Damage coupled Chaboche plasticity model and its numerical implementation☆

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Abstract

在连续损伤力学中，含损伤材料的本构模型是一个重要研究问题。损伤耦合的 Chaboche 弹塑性本构模型是一种常用的本构模型，该模型能够较好的描述金属材料的循环塑性行为，因而得到了广泛的应用。本文详细地介绍了该模型，并给出了求解该模型的数值算法，其中包括径向回溯算法和一致切线模量算法，以便于 UMAT 的编写。

Keywords: 损伤耦合的 Chaboche 弹塑性本构模型, 径向回溯算法, 一致切线模量算法, UMAT

1. Damage coupled Chaboche plasticity model

损伤耦合的 Chaboche 弹塑性本构模型是在 Chaboche 弹塑性本构模型的基础上，考虑损伤对材料力学性能的影响而建立的。建立方法有两种：其一为内变量方法，选取的 Helmholtz 自由能 $\Psi(\boldsymbol{\varepsilon}^e, T, r, \boldsymbol{\alpha}, D)$ 由两部分组成，一部分是含损伤材料的弹性应变能 W_e，另一部分是与塑性强化有关的能量[1, 2]。

$$\rho\Psi = W_e + R_\infty[r + \frac{1}{b}exp(-br)] + \frac{C}{3}\boldsymbol{\alpha} : \boldsymbol{\alpha}$$

上式最后两项具有一定的假设性。根据状态函数可得

$$\boldsymbol{\sigma} = \rho\frac{\partial\Psi}{\partial\boldsymbol{\varepsilon}^e}$$

☆Fully documented templates are available in the elsarticle package on CTAN.

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$$R = \rho\frac{\partial\Psi}{\partial r} = R_\infty(1 - exp(-br))$$
$$\mathbf{X} = \rho\frac{\partial\Psi}{\partial\boldsymbol{\alpha}} = \frac{2}{3}C\boldsymbol{\alpha}$$
$$Y = -\rho\frac{\partial\Psi}{\partial D}$$

选取耗散势函数

$$F = (\frac{\boldsymbol{\sigma}}{1-D} - \mathbf{X})_{eq} - \boldsymbol{\sigma}_Y - R + F_\mathbf{X} + F_D$$

$$F_\mathbf{X} = \frac{3\gamma}{4C}\mathbf{X} : \mathbf{X}$$

其中 $\boldsymbol{\sigma}$ 为 Cauchy 应力，D 为损伤度，$\mathbf{X}$ 为背应力，下标 $_{eq}$ 表示的是 Von Mises 应力，$\boldsymbol{\sigma}_Y$ 为初始屈服极应力，R 为屈服极限的增加量，$F_\mathbf{X}$，F_D 分别代表随动强化耗散势和损伤耗散势。根据正交流动准则可得损伤耦合的 Chaboche 弹塑性本构模型[3, 4]；另一种方法是直接根据应变等效原理，将不含损伤的 Chaboche 弹塑性本构模型中的应力量替换为有效应力量。两种方法得到的模型是一致的，具体公式如下所示：

- 小变形或小应变情况下应变分解

$$\boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}_{ij}^e + \boldsymbol{\varepsilon}_{ij}^p$$

- 损伤耦合的弹性本构模型

$$\boldsymbol{\varepsilon}_{ij}^e = \frac{1+v}{E}\Big(\frac{\boldsymbol{\sigma}_{ij}}{1-D}\Big) - \frac{v}{E}\Big(\frac{\boldsymbol{\sigma}_{kk}\delta_{ij}}{1-D}\Big)$$

- 损伤耦合的屈服条件

$$f = \sqrt{\frac{3}{2}\Big(\frac{\mathbf{s}_{ij}}{1-D} - \mathbf{X}_{ij}\Big)\Big(\frac{\mathbf{s}_{ij}}{1-D} - \mathbf{X}_{ij}\Big)} - \boldsymbol{\sigma}_Y - R$$

