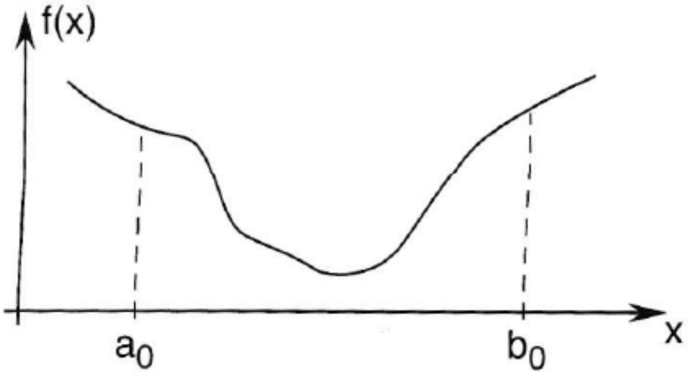
Ch.7 One-Dimensional Search Methods

7.1 Introduction

- Problem of minimizing an objective function $f : \Re \to \Re$ (i.e., one-dimensional problem).
- The approach is to use an iterative search algorithm (line-search algorithm).
  1. Golden section method (use only f)
  2. Fibonacci method (use only f)
  3. Bisection method (use only f')
  4. Secant method (use only f')
  5. Newton's method (use f' and f'')
- With an initial candidate solution $x^{(0)}$, generate a sequence of iterates $x^{(1)}, x^{(2)}, \cdots$.

7.2 Golden Section Search

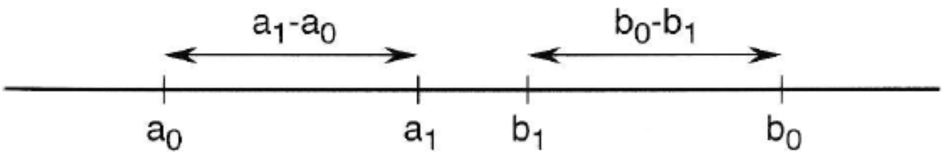
- We assume that the objective function f is unimodal, which means that f has only one local minimizer.
  ♣ Fig. 7.1 on page 92.



- Consider a unimodal function f of one variable and the interval $[a_o, b_o]$.
- We choose the intermediate points in such a way that the reduction in the range is symmetric,

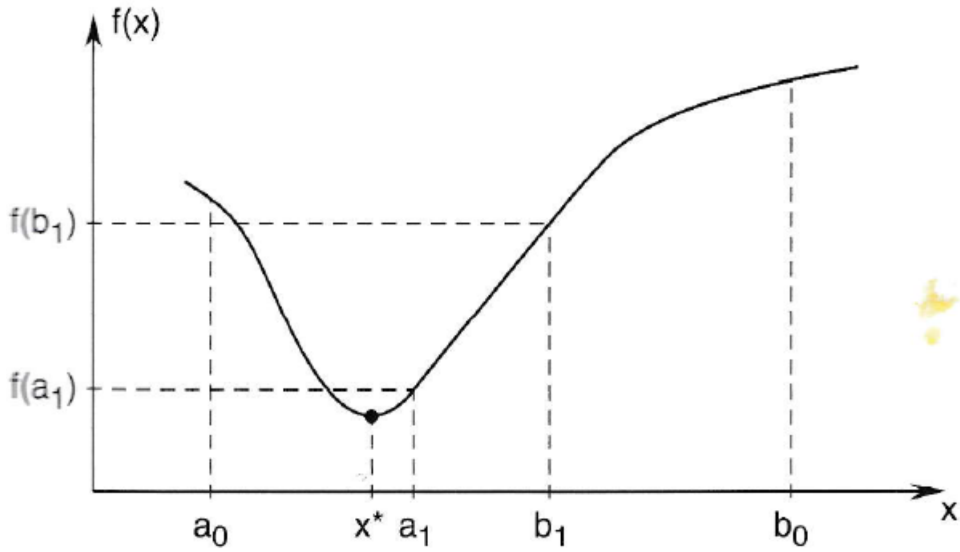
$$a_1 - a_0 = b_0 - b_1 = \rho(b_0 - a_0), \quad \rho < \frac{1}{2}$$

♣ Fig.7.2 on page 92.

