Ch.16 Simplex Method

16.1 Solving Linear Equations Using Row Operations

- An elementary row operation on a given matrix.
  1. Interchanging any two rows of the matrix.
  2. Multiplying one of its rows by a real nonzero numbers.
  3. Adding a scalar multiple of one row to another row.
- **Def. 16.1** We call E an elementary matrix of the first kind if E is obtained from the indentity matrix I by interchanging any two of its rows.
- **Def. 16.2** We call E an elementary matrix of the second kind if E is obtained from the indentity matrix I by multiplying one of its rows by a real numbers $\alpha \neq 0$.
- **Def. 16.3** We call E an elementary matrix of the third kind if E is obtained from the indentity matrix I by adding β times one row to another row of I.
- **Def. 16.4** An elementary row operation (of first, second, or third kind) on a given matrix is a premultiplication of the given matrix by a corresponding elementary matrix of the respective kind.
- If A is invertible, then, $Ax = b \ \rightarrow \ x = A^{-1}b$. Thus, the problem of solving the system of equations $Ax = b$, with

$A \in \Re^{n \times n}$ invertible is related to the problem of computing A^{-1}.

- **Theorem 16.1** *Let $A \in \Re^{n \times n}$ be a given matrix. Then, A is nonsingular (invertible) iff there exist elementary matrices E_i, $i = 1, \cdots t$. s.t.*

$$E_t \cdots E_2 E_1 A = I$$

- We first form an augmented matrix $[A,\ I]$.
We then apply elementary row operations to $[A,\ I]$ so that A is transformed into I, that is, we obtain

$$E_t \cdots E_1 [A, I] = [I, B]$$

It then follows that

$$B = E_t \cdots E_1 = A^{-1}$$

- **Ex. 16.1** Let

$$A = \begin{bmatrix} 2 & 5 & 10 & 0 \\ 1 & 1 & 1 & 0 \\ -2 & -10 & -30 & 1 \\ -1 & -2 & -3 & 0 \end{bmatrix}$$

Find A^{-1}.

We form an augumented matrix

$$[A, I] = \begin{bmatrix} 2 & 5 & 10 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & -10 & -30 & 1 & 0 & 0 & 1 & 0 \\ -1 & -2 & -3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying row operations of the first and third-kinds yields
(row1=row2, row2=row1-2 row2, row3=2 row2+row3, row4=row2+row4).

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 8 & 0 & 1 & -2 & 0 & 0 \\ 0 & -8 & -28 & 1 & 0 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

We then interchange the second and fourth rows and apply elementary row operations of the second and third kinds to get
(row1=row1+row4, row2=-row4, row3=row2+3 row4,

row4=row3-8 row4).

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & -12 & 1 & 0 & -6 & 1 & -8 \end{bmatrix}$$

Now multiplying the third row by 1/2 and then perform a sequence of the elementary operations of the third kind to obtaind
(row1=row1+1/2 row3, row2=row2-row3, row3=1/2 row3, row4=6 row3+row4)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{5}{2} & 0 & \frac{5}{2} \\ 0 & 1 & 0 & 0 & -1 & -2 & 0 & -4 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 1 & 6 & 0 & 1 & 10 \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{5}{2} & 0 & \frac{5}{2} \\ -1 & -2 & 0 & -4 \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{3}{2} \\ 6 & 0 & 1 & 10 \end{bmatrix}$$

- Suppose that $A \in \Re^{m \times n}$ where $m < n$, and rank $A = m$. A is not a square matrix. Then, $[I, D]x = \tilde{b}$, where D is an $m \times (n-m)$ matrix, which we can rewrite as $x_B + Dx_D = \tilde{b}$, or $x_B = \tilde{b} - Dx_D$.

