DE LA RECHERCHE À L'INDUSTRIE

New Acquisition strategies for Compressed Sensing in Magnetic Resonance Imaging

Philippe Ciuciu

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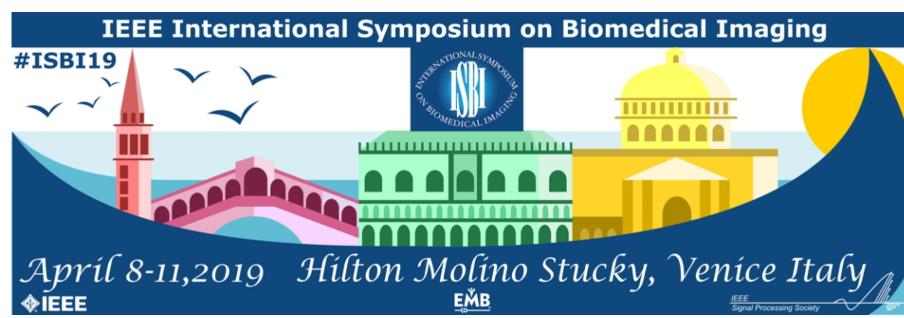

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Venice, Italy

www.cea.fr

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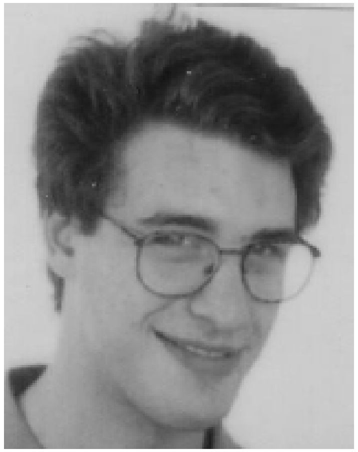
Jonas Kahn
CNRS/IMT



Outline

Part I: Background in Magnetic Resonance Imaging

- Sampling k-space & Cartesian reconstruction
- Trajectories and acquisition strategies

Part II: Compressed Sensing in MRI

- Evolution of Compressed Sensing
- (CS with blocks of measurements)

Part III: CS sampling trajectories

- Continuous Variable Density Samplers
- Image stippling techniques
- SPARKLING

Online Material

Code for MRI sampling available for reproducible research

- 7 Jupyter notebooks (Python 3.5) for Part I-II
- Matlab code for Part II (TSP+ Markov)
- Sparkling code for Part III *not yet* provided

Code for image reconstruction in Fourier imaging

- PySAP python package developed at CEA
- Off-line discussion for PR contributions, installation issues

Test data

- Numerical phantoms at different image resolution
- Real T2*-w images collected at 7 Tesla at NeuroSpin
- Sparkling trajectories provided for demos

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Part I: Background in Magnetic Resonance Imaging

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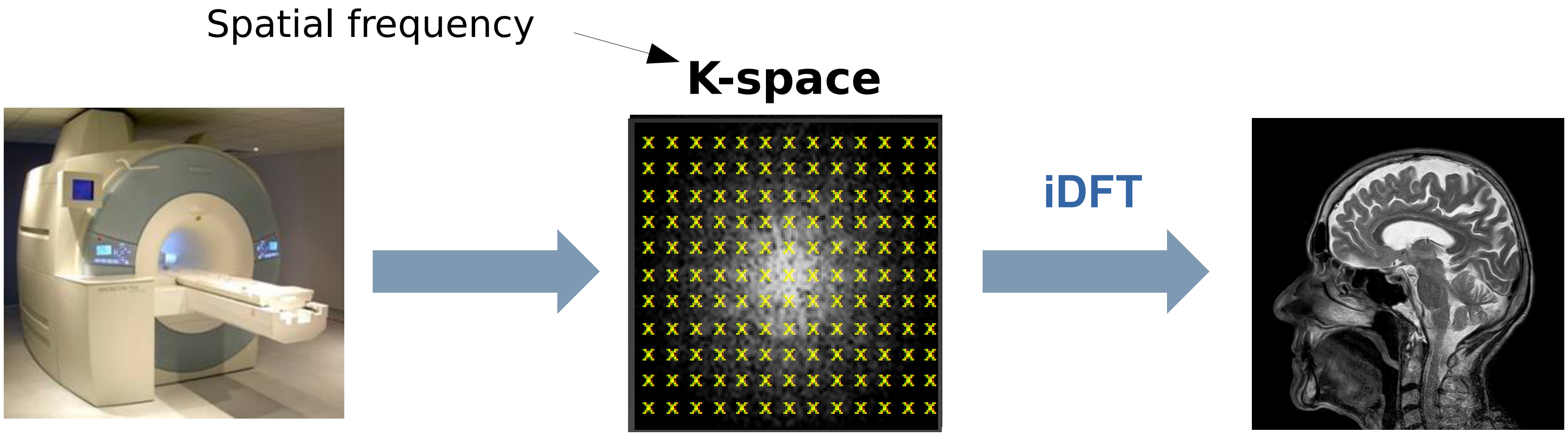
Sampling in MRI



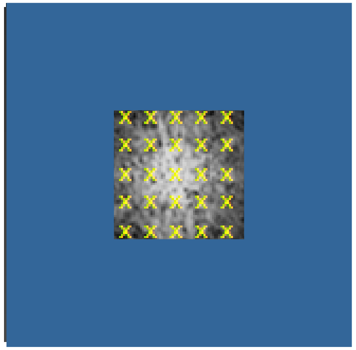
Perfect reconstruction of an object would require measurement of *all* locations
In k-space (infinite!)

Data is acquired **point-by-point** in k-space (sampling) along **curves**

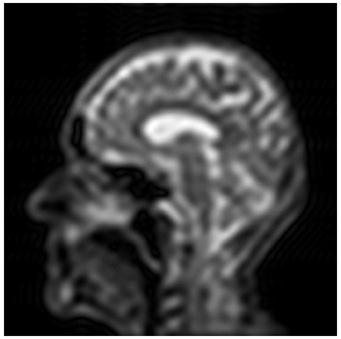
Sampling in MRI

Spatial frequency

K-space



iDFT



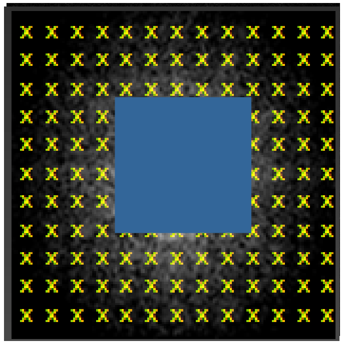
Low frequencies = Contrast

Go to 1st demo/notebook

Sampling in MRI

Spatial frequency

K-space



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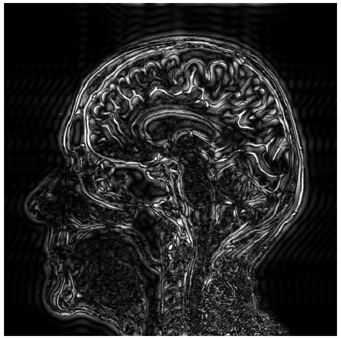
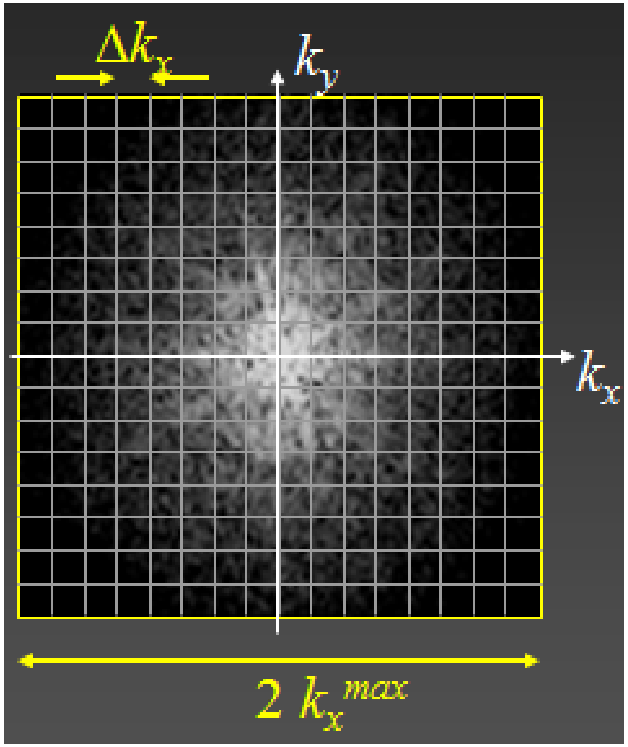


High frequencies = boundaries/edges

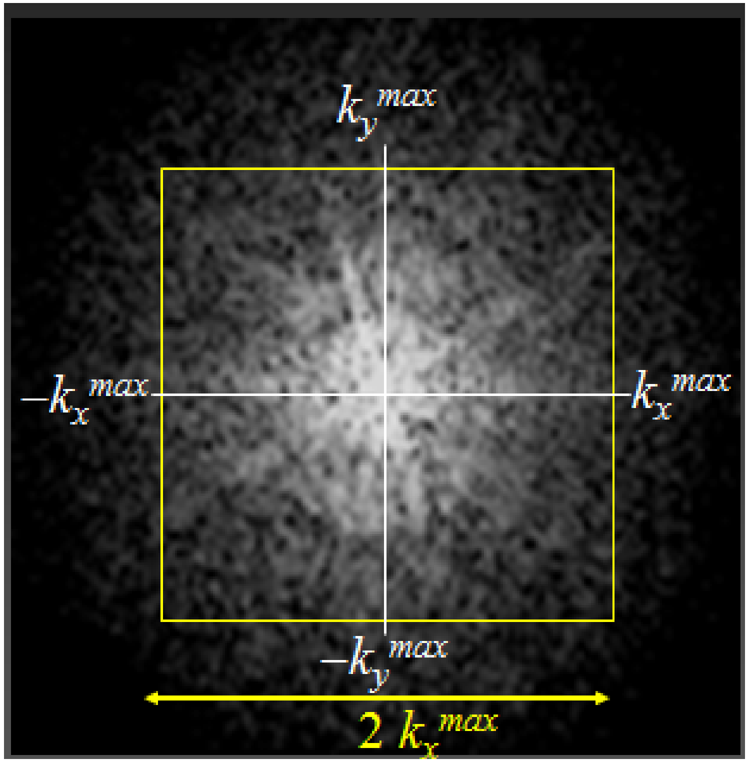
Go to 1st demo/notebook

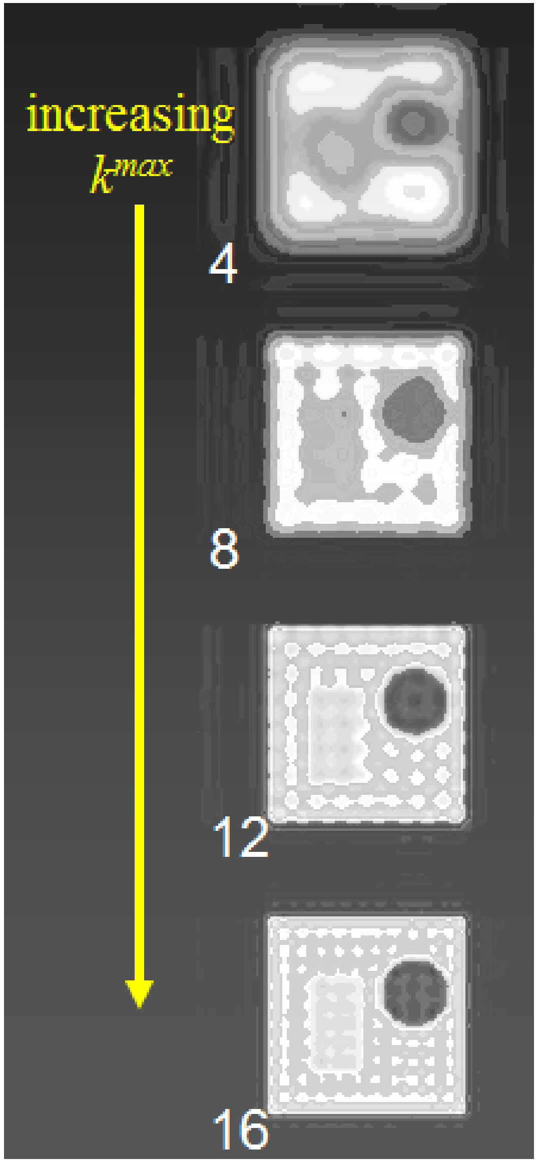
Sampling the k-space



1. What is the highest frequency we need to sample in *k*-space (k^{max})?
2. How close should the samples be in *k*-space (Δk)?

Choosing the maximal frequency





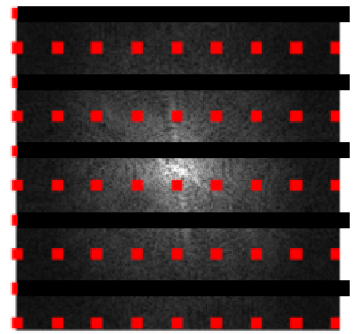
k^{max} determines image resolution

Large k^{max} means high resolution!

Go to 1st demo/notebook

Nyquist Sampling Theorem

K-space



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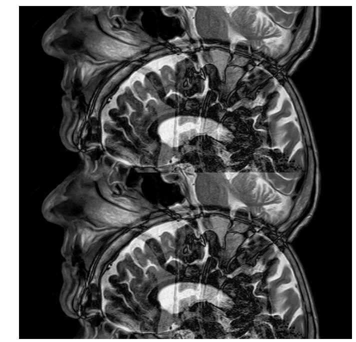


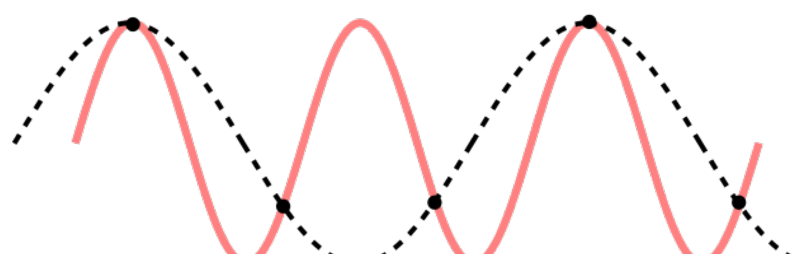
Nyquist-Shannon theory

$\uparrow$ resolution $\Rightarrow$ $\uparrow$ #samples

Long acquisition times

The sampling frequency should be at least twice the highest frequency contained in the signal

Harry Nyquist



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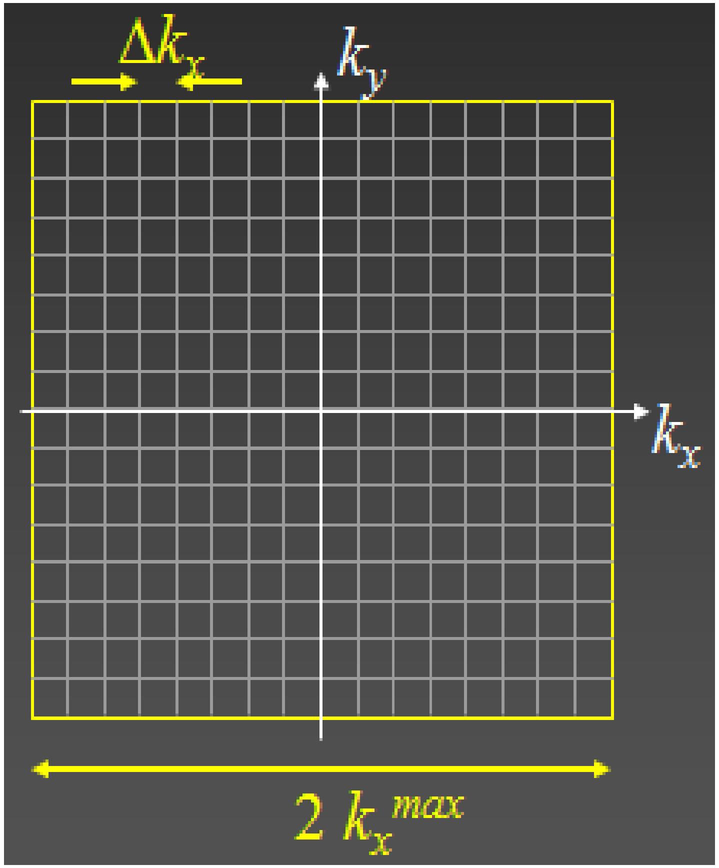
Nyquist Sampling Theorem

$$\mathrm{FOV}_x = \frac{1}{\Delta k_x}$$

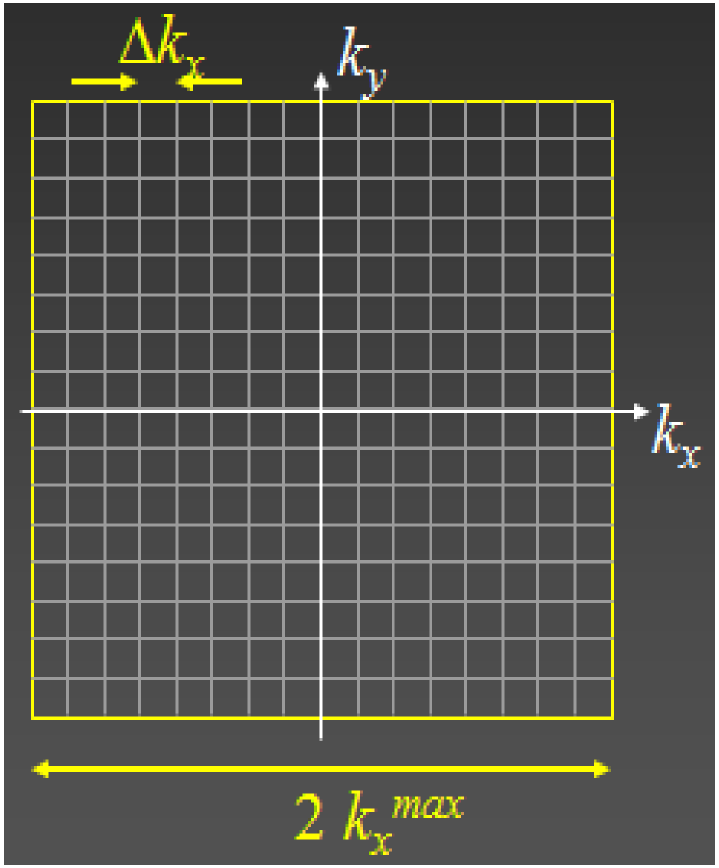
$$\Delta x = \frac{1}{2k_x^{\max}}$$



Nyquist Sampling Theorem

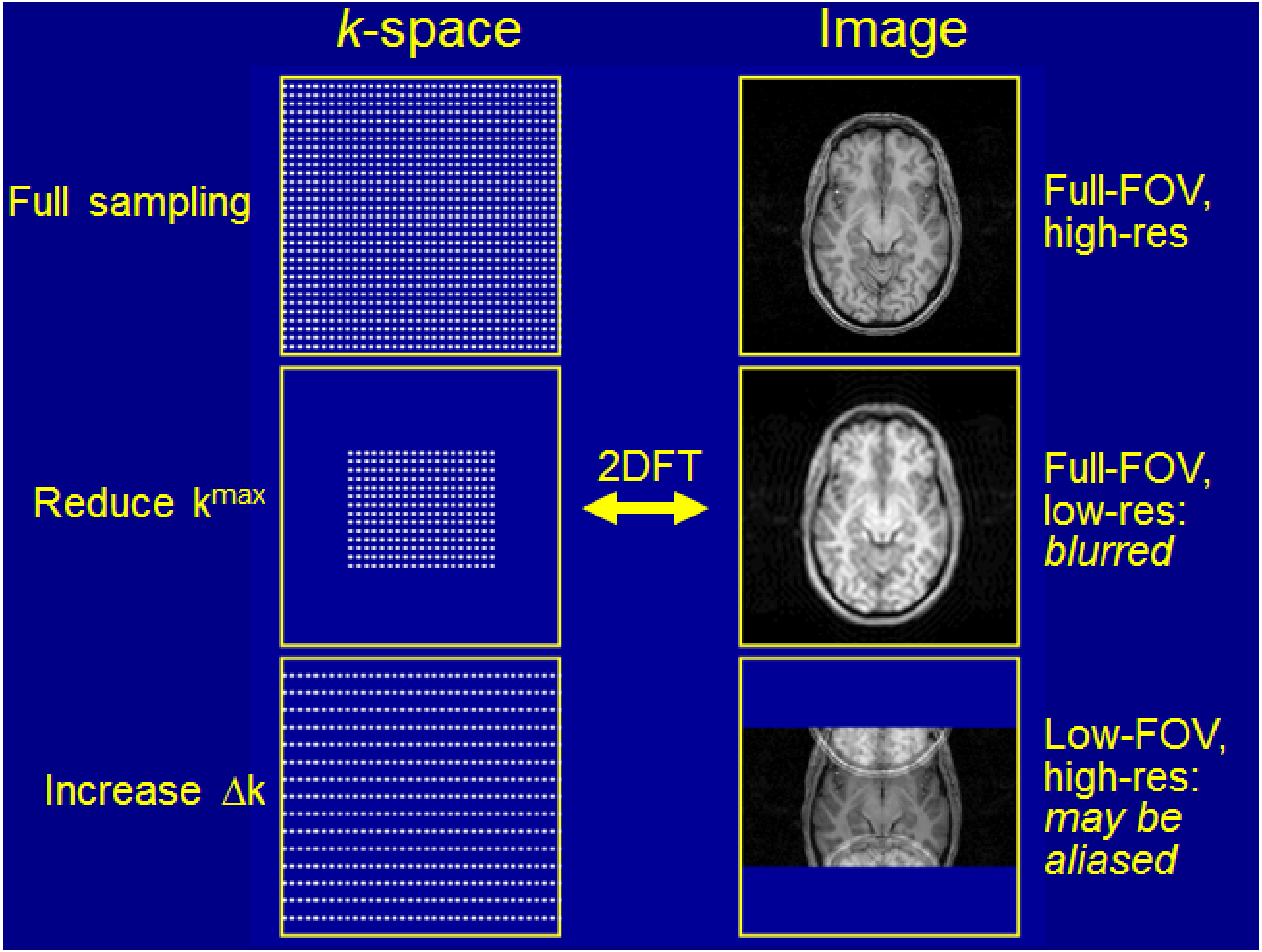
$$x_{\max} = \frac{1}{\Delta k_x}$$

$$2k_x^{\max} = \frac{1}{\Delta x}$$



k-space and image resolution are inversely related:
resolution in one domain determines extent in other.

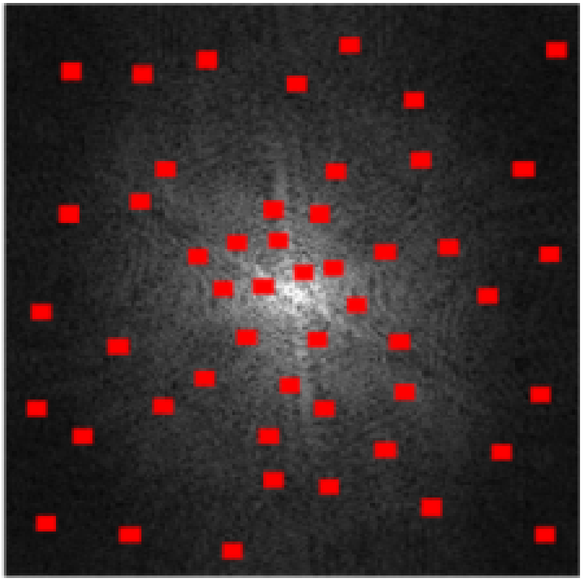
Nyquist Sampling Theorem



Go to 3rd notebook

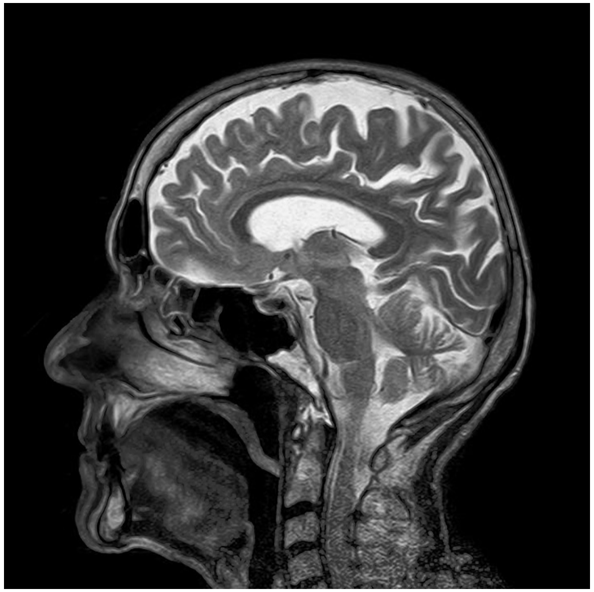
Beyond Nyquist Sampling Theorem

k-space



Nonlinear reconstructions

Image



Go to 2nd notebook

[Lustig et al, Magn Reson Med 2007]

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Part I: Background in Magnetic Resonance Imaging

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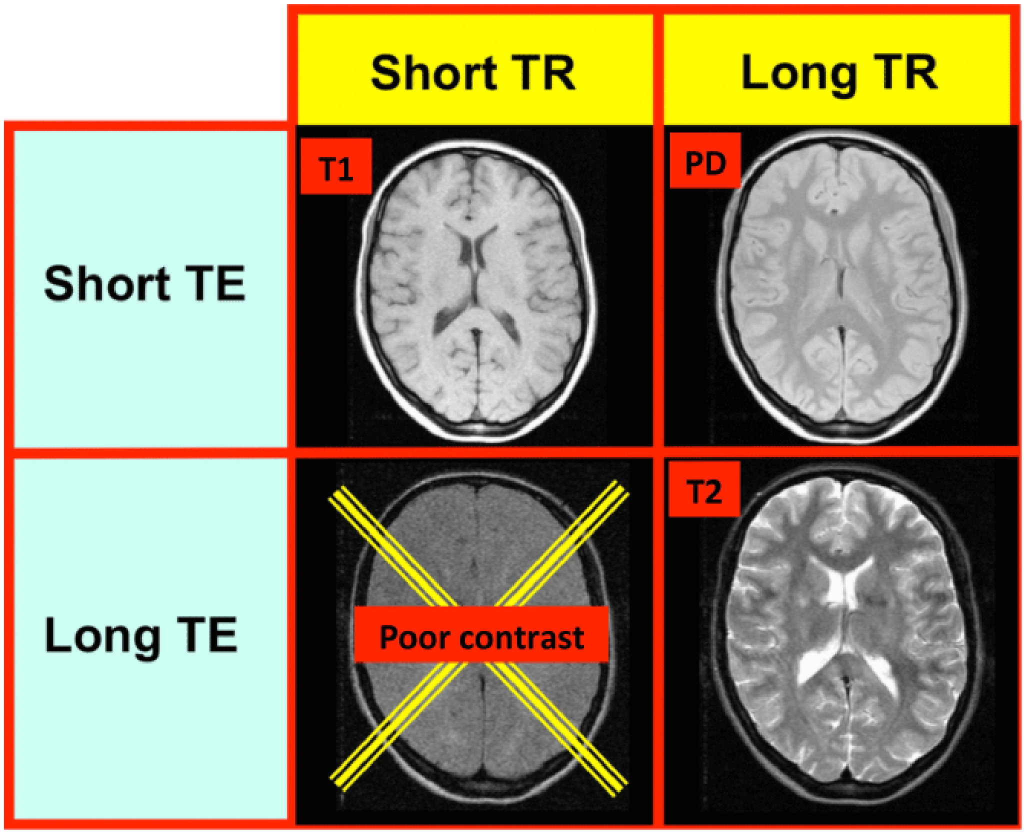
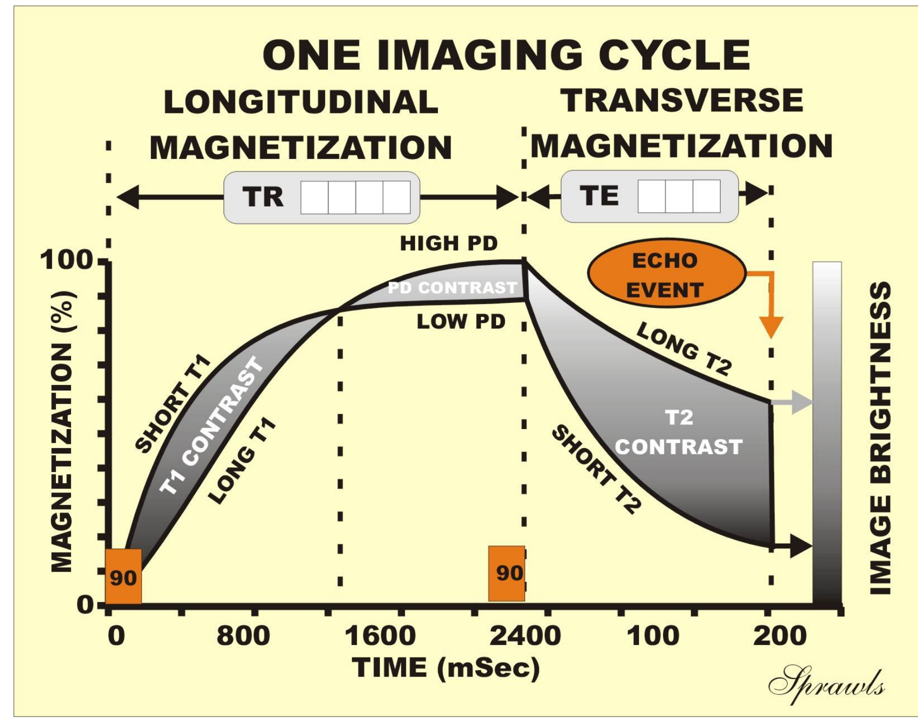
Short background in MRI

MR is versatile: multiple imaging contrasts





Simplified Bloch equations:

- Longitudinal magnetization:

$$M_z(t) \simeq M_0(1 - e^{-t/T_1})$$

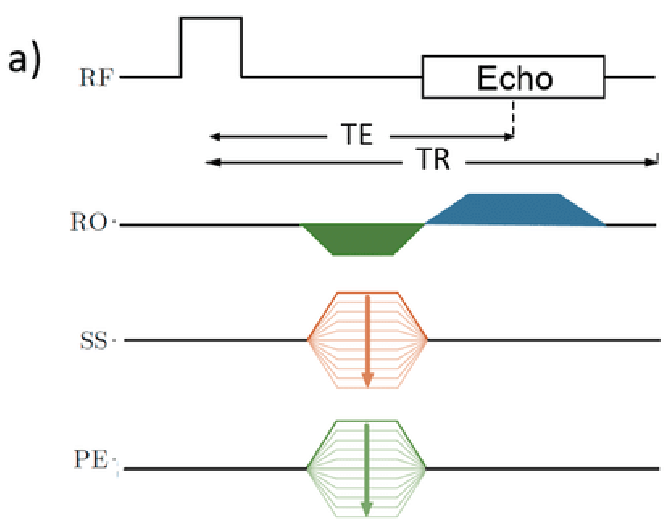
- Transverse magnetization:

$$M_{xy}(t) = M_0\, e^{-t/T_2}$$

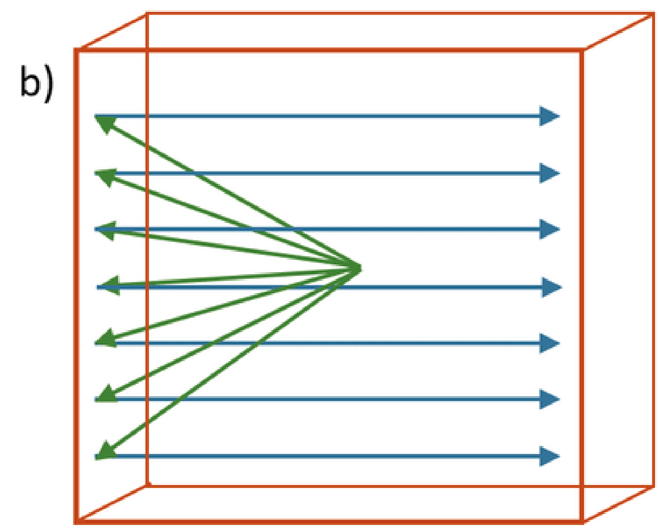
The MR signal has a short lifespan

Segmented Acquisition in MRI

Chronogram pulse sequence



K-space sampling



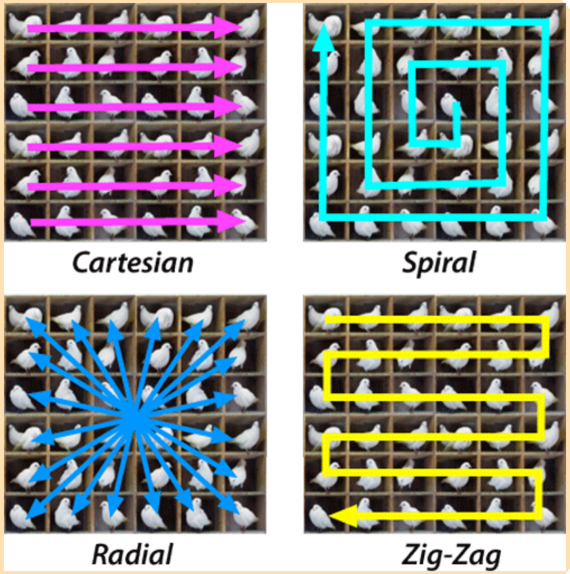
TR: Time between 2 successive RF pulses

TE: time point associated with the center of echo: $k_x(TE) = 0$

T_{obs} or readout: length of each segment/shot

Sampling Trajectories

Sampling in MRI: $k(t) = k(0) + \gamma \int_0^t G(\tau) d\tau$



- **Segmented acquisition**:
  Scan time proportional to number of shots

- **Hardware constraints on gradients**:
  $G_{max} < 40$ mT/m ; $S_{max} < 200$ T/m/s
  ➔ bounded velocity and acceleration

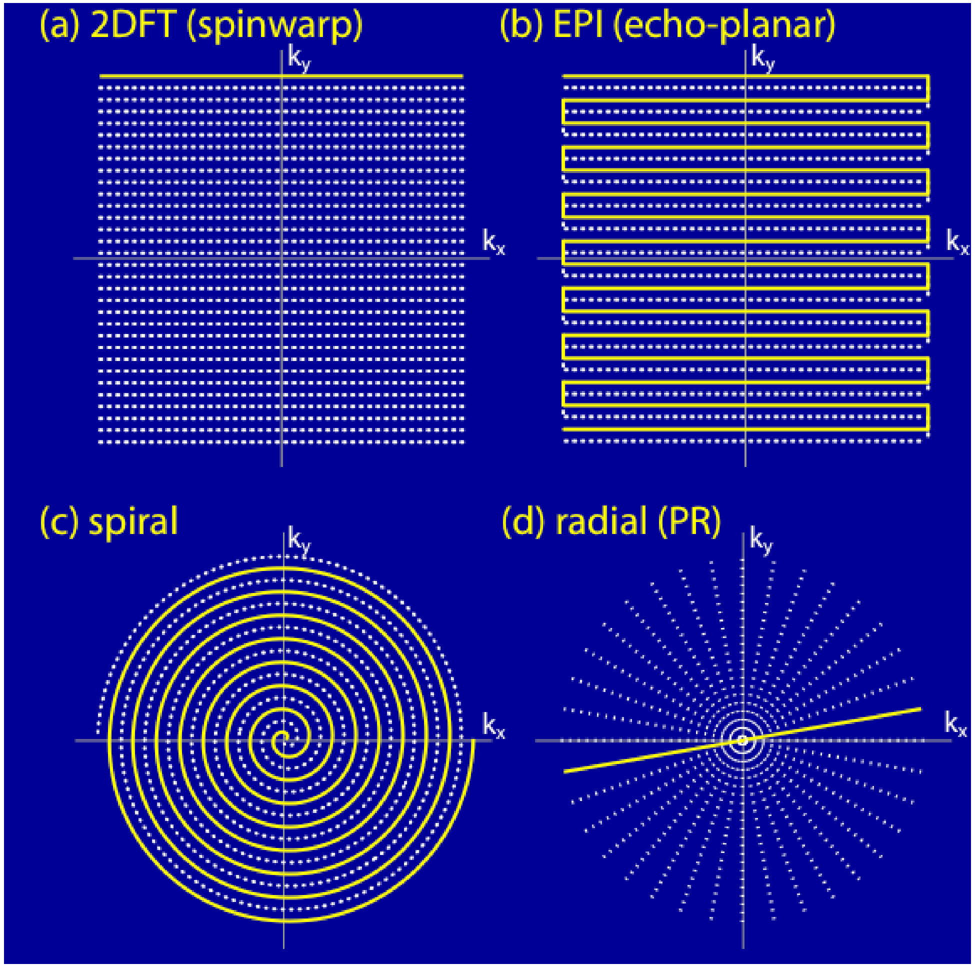
Let $\boldsymbol{k} : [0, T] \to \mathbb{R}^d, (d = 2, 3)$ denote the sampling curve:

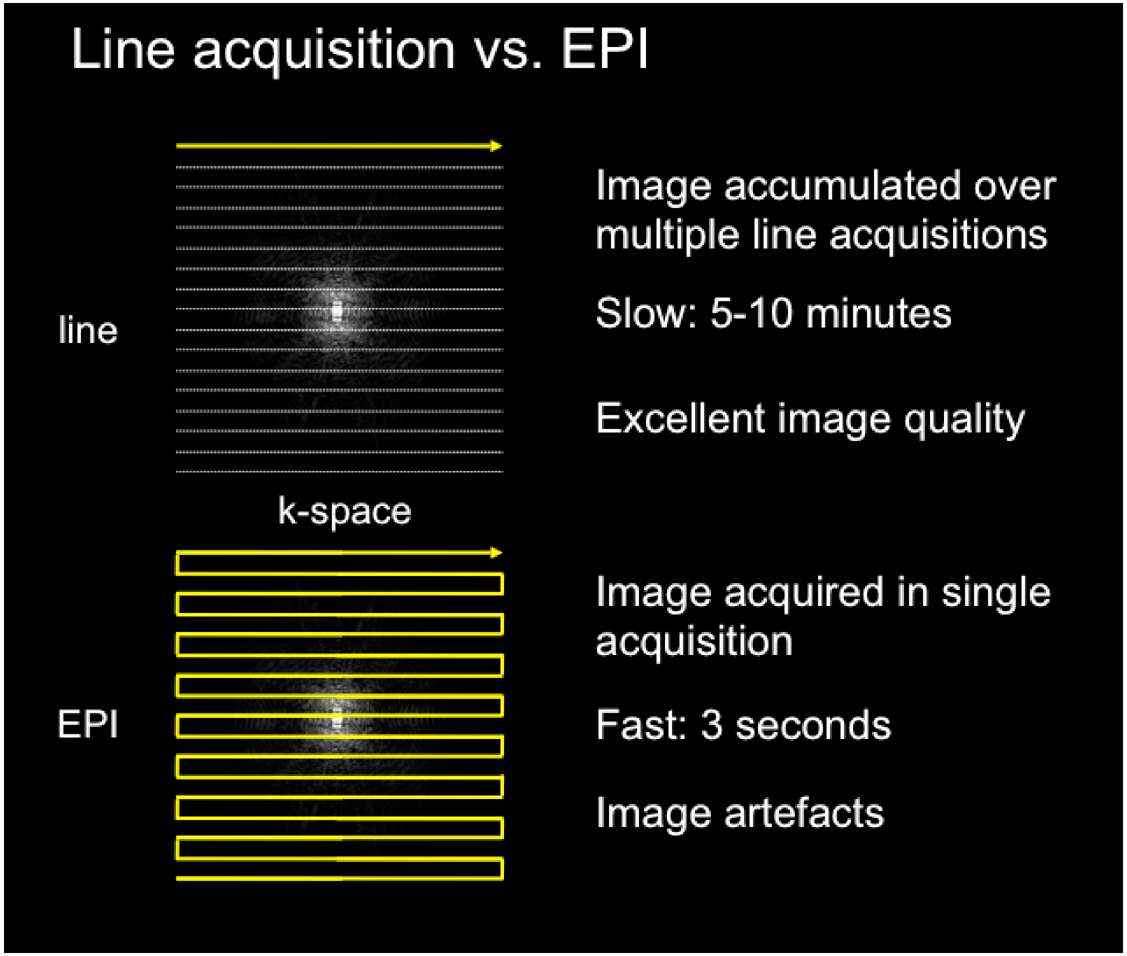
$$\boldsymbol{k}(t) = \boldsymbol{k}(0) + \gamma \int_0^t \boldsymbol{G}(\tau) d\tau \qquad d = 2 \to \boldsymbol{k} = (k_x, k_y), \ \boldsymbol{G} = (G_x, G_y)$$

$$\mathcal{S} = \begin{cases} \|\dot{\boldsymbol{k}}\|_{2,\infty} & < & \gamma G_{\max} & \text{(bounded speed)} \\ \|\ddot{\boldsymbol{k}}\|_{2,\infty} & < & \gamma S_{\max} & \text{(bounded acceleration)} \end{cases} \quad \text{where } \|\boldsymbol{c}\|_{2,\infty} = \sup_{1 \le i \le p} \left(|\boldsymbol{c}_x[i]|^2 + |\boldsymbol{c}_y[i]|^2 \right)^{1/2}$$

K-space location is proportional to accumulated area under gradient waveforms

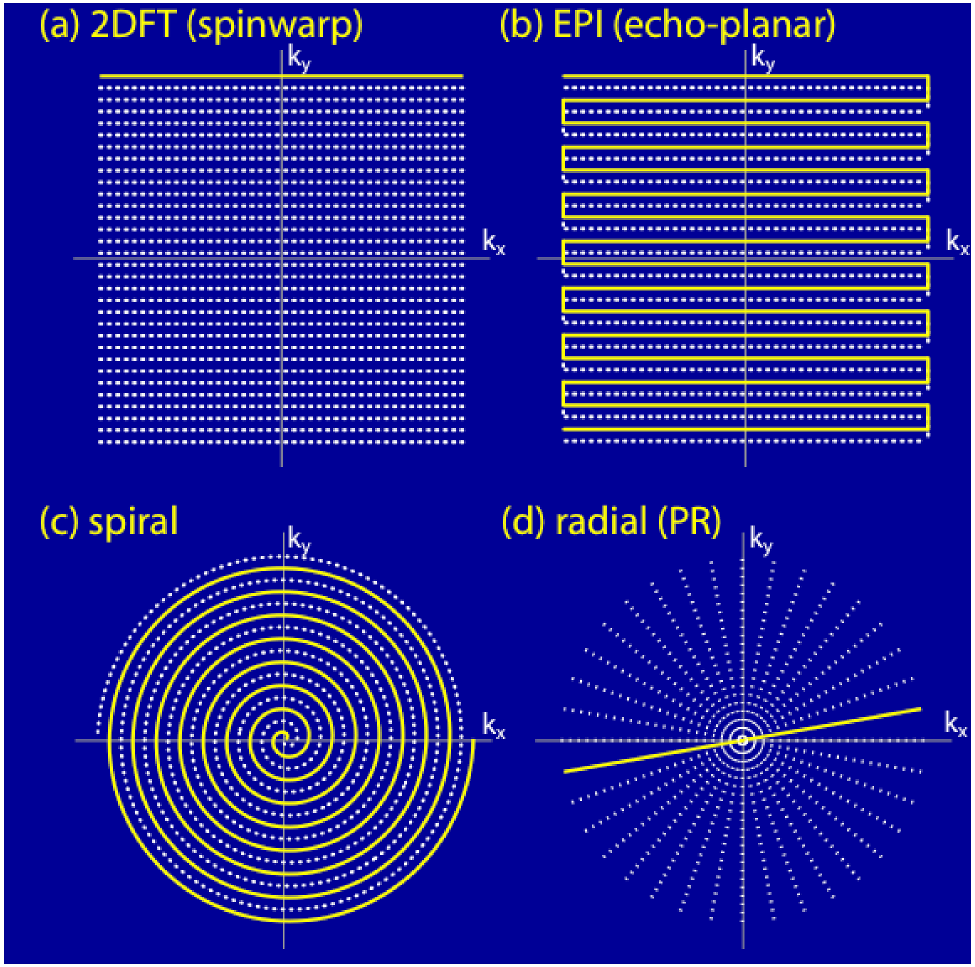
Examples of k-space Trajectories

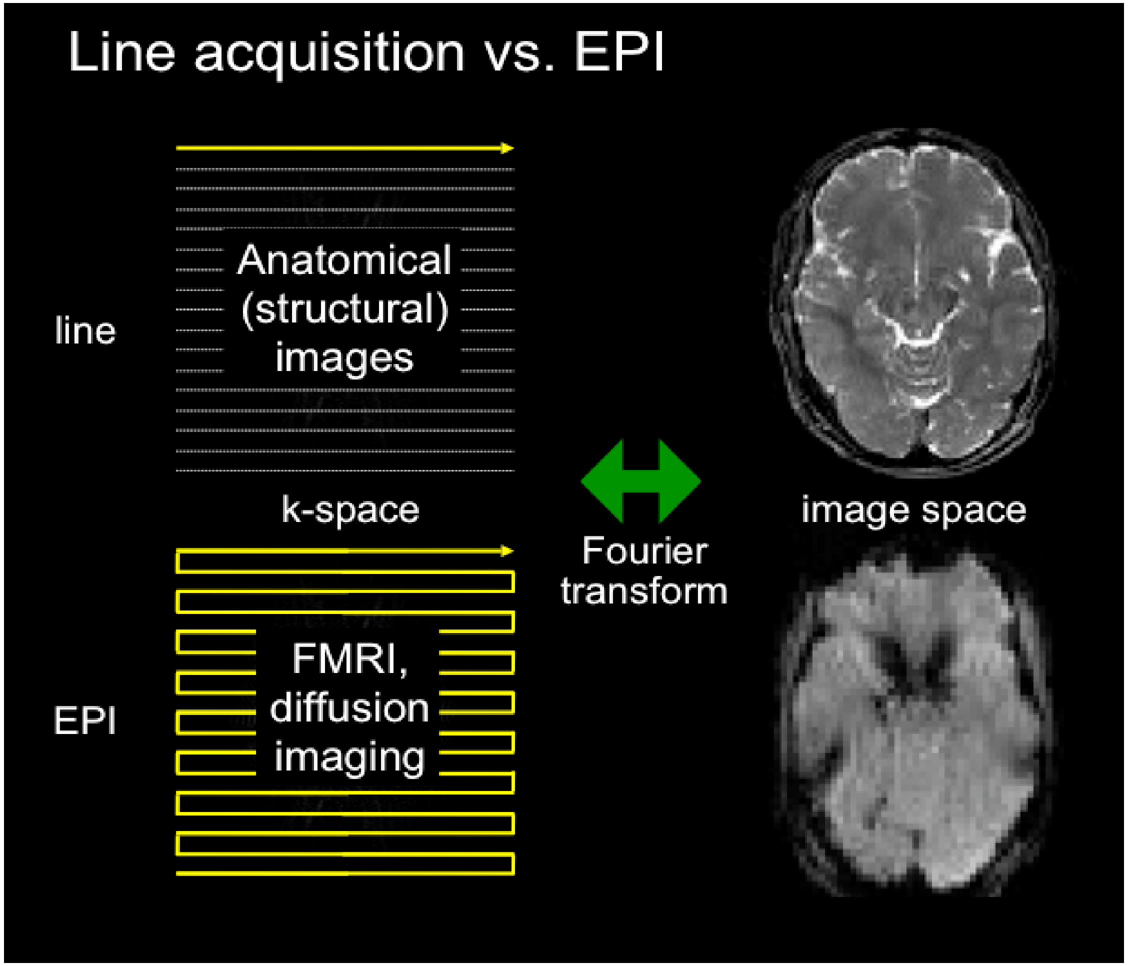




Go to the 5th and 6th demos

Examples of k-space Trajectories





Interim Summary

- **Cartesian vs non-Cartesian trajectories**
  - **Cartesian**: easier to implement less prone to gradient errors, play one gradient at a time
  - **Non-cartesian**: robust to motion, more flexible sampling, prone to gradient errors
  - The time matters: the same sampling schemes can be played in different ways (e.g. radial in-out vs radial-out)
- **Single shot vs multi-shot**
  - Multiple shot = longer acquisition times, better image quality
  - Single shot = faster acquisition, e.g. for dynamic MR imaging mainly
- **MRI acquisition constraints**
  - T_1-w: Short TE, TR → short readout and thus straight lines (Cartesian or radial) are unavoidable
  - T_2, T_2^*-w : long TE, TR → enable long readout → more fancy trajectories are feasible

Outline

Part II: Compressed Sensing in MRI

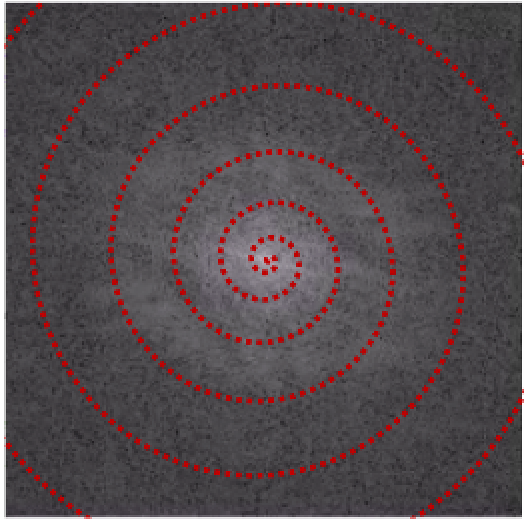
- Evolution of Compressed Sensing
- CS with blocks of measurements

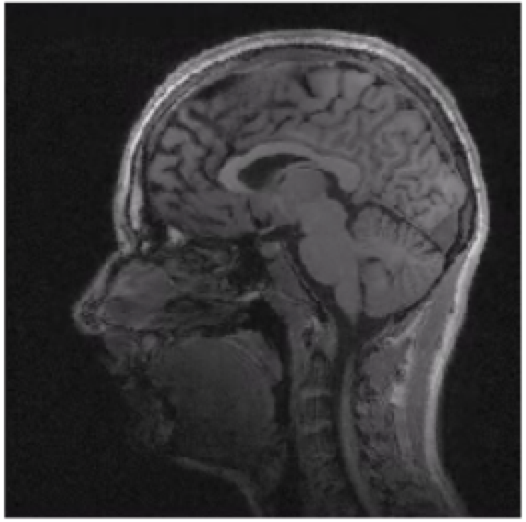
Modeling of k-space Trajectories

Sampling the curve $\boldsymbol{k}(t)$ generates a set of measurements:

$$E(\boldsymbol{k}) = \{\hat{x}(\boldsymbol{k}(p\Delta t))\}_{p\in\{0,\ldots,T/(\Delta t)\}}$$

with $\hat{x}$ the Fourier transform of x





Challenges

- **Questions in sampling theory**
  - How to choose the set of measurements E?
  - How to generate a set of segmented trajectories $\boldsymbol{k}$?
  - How to reconstruct x given E?

Jeff Fessler's part!

- **Practical consequences**
  - Shorten acquisition times
  - Increase image resolution
  - Improve medical diagnosis and reduce exam costs

A fundational result

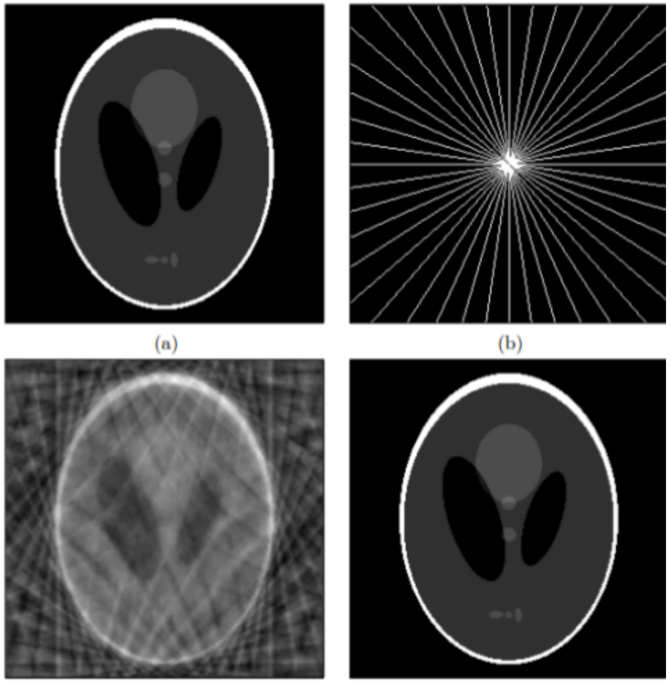
Supporting Thorem [Candes, Romberg and Tao, IEEE IT 2006]

Let $\boldsymbol{x} \in \mathbb{C}^n$ denote an s-sparse signal. Samples m values $y_k = \hat{\boldsymbol{x}}(J_k)$ where indexes $(J_k)_{1 \leq k \leq m}$ are **i.i.d. and uniformly distributed**.
If

$$m \geq C \cdot s \cdot \log(n)$$

$\boldsymbol{x}$ can be recovered **exactly**, with high probability by solving:

$$\min_{\boldsymbol{x} \in \mathbb{C}^n, \hat{\boldsymbol{x}}(J_k) = y_k} \|\boldsymbol{x}\|_1$$



As the measurements are not *independent*, this theorem does not explain this result!

Preliminaries

Wavelet expansions

- Let $\boldsymbol{\Psi} = (\Psi, \ldots, \Psi_n) \in \mathbb{C}^{n\times n}$denote an orthogonal transform. The image to reconstruct writes:

$$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{z} = \sum_{i=1}^{n} z_i \Psi_i \quad \text{where} \quad z_i = \langle \boldsymbol{x}, \Psi_i \rangle$$

- Objective: retrieve $\boldsymbol{z}$ (not $\boldsymbol{x}$)

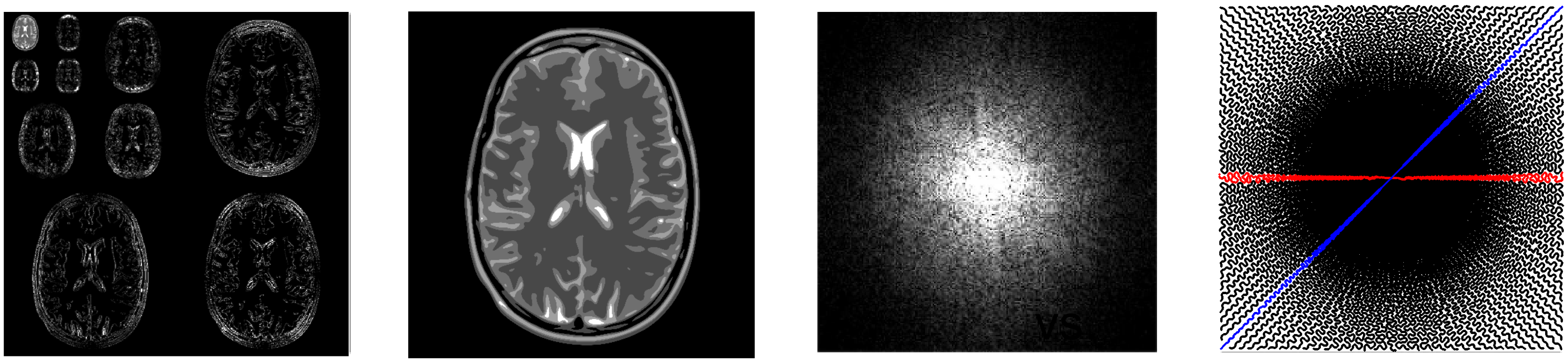
Fourier sampling

- Sensing matrix: $\boldsymbol{A}_0 = \boldsymbol{F}^*\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{a}_1^* \\ \boldsymbol{a}_2^* \\ \vdots \\ \boldsymbol{a}_n^* \end{pmatrix}$. One measurement reads $y_k = \langle \boldsymbol{a}_k, \boldsymbol{z} \rangle$
- Let $\Omega \subseteq \{1, \cdots, n\}$ and $\boldsymbol{A}_\Omega = (\boldsymbol{a}_i^*)_{i\in\Omega}$ the sampling matrix. We get:

$$\boldsymbol{y} = \boldsymbol{A}_\Omega \boldsymbol{z} \ (+\boldsymbol{b})$$

Preliminaries

$\boldsymbol{z}$ $\quad$ $\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{z}$ $\quad$ $\boldsymbol{F}^*\boldsymbol{\Psi}\boldsymbol{z} = \boldsymbol{A}\boldsymbol{z}$ $\quad$ $\boldsymbol{A}_\Omega \boldsymbol{z}$



Sparsity: Let $S = \{i, z_i \neq 0\}$ denote the support of $\boldsymbol{z}$.

We assume that: $|S| = s \ll n$

Nonlinear ℓ_1 reconstruction (synthesis formulation, noiseless case):

$$\widehat{\boldsymbol{z}} = \operatorname*{arg\,min}_{\boldsymbol{z} \in \mathbb{C}^n, \boldsymbol{A}_\Omega \boldsymbol{z} = \boldsymbol{y}} \|\boldsymbol{z}\|_1$$

Nonlinear ℓ_1 reconstruction (synthesis formulation, noisy case):

$$\widehat{\boldsymbol{z}} = \operatorname*{arg\,min}_{\boldsymbol{z} \in \mathbb{C}^n} \frac{1}{2} \|\boldsymbol{y} - \boldsymbol{A}_\Omega \boldsymbol{z}\|_2^2 + \lambda \|\boldsymbol{z}\|_1$$

$$\widehat{\boldsymbol{x}} = \boldsymbol{\Psi} \widehat{\boldsymbol{z}}$$

Evolution of Compressed Sensing

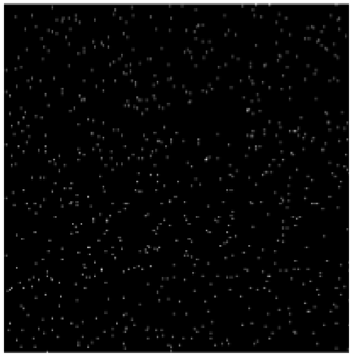
Uniform drawings: the early results

Coherence: $\kappa(\boldsymbol{A}_0) = n \max_{1 \leq k \leq n} \|\boldsymbol{a}_k\|_\infty^2$

Assume that $\boldsymbol{x}$ is s-sparse.

Draw m samples uniformly at random in $\{1, \ldots, n\}$.

$m \geq C \cdot \kappa(\boldsymbol{A}_0) \cdot s \cdot \log(\frac{n}{\epsilon}) \Rightarrow$ exact recovery with probability $1 - \epsilon$



Uniform sampling

[1] Emmanuel Candes and Terence Tao.
Near-optimal signal recovery from random projections: Universal encoding strategies?
Information Theory, IEEE Transactions on, 52(12):5406–5425, 2006.

[2] Emmanuel Candes and Yaniv Plan.
A probabilistic and ripless theory of compressed sensing.
Information Theory, IEEE Transactions on, 57(11):7235–7254, 2011.

[3] Simon Foucart and Holger Rauhut.
A mathematical introduction to compressive sensing.
Springer, 2013.

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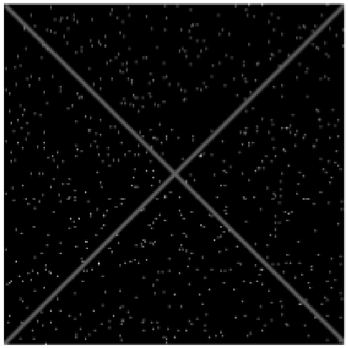

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Fourier-Wavelet system satisfies $\kappa(\boldsymbol{A}_0) \propto n$!

$\Rightarrow$ $m \gg n$

Coherence barrier!

Evolution of Compressed Sensing

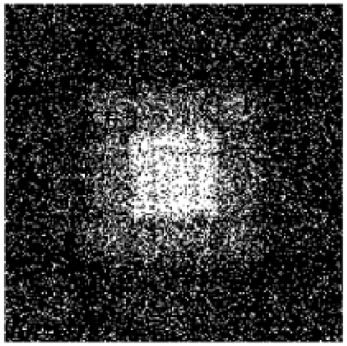
Variable density sampling

Local coherence: $\mu(\boldsymbol{A}_0) = \sum_{k=1}^{n} \|\boldsymbol{a}_k\|_\infty^2$.

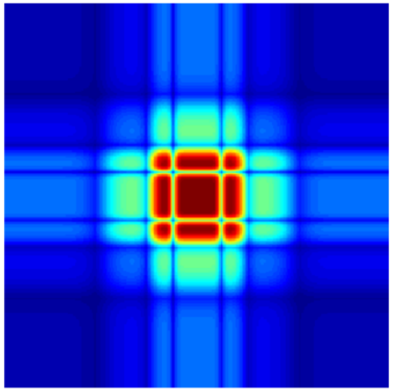
Assume that $\boldsymbol{x}$ is s-sparse.

