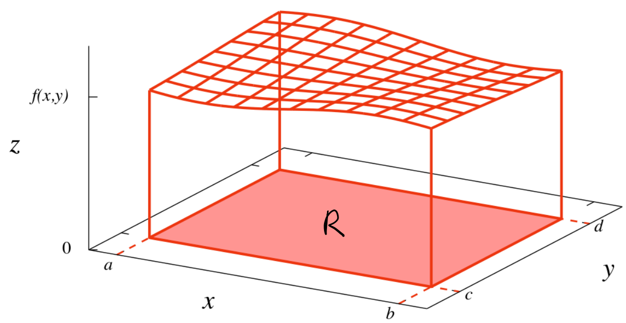
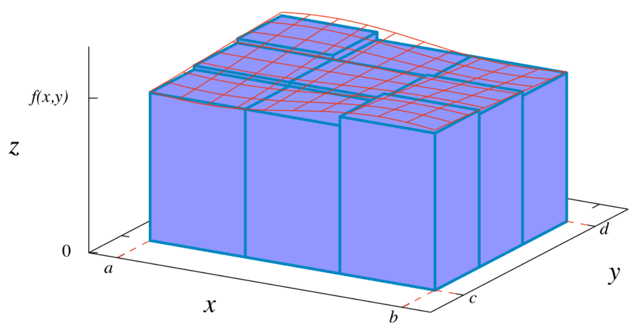
Double integrals

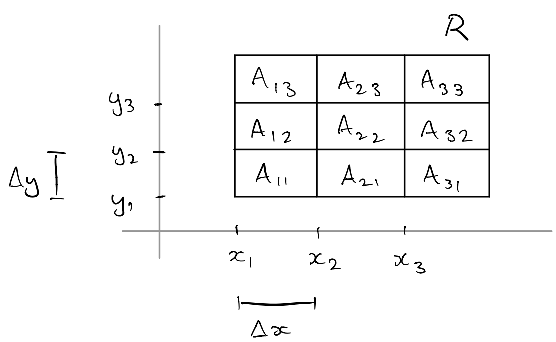
How can we find the volume below a surface $z = f(x,y)$ and above the rectangle $R = \{(x,y) \in \mathbb{R}^2 : a \leq x \leq b,\ c \leq y \leq d\}$?



First approximate by summing up volumes of rectangular prisms



The plan is to take the limit as the number of rectangular prisms $\to \infty$ and their base area $\to 0$. Need to be able to write the sums.

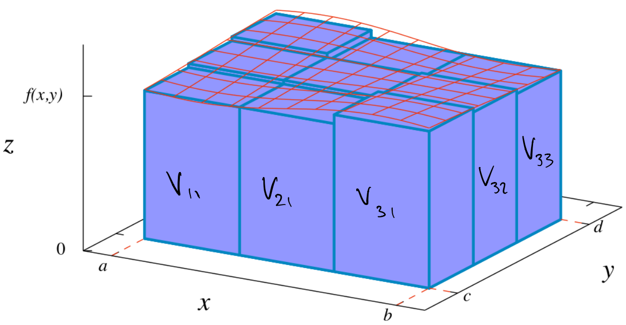


The area A_{ij} has bottom left corner at (x_i, y_j)

The total area can be written as a <u>double sum</u>

$$\sum_{i}^{3} \sum_{j}^{3} A_{ij} = \sum_{i}^{3} (A_{i1} + A_{i2} + A_{i3}) = (A_{11} + A_{12} + A_{13}) + (A_{21} + A_{22} + A_{23}) + (A_{31} + A_{32} + A_{33})$$

Write V_{11}, V_{12} ... for the volume above A_{11}, A_{12}, ...



Then $\quad V_{11} = A_{11} \times f(x_1, y_1)$ $\quad$ because $f(x_1, y_1)$ is the height of V_{11}

$$V_{21} = A_{21} \times f(x_2, y_1)$$

$$\vdots$$

$$V_{ij} = A_{ij} \times f(x_i, x_j)$$

So the total volume is also a double sum

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{3} V_{ij} = \sum_{i}^{3} \sum_{j}^{3} A_{ij} \, f(x_i, x_j)$$

Since we have chosen equally spaced intervals along the x and y axes: $\quad A_{11} = \Delta x \Delta y \quad A_{21} = \Delta x \Delta y \quad \ldots \quad A_{ij} = \Delta x \Delta y$

and therefore

$$V \approx \sum_{i}^{3} \sum_{j}^{3} f(x_i, x_j) \, \Delta x \, \Delta y$$

Inspired by our previous success with single Riemann integrals we define the double integral over a rectangular region R

$$\iint_R f(x,y)\, dA = \lim_{\substack{m,n \to \infty \\ \Delta x, \Delta y \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \, \Delta x \, \Delta y$$

if the limit exists. Here dA is called the area element.