- Any solution to $Ax = b$ has the form

$$x = \begin{bmatrix} \tilde{b} \\ 0 \end{bmatrix} + \begin{bmatrix} -Dx_D \\ x_D \end{bmatrix}$$

16.2 Canonical Augmented Matrix

- Using a sequence of elementary row operations, and reordering the variables if necessary, we transform the system $Ax = b$ into the following "canonical form":

$$\begin{array}{lcl} x_1 \qquad + y_{1m+1}x_{m+1} + \cdots + y_{1n}x_n & = & y_{10} \\ \quad x_2 \qquad + y_{2m+1}x_{m+1} + \cdots + y_{2n}x_n & = & y_{20} \\ \qquad \ddots & = & \vdots \\ \qquad x_m + y_{mm+1}x_{m+1} + \cdots + y_{mn}x_n & = & y_{m0} \end{array}$$

The above can be represented in matrix notation as

$$[I_m, Y_{m,n-m}]x = y_0$$

- **Def. 16.5** A system $Ax = b$ is said to be in canonical form if, among the n variables, there are m variables with the property that each appears in only one equation, and its coefficient in that equation is unity.
- Given a system of equation $Ax = b$, consider the

associated canonical augmented matrix

$$[I_m, Y_{m,n-m}, y_0] = \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1m+1} & \cdots & y_{1n} & y_{10} \\ 0 & 1 & \cdots & 0 & y_{2m+1} & \cdots & y_{2n} & y_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{mm+1} & \cdots & y_{mn} & y_{m0} \end{bmatrix}$$

$$b = y_{10}a_1 + y_{20}a_2 + \cdots + y_{m0}a_m$$

The entries in the last column of the canonical augmented matrix are the coordinates of the vector b with respect to the basis $\{a_1, \cdots, a_m\}$.

16.3 Updating The Augmented Matrix

- If we replace a basic variable by a nonbasic variable, what is the new canonical representation corresponding to the new set of basic variables?

$$\begin{aligned} y'_{ij} &= y_{ij} - \frac{y_{pj}}{y_{pq}} y_{iq}, \quad i \neq p, \\ y'_{pj} &= \frac{y_{pj}}{y_{pq}} \end{aligned}$$

are called as pivot equation, and y_{pq} the pivot element.

- We refer to the operation on a given matrix by the above formulas as pivoting about the (p, q)th element.

- **Ex.** Consider the system in canonical form:

$$\begin{aligned} x_1 \qquad\qquad + x_4 + x_5 - x_6 &= 5 \\ x_2 \qquad + 2x_4 - 3x_5 + x_6 &= 3 \\ x_3 - x_4 + 2x_5 - x_6 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 & 5 \\ 0 & 1 & 0 & 2 & -3 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 & -1 & -1 \end{bmatrix}$$

Replace x_1 by x_4 as a basic variable,

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 & 5 \\ -2 & 1 & 0 & 0 & -5 & 3 & -7 \\ 1 & 0 & 1 & 0 & 3 & -2 & 4 \end{bmatrix}$$

Replace x_2 by x_5 as a basic variable,

$$\begin{bmatrix} 3/5 & 1/5 & 0 & 1 & 0 & -2/5 & 18/5 \\ 2/5 & -1/5 & 0 & 0 & 1 & -3/5 & 7/5 \\ -1/5 & 3/5 & 1 & 0 & 0 & -1/5 & -1/5 \end{bmatrix}$$

Replace x_3 by x_6 as a basic variable,

$$\begin{bmatrix} 1 & -1 & -2 & 1 & 0 & 0 & 4 \\ 1 & -2 & -3 & 0 & 1 & 0 & 2 \\ 1 & -3 & -5 & 0 & 0 & 1 & 1 \end{bmatrix}$$

16.4 The Simplex Algorithm

- The essence of the simplex algorithm is to move from one basic feasible solution to another until an optimal basic solution is found.
- Suppose that we are given the basic feasible solution

$$\begin{aligned} x = [x_1, \cdots, x_m, 0, \cdots, 0]^T, & \qquad x_i \geq 0, i = 1, \cdots, m \\ x_1 a_1 + \cdots + x_m a_m &= b \end{aligned}$$

