AMATH 503: Methods for Partial Differential Equations

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Homework 6

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Problem 1

Solve the following initial-boundary value problem:

$$u_t = 4u_{xx} \text{ for all } t > 0 \text{ and } -\infty < x < \infty$$
$$u(0, x) = f(x) \text{ for all } -\infty < x < \infty$$

where

$$f(x) = \begin{cases} 1 - |x| & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Plot the solution at several representative times.

Solution.

We know the fundamental solution to the heat equation is given by

$$F(t, x; y) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{(x-y)^2}{4Dt})$$

For our problem, this is equivalent to

$$F(t, x; y) = \frac{1}{4\sqrt{\pi t}} \exp(-\frac{(x-y)^2}{16t})$$

Given a problem with an initial condition $f(x)$, a solution can be found using the convolution theorem as

$$u(t, x) = \int_{-\infty}^{\infty} f(y)F(t, x; y)\ dy$$

Plugging in our function we can evaluate the integral

$$\begin{aligned} u(t, x) &= \int_{-\infty}^{-1} (0)F(t, x; y)\ dy + \int_{-1}^{0} (1 + y)F(t, x; y)\ dy + \int_{0}^{1} (1 - y)F(t, x; y)\ dy + \int_{1}^{\infty} (0)F(t, x; y)\ dy \\ &= \int_{-1}^{0} (1 + y)F(t, x; y)\ dy + \int_{0}^{1} (1 - y)F(t, x; y)\ dy \\ &= \frac{1}{4\sqrt{\pi t}} \left[\int_{-1}^{0} (1 + y) \exp(-\frac{(x-y)^2}{16t})\ dy + \int_{0}^{1} (1 - y) \exp(-\frac{(x-y)^2}{16t})\ dy \right] \end{aligned}$$

We evaluate each of these integrals individually using Mathematica

$$\int_{-1}^{0}(1+y)\exp(-\frac{(x-y)^2}{16t})\,dy = 2\sqrt{\pi t}(x+1)\big[\operatorname{erf}(\frac{x+1}{4\sqrt{t}})-\operatorname{erf}(\frac{x}{4\sqrt{t}})\big]+8t(\exp(-\frac{(x+1)^2}{16t})-\exp(-\frac{x^2}{16t}))$$

$$\int_{0}^{1}(1-y)\exp(-\frac{(x-y)^2}{16t})\,dy = 2\sqrt{\pi t}(x-1)\big[\operatorname{erf}(\frac{x-1}{4\sqrt{t}})-\operatorname{erf}(\frac{x}{4\sqrt{t}})\big]+8t(\exp(-\frac{(x-1)^2}{16t})-\exp(-\frac{x^2}{16t}))$$

Thus, $u(t,x)$ is given as follows

$$\begin{aligned}u(t,x)=\frac{1}{4\sqrt{\pi t}}\Big[&2\sqrt{\pi t}(x+1)\big[\operatorname{erf}(\frac{x+1}{4\sqrt{t}})-\operatorname{erf}(\frac{x}{4\sqrt{t}})\big]+8t(\exp(-\frac{(x+1)^2}{16t})-\exp(-\frac{x^2}{16t}))+\\ &2\sqrt{\pi t}(x-1)\big[\operatorname{erf}(\frac{x-1}{4\sqrt{t}})-\operatorname{erf}(\frac{x}{4\sqrt{t}})\big]+8t(\exp(-\frac{(x-1)^2}{16t})-\exp(-\frac{x^2}{16t}))\Big]\end{aligned}$$

Using the FullSimplify function in Mathematica, we obtain the following result

$$\boxed{\begin{aligned}u(t,x)=\frac{1}{2}\Big[&\frac{4\sqrt{t}\exp(-\frac{(1+x)^2}{16t})(1+\exp(\frac{x}{4t})-2\exp(\frac{1+2x}{16t}))}{\sqrt{\pi}}+\\ &(x-1)\operatorname{erf}(\frac{x-1}{4\sqrt{t}})-2x\operatorname{erf}(\frac{x}{4\sqrt{t}})+(1+x)\operatorname{erf}(\frac{x+1}{4\sqrt{t}})\Big]\end{aligned}}$$

Some plots of $u(t,x)$ are shown in figure 1. We observe that as time increases, the sharp peak of the function is flattening out along with the corners at ± 1

