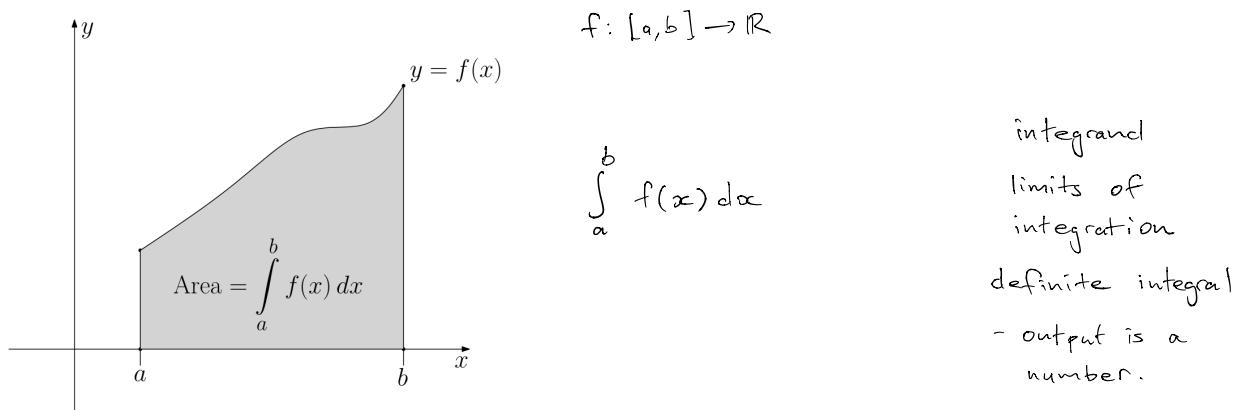
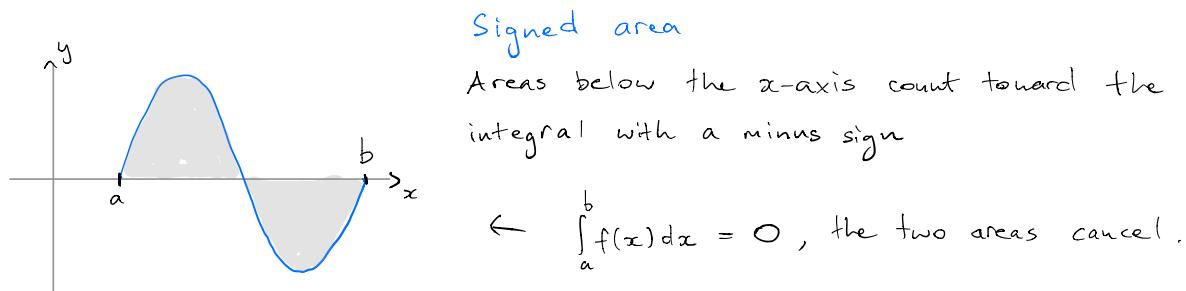


## Integration



When  $f$  is continuous on  $[a, b]$  this area is finite by the extreme value theorem: if  $K$  is the absolute maximum of  $f$  then

← area of the rectangle:



A function  $F(x)$  is called an anti-derivative for  $f(x)$  if

$$F'(x) = f(x)$$

### FUNDAMENTAL THEOREM OF CALCULUS

If  $F$  is an anti-derivative for  $f$  then

$$\int_a^b f(x) dx =$$

notation:  $=$   $=$

The indefinite integral  $\int f(x) dx$  is all the antiderivatives

i.e. if  $F(x)$  is any antiderivative for  $f$  then

$$\int f(x) dx =$$

definite integral  $\rightarrow$  signed area ETR

indefinite integral  $\rightarrow$  family of functions  
 $(+C)$

A table of things you already know

$\frac{dy}{dx}$	$y$	$\int y dx$
0	$a$ (constant)	$ax + C$
$nx^{n-1}$	$x^n$ ( $n \neq -1$ )	$\frac{x^{n+1}}{n+1} + C$
$-\frac{1}{x^2}$ or $-x^{-2}$	$\frac{1}{x}$ or $x^{-1}$	$\ln x + C$
$e^x$	$e^x$	$e^x + C$
$\frac{1}{x}$	$\ln x$	
$\cos x$	$\sin x$	$-\cos x + C$
$-\sin x$	$\cos x$	$\sin x + C$

