Homework 5

Philip Westphal
UW NetID: philw16@uw.edu

Problem 1

Consider the function

$$f(x) = \begin{cases} e^{-ax} & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

where $a \in \mathbb{R}^+$ is a positive constant.

Part 1

Calculate $F(\omega) = \mathcal{F}[f(x)]$ directly from the definition (i.e., by integrating, not using tables or properties of the Fourier transform).

Solution.
We can ignore the point value at $x = 0$ as when taking the integral of a function, changing the value of one point will not impact the results. Thus

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} \exp(-ax) \exp(-\mathrm{i}\omega x)\ dx \right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \exp(-(a + \mathrm{i}\omega)x)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \times -\frac{1}{a + \mathrm{i}\omega} \exp(-(a + \mathrm{i}\omega)x) \Big|_{-\infty}^{\infty} \\
&= -\frac{1}{\sqrt{2\pi}(a + \mathrm{i}\omega)} (0 - 1) \\
&= \frac{1}{\sqrt{2\pi}(a + \mathrm{i}\omega)}
\end{aligned}$$

Therefore

$$\boxed{F(\omega) = \frac{1}{\sqrt{2\pi}(a + \mathrm{i}\omega)}}$$

Part 2

Confirm that $f(x) = \mathcal{F}^{-1}[F(\omega)]$ directly from the definition (i.e., by integrating, not using tables or properties of the Fourier transform). **Update:** The statement is true for all x, but it is quite difficult to prove for $x = 0$. You can skip that case. That is, you only have to prove that $f(x) = \mathcal{F}^{-1}[F(\omega)]$ when $x \neq 0$.

Solution.
From the problem statement we know we can ignore finding the inverse Fourier Transform at $x = 0$. Thus, we calculate the inverse Fourier Transform for $x \neq 0$ as follows

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) \exp(\mathrm{i}\omega x)\ d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega x)}{a + \mathrm{i}\omega}\ d\omega \end{aligned}$$

For $x > 0$, we can consider the following contour integral around the singularity at $\omega = a\mathrm{i}$ in the upper half of the complex plane. The resulting integral is

$$\int_C \frac{\exp(\mathrm{i}\omega x)}{a + \mathrm{i}\omega}\ d\omega = \int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega x)}{a + \mathrm{i}\omega}\ d\omega + \int_{C_{arc}} \frac{\exp(\mathrm{i}\omega x)}{a + \mathrm{i}\omega}\ d\omega$$

We observe that for the integrand on C_{arc} is of the form

$$f(\omega) = \exp(-\mathrm{i}\omega x) g(\omega)$$

where x can be considered a constant. If we consider $g(R\exp(\mathrm{i}\omega))$, we have

$$\begin{aligned} |g(R\exp(\mathrm{i}\omega))| &= \left| \frac{1}{a + \mathrm{i}R\exp(\mathrm{i}\omega)} \right| \\ &= \frac{1}{\sqrt{R^2 - 2aR\sin(\theta) + a^2}} \text{ (simplified with Wolfram Alpha)} \\ &\leq \frac{1}{\sqrt{R^2 - 2aR + a^2}} \\ &= \frac{1}{\sqrt{(R-a)^2}} \\ &= \frac{1}{R-a} \end{aligned}$$

As this will tend to 0 as $R \to \infty$, we know Jordan's Lemma applies and the C_{arc} portion of the integral will go to 0. Thus, by solving the left hand side of the integral using the residue theorem we can obtain our solution

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega x)}{a+\mathrm{i}\omega}\ d\omega &= \int_C \frac{\exp(\mathrm{i}\omega x)}{a+\mathrm{i}\omega}\ d\omega \\
&= 2\pi\mathrm{i}\ \mathrm{Res}(\frac{\exp(\mathrm{i}\omega x)}{a+\mathrm{i}\omega},\ \omega = a\mathrm{i}) \\
&= \lim_{\omega\to a\mathrm{i}} (\omega - a\mathrm{i})\frac{\exp(\mathrm{i}\omega x)}{a+\mathrm{i}\omega} \\
&= -\mathrm{i} \lim_{\omega\to a\mathrm{i}} (a+\mathrm{i}\omega)\frac{\exp(\mathrm{i}\omega x)}{a+\mathrm{i}\omega} \\
&= -\mathrm{i}\exp(-ax)
\end{aligned}
$$

