Exploring Lagrangian Optimization

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Section 1: The Extreme Value Theorem in $\mathbb{R}^2$

Chapter 1

Hungry Joe

Our story begins with a random guy named "Joseph-Louis." Because his name is kinda long, we'll just refer to him as "Joe." Joe is pretty good at math, but he isn't really that good at making dietary choices. Despite this, Joe wants to optimize the satisfaction he gets from every meal he eats. Joe usually prefers vegetables and light snacks over meats.

Today, Joe is at Carl's Parlor (run by the arbitrarily named "Carl-Friedrich") in search for the maximum satisfaction he can get from the sweetness of ice cream. Joe won't be satisfied enough if he has too little or too much ice cream. In other words, he desires for a "Goldilocks" amount of sweetness today. If he's only considering sweetness (s) as a factor of his satisfaction, then his satisfaction S can be described as:

$$S(s) = 8e^{-\frac{(s-4)^2}{64}} \tag{1.1}$$

> **Example 1.** If Joe wants at least 1 unit of sweetness and at most 5 units, what is the maximum satisfaction that Joe can attain?

Chapter 2

Utilmaxxing

Theorem 1 (The Extreme Value Theorem in $\mathbb{R}^2$, Paul's Online Notes). Suppose that $f(x)$ is continuous on the interval $[a, b]$ then there are two numbers $a \leq c, d \leq b$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

Restated, the Extreme Value Theorem (EVT) in $\mathbb{R}^2$ guarantees for any absolute (global) maximum and minimum value for any closed-continuous interval on a function. In differential calculus, this would imply that we need to compare the endpoints of the closed interval with the local extrema of the function.

Critical numbers (values) can represent either local extrema or inflection points, but we will regardlessly test for absolute extrema because an inflection point will always exist as values between the extrema. The critical values of the function denote where the function's derivative equals 0. Keep this single variable function concept in mind, as it is important for later. Let's help Joe find his maximum util!

Because e^x is continuous for all x, we can apply the EVT to S for $s \in [1, 5]$.

$$S'(s) = -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2}{64}} = 0$$

$$\Rightarrow e^{-\frac{(s-4)^2}{64}} = 0;\ s - 4 = 0$$

Since $e^x > 0$ for all x, we can omit the first equation, giving us:

$$s = 4$$

Let's find its corresponding S value...

$$S(4) = 8e^{-\frac{(4-4)^2}{64}} = 8$$

Now, we must test the S values for the end values of the interval:

$$S(1) = 8e^{-\frac{(1-4)^2}{64}} \approx 6.95052$$

$$S(5) = 8e^{-\frac{(5-4)^2}{64}} \approx 7.87597$$

Because $\max(S(1), S(4), S(5)) = S(4) = 8$, Joe can utilize the sweetness of ice cream to attain a maximum satisfaction of 8 utils.

Section 2: The Extreme Value Theorem in $\mathbb{R}^3$

Chapter 3

Hungrier Joe

Since Joe is a math aficionado, he had already mentally precomputed that he needed 4 units of sweetness in order to achieve his maximum satisfaction of 8 utils. Because of this, Joe was fixated on a far more troubling matter...

Like other ice cream parlors, Carl's Parlor serves high-quality vegetable-based chicken strips as an ice cream topping. Unfortunately, that is the ONLY topping at Carl's.

Joe ponders the most optimal combination of cotton candy ice cream and chicken strips that will provide him with the maximum satisfaction. Joe's satisfaction S can now be represented in terms of sweetness (s) and umami (u) as:

$$S(s,u) = 8e^{-\frac{(s-4)^2+(u-4)^2}{64}} \tag{3.1}$$

Example 2. Joe desires for at least 0 units of either taste and a total sum of tastes that does not exceed 16 units.

What is the maximum satisfaction that Joe can achieve?