Calculating double integrals

Theorem 7.2. *(Fubini's theorem for rectangular regions)*

Let $f(x,y)$ be a continuous function on the rectangular region R defined by $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$. Then the double integral $\iint\limits_R f(x,y)\,dA$ exists and

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

← integrate wrt x first, treating y as constant

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

← integrate wrt y first treating x as const.

EXAMPLE

$f(x,y) = x^2 + xy \quad , \quad R = \{ (x,y) : 1 \le x \le 2 , -1 \le y \le 1 \}$

$$\iint\limits_R f(x,y)\,dA = \int_{-1}^{1} \int_{1}^{2} x^2 + xy \, dx \, dy$$

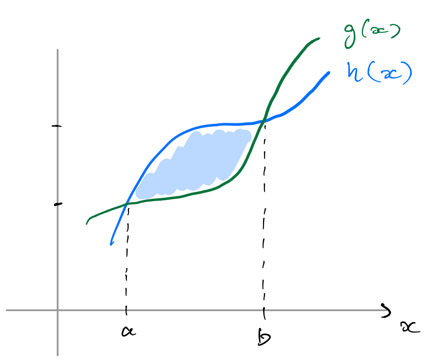
$$= \int_{-1}^{1} \left[\frac{x^3}{3} + \frac{x^2 y}{2} \right]_1^2 dy$$

$$= \int_{-1}^{1} \left[\frac{8}{3} + 2y - \frac{1}{3} - \frac{1}{2} y \right] dx$$

$$= \left[\frac{7}{3} y + \frac{3}{4} y^2 \right]_{-1}^{1}$$

$$= \frac{7}{3} + \frac{3}{4} + \frac{7}{3} - \frac{3}{4} = \frac{14}{3}$$

Double integrals over bounded regions



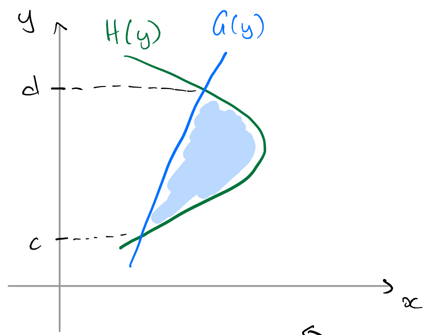
TYPE 1 : vertically simple region

$R = \{ (x,y) : a \le x \le b , g(x) \le y \le h(x) \}$

$g(x) \le h(x)$ for all $a \le x \le b$

$$\iint\limits_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

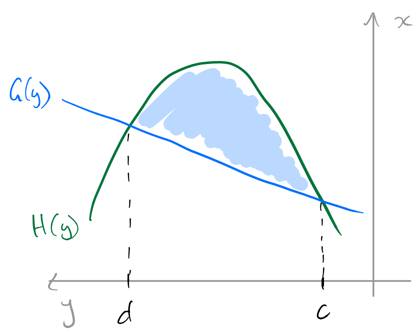
TYPE 2 : horizontally simple region



$$R = \{ (x,y) : G(y) \le x \le H(y) , c \le y \le d \}$$

where $G(y) \le H(y)$ for $c \le y \le d$

Rotate



$$\iint_R f(x,y)\, dA = \int_c^d \int_{G(y)}^{H(y)} f(x,y)\, dx\, dy$$

Note: if $g(x)$ and $h(x)$ are invertible for $a \le x \le b$ then the TYPE 1 region