Set $\boldsymbol{A}_\Omega = \begin{pmatrix} \boldsymbol{a}_{J_1} \\ \vdots \\ \boldsymbol{a}_{J_m} \end{pmatrix}$ with $\mathbb{P}(J_k = \ell) = \pi_\ell \propto \|\pi_\ell\|_\infty^2$

$m \geq C \cdot \mu(\boldsymbol{A}_0) \cdot s \cdot \log(\frac{n}{\epsilon}) \Rightarrow$ exact recovery with probability $1 - \epsilon$



Variable density sampling



π in 2D

[1] **Gilles Puy, Pierre Vandergheynst, and Yves Wiaux.**
On variable density compressive sampling.
Signal Processing Letters, IEEE, 18(10):595–598, 2011.

[2] **Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.**
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

[3] **Felix Krahmer and Rachel Ward.**
Stable and robust sampling strategies for compressive imaging.
Image Processing, IEEE Transactions on, 23(2):612–622, 2014.

Go to 2nd notebook

Evolution of Compressed Sensing

Variable density sampling

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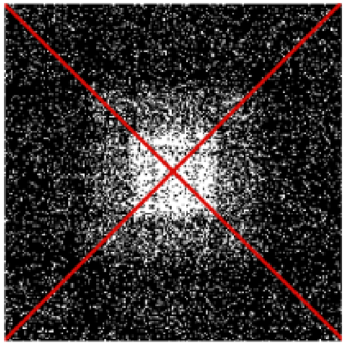

Variable density sampling

Fourier-Wavelet system satisfies $\mu(\boldsymbol{A}_0) \propto \log(n)$!

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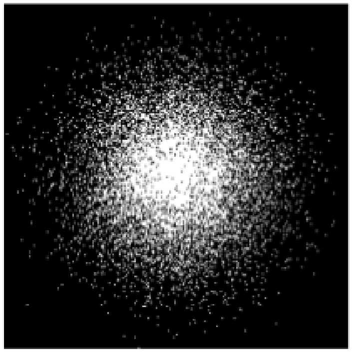
Evolution of Compressed Sensing

Variable density sampling with structure

Assume that $\boldsymbol{x}$ has a certain "sparsity structure".

Set $\boldsymbol{A}_\Omega = \begin{pmatrix} \boldsymbol{a}_{J_1} \\ \vdots \\ \boldsymbol{a}_{J_m} \end{pmatrix}$ with $\mathbb{P}(J_k = \ell) = \tilde{\pi}_\ell$

$m \geq C \cdot s \cdot \log(\frac{n}{\epsilon})$ $\Rightarrow$ exact recovery with probability $1 - \epsilon$



Variable density sampling + structure

[1] **Ben Adcock and Anders C Hansen.**
Generalized sampling and infinite-dimensional compressed sensing.
Foundations of Computational Mathematics, to appear, 2015.

[2] **Ben Adcock, Anders C. Hansen, Clarice Poon, and Bogdan Roman.**
Breaking the coherence barrier: A new theory for compressed sensing.
arXiv preprint arXiv:1302.0561, 2013.

[3] **Jérémie Bigot, Claire Boyer, and Pierre Weiss.**
Compressed sensing with structured sparsity and structured acquisition.
Applied and Computational Harmonic Analysis, 2017.

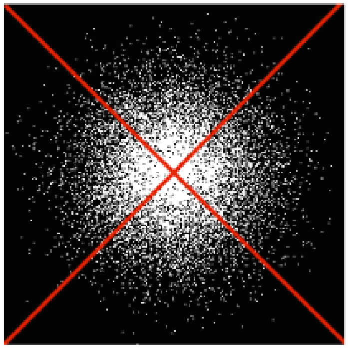
Evolution of Compressed Sensing

Variable density sampling with structure

Assume that $\boldsymbol{x}$ has a certain "sparsity structure".

Set $\boldsymbol{A}_\Omega = \begin{pmatrix} \boldsymbol{a}_{J_1} \\ \vdots \\ \boldsymbol{a}_{J_m} \end{pmatrix}$ with $\mathbb{P}(J_k = \ell) = \tilde{\pi}_\ell$

$m \geq C \cdot s \cdot \log(\frac{n}{\epsilon})$ $\Rightarrow$ exact recovery with probability $1 - \epsilon$



Variable density sampling + structure

[1] **Ben Adcock and Anders C Hansen.**
Generalized sampling and infinite-dimensional compressed sensing.
Foundations of Computational Mathematics, to appear, 2015.

[2] **Ben Adcock, Anders C. Hansen, Clarice Poon, and Bogdan Roman.**
Breaking the coherence barrier: A new theory for compressed sensing.
arXiv preprint arXiv:1302.0561, 2013.

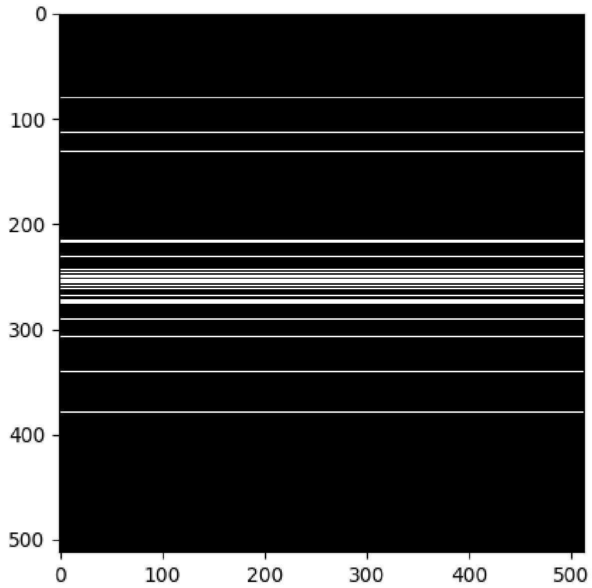
[3] **Jérémie Bigot, Claire Boyer, and Pierre Weiss.**
Compressed sensing with structured sparsity and structured acquisition.
Applied and Computational Harmonic Analysis, 2017.

Only works for isolated measurements!

Current compressed sensing theories are partially disconnected from applications!

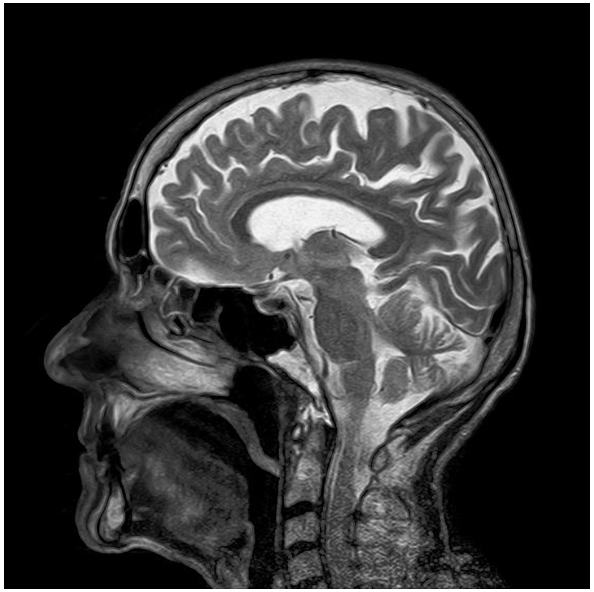
Possible 1D VDS implementation

k-space



Nonlinear reconstructions

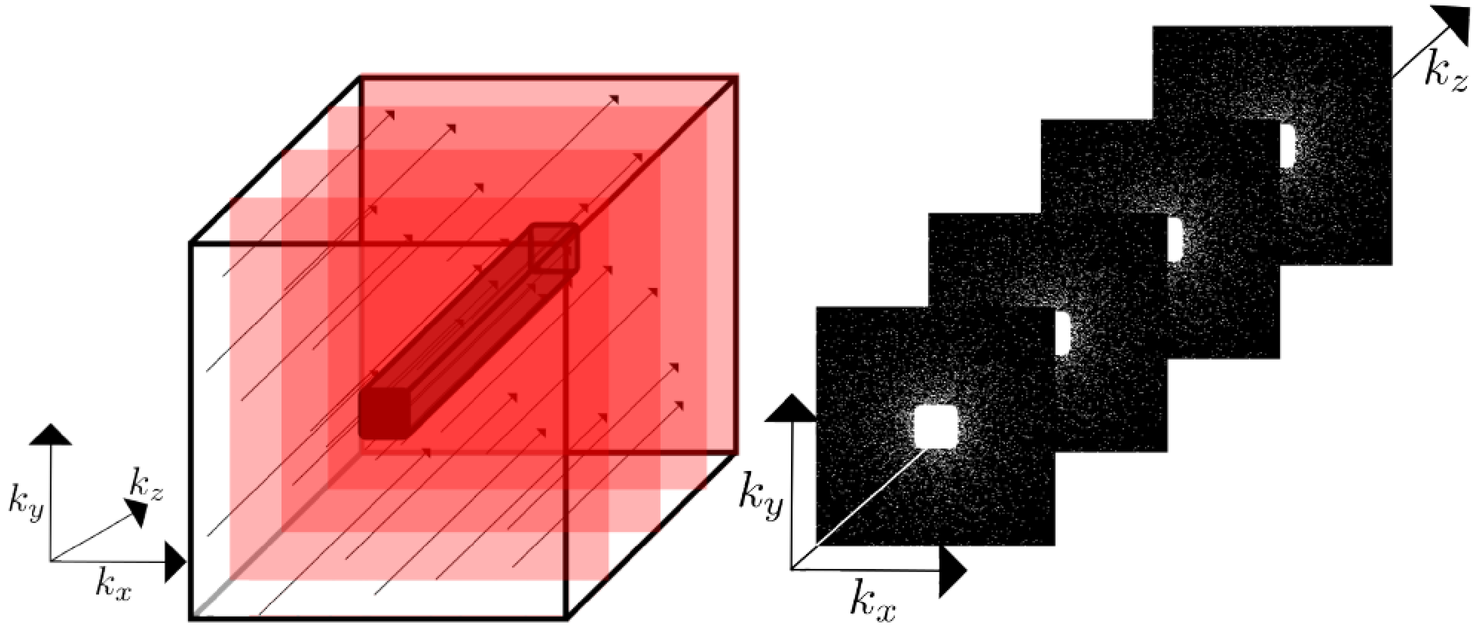
Image

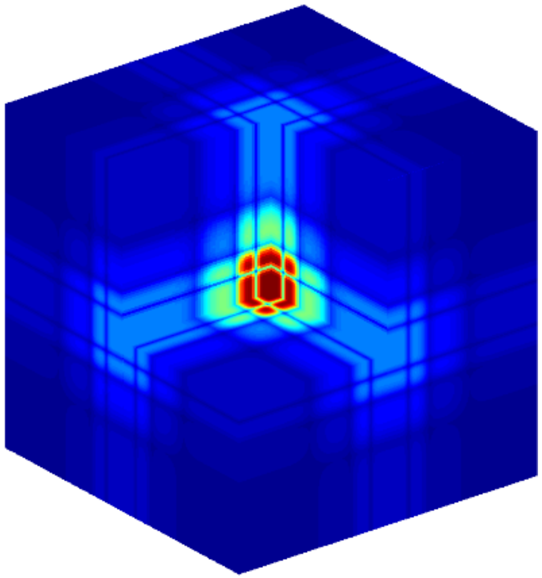


Go the 4th notebook

[Haldar et al, IEEE TMI, 2011]

Possible 2D VDS implementation





π in 3D

Perform the readout along the third dimension!
Suboptimal as compared to π in 3D.

[Lustig et al, MRM 2007, IEEE SP Mag 2008]

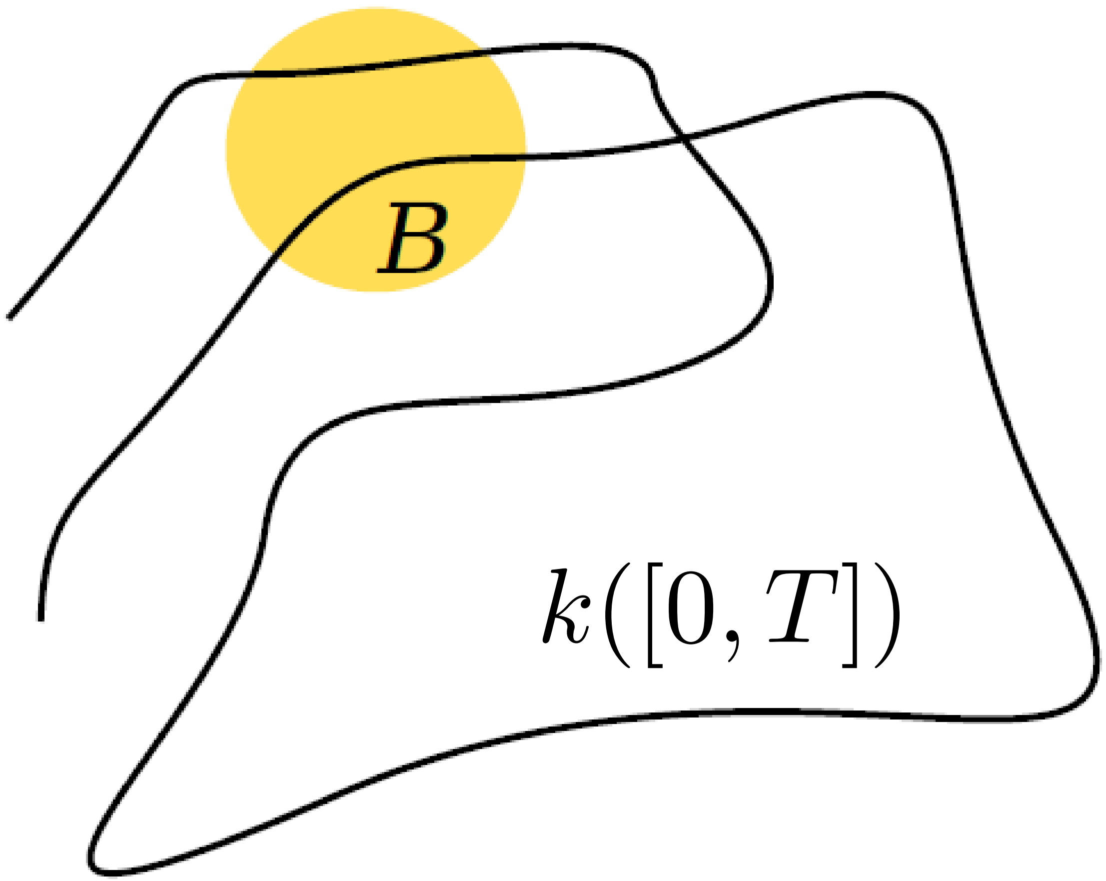
Outline

Part III: CS sampling trajectories

- Continuous Variable Density Samplers
- Image stippling techniques
- SPARKLING

Continuous Variable Density Sampler

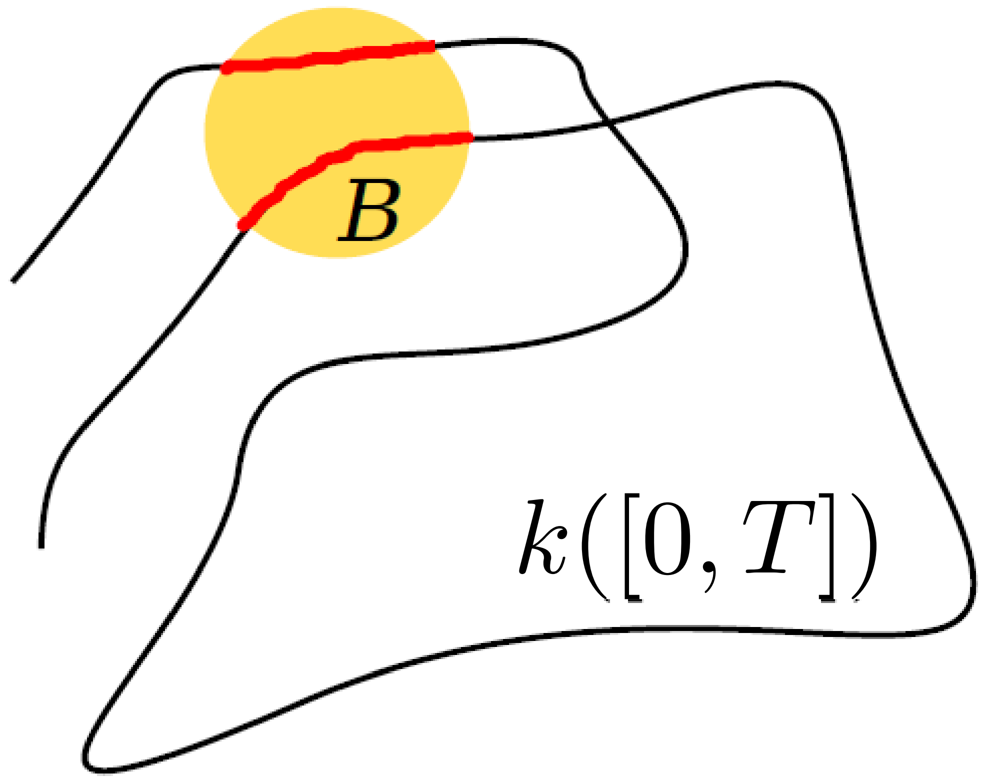
Pushforward measure - illustration



Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

Continuous Variable Density Sampler

Pushforward measure - illustration



$$\nu(B) = k_* \lambda_T(B) = \lambda_T(k^{-1}(B))$$

λ_T is the (normalized) Lebesgue measure.

Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

Push Forward Measure - Definition

Pushforward measure

Let $\Omega = [0,1]^d$, where $d = 2$ or 3 denote the space dimension. We equip Ω with the Borel algebra $\mathcal{B}$. Let (X, Σ) be a measurable space and $k : X \mapsto \Omega$ be a measurable mapping. $\mu : X \mapsto [0; +\infty)$ denote a measure. The *pushforward measure* ν of μ is defined by:

$$\nu(B) = k_* \mu(B) = \mu(k^{-1}(B)), \quad \forall \in \mathcal{B}$$

Ex. 1: Measures supported by curves

Ex. 2: Atomic measures

$k : \{1, \dots, m\} \mapsto \Omega$ where $k[i] = p_i$ denotes the i-th point. Set μ as the *counting measure* defined for any set $I \subseteq \{1, \dots, m\}$ by $\mu(I) = \frac{|I|}{m}$. Then ν is defined by

$$\nu = \frac{1}{m} \sum_{i=1}^{m} \delta_{p_i}$$

Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
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SIAM Journal on Imaging Science, 2014.

Push Forward Measure - Theorem

Weak convergence

A sequence of measures (μ_n) is said to weakly converge to μ, if for any bounded continuous function ϕ,

$$\int_\Omega \phi(x)d\mu_n(x)_{n \to +\infty}^{\longrightarrow} \int_\Omega \phi(x)d\mu(x)$$

Shorthand notation: $\mu_n \rightharpoonup \mu$.

Variable density sampler

A sequence of (random) trajectories $k_n : X_n \mapsto \Omega$ is said to be a π-Variable Density Sampler if

$$k_{n*}\mu \rightharpoonup \pi \quad \text{almost surely}$$

Examples

i.i.d. drawing, random walks ...

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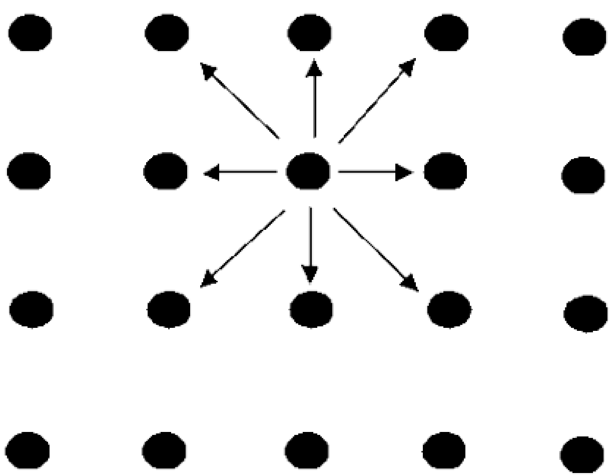
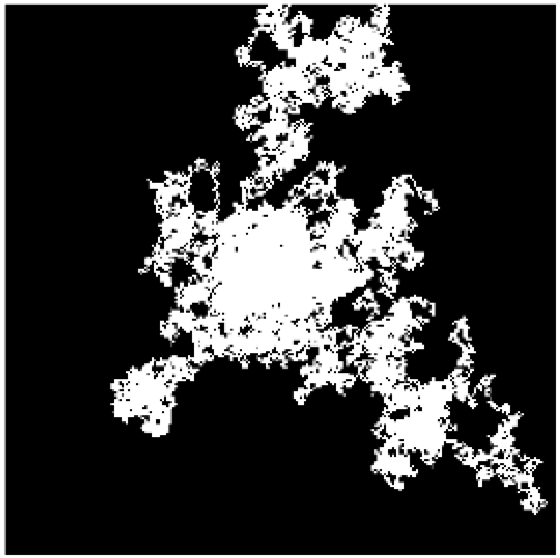
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

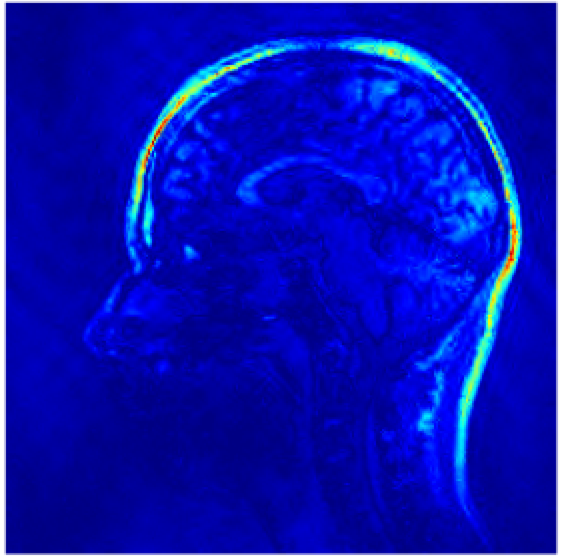
Sampling along Random Walks

Construction of a discrete Markov chain

Given a target probability distribution $\pi \in \mathbb{R}^n$.
Define a Markov chain $\boldsymbol{K} = (k_i)_{i \in \mathbb{N}}$ on the set $\{1, \ldots, n\}$. Use the Metropolis algorithm to construct a stochastic transition matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ such that π is the stationary distribution of $\boldsymbol{K}$.





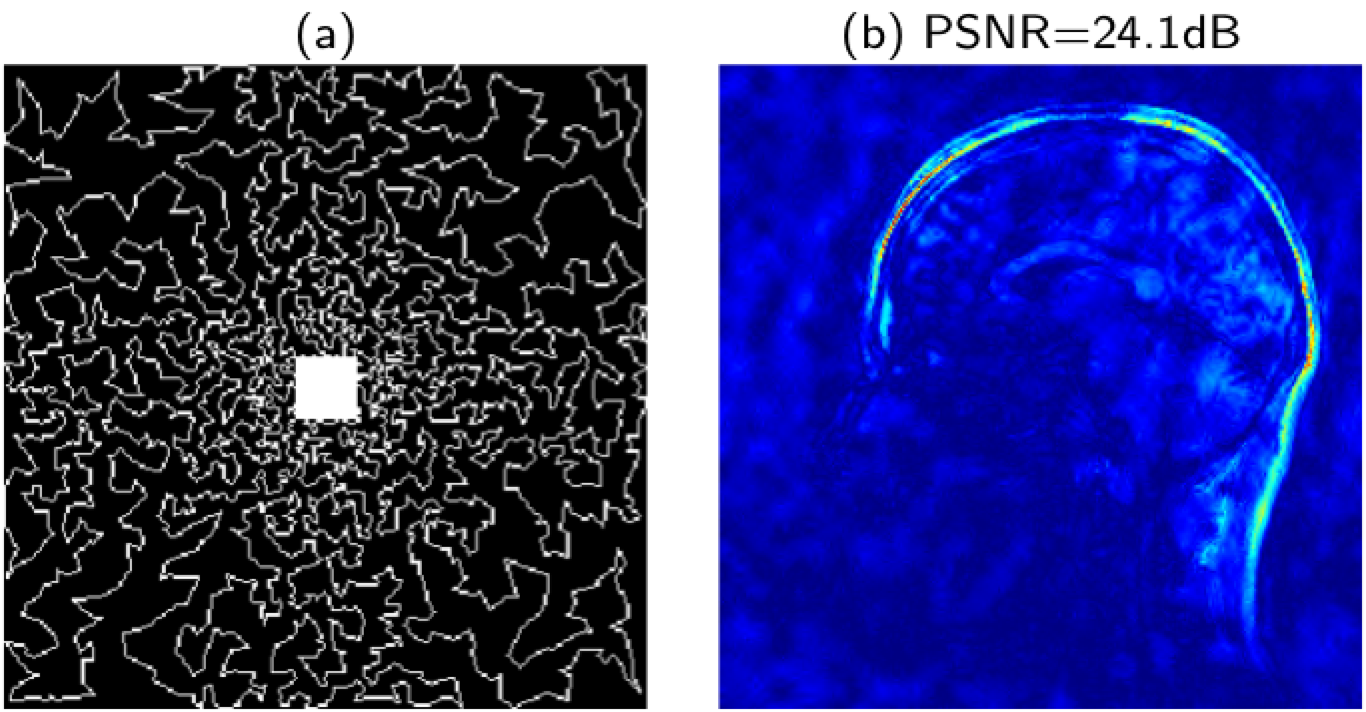


- Time to cover the k-space is slow
- Local approach

Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

Solving the Travelling Salesman Problem

Idea : cover the k-space more quickly with a global approach.



- Pushforward measure far from π. From which distribution should we sample the initial points to reach a given target distribution?

Go to Matlab code:
TSP-Markov folder

Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

TSP Sampler

- Let

$$q = \frac{\pi^{d/(d-1)}}{\int_\Omega \pi^{d/(d-1)}}$$

- $(k[i])_{i\in\mathbb{N}^*}$ a sequence of points in Ω, *i.i.d.* drawn $\sim q$.
- $\boldsymbol{k}_N = (k[i])_{i\leq N}$
- Denote $T(\boldsymbol{k}_N)$ the length of the TSP amongst $\boldsymbol{k}_N$.
- $\gamma_N : [0, T(\boldsymbol{k}_N)] \mapsto \Omega$ denotes the parametrization of the curve at speed 1.

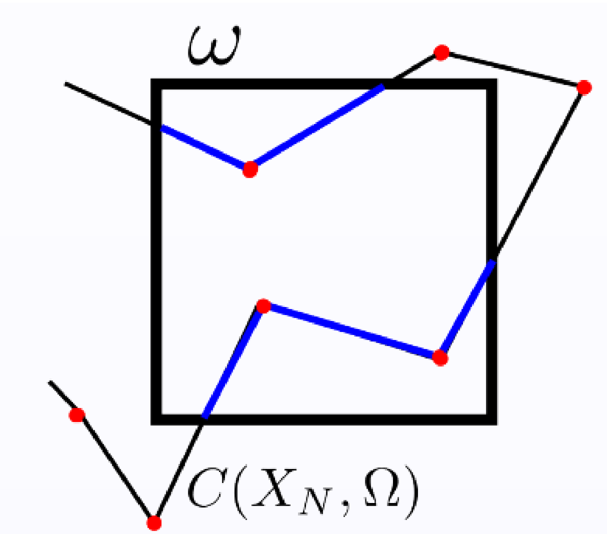
Theorem

Almost surely w.r.t. the law $q^{\otimes\mathbb{N}}$ of the sequence $(k[i])_{i\in\mathbb{N}^*}$ of random points in the hypercube, $(\gamma_N)_{N_i n\mathbb{N}}$ is a π-variable density sampler, i.e.,

$$\gamma_{N*}\lambda_{T(\boldsymbol{k}_N)} \rightharpoonup \pi$$

Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

Intuition of the Proof



Definition of the TSP distribution

For any Borelian $\omega \in \Omega$:

$$P_N(\omega) = \frac{T_{|\omega}(X_N)}{T(X_N)}$$

Intuition

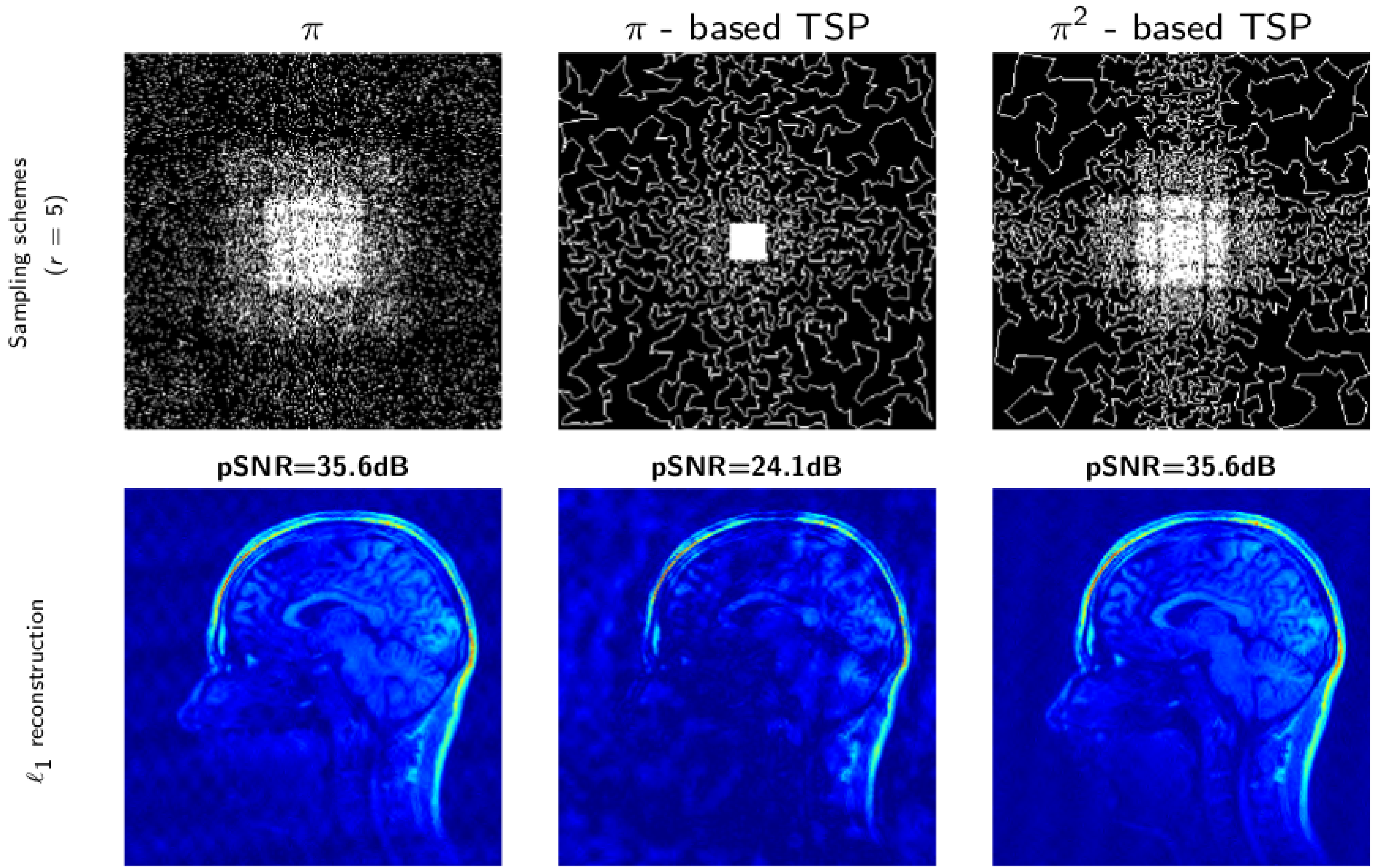
Consider a small hypercube:

- The number of point n is $\propto q$;
- The typical distance is proportional to $n^{-1/d}$ (or $q^{-1/d}$);
- $\Rightarrow$ The length of the TSP in the small cube is $\propto qq^{-1/d} = q^{(d-1)/d}$

Conclusion

To reach a target density π one should choose $q \propto \pi^{d/(d-1)}$!

