AMATH 503: Methods for Partial Differential Equations
University of Washington Spring 2022
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Homework 3

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Problem 1

Solve the following initial-boundary value problem with separation of variables:

$$\begin{aligned}
&u_{tt} = 4u_{xx} \ \text{ for all } \ 0 < x < \pi \ \text{ and } \ t > 0 \\
&u_x(t,0) = u_x(t,\pi) = 0 \ \text{ for all } \ t > 0 \\
&u(0,x) = \sin(x) \ \text{ and } \ u_t(0,x) = 0 \ \text{ for all } \ 0 < x < \pi
\end{aligned}$$

Plot the solution at several representative times. (Your solution will be written as a Fourier series - you can plot partial sums with a reasonable number of terms.)

Solution.
We assume a solution of the form

$$u(t,x) = T(t)X(x)$$

Taking derivatives and substituting into the PDE, this gives us the following relation. Since $T(t)$ and $X(x)$ are independent of each other, the relation must equal a constant $K = \lambda^2$

$$\frac{T''}{4T} = \frac{X''}{X} = K = \lambda^2$$

We solve for $X(x)$ first as our boundary conditions are for x. There will be different solutions for $X(x)$ depending on the value of K, so we must account for all cases. If $K > 0$:

$$\begin{aligned}
X'' &= KX = \lambda^2 X \\
X(x) &= A\exp(\lambda x) + B\exp(-\lambda x) \\
\therefore X'(x) &= A\lambda\exp(\lambda x) - B\lambda\exp(-\lambda x) \\
X'(0) &= 0 = A + B \longrightarrow A = -B \\
X'(\pi) &= 0 = A\lambda\exp(\lambda\pi) - B\lambda\exp(-\lambda\pi) \\
&= -B\lambda\Big(\exp(\lambda\pi) + \exp(-\lambda\pi)\Big)
\end{aligned}$$

As the exponential function will always be greater than 0, the boundary conditions will only hold if $A = B = 0$, and thus this is a trivial solution we can ignore.

If $K = 0$:

$$\begin{aligned}
X'' &= 0 \\
X(x) &= Ax + B \\
X'(x) &= A \\
X'(0) &= X'(\pi) = 0 = A \\
\therefore A &= 0
\end{aligned}$$

This gives a solution $X(x) = B$. We can solve for $T(t)$ in this case:

$$\begin{aligned}
T'' &= 0 \\
T(t) &= A + Bt
\end{aligned}$$

Therefore we have the solution

$$u(t,x) = X(x)T(t) = \frac{a_0}{2} + \frac{b_0 t}{2}$$

If $K < 0$:

$$\begin{aligned}
X'' &= -KX = -\lambda^2 X \\
X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\
X'(x) &= -A\sin(\lambda x) + B\cos(\lambda x) \\
X'(0) &= 0 = B \\
\therefore B &= 0 \\
X'(\pi) &= 0 = -A\sin(\lambda\pi) \\
\therefore \lambda\pi &= n\pi \longrightarrow \lambda = n \\
X(x) &= A\cos(nx)
\end{aligned}$$

We find $\lambda = n$ will always satisfy the boundary conditions. The corresponding function $T(t)$ can thus be found

$$T'' = -4n^2 T \longrightarrow T(t) = A\cos(2nt) + B\sin(2nt)$$

We can combine the solutions for $K = 0$ and $K < 0$ by the principle of superposition. We will also use this to express our general solution for all values of n

$$u(t,x) = \frac{a_0}{2} + \frac{b_0 t}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(2nt) + b_n \sin(2nt)\big)\cos(nx)$$

Applying our initial conditions:

$$\begin{aligned}
u(0,x) = \sin(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \\
u_t(0,x) = 0 &= \frac{b_0}{2} + \sum_{n=1}^{\infty} 2nb_n \cos(nx)
\end{aligned}$$

