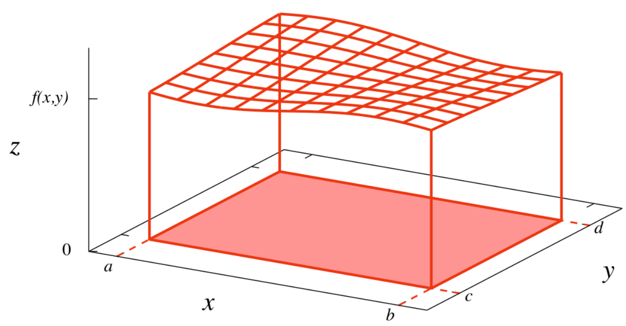
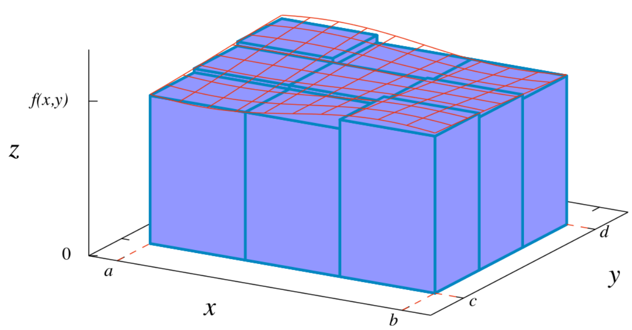
Double integrals

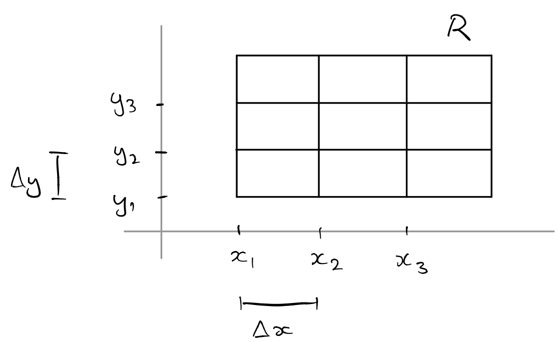
How can we find the volume below a surface $z = f(x,y)$ and above the rectangle $R = \{(x,y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$?



First approximate by summing up volumes of rectangular prisms



The plan is to take the limit as the number of rectangular prisms $\to \infty$ and their base area $\to 0$. Need to be able to write the sums.

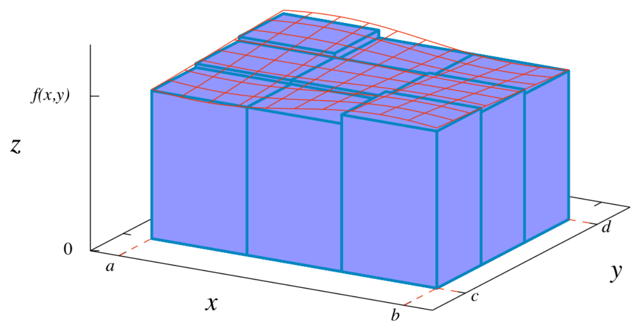


The area has bottom left corner at

The total area can be written as a <u>double sum</u>

$$\sum_{i}^{3} \sum_{j}^{3} A_{ij} = \qquad\qquad =$$

Write $V_{11}, V_{12} \ldots$ for the volume above $A_{11}, A_{12}, \ldots$



Then $V_{11} =$ because $f(x_1, y_1)$ is the height of V_{11}

$V_{21} =$

$\vdots$

$V_{ij} =$

So the total volume is also a double sum

$$V \approx \sum_{i=1}^{3} \sum_{j=1}^{3} \qquad = \sum_{i}^{3} \sum_{j}^{3}$$

Since we have chosen equally spaced intervals along the x and y axes: $A_{11} = \Delta x \Delta y \quad A_{21} = \Delta x \Delta y \quad \ldots \quad A_{ij} =$

and therefore

$$V \approx \sum_{i}^{3} \sum_{j}^{3}$$

Inspired by our previous success with single Riemann integrals we define the double integral over a rectangular region R

$$\iint_R f(x,y)\, dA = \lim_{\substack{m,n \to \infty \\ \Delta x, \Delta y \to 0}} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta x\, \Delta y$$

if the limit exists. Here dA is called the area element.

