m'}$ is derivable.

(a) If t is a closed term, then there is a derivation of $t = \overline{m}$ for some unique numeral $\overline{m}$ according to Lemma 4.1. If the vectors assigned to the premises of the *Ind*-rule are $\boldsymbol{h}$ and $\boldsymbol{g}$, then the vector of the conclusion of the induction is

$$(((\langle f_0, \dots, f_{n+1}, 2(m+1) \rangle \Box \delta^{A(x)} \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n,$$

where the length of the vector $\langle f_0, \dots, f_{n+1}, 2(m+1) \rangle$ is $n + 2 = l(A) + 2$.

(b) If on the other hand the term t contains a variable, then the vector of the conclusion of the induction is

$$(((\langle f_0, \dots, f_{n+1}, \omega \rangle \Box \delta^{A(x)} \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n,$$

where the length of $\langle f_0, \dots, f_{n+1}, \omega \rangle$ is $n + 2 = l(A) + 2$.

The vector assigned to the conclusion of a derivation is the vector assigned to the whole derivation.

Note that the delta-operation is always performed on a vector when an assumption is discharged in the subderivation to which the vector is assigned. The performed operation gives a vector not containing the variables assigned to the discarded assumption. Therefore, a formula derived without open assumptions (i.e. a theorem) must have a vector assigned to it which does not contain any variables.

In particular, the vector $\boldsymbol{f}$ used in the assignment of vectors to inductions does not contain any variables. Furthermore, since the vector $\langle 0 \rangle$ is assigned to a purely arithmetical derivation without open assumptions, the vector $\boldsymbol{f}$ does not depend on the term t', but only on the logical structure of A and the vector is well-defined.

6 Reduction procedure

The restricted reductions of [7] with a unique ordinal assignment correspond to a limitation in choice of the considered reducibility in the HA-derivation. The reducibility may not be a part of a subderivation that has open assumptions. Since all open assumptions must be discharged to derive a theorem, there would be an application of the delta-operation on the corresponding vector if there were open assumptions. The problem that arises with these general reductions is that order preservation is not necessarily provable for the delta-operation. If $f \prec g$, then $(\delta^A f)_i \prec (\delta^A g)_i$ does not follow in any obvious way when the expressions f and g differ in structure and fall under separate clauses in the definition of the delta-operation. However, even if general reductions cannot be treated, a suitable reducibility can be chosen in a derivation of falsity.

Theorem 6.1 *If there is a derivation of $\perp$ to which the vector $\boldsymbol{f}$ is assigned, then there is a derivation of $\perp$, to which the vector $\boldsymbol{g}$ is assigned and $f_0 \succ g_0$.*

Proof. Assume that there exists a derivation of $\perp$. Reduction steps are performed on the derivation in a specific order, and each step is performed as many times as possible before proceeding to the next step. First step 1 is applied as many times as possible, then step 2 if possible. If step 2 is not possible, then step 3 may apply and finally if no other reduction is possible, step 4 is performed.

Step 1. All free variables in the derivation, which are not eigenvariables, are replaced with the constant 0.

Step 2. If there is an instance of falsity elimination in the derivation below all instances of introduction rules and inductions and below which there are only arithmetical rules and major premises of elimination rules, then it is possible to eliminate the rule and the rest of the derivation below the rule. The new derivation is also a derivation of falsity, with no open assumptions.

Step 3. Assume that there is at least one induction below which there are no introduction rules, only major premises of elimination rules and arithmetical rules. Consider the lowermost (or rather one of the lowermost) of these inductions.

$$\dfrac{\begin{array}{cc} & [A(x)] \\ \vdots & \vdots \\ A(0) & A(s(x)) \end{array}}{A(t)}\,\mathit{Ind}$$

Because there are no introduction rules, and in particular no universal introduction below the induction and only major premises of elimination rules, in particular no minor premise of existential elimination, the formula $A(t)$ cannot contain eigenvariables. Furthermore, all free variables were replaced in step 1. Therefore, the term t must be closed and there exists a derivation of $t = \overline{m}$ for some numeral $\overline{m}$ according to Lemma 4.1(i). The reduction now performed depends on the numeral $\overline{m}$.

