Exploring Lagrangian Optimization

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Section 1: The Extreme Value Theorem in $\mathbb{R}^2$

Chapter 1

Hungry Joe

Our story begins with a random guy named "Joseph-Louis." Because his name is kinda long, we'll just refer to him as "Joe." Joe is pretty good at math, but he isn't really that good at making dietary choices. Despite this, Joe wants to optimize the satisfaction he gets from every meal he eats. Joe usually prefers vegetables and light snacks over meats.

Today, Joe is at Carl's Parlor (run by the arbitrarily named "Carl-Friedrich") in search for the maximum satisfaction he can get from the sweetness of ice cream. Joe won't be satisfied enough if he has too little or too much ice cream. In other words, he desires for a "Goldilocks" amount of sweetness today. If he's only considering sweetness (s) as a factor of his satisfaction, then his satisfaction S can be described as:

$$S(s) = 8e^{-\frac{(s-4)^2}{64}} \tag{1.1}$$

Example 1. If Joe wants at least 1 unit of sweetness and at most 5 units, what is the maximum satisfaction that Joe can attain?

Chapter 2

Utilmaxxing

Theorem 1 (*The Extreme Value Theorem in $\mathbb{R}^2$). Suppose that $f(x)$ is continuous on the interval $[a, b]$ then there are two numbers $a \leq c, d \leq b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

Section 2: The Extreme Value Theorem in $\mathbb{R}^3$

Chapter 3

Hungrier Joe

Since Joe is a math aficionado, he had already mentally precomputed that he needed 4 units of sweetness in order to achieve his maximum satisfaction of 8 utils. Because of this, Joe was fixated on a far more troubling matter...

Like other ice cream parlors, Carl's Parlor serves high-quality vegetable-based chicken strips as an ice cream topping. Unfortunately, that is the ONLY topping at Carl's.

Joe ponders the most optimal combination of cotton candy ice cream and chicken strips that will provide him with the maximum satisfaction. Joe's satisfaction S can now be represented in terms of sweetness (s) and umami (u) as:

$$S(s,u) = 8e^{-\frac{(s-4)^2+(u-4)^2}{64}} \tag{3.1}$$

> **Example 2.** Joe desires for at least 0 units of either taste and a total sum of tastes that does not exceed 16 units.
>
> What is the maximum satisfaction that Joe can achieve?

Chapter 4

Nerd Face Emoji

Theorem 2 (†The Extreme Value Theorem in $\mathbb{R}^3$).

Chapter 5

Metonymization, Part 1

Now, let's try a general example that requires the application of the Extreme Value Theorem in $\mathbb{R}^3$.

Example 3. Find the absolute minimum and absolute maximum of

$$f(x, y) = x^2 - y^2 + xy - 5x$$

on the region bounded by $y = 5 - x^2$ and the x-axis.

Section 3: The Method of Lagrange Multipliers

Chapter 6

Another Order

Delightful! Joe greatly enjoyed the addition of meat- the piquant umami was a new experience for his buds.

Despite already heightening his satisfaction twice, Joe was yet again deciding on another combination of a meat-topped ice cream. This time, his only constraint is that he wants the umami flavor to be inversely proportional to half the sweetness felt.

Example 4. Given that his satisfaction can again be represented by eq. (3.1), Joe desires for a nonnegative amount of each flavor and wants to try a combination where the flavor of umami he attains is inversely proportional to half the sweetness.

With what combination of (u, s) can Joe attain his maximum satisfaction?

Chapter 7

A Joe Analysis

Chapter 8

Metonymization, Part 2

Now, for what this project was all about, let's apply Joe's strategy to a generalized example.

> **Example 5.** Use the method of Lagrange multipliers to find the maximum of $2x - y$ given the following constraint:
>
> $$3x^2 - 4xy + 2y^2 = 6$$
>
> Also, find the point (x, y) where said maximum is achieved.

Section 4: The Cobb-Douglas Production Function

Chapter 9

Carl's Parlor

After Joe decided to buy everything from Carl's Parlor, Carl-Friedrich decided to not accept anymore customers until the ice cream and chicken dispensers were refilled. Also, Carl was the only worker at his ice cream shop, so he is interested in hiring more workers.

To financially plan these business plans, Carl viewed a couple YouTube videos on the **Cobb-Douglas production function**.

> **Remark.** The Cobb-Douglas production function is an economics concept that relates a firm's production output Y in terms of two variable inputs, labor (L) and capital (K).
>
> The production function can be written as:
>
> $$Y(L, K) = AL^{\alpha}K^{\beta}$$
>
> Where..
>
> - A is a constant for ‡*total-factor productivity*,
> - α is a constant for ‡*output elasticity of labor*,
> - and β is a constant for ‡*output elasticity of capital.*
>
> ‡ For the sake of keeping this paper focused on constraining the Cobb-Douglas production function, we will not derive these economic concepts. For the following example problems, we will make the following simplification:
>
> $$\beta = 1 - \alpha$$

> This derives from the economic concept of [1]elasticity that we will only briefly describe in the footnotes.

Right now, Carl is eyeing some refurbished multi-purpose dispenser machines with individual maintenance costs at around $1,500 per year. He is also looking to hire workers that will have to be paid $30,000 a year.

Carl has rich parents, but he still has a budget. He wants to spend no more than $100,000 a year on his ice cream parlor. In Carl's world, ice cream parlors can be found to have a production function similar to:

$$Y(L, K) = L^{0.35}K^{0.65} \tag{9.1}$$

> **Example 6.** With the production function given above, Carl wants to find an optimal combination of capital and labor that maximizes his parlor's output while also considering his budget.
>
> Can you help him find such combination?

Chapter 10

Money-Mouth Face Emoji

If there are any constraints on the variables L and K, like a budget, then the method of Lagrange multipliers can be utilized to find the specific quantities of capital and labor that maximizes output $Y(L, K)$ for constraint functions.

Chapter 11

Metonymization, Part 3

As shown in the above example, the method of Lagrange multipliers can be applied to the Cobb-Douglas production function to find the specific quantities of labor and capital that maximizes production output for given constraints on both labor and capital.

To relate this to a generalized case, here's another example!

Example 7. Suppose you are given a specific Cobb-Douglas function

$$f(x, y) = 50x^{0.4}y^{0.6}$$

where x is the dollar amount spent on labor and y the dollar amount spent on equipment.

Use the method of Lagrange multipliers to determine how much should be spend on labor and how much on equipment to maximize productivity if we have a total of 1.5 million dollars to invest in labor and equipment.

There could be a slight error in the example, which will be included in the Concluding Remarks section. But because we were given this example problem as a requirement for this paper, we kept it ad litteram.

Section 5: The End Stuff

Chapter 12

Concluding Remarks

"Critique my LaTeX.. and my English." - **Aaron**