Exploring Lagrangian Optimization

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Section 1: The Extreme Value Theorem in $\mathbb{R}^2$

Chapter 1

Hungry Joe

Our story begins with a random guy named "Joseph-Louis." Because his name is kinda long, we'll just refer to him as "Joe." Joe is pretty good at math, but he isn't really that good at making dietary choices. Despite this, Joe wants to optimize the satisfaction he gets from every meal he eats. Joe usually prefers vegetables and light snacks over meats.

Today, Joe is at Carl's Parlor (run by the arbitrarily named "Carl-Friedrich") in search for the maximum satisfaction he can get from the sweetness of ice cream. Joe won't be satisfied enough if he has too little or too much ice cream. In other words, he desires for a "Goldilocks" amount of sweetness today. If he's only considering sweetness (s) as a factor of his satisfaction, then his satisfaction S can be described as:

$$S(s) = 8e^{-\frac{(s-4)^2}{64}} \tag{1.1}$$

Example 1. If Joe wants at least 1 unit of sweetness and at most 5 units, what is the maximum satisfaction that Joe can attain?

Chapter 2

Utilmaxxing

Theorem 1 (The Extreme Value Theorem in $\mathbb{R}^2$ - Paul's Online Notes [1]). Suppose that $f(x)$ is continuous on the interval $[a, b]$ then there are two numbers $a \leq c, d \leq b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

Restated, the Extreme Value Theorem (EVT) in $\mathbb{R}^2$ guarantees for any absolute (global) maximum and minimum value for any closed-continuous interval on a function. In differential calculus, this would imply that we need to compare the endpoints of the closed interval with the local extrema of the function.

Critical numbers (values) can represent either local extrema or inflection points, but we will regardlessly test for absolute extrema because an inflection point will always exist as values between the extrema. The critical values of the function denote where the function's derivative equals 0. Keep this single variable function concept in mind, as it is important for later. Let's help Joe find his maximum util!

Because e^x is continuous for all x, we can apply the EVT to S for $s \in [1, 5]$.

$$S'(s) = -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2}{64}} = 0$$

$$\Rightarrow e^{-\frac{(s-4)^2}{64}} = 0;\ s - 4 = 0$$

Since $e^x > 0$ for all x, we can omit the first equation, giving us:

$$s = 4$$

Let's find its corresponding S value...

$$S(4) = 8e^{-\frac{(4-4)^2}{64}} = 8$$

Now, we must test the S values for the end values of the interval:

$$S(1) = 8e^{-\frac{(1-4)^2}{64}} \approx 6.95052$$

$$S(5) = 8e^{-\frac{(5-4)^2}{64}} \approx 7.87597$$

Because $\max(S(1), S(4), S(5)) = S(4) = 8$, Joe can utilize the sweetness of ice cream to attain a maximum satisfaction of 8 utils.

Section 2: The Extreme Value Theorem in $\mathbb{R}^3$

Chapter 3

Hungrier Joe

Since Joe is a math aficionado, he had already mentally precomputed that he needed 4 units of sweetness in order to achieve his maximum satisfaction of 8 utils. Because of this, Joe was fixated on a far more troubling matter...

Like other ice cream parlors, Carl's Parlor serves high-quality vegetable-based chicken strips as an ice cream topping. Unfortunately, that is the ONLY topping at Carl's.

Joe ponders the most optimal combination of cotton candy ice cream and chicken strips that will provide him with the maximum satisfaction. Joe's satisfaction S can now be represented in terms of sweetness (s) and umami (u) as:

$$S(s, u) = 8e^{-\frac{(s-4)^2+(u-4)^2}{64}} \tag{3.1}$$

> **Example 2.** Joe desires for at least 0 units of either taste and a total sum of tastes that does not exceed 16 units.
>
> What is the maximum satisfaction that Joe can achieve?

Chapter 4

Nerd Face Emoji

Being the second-to-highest-gold-star-sticker student in Mr. Barraza's multivariable calculus class, Joe figured that he would have to use the **Extreme Value Theorem in** $\mathbb{R}^3$ to solve this problem.

> **Theorem 2** (The Extreme Value Theorem in $\mathbb{R}^3$ - Paul's Online Notes [2]). If $f(x, y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum of the function in D.

The EVT in $\mathbb{R}^3$ is similar to the EVT in $\mathbb{R}^2$ except that, in order for the theorem to apply, the inputs (x, y) to a $\mathbb{R}^2$ function f must exist in a closed and bounded region. If the latter case suffices, the EVT states that there exists absolute extrema for a function f in such region.