Therefore

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(\mathrm{i}\omega x)}{a+\mathrm{i}\omega}\ d\omega = \frac{1}{2\pi} \times 2\pi\mathrm{i} \times -\mathrm{i}\exp(-ax) = \exp(-ax)
$$

For $x < 0$, we can apply the same contour integral strategy with a closed arc in the lower half of the complex plane. However, because $g(\omega)$ has no singularity in the lower half of the complex plane the total integral will be 0. As we have know Jordan's Lemma still applies and the integral along the arc is 0, the integral along the real line from $-\infty$ to ∞ must also be 0. Therefore, we have proved the full result ignoring the point at $x = 0$

$$
\boxed{f(x) = \begin{cases} \exp(-at), & x > 0 \\ 0, & x < 0 \end{cases}}
$$

Problem 2

Consider the initial-boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \ \text{ for all } \ t > 0 \ \text{ and } \ -\infty < x < \infty \\ u(0,x) &= 0 \ \text{ and } \ u_t(0,x) = g(x) \ \text{ for all } \ -\infty < x < \infty \end{aligned}$$

where $c > 0$ and g is a sufficiently nice function that $\mathcal{F}[g(x)] = G(\omega)$ exists and $\mathcal{F}^{-1}[G(\omega)] = g(x)$. Furthermore, suppose that $u(t,x)$ goes to zero sufficiently quickly as $x \to \pm\infty$ that we can take Fourier transforms of $u(t,x)$.

2.1 Part 1

Use Fourier transforms to show that

$$u(t,x) = \mathcal{F}^{-1}\left[G(\omega) \cdot \frac{\sin(c\omega t)}{c\omega)}\right]$$

Solution.
We know from class and the derivative property of the Fourier Transform that our problem can be rewritten as follows

$$\begin{aligned} \hat{u}_{xx} &= \mathcal{F}[u_{xx}(t,x)] = -\omega^2 c^2 \hat{u}(t,x) \\ \mathcal{F}[u_{tt}] &= \hat{u}_{tt}(t,x) \end{aligned}$$

We can therefore rewrite our problem in Fourier space as

$$\begin{aligned} \hat{u}_{tt} &= -\omega^2 c^2 \hat{u}(t,x) \\ \hat{u}(0,\omega) &= 0 \\ \hat{u}_t(0,\omega) &= G(\omega) \end{aligned}$$

This is second order ODE with a negative right hand side ($K < 0$). The general solution is therefore given as follows

$$\hat{u}(t,\omega) = A(\omega)\cos(c\omega t) + B(\omega)\sin(c\omega t)$$

We apply the initial conditions

$$\begin{aligned} \hat{u}(0,\omega) &= 0 = A(\omega) \to \therefore A(\omega) = 0 \\ \hat{u}_t(t,\omega) &= c\omega B(\omega)\cos(c\omega t) \\ \hat{u}_t(0,\omega) &= G(\omega) = c\omega B(\omega)\cos(0) \to \therefore B(\omega) = \frac{G(\omega)}{c\omega} \end{aligned}$$

Thus

$$\hat{u}(t,\omega) = \frac{G(\omega)}{c\omega}\sin(c\omega t)$$

Taking the inverse Fourier Transform gives us the desired result

$$\boxed{u(t,x) = \mathcal{F}^{-1}\left[\frac{G(\omega)}{c\omega}\sin(c\omega t)\right]}$$

Part 2

Let $p : \mathbb{R} \to \mathbb{R}$ be the function such that

$$p(x) = \begin{cases} 1/c & -ct < x < ct \\ 0 & \text{otherwise} \end{cases}$$

Show that

$$\mathcal{F}[p(x)] = \sqrt{\frac{2}{\pi}} \frac{\sin(c\omega t)}{c\omega}$$

Solution.
We use the integral evaluation of the Fourier Transform.