- In the simplex method, we want to move from one basic feasible solution to another. This means that we want to change basic columns in such a way that the last column of the canonical augmented matrix remains nonnegative ($x_i \geq 0$).

$$\begin{aligned} \text{min} \quad & c_1 x_1 + \cdots c_n x_n \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- Let us start with the basic columns $a_1, \cdots, a_m$, and assume that the corresponding basic solution $x = [y_{10}, \cdots, y_{m0}, 0, \cdots, 0]^T$ is feasible, that is, the entires y_{i0}, $i = 1 \cdots, m$, in the last column of the canonical augmented matrix are positive.
- Suppose that we now decide to make the vector a_q,

$q > m$, a basic column.

$$a_q \quad = \quad y_{1q}a_1 + y_{2q}a_2 + \cdots + y_{mq}a_m$$

Multiplying the above by $\epsilon > 0$ yields

$$\begin{aligned}\epsilon a_q = \epsilon y_{1q}a_1 + \epsilon y_{2q}a_2 + \cdots + \epsilon y_{mq}a_m \\ y_{10}a_1 + \cdots + y_{m0}a_m = b \\ (y_{10} - \epsilon y_{1q})a_1 + (y_{20} - \epsilon y_{2q})a_2 + \cdots \\ +(y_{m0} - \epsilon y_{mq})a_m + \epsilon a_q = b\end{aligned}$$

Note that the vector

$$\begin{bmatrix} y_{10} - \epsilon y_{1q} \\ \vdots \\ y_{m0} - \epsilon y_{mq} \\ 0 \\ \vdots \\ \epsilon \\ \vdots \\ 0 \end{bmatrix}$$

where ϵ appears in the qth position, is a solution to $Ax = b$.

- We choose ϵ to be the smallest value where one of the components vanishes.

$$\epsilon = \min_i \{ \frac{y_{i0}}{y_{iq}} : \; y_{iq} > 0 \}$$

- Let $r_i = 0$ for $i = 1, \cdots, m$, and $r_i = c_i - z_i$ for $i = m+1, \cdots, n$. We call r_i the ith reduced cost coefficient or relative cost coefficient.
- **Theorem 16.2** *A basic heasible solution is optimal iff the corresponding reduced cost coefficients are all nonnegative*
- **Simplex Algorithm**
  1. Form a canonical augmented matrix corresponding to an initial basic feasible solution
  2. Calculate the reduced cost coefficients corresponding to the nonbasic variables.
  3. If $r_j \geq 0$ for all j, stop - the current basic feasible solution is optimal.
  4. Select a q such that $r_q < 0$.
  5. If no $y_{iq} > 0$, stop - the problem is unbounded; else, calculate $p = \arg \min_i \{ y_{i0}/y_{iq} : \; y_{iq} > 0 \}$.
  6. Update the canonical augmented matrix by pivoting about the (p, q)th element.
  7. Go to step 2

- **Ex. 16.2** Consider the following linear program

$$
\begin{aligned}
\text{Maximize} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & x_1 \leq 4 \\
& x_2 \leq 6 \\
& x_1 + x_2 \leq 8 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

We solve this problem using the simplex method. Introducing slack variables,

$$
\begin{array}{rccccccl}
\text{minimize} & -2x_1 & -5x_2 & -0x_3 & -0x_4 & -0x_5 & & \\
\text{subject to} & x_1 & & +x_3 & & & = 4 \\
& & x_2 & & +x_4 & & = 6 \\
& x_1 & +x_2 & & & +x_5 & = 8 \\
& & & x_1, \cdots, x_5 & & & \geq 0
\end{array}
$$