The coefficients can be solved for as follows, with the integral for a_n being solved using Wolfram Alpha:

$$\begin{aligned}
\frac{a_0}{2} &= \frac{1}{L}\int_0^L f(x)\ dx = \frac{1}{\pi}\int_0^\pi \sin(x)\ dx \\
&= \frac{1}{\pi}\big[-\cos(x)\big]_0^\pi = \frac{2}{\pi} \\
\frac{b_0}{2} &= \frac{1}{L}\int_0^L g(x)\ dx = \frac{1}{\pi}\int_0^\pi 0\ dx = 0 \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{nx\pi}{L})\ dx = \frac{2}{\pi}\int_0^\pi \sin(x)\cos(nx)\ dx \\
&= \frac{2(\cos(\pi n)+1)}{\pi(1-n^2)} \\
&= \frac{2((-1)^n+1)}{\pi(1-n^2)} \\
b_n &= \frac{2}{L}\int_0^L g(x)\cos(\frac{nx\pi}{L})\ dx = \frac{2}{\pi}\int_0^\pi 0\cos(nx)\ dx = 0
\end{aligned}$$

We see that the value of a_n is not defined at $n = 1$. We calculate for this value independently via Wolfram Alpha

$$\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^\pi \sin(x)\cos(x)\ dx \\
&= 0
\end{aligned}$$

Therefore, the general solution to our problem is

$$\boxed{u(t,x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2((-1)^n+1)}{\pi(1-n^2)}\cos(2nt)\cos(nx)}$$

Figure 1 shows the solution at $t = 0, 0.75, 1.5,$ and 2.25. We see the solution oscillates as t increases and appears to be periodic.

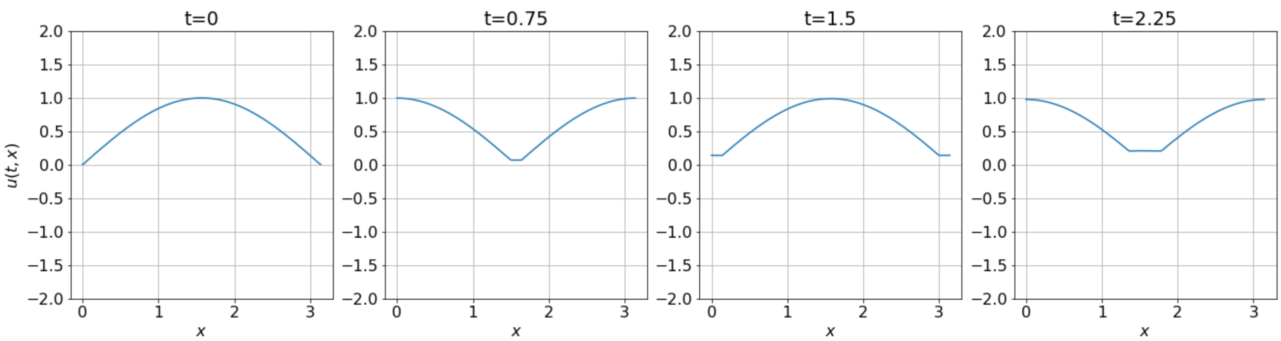

Figure 1: Solution $u(t, x)$ at various t

Problem 2

Solve the following initial-boundary value problem with separation of variables:

$$\begin{aligned} &u_t = u_{xx} \text{ for all } 0 < x < 2 \text{ and } t > 0 \\ &u(t,0) = 1 \text{ and } u(t,2) = 3 \text{ for all } t > 0 \\ &u(0,x) = 2 \text{ for all } 0 < x < 2 \end{aligned}$$

Plot the solution at several representative times. (Your solution will be written as a Fourier series - you can plot partial sums with a reasonable number of terms.)