$$\begin{aligned}
P(t,\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-ct} (0) \exp(-\mathrm{i}\omega x)\ dx + \int_{-ct}^{ct} \frac{1}{c} \exp(-\mathrm{i}\omega x)\ dx + \int_{ct}^{\infty} (0) \exp(-\mathrm{i}\omega x)\ dx \right] \\
&= \frac{1}{c\sqrt{2\pi}} \int_{-ct}^{ct} \exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{1}{c\sqrt{2\pi}} \times -\frac{1}{\mathrm{i}\omega} \times \exp(-\mathrm{i}\omega x) \Big|_{-ct}^{ct} \\
&= \frac{\mathrm{i}}{c\omega\sqrt{2\pi}} (\exp(-\mathrm{i}c\omega t) - \exp(\mathrm{i}c\omega t)) \\
&= -\frac{2\mathrm{i}}{c\omega\sqrt{2\pi}} (\frac{\exp(\mathrm{i}c\omega t) - \exp(-\mathrm{i}c\omega t)}{2}) \\
&= -\frac{2\mathrm{i}}{c\omega\sqrt{2\pi}} \mathrm{i} \sin(c\omega t) \\
&= \sqrt{\frac{2}{\pi}} \frac{\sin(c\omega t)}{c\omega}
\end{aligned}$$

We have therefore proved

$$\boxed{\mathcal{F}[p(x)] = \sqrt{\frac{2}{\pi}} \frac{\sin(c\omega t)}{c\omega}}$$

Part 3

Use the convolution theorem from class to show that

$$u(t,x) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, \mathrm{d}y$$

Solution.
Let $h(t,x)$ be the solution to problem

$$h(t,x) = \mathcal{F}^{-1}[G(\omega) P(t,\omega)]$$

where $G(\omega)$ is is the Fourier transform of our initial condition from part 1, and $P(t,\omega)$ our result from part 2. Thus, using our result from part 2 we have the following relation

$$h(t,x) = \mathcal{F}^{-1}[G(\omega)P(t,\omega)] = \mathcal{F}^{-1}[G(\omega)\sqrt{\frac{2}{\pi}}\frac{\sin(c\omega t)}{c\omega}]$$

We can prove the inverse Fourier Transform is a linear operator as follows

$$\begin{aligned}
\mathcal{F}^{-1}[aF(\omega)+bG(\omega)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(aF(\omega)+bG(\omega))\exp(\mathrm{i}\omega x)\ d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}aF(\omega)\exp(\mathrm{i}\omega x)\ d\omega + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}bG(\omega)\exp(\mathrm{i}\omega x)\ d\omega \\
&= a\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)\exp(\mathrm{i}\omega x)\ d\omega + b\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}G(\omega)\exp(\mathrm{i}\omega x)\ d\omega \\
&= a\mathcal{F}^{-1}[F(\omega)] + b\mathcal{F}^{-1}[G(\omega)]
\end{aligned}$$

We can thus use the linearity of the inverse Fourier Transform operation to pull out the constant from the Fourier Representation of $h(t,x)$. We observe that this leaves us with an expression that matches our result from part 1, and we can substitute in $u(t,x)$

$$h(t,x) = \mathcal{F}^{-1}[G(\omega)\sqrt{\frac{2}{\pi}}\frac{\sin(c\omega t)}{c\omega}] = \sqrt{\frac{2}{\pi}}\mathcal{F}^{-1}[G(\omega)\frac{\sin(c\omega t)}{c\omega}] = \sqrt{\frac{2}{\pi}}u(t,x)$$

From the convolution theorem, we also know $h(t,x)$ can be defined as follows

$$h(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(y)p(x-y)\ dy$$

We translate the initial condition for $p(x) \to p(x-y)$

$$p(x-y) = \begin{cases} \frac{1}{c} & -ct < x-y < ct \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{c} & x-ct < y < x+ct \\ 0 & \text{otherwise} \end{cases}$$

Thus

$$\begin{aligned}
h(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(y)p(x-y)\ dy \\
&= \frac{1}{\sqrt{2\pi}}\Big[\int_{-\infty}^{x-ct}g(y)(0)\ dy + \int_{x-ct}^{x+ct}g(y)\frac{1}{c}\ dy + \int_{x+ct}^{\infty}g(y)(0)\ dy\Big] \\
&= \frac{1}{c\sqrt{2\pi}}\int_{x-ct}^{x+ct}g(y)\ dy
\end{aligned}$$

We set our equations for $h(t,x)$ equal to each other and solve for $u(t,x)$

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}u(t,x) &= \frac{1}{c\sqrt{2\pi}}\int_{x-ct}^{x+ct}g(y)\ dy \\
u(t,x) &= \frac{\sqrt{\pi}}{\sqrt{2}} \times \frac{1}{c\sqrt{2\pi}}\int_{x-ct}^{x+ct}g(y)\ dy \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct}g(y)\ dy
\end{aligned}$$

Thus, we have proved that

$$\boxed{u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ dy}$$

Part 4

Plot the solution to this boundary-value problem at several representative times when $g(x) = e^{-|x|}$ and $c = 1$.