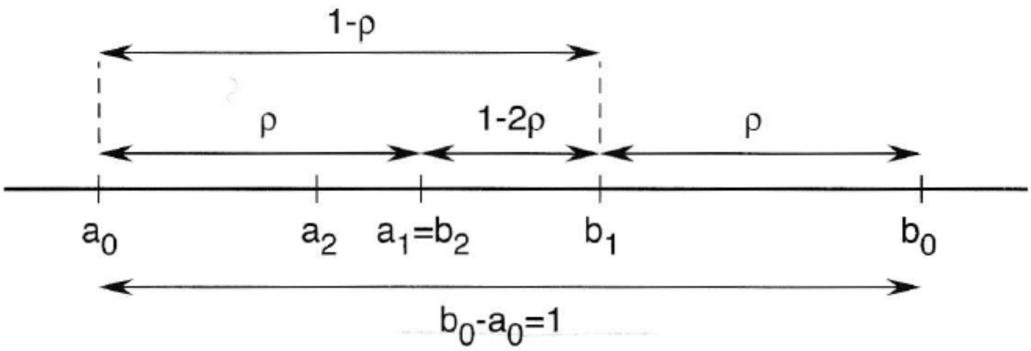


- If $f(a_1) < f(b_1)$, then the minimizer must lie in the range $[a_0, b_1]$. If $f(a_1) \geq f(b_1)$, then the minimizer is located in the range $[a_1, b_0]$.

♣ Fig.7.3 on page 92.



- To find the value of ρ that results in only one new evaluation of f, f at a_2 would be necessary.
  ♣ Fig.7.4 on page 94.



$$\rho(b_1 - a_0) = b_1 - b_2$$
$$\rho(1-\rho) = 1 - 2\rho$$
$$\rho^2 - 3\rho + 1 = 0$$

- Because we require $\rho < \frac{1}{2}$, $\rho = \frac{3-\sqrt{5}}{2} \approx 0.382$.

- Diving a range in the ratio of ρ to $1-\rho$ has the effect that the ratio of the shorter segment to the longer equals the ratio of the longer to the sum of the two. $\rightarrow$ Golden section.

$$\frac{\rho}{1-\rho} = \frac{1-\rho}{1}$$

- Example 7.1 Use the Golden Section search to find the value of x that minimizers

$$f(x) = x^4 - 14x^3 + 60x^2 - 70x$$

in the range $[0, 2]$. Locate this value of x to within a range of 0.3.
Iteration 1.

$$\begin{aligned} a_1 &= a_0 + \rho(b_0 - a_0) = 0.7639 \\ b_1 &= a_0 + (1-\rho)(b_0 - a_0) = 1.236 \end{aligned}$$

where $\rho = (3-\sqrt{5})/2$.
$f(a_1) = -24.36$, $f(b_1) = -18.96$. Thus, $f(a_1) < f(b_1)$, $[a_0, b_1] = [0, 1.236]$.
Iteration 2. We choose b_2 to coincide with a_1.

$$\begin{aligned} a_2 &= a_0 + \rho(b_1 - a_0) = 0.4721 \\ f(a_2) &= -21.10 \\ f(b_2) &= f(a_1) = -24.36 \end{aligned}$$

Now, $f(b_2) < f(a_2)$, $[a_2, b_1] = [0.4721, 1.236]$.

