Ch.15 Introduction to Linear Programming

15.1 Brief History of Linear Programming

- Any point that satisfies the constraints is called as a feasible point.
- In a linear programming problem, the objective function is linear, and the set of feasible points is determined by a set of linear equations and/or inequalities.
- Linear programming methods provide a way of choosing the best feasible point among the many possible feasible points.
- Linear programming problems (related to economy, manufacturing industry) developed in the late 1930s.
- In 1947, Dantzig developed the simplex method.

15.2 Simple Examples of Linear Programs

- A linear program is an optimization problem of the form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where $x \in \Re^n, b \in \Re^m$, and $A \in \Re^{m \times n}$.

- **Ex. 15.1** A manufacturer produces four different products X_1, X_2, X_3, and X_4.
  ♣ Table 15.1 on page 258.

Inputs	Product X_1	X_2	X_3	X_4	Input Availabilities
Person-weeks	1	2	1	2	20
Kilograms of material A	6	5	3	2	100
Boxes of material B	3	4	9	12	75
Production levels	x_1	x_2	x_3	x_4	

These constraints can be written using the data in the above table.

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 &\leq 20 \\ 6x_1 + 5x_2 + 3x_3 + 2x_4 &\leq 100 \\ 3x_1 + 4x_2 + 9x_3 + 12x_4 &\leq 75 \end{aligned}$$

And also,

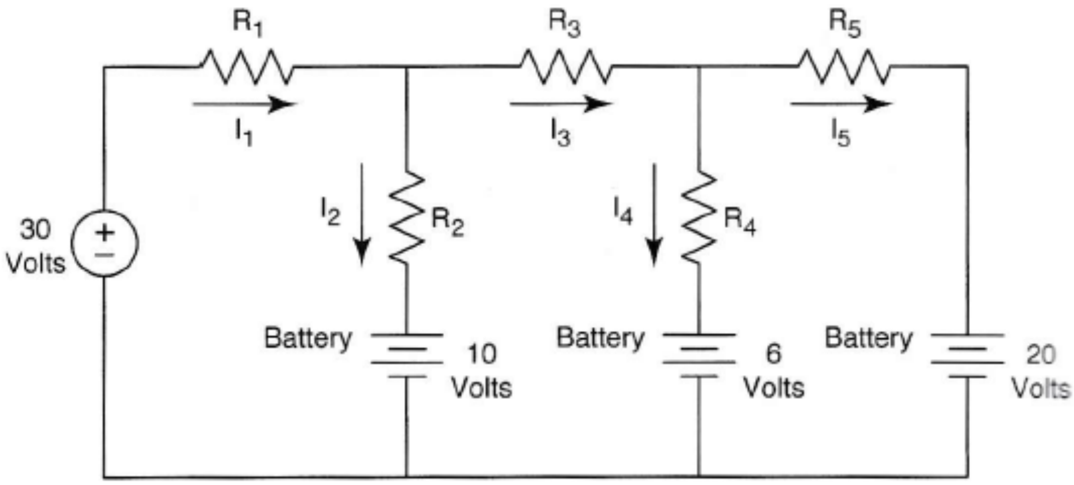
$$x_i \geq 0, \quad i = 1, 2, 3, 4.$$

Now, suppose that one unit of product X_1 sells for \$6, and X_2, X_3, and X_4 sell for \$4, \$7, and \$5, respectively. Then, the total revenue for any production decision (x_1, x_2, x_3, x_4) is

$$f(x_1, x_2, x_3, x_4) = 6x_1 + 4x_2 + 7x_3 + 5x_4$$

The problem is then to maximize f, subject to the given constraints.

- **Ex. 15.5** An electric circuit that is designed to use a 30V source to charge 10V, 6V, and 20V batteries connected in parallel.
  ♣ Fig. 15.1 on page 262.

