2.3 Continuous Dynamic Systems

Optimal programming problems for continuous systems are problems in the *calculus of variations.* They may be considered as limiting cases of optimal programming problems for discrete systems in which the time increment between steps becomes small compared to characteristic times of the continuous system. The reverse procedure is more common today; continuous systems are approximated by discrete systems for simulation on digital computers.

A continuous step dynamic system is described by an n-dimensional *state vector* $x(t)$ at time t. Choice of an m-dimensional *control vector* $u(t)$ determines the time rate of change of the state vector through the relations

$$\dot{x} = f(x, u, t) \quad , \tag{2.51}$$

A fairly general optimization problem for such a system is to find the time history of the control vector $u(t)$ for $t_0 \le t \le t_f$ to minimize a performance index of the form

$$J = \phi[x(t_f)] + \int_{t_0}^{t_f} L(x, u, t)dt \quad , \tag{2.52}$$

subject to (2.51) with

$$t_0, t_f \text{ and } x(t_0) \text{ specified.} \tag{2.53}$$

2.3.1 Necessary Conditions for a Stationary Solution

Adjoin the constraints (2.51) to the performance index (2.52) with a time varying Lagrange multiplier vector $\lambda(t)$ as follows:

$$\bar{J} = \phi[x(t_f)] + \int_{t_0}^{t_f} \left\{L[x(t), u(t), t] + \lambda^T(t)\left[f[x(t), u(t), t] - \dot{x}\right]\right\} dt \quad . \tag{2.54}$$

Define the scalar *Hamiltonian* function $H[x(t), u(t), \lambda(t), t]$ which we shall call $H(t)$ for a shorter notation:

$$H(t) \triangleq L[x(t), u(t), t] + \lambda^T(t) f[x(t), u(t), t] \quad . \tag{2.55}$$

Also, let us integrate the $\lambda^T \dot{x}$ term in (2.54) by parts, yielding

$$\bar{J} = \phi[x(t_f)] - \lambda^T(t_f)x(t_f) + \lambda^T(t_0)x(t_0) + \int_{t_0}^{t_f} \left\{H[x(t), u(t), \lambda(t), t] + \dot{\lambda}^T x(t)\right\} dt \quad . \tag{2.56}$$

Now consider an infinitesimal variation in $u(t)$ which we shall call $\delta u(t)$ such as the one shown in Fig. 2.7. Such a variation will produce variations in the state histories $\delta x(t)$ and a variation in the performance index $\delta \bar{J}$ which could be calculated from:

$$\delta \bar{J} = \left[\left(\phi_x - \lambda^T\right)\delta x\right]_{t=t_f} + \left[\lambda^T \delta x\right]_{t=t_0} + \int_{t_0}^{t_f} \left[\left(H_x + \dot{\lambda}^T\right)\delta x + H_u \delta u\right] dt \quad . \tag{2.57}$$

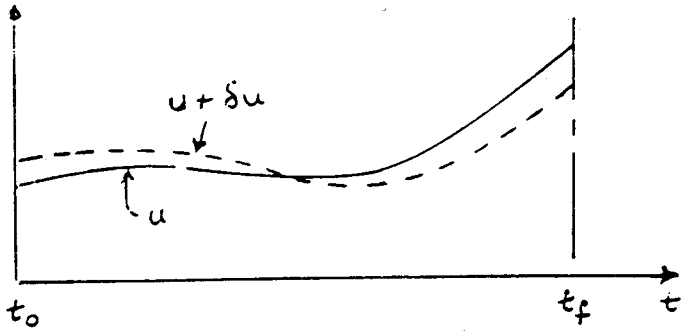

Figure 2.7: An Infinitesimal Variation in the Control History $\delta u(t)$

To avoid having to determine the functions $\delta x(t)$ produced by $\delta u(t)$, we *choose* the multiplier functions $\lambda(t)$ so that the coefficient of $\delta x(t)$ in (2.57) vanishes, i. e. we choose

$$\dot{\lambda}^T = -H_x , \tag{2.58}$$

or

$$\dot{\lambda}^T = -L_x - \lambda^T f_x ,$$

with boundary conditions

$$\lambda^T(t_f) = \phi_x(t_f) . \tag{2.59}$$

(2.57) then becomes

$$\delta \bar{J} = \lambda^T(t_0)\delta x(t_0) + \int_{t_0}^{t_f} H_u \delta u dt . \tag{2.60}$$

Thus $H_u(t)$ *is the impulse response function for* J, while holding $x(t_0)$ constant and satisfying (2.51), i. e. a unit impulse in δu at time t_1 will produce $\delta J = H_u(t_1)$. Also $\lambda^T(t_0) \equiv J_x(t_0)$, i. e. $\lambda^T(t_0)$ is the gradient of J with respect to $x(t_0)$, while holding $u(t)$ constant and satisfying (2.51). If $x(t_0)$ is specified, then $\delta x(t_0) = 0$.

For a stationary solution $\delta \bar{J} = 0$ for arbitary $\delta u(t)$; this can only happen if

$$H_u = 0 , \; t_0 \le t \le t_f . \tag{2.61}$$

(2.58), (2.59), and (2.61) are known as the *Euler-Lagrange equations* in the calculus of variations.

Hence, to find a control vector function $u(t)$ that produces a stationary value of the performance index J, we must solve the following *differential equations*

$$\dot{x} = f(x, u, t) , \tag{2.62}$$
$$\dot{\lambda} = -H_x^T \equiv -L_x^T - f_x^T \lambda , \tag{2.63}$$

where $u(t)$ is determined from (2.61), which may be written as

$$H_u \equiv L_u + \lambda^T f_u = 0 . \tag{2.64}$$