Laplace transforms

$f(t)$ function defined for all $t \geq 0$

the Laplace transform of f is the function

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

defined for all $s \in \mathbb{R}$ for which the integral exists (is convergent).

Notation: often write $\mathcal{L}(f)$ or $\mathcal{L}(f)(s)$ for $F(s)$.

What are they for? Solving linear differential equations

eg: $$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \cos x$$

we will see how they help with this later.

EXAMPLES

$f(t) = 1$

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} dt = \lim_{x \to \infty} \left[-\frac{e^{-st}}{s} \right]_0^x$$

$$= \lim_{x \to \infty} -\frac{e^{-sx}}{s} + \frac{1}{s}$$

limit exists only if $s > 0$

$$= \frac{1}{s}$$

$g(t) = t$ recall: integration by parts $\int_a^b u v' = uv\Big|_a^b - \int_a^b u'v$

$$G(t) = \int_0^\infty t e^{-st} dt = \lim_{x \to \infty} \left(-t\frac{e^{-st}}{s}\Big|_0^x + \int_0^x \frac{e^{-st}}{s} dt \right)$$

u v'

$v = -\frac{e^{-st}}{s}$

$$= \lim_{x \to \infty} \left(-x\frac{e^{-sx}}{s} + \left[-\frac{e^{-st}}{s^2} \right]_0^x \right)$$

$$= \lim_{x \to \infty} \left(-x\frac{e^{-sx}}{s} - \frac{e^{-sx}}{s^2} + \frac{1}{s^2} \right)$$

$$= 0^{*} + 0 + \frac{1}{s^2} \quad \underline{\text{for } s > 0}$$

$$= \frac{1}{s^2}$$

$$* \quad \lim_{x \to \infty} \frac{-xe^{-sx}}{s} = \lim_{x \to \infty} \frac{-x}{se^{sx}} \underset{\substack{\uparrow \\ \text{L'Hospital's} \\ \text{rule}}}{=} \lim_{x \to \infty} \frac{-1}{s^2 e^{sx}} = 0$$

Function $f(t)$	Laplace transform $\mathcal{L}(f)(s) = F(s)$
1	$1/s$
t	$1/s^2$

If $F(s)$ is the Laplace transform of $f(t)$, we say that $f(t)$ is the inverse Laplace transform of $F(s)$.

Notation: $f(t) = \mathcal{L}^{-1}(F(s))(t)$

or $f = \mathcal{L}^{-1}(F)$

eg: from above, $\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$

Laplace transforms – more examples

Consider $f(t) = e^{-at}$, $t \geq 0$ and $a \in \mathbb{R}$ a constant.

$$F(s) = \int_0^\infty e^{-st} e^{-at} dt = \lim_{x \to \infty} \int_0^x e^{-(a+s)t} dt$$

$$= \lim_{x \to \infty} \left[\frac{e^{-(a+s)t}}{a+s} \right]_0^x$$

$$= \lim_{x \to \infty} \left(\frac{e^{-(a+s)x}}{a+s} - \frac{1}{a+s} \right)$$

limit exists only if $a+s > 0$

$s > -a$

$$= 0 - \frac{1}{a+s}$$

$$= \frac{1}{a+s}$$

Consider $g(t) = \sin(at)$, $a \neq 0$ constant

$$G(s) = \int_0^\infty \underbrace{e^{-st}}_{v'} \underbrace{\sin(at)}_{u} dt$$

$v = -\frac{e^{-st}}{s}$

$$= -\frac{e^{-st}}{s} \sin(at) \Big|_0^\infty + \frac{a}{s} \int_0^\infty \underbrace{e^{-st}}_{v'} \underbrace{\cos at}_{u} \, dt$$

$$= -\frac{e^{-st}}{s} \sin(at) \Big|_0^\infty - \frac{a e^{-st}}{s^2} \cos at \Big|_0^\infty + \frac{a^2}{s^2} \int_0^\infty e^{-st} \sin(at) \, dt$$

now $-\frac{e^{-sx}}{s} \leq \frac{e^{-sx} \sin(ax)}{s} \leq \frac{e^{-sx}}{s}$ for $s > 0$

$x \to \infty$ $\searrow$ $\downarrow$ $\swarrow$

0 (squeeze theorem)

similarly $\dfrac{e^{-sx}\cos(ax)}{s} \to 0$, and therefore

$$G(s) = \frac{a}{s^2} e^0 \cos(0) + \frac{a^2}{s^2} G(s)$$

rearrange $G(s)\left(1 - \frac{a^2}{s^2}\right) = \frac{a}{s^2}$

$$G(s) = \frac{a}{s^2\left(1 - a^2/s^2\right)} = \frac{a}{s^2 + a^2} \quad (s>0)$$

Now we know:

$f(t)$	1	t	e^{-at}	$\sin(at)$
$F(s)$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s+a}$	$\frac{1}{s^2+a^2}$

Linearity of the Laplace transform

If $f(t)$ and $g(t)$ have Laplace transforms $\mathcal{L}(f)$ and $\mathcal{L}(g)$ which exist for all $s \geq a \in \mathbb{R}$, then for any $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}
\mathcal{L}(\alpha f(t) + \beta g(t))(s) &= \int_0^\infty e^{-st}(\alpha f(t) + \beta g(t))\,dt \\
&= \alpha \int_0^\infty e^{-st} f(t)\,dt + \beta \int_0^\infty e^{-st} g(t)\,dt \\
&= \alpha \mathcal{L}(f)(s) + \beta \mathcal{L}(g)(s)
\end{aligned}$$

i.e. the Laplace transform is a linear transformation (but on an infinite dimensional vector space of functions - not $\mathbb{R}^n$)

It can be shown from this that the inverse Laplace transform is also linear.

EXAMPLES we will use our table

$f(t)$	1	t	e^{-at}	$\sin(at)$
$F(s)$	$\frac{1}{s}$	$\frac{1}{s^2}$	$\frac{1}{s+a}$	$\frac{1}{s^2+a^2}$

Let $f(t) = 2t + e^{5t}$, then

$$\begin{aligned}
F(s) = \mathcal{L}(f)(s) &= \mathcal{L}(2t + e^{5t})(s) \\
&= 2\mathcal{L}(t)(s) + \mathcal{L}(e^{5t})(s) \\
&= 2\frac{1}{s^2} + \frac{1}{s-5}
\end{aligned}$$

If $G(s) = \frac{1}{s} - \frac{1}{s^2+2}$ then

$$g(t) = 1 - \sin\sqrt{2}\,t$$

Inverse Laplace transforms of rational functions

Suppose $F(s) = \dfrac{2s-1}{(s^2-1)(s+3)}$, we can't use linearity of $\mathcal{L}^{-1}$ directly – need to write $F(s)$ in partial fractions

$$\frac{2s-1}{(s-1)(s+1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$2s-1 = A(s+1)(s+3) + B(s-1)(s+3) + C(s-1)(s+1)$$

sub $s = 1$: $\quad 2-1 = A(2)(4)$

$$A = \frac{1}{8}$$

$s = -1$ $\quad 2(-1)-1 = B(-2)(2)$

$$B = \frac{3}{4}$$

$s = -3$ $\quad -6-1 = C(-4)(2)$

$$C = \frac{7}{8}$$

$$F(s) = \frac{1}{8(s-1)} + \frac{3}{4(s+1)} + \frac{7}{8(s+3)}$$

so

$$f(t) = \frac{1}{8}e^{t} + \frac{3}{4}e^{-t} + \frac{7}{8}e^{3t}$$