Case 1. If $\overline{m} \equiv 0$, then according to Lemma 4.1(iv) there is a derivation of $A(0) \supset A(t)$ without inductions. The reduced derivation is composed by implication elimination with the first premise of the induction as minor premise.

$$\dfrac{\begin{array}{cc} \vdots & \vdots \\ A(0) \supset A(t) & A(0) \end{array}}{A(t)}\,\supset E$$

Case 2. If $\overline{m} \equiv s(\overline{m'})$ for some numeral $\overline{m'}$, then according to Lemma 4.1(iv) there is a derivation of $A(s\overline{m'}) \supset A(t)$ without inductions and the following reduction on the derivation is performed:

$$\dfrac{\begin{array}{c}\vdots\\ A(s(\overline{m'}))\supset A(t)\end{array} \quad \dfrac{\dfrac{\begin{array}{c}[A(\overline{m'})]\\ \vdots \\ A(s(\overline{m'}))\end{array}}{A(\overline{m'})\supset A(s(\overline{m'}))}\,\supset I \quad \dfrac{\begin{array}{cc} & [A(x)]\\ \vdots & \vdots \\ A(0) & A(s(x))\end{array}}{A(\overline{m'})}\,\mathit{Ind}}{A(s(\overline{m'}))}\,\supset E}{A(t)}\,\supset E$$

The derivation of $A(s(\overline{m'}))$ from $A(\overline{m'})$ is the second premise of the original induction with $\overline{m'}$ substituted for x.

Step 4. According to Lemma 4.2 there is no purely arithmetical derivation of falsity. Therefore, the derivation must contain an instance of induction, falsity elimination, or another logical rule. Consider a lowermost instance of such a rule. If it is falsity elimination or induction, then either step 2 or step 3 applies. Because the conclusion of the derivation is not compound, there must be an elimination rule below each introduction. Hence, it may be assumed that the rule is an elimination rule. If the major premise of the elimination rule is the conclusion of another elimination rule, then it is possible to trace up through the major premises of elimination rules, until a formula concluded by some other rule is reached. The major premise under consideration is a compound formula and therefore not concluded by an arithmetical rule, neither can the formula be a discharged assumption because no rule discharges assumptions above major premises of elimination rules. Three possibilities remain each suitable for reduction: the formula is concluded by falsity elimination, induction or an introduction rule. In the first two cases step 2, or step 3 applies. In the third case an operational reduction is performed depending on the outermost logical connective of the formula.

Case 1. If the formula is an implication, then the derivation has the form:

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\supset I \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B}\supset E$$

and this is reduced into the following derivation:

$$\begin{matrix}\vdots\\ A\\ \vdots\\ B\end{matrix}$$

Case 2 to case 5 on conjunctions, disjunctions, universally and existentially quantified formulas respectively are similar standard detour conversions.

Thus, in all cases the derivation has been reduced. The vector calculations associated with the reductions are performed in Lemma 7.10. □

7 Vector calculations

Lemmas 7.1 – 7.7, stating properties of the box- and delta-operations, are all proven or simply stated in [7]. The Lemmas 7.1 – 7.4 are proven by downward induction on i from the axioms of the theory $\mathcal{E}$. The expression denoted by $f[\boldsymbol{g}/\boldsymbol{x}^A]$ is obtained from the expression f by substitution of each occurrence of x_i^A with $(\boldsymbol{g})_i$. The notation also applies to vectors $\boldsymbol{f}[\boldsymbol{g}/\boldsymbol{x}^A]$, where variable occurrences are replaced in each component.

Lemma 7.1 $(\boldsymbol{f}\Box\boldsymbol{g})_i \succcurlyeq f_i$ *for all* i.

Lemma 7.2 *Assume that* $\mathrm{length}(\boldsymbol{f}) = \mathrm{length}(\boldsymbol{g}) = n$ *and* $f_i \succcurlyeq g_i$ *for* $0 \leqslant i \leqslant n$*, then* $(\boldsymbol{f}\Box\boldsymbol{h})_i \succcurlyeq (\boldsymbol{g}\Box\boldsymbol{h})_i$ *for all* i.

Lemma 7.3 *Under the assumption of Lemma* 7.2 *and the additional assumption* $f_i \succ g_i$ *for* $0 \leqslant i \leqslant k$ *and some* $k \leqslant n$*, the inequality* $(\boldsymbol{f}\Box\boldsymbol{h})_i \succ (\boldsymbol{g}\Box\boldsymbol{h})_i$ *holds for* $0 \leqslant i \leqslant k$.

Lemma 7.3 can be generalized for vectors that differ in length. Because if $\mathrm{length}(\boldsymbol{f}) = n$, $\mathrm{length}(\boldsymbol{g}) = m$, $n > m$ and $f_i \succ g_i$ for all $i \leqslant m$, then $(\boldsymbol{f}\Box\boldsymbol{h})_i \succ (\langle g_0, \dots, g_m, 0, \dots, 0\rangle\Box\boldsymbol{h})_i = (\boldsymbol{g}\Box\boldsymbol{h})_i$ for all $i \leqslant m$.

