Dynamic Optimization (In Prints)

Dynamic Optimization

- A nonlinear dynamical system in state space (time domain)

$$\text{State vector } \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{Control vector } \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\dot{\bar{x}} = f(\bar{x}, \bar{u}, t)$$

- A very special case,

$$\dot{\bar{x}} = A\bar{x} + B\bar{u} \qquad \text{linear system}$$

- Cost functional (performance index)

$$I = \Phi(\bar{x}_f, t_f) + \int_{t_0}^{t_f} L(\bar{x}, \bar{u}, t)dt$$

Φ, L are scalars

- Constraints

$$|u| \leq u_{max}$$

$$\text{or } |\dot{u}| \leq M$$

or path contraints are possible

- A general optimal control problem:
  We seek to find $u^*(t)$ with fixed t_f,

$$\min_{u(t)} I = \Phi(x_f, t_f) + \int_{t_0}^{t_f} L(x, u, t)dt$$

such that

$$\dot{x} = f(x, u, t) \;\; x(t_0), \text{ given}$$

path constraints

$$S(x) \leq 0$$
$$P(x, u) \leq 0$$

interior constraints

$$c_i(x_i, t_i) = 0, \;\; i = 1, 2, \cdots, k$$

and terminal constraints

$$\Psi(x_f, t_f) = 0$$

- Fundamental Lemma of Calculus of Variations

$$\text{If } \int_{t_0}^{t_f} K(t)\eta(t)dt = 0$$
$$\text{and } \eta(t) \text{ is arbitrary} \Rightarrow K(t) = 0$$

- Functions: $x(t)$, $u(t)$ functions of t.

- Functional: function of functions

$$\int_{t_0}^{t_f} L(x,u,t)dt$$

- Variation of a function (differential of a variable)
  $u(t) = u_o(t) + \delta u(t)$

$$\begin{aligned} \min_{u(t)} I &= \Phi(x_f) + \int L(x,u,t)dt \\ \dot{x} &= f(x,u,t), \quad x(t_0) \end{aligned}$$

- Effects of variation

$$\begin{aligned} u_o(t) \to x_o(t) \quad \dot{x} = f(x,u,t) \\ x(t_0) = x_o \\ u_o(t) + \delta u(t) \to x_o(t) + \Delta x(t) \\ \delta u(t) \quad \text{first order variation} \\ \delta^2 u(t) \quad \text{2nd-order variation} \\ \dot{x}^o = f(x^o, u^o, t), \quad \text{nominal solution} \\ x^o(t_0) = x_o \end{aligned}$$

$$\begin{aligned} (x^o + \Delta x)' &= f(x^o + \Delta x, u^o + \delta u, t) \\ \dot{x}^o + \Delta \dot{x} &= f(x^o + \Delta x, u^o + \delta u, t) \\ \Delta \dot{x} &= f(x^o + \Delta x, u^o + \delta u, t) - f(x^o, u^o, t) \\ &= f_x(x^o, u^o, t)\Delta x + \frac{1}{2}\Delta x^T f_{xx}(x^o, u^o, t)\Delta x \\ &\quad + \cdots + f_u(x^o, u^o, t)\delta u \end{aligned}$$

Consider

$$\Delta x = \delta x(t) + \frac{1}{2!}\delta^2 x(t) + \frac{1}{3!}\delta^3 x + \cdots$$

Assuming $\delta u(t)$ is small

$$\delta \dot{x} = f_x|_o \delta x + f_u|_o \delta u, \quad \delta x(t_0) = 0$$

- First variation

$$\delta I = \Phi_x \delta x(t_f) + \int_{t_0}^{t_f} (L_x \delta x + L_u \delta u) dt$$

