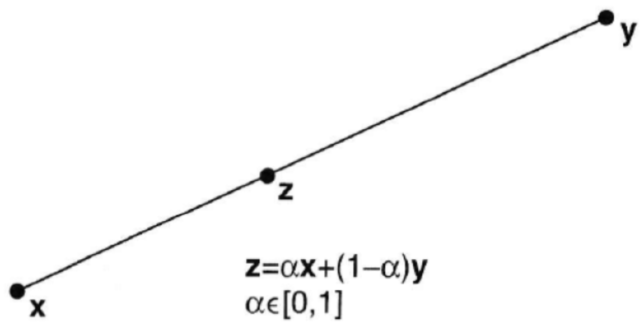
Ch.4 Concepts from Geometry

4.1 Line Segments

- The line segment between two points x and y in $\Re^n$ is the set of points on the straight line joining points x and y.

$$\{\alpha x + (1-\alpha)y : \alpha \in [0,1]\}$$

♣ Fig.4.1 on page 46.



4.2 Hyperplanes and Linear Varieties

- Let $u_1, u_2, \cdots, u_n, v \in \Re$, where at least one of the u_i is nonzero. The set of all points $x = [x_1, x_2, \cdots, x_n]^T$ that satisfy the linear equation

$$u_1x_1 + u_2x_2 + \cdots + u_nx_n = v$$

is called a hyperplane of the space $\Re^n$.

$$\{x \in \Re^n : u^T x = v\}$$

- The hyperplane $H = \{x : u_1x_1 + \cdots + u_nx_n = v\}$ divides $\Re^n$ into two half-spaces.

$$\begin{aligned} H_+ &= \{x \in \Re^n : u^T x \geq v\} \\ H_- &= \{x \in \Re^n : u^T x \leq v\} \end{aligned}$$

- The half-space H_+ is called the positive half-space, and the half-space H_- is called the negative half-space.
- A linear variety is a set of the form

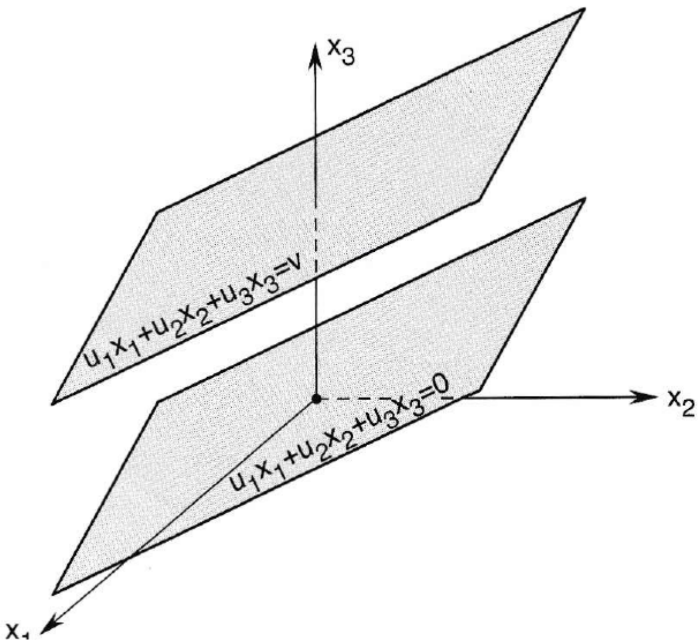
$$\{x \in \Re^n : Ax = b\}$$

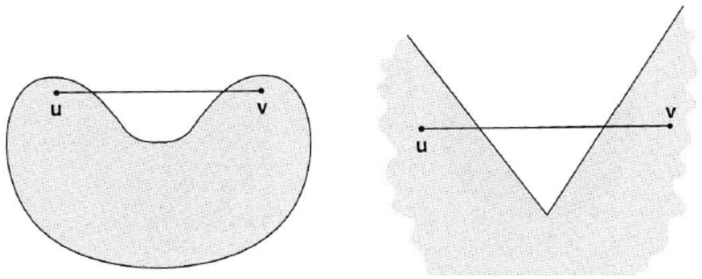
for some matrix $A \in \Re^{m \times n}$ and vector $b \in \Re^n$.