Solution.
We know the solution to the boundary value problem will be a combination of a single solution to the inhomogenous problem and the general solution to the homogeneous problem. We consider a simple solution to the inhomogenous problem

$$\begin{aligned} u_{xx}(t,x) &= 0 \\ u_x(t,x) &= c_1 \\ u(t,x) &= c_1 x + c_2 \end{aligned}$$

Using the boundary conditions, we can find an equation of the line between the two points to get our inhomogeneous solution. We will call this solution $v^*(t,x)$

$$\begin{aligned} v^*(t,x) &= x_0 + \frac{x_1 - x_0}{2} x \\ &= 1 + \frac{3-1}{2} x \\ &= 1 + x \end{aligned}$$

We now consider the homogeneous boundary conditions. As these are Dirichlet conditions, from class we know the general solution will be

$$v(t,x) = \sum_{n=1}^{\infty} b_n \exp(-(\frac{n\pi}{L})^2 t) \sin(\frac{n\pi x}{L})$$

with $L = 2$ in our problem. We now combine our inhomogenous solution and the homogeneous solution and apply the initial condition

$$\begin{aligned} u(t,x) &= v^*(t,x) + v(t,x) \\ &= 1 + x + \sum_{n=1}^{\infty} b_n \exp(-(\frac{n\pi}{2})^2 t) \sin(\frac{n\pi x}{2}) \\ u(0,x) &= 1 + x + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{2}) \end{aligned}$$

We calculate the coefficients b_n as follows, noting to account for the inhomogeneous solution to the

PDE

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L (f(x) - v^*(t,x))\sin(\frac{n\pi x}{L})\ dx \\ &= \frac{2}{2}\int_0^2 (2-(1+x))\sin(\frac{n\pi x}{2})\ dx \\ &= \int_0^2 (1-x)\sin(\frac{n\pi x}{2})\ dx \end{aligned}$$

Evaluation this integral in Wolfram Alpha

$$\begin{aligned} b_n &= \frac{2(\pi n - 2\sin(\pi n) + \pi n\cos(\pi n))}{\pi^2 n^2} \\ &= \frac{2(1-\cos(\pi n))}{\pi n} \\ b_n &= \frac{2(1-(-1)^n)}{\pi n} \end{aligned}$$

The final solution to the PDE is therefore

$$\boxed{u(t,x) = 1 + x + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{\pi n}\exp(-(\frac{n\pi}{2})^2 t)\sin(\frac{n\pi x}{2})}$$

Figure 2 show the solution at $t = 0, 0.25, 0.5$, and 1. We see the solution is converging to $v^*(t,x)$ as t increases

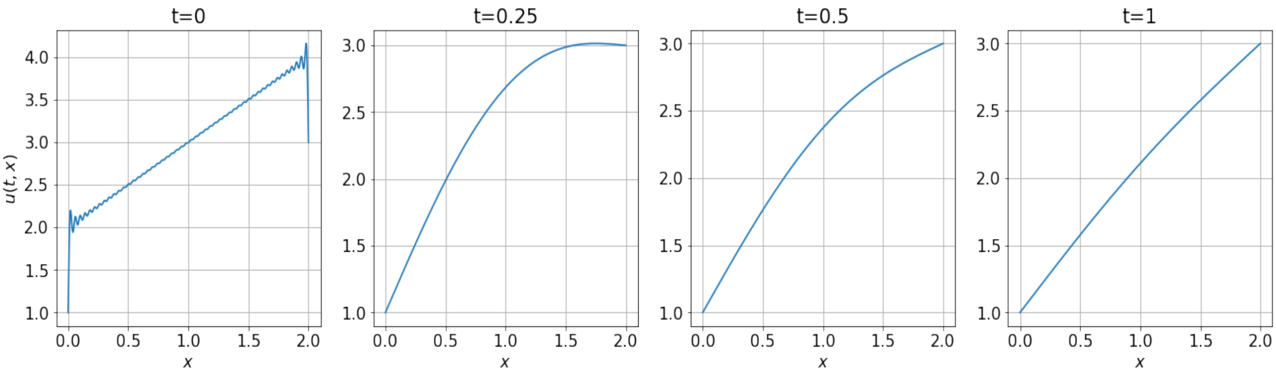

Figure 2: Solution of $u(t,x)$ at various t with inhomogeneous BCs

Problem 3

Note: All of the functions you plot in this problem should be real-valued. This will happen automatically if you plot partial sums of the form

$$\sum_{n=-N}^{N} c_n e^{inx}$$

Part 1

Find the complex Fourier series for the function

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

Plot this Fourier series (using a reasonable number of terms in a partial sum). Note that f is not defined at $x = 0$. This value is irrelevant, because the integral of a function does not depend on the value at a single point.