Calculating double integrals

Theorem 7.2. *(Fubini's theorem for rectangular regions)*
Let $f(x,y)$ be a continuous function on the rectangular region R defined by $R = \{(x,y) \mid a \le x \le b, c \le y \le d\}$. Then the double integral $\iint\limits_R f(x,y)\,dA$ exists and

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

← integrate wrt x first, treating y as constant

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

← integrate wrt y first treating x as const.

EXAMPLE

$f(x,y) = x^2 + xy$, $R = \{(x,y) : 1 \le x \le 2, -1 \le y \le 1\}$

$$\iint\limits_R f(x,y)\,dA =$$

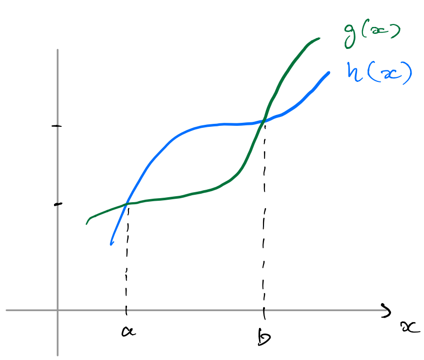
$=$

$=$

$=$

$=$ $\qquad = \frac{14}{3}$

Double integrals over bounded regions

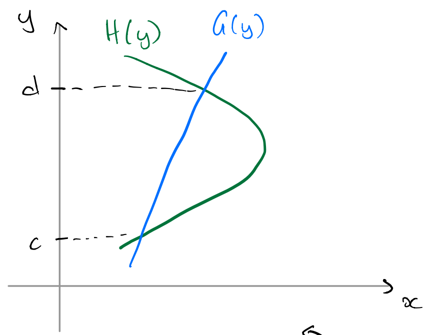


TYPE 1 : vertically simple region

$R = \{(x,y) : a \le x \le b, g(x) \le y \le h(x)\}$

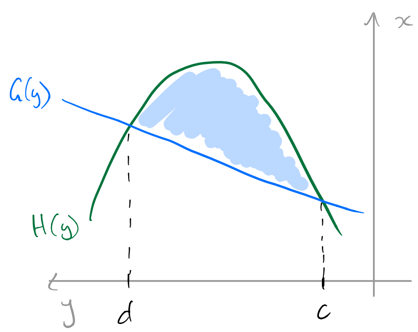
$$\iint\limits_R f(x,y)\,dA =$$

TYPE 2 : horizontally simple region



$$R = \{ (x,y) : G(y) \leq x \leq H(y) \, , \, c \leq y \leq d \}$$

Rotate



$$\iint_R f(x,y)\, dA =$$

Note: if $g(x)$ and $h(x)$ are invertible for $a \leq x \leq b$ then the TYPE 1 region

$$R = \{ (x,y) : a \leq x \leq b \, , g(x) \leq y \leq h(x) \}$$

is also a TYPE 2 region

$$R = \left\{ (x,y) : c \leq y \leq d \, , \begin{array}{ll} g^{-1}(y) \leq x \leq h^{-1}(y) & \text{if } g^{-1}(y) \leq h^{-1}(y) \\ \text{OR} & \\ h^{-1}(y) \leq x \leq g^{-1}(y) & \text{if } h^{-1}(y) \leq g^{-1}(y) \end{array} \right\}$$

sim. if $G(y)$ and $H(y)$ are invertible...