$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	
$-\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	
$\frac{1}{1+x^2}$	$\tan^{-1} x$	

## Integration techniques

Substitution: undoing the chain rule:

if  $F(x)$  is an antiderivative for  $f(x)$ , then  $F'(x) = f(x)$ , so

chain rule:  $\frac{d}{dx} F(u(x)) =$   $=$

i.e.  $f(u(x)) u'(x) = \frac{d}{dx} F(u(x))$

now integrating each side:

indefinite  $\int f(u(x)) u'(x) dx =$

definite  $\int_{x=a}^{x=b} f(u(x)) u'(x) dx = \int_{u(a)}^{u(b)} f(u) du$

Example  $\int \sqrt{x^2 + 4x + 7} (x+2) dx$

Let  $u = x^2 + 4x + 7$  so  $\frac{du}{dx} =$   
 $dx =$

substitute  $u$  and  $dx$ :

$$\int \sqrt{x^2 + 4x + 7} (x+2) dx = \int$$

$$= \int$$

$$= \int$$

$$= =$$

Example  $\int_0^{\pi/2} \sin^2 x \cos x \, dx$

$$\text{let } u = \sin x \quad \frac{du}{dx} = , \quad dx =$$

limits  $x =$  ;

$$x = \dots$$

subst.  $u, dx$ , limits

$$\int_0^{\pi/2} \sin^2 x \cos x \, dx =$$

二

## Using trig identities

how to integrate  $\int \sin^2 x \, dx$  or  $\int \cos^2 x \, dx$ .

$$\text{identities : } \sin^2 x + \cos^2 x = 1 \quad \rightarrow \quad \begin{aligned} \sin^2 x &= \\ \cos^2 x &= \end{aligned}$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

二

二

rearrange:  $\cos^2 x =$

$$\sin^2 x =$$

$$\text{so eg: } \int \sin^2 x \, dx = \int \quad \quad \quad dx$$

二

In general: integrating  $\int \sin^m x \cos^n x dx$  requires some combination of the above identities with a substitution.

other identities: eg:  $\int \sin(4x) \cos(5x) dx$

use the product formula. (derived from addition formula)

$$\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$$

$$\sin(4x) \cos(5x) = \quad = \quad \text{now integrate...}$$

## Inverse trig substitutions

Example integrate  $\int \frac{2x}{\sqrt{4-x^2}} dx$

$$u = 4 - x^2 \quad \frac{du}{dx} = -2x \quad dx = \frac{-1}{2x} du$$

subst.

$$\begin{aligned} \int \frac{2x}{\sqrt{u}} \cdot \left( \frac{-1}{2x} \right) du &= \int \frac{-1}{\sqrt{u}} du = \int -u^{-\frac{1}{2}} du \\ &= -2u^{\frac{1}{2}} = -2\sqrt{u} \\ &= -2\sqrt{4-x^2} \end{aligned}$$

what about  $\int \frac{x^2}{\sqrt{4-x^2}} dx$  ?

the substitution  $u = 4 - x^2$  doesn't work with  $x^2$  in the numerator. ( $\frac{du}{dx}$  doesn't cancel out)

solution: substitute  $u = \sin^{-1}\left(\frac{x}{2}\right)$  (not obvious)

for this substitution it is easier to calculate  $\frac{dx}{du}$  instead of  $\frac{du}{dx}$

so from  $u = \sin^{-1}\left(\frac{x}{2}\right)$  take sin of each side:

$$\sin u = \frac{x}{2}$$

$$x = 2 \sin u$$

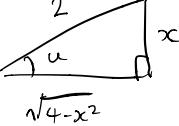
$$\frac{dx}{du} = 2 \cos u \Rightarrow dx = 2 \cos u du$$