The Traveling Salesman Sampler in 2D



Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

Go to Matlab code:
TSP-Markov folder

The Traveling Salesman Sampler in 3D

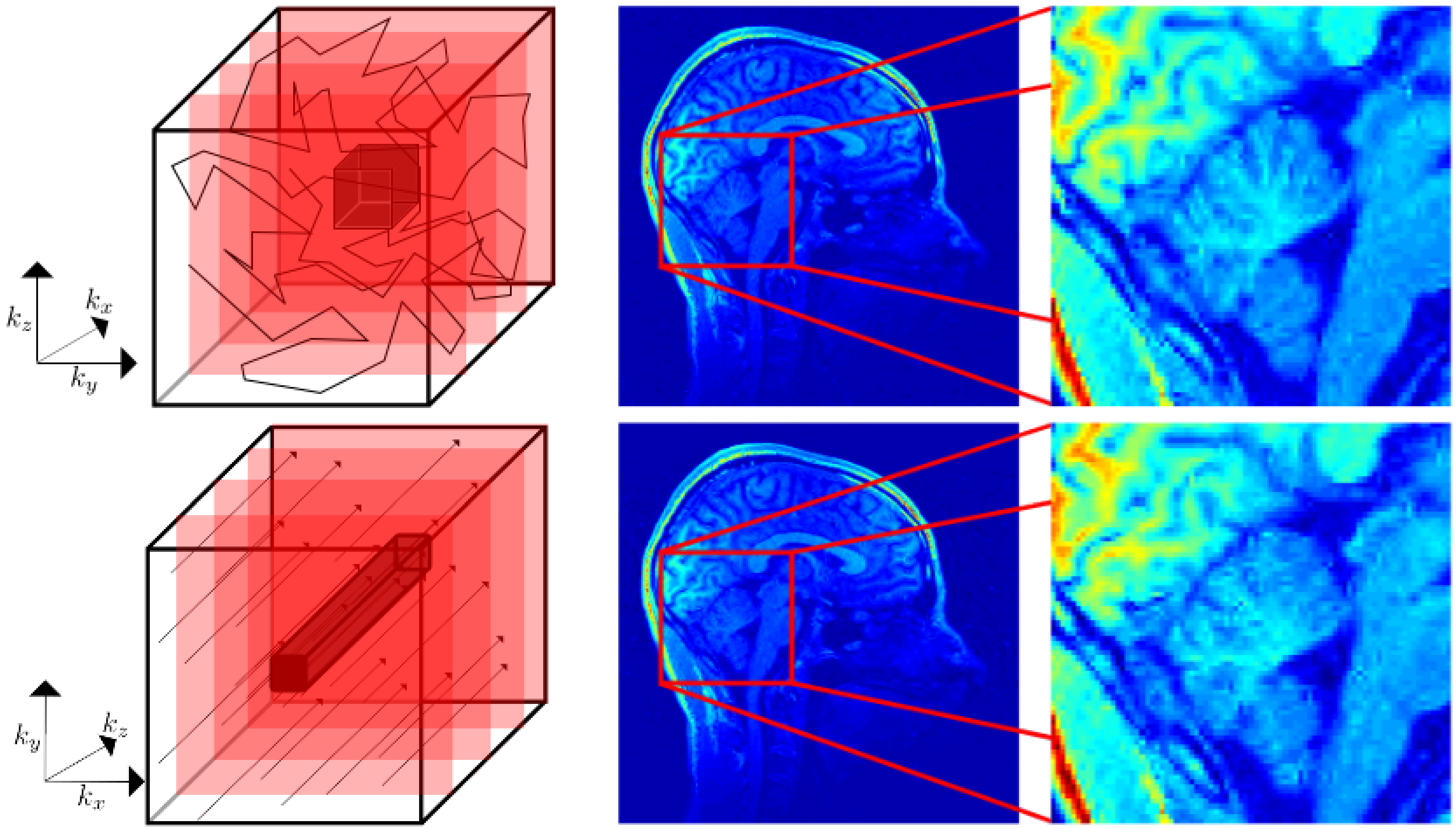
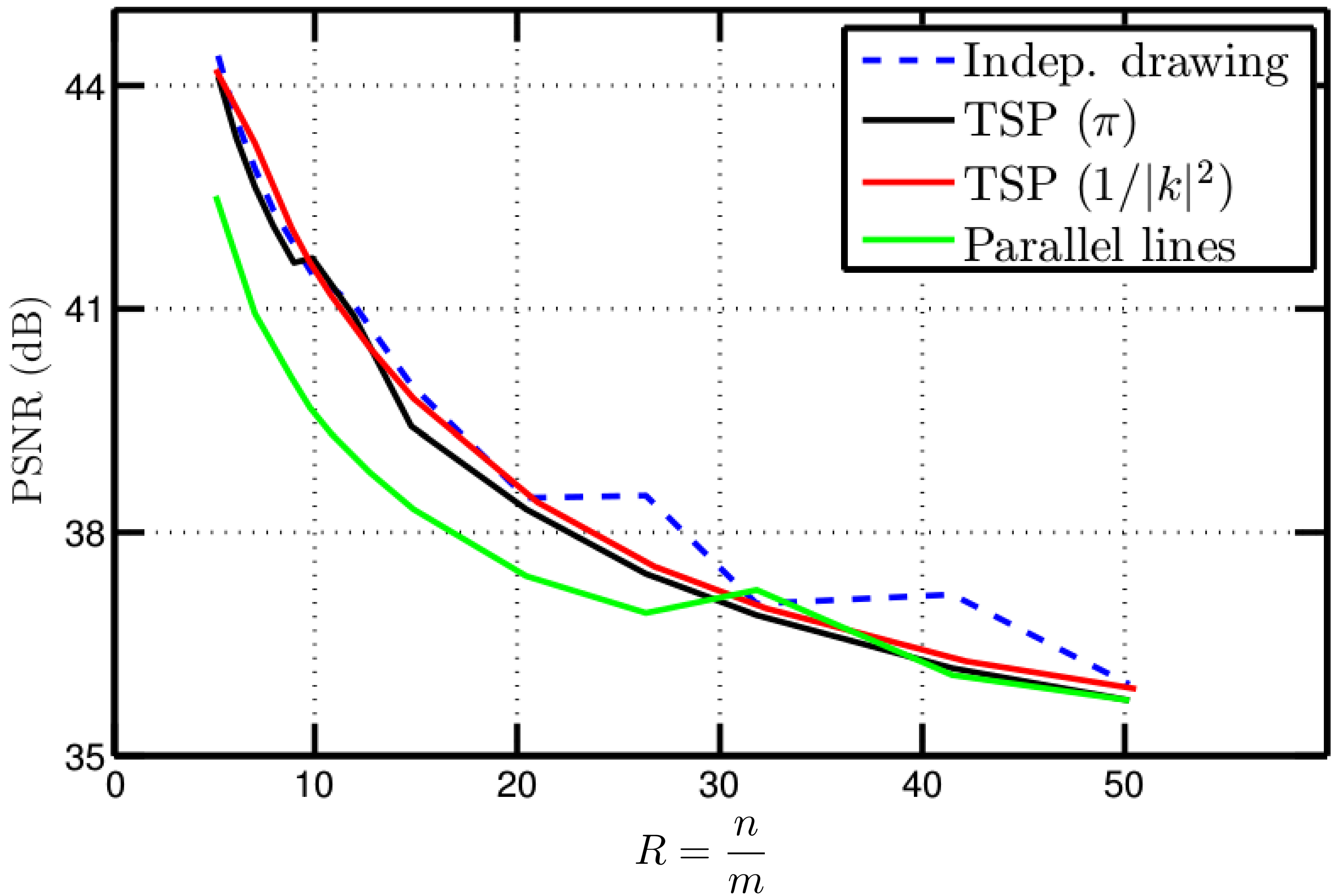


Figure: 3D reconstruction results for $r = 8.8$ for various sampling strategies. **Top row:** TSP-based sampling schemes (PSNR=42.1 dB). **Bottom row:** 2D random drawing and acquisitions along parallel lines [Lustig et al., 2007] (PSNR=40.1 dB).

Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

The Traveling Salesman Sampler in 3D



Nicolas Chauffert, Philippe Ciuciu, Jonas Kahn, and Pierre Weiss.
Variable density sampling with continous sampling trajectories.
SIAM Journal on Imaging Science, 2014.

Back to MRI constraints

MRI constraints:

An **admissible sampling curve in MRI** is a curve belonging either to the set:

$$\mathcal{S}_{\text{MRI}} = \left\{ \boldsymbol{k} : [0,T] \mapsto \mathbb{R}^2, \|\dot{\boldsymbol{k}}\|_\infty \leqslant \alpha, \|\ddot{\boldsymbol{k}}\|_\infty \leqslant \beta \right\}$$

or

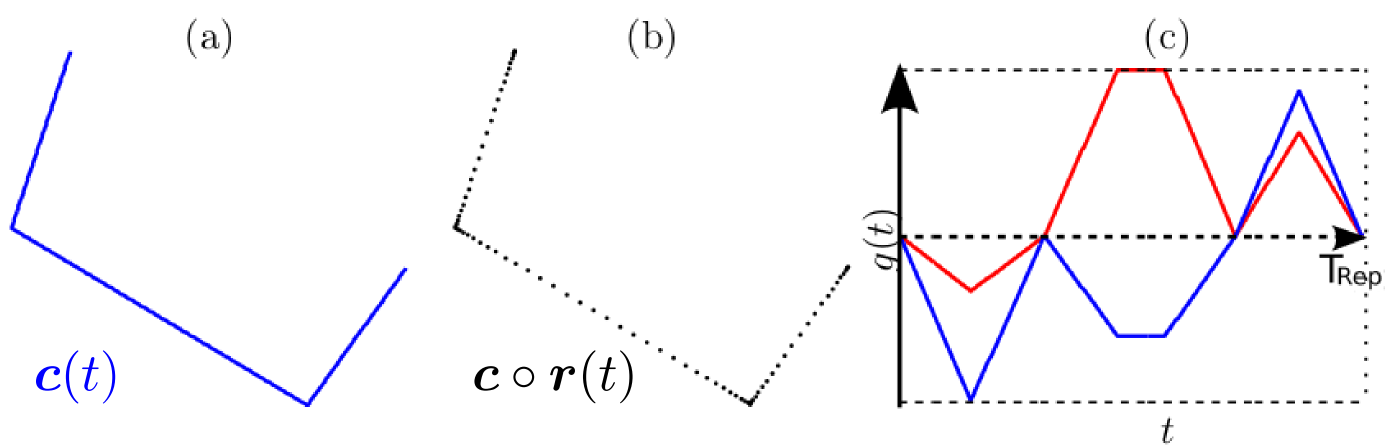
$$\mathcal{S}_{\text{MRI}} = \left\{ \boldsymbol{k} : [0,T] \mapsto \mathbb{R}^2, \|\dot{\boldsymbol{k}}\|_{2,\infty} \leqslant \alpha, \|\ddot{\boldsymbol{k}}\|_{2,\infty} \leqslant \beta \right\}$$

depending on the scanner.

Finding a parameterization in $\mathcal{S}_{\text{MRI}}$ corresponding to a curve support is not easy !

- Classical approach, find an admissible parameterization [Hargreaves et al., 2004, Lustig et al., 2008]:

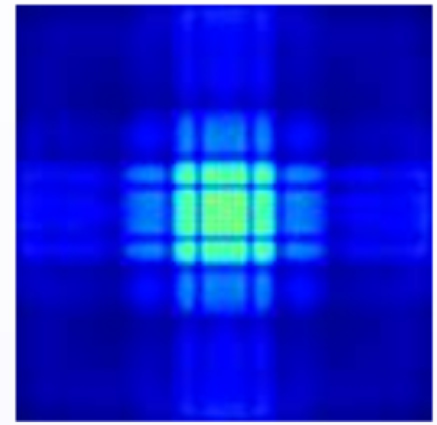
$$T_{\text{rep}} = \min T' \;\text{ such that } \exists\, \boldsymbol{r} : [0, T'] \mapsto [0,T],\; \boldsymbol{c} \circ \boldsymbol{r}(t) \in \mathcal{S}_{\text{MRI}}$$



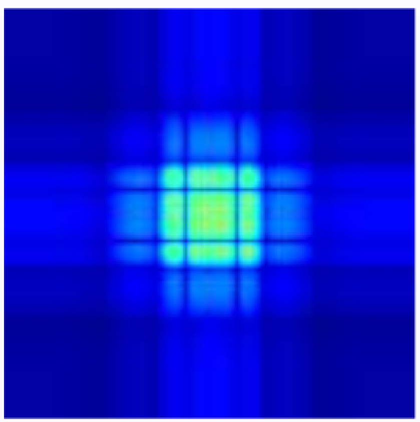
Issues with reparametrization

Monte Carlo study over 4000 independent drawings:

constant speed

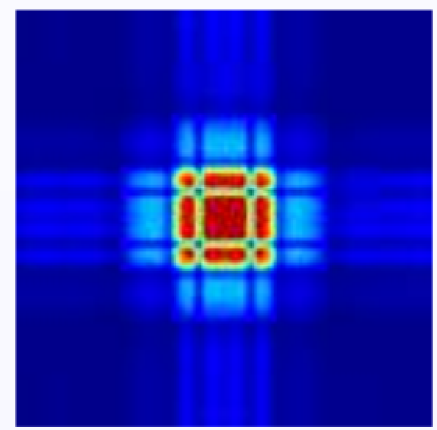


empirical distribution

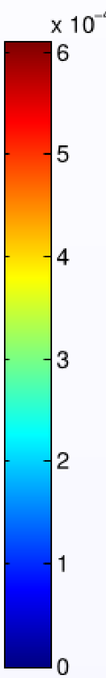


target distribution

reparameterization
[Lustig et al., 2008]

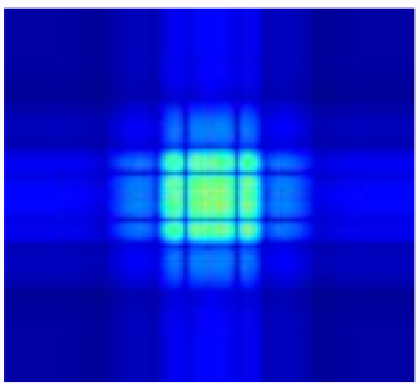


empirical distribution

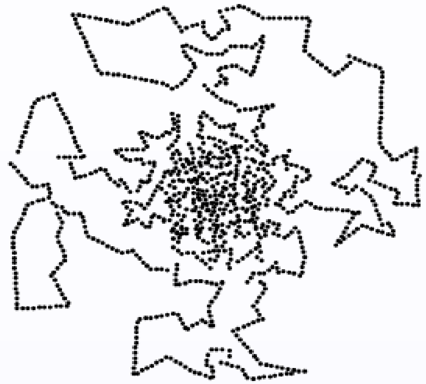


Issues with reparametrization

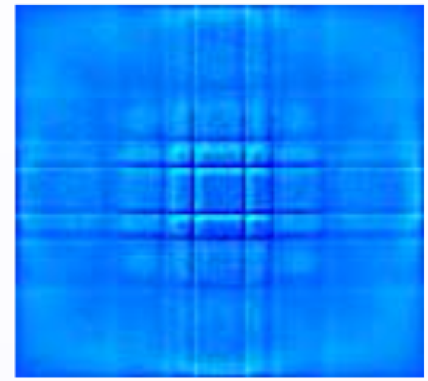
Monte Carlo study over 4000 independent drawings:



target distribution

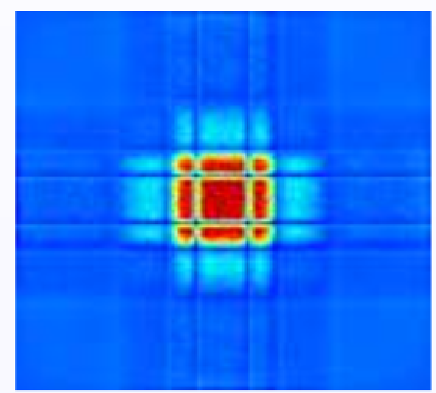


constant speed

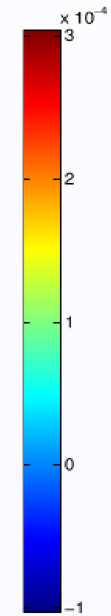


normalized error: 14%

reparameterization [Lustig et al., 2008]



normalized error: 60%

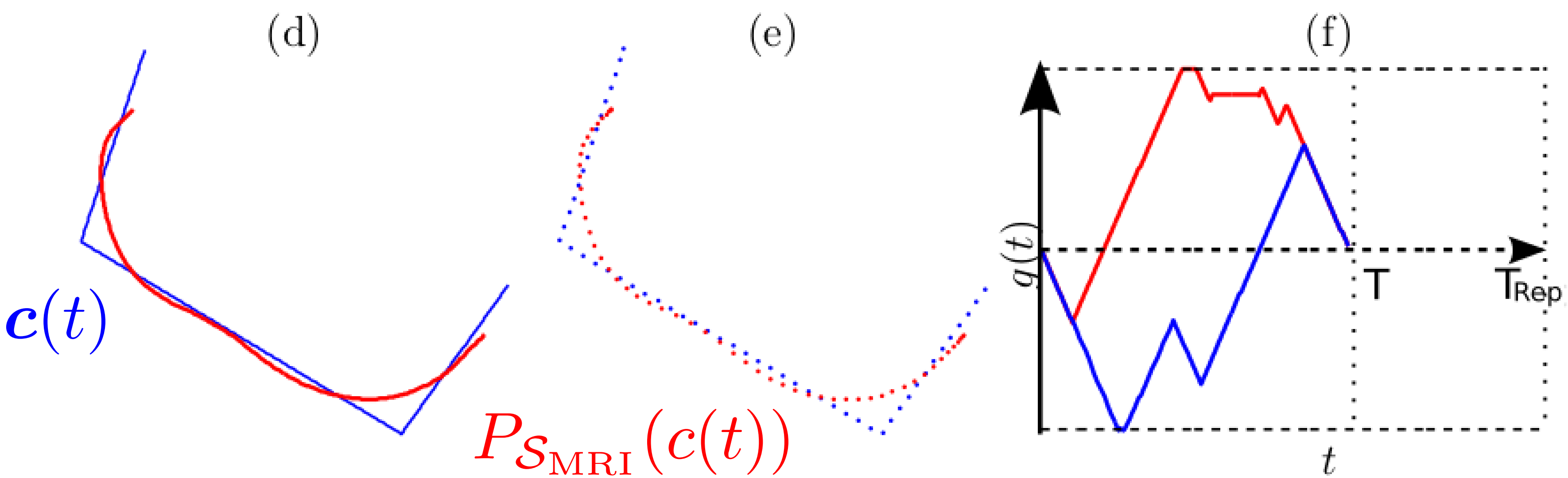


Normalized error

$$\frac{\sum_i (\pi(\boldsymbol{k}(i)) - \widehat{\pi}(\boldsymbol{k}(i)))^2}{\sum_i \pi(\boldsymbol{k}(i))^2}$$

Projection onto the MRI constraints

- Projection onto $\mathcal{S}_{\text{MRI}}$ [Chauffert et al, IEEE TMI 2016]



Leave the support of c but keep control on the sampling density

Go to Matlab code:
Gradient_waveform_design

The Projection Algorithm

For an input parameterized curve $c : [0, T] \mapsto \Omega$, define:

$$P_{\mathcal{S}_{\mathrm{MRI}}}(c) = \operatorname*{arg\,min}_{\boldsymbol{k} \in \mathcal{S}_{\mathrm{MRI}}} \int_0^T (\boldsymbol{k}(t) - \boldsymbol{c}(t))^2$$

Main properties

- Fast resolution using accelerated proximal gradient descent on the dual.
- The sampling time is fixed (equal to T).
- The sampling distribution is well preserved (approximation of Wasserstein distance $\mathcal{W}_2$).

Leave the support of c but keep control on the sampling density

[Chauffert et al, IEEE TMI 2016]

Mathematical details: Discretization

- A discrete curve $\boldsymbol{k} = \begin{pmatrix} \boldsymbol{k}[1] \\ \vdots \\ \boldsymbol{k}[n] \end{pmatrix}$ defined as a vector in $\mathbb{R}^{n \cdot d}$.
- $\boldsymbol{k}[i] \in \mathbb{R}^d$ denotes the curve location at time $(i-1)\delta t$ with $\delta t = \frac{T}{n-1}$ and $i \in \{1, \dots, n\}$.
- The discrete derivative $\dot{\boldsymbol{k}} \in \mathbb{R}^{nd}$ is defined using first order differences:

$$\dot{\boldsymbol{k}}[i] = \begin{cases} 0 & \text{if } i = 1 \\ (\boldsymbol{k}[i] - \boldsymbol{k}[i-1])/\delta t & \text{if } i \in \{2, \dots, n\} \end{cases}$$

- Let $\boldsymbol{A}_1 \in \mathbb{R}^{nd \times nd}$ be such that $\dot{\boldsymbol{k}} = \boldsymbol{A}_1 \boldsymbol{k}$
- We define the discrete second order differential operator by $\boldsymbol{A}_2 = -\boldsymbol{A}_1^* \boldsymbol{A}_1 \in \mathbb{R}^{nd \times nd}$
- More generally, the m-th order differential operator will be denoted $\boldsymbol{A}_m$

[Chauffert et al, IEEE TMI 2016]

Objective function

[Chauffert et al, IEEE TMI 2016]

Constraint set

$$\mathcal{S} = \{\boldsymbol{k} \in \mathbb{R}^{nd}, \|\boldsymbol{A}_j \boldsymbol{k}\|_{N_j} \leqslant \alpha_j,\ j = 1, \dots J\}$$

Input: $\boldsymbol{k}_0 \in \mathbb{R}^{nd}$.
Problem: find

$$\widehat{\boldsymbol{k}} = \operatorname*{Arg\,min}_{\boldsymbol{k} \in \mathcal{S}} \frac{1}{2}\|\boldsymbol{k} - \boldsymbol{k}_0\|_2^2$$

Using **Fenchel duality**:

$$\min_{\boldsymbol{k} \in \mathcal{S}} \frac{1}{2}\|\boldsymbol{k} - \boldsymbol{k}_0\|_2^2 = \min_{\boldsymbol{k} \in \mathbb{R}^{nd}} \frac{1}{2}\|\boldsymbol{k} - \boldsymbol{k}_0\|_2^2 + \sum_{j=1}^{J} \sup_{\boldsymbol{k}_j^* \in \mathbb{R}^{nd}} \langle \boldsymbol{A}_j \boldsymbol{k}, \boldsymbol{k}_j^* \rangle - \alpha_j \|\boldsymbol{k}_j^*\|_{N_j^*}$$

$$= \sup_{\boldsymbol{k}_j^* \in \mathbb{R}^{nd}} -\alpha_j \|\boldsymbol{k}_j^*\|_{N_j^*} + \underbrace{\min_{\boldsymbol{k} \in \mathbb{R}^{nd}} \left(\frac{1}{2}\|\boldsymbol{k} - \boldsymbol{k}_0\|_2^2 + \sum_{j=1}^{J} \langle \boldsymbol{A}_j \boldsymbol{k}, \boldsymbol{k}_j^* \rangle \right)}$$

We have the duality relation:

$$\boldsymbol{k} = \boldsymbol{k}_0 - \sum_{j=1}^{K} \boldsymbol{A}_j^* \boldsymbol{k}_j^*$$

Optimization in the dual space

The dual criterion to be minimized reads: [Chauffert et al, IEEE TMI 2016]

$$\operatorname*{Arg\,min}_{\boldsymbol{k}^*=(\boldsymbol{k}_j^*)} \underbrace{\frac{1}{2}\Big\|\sum_{j=1}^{J}\boldsymbol{A}_j^*\boldsymbol{k}_j^*\Big\|_2^2 - \sum_{j=1}^{J}\langle \boldsymbol{A}_j^*\boldsymbol{k}_j^*, \boldsymbol{k}_0\rangle}_{g(\boldsymbol{k}^*)} + \underbrace{\sum_{j=1}^{J}\alpha_j\|\boldsymbol{k}_j^*\|_{N_j^*}}_{h(\boldsymbol{k}^*)}$$

- g is a smooth convex function with L-Lipschitz gradient.
- h is a non-smooth convex function.

Algorithm 2 Accelerated proximal gradient method [Nesterov, 1983]

Initialize $\boldsymbol{k}_j^{(0)}, \boldsymbol{y}_j^{(0)}, \quad \text{for } 1 \leqslant j \leqslant J\,.$

Set $\eta = 1/L$.

for $n = 1, \dots, n_{\text{iter}}$ **do**

$$\boldsymbol{k}_j^{(n)} = \operatorname*{Prox}_{\mu h_j}\left(\boldsymbol{y}_j^{(n-1)} - \nu\nabla_j g(\boldsymbol{y}^{(n-1)})\right)$$

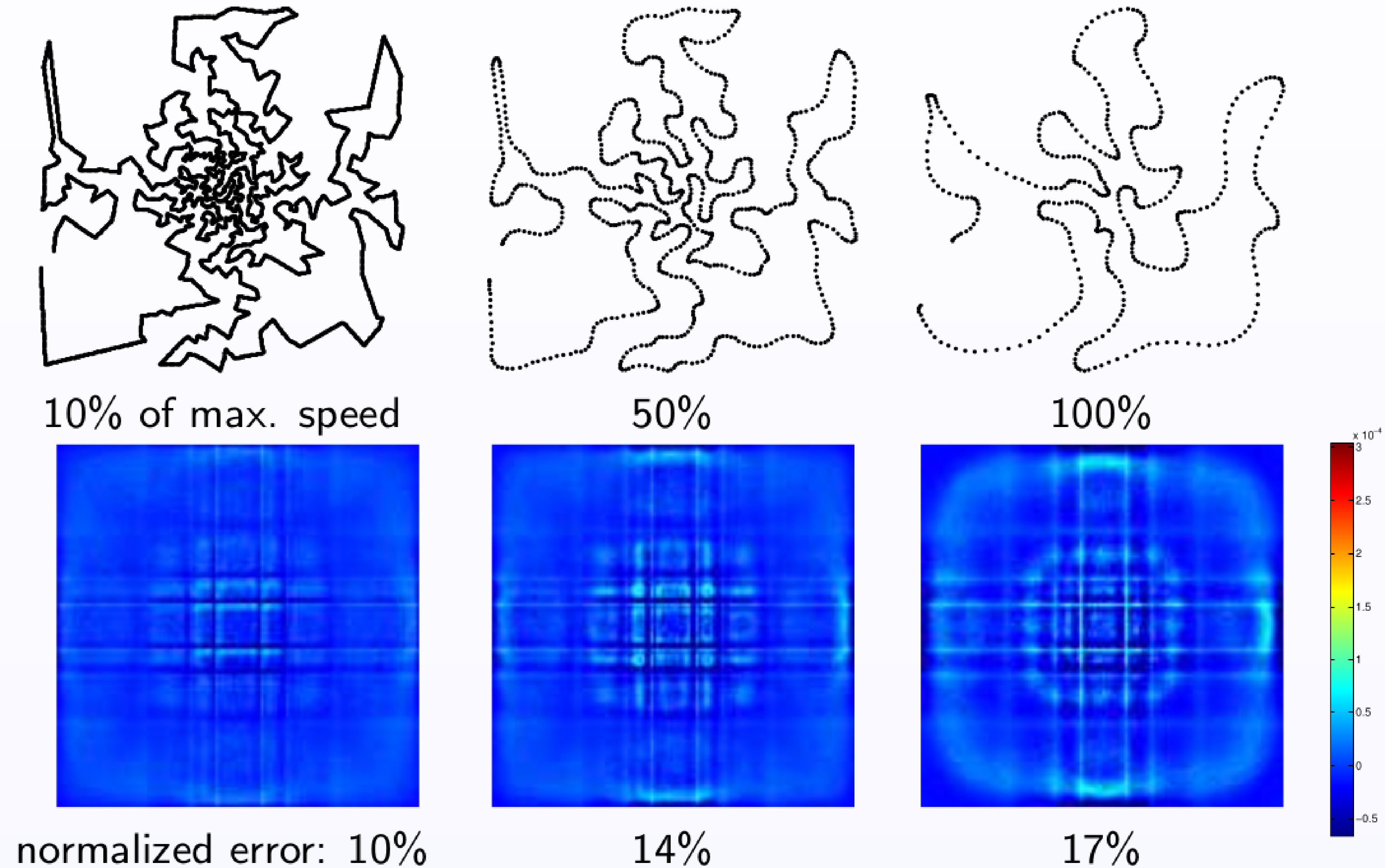
$$\boldsymbol{y}_j^{(n)} = \boldsymbol{k}_j^{(n)} + \frac{n-1}{n+2}\left(\boldsymbol{k}_j^{(n)} - \boldsymbol{k}_j^{(n-1)}\right)$$

return $\boldsymbol{k} = \boldsymbol{k}_0 - \sum_{j=1}^{J}\boldsymbol{A}_j^*\boldsymbol{k}_j^{(n_{\text{iter}})}$

The convergence rate of this algorithm is $\mathcal{O}(1/n^2)$.

Sampling density of projected trajectories

[Chauffert et al, IEEE TMI 2016]



For optimal control, error was 60%!