4.3 Convex Sets

- Line segment between two points u, $v \in \Re^n$ is the set $\{w \in \Re^n : w = \alpha u + (1-\alpha)v,\ \alpha \in [0,1]\}$. A point $w = \alpha u + (1-\alpha)v$ (where $\alpha \in [0,1]$) is called a convex combination of the points u and v.
- A set $\Theta \subset \Re^n$ is convex if for all u, $v \in \Theta$, the line segment between u and v is in Θ.

  ♣ Fig.4.4 and Fig. 4.5 on page 42.





- **Theorem 4.1** *Convex subsets of $\Re^n$ have the following properties:*
  1. *If Θ is a convex set and β is a real number, then the set*
  $$\beta\Theta = \{x : x = \beta v, v \in \Theta\}$$
  is also convex.
  2. *If Θ_1 and Θ_2 are convex sets, then the set*
  $$\Theta_1 + \Theta_2 = \{x : x = v_1 + v_2,\ v_1 \in \Theta_1, v_2 \in \Theta_2\}$$
  is also convex.

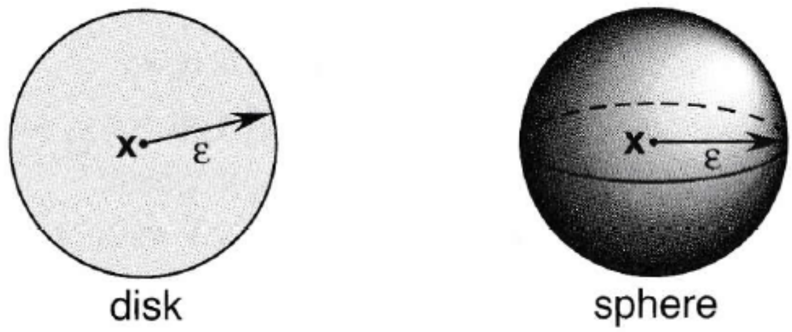
3. *The intersection of any collection of convex sets is convex.*

4.4 Neighborhoods

- A neighborhood of a point $x \in \Re^n$ is the set

$$\{y \in \Re^n : \|y - x\| < \epsilon\}$$

where ϵ is some positive number.
♣ Fig. 4.7 on page 51.



- A set S is said to be open if S contains no boundary points.
- A set S is said to be closed if it contains its boundary.
- A set that is contained in a ball of finite radius is said to be bounded. A set is compact if it is both closed and bounded.

4.5 Polytopes and Polyhedra

- A set that can be expressed as the intersection of a finite number of half-spaces is called a convex polytope.

- A nonempty bounded polytope is called a polyhedron
  ♣ Fig. 4.10 and Fig. 4.11 on page 46.

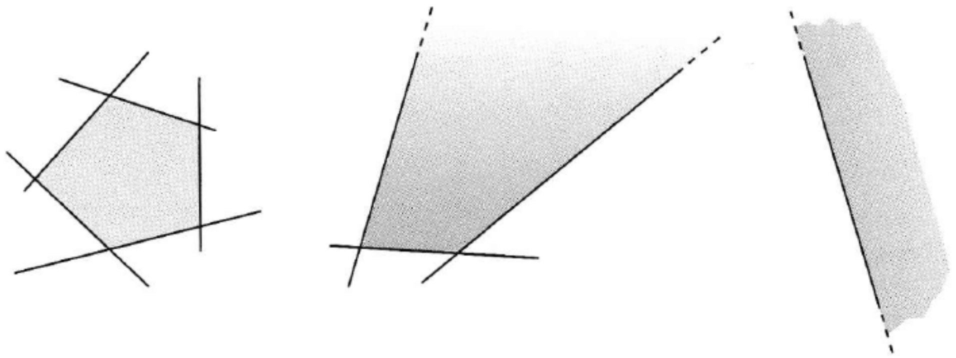


Figure 4.10 Polytopes.