- 塑性流动

$$\dot{\boldsymbol{\varepsilon}}_{ij}^p = \dot{\lambda}\frac{\partial F}{\partial\boldsymbol{\sigma}_{ij}} = \frac{3}{2}\frac{\dot{\lambda}}{1-D}\frac{\mathbf{s}_{ij}/(1-D) - \mathbf{X}_{ij}}{(\boldsymbol{\sigma}/(1-D) - \mathbf{X})_{eq}} = \frac{\dot{\lambda}}{1-D}\mathbf{r}$$

$$\dot{p} = \sqrt{\frac{2}{3}\dot{\boldsymbol{\varepsilon}}_{ij}^p\dot{\boldsymbol{\varepsilon}}_{ij}^p} = \frac{\dot{\lambda}}{1-D}$$

- 非线性随动强化

$$\mathbf{X} = \frac{2}{3} C \boldsymbol{\alpha}$$

$$\dot{\boldsymbol{\alpha}} = -\dot{\lambda}\frac{\partial F}{\partial \mathbf{X}} = \dot{\lambda}\Big(\frac{3}{2}\frac{\mathbf{s}/(1-D)-\mathbf{X}}{(\boldsymbol{\sigma}/(1-D)-\mathbf{X})_{eq}} - \frac{3\gamma}{2C}\mathbf{X}\Big)$$

在 Chaboche 弹塑性本构模型中，采用多个背应力叠加，从而更为准确地描述材料的非线性随动强化行为。

$$\mathbf{X}_{ij} = \sum_{k=1}^{M} \mathbf{X}_{ij}^{(k)}$$

$$\dot{\mathbf{X}}_{ij}^{(k)} = (1-D)(\frac{2}{3} C_k \dot{\boldsymbol{\varepsilon}}_{ij}^{p} - \gamma_k \mathbf{X}_{ij}^{(k)} \dot{p})$$

- 各向同性强化

$$\dot{r} = -\dot{\lambda}\frac{\partial F}{\partial R} = \dot{\lambda}$$

$$\dot{R} = b(R_{\infty} - R)\dot{r}$$

- 加载、卸载以及中性加载条件

$$\dot{\lambda} \geq 0\,,\; f \leq 0\,,\; \dot{\lambda} f = 0$$

其中 E，v 为弹性模量和泊松比， C_k，γ_k，b，R_∞ 为材料参数。

2. Radial return mapping algorithms

对率形式的本构模型，需要采用相应的应力更新算法来求解[5, 6, 7, 8]。具体而言，在一次循环内 n 时刻变量 $\boldsymbol{\sigma}_n$，$\boldsymbol{\varepsilon}_n$，$\boldsymbol{\varepsilon}_n^p$，$\mathbf{X}_n^{(k)}$，$R_n$，$p_n$，$D_n$ 都已知，根据率形式的本构模型求解 $n+1$ 时刻的变量。此一过程需要几次迭代，才能使得最终的应力达到给定的载荷 $\boldsymbol{\sigma}_{n+1}$。

迭代开始时，首先根据 $\Delta\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_n$ 和 n 时刻的一致切向模量 $\mathbf{C}_4^{alg(0)}$ 计算出 $\Delta\boldsymbol{\varepsilon}_{n+1}^{(0)}$。

$$\boldsymbol{\varepsilon}_{n+1}^{(i+1)} = \boldsymbol{\varepsilon}_{n+1}^{(i)} + \Delta\boldsymbol{\varepsilon}_{n+1}^{(i)}$$

$$\boldsymbol{\varepsilon}_{n+1}^{p(i+1)} = \boldsymbol{\varepsilon}_{n+1}^{p(i)} + \Delta p_{n+1}^{(i)}\,\mathbf{r}_{n+1}^{(i+1)}$$