Chapter 4

Nerd Face Emoji

Being the second-to-highest-gold-star-sticker student in Mr. Barraza's multivariable calculus class, Joe figured that he would have to use the **Extreme Value Theorem in** $\mathbb{R}^3$ to solve this problem.

> **Theorem 2** (The Extreme Value Theorem in $\mathbb{R}^3$, Paul's Online Notes). If $f(x, y)$ is continuous in some closed, bounded set D in $\mathbb{R}^2$ then there are points in D, (x_1, y_1) and (x_2, y_2) so that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum of the function in D.

The EVT in $\mathbb{R}^3$ is similar to the EVT in $\mathbb{R}^2$ except that, in order for the theorem to apply, the inputs (x, y) to a $\mathbb{R}^2$ function f must exist in a closed and bounded region. If the latter case suffices, the EVT states that there exists absolute extrema for a function f in such region.

Firstly, we have to find the 3D critical points for the function S. In order to do this, we must find where the partial derivatives of the function equal zero.

$$\frac{\partial}{\partial s}[S(s,u)] = -\frac{1}{4}(u-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} = 0$$
$$\frac{\partial}{\partial u}[S(s,u)] = -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2+(u-4)^2}{64}} = 0$$

From a similar expression (see Chapter 2, Utilmaxxing), we can simplify this to see that:

$$s = 4;\ u = 4$$

Doing some algebra and finding the corresponding values for each variable's solution in (s, u) will yield one critical point solution, $(4, 4)$.

Then, we have to test for points of absolute extrema on the bounds of the region. Here, we'll use what we know about the EVT in $\mathbb{R}^2$ to test for absolute extrema. Though, we must first describe the bounding functions!

The problem implies the following relations:

1. $s \geq 0$
2. $u \geq 0$
3. $s + u \leq 16$

To find a single variable function for each bound, given the satisfaction function, we must make some substitutions at the extreme/boundary cases...

$$(s = 0;\ u = 0;\ s + u = 16)$$

For relation 1, we get:

$$S(0, u) = 8e^{-\frac{(-4)^2+(u-4)^2}{64}}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}u}[S(0, u)] = -\frac{1}{4}(u-4)e^{-\frac{(u-4)^2}{64}-\frac{1}{4}} = 0$$
$$\Rightarrow u = 4$$

Regarding relation 2:

$$S(s, 0) = 8e^{-\frac{(s-4)^2+(-4)^2}{64}}$$
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}s}[S(s, 0)] = -\frac{1}{4}(s-4)e^{-\frac{(s-4)^2}{64}-\frac{1}{4}} = 0$$
$$\Rightarrow s = 4$$

And lastly, relation 3:

$$S(s, 16-s) = 8e^{-\frac{(s-4)^2+(12-s)^2}{64}}$$
$$\frac{\mathrm{d}}{\mathrm{d}s}[S(s, 16-s)] = -\frac{1}{2}(s-8)e^{-\frac{s^2}{32}+\frac{s}{2}-\frac{5}{2}} = 0$$
$$\Rightarrow s = 8$$

Doing a bit of algebra (or not.. for the first two relations) to find the complement in each coordinate pair using the solved values just above, we get:

1. $s \geq 0 \Rightarrow (0, 4)$
2. $u \geq 0 \Rightarrow (4, 0)$
3. $s + u \leq 16 \Rightarrow (8, 8)$

Finally, we can compare the found points' satisfaction outputs...

$$S(4,4) = 8$$
$$S(0,4) = \frac{8}{\sqrt[4]{e}} \approx 6.23041$$
$$S(4,0) = \frac{8}{\sqrt[4]{e}} \approx 6.23041$$
$$S(8,8) = \frac{8}{\sqrt{e}} \approx 4.85225$$

Through all of this work, we found that Joe can still only achieve a maximum satisfaction of 8 utils!

Do note that, unlike when sweetness was the only option in the $\mathbb{R}^2$ case, Joe does not just want either all sweetness or all umami. Also note that this is NOT a pattern that holds true for all functions you decide to optimize using the EVT in $\mathbb{R}^3$.