Illustration & Results

[Chauffert et al, IEEE ISBI 2015]

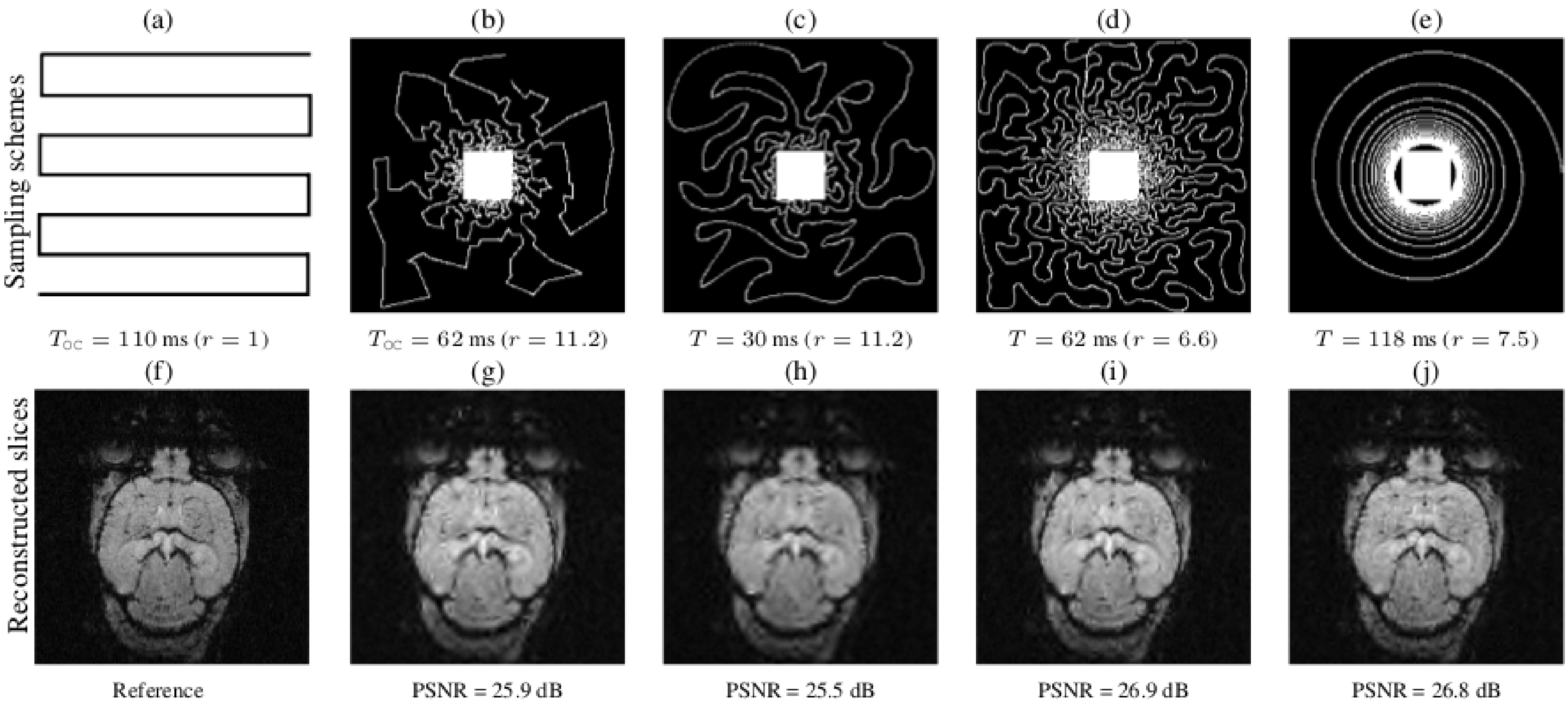
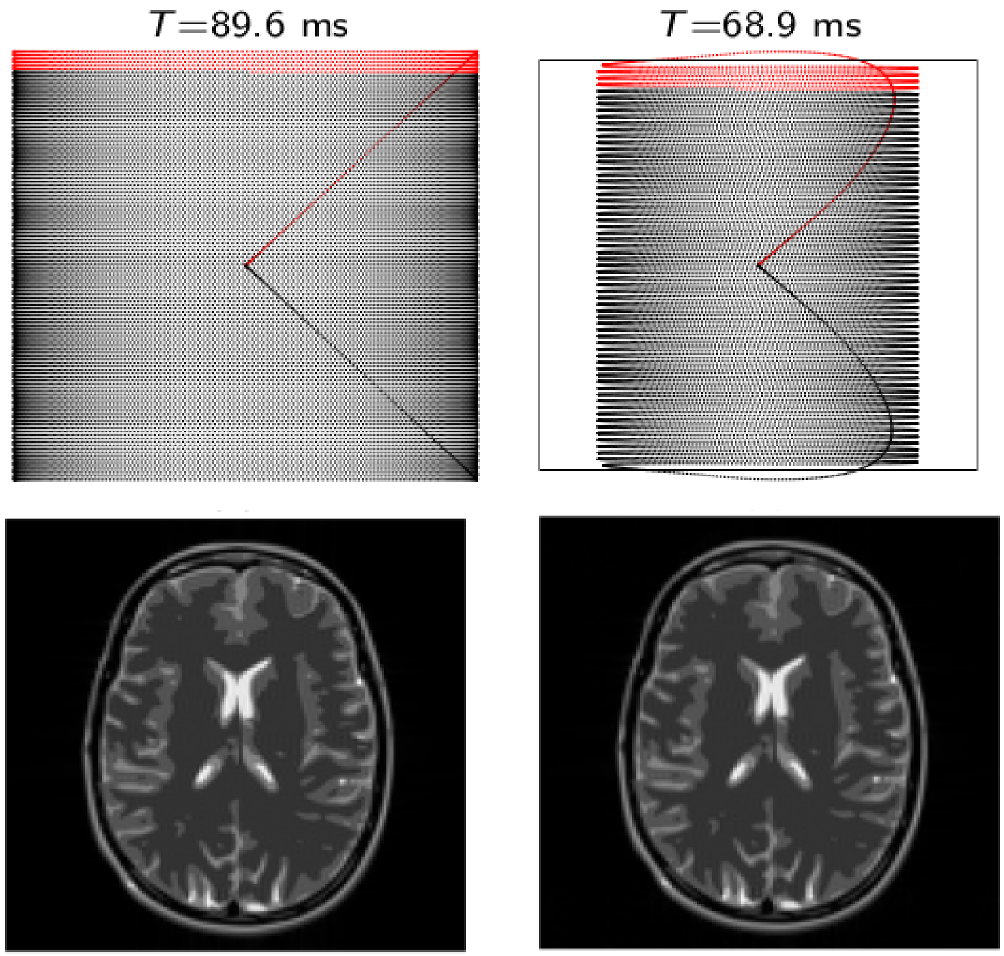


Illustration & Results

[Chauffert et al, IEEE TMI 2016]

EPI trajectories



Interim Summary

- **Heuristic conconclusions**

  The main features characterizing the efficiency of a given sampler are :

  – Its empirical (pushforward) measure/density

  – Its mixing time, i.e. its ability to cover k-space rapidly

- **The new quest …**

  Design a variable density sampler with:

  1) Prescribed density

  2) Good mixing properties

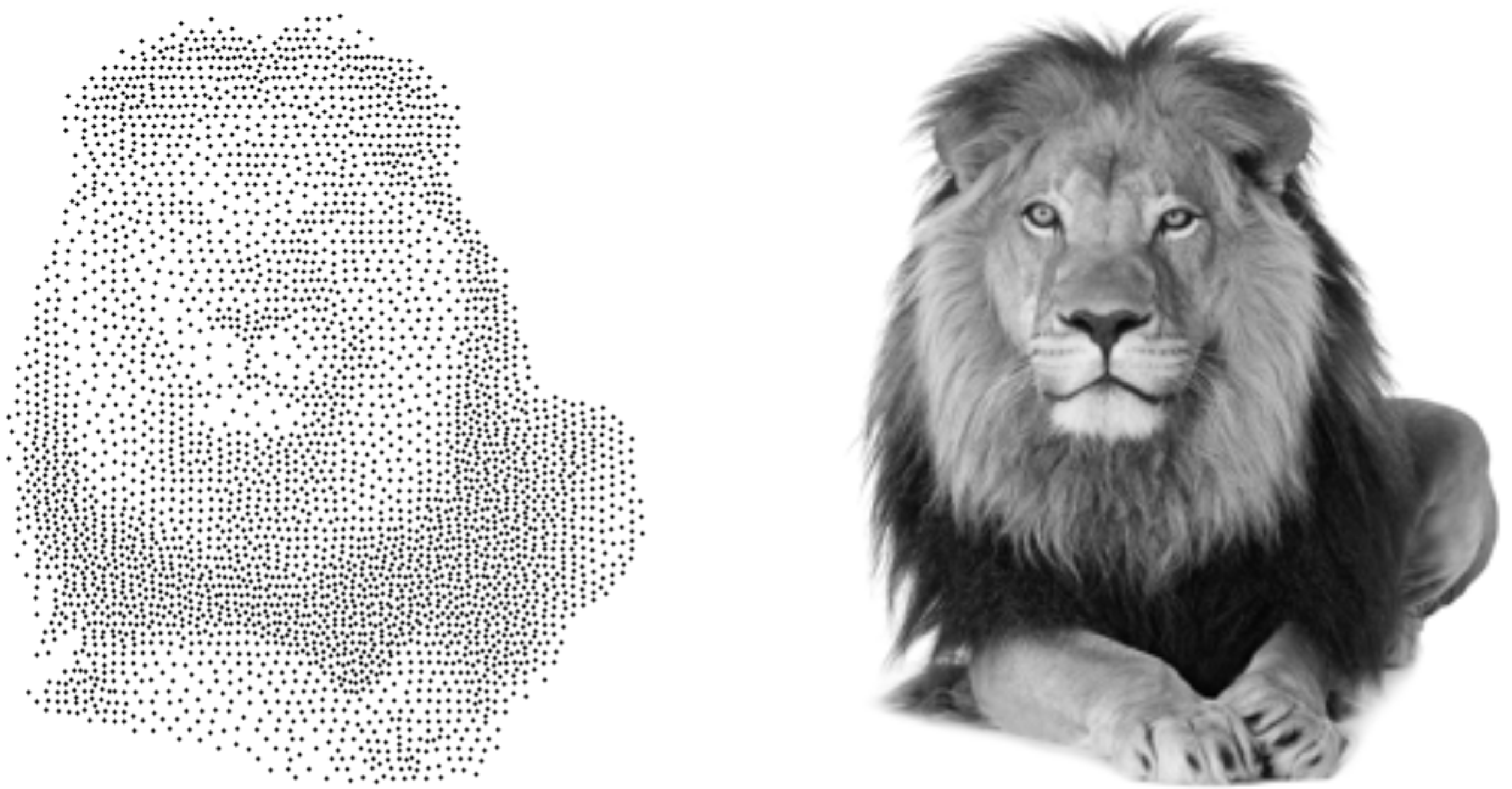
  – Belonging to a certain class, i.e. being compliant with MR hardware constraints and a given imaging contrast (e.g. T_1, T_2, etc)

Outline

Part III: CS sampling trajectories

- Continuous Variable Density Samplers
- Image stippling techniques
- SPARKLING

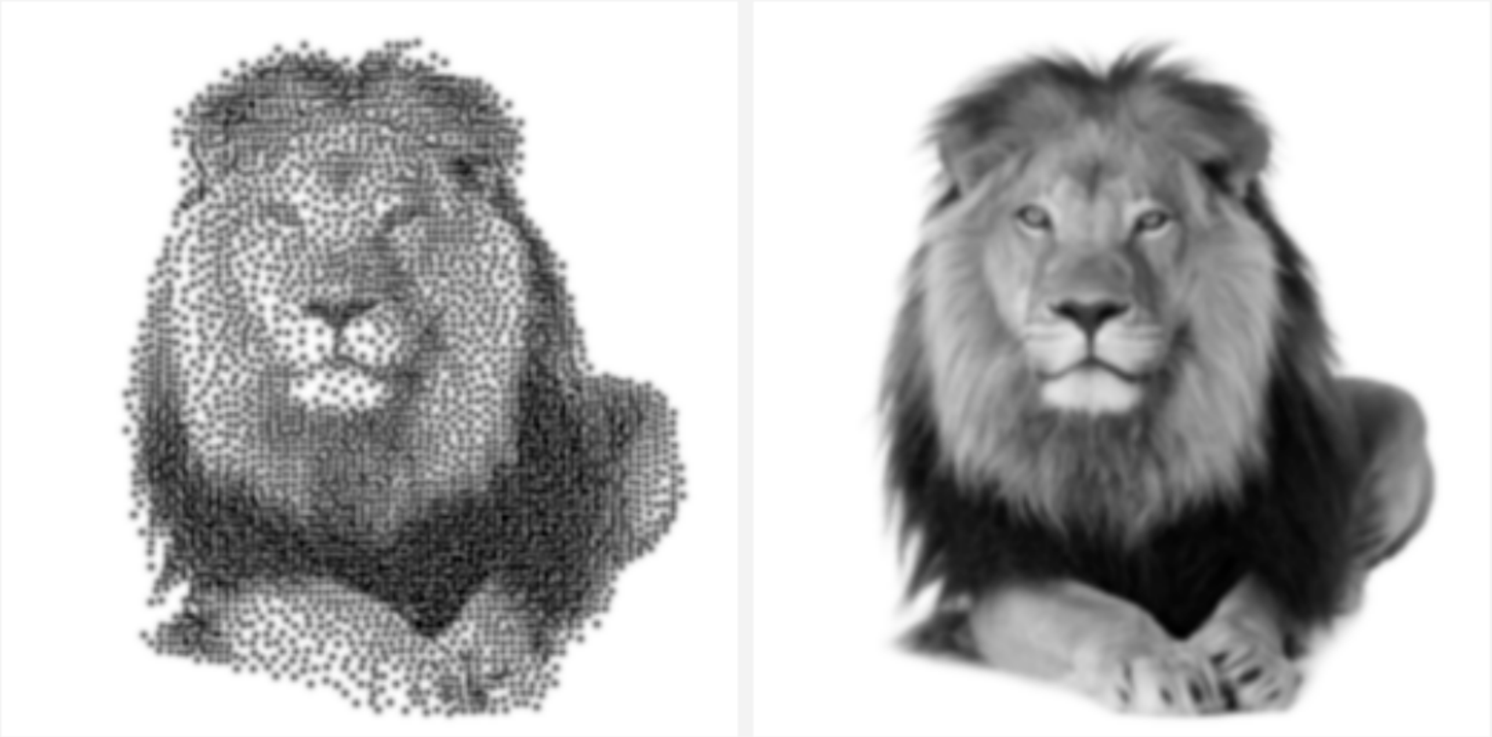
How to measure similarity between images?



Halftoned and reference images

Why are the images alike?

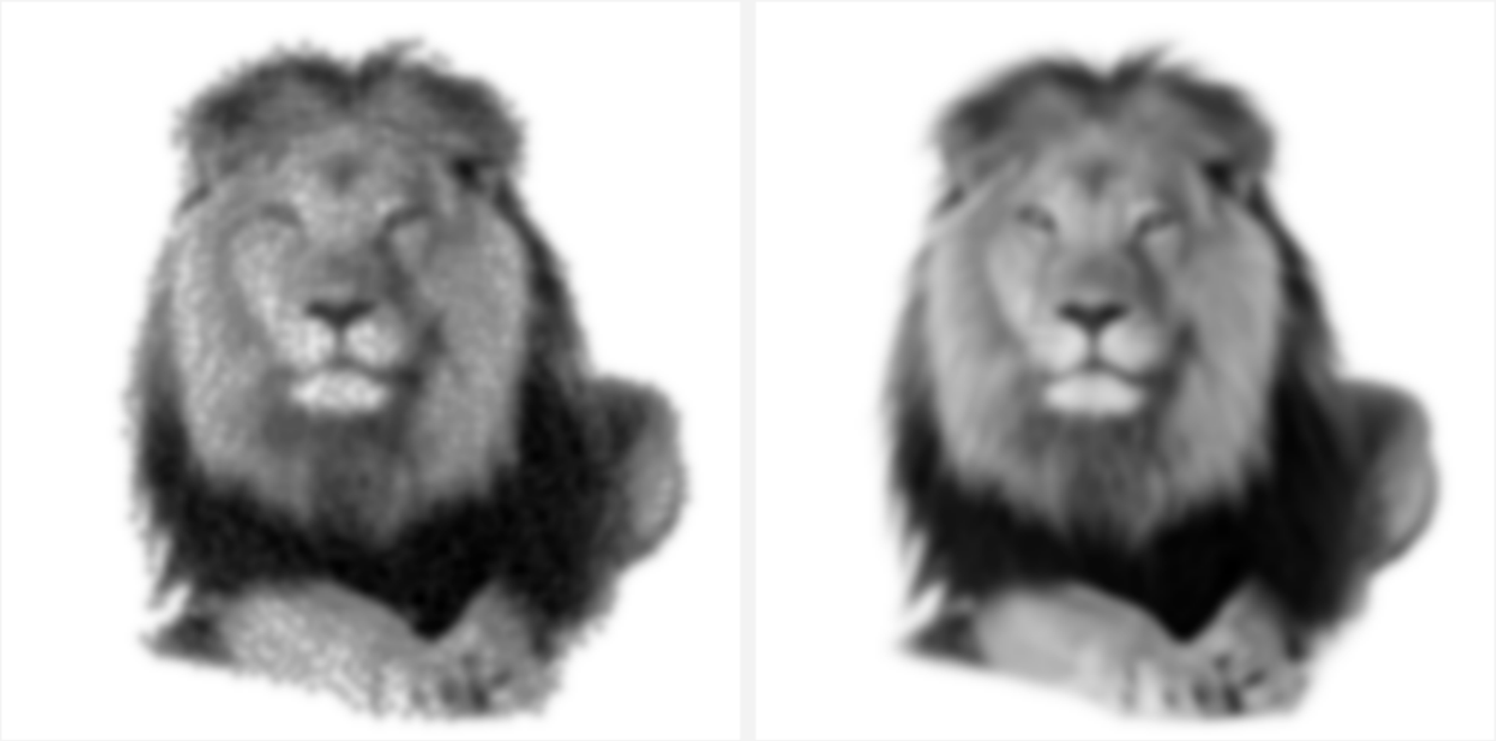
The multiresolution feature of the human visual system.



Convolution with a Gaussian $\sigma = 1$.

Why are the images alike?

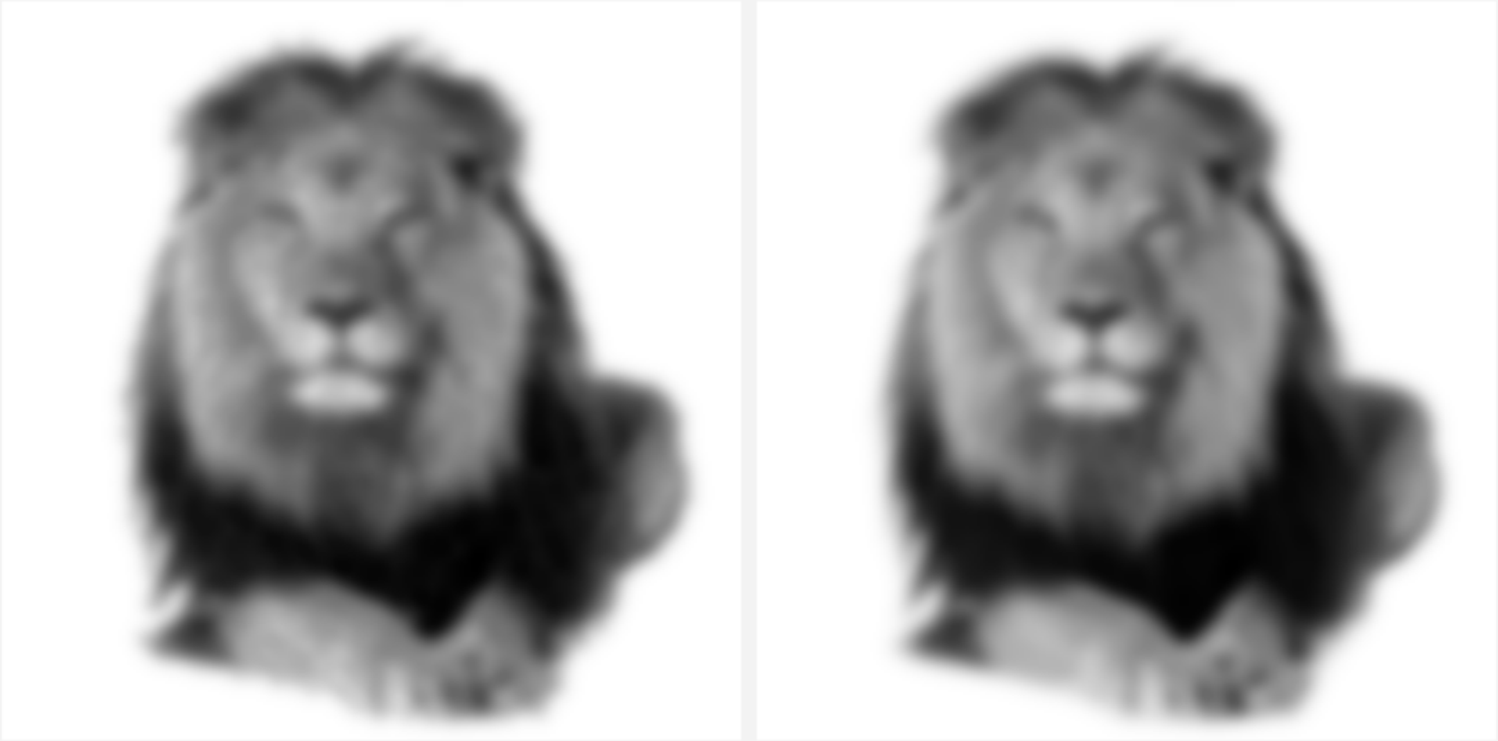
The multiresolution feature of the human visual system.



Convolution with a Gaussian $\sigma = 2$.

Why are the images alike?

The multiresolution feature of the human visual system.



Convolution with a Gaussian $\sigma = 3$.

➡ The images are similar only at **low resolution**

Halftoning

Constructing a metric

Let π denote the target image (the lion).
Let ν denote the atomic measure (the dots).

A natural distance to quantify image similarity reads:

$$\mathcal{D}_h(\nu - \pi) = \|h \star \nu - h \star \pi\|_2 \,.$$

The kernel h can be chosen as the impulse response of the visual system.

[Chauffert et al, Construct Approx, 2017]

Halftoninig as a projection problem

How to find the point locations?

Let $\Omega \subseteq \mathbb{R}^d$ denote the image domain.
Let $\mathcal{M}(\Omega^N)$ denote the set of atomic measures with N points.

$$\mathcal{M}(\Omega^N) = \left\{ \nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{p_i},\ p_i \in \Omega \right\}$$

The halftoning problem can be formulated as a projection problem:

$$\min_{\nu \in \mathcal{M}(\Omega^N)} \mathcal{D}_h(\nu - \pi)\,.$$

[Chauffert et al, Construct Approx, 2017]

Link with Attraction-Repulsion

Proposition

The projection problem

$$\min_{\nu \in \mathcal{M}(\Omega^N)} \frac{1}{2} \|h \star \nu - h \star \pi\|_2^2$$

can be rewritten

$$\min_{(p_1,\ldots,p_N)\in\Omega^N} \underbrace{\frac{1}{N} \sum_{i=1}^{N} \int_\Omega H(p_i - x)\pi(x)\, dx}_{\text{Attraction potential}} - \underbrace{\frac{1}{2N^2} \sum_{1\leq i,j\leq N} H(p_i - p_j)}_{\text{Repulsion potential}}$$

where $\hat{H} = |\hat{h}|^2$.

[Chauffert et al, Construct Approx, 2017]

Modeling MRI kinematic constraints

$$\boxed{\min_{\boldsymbol{k}\in\mathcal{Q}_N} \frac{1}{2}\|h\star\nu(\boldsymbol{k}) - h\star\pi\|_2^2 \qquad \text{with } \nu(\boldsymbol{k}) = \frac{1}{N}\sum_{i=1}^{N}\delta_{k[i]},\ \ \boldsymbol{k} = (k[i])_{1\le i\le N}\in\mathcal{Q}_N}$$

$$\text{where } \mathcal{Q}_N = \{\|\dot{\boldsymbol{k}}\|_{2,\infty}\le\gamma G_{\max},\ \|\ddot{\boldsymbol{k}}\|_{2,\infty}\le\gamma S_{\max}\}$$

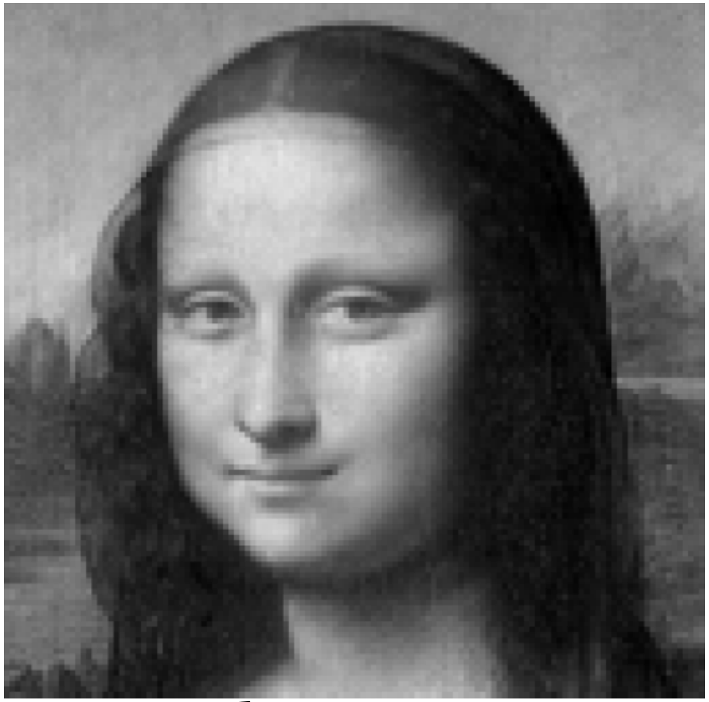
$$\min_{\boldsymbol{k}\in\mathcal{Q}_N} \underbrace{\frac{1}{N}\sum_{i=1}^{N}\int_\Omega H(x-k[i])\pi(x)\,dx}_{\substack{F_{\mathrm{a}}(\boldsymbol{k}) \\ \textit{attraction}}} - \underbrace{\frac{1}{2N^2}\sum_{1\le i,j\le N} H(k[i]-k[j])}_{\substack{F_{\mathrm{r}}(\boldsymbol{k}) \\ \textit{repulsion}}}$$

Projected gradient descent: $\boldsymbol{k}^{(t+1)} = P_{\mathcal{Q}_N}(\boldsymbol{k}^{(t)} - \tau^{(k)}\nabla(F_a - F_r)(\boldsymbol{k}^{(t)}))$

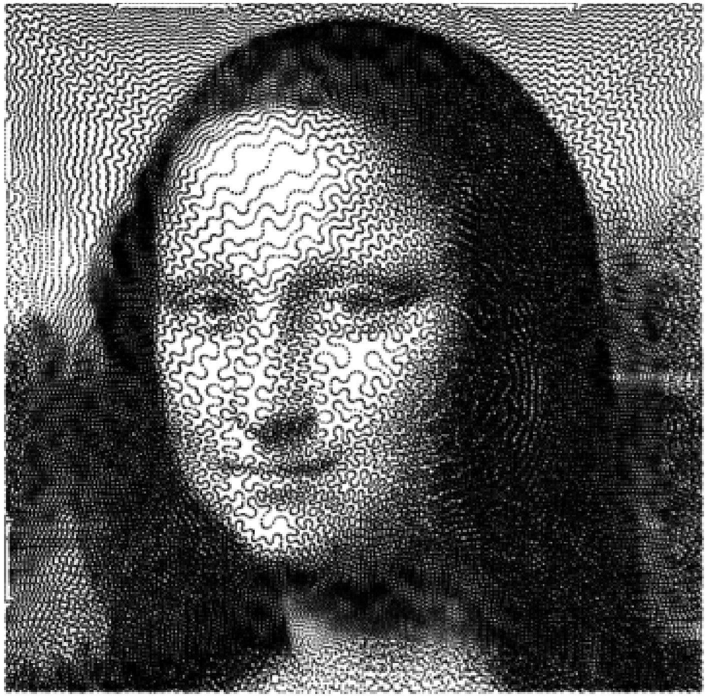
[Chauffert et al, IEEE TMI 2016]

Inertial adaptive step-size (BB update)

Image stippling with projection onto MRI kinematic constraints



$h_\sigma \star \pi$



$h_\sigma \star \nu$

[Chauffert et al, Construct Approx, 2017]