Lemma 7.4 *Assume that* $\mathrm{length}(\boldsymbol{f}) = \mathrm{length}(\boldsymbol{g}) = n + 1 > \mathrm{length}(\boldsymbol{h})$ *and* $f_i \succ 0$ *and* $g_i \succ 0$ *for* $0 \leqslant i \leqslant n+1$*. Let* $\boldsymbol{c}$ *be a vector such that* $2f_{n+1} + 2g_{n+1} \prec c_{n+1}$ *and* $f_i + g_i \preccurlyeq c_i$ *for all* $i \leqslant n$*. Then*

$$2((\boldsymbol{f}\Box\boldsymbol{h})\Box(\boldsymbol{g}\Box\boldsymbol{h}))_i \prec (\boldsymbol{c}\Box\boldsymbol{h})_i \textit{ for all } i \leqslant n+1.$$

Lemma 7.5 *Let* $\mathbf{e}$ *be a vector of length* $l(A)$ *and assume* h *is in* C_i*. Then* $((\delta^A h)\Box\boldsymbol{e})_i \succ h[\boldsymbol{e}/\boldsymbol{x}^A]$.

Proof. The lemma is proven by induction on the number of times clauses 3 and 4 in the definition of the delta-operation are applied in $\delta^A h$. A complete proof is found in [7, Lemma 2.11]. □

Corollary 7.6 *If* $\boldsymbol{h}$ *is in* $\mathbf{C}$ *and* $\mathbf{e}$ *has length* $l(A)$*, then* $((\delta^A\boldsymbol{h})\Box\boldsymbol{e})_i \succ (\boldsymbol{h}[\boldsymbol{e}/\boldsymbol{x}^{\boldsymbol{A}}])_i$ *for all* $i \leqslant \mathrm{length}(\boldsymbol{h})$.

The next lemma is proven by induction on the length of the derivation.

Lemma 7.7 *Assume that there is a derivation of* A *to which the vector* $\boldsymbol{f}$ *is assigned and a derivation of* B *to which the vector* $\mathbf{g}$ *is assigned and that* A *is an open assumption in the latter derivation. Assume furthermore that no open assumption in the derivation of* A *becomes discarded in the derivation of* B*, where all assumptions* A *have been replaced with the derivation of* A*. Then the vector assigned to this derivation is* $\boldsymbol{g}[\boldsymbol{f}/\boldsymbol{x}^A]$.

Now the aim becomes to prove that the reduction performed in step 1, substituting a constant for each free variable, does not increase the components of the vector.

Lemma 7.8 *If one derivation, to which the vector $\boldsymbol{g}$ is assigned, is received by substituting a term for a free variable in another derivation, to which the vector $\boldsymbol{h}$ is assigned, then $h_i \succcurlyeq g_i$ for $0 \leqslant i \leqslant \operatorname{length}(\boldsymbol{h})$.*

Proof. The vector assignment is otherwise the same, but for the fact that some inductions concluding a term with a variable may now have become inductions with a closed term, which fall under the first clause of the vector assignments to inductions. Assume that this is the case for some induction. Now it holds for all i $(\langle f_0, \ldots, f_{n+1}, \omega\rangle)_i \succcurlyeq (\langle f_0, \ldots, f_{n+1}, 2(m+1)\rangle)_i$. By induction on the number of times the box- and delta-operations are applied, it is possible to show that the inequalities are preserved for the components of the vectors. The induction hypothesis is $h_i \succcurlyeq h'_i$, $g_i \succcurlyeq g'_i$ for all i, and the inductive step gives $(\boldsymbol{h} \Box \boldsymbol{g})_i \succcurlyeq (\boldsymbol{h} \Box \boldsymbol{g'})_i \succcurlyeq (\boldsymbol{h'} \Box \boldsymbol{g'})_i$ by Lemma 7.2.

What remains to be shown is that a similar inequality holds for the delta-operation, $(\delta^A \boldsymbol{h})_i \succcurlyeq (\delta^{A'} \boldsymbol{h'})_i$, where A' comes from A by substitution of the term for the free variable.

1. If h_j contains no component of $\boldsymbol{x}^A$, then h'_j contains no component of $\boldsymbol{x}^{A'}$ and from the induction hypothesis follows the inequality $(\delta^A h_j)_j = h_j + 1 \succcurlyeq h'_j + 1 = (\delta^{A'} h'_j)_j$. The other components of the vectors $\delta^A h_j$ and $\delta^{A'} h'_j$ are 1.

2. If h_j is x^A_j, then h'_j is $x^{A'}_j$ and the components of the vectors $\delta^A h_j$ and $\delta^{A'} h'_j$ are 1.

3. If h_j contains a component of $\boldsymbol{x}^A$ and $h_j = u + v$, then h'_j contains a component of $\boldsymbol{x}^{A'}$ and $h'_j = u' + v'$. Then $\delta^A h_j = \delta^A u + \delta^A v \succcurlyeq \delta^{A'} u' + \delta^A v \succcurlyeq \delta^{A'} u' + \delta^{A'} v' = \delta^{A'} h'_j$.