now subst.  $x$  and  $dx$

$$\begin{aligned} \int \frac{x^2}{\sqrt{4-x^2}} dx &= \int \frac{4 \sin^2 u}{\sqrt{4 - 4 \sin^2 u}} 2 \cos u du \\ &= \int \frac{4 \sin^2 u}{\sqrt{4(1 - \sin^2 u)}} 2 \cos u du \end{aligned}$$

$$\begin{aligned}
&= \int \frac{4 \sin^2 u}{2 \sqrt{\cos^2 u}} 2 \cos u \, du \\
&= \int 4 \sin^2 u \, du \\
&= \int \frac{4}{2} (1 - \cos 2u) \, du \\
&= 2u - \sin 2u + C \\
&= 2u - 2 \sin u \cos u + C
\end{aligned}$$

to get this in terms of  $x$  (neatly):

$$\begin{aligned}
x &= 2 \sin u \\
\sin u &= \frac{x}{2}
\end{aligned}$$


$$\rightarrow \cos u = \frac{\sqrt{4-x^2}}{2}$$

so

$$2u - 2 \sin u \cos u = 2 \sin^{-1}\left(\frac{x}{2}\right) - 2 \left(\frac{x}{2}\right) \frac{\sqrt{4-x^2}}{2}$$

## Partial fractions

Example  $f(x) = \frac{3x-1}{x^2-1}$  is not easy to integrate (try!)

however, observe that

$$\frac{2}{x+1} + \frac{1}{x-1} = -$$

$$\int \text{LHS } dx = \leftarrow \text{easy (ish)}$$

So how do we reverse the common denominator technique?

## Partial fractions

This applies to functions of the form  $\frac{P(x)}{Q(x)}$  where  $P(x), Q(x)$

are polynomials and the degree (highest power) of  $P$  is less than the degree of  $Q$ :

### Case 1: Denominator has distinct linear factors

$$f(x) = \frac{P(x)}{(x-a_1)\cdots(x-a_k)} = \frac{A_1}{x-a_1} + \cdots + \frac{A_k}{x-a_k},$$

where  $a_1, \dots, a_k$  are pairwise distinct.

### Case 2: Denominator has repeated linear factors

$$f(x) = \frac{P(x)}{(x-a)^c} = \frac{B_1}{x-a} + \frac{B_2}{(x-a)^2} + \cdots + \frac{B_{c-1}}{(x-a)^{c-1}} + \frac{B_c}{(x-a)^c}.$$

### Case 3: Denominator has an irreducible factor of degree 2

$$f(x) = \frac{P(x)}{(x-a)(x^2+bx+c)} = \frac{A_1}{x-a} + \frac{C_1x+C_2}{x^2+bx+c}.$$

irreducible means it can't be factorised - the quadratic formula will give complex solutions.

Example

$$\frac{3x-1}{x^2-1} =$$

factorise denominator

$$= (\text{Case 1})$$

now multiply each side by the factorised denominator

$$3x - 1 =$$

$$3x - 1 =$$

At this point there are two ways to proceed, use whichever you prefer

- (1) substitute  $x = -1$  (so that  $(x+1) = 0$ )

$$3(-1) - 1 = 0 + B(-2)$$

$$-4 = -2B$$

$$B = 2$$

substitute  $x = 1$  (so that  $(x-1) = 0$ )

$$3(1) - 1 = A(2) + 0$$

$$2 = 2A$$

$$A = 1$$

Therefore

$$\frac{3x-1}{x^2-1} = \frac{1}{x-1} + \frac{2}{x+1}$$

(2)  $3x - 1 = A(x+1) + B(x-1)$

$$3x - 1 = Ax + A + Bx - B$$

equate constants ( $x=0$ )

equate coefficients of  $x$  } solve simultaneously:  
adding eqns:

more examples in the unit reader

## Integration by parts

Undoing the product rule

$$\text{Product rule: } \frac{d}{dx} (u(x)v(x)) =$$

integrate each side wrt  $x$ ,

$$\text{indefinite } u(x)v(x) =$$

$$\text{definite } [u(x)v(x)]_a^b =$$

These formulas are actually most useful after rearranging:

Integration by parts formula:

$$\text{indefinite form } \int u v' dx =$$

$$\text{definite } \int_a^b u v' dx =$$

How to use:

$$\int x \sin x dx =$$

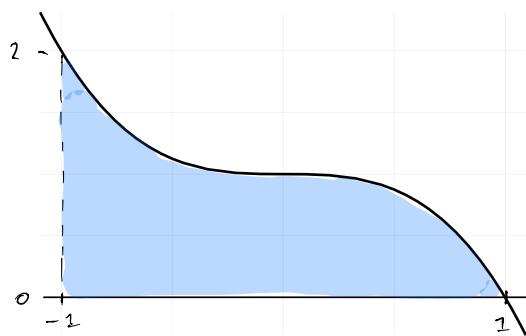
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$$\int x \ln x dx =$$

=

Plan ahead when assigning  $u$  and  $v'$  so that  $v$  is easy to find and  $\int u'v dx$  is easier than the original integral.

## Riemann sums and integrals



$$f(x) = 1 - x^3$$

$$\int_{-1}^1 1 - x^3 = \left[ x - \frac{x^4}{4} \right]_{-1}^1 = 2$$

↗  
antiderivative

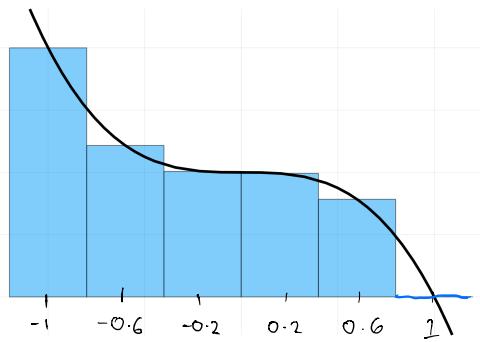
SOME FUNCTIONS DON'T HAVE ANTIDERIVATIVES!

For example

So how do we compute eg:

?

Back to basics:



$$A_1 = 2 \times 0.4 = 0.8$$

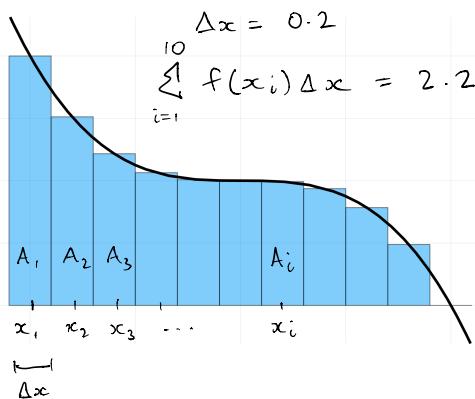
$$A_2 = \approx 0.49$$

$$A_3 \approx 0.4 \quad A_4 \approx 0.4 \quad A_5 \approx 0.3$$

$$\text{Total area} = 2.4$$

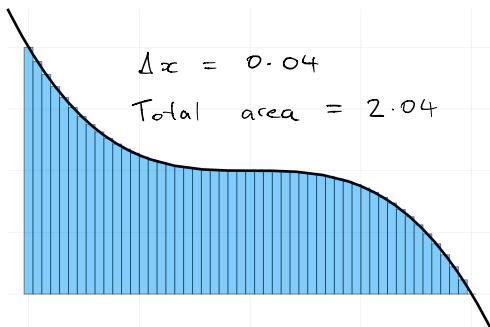
Formula:  $A_i =$

Total Area =



$$\Delta x = 0.2$$

$$\sum_{i=1}^{10} f(x_i) \Delta x = 2.2$$



$$\Delta x = 0.04$$

$$\text{Total area} = 2.04$$

Riemann sum  $\sum_{i=1}^n f(x_i) \Delta x$  (total area of the bars)

Riemann integral  $\int_a^b f(x) dx = \lim_{\substack{n \rightarrow \infty \\ \Delta x \rightarrow 0}} \sum_{i=1}^n f(x_i) \Delta x$

if the limit exists  $f$  is called Riemann integrable over  $[a, b]$

Notes midpoint of the interval

Here we have used intervals of the form  $[x_i, x_{i+1}]$  and we evaluate  $f(x_i)$  in the Riemann sum. There are several alternatives which turn out to give the same result:

- intervals  $[x_i, x_{i+1}]$
- intervals of varying size:  $[x_i, x_i + \Delta x_i]$
- $\sum f(c_i) \Delta x_i$  where  $c_i$  is any element in the  $i^{\text{th}}$  interval.