Iteration 3. We set $a_3 = b_2$,

$$\begin{aligned} b_3 &= a_2 + (1-\rho)(b_1 - a_2) = 0.9443 \\ f(a_3) &= f(b_2) = -24.36 \\ f(b_3) &= -23.59 \end{aligned}$$

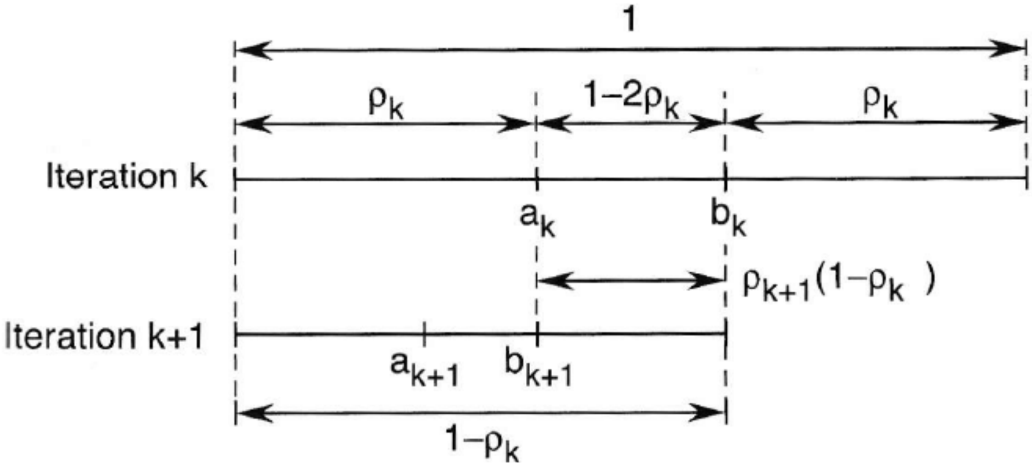
So $f(b_3) > f(a_3)$. $[a_2, b_3] = [0.4721, 0.9443]$.
Iteration 4. We set $b_4 = a_3$,

$$\begin{aligned} a_4 &= a_2 + \rho(b_3 - a_2) = 0.6525 \\ f(a_4) &= -23.84 \\ f(b_4) &= f(a_3) = -24.36 \end{aligned}$$

Hence, $f(a_4) > f(b_4)$. $[a_4, b_3] = [0.6525, 0.9443]$. And $b_3 - a_4 = 0.292 < 0.3$.

7.3 Fibonacci Search

- Golden Section methods uses the same value of ρ throughout.
- At each stage, we need to change ρ_k.
  ♣ Fig.7.5 on page 96.



$$\begin{aligned}\rho_{k+1}(1-\rho_k) &= 1-2\rho_k \\ \rho_{k+1} &= 1-\frac{\rho_k}{1-\rho_k}\end{aligned}$$

- After N iterations of the algorithm, the uncertainty range is reduced by a factor of

$$(1-\rho_1)(1-\rho_2)\cdots(1-\rho_N)$$

- This problem is a constrained optimization problem that can be formally stated:

$$\begin{aligned}\text{minimize} \quad & (1-\rho_1)(1-\rho_2)\cdots(1-\rho_N) \\ \text{subject to} \quad & \rho_{k+1} = 1-\frac{\rho_k}{1-\rho_k},\ k=1,\cdots,N-1. \\ & 0 \le \rho_k \le \frac{1}{2},\ k=1,\cdots,N.\end{aligned}$$

- Fibonacci sequence, let $F_{-1}=0,\ F_0=1$. Then for

$k \geq 0$,

$$F_{k+1} = F_k + F_{k-1}$$

$$\begin{array}{ccccccc} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 \cdots \\ 1 & 2 & 3 & 5 & 8 & 13 & 21 \cdots \end{array}$$

- The solution of the above optimization problem is:

$$\begin{aligned} \rho_1 &= 1 - \frac{F_N}{F_{N+1}} \\ \rho_2 &= 1 - \frac{F_{N-1}}{F_N} \\ \vdots & \quad \vdots \\ \rho_N &= 1 - \frac{F_1}{F_2} \end{aligned}$$

- The resulting algorithm is called the Fibonacci search method.
- The Fibonacci method is better than the Golden Section method in that it gives a smaller final uncertainty range.
- With $\rho_N = 1/2$, the two intermediate points coincide in the middle of the uncertainty interval, and therefore we cannot further reduce the uncertainty range.
- We perform the new evaluation for the last iteration using $\rho_N = 1/2 - \epsilon$.