4. If h_j contains a component of $\boldsymbol{x}^A$ and $h_j = (u, v)$, then h'_j contains a component of $\boldsymbol{x}^{A'}$ and $h'_j = (u', v')$. The calculations of the inequalities are similar to those in case 3.

Thus in all cases $(\delta^A h_j)_i \succcurlyeq (\delta^{A'} h'_j)_i$ and therefore $(\delta^A \boldsymbol{h})_i \succcurlyeq (\delta^{A'} \boldsymbol{h'})_i$. □

The following lemma calculates the vectors of the reduction performed in step 3 (case 2) dealing with the inductions.

Lemma 7.9 *Let*

$$\boldsymbol{e} = ((\langle f_0, \ldots, f_{n+1}, 2m+4\rangle \Box \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n$$

and

$$\boldsymbol{e'} = (((((\langle f_0, \ldots, f_{n+1}, 2m+2\rangle \Box \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n, \Box \boldsymbol{g}) \restriction_n \Box \boldsymbol{f}) \restriction_n,$$

where $\boldsymbol{f} = \delta^A \boldsymbol{f'}$, $\boldsymbol{g} = \delta^A \boldsymbol{g'}$ and furthermore $\operatorname{length}(\boldsymbol{f}) = \operatorname{length}(\boldsymbol{g}) = n+1$ and $\operatorname{length}(\boldsymbol{h}) = n$. Then $e_i \succ e'_i$ for $0 \leqslant i \leqslant n$.

Proof. First, the vectors $\boldsymbol{r}$, $\boldsymbol{t}$ and $\boldsymbol{b}$ are defined by

$$\begin{aligned}
\boldsymbol{r} &= ((((\langle f_0, \ldots, f_{n+1}, 2m+2\rangle \Box \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n \Box \boldsymbol{g}) \restriction_n, \\
\boldsymbol{t} &= (((\langle f_0, \ldots, f_{n+1}, 2m+2\rangle \Box \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n, \\
\boldsymbol{b} &= ((\langle f_0, \ldots, f_{n+1}, 2m+2\rangle \Box \boldsymbol{g}) \restriction_{n+1} .
\end{aligned}$$

Let $\boldsymbol{a}$ be the vector of length $n+1$ defined as follows: $a_{n+1} = g_{n+1} + f_{n+1} + 1$ and $a_i = (a_{i+1}, g_i + f_i)$ for $0 \leqslant i \leqslant n$. Relying on these definitions, it is possible to prove that $e'_i \preccurlyeq (\boldsymbol{a} \Box \boldsymbol{t})_i$, by downward induction on $i \leqslant n$. In fact, the stronger claim $2e'_i \prec (\boldsymbol{a} \Box \boldsymbol{t})_i$ is proven.

For $i = n$ the equation $2e'_n = 2(f_{n+1}, r_n + f_n) = 2(f_{n+1}, (\boldsymbol{t} \Box \boldsymbol{g})_n + f_n) = 2(f_{n+1}, (g_{n+1}, t_n + g_n) + f_n)$ holds. Furthermore, because $\boldsymbol{f} = \delta^A \boldsymbol{f'}$ for some vector $\boldsymbol{f'}$, the strict inequality $f_j \succcurlyeq 1 \succ 0$ holds for all $j \leqslant n+1$ by the definition of the delta-operation. By a similar argument: $g_j \succcurlyeq 1 \succ 0$ for all $j \leqslant n+1$.

Thus, by inequality (5.1), $2(f_{n+1}, (g_{n+1}, t_n + g_n) + f_n) \prec 2(f_{n+1}, (g_{n+1}, t_n + g_n + f_n))$. By axiom 13 this equals $2(f_{n+1} + g_{n+1}, t_n + g_n + f_n)$. Because also the inequality $g_n + f_n = (0, g_n + f_n) \preccurlyeq (a_{n+1}, g_n + f_n) = a_n$

holds, $2(f_{n+1}+g_{n+1},t_n+g_n+f_n) \preccurlyeq 2(f_{n+1}+g_{n+1},t_n+a_n)$. Using axiom 9 and the fact $f_{n+1}+g_{n+1} \prec a_{n+1}$ the proof of the base case is completed by the calculation $2(f_{n+1}+g_{n+1},t_n+a_n) \preccurlyeq (a_{n+1},t_n+a_n) = (\boldsymbol{a}\Box\boldsymbol{t})_n$.