Solution.
We evaluate the inetgral from part 3 using the given $g(x)$, noting that we can break apart the integral into two pieces. The first is for $x < 0$ and the second for $x > 0$

$$\begin{aligned}
u(t,x) &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ dy = \frac{1}{2c}\left[\int_{x-ct}^{0} g(y)\ dy + \int_{0}^{x+ct} g(y)\ dy\right] \\
&= \frac{1}{2c}\left[\int_{x-ct}^{0} \exp(y)\ dy + \int_{0}^{x+ct} \exp(-y)\ dy\right] \\
&= \frac{1}{2c}\left[\exp(y)\Big|_{x-ct}^{0} - \exp(-y)\Big|_{0}^{x+ct}\right] \\
&= \frac{1}{2c}\Big[(1-\exp(x-ct)) - (\exp(-(x+ct)) - 1)\Big] \\
&= \frac{1}{2c}\Big[2 - \exp(x-ct) - \exp(-(x+ct))\Big] \\
&= \frac{1}{2c}\Big[2 - (\exp(x-ct) + \exp(-x-ct))\Big] \\
&= \frac{1}{2c}\Big[2 - \exp(-ct)(\exp(x) + \exp(-x))\Big] \\
&= \frac{1}{2c}\Big[2 - 2\exp(-ct)\cosh(x)\Big]
\end{aligned}$$

Our general solution is thus

$$\boxed{u(t,x) = \frac{1}{c}(1 - \exp(-ct)\cosh(x))}$$

Some plots of $u(t,x)$ are shown in figure 1 with $c = 1$. We observe that the function flattens out towards a constant value of 1 as time increases.