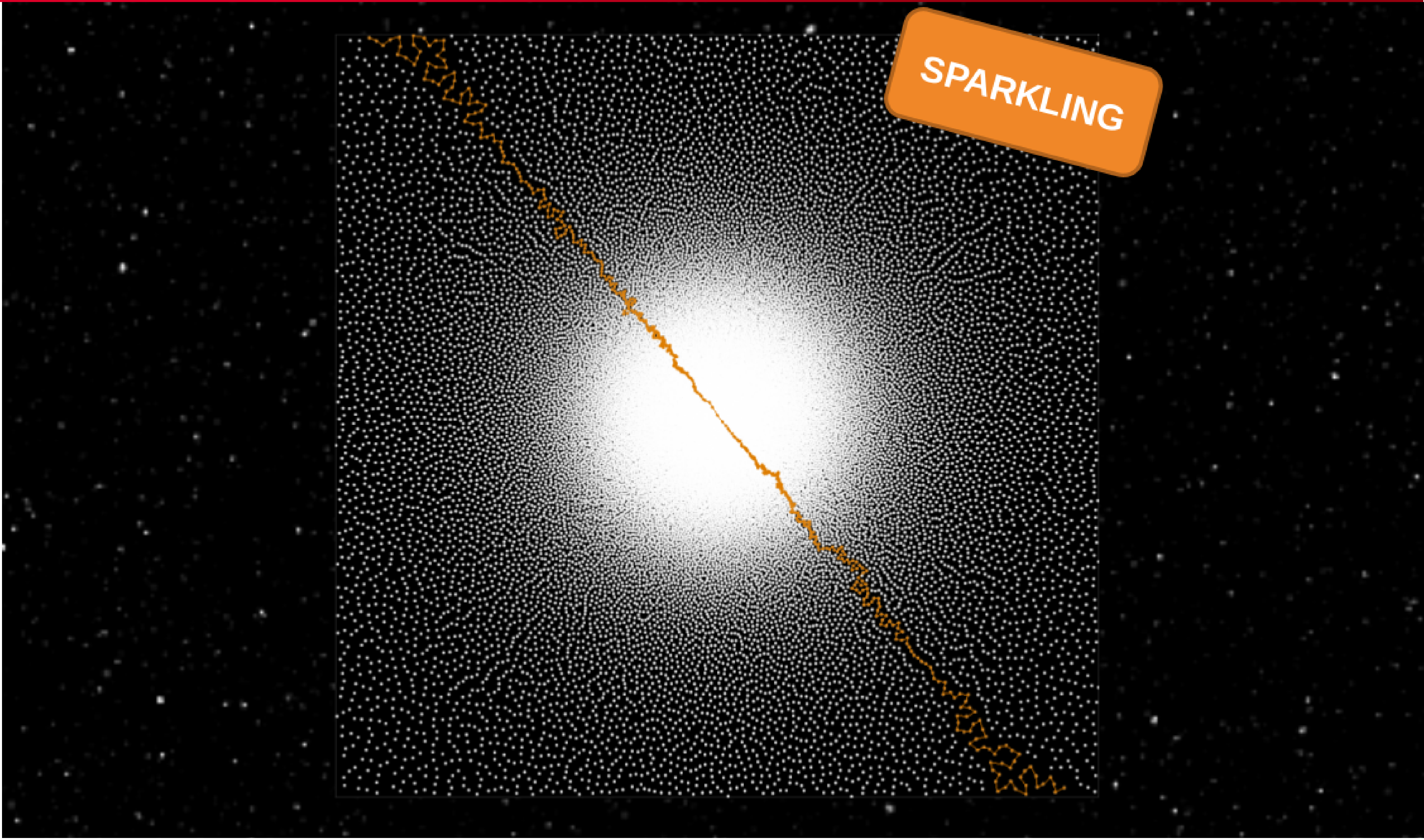
Outline

Part III: CS sampling trajectories

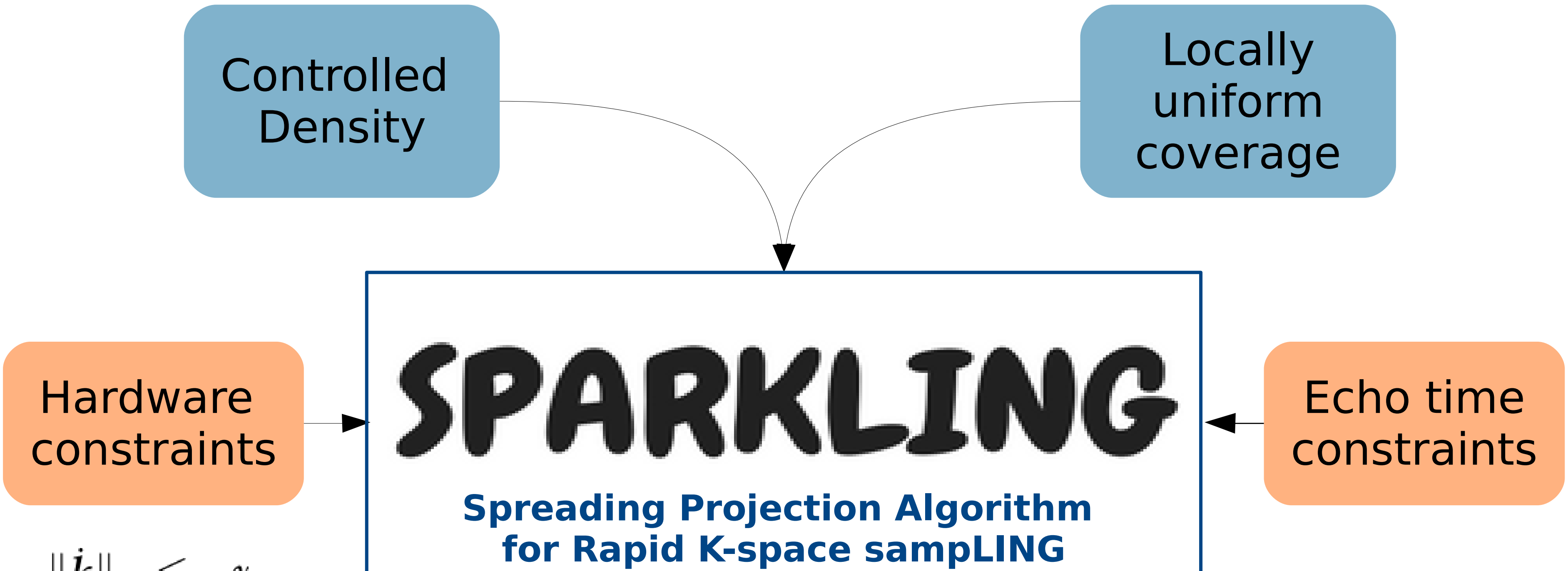
- Continuous Variable Density Samplers
- Image stippling techniques
- SPARKLING

SPARKLING

Spreading Projection Algorithm for Rapid K-space sampLING

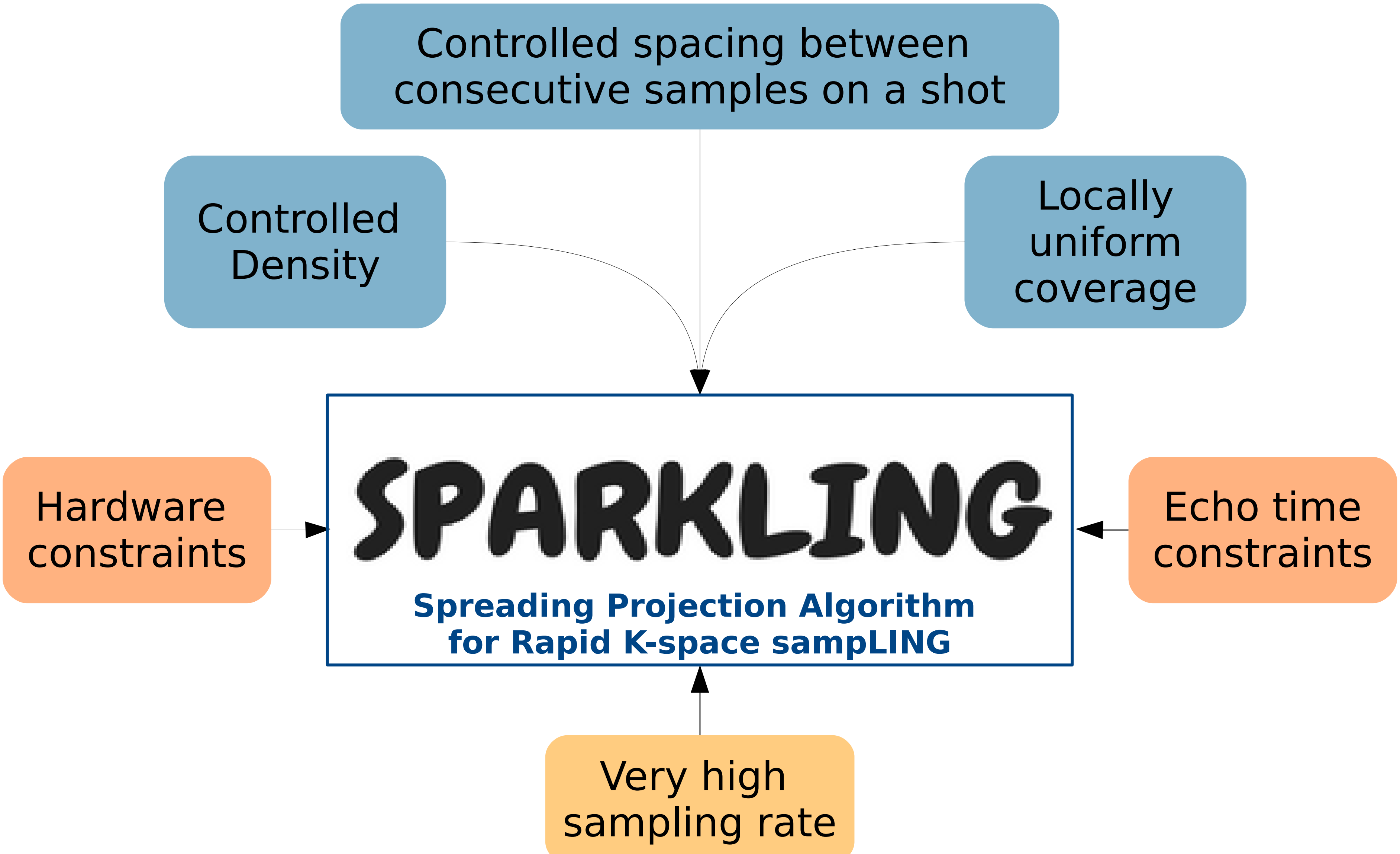


Sparkling recipe

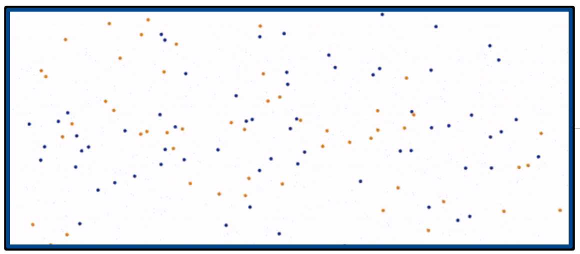


$$
\begin{aligned}
||\dot{\boldsymbol{k}}|| &\leq \alpha \\
||\ddot{\boldsymbol{k}}|| &\leq \beta
\end{aligned}
$$

Sparkling recipe



Sparkling acceleration



Reduce #shots → Reduce Acquisition time

Acceleration Factor AF = N/ns
Subsampling Factor R = n/m

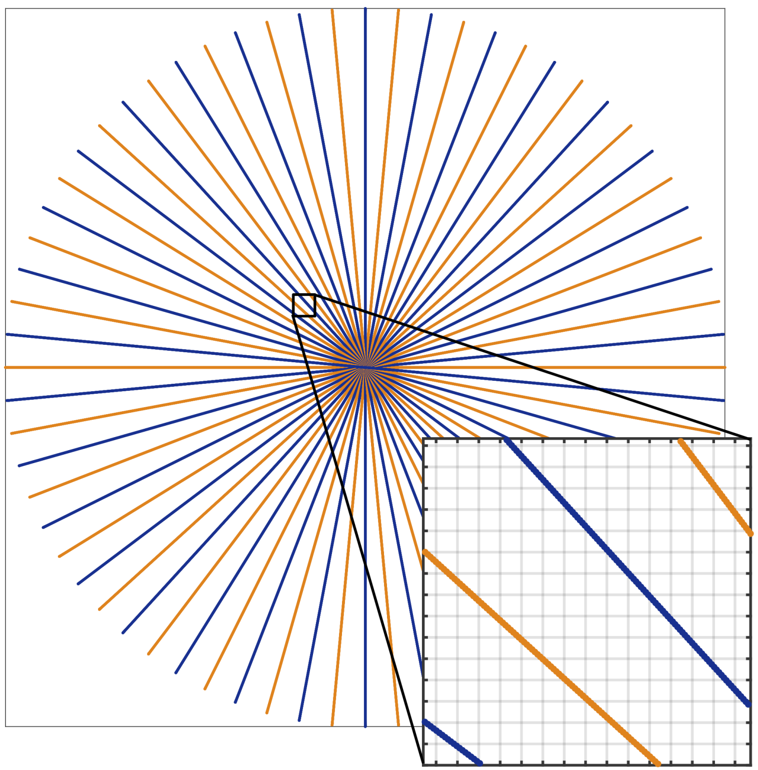
(with n=NxN=#pixels, ns=#shots and m=#samples)

AF ≥ R

Radial VDS Sparkling

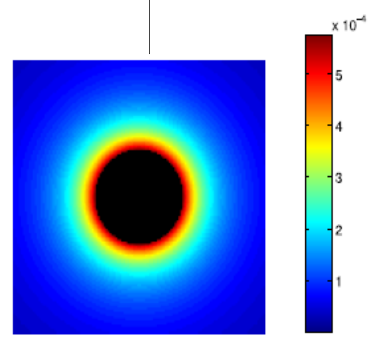
Tobs = 30 ms

Input



Gmax = 40 mT/m
Smax = 200 T/m/s
Δt = 10μs

SPARKLING

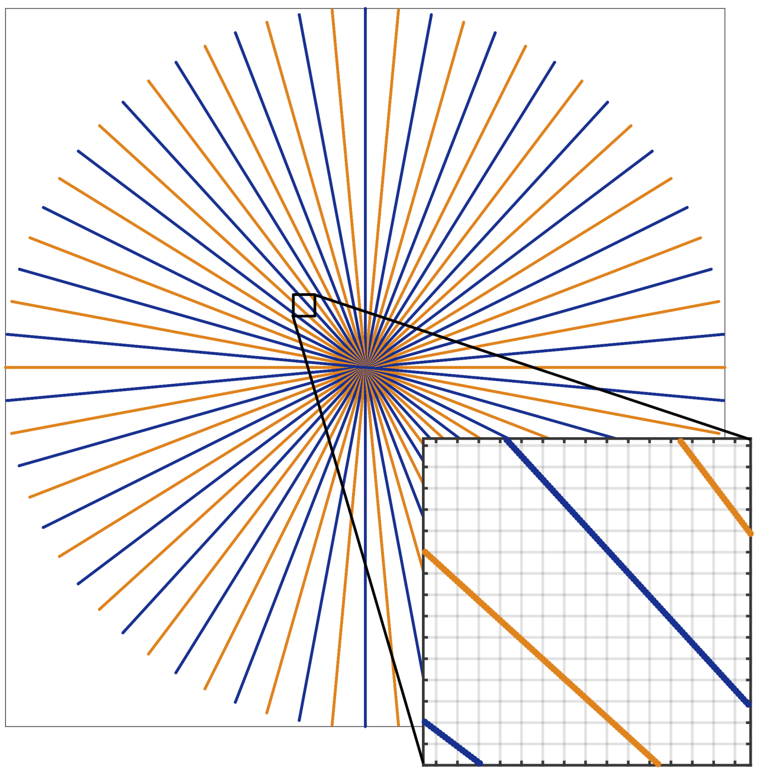


Acceleration factor in time:
AF = N/nc = 512/34 = 15

Radial VDS Sparkling

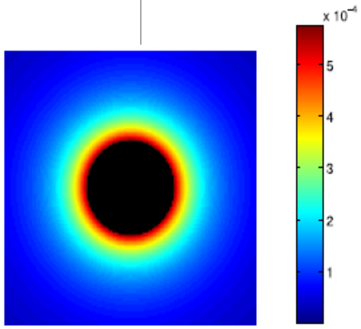
Tobs = 30 ms

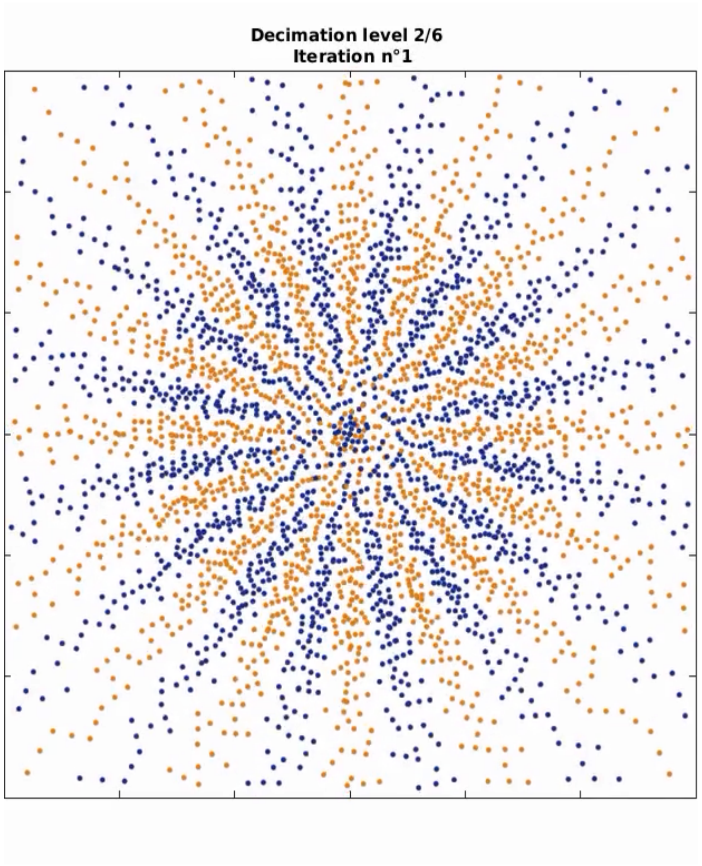
Input



Gmax = 40 mT/m
Smax = 200 T/m/s
Δt = 10μs

SPARKLING



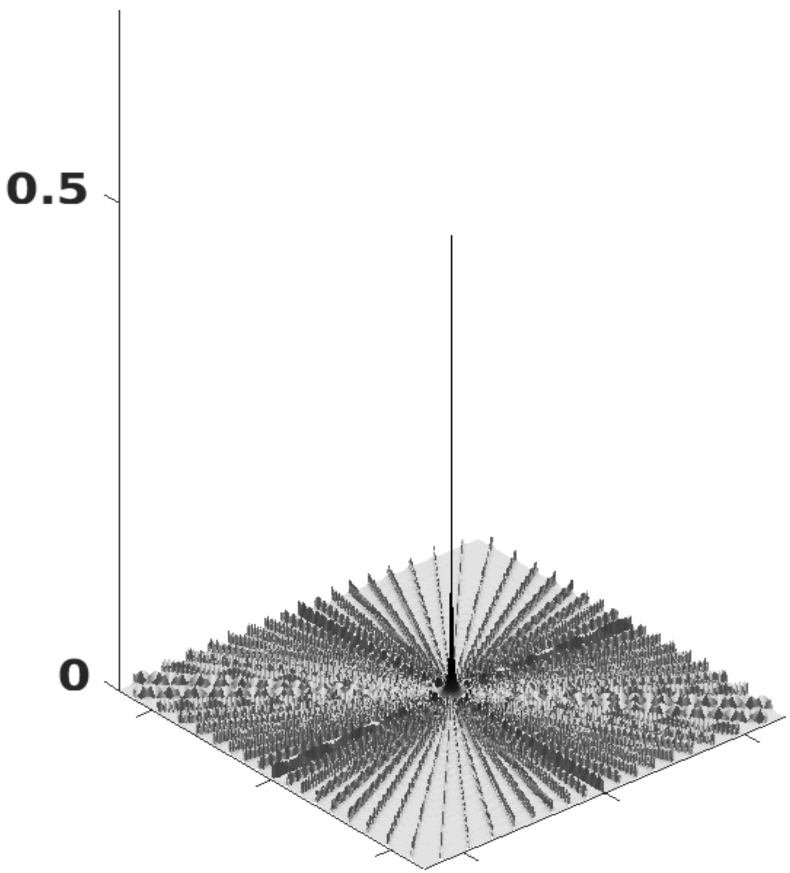


Acceleration factor in time:
AF = N/nc = 512/34 = 15

Radial VDS Sparkling

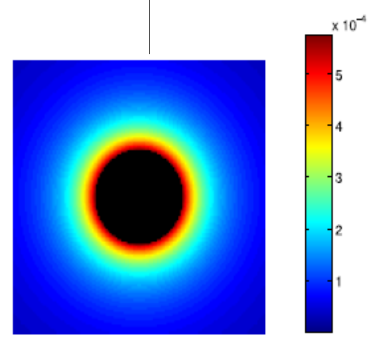
Tobs = 30 ms

Input

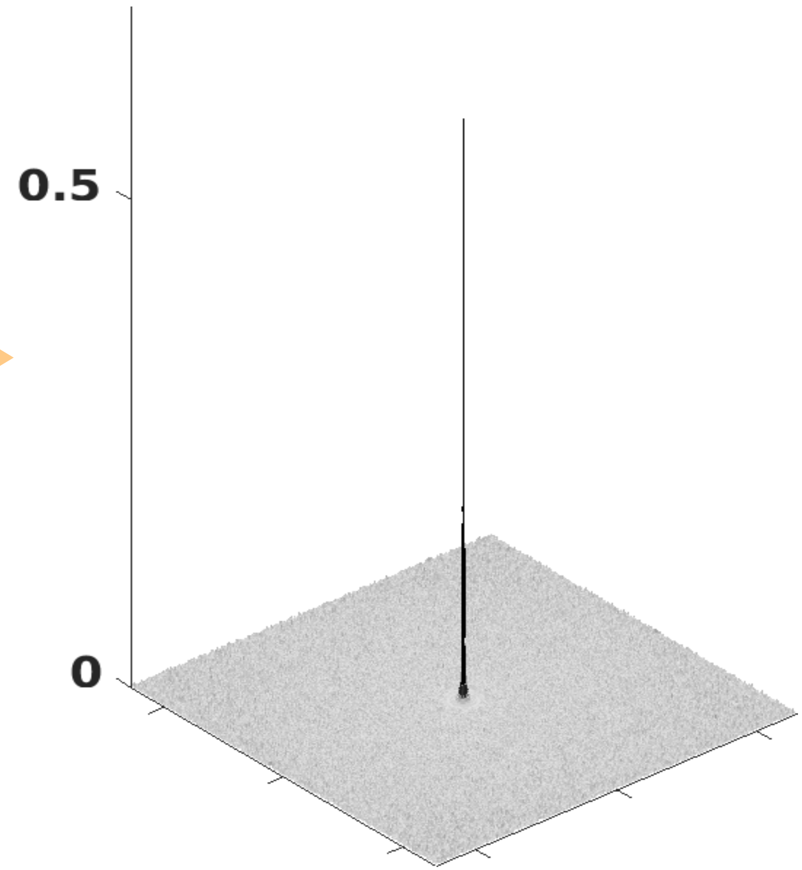


Gmax = 40 mT/m
Smax = 200 T/m/s
Δt = 10μs

SPARKLING



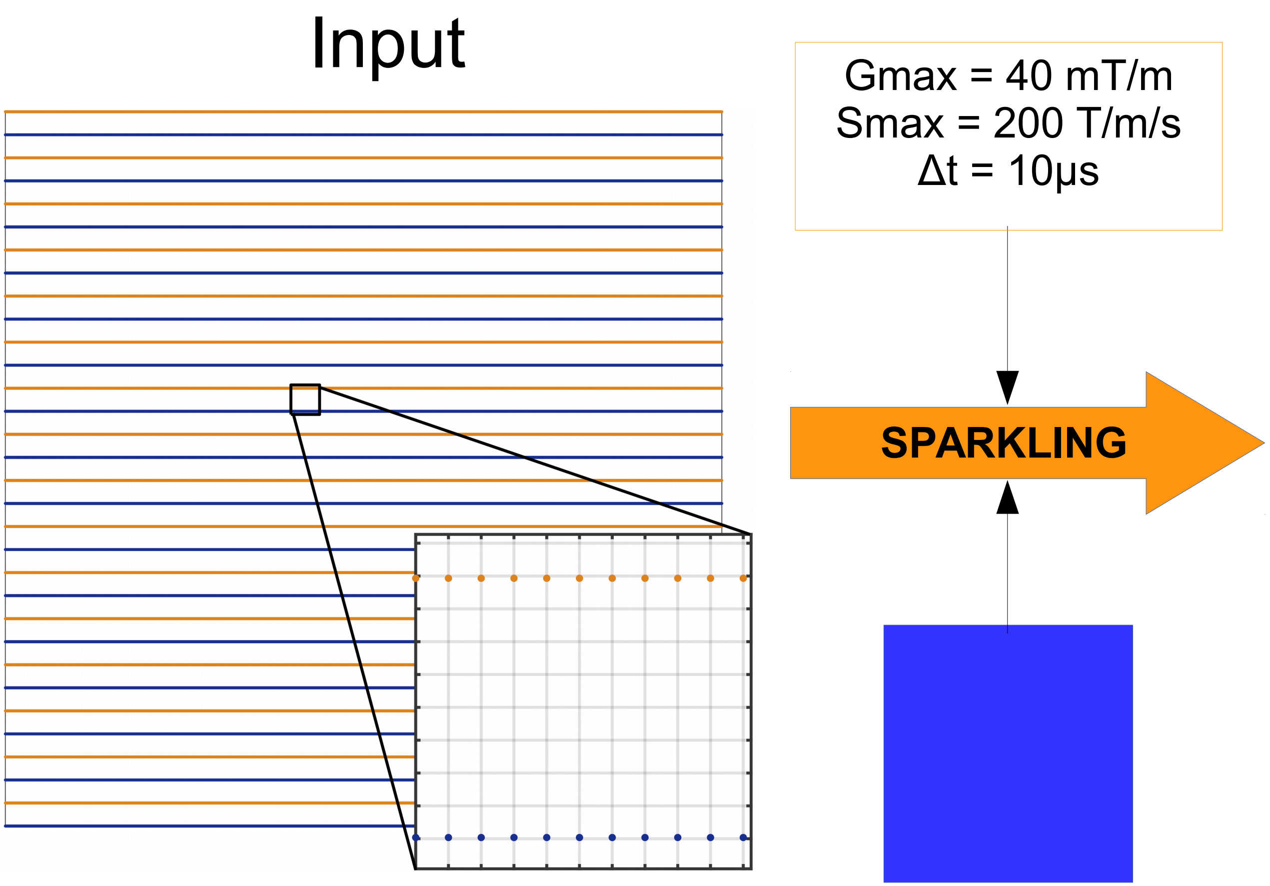
Output



Acceleration factor in time:
AF = N/nc = 512/34 = 15

Uniform Sparkling

Tobs = 20 ms

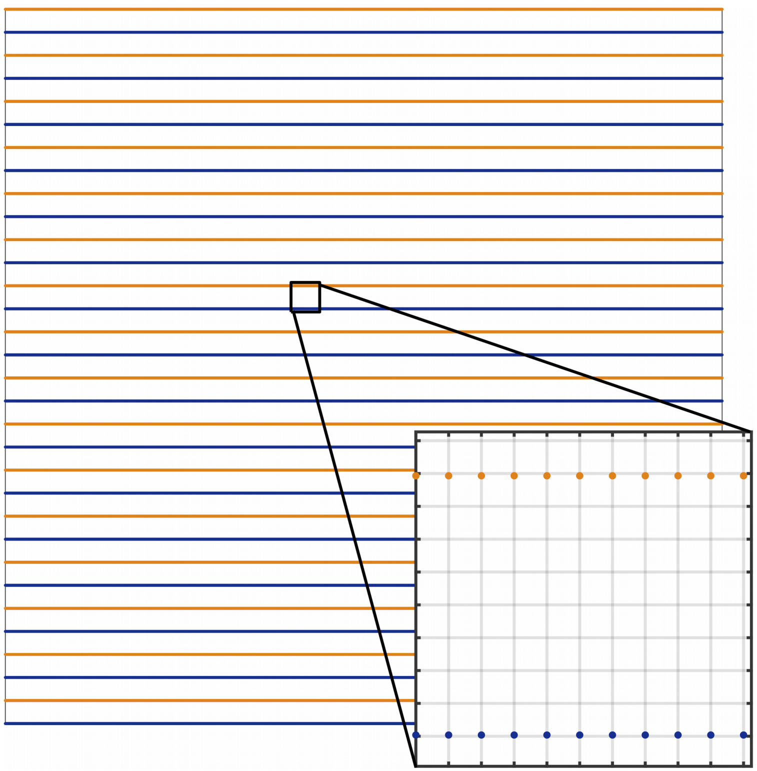


Acceleration factor in time:
AF = N/nc = 256/32 = 8

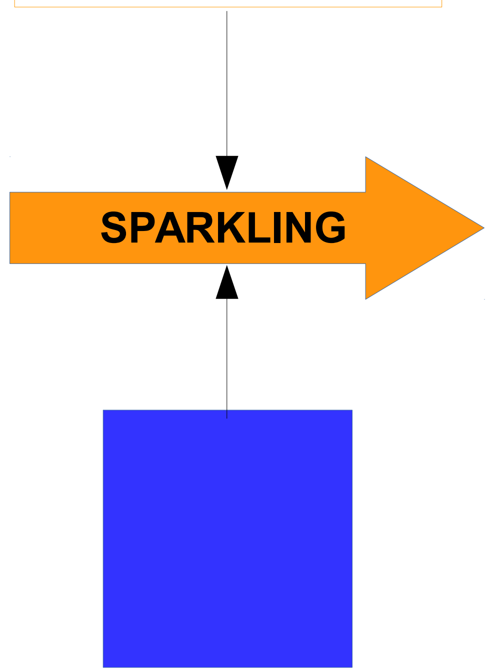
Uniform Sparkling

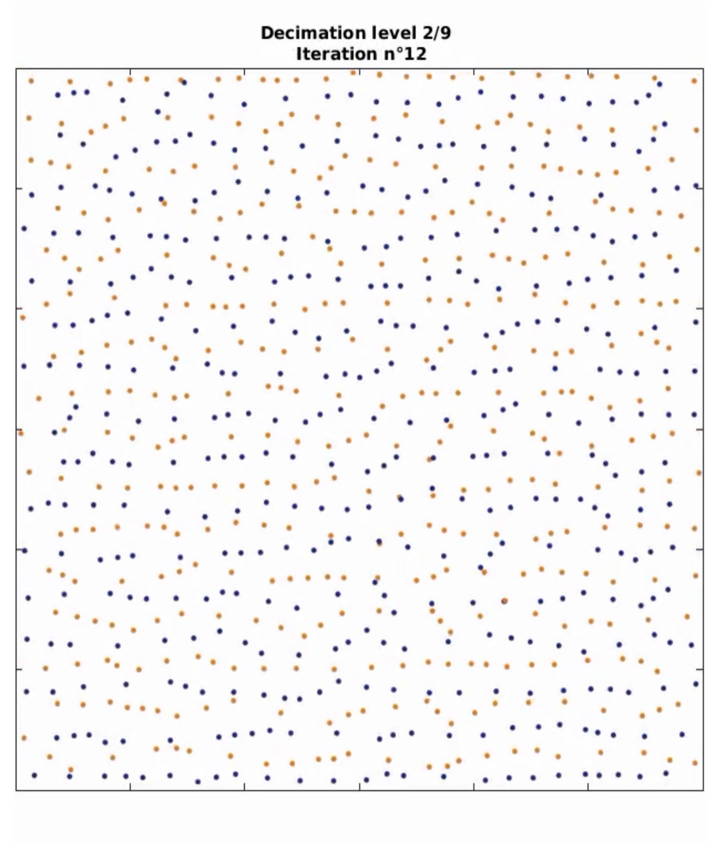
Tobs = 20 ms

Input



Gmax = 40 mT/m
Smax = 200 T/m/s
Δt = 10µs



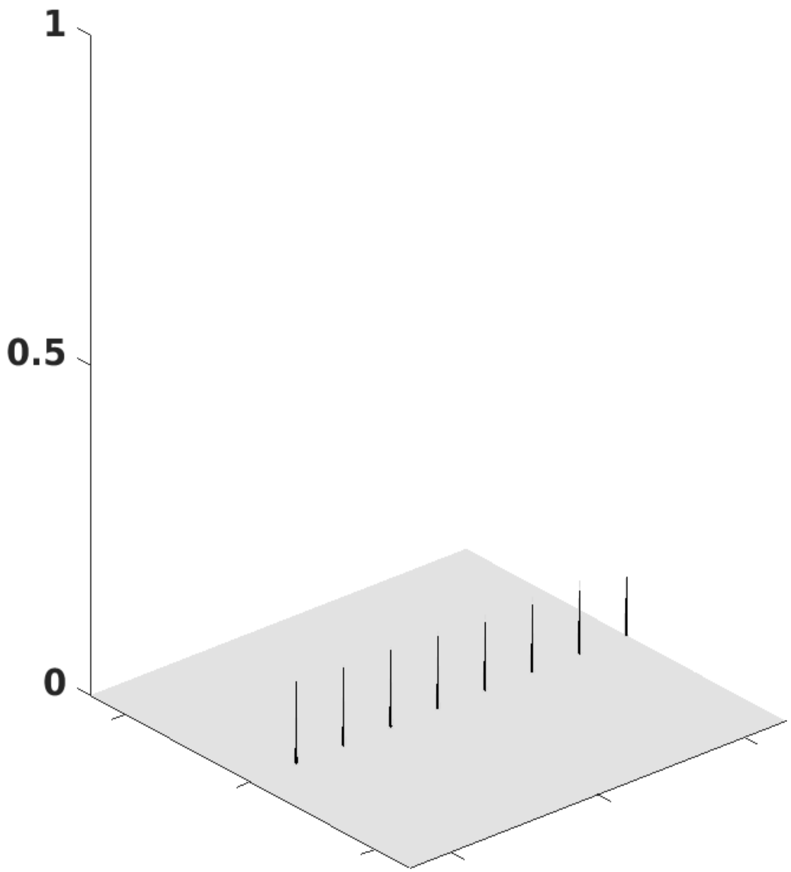


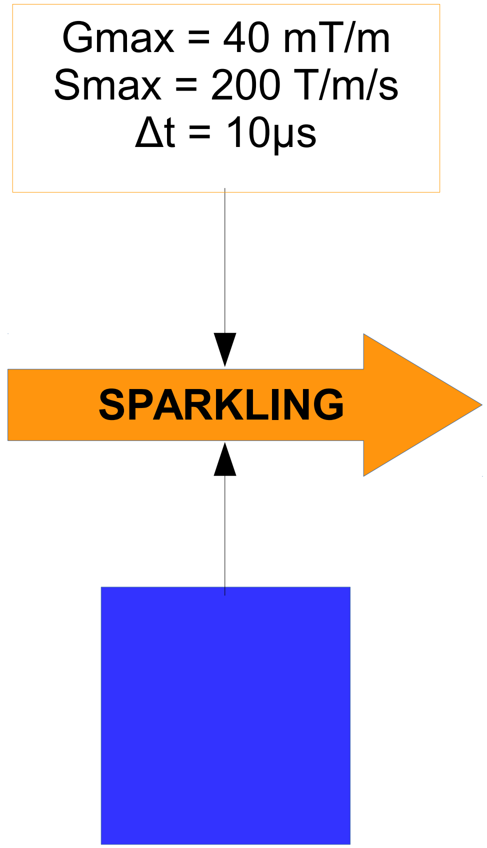
Acceleration factor in time:
AF = N/nc = 256/32 = 8