Firstly, we have to find the 3D critical points for the function S. In order to do this, we must find where the partial derivatives of the function equal zero.

$$\frac{\partial}{\partial s}[S(s,u)] = -\frac{1}{4}(u-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} = 0$$
$$\frac{\partial}{\partial u}[S(s,u)] = -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} = 0$$

From a similar expression (see Chapter 2, Utilmaxxing), we can simplify this to see that:

$$s = 4;\ u = 4$$

Doing some algebra and finding the corresponding values for each variable's solution in (s, u) will yield one critical point solution, $(4, 4)$.

Then, we have to test for points of absolute extrema on the bounds of the region. Here, we'll use what we know about the EVT in $\mathbb{R}^2$ to test for absolute extrema. Though, we must first describe the bounding functions!

The problem implies the following relations:

1. $s \geq 0$
2. $u \geq 0$
3. $s + u \leq 16$

To find a single variable function for each bound, given the satisfaction function, we must make some substitutions at the extreme/boundary cases...

$$(s = 0;\ u = 0;\ s + u = 16)$$

For relation 1, we get:

$$S(0, u) = 8e^{-\frac{(-4)^2+(u-4)^2}{64}}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}u}[S(0, u)] = -\frac{1}{4}(u-4)e^{-\frac{(u-4)^2}{64}-\frac{1}{4}} = 0$$
$$\Rightarrow u = 4$$

Regarding relation 2:

$$S(s, 0) = 8e^{-\frac{(s-4)^2+(-4)^2}{64}}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}[S(s, 0)] = -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2}{64}-\frac{1}{4}} = 0$$
$$\Rightarrow s = 4$$

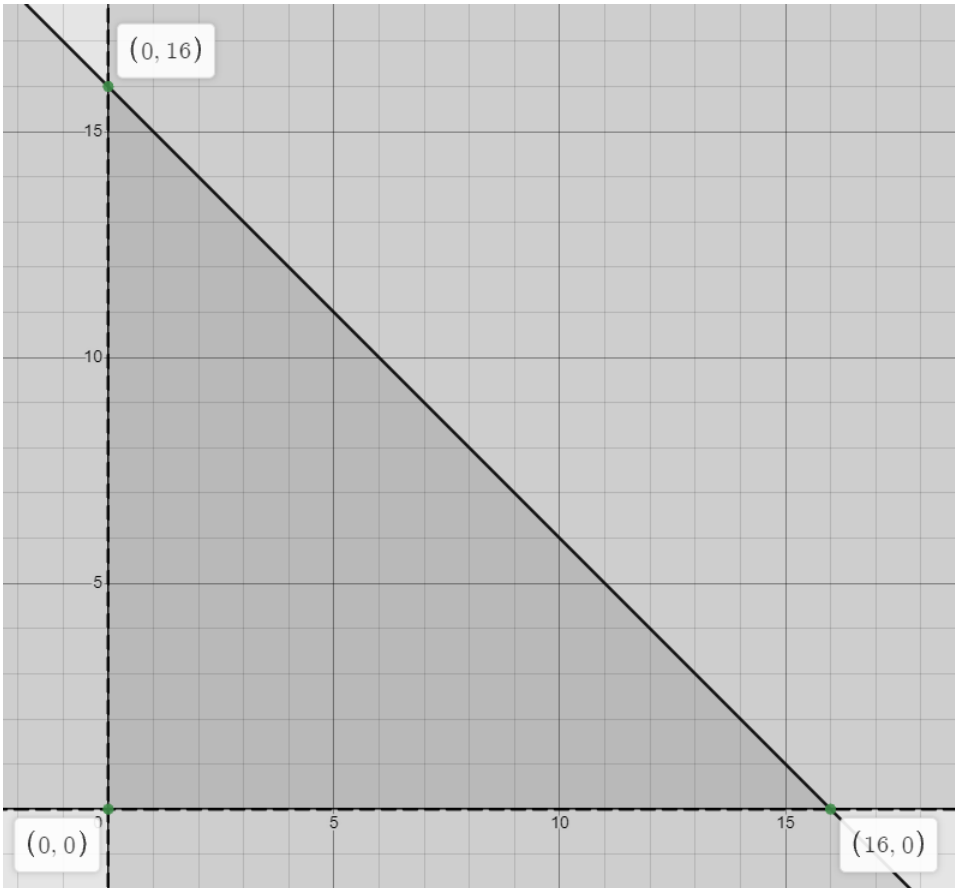
And lastly, relation 3:

$$S(s, 16-s) = 8e^{-\frac{(s-4)^2+(12-s)^2}{64}}$$
$$\frac{\mathrm{d}}{\mathrm{d}s}[S(s, 16-s)] = -\frac{1}{2}(s-8)e^{-\frac{s^2}{32}+\frac{s}{2}-\frac{5}{2}} = 0$$
$$\Rightarrow s = 8$$