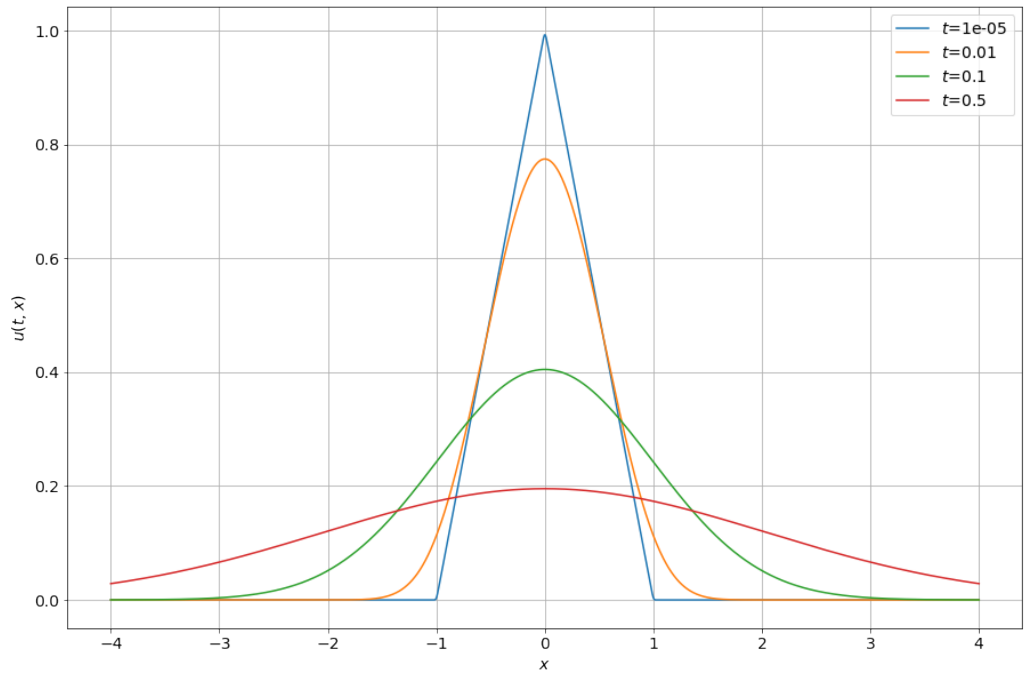

Figure 1: Solution $u(t,x)$ at various t for $x\in[-4,4]$

Problem 2

Solve the following initial-boundary value problem:

$$\begin{aligned}&u_t = u_{xx} \ \text{ for all } \ t > 0 \ \text{ and } \ -\infty < x < 0\\&u(t,0) = 0 \ \text{ for all } \ t > 0\\&u(0,x) = e^x \ \text{ for all } \ -\infty < x < 0\end{aligned}$$

Plot the solution at several representative times.

Solution.
We again know that the fundamental solution to the heat equation is given by

$$F(t,x;y) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{(x-y)^2}{4Dt})$$

For our problem, this is equivalent to

$$F(t,x;y) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{(x-y)^2}{4t})$$

We utilize the method of images to solve this problem on half of the real line. On the full real line, the solution to this PDE by the convolution theorem with initial condition $g(x)$ will be

$$u(t,x) = \int_{-\infty}^{\infty} g(y)F(t,x;y)\ dy$$

For our problem, we have $g(x) = \exp(x)\ \forall x < 0$. However, for our problem we also need to satisfy the boundary condition at $x = 0$. We know this boundary condition can be represented in integral form as

$$u(t,0) = \int_{-\infty}^{\infty} g(y)F(t,0;y)\ dy = \int_{-\infty}^{\infty} g(y)\frac{1}{\sqrt{4\pi t}} \exp(-\frac{y^2}{4t})\ dy$$

We observe that $F(t,0;y)$ is an even function of y. Thus, we can make this integral evaluate to 0 by choosing an odd function $g(x)$ with the condition that $g(x) = \exp(x)\ \forall x < 0$. Therefore

$$g(x) = \begin{cases} f(x), & \forall x < 0 \\ -f(-x), & \forall x > 0 \end{cases} = \begin{cases} \exp(x), & \forall x < 0 \\ -\exp(-x), & \forall x > 0 \end{cases}$$