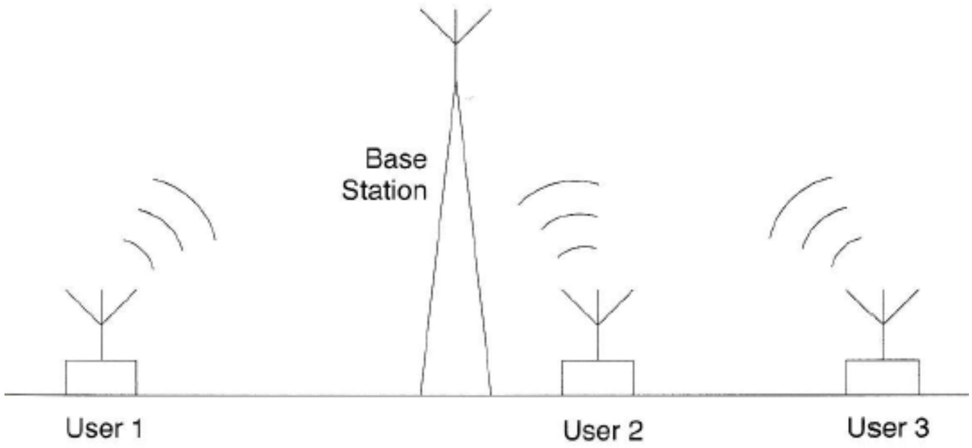


Physical constraints limit the currents I_1, I_2, I_3, I_4 and I_5 to a maximum of 4A, 3A, 3A, 2A, and 2A, respectively.
We wish to find the values of the currents $I_1, \cdots, I_5$ such that the total power transferred to the batteries is maximized.

$$\begin{array}{rl} \text{maximize} & 10I_2 + 6I_4 + 20I_5 \\ \text{subject to} & I_1 = I_2 + I_3 \\ & I_3 = I_4 + I_5 \\ & I_1 \leq 4, \quad I_2 \leq 3 \end{array}$$

$$I_3 \le 3, \quad I_4 \le 2$$
$$I_5 \le 2$$
$$I_1, I_2, I_3, I_4, I_5 \ge 0$$

- **Ex. 15.6** Consider a wireless communication system. ♣ Fig. 15.2 on page 262.



There are n mobile uses. For each $i = 1, \cdots, n$, user i transmits a signal to the base station with power p_i and an attenuation factor of h_i (i.e., the actual received signal power at the base station from user i is $h_i p_i$). When the base station is receiving from user i, the total received power from all other users is considered "interference" (i.e., the interference for user i is $\sum_{j \ne i} h_j p_j$). For the communicaiton with user i to be reliable, the signal-to-interference ratio must exceed a threshold γ_i, where the signal is the reveiced power for user i.

The problem can be written as

$$\begin{aligned} \text{minimize} \quad & p_1 + \cdots + p_n \\ \text{subject to} \quad & \frac{h_i p_i}{\sum_{j \neq i} h_j p_j} \geq \gamma_i, \; i = 1, \cdots, n \\ & p_1, \cdots, p_n \geq 0 \end{aligned}$$

We can write the above as the linear programming problem

$$\begin{aligned} \text{minimize} \quad & p_1 + \cdots + p_n \\ \text{subject to} \quad & h_i p_i - \gamma_i \sum_{j \neq i} h_j p_j \geq 0, \quad i = 1, \cdots, n \\ & p_1, \cdots, p_n \geq 0 \end{aligned}$$

Or in matrix form,

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} c &= [1, \cdots, 1]^T \\ A &= \begin{bmatrix} h_1 & -\gamma_1 h_2 & \cdots & -\gamma_1 h_n \\ -\gamma_2 h_1 & h_2 & \cdots & -\gamma_2 h_n \\ \cdots & \cdots & \cdots & \cdots \\ -\gamma_n h_1 & -\gamma_n h_2 & \cdots & h_n \end{bmatrix} \\ b &= 0 \end{aligned}$$