Doing a bit of algebra (or not.. for the first two relations) to find the complement in each coordinate pair using the solved values just above, we get:

1. $s \geq 0 \Rightarrow (0, 4)$
2. $u \geq 0 \Rightarrow (4, 0)$
3. $s + u \leq 16 \Rightarrow (8, 8)$

Now, we must find the S values for each of the corner points. There are 3, given the intersections that form from the bounding equations. We can find each point using trivial algebra, but here's a graph to help visualize this:



Finally, we can compare the found points' satisfaction outputs...

$$S(4,4) = 8$$

$$S(0,4) = \frac{8}{\sqrt[4]{e}} \approx 6.23041$$

$$S(4,0) = \frac{8}{\sqrt[4]{e}} \approx 6.23041$$

$$S(8,8) = \frac{8}{\sqrt{e}} \approx 4.85225$$

$$S(0,0) = \frac{8}{\sqrt{e}} \approx 4.85225$$

$$S(0,16) = \frac{8}{e^{\frac{5}{2}}} \approx 0.65668$$

$$S(16,0) = \frac{8}{e^{\frac{5}{2}}} \approx 0.65668$$

Through all of this work, we found that Joe can still only achieve a maximum satisfaction of 8 utils!

Do note that, unlike when sweetness was the only option in the $\mathbb{R}^2$ case, Joe does not just want either all sweetness or all umami. Also note that this is NOT a pattern that holds true for all functions you decide to optimize using the EVT in $\mathbb{R}^3$.

To make this more clear, let's reiterate this multivariable problem-solving concept with another problem.. in the next chapter!

Chapter 5

Metonymization, Part 1

Now, let's try a general example that requires the application of the Extreme Value Theorem in $\mathbb{R}^3$.

Example 3. Find the absolute minimum and absolute maximum of

$$f(x, y) = x^2 - y^2 + xy - 5x$$

on the region bounded by $y = 5 - x^2$ and the x-axis.

Like when we had to find Joe's maximum satisfaction, we can apply the same techniques!

First, we must find the critical points for f, so we set its partial derivatives equal to zero:

$$\begin{aligned} f_x(x, y) &= 2x + y - 5 = 0 \\ f_y(x, y) &= -2y + x = 0 \end{aligned}$$

Then, we solve the system of two equations for two unknowns.

Here, we'll use elimination:

$$\begin{aligned} 4x + 2y &= 10 \\ -2y + x &= 0 \\ \Rightarrow 5x &= 10 \\ x &= 2 \end{aligned}$$

Doing a bit of algebra yields us with the following:

$$y = 1$$

So, we end up with the critical point $(2, 1)$.

Now, we have to find the absolute extrema candidates on the bounds (which we'll infer from the problem statement)!

1. $y \leq 5 - x^2$
2. $y \geq 0$

For 1:

$$\begin{aligned} f(x, 5 - x^2) &= x^2 - (5 - x^2)^2 + x(5 - x^2) - 5x \\ &= -x^4 - x^3 + 11x^2 - 25 \end{aligned}$$

Find the critical values for 1:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[f(x, 5 - x^2)] &= -4x^3 - 3x^2 + 22x = 0 \\ &\Rightarrow x \in \left\{-\frac{11}{4}, 0, 2\right\} \end{aligned}$$

And their implied points:

$$\Rightarrow \left\{\left(-\frac{11}{4}, -\frac{41}{16}\right), (0, 5), (2, 1)\right\}$$

Note that we can omit $\left(-\frac{11}{4}, -\frac{41}{16}\right)$ because it provides a y-value that does not follow relation #2. We can also ignore the repeated $(2, 1)$ because it already appeared when we used partial derivatives to find the critial point of f.