Uniform Sparkling

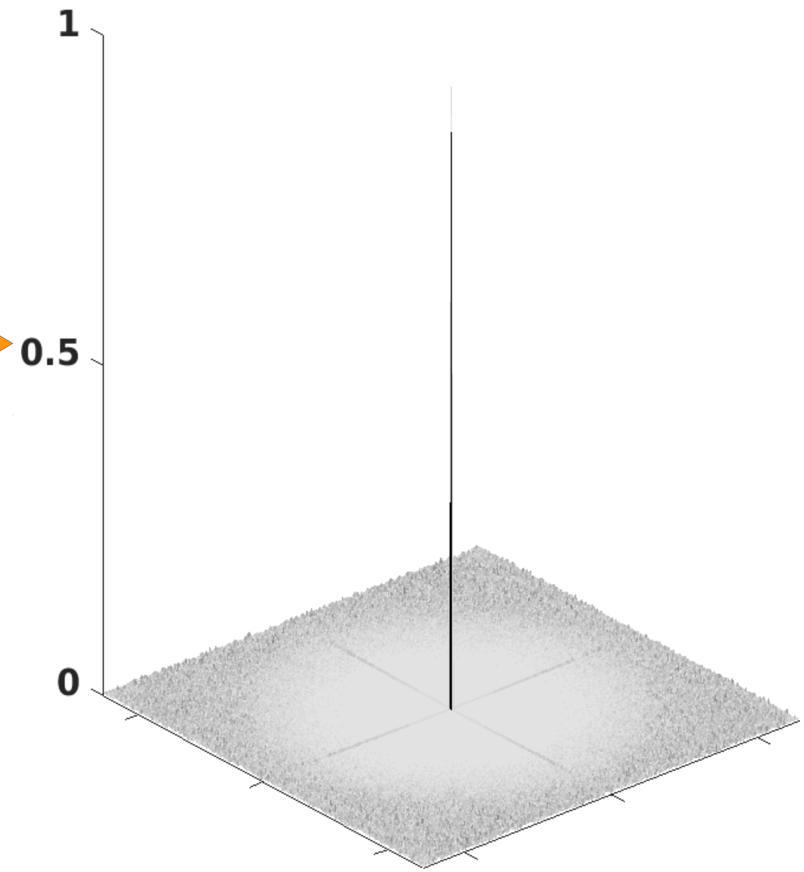
Tobs = 20 ms

Input





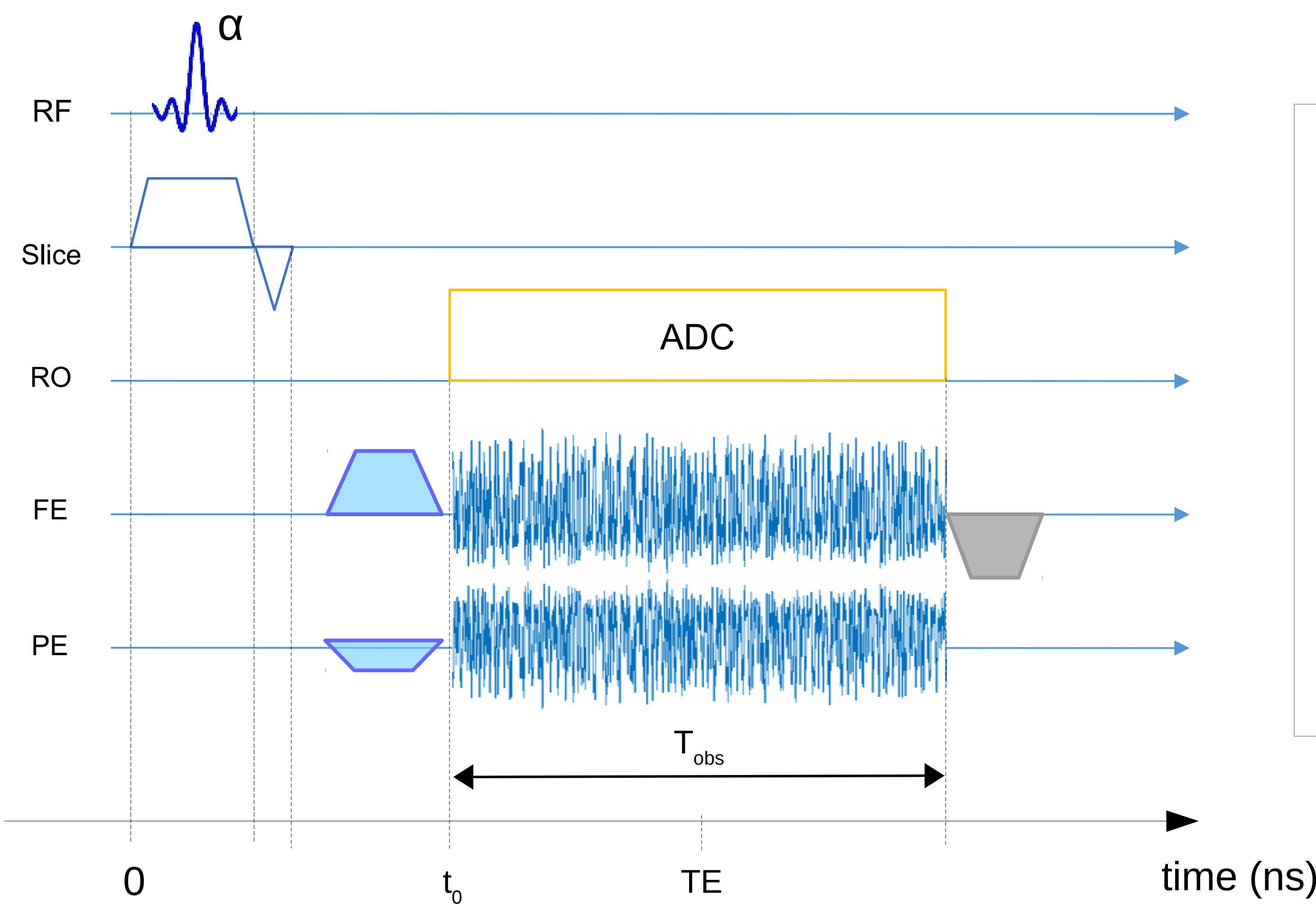
Output



Acceleration factor in time:
AF = N/nc = 256/32 = 8

Sequence Implementation

- T_2*-weighted Gradient-recalled echo sequence (GRE)
- Based on Siemens FLASH sequence



Sequence parameters:

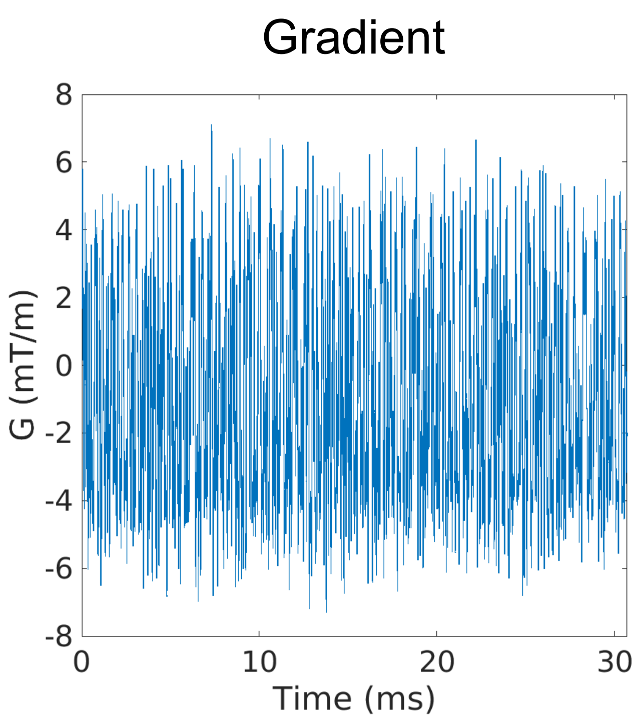
- N=512
- FOV=200x200 mm²
- **TR=550 ms → interleaved**
- TE=30 ms
- α=25°
- BW=32.55 Hz/px
- Tobs=30.72 ms
- Slice thickness: 3 mm
- 32-channel receiver coil

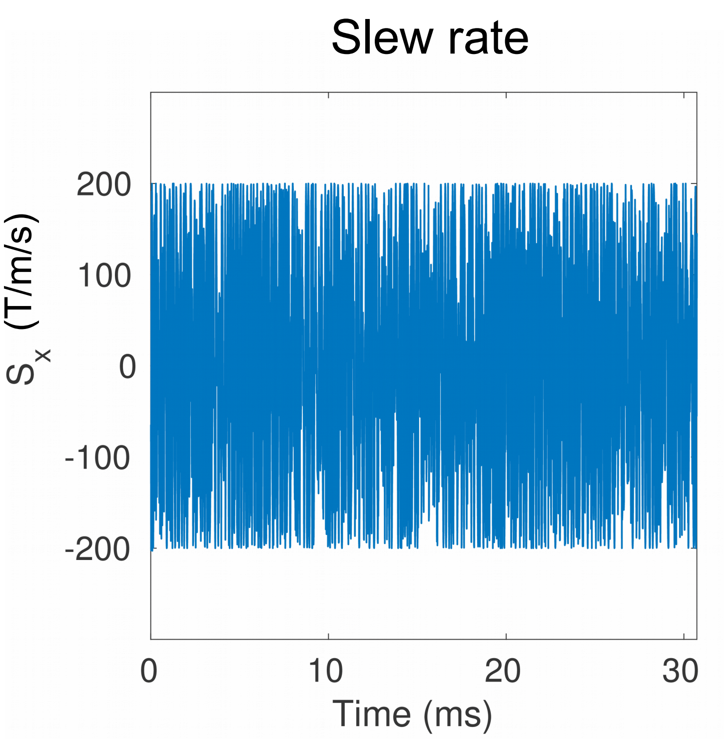
- Gradient raster time = 10 µs

Kasper et al. 2018. NeuroImage.

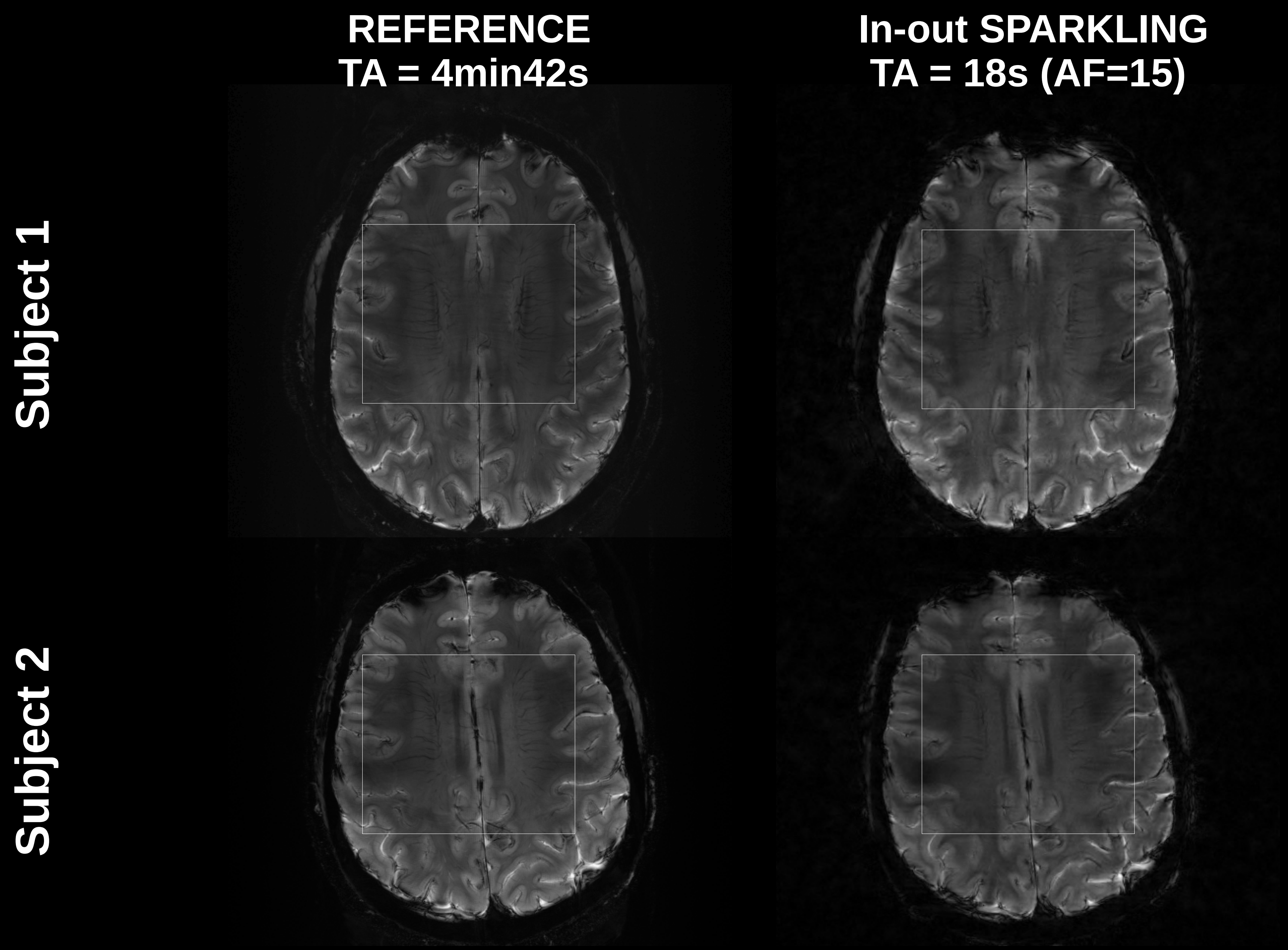
Sequence Implementation

Siemens 7 Tesla scanner





In vivo results: 0.39 mm – 11 slices of 3-mm thickness



Lazarus et al ,in revision to MRM 2018

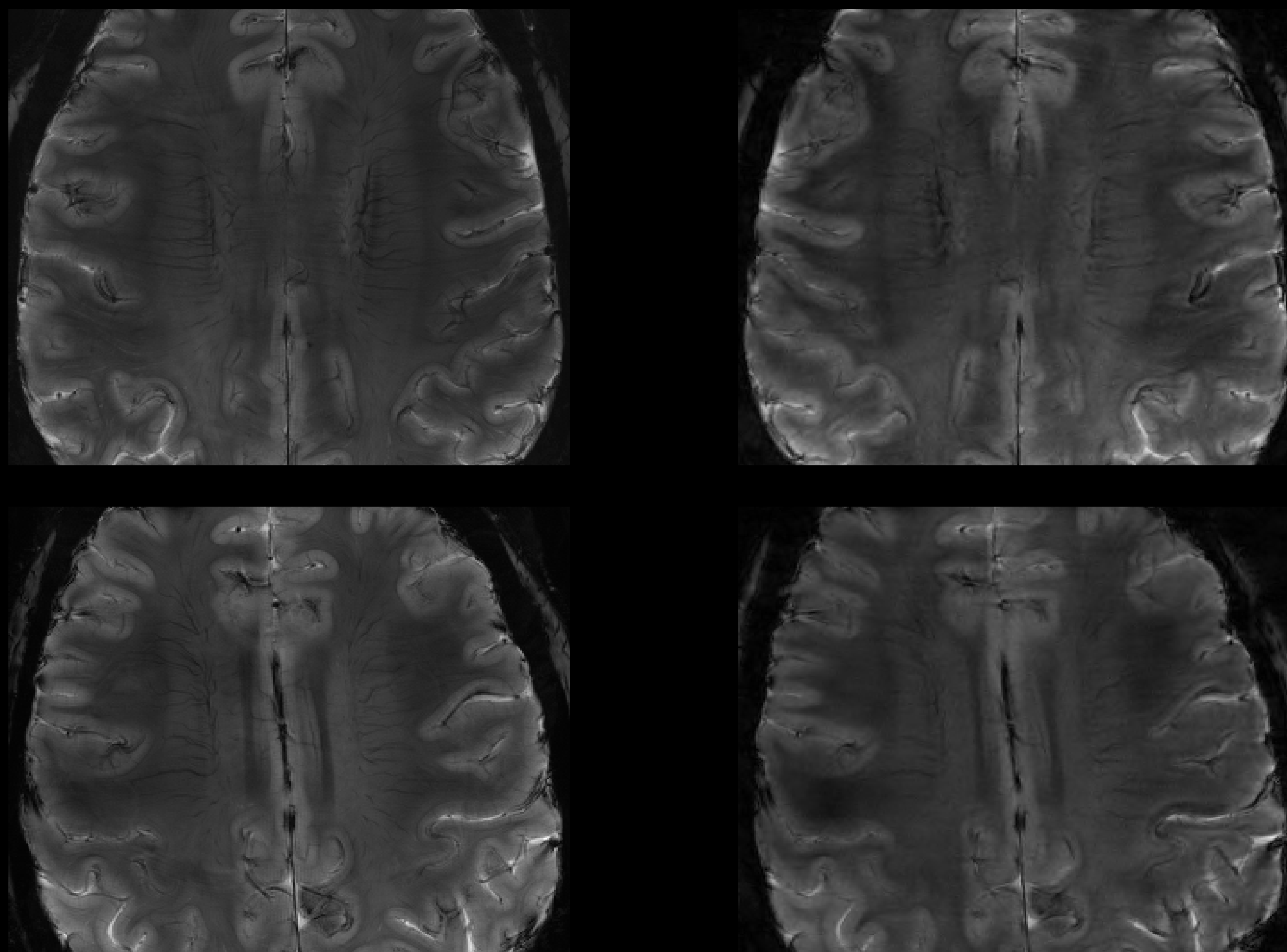
In vivo results: 0.39 mm – 11 slices of 3-mm thickness

REFERENCE
TA = 4 min42s

In-out SPARKLING
TA = 18s (AF=15)

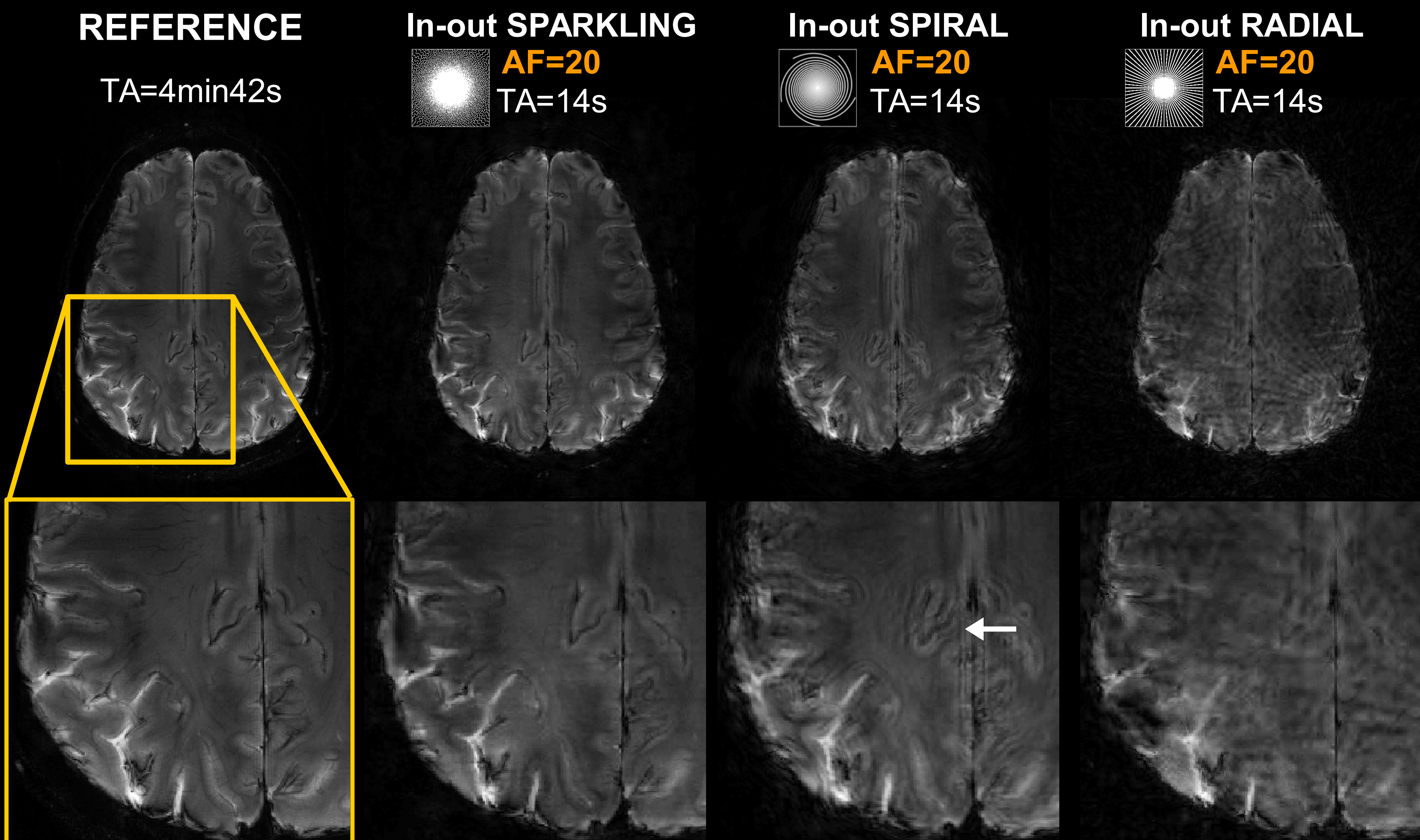
Subject 1

Subject 2



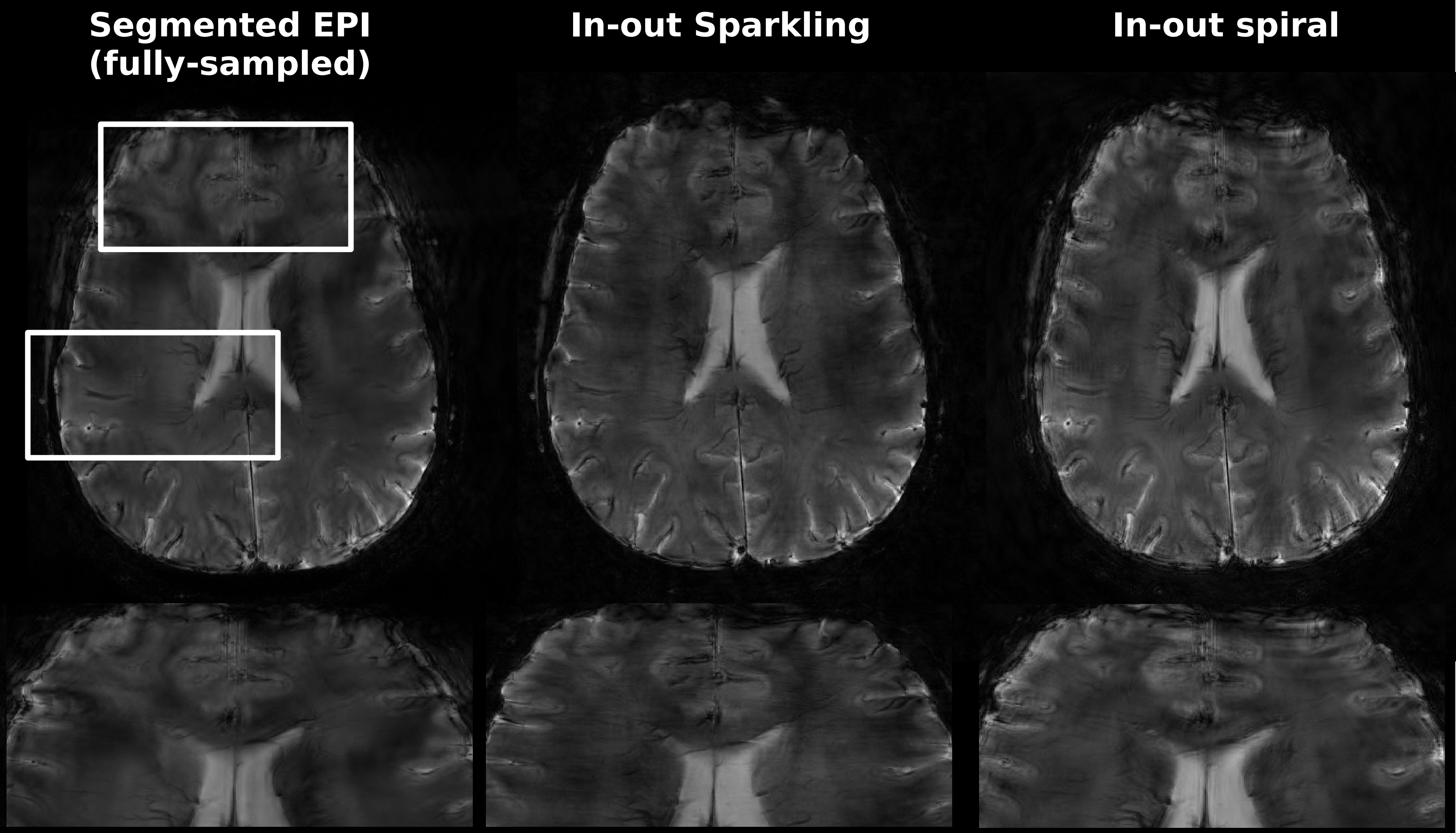
Lazarus et al ,in revision to MRM 2018

In vivo results at 0.39mm resolution
26 shots – 11 slices



Lazarus et al ,in revision to MRM 2018

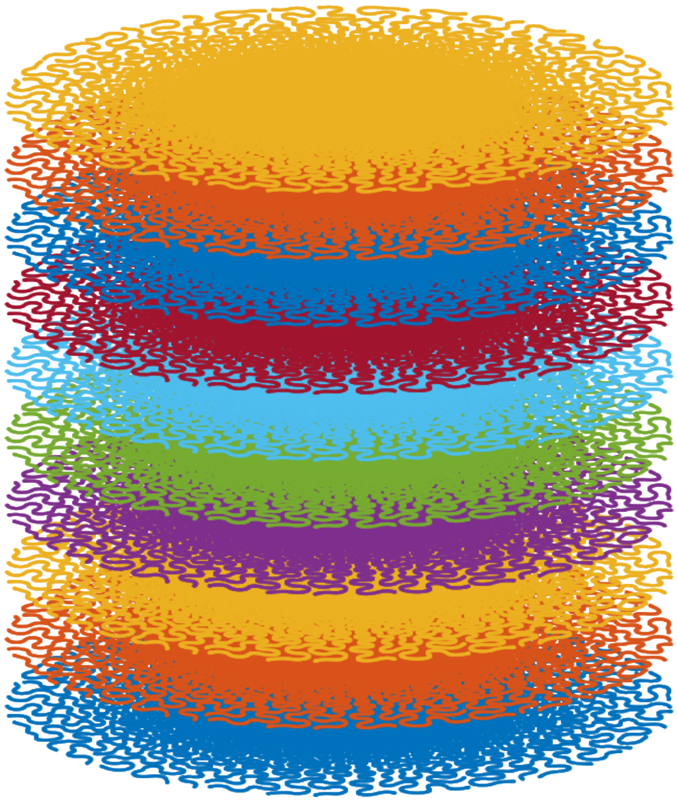
In vivo results at 0.39mm – 43 shots – 11 slices – BW=400kHz – TA=23s



Lazarus et al ,in revision to MRM 2018

Stack-of-Sparkling (SOS)