For the inductive step $i < n$ the vector $\boldsymbol{e'}$ can be analyzed as follows:

$$2e'_i = 2(e'_{i+1}, r_i + f_i) = 2(e'_{i+1}, (r_{i+1}, t_i + g_i) + f_i).$$

Since $i < n$, $r_{i+1} = (\boldsymbol{t}\Box\boldsymbol{g})_{i+1} \succcurlyeq g_{i+1} \succcurlyeq 1$ by Lemma 7.1. Therefore, it is possible to use inequality (5.1) followed by axiom 13 to get

$$2(e'_{i+1}, (r_{i+1}, t_i + g_i) + f_i) \prec 2(e'_{i+1}, (r_{i+1}, t_i + g_i + f_i)) = 2(e'_{i+1} + r_{i+1}, t_i + g_i + f_i).$$

By Lemma 7.1 $r_{i+1} \preccurlyeq (\boldsymbol{r}\Box\boldsymbol{f})_{i+1} = e'_{i+1}$, which gives $r_{i+1} + e'_{i+1} \preccurlyeq 2e'_{i+1}$. By axiom 9 and the induction hypothesis the inductive step is completed

$$2(e'_{i+1} + r_{i+1}, t_i + g_i + f_i) \preccurlyeq ((\boldsymbol{a}\Box\boldsymbol{t})_{i+1}, t_i + g_i + f_i) \preccurlyeq ((\boldsymbol{a}\Box\boldsymbol{t})_{i+1}, t_i + a_i) = (\boldsymbol{a}\Box\boldsymbol{t})_i.$$

Thus, $e'_i \preccurlyeq (\boldsymbol{a}\Box\boldsymbol{t})_i \preccurlyeq (\boldsymbol{a}\Box(\boldsymbol{b}\Box\boldsymbol{h}))_i$ for $i \leqslant n+1$.

By Lemma 7.1 $a_i \preccurlyeq (\boldsymbol{a}\Box\boldsymbol{h})_i$ for $i \leqslant n+1$ and hence $e'_i \preccurlyeq ((\boldsymbol{a}\Box\boldsymbol{h})\Box(\boldsymbol{b}\Box\boldsymbol{h}))_i$. It is therefore sufficient to prove that $((\boldsymbol{a}\Box\boldsymbol{h})\Box(\boldsymbol{b}\Box\boldsymbol{h}))_i \prec (\boldsymbol{c}\Box\boldsymbol{h})_i$ for all $i \leqslant n+1$ where

$$\boldsymbol{c} = (\langle f_0, \ldots, f_{n+1}, 2m+4\rangle \Box \boldsymbol{g}) \restriction_{n+1} .$$

Clearly, $a_i \succ 0$ for $i \leqslant n+1$ and as stated above $g_i \succcurlyeq 1 \succ 0$ for $i \leqslant n+1$. Therefore, by Lemma 7.1, $b_i = (\langle f_0, \ldots, f_{n+1}, 2m+2\rangle \Box \boldsymbol{g})_i \succcurlyeq g_i \succcurlyeq 1$ for $i \leqslant n+1$ and also $b_i \succ 0$ holds. By Lemma 7.4 it is sufficient to prove

$$2a_{n+1} + 2b_{n+1} \prec c_{n+1}, \tag{7.1}$$

$$a_i + b_i \preccurlyeq c_i, \text{ for } i < n+1. \tag{7.2}$$

The first goal is to prove inequality (7.1). From $f_{n+1} \succcurlyeq 1$ and $g_{n+1} \succcurlyeq 1$ follow that

$$2a_{n+1} = 2(f_{n+1} + g_{n+1} + 1) \preccurlyeq 3(f_{n+1} + g_{n+1}).$$

By axioms 12, 9 and 11,

$$3(f_{n+1} + g_{n+1}) = 3(0, f_{n+1} + g_{n+1}) \preccurlyeq (2, f_{n+1} + g_{n+1}) \prec (2m+3, f_{n+1} + g_{n+1}).$$

On the other hand by axiom 9

$$2b_{n+1} = 2(2m+2, f_{n+1} + g_{n+1}) \preccurlyeq (2m+3, f_{n+1} + g_{n+1}).$$

Thus,

$$2a_{n+1} + 2b_{n+1} \prec 2(2m+3, f_{n+1} + g_{n+1})$$

and by axiom 9

$$2a_{n+1} + 2b_{n+1} \prec (2m+4, f_{n+1} + g_{n+1}) = c_{n+1}.$$

The remaining goal is to prove inequality (7.2). Because it was proved above, that $a_{i+1} \succ 0$ and $b_{i+1} \succ 0$, it follows that $a_{i+1} + b_{i+1} \succ a_{i+1}$ and $a_{i+1} + b_{i+1} \succ b_{i+1}$. Therefore, by axiom 9,

$$a_i + b_i = (a_{i+1}, f_i + g_i) + (b_{i+1}, f_i + g_i) \preccurlyeq (a_{i+1} + b_{i+1}, f_i + g_i).$$

Hence, by induction hypothesis $a_i + b_i \preccurlyeq (c_{i+1}, f_i + g_i) = c_i$, which proves the claim. $\square$

The proofs above are sufficient preparation for calculating the vectors of the derivations in the reduction procedure.