The starting canonical augmented matrix

$$
\begin{bmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & b \\
1 & 0 & 1 & 0 & 0 & 4 \\
0 & 1 & 0 & 1 & 0 & 6 \\
1 & 1 & 0 & 0 & 1 & 8 \\
-2 & -5 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The starting feasible solution to the problem in standard form is

$$x = [0, 0, 4, 6, 8]^T$$

The columns a_3, a_4, and a_5 corresponding to x_3, x_4, and x_5 are basic, and the basis matrix is
$B = [a_3, a_4, a_5] = I_3$.
We next compute the reduced cost coefficients corresponding to the nonbasic variables x_1 and x_2.

$$\begin{aligned} r_1 &= -2 \\ r_2 &= -5 \end{aligned}$$

A common practive is to select the most negative value of r_j and then to bring the corresponding column into the basis. We bring a_2 into the basis, that is, we choose a_2 as the new basic column. $(q = 2)$
$p = \arg\min_i \{y_{i0}/y_{i2} : \ y_{i2} > 0\} =$
$\arg\min_i [4/0, 6/1, 8/1] = 2$. So we pivote about the $(2, 2)$th entry using the pivot equations:

$$\begin{aligned} y'_{ij} &= y_{ij} - \frac{y_{2j}}{y_{22}} y_{i2}, \quad i \neq 2 \\ y'_{2j} &= \frac{y_{2j}}{y_{22}} \end{aligned}$$

The resulting updated canonical augmented matrix is

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & b \\ 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \\ -2 & 0 & 0 & 5 & 0 & 30 \end{array}$$

a_2 entered the basis, and a_4 left the basis. The corresponding basic feasible solution is $x = [0, 6, 4, 0, 2]^T$. The reduced cost coefficients for the nonbasic columns:

$$\begin{aligned} r_1 &= -2 \\ r_4 &= 5 \end{aligned}$$

Because $r_1 = -2 < 0$, the current solution is not optimal.($q = 1$) Proceeding to update the canonical augmented matrix by pivoting about the (3,1)th element, ($p = 3$)

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & b \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 & 0 & 6 \\ 1 & 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 3 & 2 & 34 \end{array}$$

The corresponding basic feasible solution is $x = [2, 6, 2, 0, 0]^T$. The reduced cost coefficients are

$$\begin{aligned} r_4 &= 3 \\ r_5 &= 2 \end{aligned}$$

The current basic feasible solution $x = [2, 6, 2, 0, 0]^T$ is optimal and the objective function value is 34.

16.5 Matrix Form of The Simplex Method

- Consider a linear programming problem in standard form

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- We partition the cost vector, c and A using basic columns
- We add the cost coefficient vector c^T to the bottom of the augmented matrix $[A, b]$ as follows:

$$\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix} = \begin{bmatrix} B & D & b \\ c_B^T & c_D^T & 0 \end{bmatrix}$$

$\Rightarrow$ tableau of the given LP problem.

- We now apply elementary row operations to the tableau such that the top part of the tableau corresponding to the augmented matrix $[A, b]$ is transformed into canonical form.

$$\begin{bmatrix} B^{-1} & 0 \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} B & D & b \\ c_B^T & c_D^T & 0 \end{bmatrix} = \begin{bmatrix} I_m & B^{-1}D & B^{-1}b \\ c_B^T & c_D^T & 0 \end{bmatrix}$$

- We now apply elementary row operations to the above tableau so that the entries of the last row corresponding to the basic columns become zero.

$$\begin{bmatrix} I_m & 0 \\ -c_B^T & 1 \end{bmatrix} \begin{bmatrix} I_m & B^{-1}D & B^{-1}b \\ c_B^T & c_D^T & 0 \end{bmatrix}$$
$$= \begin{bmatrix} I_m & B^{-1}D & B^{-1}b \\ 0^T & c_D^T - c_B^T B^{-1}D & -c_B^T B^{-1}b \end{bmatrix}$$

$\Rightarrow$ canonical tableau corresponding to the basis B.