Now for 2:

$$f(x, 0) = x^2 - 5x$$

Find its critical values:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[f(x, 0)] &= 2x - 5 = 0 \\ &\Rightarrow x = \frac{5}{2} \end{aligned}$$

And its implied point:

$$\Rightarrow \left(\frac{5}{2}, 0\right)$$

Then, find the corner points:

$$0 = 5 - x^2 \Rightarrow x \in \{-\sqrt{5}, \sqrt{5}\} \Rightarrow \{(-\sqrt{5}, 0), (\sqrt{5}, 0)\}$$

And, finally, we must compare the f values for all found points:

$$\begin{aligned} f(2,1) &= -5 \\ f(0,5) &= -25 \\ f\left(\frac{5}{2}, 0\right) &= -\frac{25}{4} \approx -6.25 \\ f(-\sqrt{5}, 0) &= 5 + 5\sqrt{5} \approx 16.18034 \\ f(\sqrt{5}, 0) &= 5 - 5\sqrt{5} \approx -6.18034 \end{aligned}$$

The amount of work that went into this problem is absurd, but that's fine. In the end, we find that the absolute minimum of f is -25 and the absolute maximum is $5 + 5\sqrt{5}$, or 16.18034.

This method of finding absolute extrema, given the conditions for EVT in $\mathbb{R}^3$ are held, can be used to:

- Optimize stress and strain in 3D engineering models,
- minimize drag and minimize lift in aerodynamic designs,
- and analyze historical data patterns regarding natural resource extraction and help companies locate locations with optimal resource quantities.

Moreover, there are many more applications for EVT in 3D, so long as the 2D constrained regions of optimization are closed and bounded.

Now wait a second.. what would happen if the 2D region describing the input of the 3D function.. is not bounded? Obviously, we can't apply the Extreme Value Theorem in $\mathbb{R}^3$ in this case.. unless you know how to test all possible points on an unbounded region! Which, unfortunately, you can't.

Instead, read on...

Section 3: The Method of Lagrange Multipliers

Chapter 6

Another Order

Delightful! Joe greatly enjoyed the addition of meat- the piquant umami was a new experience for his buds.

Despite already heightening his satisfaction twice, Joe was yet again deciding on another combination of a meat-topped ice cream. This time, his only constraint is that he wants the umami flavor to be inversely proportional to half the sweetness felt.

> **Example 4.** Given that his satisfaction can again be represented by eq. (3.1), Joe desires for a nonnegative amount of each flavor and wants to try a combination where the flavor of umami he attains is inversely proportional to half the sweetness.
>
> Can you find a combination of (s, u) where Joe can attain his maximum satisfaction?

Chapter 7

A Joe Analysis

To solve this problem, we must utilize Lagrange multipliers in order to find the extrema of an arbitrary function f constrained by another function that we'll call g.

The method of Lagrange multipliers for solving constrained optimization problems utilizes the following formula:

$$\nabla f(x, y) = \lambda \nabla g(x, y) \tag{7.1}$$

In the above equation, ∇ represents the gradient operator on a function, which produces a vector where each component represents the partial derivative of the function with respect to the corresponding variable. This formula displays that the gradient vector of f is equal to the gradient vector of g times an unknown multiple λ. λ is called a Lagrange multiplier and helps constrain our possible solutions such that f is a multiple of g. If we were to separate this vector equation into partially differentiated functions with respect to either x and y components, we'd get the following equations:

$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$

Upon solving for x and y in this equation, you'd get the values of x and y that are candidates for absolute extrema outputs of the function f. Now, let's help Joe.

We are tasked with optimizing a function S constrained to $u = \frac{2}{s}$. We can describe our constraint function g to be:

$$g(s, u) = su - 2 \tag{7.2}$$

Taking some partial derivatives and rewriting the vectors to be equations for each component:

$$\begin{aligned} -\frac{1}{4}(u-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} &= \lambda u \\ -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} &= \lambda s \end{aligned}$$

Okay, we have 3 unknowns.. but only 2 equations. To address this, we will utilize our constraint equation.

$$\begin{aligned} -\frac{1}{4}(u-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} &= \lambda u \\ -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} &= \lambda s \\ su &= 2 \end{aligned}$$

Now, we can use basic algebra to find solutions for (s, u, λ). Because we are only concerned with which (s, u) that produce the maximum S for Joe, we will ignore using λ once found.

Because the algebra will be gruesome to type out for this calculus-based research paper, here is the solution for (s, u):

$$\left(\frac{8}{(2\sqrt{2}-1)e^{\frac{1}{2\sqrt{2}}-\frac{9}{16}}}, \frac{1}{4}(2\sqrt{2}-1)e^{\frac{1}{2\sqrt{2}}-\frac{9}{16}} \right) \approx (5.39208, 0.37091)$$

If you were to do this by hand, you'd find a solution with a u value that is less than 0. Because Joe wants a nonnegative amount of each flavor, we can omit the corresponding solution.