Part 2

Integrate the Fourier series you found in part 1 term-by-term. You can ignore the constant of integration. Plot this new series (using a reasonable number of terms in a partial sum). Does this series appear to converge? How does it compare to $\int f(x)\,\mathrm{d}x$?

Part 3

Differentiate the Fourier series you found in part 1 term-by-term and plot this new series (using a reasonable number of terms in a partial sum). Does this series appear to converge? How does it compare to $f'(x)$ (which, just like f, is defined on $(-\pi, 0) \cup (0, \pi)$, but not at $x = 0$).

3.1 Part 1

Solution.
From class we know a complex Fourier Series for a function can be expressed as follows

$$f(x) = \sum_{-\infty}^{\infty} C_n \exp(inx) \qquad\qquad C_n = \int_{-\pi}^{\pi} f(x) \exp(-inx)\ dx$$

If $n = 0$:

$$\begin{aligned} C_0 &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\ dx + \int_{0}^{\pi} dx\right] \\ &= \frac{1}{2\pi}\Big[x\Big]_0^{\pi} = \frac{1}{2} \end{aligned}$$

If $n \neq 0$:

$$\begin{aligned}
C_0 &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0\exp(-\mathrm{i}nx)\ dx + \int_{0}^{\pi} \exp(-\mathrm{i}nx)\ dx\right] \\
&= \frac{1}{2\pi}\left[\frac{1}{-\mathrm{i}n}\exp(-\mathrm{i}nx)\right]_0^{\pi} \\
&= \frac{\mathrm{i}}{2\pi n}(\exp(-\mathrm{i}n\pi) - 1) \\
&= \frac{\mathrm{i}}{2\pi n}((-1)^n - 1)
\end{aligned}$$

Therefore, our solution is

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}nx) \times \begin{cases} \frac{1}{2}, & n = 0 \\ \frac{\mathrm{i}}{2\pi n}((-1)^n - 1), & n \neq 0 \end{cases}}$$

Figure 3 shows the solution using various number of terms n. We see it converges to something similar to $f(x)$, but Gibbs Phenomenon occurs at the boundary.

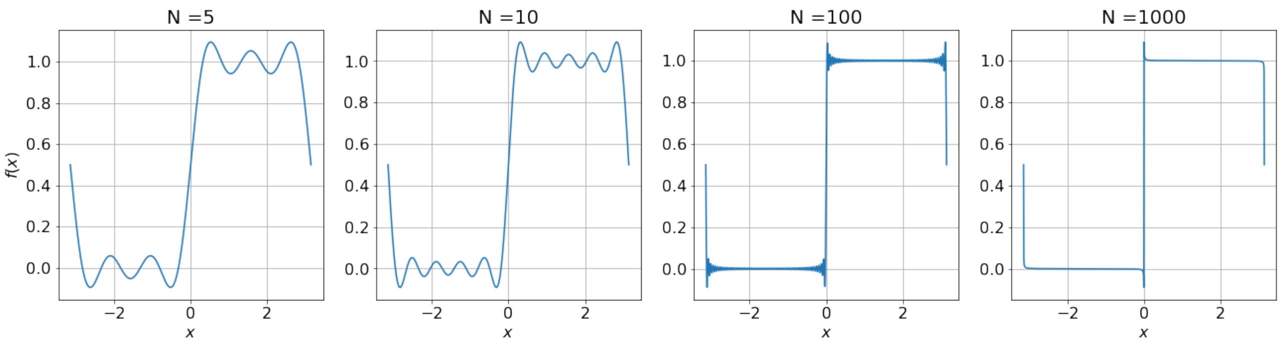

Figure 3: Complex Fourier Series of $f(x)$

3.2 Part 2

Solution.