- First m entries of the last column of the canonical tableau, $B^{-1}b$, are the values of the basic variables corresponding to the basis B.
- The entries of $c_D^T - c_B^T B^{-1}D$ in the last row are the reduced cost coefficients.
- The last element in the last row of the tableau, $-c_B^T B^{-1}b$, is the negative of the value of the objective function corresponding to the basic feasible solution.

- Updating of the tableau,

$$
\begin{aligned}
y'_{ij} &= y_{ij} - \frac{y_{pj}}{y_{pq}}, \quad i \neq p \\
y'_{pj} &= \frac{y_{pj}}{y_{pq}}
\end{aligned}
$$

- **Ex. 16.3** Linear programming problems:

$$
\begin{aligned}
\text{maximize} \quad & 7x_1 + 6x_2 \\
\text{subject to} \quad & 2x_1 + x_2 \leq 3 \\
& x_1 + 4x_2 \leq 4 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

We introduce two nonnegative slack variables, x_3 and x_4, and the tableau for the problem is

$$
\begin{array}{cccccc}
 & a_1 & a_2 & a_3 & a_4 & b \\
 & 2 & 1 & 1 & 0 & 3 \\
 & 1 & 4 & 0 & 1 & 4 \\
c^T & -7 & -6 & 0 & 0 & 0
\end{array}
$$

Because $r_1 = -7$ is the most negative reduced cost coefficient, we bring a_1 into the basis.($q = 1$) We then compute the ratios $y_{10}/y_{11} = 3/2$ and $y_{20}/y_{21} = 4$. We get $p = arg\ min_i\{y_{i0}/y_{i1} : y_{i1} > 0\} = 1.(p = 1)$ We pivot about the (1,1)th element of the tableau to obtain

$$\begin{array}{ccccc} 1 & 1/2 & 1/2 & 0 & 3/2 \\ 0 & 7/2 & -1/2 & 1 & 5/2 \\ 0 & -5/2 & 7/2 & 0 & 21/2 \end{array}$$

Now only r_2 is negative. Therefore, $q = 2$(i.e., we bring a_2 into the basis)

$$\frac{y_{10}}{y_{12}} = 3, \quad \frac{y_{20}}{y_{22}} = \frac{5}{7}$$

we have $p = 2$. We thus pivot about the (2,2)th element.

$$\begin{array}{ccccc} 1 & 0 & 4/7 & -1/7 & 8/7 \\ 0 & 1 & -1/7 & 2/7 & 5/7 \\ 0 & 0 & 22/7 & 5/7 & 86/7 \end{array}$$

Because the last row of the third tableau above has no negative elements, we conclude that the basic feasible solution corresponding to the third tableau is optimal. $x_1 = 8/7,\ x_2 = 5/7,\ x_3 = 0,\ x_4 = 0$ is the solution to our LP in standard form, and the corresponding objective value is -86/7. The solution to the original problem is simply $x_1 = 8/7,\ x_2 = 5/7$, and the corresponding objective value is 86/7.

- Simple rule for choosing q and p,

$$q \quad = \quad \min\{i : r_i < 0\}$$

$$p = \min\{j : y_{j0}/y_{jq}\}$$

16.6 Two-Phase Simplex Method

- Linear program in standard form:

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

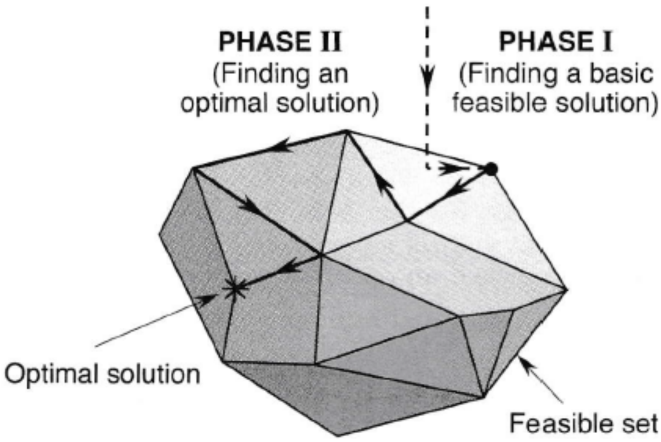
- In general, an initial basic feasible solution is not always apparent. A systematic method for finding an initial basic feasible solution for general LP problem, so that the simplex method can be initialized.
- Consider the following associated artificial problem:

$$\begin{array}{rl} \text{minimize} & y_1 + y_2 + \cdots + y_m \\ \text{subject to} & [A, I_m] \begin{bmatrix} x \\ y \end{bmatrix} = b \\ & \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \end{array}$$

  where $y = [y_1, \cdots, y_m]^T$. y: vector of artificial variables.
- **Proposition 16.1** *The original LP problem has a basic feasible solution iff the associated artificial problem has*

an optimal feasible solution with objective function value zero

- Two-phase simplex method: in phase I, we introduce artificial variables and the artificial objective function, and find a basic feasible solution. In phase II, we use the basic feasible solution resulting from phase I to initialize the simplex algorithm to solve the original LP problem.



- **Ex. 16.4** Linear programming problem:

$$\begin{array}{rl} \text{minimize} & 2x_1 + 3x_2 \\ \text{subject to} & 4x_1 + 2x_2 \geq 12 \\ & x_1 + 4x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{rl} \text{minimize} & 2x_1 + 3x_2 \\ & 4x_1 + 2x_2 - x_3 = 12 \end{array}$$

$$x_1 + 4x_2 - x_4 = 6$$
$$x_1, \cdots, x_4 \geq 0$$

Phase I. We introduce artificial variables $x_5, x_6 \geq 0$, and an artificial objective function $x_5 + x_6$. The corresponding tableau

	a_1	a_2	a_3	a_4	a_5	a_6	b
	4	2	-1	0	1	0	12
	1	4	0	-1	0	1	6
c^T	0	0	0	0	1	1	0

To initiate the simplex procedure, we must update the last row of the above tableau to transform it into canonical form,

a_1	a_2	a_3	a_4	a_5	a_6	b
4	2	-1	0	1	0	12
1	4	0	-1	0	1	6
-5	-6	1	1	0	0	-18

We proceed with the simplex method to obtain the next tableau:

$7/2$	0	-1	$1/2$	1	$-1/2$	9
$1/4$	1	0	$-1/4$	0	$1/4$	$3/2$
$-7/2$	0	1	$-1/2$	0	$3/2$	-9

Performing another iteration,

$$\begin{array}{ccccccc} 1 & 0 & -2/7 & 1/7 & 2/7 & -1/7 & 18/7 \\ 0 & 1 & 1/14 & -2/7 & -1/14 & 2/7 & 6/7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}$$

We now proceed to phase II.

Phase II. We start by deleting the columns corresponding to the artificial variables in the last tableau in phase I, and revert back to the original objective function.

$$\begin{array}{cccccc} & a_1 & a_2 & a_3 & a_4 & b \\ & 1 & 0 & -2/7 & 1/7 & 18/7 \\ & 0 & 1 & 1/14 & -2/7 & 6/7 \\ c^T & 2 & 3 & 0 & 0 & 0 \end{array}$$

We transform the last row so that the zeros appear in the basis columns, that is, we transform the above tableau into canonical form:

$$\begin{array}{ccccc} 1 & 0 & -2/7 & 1/7 & 18/7 \\ 0 & 1 & 1/14 & -2/7 & 6/7 \\ 0 & 0 & 9/14 & 4/7 & -54/7 \end{array}$$

The optimal solution is

$$x = \left[\frac{18}{7}, \frac{6}{7}, 0, 0\right]^T$$

and the optiumal cost is 54/7.