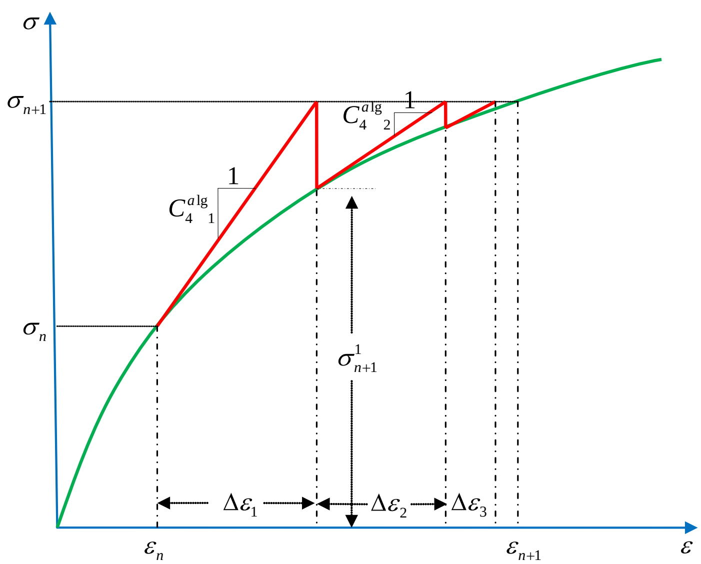
$$p_{n+1}^{(i+1)} = p_{n+1}^{(i)} + \Delta p_{n+1}^{(i)}$$

$$\boldsymbol{\sigma}_{n+1}^{(i+1)} = \boldsymbol{\sigma}_{n+1}^{(i)} + \mathbf{C}_4(1 - D_{n+1}^{(i+1)}) : (\Delta\boldsymbol{\varepsilon}_{n+1}^{(i)} - \Delta\boldsymbol{\varepsilon}_{n+1}^{p(i)})$$

$$f_{n+1}^{(i+1)} = 0$$

$$\Delta\boldsymbol{\varepsilon}_{n+1}^{(i+1)} = \frac{\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_{n+1}^{(i+1)}}{\mathbf{C}_4^{alg(i+1)}}$$

直到 $\boldsymbol{\sigma}_{n+1}^{(i+1)} = \boldsymbol{\sigma}_{n+1}$，迭代结束。求解上述增量方程的算法即为径向映射算法，而一致切向模量 $\mathbf{C}_4^{alg(i+1)}$ 在第 $i+1$ 次应力应变计算结束之后进行更新。



在 UMAT 中不需要处理上述迭代，这些迭代由商用有限元软件来完成，而仅仅需要给出一次迭代的应力和一致切向模量的更新算法。考虑第 n 和 n+1 次迭代，假设在第 n 次迭代时变量 $\boldsymbol{\sigma}_n$，$\boldsymbol{\varepsilon}_n$，$\boldsymbol{\varepsilon}_n^p$，$\mathbf{X}_n^{(k)}$，$R_n$，$p_n$ 都已知，应变增量 $\Delta\boldsymbol{\varepsilon}$ 也已知。需要注意的是，在每次应力更新之前应该先更新损伤变量 D_{n+1}（如果此时一个循环结束，则根据损伤演化方程更新损伤变量，否则保持不变），所以在每次迭代中损伤变量保持不变。

$$\boldsymbol{\varepsilon}_{n+1} = \boldsymbol{\varepsilon}_n + \Delta\boldsymbol{\varepsilon}$$

$$\boldsymbol{\varepsilon}_{n+1}^p = \boldsymbol{\varepsilon}_n^p + \Delta\boldsymbol{\varepsilon}^p = \boldsymbol{\varepsilon}_n^p + \mathbf{r}_{n+1}\Delta p$$

$$\boldsymbol{\sigma}_{n+1} = \boldsymbol{\sigma}_n + \mathbf{C}_4(1 - D_{n+1}) : (\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}^p)$$

$$\mathbf{X}_{n+1}^{(k)} = \mathbf{X}_n^{(k)} + (1-D_{n+1})(\frac{2}{3}C_k\Delta\boldsymbol{\varepsilon}^p - \gamma_k \mathbf{X}_{n+1}^{(k)}\Delta p)$$

$$\begin{aligned}
\mathbf{X}_{n+1} &= \sum_{k=1}^{M}\frac{1}{1+(1-D_{n+1})\gamma_k\Delta p}\Big(\mathbf{X}_n^{(k)} + \frac{2}{3}(1-D_{n+1})C_k\Delta\boldsymbol{\varepsilon}^p\Big)\\
&= \sum_{k=1}^{M}\theta_{n+1}^{(k)}\Big(\mathbf{X}_n^{(k)} + \frac{2}{3}(1-D_{n+1})C_k\Delta\boldsymbol{\varepsilon}^p\Big)
\end{aligned}$$