It is possible to prove (see unit reader for a sketch-proof) that the Riemann integral satisfies the Fundamental Theorem of Calculus

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

where  $F(x)$  is an antiderivative of  $f(x)$

So all the integration you have done so far (i.e. using antiderivatives) has been Riemann integration.

It is also possible to prove that if  $f$  is bounded and piecewise continuous (i.e. has only finitely many points of discontinuity) then it is Riemann integrable.

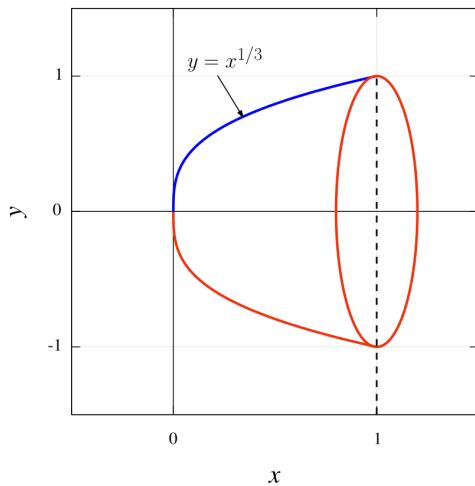
So alternative definitions of integration (eg: Lebesgue integration) are only needed when considering weird functions

## Applications of Riemann Sums.

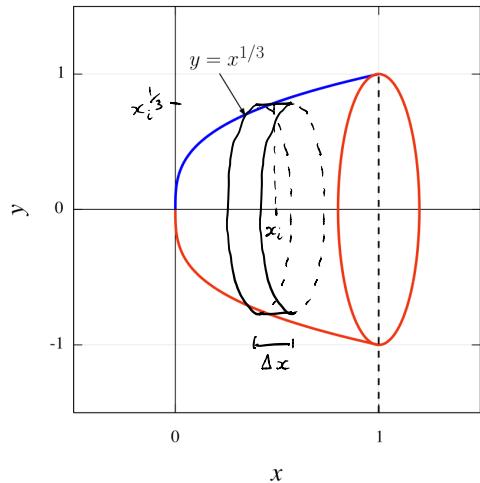
Thinking about integrals in terms of Riemann sums gives us a good way of figuring out how to solve real problems using integrals (as opposed to just finding areas under curves). As usual, this is best demonstrated with examples.

### Volumes by cross-sections (rotational solids)

Suppose a piece of metal is cut using a lathe along the curve  $y = x^{1/3}$ ,  $0 < x < 1$  so that the remaining object has the shape of the solid obtained by rotating the area below  $y = x^{1/3}$  about the  $x$ -axis:



we can find the volume of this object by setting up a Riemann integral.



The cylinder of width  $\Delta x$  at  $x_i$  has radius and therefore volume

A Riemann sum of  $n$  such cylinders:

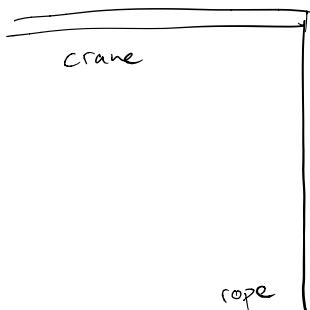
$$V \approx$$

In the limit  $n \rightarrow \infty$  and  $\Delta x \rightarrow 0$

so the volume of the remaining solid is

### Work done by a force

Consider a rope with uniform density  $0.1 \text{ kg/m}$  and length  $50\text{m}$  which is hanging from a crane. Suppose we need to calculate how much work it will take to lift all the rope to the top of the crane. Remember  $\text{work} = \text{force} \times \text{displacement}$