We integrate each term of the complex Fourier series with respect to x. As a result, n can be considered a constant. From the problem statement, we can ignore the constant of integration. First, if $n = 0$:

$$\int f(x) = \int \frac{1}{2}\ dx = \frac{1}{2}x$$

If $n \neq 0$:

$$\begin{aligned}
\int f(x) &= \frac{\mathrm{i}}{2\pi n}((-1)^n - 1)\int \exp(\mathrm{i}nx)\ dx \\
&= \frac{\mathrm{i}}{2\pi n}((-1)^n - 1)\frac{1}{\mathrm{i}n}\exp(\mathrm{i}nx) \\
&= \frac{1}{2\pi n^2}((-1)^n - 1)\exp(\mathrm{i}nx)
\end{aligned}$$

Therefore

$$\boxed{\int f(x)\,dx = \sum_{n=-\infty}^{\infty} \exp(inx) \times \begin{cases} \frac{x}{2}, & n = 0 \\ \frac{1}{2\pi n^2}((-1)^n - 1), & n \neq 0 \end{cases}}$$

Figure 4 shows the solution using various number of terms n. We see it converges to the piecewise of a constant and a linear segment. This appears to match with what we would expect from the integral of $f(x)$, and additionally no Gibbs Phenomenon is observed at the boundary.

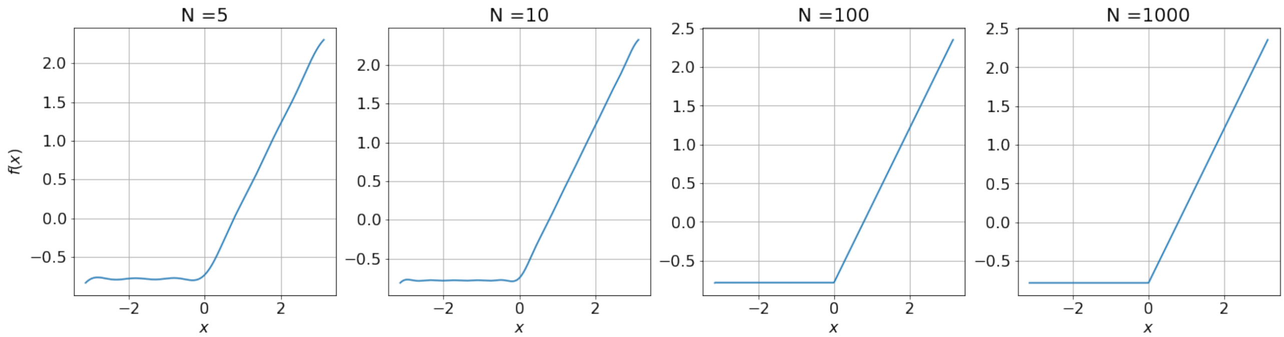

Figure 4: Complex Fourier Series of $\int f(x)$

3.3 Part 3

Solution.

We differentiate each term of the complex Fourier series with respect to x. As a result, n can be considered a constant. First, if $n = 0$:

$$\frac{df(x)}{dx} = \frac{d}{dx}(\frac{1}{2}) = 0$$

If $n \neq 0$:

$$\begin{aligned} \frac{df(x)}{dx} &= \frac{d}{dx}\left(\frac{\mathrm{i}}{2\pi n}((-1)^n - 1)\exp(\mathrm{i}nx)\right) \\ &= \frac{\mathrm{i}}{2\pi n}((-1)^n - 1)\frac{d}{dx}(\exp(\mathrm{i}nx)) \\ &= \frac{\mathrm{i}^2 n}{2\pi n}((-1)^n - 1)\exp(\mathrm{i}nx) \\ &= \frac{1}{2\pi}(1 - (-1)^n)\exp(\mathrm{i}nx) \end{aligned}$$

Therefore

$$\boxed{\frac{df(x)}{dx} = \sum_{n=-\infty}^{\infty} \exp(\mathrm{i}nx) \times \begin{cases} 0, & n = 0 \\ \frac{1}{2\pi}(1 - (-1)^n), & n \neq 0 \end{cases}}$$

Figure 5 shows the solution using various number of terms n. We see the derivative of $f(x)$ does not converge well as there are a large number of oscillations that occur with high values of n. Additionally, while we expect a constant zero from the plot we see for $n = 1000$ this is not the case.