Regular SOS

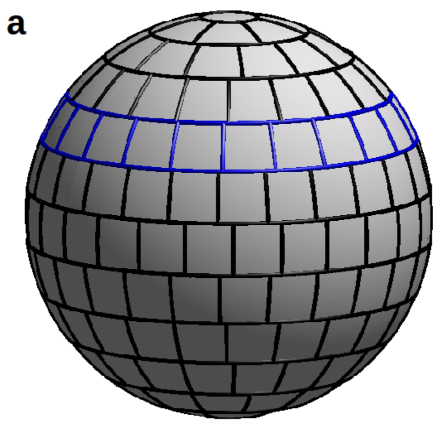


Z-vd SOS

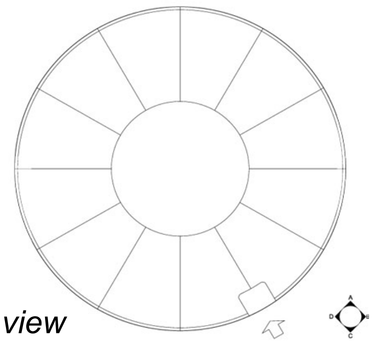
$$n(k_z) = n(0)\frac{\int \pi(:,:,k_z)}{\int \pi(:,:,0)},$$

Fully 3D Sparkling

- Shot-by-shot generation



Equal area tessellation



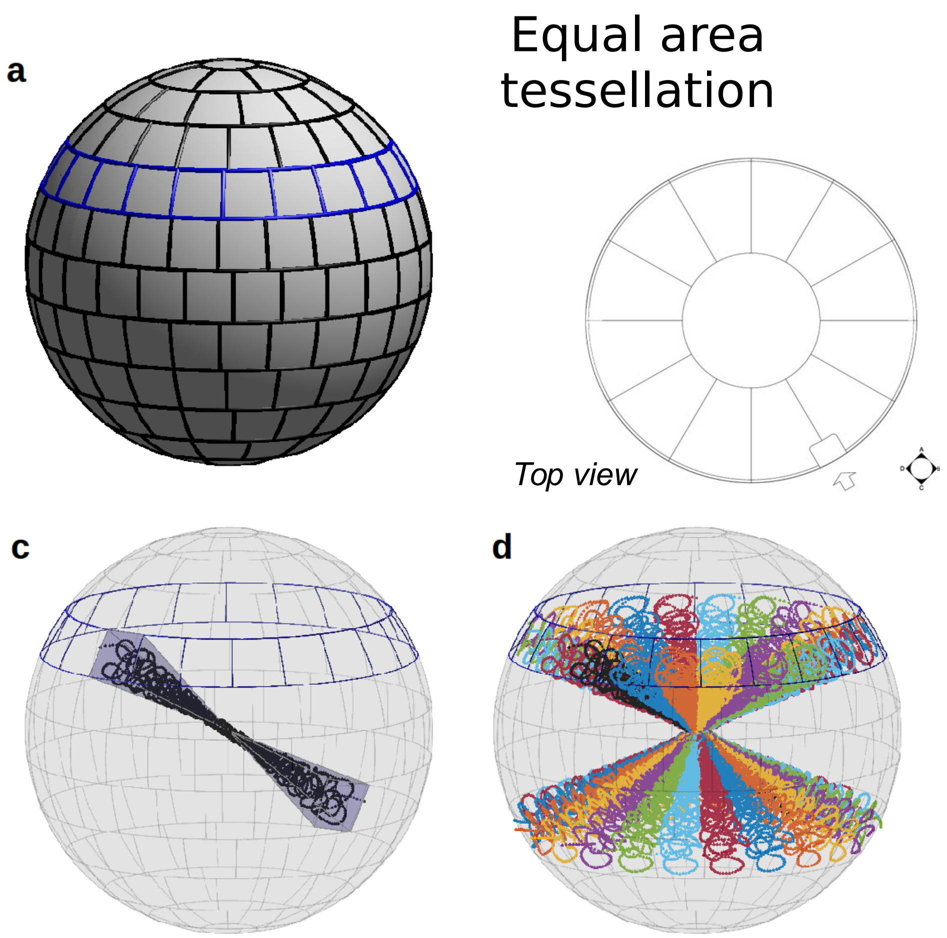
From 1 year to 20 min of computation time

Leopardi et al. 2016. Electronic Transactions on Numerical Analysis

Fully 3D Sparkling

- Shot-by-shot generation

From 1 year to 20 min of computation time

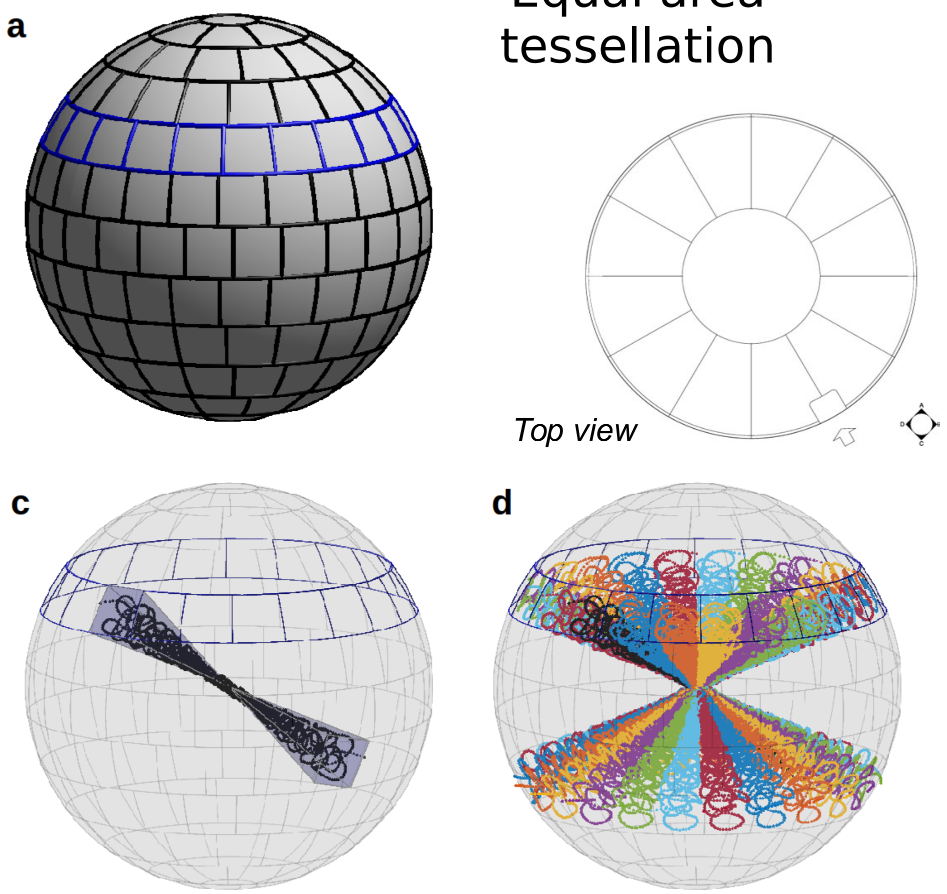


Leopardi et al. 2016. Electronic Transactions on Numerical Analysis

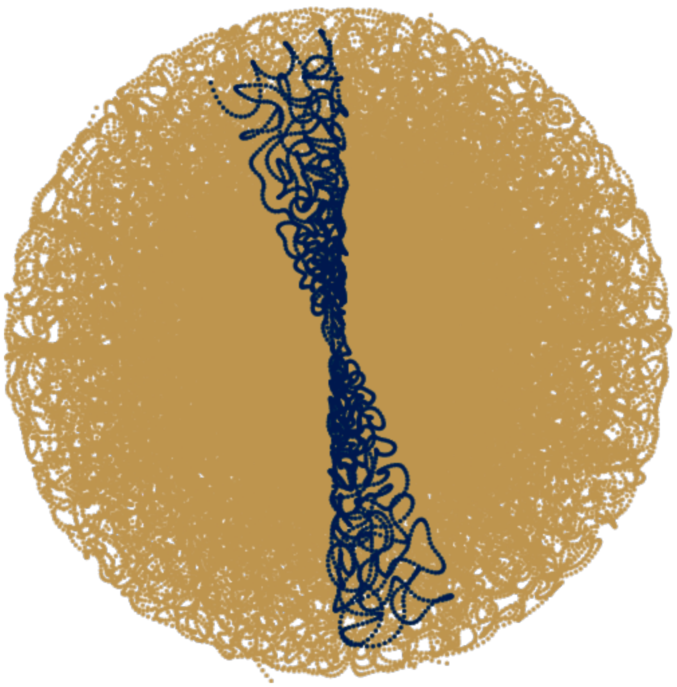
Fully 3D Sparkling

- Shot-by-shot generation

Equal area tessellation



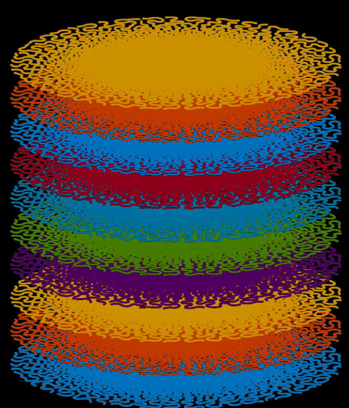
From 1 year to 20 min of computation time



Leopardi et al. 2016. Electronic Transactions on Numerical Analysis

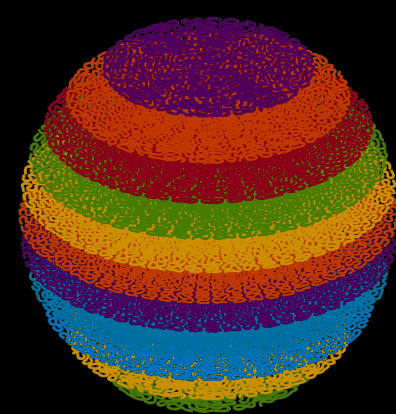
Ex vivo comparison of different Sparkling strategies

T2* contrast
Isotropic resolution of 0.6 mm
Tobs=15 ms – FOV=20x20x14 cm^3



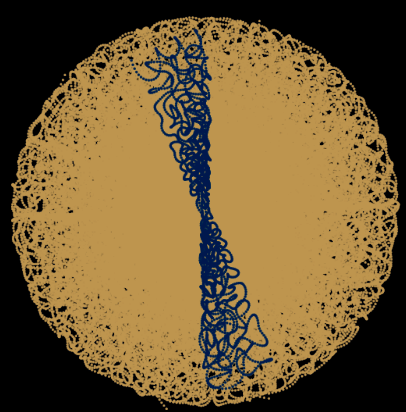
Regular SOS

vs.



z-vd SOS

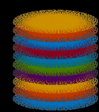
vs.

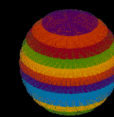


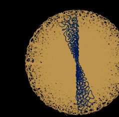
Fully 3D

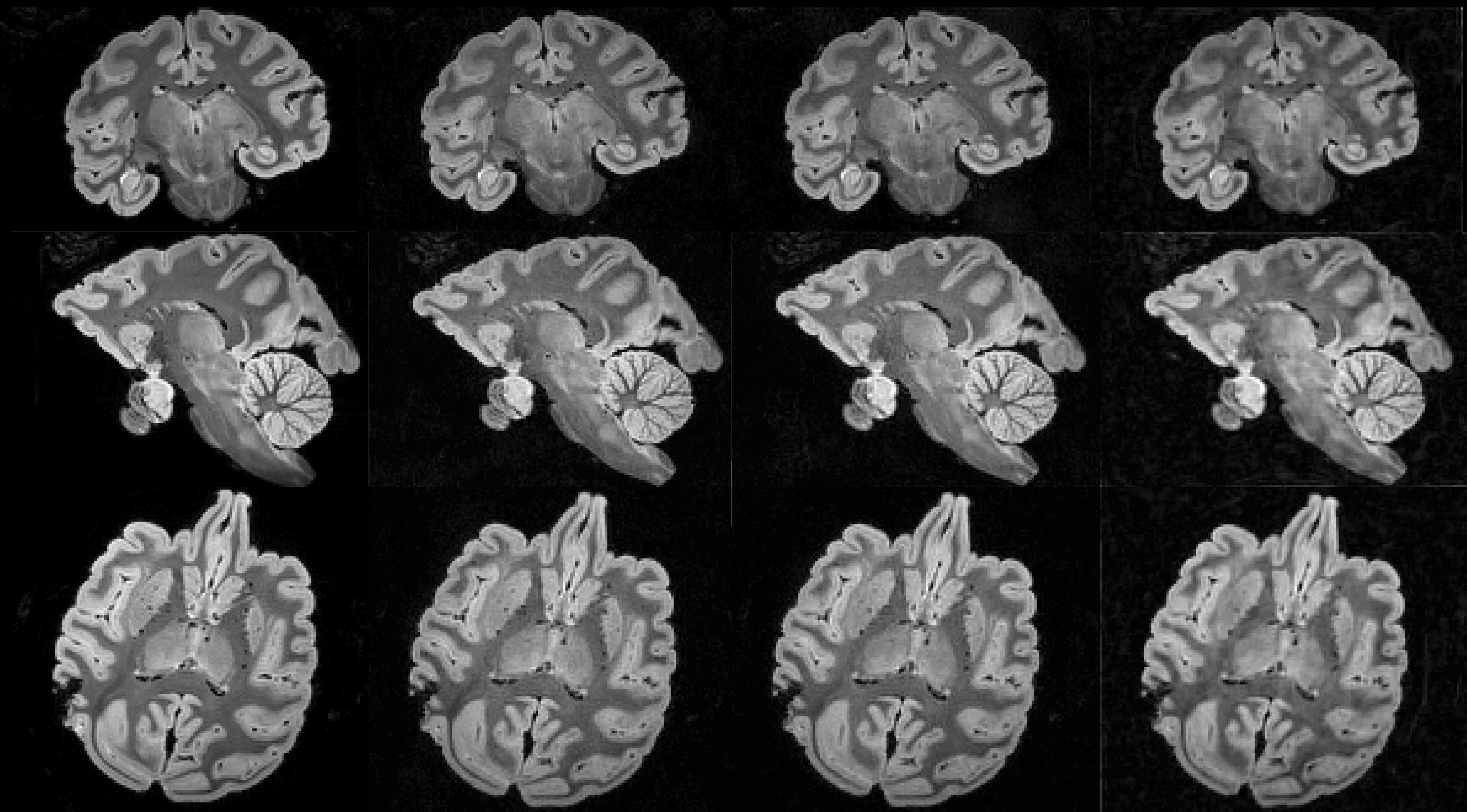
Comparison of different Sparkling methods 4000 shots – AF=20

IPAT 4
TA=14min31s


Regular SOS
TA=2min40s
SSIM=0.87

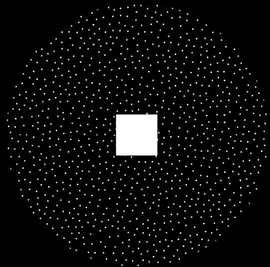

z-vd SOS
TA=2min40s
SSIM=0.88

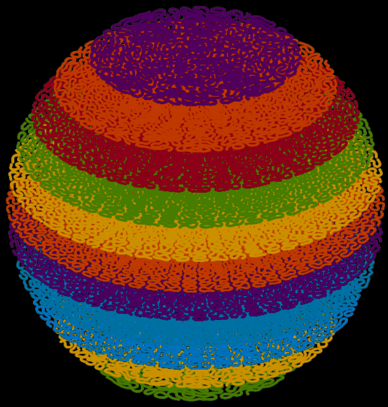

Fully 3D
TA=2min40s
SSIM=0.79



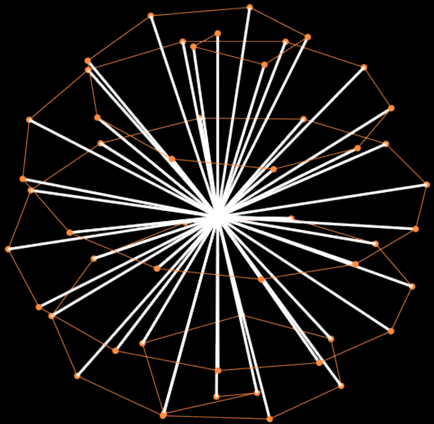
Ex vivo comparison with other strategies

T2* contrast
Isotropic resolution of 0.6 mm
TA=45s



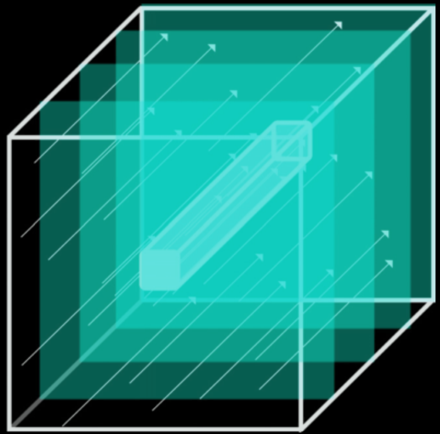


VS.



Larson et al. 2007

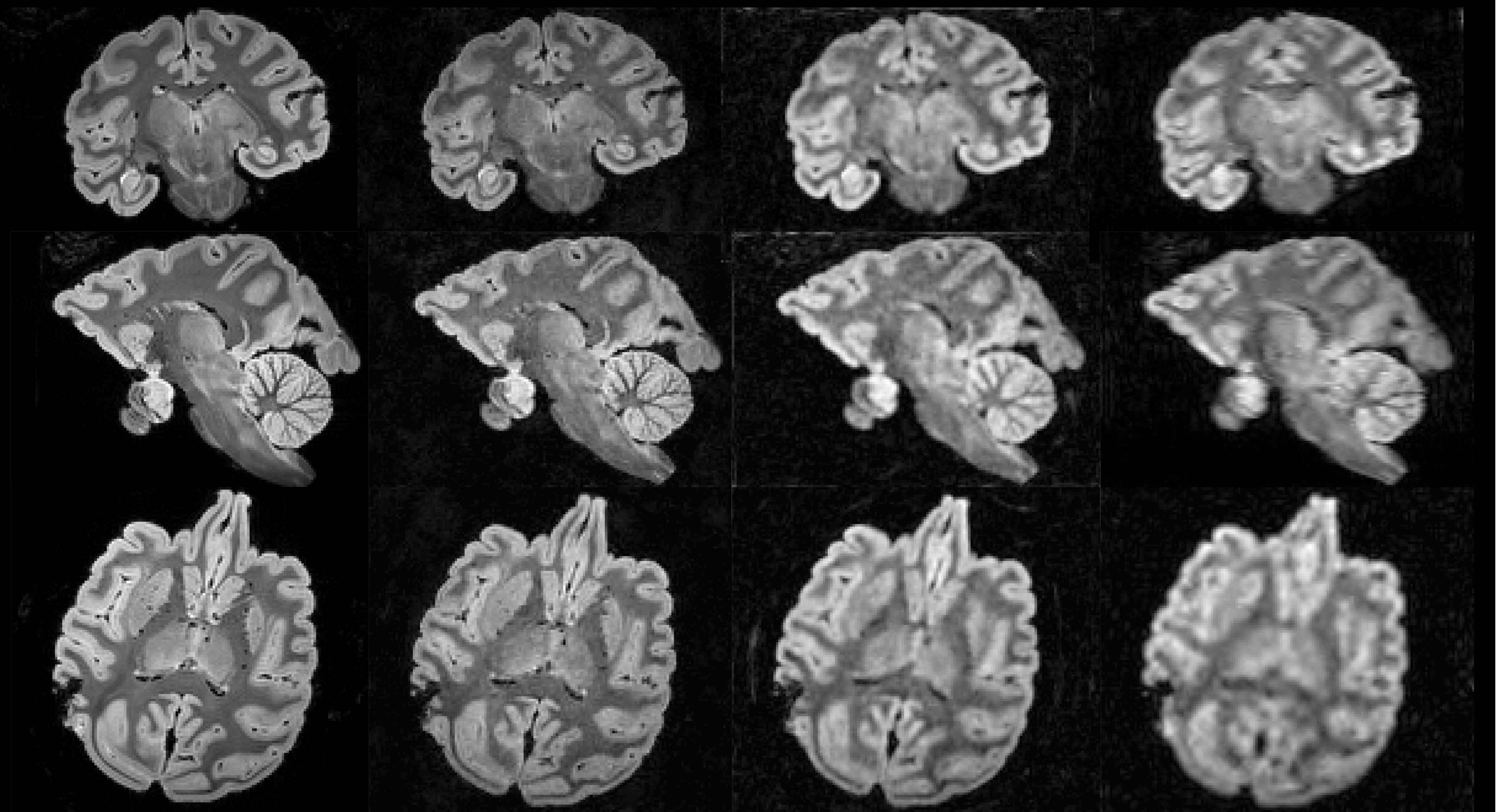
VS.



Lustig et al. 2008

SPARKLING vs. other strategies
1140 shots - AF=69

IPAT 4	SPARKLING	RADIAL	Poisson disk lines
TA=14min31s	TA=45s	TA=45s	TA=45s
	SSIM=0.86	SSIM=0.72	SSIM=0.58



Conclusion

- **SPARKLING 2D**
  - *Ex vivo* and *in vivo* validation of SPARKLING 2D:

    Demonstration of its **superiority** over both radial and spiral trajectories
  - Sampling **efficiency**, robustness to subsampling
  - Adaptability to hardware constraints and various densities (e.g. anisotropic)
- **SPARKLING 3D**
  - Higher acceleration factors
  - *Ex vivo* validation of SPARKLING 3D
- **MR image reconstruction**
  - 2D **offline/online** multichannel image recon with sensitivity maps extraction
  - 3D **offline** single/multi-channel image recon
  - **2D/3D calibrationless** image recon with group sparsity promotion
  - **Analysis-based** regularization using redundant decompositions (e.g. curvelets) and primal/dual recon algorithms

Acknowledgements:

NeuroSpin:
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Pierre Weiss
Jonas Kahn
Claire Boyer

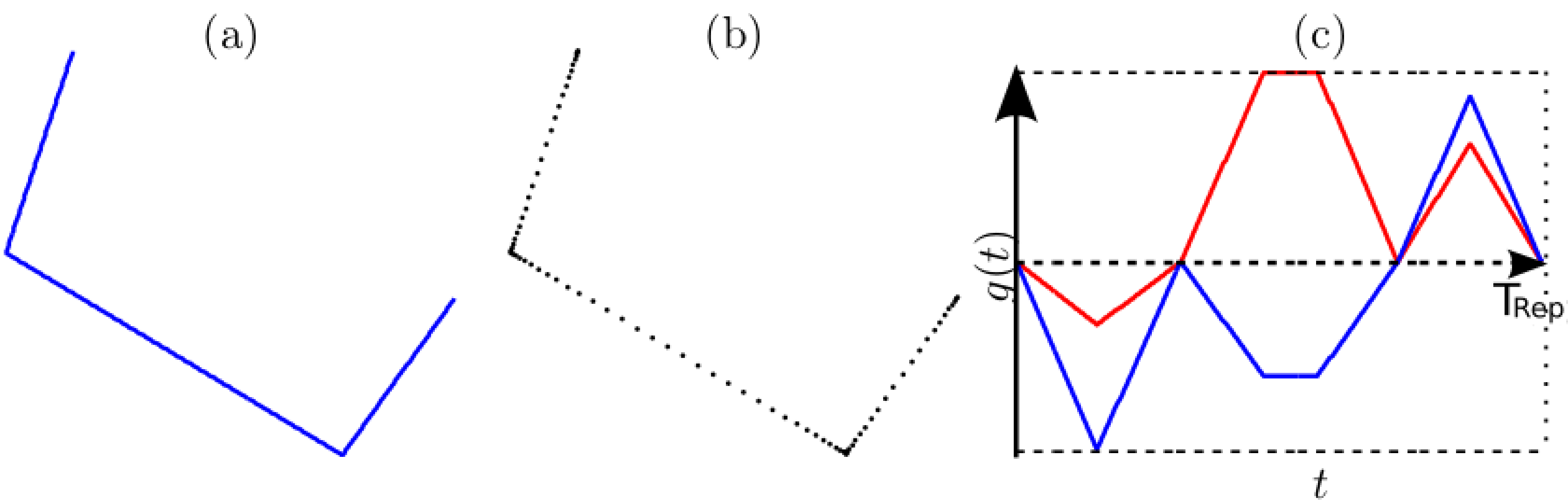
Siemens Healthineers:
Franck Mauconduit

The Parameterization Problem

Finding a parameterization in $\mathcal{S}$ corresponding to a curve support is not easy !

- Classical approach, find an admissible parameterization [Hargreaves et al., 2004, Lustig et al., 2008]:

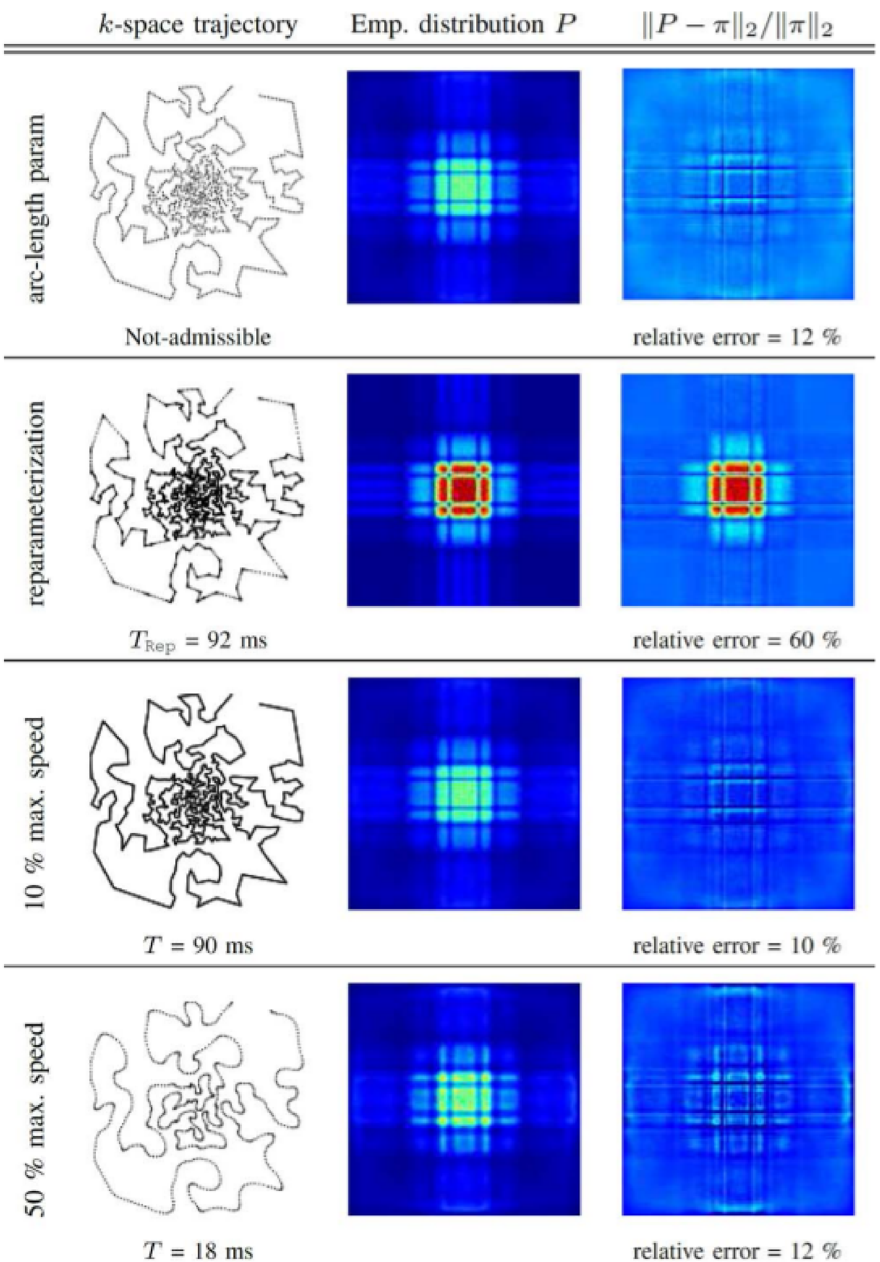
$$T_{\text{rep}} = \min T' \text{ such that } \exists r : [0, T'] \mapsto [0, T], c \circ r \in \mathcal{S}$$



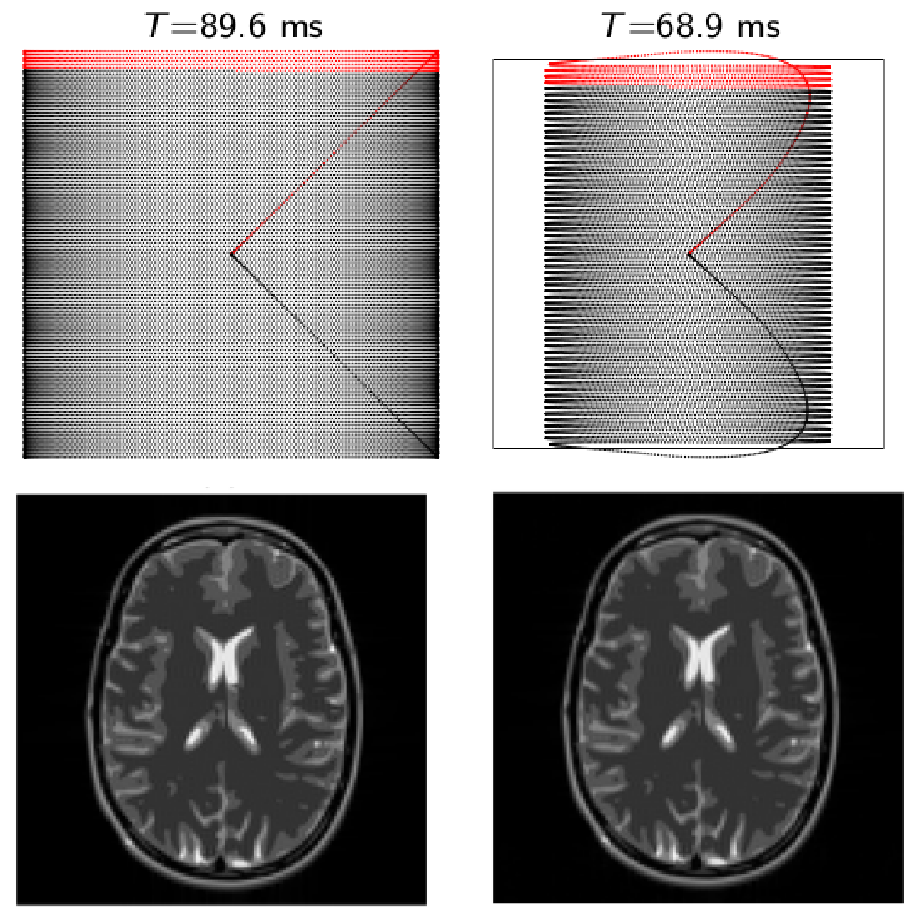
Stick to the support of c

[Chauffert et al, IEEE TMI 2016]

Illustration & Results



EPI trajectories



[Chauffert et al, IEEE TMI 2016]