We can now substitute this $g(x)$ into our convolution equation and solve for $u(t,x)$ that will satisfy the initial condition integral and thus our problem. The integrals were solved using Wolfram Mathematica

$$
\begin{aligned}
u(t,x) &= \int_{-\infty}^{\infty} g(y)F(t,x;y)\ dy \\
&= \frac{1}{\sqrt{4\pi t}}\left[\int_{-\infty}^{0} \exp(y)\exp(-\frac{(x-y)^2}{4t})\ dy - \int_{0}^{\infty} \exp(-y)\exp(-\frac{(x-y)^2}{4t})\ dy\right] \\
&= \frac{1}{\sqrt{4\pi t}}\left[\sqrt{\pi t}\exp(t)\exp(x)(1-\operatorname{erf}(\frac{2t+x}{2\sqrt{t}})) - \sqrt{\pi t}\exp(t)\exp(-x)(1-\operatorname{erf}(\frac{2t-x}{2\sqrt{t}}))\right] \\
&= \frac{\exp(t)}{2}\left[(\exp(x)-\exp(-x)) - \exp(x)\operatorname{erf}(\frac{2t+x}{2\sqrt{t}}) + \exp(-x)\operatorname{erf}(\frac{2t-x}{2\sqrt{t}})\right]
\end{aligned}
$$

$$
\boxed{u(t,x) = \frac{\exp(t)}{2}\left[2\sinh(x) - \exp(x)\operatorname{erf}(\frac{2t+x}{2\sqrt{t}}) + \exp(-x)\operatorname{erf}(\frac{2t-x}{2\sqrt{t}})\right]}
$$

Some plots of $u(t,x)$ are shown in figure 2. We observe that as time increases, the initial exponential becomes more of a peak that is flattening out while decaying towards 0.

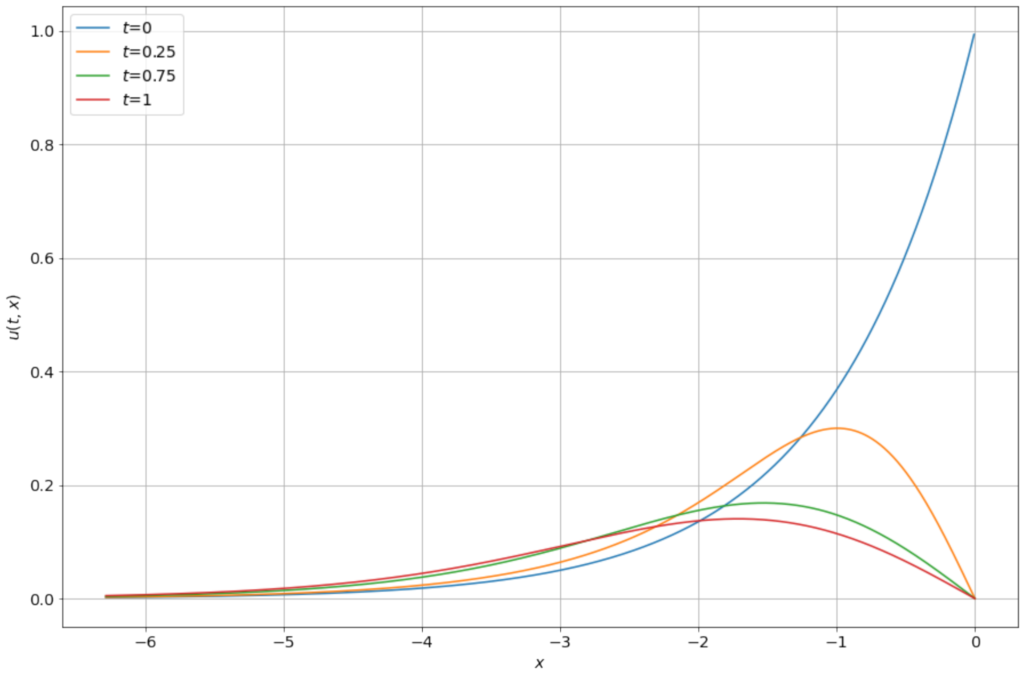

Figure 2: Solution $u(t,x)$ at various t for $x \in [-2\pi, 0]$

Problem 3

Consider the cable equation

$$u_t = Du_{xx} - \alpha u$$

where $t > 0$ and $-\infty < x < \infty$.

Part 1

Show that the general solution to the cable equation is

$$u(t,x) = e^{-\alpha t} v(t,x)$$

where $v(t,x)$ is a solution to the heat equation.