Lemma 7.10 *Let P be a derivation of $\perp$ to which the vector $\boldsymbol{f}$ is assigned. If P' is received from P by performing the reduction described in Theorem* 6.1 *and $\boldsymbol{g}$ is the vector assigned to P', then $f_0 \succ g_0$.*

Proof.

Step 1. According to Lemma 7.8 the expressions of the vector are not increased, by the procedure of substituting a constant for free variables in the derivation.

Step 2. The level of falsity is 0, so the vector assigned to the premise of the falsity elimination has one component, $\boldsymbol{f} = \langle f_0 \rangle$. By induction on the number of rules below the falsity elimination it can be shown that the first component of the vector assigned to the derivation P is greater than f_0. For the base case of the induction it can be concluded that the rule of falsity elimination increases the vector, because $f_0 + 1 \succ f_0$. Now assume as the induction hypothesis that $g_0 \succ f_0$ for some vector $\boldsymbol{g}$ assigned to a formula below the falsity elimination. Below the falsity elimination rule there are no rules that discharge assumptions that falsity depends on, so there are no delta-operations on the vector $\boldsymbol{g}$, but only box-operations and additions of 1. For the case of the box-operation a simple calculation gives $(\boldsymbol{g} \Box \boldsymbol{h})_0 \succcurlyeq g_0 \succ f_0$ by Lemma 7.1 and the induction hypothesis. If the elimination rule only adds 1 to the components of the vector, then the statement is clear. This proves the claim.

Step 3. In this step, where an induction is reduced, there are two cases.

Case 1. The first case considered is when the term in the conclusion of the induction is equal to 0. Let $\boldsymbol{h}$ and $\boldsymbol{g'}$ be the vectors assigned to the premises of the induction in P and let $\boldsymbol{f}$ be the vector assigned to the derivation of $A(\overline{m}) \supset A(t)$. Furthermore, let $\boldsymbol{g} = \delta^{A(x)} \boldsymbol{g'}$ and denote $\boldsymbol{e} = (\langle f_0, \dots, f_{n+1}, 2 \rangle \Box \boldsymbol{g}) \restriction_{n+1}$. Then the vector assigned to the conclusion of the induction rule in P is $(\boldsymbol{e} \Box \boldsymbol{h}) \restriction_n$. The vector assigned to the reduced derivation is $(\boldsymbol{f} \Box \boldsymbol{h}) \restriction_n$.

By downward induction on i, it can be proved that $e_i \succ f_i$. For $i = n+1$ the component of the vector $\boldsymbol{e}$ is $e_{n+1} = (2, g_{n+1} + f_{n+1})$. Since $\boldsymbol{f} = \delta^A \boldsymbol{f'}$ for some vector $\boldsymbol{f'}$, the components of the vector are positive, $f_i \succcurlyeq 1 \succ 0$ for $0 \leqslant i \leqslant n+1$. Thus with axiom 11 of the theory $\mathcal{E}$, the following calculation holds:

$$e_{n+1} \succ (0, g_{n+1} + f_{n+1}) = g_{n+1} + f_{n+1} \succcurlyeq f_{n+1}.$$

For the inductive step the component of $\boldsymbol{e}$ is $e_i = (e_{i+1}, g_i + f_i)$. By the induction hypothesis $e_{i+1} \succ f_{i+1} \succ 0$. Again by axiom 11 and the fact that $f_i \succ 0$ it can be concluded that $e_i \succ (0, g_i + f_i) = g_i + f_i \succcurlyeq f_i$. This concludes the inductive proof.

By Lemma 7.3 and the claim proved above $(\boldsymbol{e} \Box \boldsymbol{h})_i \succ (\boldsymbol{f} \Box \boldsymbol{h})_i$ for $0 \leqslant i \leqslant n+1$ and the vectors of the reduced part of the derivation have been calculated.