$$R = \{ (x,y) : a \le x \le b , g(x) \le y \le h(x) \}$$

is also a TYPE 2 region

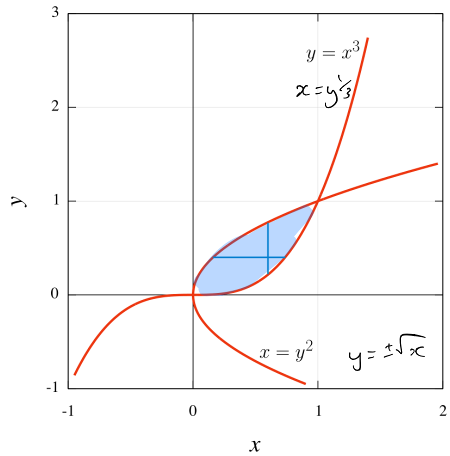
$$R = \left\{ (x,y) : c \le y \le d , \begin{array}{ll} g^{-1}(y) \le x \le h^{-1}(y) & \text{if } g^{-1}(y) \le h^{-1}(y) \\ \text{OR} & \\ h^{-1}(y) \le x \le g^{-1}(y) & \text{if } h^{-1}(y) \le g^{-1}(y) \end{array} \right\}$$

sim. if $G(y)$ and $H(y)$ are invertible...

EXAMPLE Let $f(x,y) = 1$ and R the region bounded by $y = x^3$ and $x = y^2$. Evaluate $\iint_R f(x,y)\, dA$

$$R = \{ (x,y) : 0 \le x \le 1 , x^3 \le y \le \sqrt{x} \}$$

$$= \{ (x,y) : 0 \le y \le 1 , y^{1/3} \le x \le y^2 \}$$



Two options

$$\iint_R f \, dA = \int_0^1 \int_{x^3}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^1 \int_{y^2}^{y^{1/3}} 1 \, dx \, dy$$

$$\int_0^1 \int_{x^3}^{\sqrt{x}} 1 \, dy \, dx = \int_0^1 \left[y \right]_{x^3}^{\sqrt{x}} dx$$

$$= \int_0^1 \left(\sqrt{x} - x^3 \right) dx$$

$$= \left[\frac{2x^{3/2}}{3} - \frac{x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

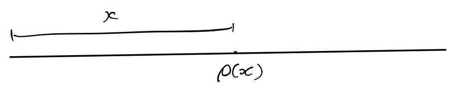
$$\int_0^1 \int_{y^2}^{y^{1/3}} 1 \, dx \, dy = \int_0^1 \left[x \right]_{y^2}^{y^{1/3}} dy$$

$$= \int_0^1 y^{\frac{1}{3}} - y^2 \, dy$$

$$= \left[\frac{3}{4} y^{4/3} - \frac{y^3}{3} \right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

The integral as a weighted sum

We have seen that integration in one variable is not just for finding areas. For example; consider a thin wire with varying density $\rho(x)$ kg/m



The mass of the wire is given by

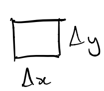
$$\sum_i^n \rho(x_i)\,\Delta x \longrightarrow \int_0^L \rho(x)\,dx = M$$

"weighted sum" - we are adding up small intervals weighted by their density.

The weighting need not be a density (of the kg / ?) kind. It can be an area (eg: finding volumes of rotational solids), a probability density, electric charge density... depending on the application at hand.

Mental picture: a single integral is a (limit of a) weighted sum intervals. Similarly, a <u>double</u> integral is a (limit of a) weighted sum of areas

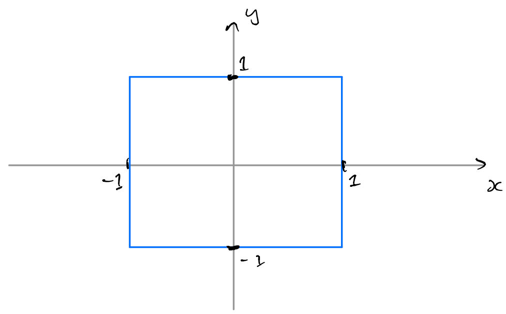
$$\sum_i \sum_j f(x_i, y_j)\,\Delta x\,\Delta y \longrightarrow \iint_R f(x,y)\,dA$$



and the weighting need not be a height (as in finding the volume under a surface) - it could be mass density, energy density, electric charge density, the speed of a fluid flowing through the region R...

In particular, if the weighting is $f(x,y) = 1$, we are just summing up areas, so $\iint_R 1\,dA = \text{Area}(R)$

Example consider a square metal plate



with density $\rho(x,y) = 1 + x^2 + y^2$ kg/m^2 i.e. density increasing with distance from $\underset{\sim}{0}$, maybe it gets thicker away from zero.

We get the mass of the plate by summing up small areas weighted by density:

$$\sum_i \sum_j \rho(x_i, x_j)\,\Delta x\,\Delta y \longrightarrow \int_{-1}^{1}\int_{-1}^{1} \rho(x,y)\,dA$$