To make this more clear, let's reiterate this multivariable problem-solving concept with another problem.. in the next chapter!

Chapter 5

Metonymization, Part 1

Now, let's try a general example that requires the application of the Extreme Value Theorem in $\mathbb{R}^3$.

> **Example 3.** Find the absolute minimum and absolute maximum of
> $$f(x, y) = x^2 - y^2 + xy - 5x$$
> on the region bounded by $y = 5 - x^2$ and the x-axis.

Like when we had to find Joe's maximum satisfaction, we can apply the same techniques!

First, we must find the critical points for f, so we set its partial derivatives equal to zero:

$$\begin{aligned} f_x(x, y) &= 2x + y - 5 = 0 \\ f_y(x, y) &= -2y + x = 0 \end{aligned}$$

Then, we solve the system of two equations for two unknowns.

Here, we'll use elimination:

$$\begin{aligned} 4x + 2y &= 10 \\ -2y + x &= 0 \\ \Rightarrow 5x &= 10 \\ x &= 2 \end{aligned}$$

Doing a bit of algebra yields us with the following:

$$y = 1$$

So, we end up with the critical point $(2,1)$.

Now, we have to find the absolute extrema candidates on the bounds (which we'll infer from the problem statement)!

1. $y \leq 5 - x^2$
2. $y \geq 0$

For 1:

$$\begin{aligned} f(x, 5-x^2) &= x^2 - (5-x^2)^2 + x(5-x^2) - 5x \\ &= -x^4 - x^3 + 11x^2 - 25 \end{aligned}$$

Find the critical values for 1:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x, 5-x^2)] = -4x^3 - 3x^2 + 22x = 0$$

$$\Rightarrow x \in \left\{-\frac{11}{4}, 0, 2\right\}$$

And their implied points:

$$\Rightarrow \left\{\left(-\frac{11}{4}, -\frac{41}{16}\right), (0,5), (2,1)\right\}$$

Note that we can omit $\left(-\frac{11}{4}, -\frac{41}{16}\right)$ because it provides a y-value that does not follow relation #2. We can also ignore the repeated $(2,1)$ because it already appeared when we used partial derivatives to find the critial point of f.

Now for 2:

$$f(x, 0) = x^2 - 5x$$

Find its critical values:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x, 0)] = 2x - 5 = 0$$

$$\Rightarrow x = \frac{5}{2}$$

And its implied point:

$$\Rightarrow \left(\frac{5}{2}, 0\right)$$

Then, find the corner points:

$$0 = 5 - x^2 \Rightarrow x \in \{-\sqrt{5}, \sqrt{5}\} \Rightarrow \{(-\sqrt{5}, 0), (\sqrt{5}, 0)\}$$

And, finally, we must compare the f values for all found points:

$$\begin{aligned} f(2,1) &= -5 \\ f(0,5) &= -25 \\ f\left(\frac{5}{2}, 0\right) &= -\frac{25}{4} \approx -6.25 \\ f(-\sqrt{5}, 0) &= 5 + 5\sqrt{5} \approx 16.18034 \\ f(\sqrt{5}, 0) &= 5 - 5\sqrt{5} \approx -6.18034 \end{aligned}$$

The amount of work that went into this problem is absurd, but that's fine. In the end, we find that the absolute minimum of f is -25 and the absolute maximum is $5 + 5\sqrt{5}$, or 16.18034.

Now wait a second.. what would happen if the 2D region describing the input of the 3D function.. is not bounded? Obviously, we can't apply the Extreme Value Theorem in $\mathbb{R}^3$ in this case.. unless you know how to test all possible points on an unbounded region! Which, unfortunately, you can't.

Instead, read on...

Section 3: The Method of Lagrange Multipliers

Chapter 6

Another Order

Delightful! Joe greatly enjoyed the addition of meat- the piquant umami was a new experience for his buds.