What remains to be shown is that the rules below the reduced part of the derivation preserve the inequality. This claim is proved by induction on the number of rules. As in step 2, it can be concluded that there are no rules below discharging assumptions that the conclusion of the induction rule depends on, so there are only box-operations and additions of 1. The base case, that the inequality holds if there are no rules below, is already proved above. Now assume as the induction hypothesis, that $a_0 \succ b_0$ and $a_i \succcurlyeq b_i$ for $i \leqslant \mathrm{length}(\boldsymbol{a}) = \mathrm{length}(\boldsymbol{b})$. Then Lemma 7.3 can be used to get $(\boldsymbol{a} \Box \boldsymbol{c})_0 \succ (\boldsymbol{b} \Box \boldsymbol{c})_0$ for some vector $\boldsymbol{c}$ and $(\boldsymbol{a} \Box \boldsymbol{c})_i \succcurlyeq (\boldsymbol{b} \Box \boldsymbol{c})_i$ for $i > 0$. On the other hand, also $a_0 + 1 \succ b_0 + 1$ follows from the induction hypothesis as well as $a_i + 1 \succcurlyeq b_i + 1$ for $i > 0$. This proves the claim.

Case 2. The second case of step 3, considers an induction, for which the term in the conclusion equals a successor. Let $\boldsymbol{h}$, $\boldsymbol{g}$ and $\boldsymbol{f}$ be as in case 1. Then the vector assigned to the conclusion of the induction rule in P is

$$((\langle f_0, \dots, f_{n+1}, 2m+4 \rangle \Box \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n,$$

and this is in P' reduced to

$$((((\langle f_0, \dots, f_{n+1}, 2m+2 \rangle \Box \boldsymbol{g}) \restriction_{n+1} \Box \boldsymbol{h}) \restriction_n \Box \boldsymbol{g}) \restriction_n \Box \boldsymbol{f}) \restriction_n .$$

The claim that the components of the vector in P are greater than the components of the vector in P' for $i \leqslant n$ is proven by Lemma 7.9. As in case 1 the rules below preserve the inequality.

Step 4. This step is divided into cases according to the logical rules in the detour conversion.

Case 1. Let the vector assigned to the premise of the implication introduction rule in P be $\boldsymbol{f}$ and the vector assigned to the minor premise of the implication elimination rule be $\boldsymbol{g}$. Then the vector assigned to the conclusion of the elimination is $(\delta^A \boldsymbol{f} \Box \boldsymbol{g}) \restriction_{l(B)}$. This vector is reduced to the vector $\boldsymbol{f}[\boldsymbol{g}/\boldsymbol{x}^A]$ according to Lemma 7.7. Corollary 7.6 gives the desired result for these two vectors. As in step 3 (case 1) the rules below preserve the inequality.

Case 2. Let the vectors assigned to the premises of the conjunction introduction rule in P be $\boldsymbol{f}$ and $\boldsymbol{g}$. Assume that A is the conclusion of the elimination rule. Then the vector assigned to the conclusion of the elimination is $(\langle f_0 + g_0 + 1, \dots, f_n + g_n + 1\rangle) \restriction_{l(A)}$, where $n = \max\{\text{length}(\boldsymbol{f}), \text{length}(\boldsymbol{g})\}$. This vector is reduced to the vector $\boldsymbol{f}$.

By the axioms of $\mathcal{E}$ it is easily concluded that $f_i + g_i + 1 \succ f_i$ for all $i \leqslant n$ and as in step 3 (case 1) the rules below preserve the inequality.

Case 3. Let the vector assigned to the premise of the disjunction introduction rule in P be $\boldsymbol{f}$ and let $\boldsymbol{g}$ and $\boldsymbol{h}$ be the vectors assigned to the minor premises of the elimination rule. Assume that the premise of the introduction rule is A, since the other case is dual. Then the vector assigned to the conclusion of the elimination is $(\langle f_0 + 1, \dots, f_n + 1\rangle \Box (\delta^A \boldsymbol{g} + \delta^B \boldsymbol{h})) \restriction_{l(C)}$, where $n = l(A \vee B) \geqslant \text{length}(\boldsymbol{f})$, an this is reduced to $\boldsymbol{g}[\boldsymbol{f}/\boldsymbol{x}^A]$ according to Lemma 7.7.

Now, $(\delta^A \boldsymbol{g} + \delta^B \boldsymbol{h})_i \succcurlyeq (\delta^A \boldsymbol{g})_i$ for all i by the axioms of $\mathcal{E}$ and thus

$$(\langle f_0 + 1, \dots, f_n + 1\rangle \Box (\delta^A \boldsymbol{g} + \delta^B \boldsymbol{h}))_i \succcurlyeq (\langle f_0 + 1, \dots, f_n + 1\rangle \Box \delta^A \boldsymbol{g})_i$$

for all i by Lemma 7.2. Furthermore,

$$(\langle f_0 + 1, \dots, f_n + 1\rangle \Box \delta^A \boldsymbol{g})_i \succcurlyeq (\langle f_0, \dots, f_n\rangle \Box \delta^A \boldsymbol{g})_i = (\boldsymbol{f} \Box \delta^A \boldsymbol{g})_i$$

for all i. According to Corollary 7.6 the desired result $(\boldsymbol{f} \Box \delta^A \boldsymbol{g})_i \succ (\boldsymbol{g}[\boldsymbol{f}/\boldsymbol{x}^{\boldsymbol{A}}])_i$ holds for all $i \leqslant l(C)$. As in step 3 (case 1) the rules below preserve the inequality.