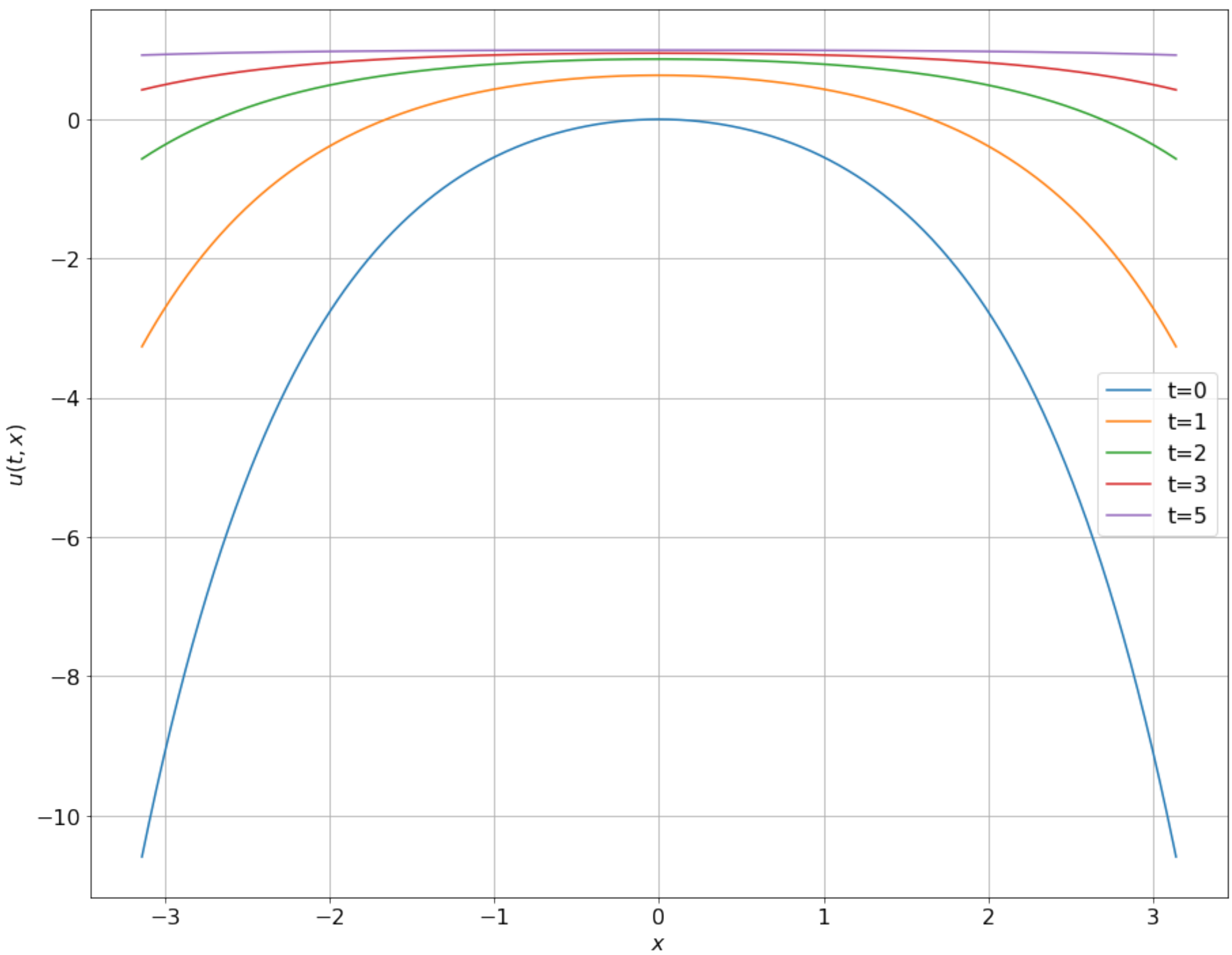

Figure 1: Solution $u(t, x)$ at various t

Problem 3

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an even function (i.e., $f(-x) = f(x)$) and that $g : \mathbb{R} \to \mathbb{R}$ is an odd function (i.e., $g(-x) = -g(x)$) and $h : \mathbb{R} \to \mathbb{R}$ is an arbitrary (not necessarily even or odd) function. Definethe functions $F, G, H : \mathbb{R} \to \mathbb{C}$ such that $F(\omega) = \mathcal{F}[f(x)]$ and $G(\omega) = \mathcal{F}[g(x)]$ and $H(\omega) = \mathcal{F}[h(x)]$. (This is the standard notation from class.)

Part 1

Show that F is even and real-valued.

Solution.
From the solution to problem 4 we know that the Fourier Transform can be written as

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx - \mathrm{i}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\sin(\omega x)\ dx$$

If $f(x)$ is an even function, then $f(x)\sin(\omega x)$ will be an odd function as sine is an odd function. The second piece of the Fourier Transform is thus the integral of an odd function over a symmetric interval, which will evaluate to 0. Therefore for an even $f(x)$, the Fourier Transform is given by

$$\mathcal{F}[f(x)] = F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx$$

There are no complex values inside the integral as from the problem statement $f : \mathbb{R} \to \mathbb{R}$. Thus, we know $\mathcal{F}[f(x)]$ is purely real-valued. To show the Fourier Transform is even, we consider $F(-\omega)$

$$\begin{aligned} F(-\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos(-\omega x)\ dx \\ &\text{Let } y = -x,\ dy = -dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} -f(-y)\cos(\omega y)\ dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(y)\cos(\omega y)\ dy \\ &= F(\omega) \end{aligned}$$

As $F(\omega) = F(-\omega)$, we know the Fourier Transform of an even function is even.

Part 2

Show that G is odd and purely imaginary-valued.

Solution.
From the solution to problem 4 we know that the Fourier Transform can be written as

$$\mathcal{F}[g(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\cos(\omega x)\ dx - \mathrm{i}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\sin(\omega x)\ dx$$

If $g(x)$ is an odd function, then $g(x)\cos(\omega x)$ will be an odd function as cosine is an even function. The first piece of the Fourier Transform is thus the integral of an odd function over a symmetric interval, which will evaluate to 0. Therefore for an odd $g(x)$, the Fourier Transform is given by

$$\mathcal{F}[g(x)] = G(\omega) = -\frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\sin(\omega x)\ dx$$

There are no complex values inside the integral as from the problem statement $g : \mathbb{R} \to \mathbb{R}$. Thus, we know the integral will portion of $\mathcal{F}[g(x)]$ purely real-valued. As we are then multipling this by $-\mathrm{i}$, the result will be a purely imaginary value. Therefore, $\mathcal{F}[g(x)]$ is purely imaginary valued. To show the Fourier Transform is odd, we consider $F(-\omega)$

$$\begin{aligned}
F(-\omega) &= -\frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\sin(-\omega x)\ dx \\
&\text{Let } y = -x,\ dy = -dx \\
&= -\frac{\mathrm{i}}{\sqrt{2\pi}} \int_{\infty}^{-\infty} -g(-y)\sin(\omega y)\ dy \\
&= -\frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(-y)\sin(\omega y)\ dy \\
&= -\frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -g(y)\sin(\omega y)\ dy \\
&= \frac{\mathrm{i}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\sin(\omega y)\ dy \\
&= -F(\omega)
\end{aligned}$$

As $F(-\omega) = -F(\omega)$, we have proved the Fourier Transform of an odd function is odd.