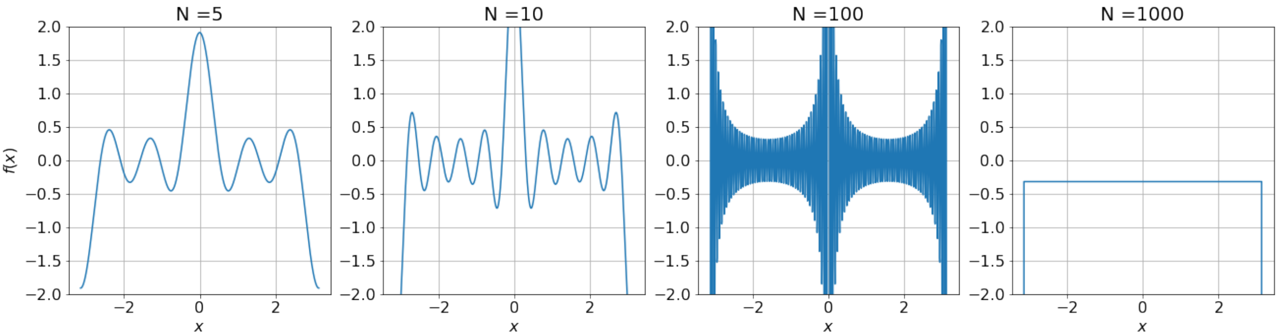

Figure 5: Complex Fourier Series of $\frac{d}{dx}f(x)$

Problem 4

Find all of the non-trivial solutions to the boundary value problem

$$X'' = kX \text{ and } X(0) = 0 \text{ and } X'(\pi) = 0$$

where K is a constant. You should consider the cases where $k > 0$, $k = 0$ and $k < 0$.

Solution.
If $k > 0$:

$$\begin{aligned} X'' &= kX = \lambda^2 X \\ X(x) &= A\exp(\lambda x) + B\exp(-\lambda x) \\ X'(x) &= A\lambda\exp(\lambda x) - B\lambda\exp(-\lambda x) \end{aligned}$$

Applying the boundary conditions

$$\begin{aligned} X(0) &= 0 = A + B \\ X'(\pi) &= 0 = A\lambda\exp(\lambda\pi) - B\lambda\exp(-\lambda\pi) \\ &= -B\lambda(\exp(\lambda\pi) + \exp(-\lambda\pi)) \end{aligned}$$

This system of equations can only be solved if $A = B = 0$ or $\lambda = 0$, which are trivial solutions. If $k = 0$:

$$\begin{aligned} X''(x) &= 0 \\ X'(x) &= A \\ X(x) &= Ax + B \end{aligned}$$

Applying the boundary conditions

$$\begin{aligned} X(0) &= 0 = A(0) + B = B \\ \therefore B &= 0 \\ X'(\pi) &= 0 = A \\ \therefore A &= 0 \end{aligned}$$

Therefore, the only solution for $k = 0$ is the trivial solution. If $k < 0$

$$\begin{aligned} X'' &= -kX = -\lambda^2 X \\ X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\ X'(x) &= -A\sin(\lambda x) + B\cos(\lambda x) \end{aligned}$$

Applying the boundary conditions

$$\begin{aligned}
X(0) &= 0 = A\cos(0) + B\sin(0) = A \\
\therefore A &= 0 \\
X'(\pi) &= 0 = B\cos(\lambda\pi) \\
\therefore \lambda\pi &= n\pi - \frac{\pi}{2} \\
\lambda &= n - \frac{1}{2}
\end{aligned}$$

There is therefore only one non-trivial solution, given for a value of n as

$$\boxed{X(x) = B\sin((n - \frac{1}{2})x),\ \ B \neq 0}$$

Considering all values of n, via the principle of superposition

$$\boxed{X(x) = \sum_{n=0}^{\infty} b_n \sin((n - \frac{1}{2})x)}$$

Problem 5

Suppose that

$$X_1'' = k_1 X_1$$

with

$$aX_1(0) + bX_1'(0) = 0 \text{ and } cX_1(L) + dX_1'(L) = 0$$

and

$$X_2'' = k_2 X_2$$

with

$$aX_2(0) + bX_2'(0) = 0 \text{ and } cX_2(L) + dX_2'(L) = 0$$

where a, b, c, d, L, k_1 and k_2 are constants and $L > 0$.
Prove that if $k_1 \neq k_2$, then

$$\int_0^L X_1(x)X_2(x)\,\mathrm{d}x = 0$$

Hint: Multiply the first ODE by X_2 and multiply the second ODE by X_1 and then subtract the two equations.