At optimum $\delta I = 0$ with $\delta \dot{x} = f_x \delta x + f_u \delta u$

- Lagrange multiplier functions: $\lambda(t) \in \Re^n$

$$\begin{aligned} \delta J &= \Phi_x \delta x(t_f) + \int_{t_0}^{t_f} [L_x \delta x + L_u \delta u \\ &\quad + \lambda^T (f_x \delta x + f_u \delta u - \delta \dot{x})] dt \\ &= \delta I \\ &= \Phi_x \delta x_f + \int_{t_o}^{t_f} [(L_x + \lambda^T f_x)\delta x \\ &\quad + (L_u + \lambda^T f_u)\delta u - \lambda^T \delta \dot{x}] dt \end{aligned}$$

Integration by parts

$$\begin{aligned} \delta J &= \Phi_x \delta x_f - \lambda^T \delta x|_{t_o}^{t_f} + \int_{t_o}^{t_f} [(L_x + \lambda^T f_x)\delta x \\ &\quad + (L_u + \lambda^T f_u)\delta u + \dot{\lambda}^T \delta x] dt \end{aligned}$$

$$
\begin{aligned}
\delta J &= (\Phi_x - \lambda^T)\delta x(t_f) + \int_{t_o}^{t_f} [(\dot{\lambda}^T + L_x + \lambda^T f_x)\delta x \\
&\quad + [L_u + \lambda^T f_u)\delta u] dt \\
&= \delta I
\end{aligned}
$$

Choose as

$$
\begin{aligned}
\dot{\lambda}^T &= -L_x - \lambda^T f_x \\
\lambda(t_f) &= (\Phi_x)^T
\end{aligned}
$$

Then

$$
\delta J = \int_{t_o}^{t_f} (L_u + \lambda^T f_u)\delta u dt = 0
$$

$$
L_u + \lambda^T f_u = 0
$$

- Hamiltonian function

$$
\begin{aligned}
H &\equiv L + \lambda^T f \\
H(x, u, t, \lambda) &= L(x, u, t) + \lambda^T f(x, u, t)
\end{aligned}
$$

- 1st-order necessary conditions:

$$
\begin{aligned}
H &= L + \lambda^T f \\
\dot{\lambda} &= -H_x^T = -L_x^T - f_x^T \lambda, \\
\lambda(t_f) &= \left(\frac{\partial \Phi}{\partial x_f} \right)^T \\
\dot{x} &= f(x, u, t), \quad x(t_0) \text{ given} \\
H_u &= L_u + \lambda^T f_u = 0
\end{aligned}
$$

Euler-Lagrange Eqs.

- **Example**

$$\dot{x} = u, \quad x(t_o) = 0$$

$$\min_u I = K[x(t_f) - 1]^2 + \int_{t_0}^{t_f} u^2 dt$$

Following the 1st-order necessary conditions;

$$
\begin{aligned}
(1) \qquad & H = u^2 + \lambda u, \quad \Phi = K[x_f - 1]^2 \\
& \dot{\lambda} = -\frac{\partial H}{\partial x} = 0, \quad \lambda = \text{const} \\
& \lambda_f = \frac{\partial \Phi}{\partial x_f} = 2K(x_f - 1) \\
& \lambda = 2K(x_f - 1) \\
(2) \qquad & H_u = 2u + \lambda = 0 \\
& u = -\frac{\lambda}{2} = -K(x_f - 1) \\
(3) \qquad & \dot{x} = -K(x_f - 1), \quad x(t_o) = 0 \\
& x(t) - 0 = -K(x_f - 1)t \\
& x(t) = -K[x_f - 1]t
\end{aligned}
$$

At $t = t_f$

$$
\begin{aligned}
x_f &= -K[x_f - 1]t_f \\
x_f &= \frac{Kt_f}{1 + Kt_f}
\end{aligned}
$$

$$
\begin{aligned}
u^*(t) &= -K\left[\frac{Kt_f}{1+Kt_f} - 1\right] \\
&= \frac{K}{1+Kt_f}
\end{aligned}
$$

$$x^*(t) = \frac{Kt}{1+Kt_f}$$