- Example 7.2 Consider the function

$$f(x) = x^4 - 14x^3 + 60x^2 - 70x$$

Use the Fibonacci search method to find the value of x that minimizes f over the range $[0, 2]$. Locate this value of x to within a range 0.3.

Iteration 1.

$$\begin{aligned}
1 - \rho_1 &= \frac{F_4}{F_5} = \frac{5}{8} \\
a_1 &= a_0 + \rho_1(b_0 - a_0) = \frac{3}{4} \\
b_1 &= a_0 + (1 - \rho_1)(b_0 - a_0) = \frac{5}{4} \\
f(a_1) &= -24.34 \\
f(b_1) &= -18.65 \\
f(a_1) &< f(b_1)
\end{aligned}$$

$[a_0, b_1] = [0, \frac{5}{4}]$.

Iteration 2

$$\begin{aligned}
1 - \rho_2 &= \frac{F_3}{F_4} = \frac{3}{5} \\
a_2 &= a_0 + \rho_2(b_1 - a_0) = \frac{1}{2} \\
b_2 &= a_1 = \frac{3}{4} \\
f(a_2) &= -21.69 \\
f(b_2) &= f(a_1) = -24.34
\end{aligned}$$

$$f(a_2) > f(b_2)$$

$[a_2, b_1] = [\frac{1}{2}, \frac{5}{4}]$
Iteration 3.

$$\begin{aligned}
1-\rho_3 &= \frac{F_2}{F_3} = \frac{2}{3} \\
a_3 &= b_2 = \frac{3}{4} \\
b_3 &= a_2 + (1-\rho_3)(b_1 - a_2) = 1 \\
f(a_3) &= f(b_2) = -24.34 \\
f(b_3) &= -23 \\
f(a_3) &< f(b_3)
\end{aligned}$$

$[a_2, b_3] = [\frac{1}{2}, 1]$.
Iteration 4. We choose $\epsilon = 0.05$.

$$\begin{aligned}
1-\rho_4 &= \frac{F_1}{F_2} = \frac{1}{2} \\
a_4 &= a_2 + (\rho_4 - \epsilon)(b_3 - a_2) = 0.725 \\
b_4 &= a_3 = \frac{3}{4} \\
f(a_4) &= -24.27 \\
f(b_4) &= f(a_3) = -24.34 \\
f(a_4) &> f(b_4)
\end{aligned}$$

$[a_4, b_3] = [0.725, 1]$.
$b_3 - a_4 = 0.275 < 0.3$.

7.4 Bisection Method

- Find the minimizer of an objective function $f : \Re \rightarrow \Re$ over an interval $[a_0,\ b_0]$.
- Bisection method is a simple algorithm for successively reducing the uncertainty interval based on evaluation of the derivative.
- Let $x^{(0)} = (a_0 + b_0)/2$ be the midpoint of the initial uncertainty interval.
- If $f'(x^{(0)}) > 0$, the minimizer lies to the left of $x^{(0)}$. We reduce the uncertainty interbal to $[a_0,\ x^{(0)}]$.
- If $f'(x^{(0)}) < 0$, the minimizer lies to the right of $x^{(0)}$. We reduce the uncertainty interbal to $[x^{(0)},\ b_0]$.
- *Example 7.3* We wish to find the minimizer of

$$f(x) = x^4 - 14x^3 + 60x^2 - 70x$$

in the interval $[0,\ 2]$ to within a range of 0.3.
<u>*Sol.*</u> The golden section method requires at least four stages of reduction. If we use the bisection method, we would choose N so that

$$(0.5)^N < 0.3/2$$

Only 3 stages of reduction are needed.