consider a small section of rope with length  $\Delta x$  which is  $x_i$  metres from the top. The mass of this piece is  $m_i =$ , so the force required to lift it is

$$F_i = m_i g =$$

and the work required is

$$W_i = \cdot =$$

=

we divide the rope into  $n$  such pieces, and sum up the work required:

$$W \approx \longrightarrow = W$$

$$= [0.05g x^2]_0^{50}$$

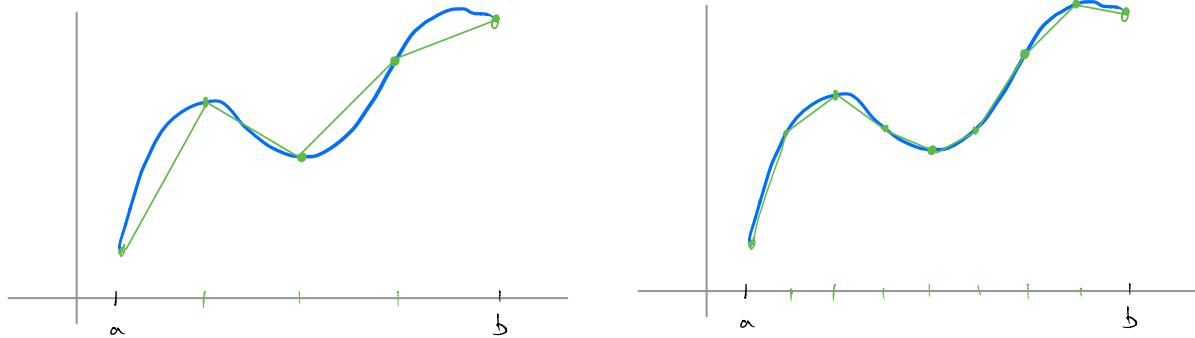
$$\approx 1225$$

## Lengths of curves

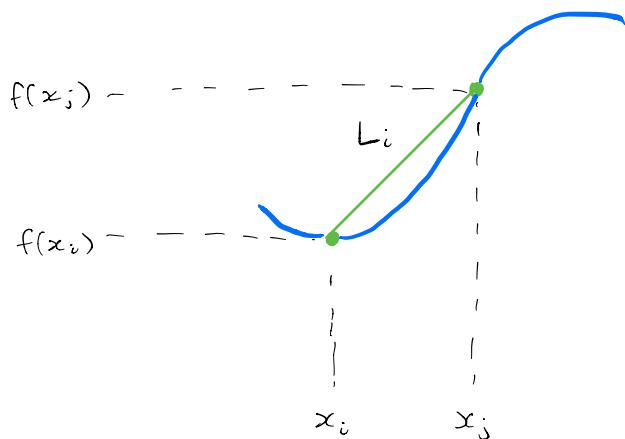
$f(x)$  continuously differentiable on  $[a, b]$

We can find the length of the curve  $\{(x, f(x)) : x \in [a, b]\}$  by setting up a Riemann integral.

Approximate with lengths of line segments



For a segment between  $x_i$  and  $x_j$ :



The length  $L_i =$

Since  $f(x)$  is continuously differentiable there is a theorem (called the Mean Value Theorem) which states that there exists  $c_i$  between  $x_i$  and  $x_j$  such that

$$f'(c_i) = \frac{f(x_j) - f(x_i)}{x_j - x_i}$$

i.e.  $f(x_j) - f(x_i) =$

substituting this into the expression for  $L_i$ :

$$L_i =$$

so the total length

$$\sum_{i=1}^n L_i =$$

=

(Recall that the Riemann integral actually doesn't depend on which value in the interval  $\Delta x$  is used in the Riemann sum - intuitively: as the width  $\Delta x \rightarrow 0$ , every point in the interval approaches  $x_i$ )

so now taking the limit  $n \rightarrow \infty$   
 $\Delta x \rightarrow 0$

$$L =$$