$$R_{n+1} = R_\infty(1-e^{-(1-D_{n+1})bp_{n+1}})$$

$$p_{n+1} = p_n + \Delta p$$

结合屈服函数，可得

$$\begin{aligned}
\frac{\boldsymbol{\sigma}_{n+1}}{1-D_{n+1}} &= \frac{\boldsymbol{\sigma}_n}{1-D_{n+1}} + \mathbf{C}_4\Delta\boldsymbol{\varepsilon} - \mathbf{C}_4\mathbf{r}_{n+1}\Delta p\\
&= \frac{\boldsymbol{\sigma}_n}{1-D_{n+1}} + \mathbf{C}_4\Delta\boldsymbol{\varepsilon} - 2G\mathbf{r}_{n+1}\Delta p
\end{aligned}$$

$$\frac{\mathbf{s}_{n+1}}{1-D_{n+1}} = (\frac{\boldsymbol{\sigma}_n}{1-D_{n+1}} + \mathbf{C}_4\Delta\boldsymbol{\varepsilon})_{dev} - 2G\mathbf{r}_{n+1}\Delta p$$

$$\begin{aligned}
\frac{\mathbf{s}_{n+1}}{1-D_{n+1}} - \mathbf{X}_{n+1} &= (\frac{\boldsymbol{\sigma}_n}{1-D_{n+1}} + \mathbf{C}_4\Delta\boldsymbol{\varepsilon})_{dev} - 2G\mathbf{r}_{n+1}\Delta p\\
&\quad -\sum_{k=1}^{M}\theta_{n+1}^{(k)}\Big(\mathbf{X}_n^{(k)} + \frac{2}{3}(1-D_{n+1})C_k\Delta\boldsymbol{\varepsilon}^p\Big)\\
&= (\frac{\boldsymbol{\sigma}_n}{1-D_{n+1}} + \mathbf{C}_4\Delta\boldsymbol{\varepsilon})_{dev} - \sum_{k=1}^{M}\theta_{n+1}^{(k)}\mathbf{X}_n^{(k)}\\
&\quad -\Big(2G + \sum_{k=1}^{M}\theta_{n+1}^{(k)}\frac{2}{3}(1-D_{n+1})C_k\Big)\mathbf{r}_{n+1}\Delta p
\end{aligned}$$

由于上式中的三项均包含 $\mathbf{r}_{n+1}$，将其转化为标量方程则有

$$\begin{aligned}
\Big(\frac{\mathbf{s}_{n+1}}{1-D_{n+1}} - \mathbf{X}_{n+1}\Big)_{eq} &= \Big((\frac{\boldsymbol{\sigma}_n}{1-D_{n+1}} + \mathbf{C}_4\Delta\boldsymbol{\varepsilon})_{dev} - \sum_{k=1}^{M}\theta_{n+1}^{(k)}\mathbf{X}_n^{(k)}\Big)_{eq}\\
&\quad -\Big(3G + \sum_{k=1}^{M}\theta_{n+1}^{(k)}(1-D_{n+1})C_k\Big)\Delta p\\
&= \boldsymbol{\sigma}_Y + R_\infty(1-e^{-(1-D_{n+1})bp_{n+1}})
\end{aligned}$$

上式是一个关于变量 Δp 的非线性方程，采用牛顿迭代法可以求解。具体算法为：

(1) 当 $i=0$ 时，$\Delta p_0=0$，$p_{n+1,0}=p_n$，$\boldsymbol{\sigma}_{n+1,0}=\boldsymbol{\sigma}_n/(1-D_{n+1})+\mathbf{C}_4\Delta\boldsymbol{\varepsilon}$，该标量方程可以写成

$$\begin{aligned} g(\Delta p) \quad &= \quad \Big(\boldsymbol{\sigma}_{n+1,0}-\sum_{k=1}^{M}\theta_{n+1}^{(k)}\mathbf{X}_n^{(k)}\Big)_{eq} \\ & \qquad -\Big(3G+\sum_{k=1}^{M}\theta_{n+1}^{(k)}(1-D_{n+1})C_k\Big)\Delta p \\ & \qquad -\boldsymbol{\sigma}_Y-R_\infty(1-e^{-(1-D_{n+1})b(p_n+\Delta p)}) \end{aligned}$$