Part 3

Show that $\mathrm{Re}[H]$ is even and $\mathrm{Im}[H]$ is odd. **Hint:** $h(x) = \frac{1}{2}(h(x) + h(-x)) + \frac{1}{2}(h(x) - h(-x))$

Solution.
We assume $h(x)$ can be written as the sum of an even and an odd function. Thus

$$\begin{aligned}
h(x) &= h_e(x) + h_o(x) \\
h(-x) &= h_e(x) - h_o(x)
\end{aligned}$$

If we consider adding and subtracting these two equations, we get the following relations

$$\begin{aligned}
2h_e(x) = h(x) + h(-x) &\to h_e(x) = \frac{1}{2}(h(x) + h(-x)) \\
2h_o(x) = h(x) - h(-x) &\to h_o(x) = \frac{1}{2}(h(x) - h(-x))
\end{aligned}$$

We see that

$$h_e(x) + h_o(x) = \frac{1}{2}(h(x) + h(-x)) + \frac{1}{2}(h(x) - h(-x)) = h(x)$$

We have thus proved that $h(x)$ can be written as the sum of an even and odd function. We now can use this fact in the Fourier Transform formula

$$\begin{aligned}
\mathcal{F}[h(x)] &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}(h_e(x)+h_o(x))\cos(\omega x)\ dx - \mathrm{i}\int_{-\infty}^{\infty}(h_e(x)+h_o(x))\sin(\omega x)\ dx\right] \\
&= \frac{1}{\sqrt{2\pi}}\Bigg[\int_{-\infty}^{\infty} h_e(x)\cos(\omega x)\ dx + \int_{-\infty}^{\infty} h_o(x)\cos(\omega x)\ dx - \mathrm{i}\int_{-\infty}^{\infty} h_e(x)\sin(\omega x)\ dx \\
&\qquad - \mathrm{i}\int_{-\infty}^{\infty} h_o(x)\sin(\omega x)\ dx\Bigg]
\end{aligned}$$

We observe that the second and third integrals will result in integrals of odd functions over symmetric intervals (from parts 1 and 2). As a result, They will go to 0 and the Fourier Transform will be given as

$$\mathcal{F}[h(x)] = H(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_e(x)\cos(\omega x)\ dx - \frac{\mathrm{i}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_o(x)\sin(\omega x)\ dx$$

We see that

$$\mathrm{Re}[H(\omega)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_e(x)\cos(\omega x)\ dx$$

This matches the problem from part 1. As $h_e(x)$ is an even function, we therefore know $\mathrm{Re}[H(\omega)]$ must be even. Similarly

$$\mathrm{Im}[H(\omega)] = -\frac{\mathrm{i}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} h_o(x)\sin(\omega x)\ dx$$

This matches the problem from part 2. As $h_0(x)$ is an odd function, we therefore know that $\mathrm{Im}[H(\omega)]$ must be odd. We have thus proved both cases.

Problem 4

The Fourier cosine transform is defined by

$$\mathcal{F}_c[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos(\omega x)\,\mathrm{d}x$$

and the Fourier sine transform is defined by

$$\mathcal{F}_s[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\sin(\omega x)\,\mathrm{d}x$$

Part 1

Show that $\mathcal{F}[f(x)] = \mathcal{F}_c[f(x)] - i\mathcal{F}_s[f(x)]$

Solution.
Eulers formula is defined as

$$\exp(\mathrm{i}x) = \cos(x) + \mathrm{i}\sin(x)$$

Letting $y = -x$, we can show this relationship for Negative exponents as well using the even and odd function properties of sine and cosine

$$\begin{aligned}\exp(\mathrm{i}(-y)) &= \cos(-y) + \mathrm{i}\sin(-y)\\ &= \cos(y) - \mathrm{i}\sin(y)\end{aligned}$$

We can substitute this into the Fourier Transform equation

$$\begin{aligned}\mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}\omega x)\ dx\\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)(\cos(\omega x) - \mathrm{i}\sin(\omega x))\ dx\\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx - \mathrm{i}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\sin(\omega x)\ dx\end{aligned}$$

We see the integrals match given formulas, and by substituting we have obtained our solution

$$\boxed{\mathcal{F}[f(x)] = \mathcal{F}_c[f(x)] - \mathrm{i}\mathcal{F}_s[f(x)]}$$

Part 2

Show that $\mathcal{F}[f(x)] = \mathcal{F}_c[f(x)]$ if f is an even function and that $\mathcal{F}[f(x)] = -i\mathcal{F}_s[f(x)]$ if f is an odd function. (The results of the previous problem may be useful.)