Chapter 8

Metonymization, Part 2

The method of Lagrange multipliers for constrained optimization is useful for finding where the extrema of 3D functions occur, especially if the constraint functions are unbounded. This is because the method utilizing EVT in $\mathbb{R}^3$, as mentioned in a previous chapter, cannot suffice for unbounded constraints. Do note that the method of Lagrange multipliers will find the maximum of a function. Finding the minimum would be finding the maximum of the negated function.

Now, for what this project was all about, let's apply Joe's strategy to a general example...

> **Example 5.** Use the method of Lagrange multipliers to find the maximum of $2x - y$ given the following constraint:
>
> $$3x^2 - 4xy + 2y^2 = 6$$
>
> Also, find the point (x, y) where said maximum is achieved.

Take the partial derivatives and set the equations up:

$$\begin{aligned} 2 &= \lambda(6x - 4y) \\ -1 &= \lambda(-4x + 4y) \\ 3x^2 - 4xy + 2y^2 &= 6 \end{aligned}$$

We'll leave the algebra as an exercise to the meticulous reader. Here is the candidate for absolute extrema:

$$(x, y, \lambda) : \left(2, 1, \frac{1}{4}\right)$$

Therefore, the maximum of f is $\boxed{3 \text{ at } (2, 1)}$.

Section 4: The Cobb-Douglas Production Function

Chapter 9

Carl's Parlor

After Joe decided to buy everything from Carl's Parlor, Carl-Friedrich decided to not accept anymore customers until the ice cream and chicken dispensers were refilled. Also, Carl was the only worker at his ice cream shop, so he is interested in hiring more workers.

To financially plan these business plans, Carl viewed a couple YouTube videos on the **Cobb-Douglas production function**.

Remark. The Cobb-Douglas production function is an economics concept that relates a firm's production output Y in terms of two variable inputs, labor (L) and capital (K).

The production function can be written as:

$$Y(L, K) = AL^{\alpha}K^{\beta}$$

Where..

- A is a constant for †*total-factor productivity*,
- α is a constant for †*output elasticity of labor*,
- and β is a constant for †*output elasticity of capital.*

† For the sake of keeping this paper focused on constraining the Cobb-Douglas production function, we will not derive these economic concepts. For the following example problems, we will make the following simplification:

$$\beta = 1 - \alpha$$

> This derives from the economic concept of elasticity and how $\alpha + \beta = 1$ in this context. Note that we will not further explain the derivation of this in our paper.

Right now, Carl is eyeing some refurbished multi-purpose dispenser machines with individual maintenance costs at around $\$1,500$ per year. He is also looking to hire workers that will have to be paid $\$30,000$ a year.

Carl has rich parents, but he still has a budget. He wants to spend no more than $\$100,000$ a year on his ice cream parlor. In Carl's world, ice cream parlors can be found to have a production function similar to:

$$Y(L, K) = L^{0.35}K^{0.65} \tag{9.1}$$

> **Example 6.** With the production function given above, Carl wants to find an optimal combination of capital and labor that maximizes his parlor's output while also considering his budget.
>
> Can you help him find such combination?

Chapter 10

Money-Mouth Face Emoji

If there are any constraints on the variables L and K, like a budget, then the method of Lagrange multipliers can directly be utilized to find the specific quantities of capital and labor that maximizes output $Y(L, K)$ for constraint functions.

In this problem, the constraint function would be the budget. The budget function would look something like this:

$$30,000L + 1,500K = 100,000 \tag{10.1}$$

If we treat L and K as nonnegative, we can see that these constraints form a closed and bounded region. Since Y is differentiable for all combinations of nonnegative inputs, we can apply EVT to show that there does exist an absolute extrema in the bounded region.

So, we can take a couple partial derivatives and set the equations up:

$$\begin{aligned} 0.35L^{-0.65}K^{0.65} &= 30,000\lambda \\ 0.65L^{0.35}K^{-0.35} &= 1,500\lambda \\ 30,000L + 1,500K &= 100,000 \end{aligned}$$

Solving the algebra with a calculator, we get an (L, K) pair that maximizes Y for the given budget:

$$(1.16667, 43.33333)$$

Because a unit of L or K in this example should be integers, we would round down to get the optimal combination for Carl.

In the end, Carl should $\boxed{\text{hire 1 worker and buy 43 dispensers}}$!