(2) 假设 $g(\Delta p_{i+1})=0$，根据泰勒展开则有

$$g(\Delta p_{i+1})=g(\Delta p_i)+g'(\Delta p_i)(\Delta p_{i+1}-\Delta p_i)$$

$$\Delta p_{i+1}=\Delta p_i-\frac{g(\Delta p_i)}{g'(\Delta p_i)}$$

其中，

$$\begin{aligned} g'(\Delta p) \quad &= \quad \frac{3}{2}\frac{\mathbf{s}_{n+1,0}-\sum_{k=1}^{M}\theta_{n+1}^{(k)}\mathbf{X}_n^{(k)}}{\Big(\boldsymbol{\sigma}_{n+1,0}-\sum_{k=1}^{M}\theta_{n+1}^{(k)}\mathbf{X}_n^{(k)}\Big)_{eq}}:\sum_{k=1}^{M}\frac{(1-D_{n+1})\gamma_k\mathbf{X}_n^{(k)}}{\Big(1+(1-D_{n+1})\gamma_k\Delta p\Big)^2} \\ & \qquad -3G-\sum_{k=1}^{M}\frac{(1-D_{n+1})C_k}{\Big(1+(1-D_{n+1})\gamma_k\Delta p\Big)^2} \\ & \qquad -(1-D_{n+1})bR_\infty e^{-(1-D_{n+1})b(p_n+\Delta p)} \end{aligned}$$

(3) 判断 $g(\Delta p_{i+1})$ 是否等于 0，如果是则退出，Δp_{i+1} 即为所求，否则继续步骤(2)和(3)。

在得到 Δp 之后，更新第 $n+1$ 次迭代时的相关变量 $\boldsymbol{\sigma}_{n+1}$，$\boldsymbol{\varepsilon}_{n+1}$，$\boldsymbol{\varepsilon}_{n+1}^p$，$\mathbf{X}_{n+1}^{(k)}$，$R_{n+1}$，$p_{n+1}$。

3. Consistent tangent modulus

在应力更新完成之后，需要计算此时的切向模量，即一致切向模量。一致切向模量的更新是在应力应变计算之后进行的，根据新得到的力学量求解此时的应力应变曲线斜率，主要是为下一次迭代服务。采用一致切向模量可以获得准二次收敛速度。如果不是采用一致切向模量，同样可以得到最终的应力应变

值，只是收敛的速度较慢而已，因为最终的收敛准则是计算得到的应力和给定的应力值一样。

$$\mathbf{C}_4^{alg} = \frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}}$$

对径向映射算法中的各式和屈服函数求全微分，得到一组关于 $\mathrm{d}\boldsymbol{\sigma}_{n+1}$，$\mathrm{d}\boldsymbol{\varepsilon}_{n+1}$，$\mathrm{d}\boldsymbol{\varepsilon}^p_{n+1}$，$\mathrm{d}\mathbf{X}_{n+1}$，$\mathrm{d}p_{n+1}$，$\mathrm{d}R_{n+1}$ 的方程组，继而得到一致切向模量。

$$\mathrm{d}\boldsymbol{\sigma}_{n+1} = \mathbf{C}_4(1-D_{n+1})(\mathrm{d}\boldsymbol{\varepsilon}_{n+1} - \mathrm{d}\boldsymbol{\varepsilon}^p_{n+1})$$

$$\mathrm{d}\boldsymbol{\varepsilon}^p_{n+1} = \mathrm{d}(\Delta p)\mathbf{r}_{n+1} + \Delta p \mathrm{d}(\mathbf{r}_{n+1})$$

$$\begin{aligned}
\mathrm{d}(\mathbf{r}_{n+1}) &= \mathrm{d}\Big(\frac{3}{2}\frac{\mathbf{s}_{n+1}/(1-D_{n+1}) - \mathbf{X}_{n+1}}{(\boldsymbol{\sigma}_{n+1}/(1-D_{n+1}) - \mathbf{X}_{n+1})_{eq}}\Big) \\
&= \frac{3}{2}\frac{\mathbf{I}_4 - \mathbf{n}\otimes\mathbf{n}}{(\boldsymbol{\sigma}_{n+1}/(1-D_{n+1}) - \mathbf{X}_{n+1})_{eq}}\big(\frac{\mathrm{d}\mathbf{s}_{n+1}}{1-D_{n+1}} - \mathrm{d}\mathbf{X}_{n+1}\big)
\end{aligned}$$