Case 4. Let the vector assigned to the premise of the universal introduction rule in P be $\boldsymbol{f}$, then the vector assigned to the conclusion of the elimination rule is $\langle f_0 + 1, \dots, f_n + 1\rangle$. In the derivation of the premise of the introduction rule, $A(y/x)$, the term t can be substituted for x. If t contains a variable, then the vector of the derivation received, $\boldsymbol{f'}$, remains unchanged and equal to $\boldsymbol{f}$ and if t is closed, then by Lemma 7.8 the components of the vector are not increased. Thus, $f_i + 1 \succ f_i \succcurlyeq f'_i$ for all $i \leqslant n$. As in step 3 (case 1) the rules below preserve the inequality.

Case 5. Let the vector assigned to the premise of the existential introduction rule in P be $\boldsymbol{f}$ and let $\boldsymbol{g}$ be the vector assigned to the minor premise, C, of the elimination rule. Then the vector assigned to the conclusion of the elimination is $(\boldsymbol{f} \Box \delta^A \boldsymbol{g}) \restriction_{l(C)}$. Let $\boldsymbol{g'}$ be the vector received by substituting the term t in the premise of the introduction rule for x in the derivation of the minor premise of the elimination rule. Then, $(\delta^A \boldsymbol{g})_i \succcurlyeq (\delta^A \boldsymbol{g'})_i$ and the vector assigned to the reduced derivation is $\boldsymbol{g'}[\boldsymbol{f}/\boldsymbol{x}^A]$. By Lemma 7.2 follows that $(\boldsymbol{f} \Box \delta^A \boldsymbol{g})_i \succcurlyeq (\boldsymbol{f} \Box \delta^A \boldsymbol{g'})_i$ for all i. >From Corollary 7.6 the desired result $(\boldsymbol{f} \Box \delta^A \boldsymbol{g'})_i \succ (\boldsymbol{g'}[\boldsymbol{f}/\, \boldsymbol{x}^A])_i$ follows for $i \leqslant l(C)$. As in step 3 (case 1) the rules below preserve the inequality. □

8 The theorem of consistency

8.1 Interpretation of $\mathcal{E}$

To finalize the argument of the consistency proof, expressions of the theory $\mathcal{E}$ are to be interpreted as ordinals. In general, expressions can be interpreted as functions of the variables contained in them. From the first component of the vector assigned to a derivation a function is obtained, which applied to a suitable constant, say 0, will give an ordinal.

The relation $a \succ b$ is interpreted as $a > b$ and $a + b$ as the natural sum $a \# b$. The natural sum of two ordinals a and b represented in Cantor normal form $a = \omega^{a_1} + \dots + \omega^{a_n}$ and $b = \omega^{b_1} + \dots + \omega^{b_m}$, where $a_1 \geqslant \dots \geqslant a_n$ and $b_1 \geqslant \dots \geqslant b_m$, is defined as $a \# b = \omega^{c_1} + \dots + \omega^{c_{n+m}}$, where $c_1 \geqslant \dots \geqslant c_{n+m}$ is a rearrangement of the sequence $a_1, \dots, a_n, b_1, \dots, b_m$. The definition of (a, b) is separated into two cases depending on whether $b = 0$. The pair $(a, 0)$ is interpreted as 0. On the other hand, assume that $b \succ 0$ and represent b in Cantor normal form to the base 2, $b = 2^{b_1} + \dots + 2^{b_n}$, where $b_1 > \dots > b_n$. Then (a, b) is $2^{c_1} + \dots + 2^{c_n}$, where $c_i = a \# b_i$ and $1 \leqslant i \leqslant n$. The described interpretation satisfies axioms 1 to 13 of the theory $\mathcal{E}$ given in Section 5.1.

8.2 Consistency of Heyting arithmetic

Definition 8.1 A system is said to be *inconsistent* if falsity is derivable. If the system is not inconsistent it is *consistent*.

Theorem 8.2 (The consistency of Heyting arithmetic) *Falsity is not derivable in the system HA, that is, it is consistent.*

Proof. Assume that the system HA is inconsistent and that there is a derivation of falsity. According to Theorem 6.1 there is a reduced derivation with a lower ordinal and another reduced derivation and so on. This produces an infinite succession of decreasing ordinals all less than ε_0, but this is impossible because the well-ordering of ε_0 implies that the reduction procedure must terminate. Thus, there cannot exist a derivation of falsity and the system of Heyting arithmetic is consistent. □

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