15.3 Two-Dimensional Linear Programs

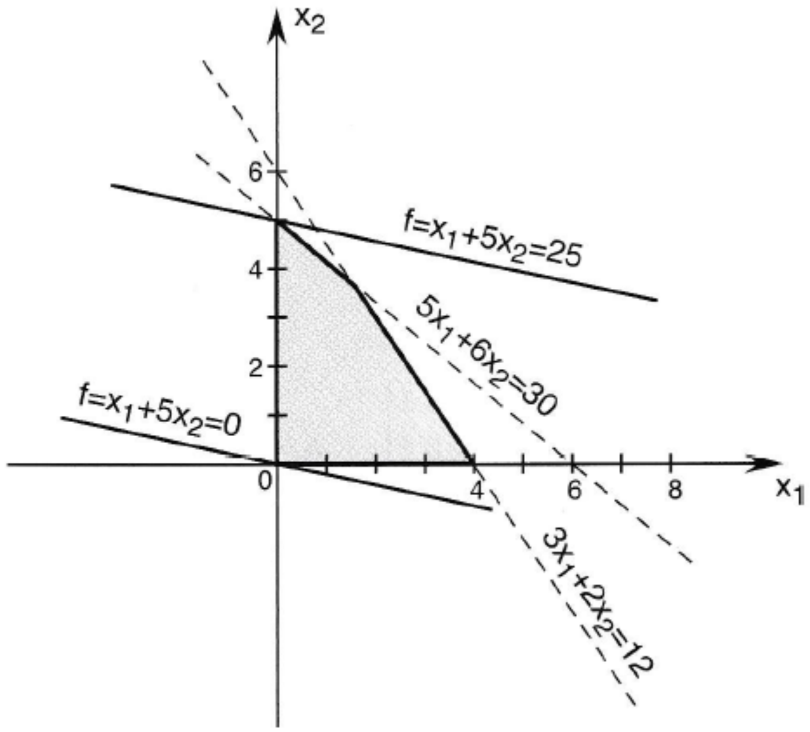
- Consider the following linear program

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

  where

$$\begin{aligned} c^T &= [1, 5] \\ x &= [x_1, x_2]^T \\ A &= \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} \\ b &= [30, 12]^T \end{aligned}$$

- We can solve easily the problem using geometric argumetns.
  ♣ Fig. 15.3 on page 265.



- For more complicated problems, there could be a limit.

15.5 Standard Form of Linear Programs

- We refer to a linear program of the form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

  as a linear program in standard form. Here A is an $m \times n$ matrix composed of real entries, $m < n$, rank $A = m$. Without loss of generality, we assume $b \geq 0$.

- Other forms of linear programs can be converted to the standard form. If a linear program is in the form,

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

We convert the original problem into the standard form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax - I_m y = [A, -I_m] \begin{bmatrix} x \\ y \end{bmatrix} = b \\ & x \geq 0, \;\; y \geq 0 \end{aligned}$$

- **Ex. 15.8** Suppose that we are given the inequality constraint

$$x_1 \leq 7$$

We convert this to an equality constraint by introducing a slack variable $x_2 \geq 0$ to obtain

$$\begin{aligned} x_1 + x_2 &= 7 \\ x_2 &\geq 0 \end{aligned}$$

- **Ex. 15.9** Consider the inequality constraints,

$$\begin{aligned} a_1 x_1 + a_2 x_2 \leq b \\ x_1, \; x_2 \geq 0 \end{aligned}$$

where a_1, a_2, and b are positive numbers. We introduce a slack variable $x_3 \geq 0$ to get

$$\begin{aligned} a_1x_1 + a_2x_2 + x_3 &= b \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

15.6 Basic Solutions

- Consider the system of equalities

$$Ax = b$$

- Let B be a square matrix whose columns are m linearly independent columns of A.
- A has the form $A = [B, D]$, where D is an $m \times (n - m)$ matrix whose columns are the remaining columns of A. The matrix B is nonsingular, and thus we can solve the equation

$$Bx_B = b$$

  The solution is $x_B = B^{-1}b$.
- **Definition 15.1**
  (1) We call $[x_B^T, 0^T]^T$ a basic solution to $Ax = b$ with respect to the basis B. We refer to the components of the vector x_B as basic variables, and the columns of B as basic columns.
  (2) If some of the basic variables of a basic solution are zero, then the basic solution is said to be a degenerate

basic solution.
(3) A vector x satisfying $Ax = b,\ x \geq 0$, is said to be a feasible solution.
(4) A feasible solution that is also basic is called a basic feasible solution.
(5) If the basic feasible solution is a degenerate basic solution, then it is called a degenerate basic feasible solution.

- **Ex. 15.12** Consider the equation $Ax = b$ with

$$A = [a_1, a_2, a_3, a_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix},\ b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

(1) $x = [6, 2, 0, 0]^T$ is a basic feasible solution with respect to the basis $B = [a_1, a_2]$.
(2) $x = [0, 0, 0, 2]^T$ is a degenerate basic feasible solution with respect to the basis $B = [a_3, a_4]$.
(3) $x = [3, 1, 0, 1]^T$ is a feasible solution that is not basic.
(4) $x = [0, 2, -6, 0]^T$ is a basic solution with respect to the basis $B = [a_2, a_3]$, but is not feasible.

15.7 Properties of Basic Solutions

- The importance of basic feasible solutions in solving linear programming (LP) problems.
- **Definition 15.2**

(1) Any vector x that yields the minimum value of the objective function $c^T x$ over the set of vectors satisfying the constraints $Ax = b, x \geq 0$, is said to be an optimal feasible solution.
(2) An optimal feasible solution that is basic is said to be an optimal basic feasible solution.

- **Theorem 15.1** *Fundamental Theorem of LP. Consider a linear program in standard form.*
  1. *If there exists a feasible solution, then there exists a basic feasible solution.*
  2. *If there exists an optimal fesible solution, then there exists an optimal basic feasible solution.*