其中，

$$\mathbf{n} = \sqrt{\frac{2}{3}}\mathbf{r}_{n+1}$$

$$\begin{aligned}
\mathrm{d}\boldsymbol{\varepsilon}^p_{n+1} &= \mathrm{d}(\Delta p)\mathbf{r}_{n+1} \\
&\quad + \Delta p\frac{3}{2}\frac{\mathbf{I}_4 - \mathbf{n}\otimes\mathbf{n}}{(\boldsymbol{\sigma}_{n+1}/(1-D_{n+1}) - \mathbf{X}_{n+1})_{eq}} : \big(\frac{\mathrm{d}\mathbf{s}_{n+1}}{1-D_{n+1}} - \mathrm{d}\mathbf{X}_{n+1}\big) \\
&= \mathrm{d}(\Delta p)\mathbf{r}_{n+1} + \mathbf{D} : \big(\frac{\mathrm{d}\mathbf{s}_{n+1}}{1-D_{n+1}} - \mathrm{d}\mathbf{X}_{n+1}\big)
\end{aligned}$$

$$\begin{aligned}
\mathrm{d}\mathbf{X}_{n+1} &= \sum_{k=1}^{M}\mathrm{d}\theta^{(k)}_{n+1}\mathbf{X}^{(k)}_n \\
&\quad + \frac{2}{3}(1-D_{n+1})\sum_{k=1}^{M}C_k(\mathrm{d}\theta^{(k)}_{n+1}\Delta\boldsymbol{\varepsilon}^p_{n+1} + \theta^{(k)}_{n+1}\mathrm{d}(\Delta\boldsymbol{\varepsilon}^p)) \\
&= \sum_{k=1}^{M}\Big(\frac{-(1-D_{n+1})\gamma_k\mathbf{X}^{(k)}_n}{(1+(1-D_{n+1})\gamma_k\Delta p)^2} \\
&\quad + \frac{2}{3}(1-D_{n+1})C_k\frac{-(1-D_{n+1})\gamma_k\Delta\boldsymbol{\varepsilon}^p}{(1+(1-D_{n+1})\gamma_k\Delta p)^2}\Big)\mathrm{d}(\Delta p) \\
&\quad + \frac{2}{3}(1-D_{n+1})\sum_{k=1}^{M}C_k\theta^{(k)}_{n+1}\mathrm{d}(\Delta\boldsymbol{\varepsilon}^p) \\
&= \Big(\sum_{k=1}^{M}(-1)(1-D_{n+1})\gamma_k\theta^{(k)}_{n+1}\mathbf{X}^{(k)}_{n+1}\Big)\mathrm{d}(\Delta p) +
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{2}{3}(1-D_{n+1})\sum_{k=1}^{M}C_k\theta_{n+1}^{(k)}\Big)\mathrm{d}(\Delta\boldsymbol{\varepsilon}^p)\\
=\quad &\mathbf{A}\mathrm{d}(\Delta p)+B\mathrm{d}(\Delta\boldsymbol{\varepsilon}^p)
\end{aligned}
$$

对屈服函数进行微分可得

$$
\begin{aligned}
\mathbf{r}_{n+1}:(\frac{\mathrm{d}\mathbf{s}_{n+1}}{1-D_{n+1}}-\mathrm{d}\mathbf{X}_{n+1}) &= (1-D_{n+1})bR_\infty e^{-(1-D_{n+1})b(p_n+\Delta p)}\mathrm{d}(\Delta p)\\
&= H\mathrm{d}(\Delta p)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{d}\mathbf{s}_{n+1} &= \mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\sigma}_{n+1}\\
&= 2G(1-D_{n+1})\mathbf{I}_4^{dev}\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-2G(1-D_{n+1})\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p
\end{aligned}
$$

$$
\frac{\mathrm{d}\mathbf{s}_{n+1}}{1-D_{n+1}}-\mathrm{d}\mathbf{X}_{n+1}=2G\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-(2G+B)\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p-\mathbf{A}\mathrm{d}(\Delta p)
$$