Solution.
We first utilize the hint given in the problem

$$\begin{aligned}
X_1''X_2 = k_1X_1X_2 &\implies X_1''X_2 - k_1X_1X_2 = 0 \\
X_2''X_1 = k_2X_2X_1 &\implies X_2''X_1 - k_2X_1X_2 = 0 \\
0 &= X_1''X_2 - k_1X_1X_2 - X_2''X_1 + k_2X_1X_2 \\
0 &= (X_1''X_2 - X_2''X_1) - (k_1X_1X_2 - k_2X_1X_2) \\
(k_1 - k_2)X_2X_1 &= X_1''X_2 - X_2''X_1
\end{aligned}$$

We see the problem is asking about the integral of X_1X_2, so we will take the integral of both sides

$$\begin{aligned}
(k_1 - k_2)\int_0^L X_2X_1 \; dx &= \int_0^L (X_1''X_2 - X_2''X_1) \; dx \\
&= \int_0^L X_1''X_2 \; dx - \int_0^L X_2''X_1 \; dx
\end{aligned}$$

We can perform integration by parts on both of the integrals on the right side using the following substitutions

$$\begin{aligned}
u_1 = X_2 \quad du_1 = X_2' \quad dv = X_1''dx \quad v = X_1' \\
u_2 = X_1 \quad du_2 = X_1' \quad dv = X_2''dx \quad v = X_2'
\end{aligned}$$

Our problem is thus as follows

$$\begin{aligned}
\int_0^L X_2X_1\ dx &= \frac{1}{k_1-k_2}\left[X_2X_1'\Big|_0^L - \int_0^L X_1'X_2'\ dx - X_1X_2'\Big|_0^L + \int_0^L X_1'X_2'\ dx\right] \\
&= \frac{1}{k_1-k_2}\left[X_2X_1'\Big|_0^L - X_1X_2'\Big|_0^L\right] \\
&= \frac{1}{k_1-k_2}\left[X_2(L)X_1'(L) - X_2(0)X_1'(0) - X_1(L)X_2'(L) + X_1(0)X_2'(0)\right]
\end{aligned}$$

Using the boundary conditions, we have the following equalities

$$\begin{aligned}
aX_2(0) &= -bX_2'(0) &\qquad cX_2(L) &= -dX_2'(L) \\
aX_1(0) &= -bX_1'(0) &\qquad cX_1(L) &= -dX_1'(L)
\end{aligned}$$

Using these relations, we can substitute and show the following are all equal to zero

$$\begin{aligned}
\int_0^L X_2X_1\ dx &= \frac{1}{k_1-k_2}\Big[X_2(L)(-\frac{c}{d}X_1(L)) - X_1(L)(-\frac{c}{d}X_2(L)) - \\
&\qquad X_2(0)(-\frac{a}{b}X_1(0)) + X_1(0)(-\frac{a}{b}X_2(0))\Big] = 0 \\
\int_0^L X_2X_1\ dx &= \frac{1}{k_1-k_2}\Big[(-\frac{d}{c}X_2'(L))X_1'(L) - (-\frac{d}{c}X_1'(L))X_2'(L) - \\
&\qquad (-\frac{b}{a}X_1'(0))X_2'(0) + (-\frac{b}{a}X_2'(0))X_1'(0)\Big] = 0
\end{aligned}$$

These two relations show that either substitution will lead to the cancellation of the right hand side evaluated at 0 and L. Thus, any combination of these substitutions will work in case one of $a, b = 0$ or one of $c, d = 0$. Therefore, as long as $k_1 \neq k_2$:

$$\boxed{\int_0^L X_2X_1\ dx = 0}$$