7.5 Newton's Method

- We can fit a quadratic function through each measurement point $x^{(k)}$ that matches its first and second derivatives with that of the function f.

$$q(x) = f(x^{(k)}) + f'(x^{(k)})(x - x^{(k)}) + \frac{1}{2}f''(x^{(k)})(x - x^{(k)})^2$$

- The first-order necessary condition for a minimizer of q yields,

$$0 = q'(x) = f'(x^{(k)}) + f''(x^{(k)})(x - x^{(k)})$$

- Setting $x = x^{(k+1)}$, we obtain

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}$$

- *Example 7.4* Using Newton's method, find the minimizer of

$$f(x) = \frac{1}{2}x^2 - \sin x.$$

The initial value is $x^{(0)} = 0.5$. The required accuracy is $\epsilon = 10^{-5}$, in the sense that we stop when $|x^{(k+1)} - x^{(k)}| < \epsilon$.

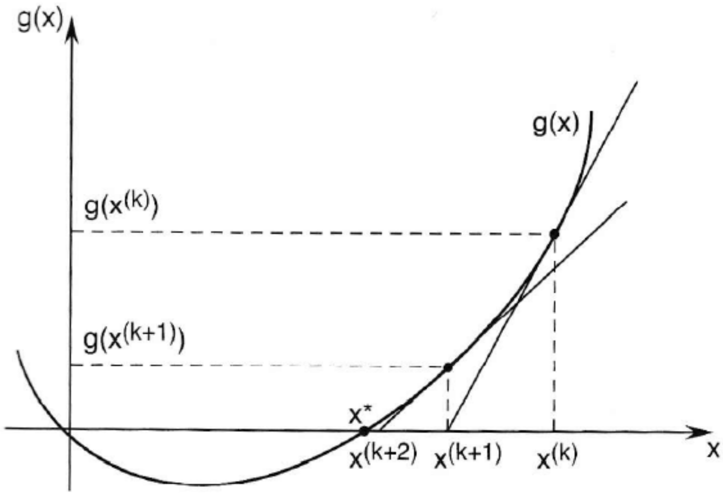
$$\begin{aligned} f'(x) &= x - \cos x, \;\; f''(x) = 1 + sinx \\ x^{(1)} &= 0.5 - \left[\frac{0.5 - cos0.5}{1 + sin0.5}\right] = 0.7552 \\ x^{(2)} &= x^{(1)} - \frac{f'(x^{(1)})}{f''(x^{(1)})} = 0.7391 \end{aligned}$$

$$x^{(3)} = x^{(2)} - \frac{f'(x^{(2)})}{f''(x^{(2)})} = 0.7390$$

$$x^{(4)} = x^{(3)} - \frac{f'(x^{(3)})}{f''(x^{(3)})} = 0.7390$$

Note that $|x^{(4)} - x^{(3)}| < \epsilon = 10^{-5}$. $x^* \approx x^{(4)}$ is a strict minimizer.

- Newton's method for solving equations of the form $f'(x) = g(x) = 0$ is also referred to as Newton's method of tangents.
  ♣ Fig.7.8 on page 106.



- If we draw a tangent to $g(x)$ at the given point $x^{(k)}$, then the tangent line intersects the x-axis at the point $x^{(k+1)}$, which we expect to be closer to the root x^* of

$g(x) = 0$.

$$\begin{aligned} g'(x^{(k)}) &= \frac{g(x^{(k)})}{x^{(k)} - x^{(k+1)}} \\ x^{(k+1)} &= x^{(k)} - \frac{g(x^{(k)})}{g'(x^{(k)})} \end{aligned}$$

7.6 Secant Method

- If the second derivative is not available in Newton's method, we may approximate $f''(x^{(k)})$ above with

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - \frac{x^{(k)} - x^{(k-1)}}{f'(x^{(k)}) - f'(x^{(k-1)})} f'(x^{(k)}) \\ x^{(k+1)} &= \frac{f'(x^{(k)})x^{(k-1)} - f'(x^{(k-1)})x^{(k)}}{f'(x^{(k)}) - f'(x^{(k-1)})} \end{aligned}$$

- The above algorithm is called the secant method.
- *Example 7.6*

$$g(x) = x^3 - 12.2x^2 + 7.45x + 42 = 0$$

We perform two iterations, with starting points $x^{(-1)} = 13$ and $x^{(0)} = 12$.

$$x^{(1)} = 11.40, \quad x^{(2)} = 11.25$$