由此可得

$$
\mathrm{d}(\Delta p)=\frac{2G\mathbf{r}_{n+1}:\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-(2G+B)\mathbf{r}_{n+1}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p}{H+\mathbf{r}_{n+1}:\mathbf{A}}
$$

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p &= (\mathbf{r}_{n+1}-\mathbf{D}_4:\mathbf{A})\mathrm{d}(\Delta p)\\
&\quad +\mathbf{D}_4:(2G\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-(2G+B)\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p)\\
&= \Big(\frac{\mathbf{r}_{n+1}-\mathbf{D}_4:\mathbf{A}}{H+\mathbf{r}_{n+1}:\mathbf{A}}\otimes\mathbf{r}_{n+1}+\mathbf{D}_4\Big):(2G\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-(2G+B)\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p)\\
&= \mathbf{L}_4:(2G\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-(2G+B)\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p)
\end{aligned}
$$

$$
\Big(\mathbf{I}_4+(2G+B)\mathbf{L}_4\Big):\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p=2G\mathbf{L}_4:\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}
$$

$$
\mathbf{J}_4:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p=2G\mathbf{L}_4:\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}
$$

$$
\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p=2G\mathbf{J}_4^{-1}:\mathbf{L}_4:\mathbf{I}_4^{dev}:\mathrm{d}\boldsymbol{\varepsilon}_{n+1}
$$

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\sigma}_{n+1} &= \mathbf{C}_4(1-D_{n+1})(\mathrm{d}\boldsymbol{\varepsilon}_{n+1}-\mathrm{d}\boldsymbol{\varepsilon}_{n+1}^p)\\
&= \Big(\mathbf{C}_4(1-D_{n+1})-2G(1-D_{n+1})\mathbf{C}_4:\mathbf{J}_4^{-1}:\mathbf{L}_4:\mathbf{I}_4^{dev}\Big)\mathrm{d}\boldsymbol{\varepsilon}_{n+1}
\end{aligned}
$$

$$
\mathbf{C}_4^{alg}=\frac{\mathrm{d}\boldsymbol{\sigma}_{n+1}}{\mathrm{d}\boldsymbol{\varepsilon}_{n+1}}=\mathbf{C}_4(1-D_{n+1})-4G^2(1-D_{n+1})\mathbf{J}_4^{-1}:\mathbf{L}_4:\mathbf{I}_4^{dev}
$$

其中，

$$
\begin{aligned}
\mathbf{J}_4 &= \mathbf{I}_4 + (2G + B)\mathbf{L}_4 \\
\mathbf{L}_4 &= \frac{\mathbf{r}_{n+1} - \mathbf{D}_4 : \mathbf{A}}{H + \mathbf{r}_{n+1} : \mathbf{A}} \otimes \mathbf{r}_{n+1} + \mathbf{D}_4 \\
\mathbf{D}_4 &= \Delta p \frac{3}{2} \frac{\mathbf{I}_4 - \mathbf{n} \otimes \mathbf{n}}{(\boldsymbol{\sigma}_{n+1}/(1 - D_{n+1}) - \mathbf{X}_{n+1})_{eq}} \\
H &= (1 - D_{n+1}) b R_\infty e^{-(1-D_{n+1})b(p_n+\Delta p)} \\
\mathbf{A} &= \sum_{k=1}^{M} (-1)(1 - D_{n+1}) \gamma_k \theta_{n+1}^{(k)} \mathbf{X}_{n+1}^{(k)} \\
B &= \frac{2}{3}(1 - D_{n+1}) \sum_{k=1}^{M} C_k \theta_{n+1}^{(k)}
\end{aligned}
$$

最后需要的是，在 ABAQUS 中采用的是工程应变，而本文推导时采用的应变张量，因此需要在 UMAT 中适当修改上述刚度矩阵和一致切向模量矩阵。

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Appendix

$$\mathbf{C}_4 = \begin{pmatrix} \lambda+2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{pmatrix}$$

$$\mathbf{I}_2 \otimes \mathbf{I}_2 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{I}_4^{dev} = \mathbf{I}_4 - \frac{1}{3}\mathbf{I}_2 \otimes \mathbf{I}_2 = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$