Chapter 11

Metonymization, Part 3

As shown in the above example, the method of Lagrange multipliers can be applied to the Cobb-Douglas production function to find the specific quantities of labor and capital that maximizes production output for given constraints on both labor and capital.

To relate this to a more general case, here's another example!

> **Example 7.** Suppose you are given a specific Cobb-Douglas function
>
> $$f(x, y) = 50x^{0.4}y^{0.6}$$
>
> where x is the amount of labor units and y is the amount of equipment units. You are also given that each unit of labor costs 1 and each unit of equipment costs 1.
>
> Use the method of Lagrange multipliers to determine how much should be spend on labor and how much on equipment to maximize productivity if we have a total of 1.5 million dollars to invest in labor and equipment.
>
> We believe there could be a slight logic error in the Cobb-Douglas example that was stated as a requirement for our project (see Ch. 13).

First, we set the partial derivatives of the production function equal to their $\lambda \nabla g(...)$ counterparts, where $g(...)$ represents our constraint/budget ($x + y = 1,500,000$):

$$50(0.4)x^{-0.6}y^{0.6} = \lambda$$
$$50(0.6)x^{0.4}y^{-0.4} = \lambda$$
$$x + y = 1,500,000$$

Without showing the algebra (to keep this paper from getting any longer), the algebra concludes to the following solution:

$$x = 600,000$$
$$y = 900,000$$

Because a unit of labor or equipment costs 1, we can maximize our Cobb-Douglas production output f with $(x, y) = (600000, 900000)$.

In general, the Cobb-Douglas production function is used to determine the production output of a firm, given units of labor and capital. This value is modified by the elasticities (efficacy of either labor or capital on the production output) of each labor and capital.

In optimizing the production function with given constraints, you can find a combination of labor and capital that suffices a constraint (like a budget) while maximizing the production output for such budget. In conclusion, the mathematical method of Lagrange multipliers can help you perform constrained optimization on not only the Cobb-Douglas production function, but various other functions that have continuous partial derivatives.

Section 5: Concluding Remarks

Chapter 12

An Important Distinction

Despite both being methods to find absolute extrema in $\mathbb{R}^3$, the method of Lagrange multipliers can not be classified as a "special case" of the Extreme Value Theorem (in $\mathbb{R}^3$) [3]. This is because the constraint function that determines the bounds of the $\mathbb{R}^2$ region to search for absolute extrema may not be bounded. In this case, EVT will not be applicable.

Do note that most problems may allow for the usage of the method of Lagrange multipliers and the Extreme Value Theorem, but some problems may not. For example, a function with discontinuous partial derivatives on an interval may not permit the method of Lagrange multipliers, but the EVT could work to guarantee an absolute extrema for known values of the function (if it's a absolute-value, step, other piecewise, or etc.. function).

Chapter 13

A Possible Error

From what we researched, the Cobb-Douglas production function's two inputs are units of capital and labor. [4] In the problem statement for this assignment, we are given a Cobb-Douglas function in terms of x and y as well as "x is the dollar amount spent on labor and y is the dollar amount spent on equipment."

Because of this inconsistency, we treated x as units of labor and y as units of equipment.

To clear any confusion in the example we put this paper, we labeled the cost for each unit of labor and equipment to be \$1. Sorry if this problem seems a bit weirdly unrealistic due to the altered constants; we were quite confused. :pensive:

Following this instruction, we used a general Cobb-Douglas production function with different constants, but implemented the budget-constraint equation as:

$$(1)x + (1)y = 1,500,000$$

Chapter 14

Acknowledgements

The "Joe/Carl" problems and their solutions were created and written in LaTeX by Aaron, but much of the content we used to assist us in understanding the Extreme Value Theorem, Lagrange multipliers, and the Cobb-Douglas production function were also helpfully compiled and explained by the other group members. Namely, Kerem and Oliver focused on the engineering applications of the method of Lagrange multipliers (which will be discussed in the presentation). Brennan and Jordan greatly helped with the understanding and problemsetting for the examples used to explain the method of Lagrange multipliers applied to the Cobb-Douglas production function.

Oh, and Aaron had a hard time figuring out the themes to the problems, but ended up choosing the names "Joe" and "Carl" after Joseph-Louis Lagrange and Carl-Friedrich Gauss, respectively. Also, the S functions were Aaron's attempted variations of the **Gauss**ian Distribution functions but nobody needs to know that (the max is the coefficient lol).

Chapter 15

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