Despite already heightening his satisfaction twice, Joe was yet again deciding on another combination of a meat-topped ice cream. This time, his only constraint is that he wants the umami flavor to be inversely proportional to half the sweetness felt.

> **Example 4.** Given that his satisfaction can again be represented by eq. (3.1), Joe desires for a nonnegative amount of each flavor and wants to try a combination where the flavor of umami he attains is inversely proportional to half the sweetness.
>
> With what combination of (u, s) can Joe attain his maximum satisfaction?

Chapter 7

A Joe Analysis

Chapter 8

Metonymization, Part 2

Now, for what this project was all about, let's apply Joe's strategy to a general example...

> **Example 5.** Use the method of Lagrange multipliers to find the maximum of $2x - y$ given the following constraint:
>
> $$3x^2 - 4xy + 2y^2 = 6$$
>
> Also, find the point (x, y) where said maximum is achieved.

Section 4: The Cobb-Douglas Production Function

Chapter 9

Carl's Parlor

After Joe decided to buy everything from Carl's Parlor, Carl-Friedrich decided to not accept anymore customers until the ice cream and chicken dispensers were refilled. Also, Carl was the only worker at his ice cream shop, so he is interested in hiring more workers.

To financially plan these business plans, Carl viewed a couple YouTube videos on the **Cobb-Douglas production function**.

Remark. The Cobb-Douglas production function is an economics concept that relates a firm's production output Y in terms of two variable inputs, labor (L) and capital (K).

The production function can be written as:

$$Y(L, K) = AL^{\alpha}K^{\beta}$$

Where..

- A is a constant for †*total-factor productivity*,
- α is a constant for †*output elasticity of labor*,
- and β is a constant for †*output elasticity of capital*.

† For the sake of keeping this paper focused on constraining the Cobb-Douglas production function, we will not derive these economic concepts. For the following example problems, we will make the following simplification:

$$\beta = 1 - \alpha$$

This derives from the economic concept of elasticity and how $\alpha + \beta = 1$ in this context. Note that we will not further explain the derivation of this in our paper.

Right now, Carl is eyeing some refurbished multi-purpose dispenser machines with individual maintenance costs at around $\$1,500$ per year. He is also looking to hire workers that will have to be paid $\$30,000$ a year.

Carl has rich parents, but he still has a budget. He wants to spend no more than $\$100,000$ a year on his ice cream parlor. In Carl's world, ice cream parlors can be found to have a production function similar to:

$$Y(L, K) = L^{0.35}K^{0.65} \tag{9.1}$$

Example 6. With the production function given above, Carl wants to find an optimal combination of capital and labor that maximizes his parlor's output while also considering his budget.

Can you help him find such combination?

Chapter 10

Money-Mouth Face Emoji

If there are any constraints on the variables L and K, like a budget, then the method of Lagrange multipliers can be utilized to find the specific quantities of capital and labor that maximizes output $Y(L, K)$ for constraint functions.

Chapter 11

Metonymization, Part 3

As shown in the above example, the method of Lagrange multipliers can be applied to the Cobb-Douglas production function to find the specific quantities of labor and capital that maximizes production output for given constraints on both labor and capital.

To relate this to a more general case, here's another example!

Example 7. Suppose you are given a specific Cobb-Douglas function

$$f(x, y) = 50x^{0.4}y^{0.6}$$

where x is the dollar amount spent on labor and y the dollar amount spent on equipment.

Use the method of Lagrange multipliers to determine how much should be spend on labor and how much on equipment to maximize productivity if we have a total of 1.5 million dollars to invest in labor and equipment.

We believe there could be a slight logic error in the example (see Ch. 12, Concluding Remarks). But because we were given this example problem as a requirement for this paper, we kept it ad litteram.

Section 5: The End Stuff

Chapter 12

Concluding Remarks

"Critique my LaTeX.. and my English." - **Aaron**
"Theory can only take you so far..." - **Kerem**