EXAMPLE Let $f(x,y) = 1$ and R the region bounded by $y = x^3$ and $x = y^2$. Evaluate $\iint_R f(x,y)\, dA$

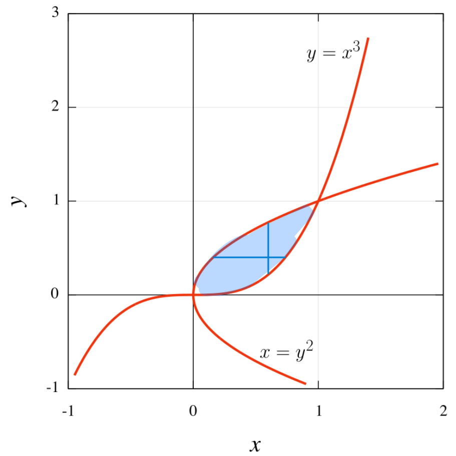
$$R = \{ (x,y) : \qquad \qquad \}$$

$$= \{ (x,y) : \qquad \qquad \}$$

Two options

$$\iint_R f \, dA =$$

$$=$$



$$\int_0^1 \int_{x^3}^{\sqrt{x}} 1 \, dy \, dx = \int_0^1$$

$$= \int_0^1$$

$$= \Big[\qquad \Big]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

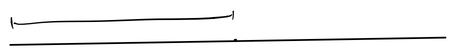
$$\int_0^1 \int_{y^2}^{y^{1/3}} 1 \, dx \, dy = \int_0^1 \Big[x \Big]_{y^2}^{y^{1/3}} dy$$

$$= \int_0^1 y^{\frac{1}{3}} - y^2 \; dy$$

$$= \Big[\frac{3}{4} y^{4/3} - \frac{y^3}{3} \Big]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$$

The integral as a weighted sum

We have seen that integration in one variable is not just for finding areas. For example; consider a thin wire with varying density $\rho(x)$ kg/m



The mass of the wire is given by

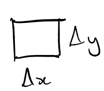
$$\sum_{i}^{n} \qquad \longrightarrow \qquad = M$$

"weighted sum" - we are adding up small intervals weighted by their density.

The weighting need not be a density of the kg/? kind. It can be an area (eg: finding volumes of rotational solids), a probability density, electric charge density... depending on the application at hand.

Mental picture: a single integral is a (limit of a) weighted sum intervals. Similarly, a <u>double</u> integral is a (limit of a) weighted sum of areas

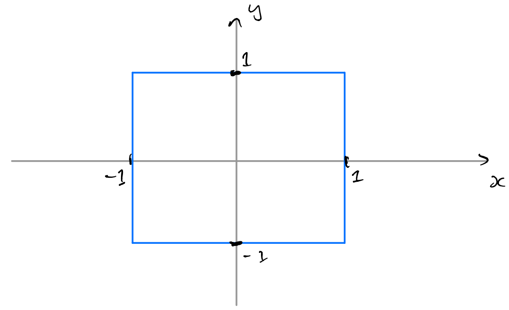
$$\sum_{i} \sum_{j} f(x_i, y_j)\, \Delta x\, \Delta y \quad \longrightarrow \quad \iint_R f(x,y)\, dA$$



and the weighting need not be a height (as in finding the volume under a surface) - it could be mass density, energy density, electric charge density, the speed of a fluid flowing through the region R...

In particular, if the weighting is $f(x,y) = 1$, we are just summing up areas, so $\iint_R 1\, dA = \text{Area}(R)$

Example consider a square metal plate



with density $\rho(x,y) = 1 + x^2 + y^2$ kg/m^2 i.e. density increasing with distance from $\underset{\sim}{0}$, maybe it gets thicker away from zero.

We get the mass of the plate by summing up small areas weighted by density:

$$\sum_i \sum_j \rho(x_i, x_j)\,\Delta x\,\Delta y \longrightarrow \int_{-1}^{1}\int_{-1}^{1} \rho(x,y)\,dA$$