Solution.
If f is an even function, then $f(x)\cos(\omega x)$ will also be an even function as cosine is an even function. Conversely, $f(x)\sin(\omega x)$ will be odd as sine is an odd function. As each integral from part 1 is over a symmetric range, the sine portion of the integral will evaluate to 0 due to being an odd function.

Thus

$$\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx - \mathrm{i}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\ dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx \\
&\boxed{\mathcal{F}[f(x)] = \mathcal{F}_c[f(x)]}
\end{aligned}$$

Similarly, the reverse if true if $f(x)$ is an odd function. The function $f(x)\sin(\omega x)$ will be even as sine is an odd function, and the function $f(x)\cos(\omega x)$ will be odd as cosine is an even function. Thus, when integrating over a symmetric interval the following must be true

$$\begin{aligned}
\mathcal{F}[f(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos(\omega x)\ dx - \mathrm{i}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\ dx \\
&= -\mathrm{i}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\sin(\omega x)\ dx \\
&\boxed{\mathcal{F}[f(x)] = -\mathrm{i}\mathcal{F}_s[f(x)]}
\end{aligned}$$

Problem 5

It is possible to define the derivative of the Dirac delta "function" $\delta(x)$. Since δ is a distribution and not really a function, we should expect its derivative to be a distribution as well. Let $\delta'(x)$ denote the derivative of $\delta(x)$. You can assume that both δ' and δ are well-defined and that their Fourier transforms follow all of the rules we have derived in class.

Part 1

What is $\mathcal{F}[\delta'(x)]$?

Solution.
We know from class that the Fourier Transform of $\delta(x)$ is given as

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}$$

From the derivative property of the Fourier Transform we know that

$$\mathcal{F}[f'(x)] = \mathrm{i}\omega\mathcal{F}[f(x)]$$

Thus, we can find $\mathcal{F}[\delta'(x)]$ as follows

$$\begin{aligned}\mathcal{F}[\delta'(x)] &= \mathrm{i}\omega\mathcal{F}[\delta(x)] \\ &= \mathrm{i}\omega \times \frac{1}{\sqrt{2\pi}} \\ \boxed{\mathcal{F}[\delta'(x)] = \frac{\mathrm{i}\omega}{\sqrt{2\pi}}}\end{aligned}$$

Part 2

If f is a smooth function, what is $\delta' \star f$?

Solution.
Let

$$h(x) = \mathcal{F}^{-1}[G(\omega)F(\omega)]$$

where $G(\omega) = \mathcal{F}[\delta'(x)]$ and $F(\omega) = \mathcal{F}[f(x)]$. Substituting for $G(\omega)$ and using the linearity of the inverse Fourier Transform, we can write $h(x)$ as follows

$$\begin{aligned}h(x) &= \mathcal{F}^{-1}[G(\omega)F(\omega)] \\ &= \mathcal{F}^{-1}[\frac{i\omega}{\sqrt{2\pi}}F(\omega)] \\ &= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[i\omega F(\omega)]\end{aligned}$$

From the derivative property of the Fourier Transform derived in class, we know that

$$\mathcal{F}^{-1}[i\omega F(\omega)] = f'(x)$$

Thus

$$h(x) = \frac{1}{\sqrt{2\pi}} f'(x)$$

From the convolution theorem, we also know that $h(x)$ can be written as

$$h(x) = \frac{1}{\sqrt{2\pi}}(g * f) = \frac{1}{\sqrt{2\pi}}(\delta' * f)$$

Setting these relationships equal, we obtain a solution for $\delta' * f$

$$\frac{1}{\sqrt{2\pi}}(\delta' * f) = \frac{1}{\sqrt{2\pi}} f'(x)$$
$$\boxed{\delta' * f = f'(x)}$$