Solution.
We can utilize Fourier Transforms to prove $u(t,x)$ is the general solution to the cable equation. Let

$$U(t,\omega) = \mathcal{F}[u(t,x)]$$

The we have

$$\begin{aligned}
\mathcal{F}[u_t(t,x)] &= U_t(t,\omega) \\
\mathcal{F}[u_{xx}(t,x)] &= (\mathrm{i}\omega)^2 DU(t,\omega) = -D\omega^2 U(t,\omega)
\end{aligned}$$

Our PDE in Fourier Space is thus

$$U_t = -D\omega^2 U - \alpha U = -(D\omega^2 U + \alpha)U$$

This is a first order ODE which has the following solution

$$U(t,\omega) = A(\omega)\exp(-(D\omega^2 + \alpha)t) = A(\omega)\exp(-D\omega^2 t)\exp(-\alpha t)$$

We know from class that with initial condition $u(0,x) = f(x)$, this problem is equivalent to

$$U(t,\omega) = F(\omega)\exp(-D\omega^2 t)\exp(\alpha t)$$

where $F(\omega) = \mathcal{F}[f(x)]$. The solution in the problem space can be given as

$$u(t,x) = \mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)\exp(-\alpha t)]$$

This problem can be rewritten and simplified as follows

$$\begin{aligned}
u(t,x) &= \mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)\exp(-\alpha t)] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(F(\omega)\exp(-D\omega^2 t)\exp(-\alpha t))\exp(\mathrm{i}\omega x)\ d\omega \\
&= \frac{\exp(-\alpha t)}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\omega)\exp(-D\omega^2 t)\exp(\mathrm{i}\omega x)\ d\omega \\
&= \exp(-\alpha t)\mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)]
\end{aligned}$$

From class, we know that $\mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)]$ is the general solution to the heat equation. If we let this be defined as $v(t,x)$, then we have

$$u(t,x) = \exp(-\alpha t)v(t,x)$$

We have thus proved that $u(t,x)$ is the general solution to the cable equation.

Part 2

Find the fundamental solution to the cable equation.

Solution.
The general form Fourier representation is given as

$$u(t,x) = \mathcal{F}^{-1}[F(\omega)G(t,\omega)]$$

From our solution to part 1, we know that

$$u(t,x) = \mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)\exp(-\alpha t)]$$

We let

$$G(t,\omega) = \exp(-D\omega^2 t)\exp(-\alpha t)$$

The inverse Fourier Transform can be found as follows, noting that the $\exp(-\alpha t)$ term can come out of the of the inverse transform as it does not depend on ω

$$\begin{aligned}
g(t,x) &= \mathcal{F}^{-1}[G(t,\omega)] \\
&= \exp(-\alpha t)\mathcal{F}^{-1}[\exp(-D\omega^2 t)] \leftarrow \text{Can sub from solution in class} \\
&= \frac{\exp(-\alpha t)}{\sqrt{2Dt}}\exp(\frac{-x^2}{4Dt})
\end{aligned}$$

From the convolution theorem, we know that given initial condition $f(x)$

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)g(x-z)\ dz$$

where the fundamental solution is found using $f(x) = \delta(x-y)$. We can therefore solve for the

Fundamental solution of the cable equation as follows

$$\begin{aligned}
F(t,x;y) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(z-y) \frac{\exp(-\alpha t)}{\sqrt{2Dt}} \exp(\frac{-(x-z)^2}{4Dt})\ dz \\
&= \frac{\exp(-\alpha t)}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \delta(z-y) \exp(\frac{-(x-z)^2}{4Dt})\ dz
\end{aligned}$$

$$\boxed{F(t,x;y) = \frac{\exp(-\alpha t)}{\sqrt{4\pi Dt}} \exp(\frac{-(x-y)^2}{4Dt})}$$

Problem 4

Define

$$u(t,x) = \frac{\partial F}{\partial x}(t,x;0)$$

where F is the fundamental solution to the heat equation $u_t = Du_{xx}$.

Part 1

Show that u satisfies the heat equation $u_t = Du_{xx}$ for all $x \in \mathbb{R}$ and $t > 0$.

Solution.
We know for the heat equation, the fundamental solution is given as

$$F(t,x;y) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{(x-y)^2}{4Dt})$$

Therefore

$$F(t,x;0) = \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{x^2}{4Dt})$$

We can calculate the derivative F_x as follows

$$\begin{aligned} u(t,x) = F_x(t,x;0) &= \frac{1}{\sqrt{4\pi Dt}} \times \frac{-2x}{4Dt} \times \exp(-\frac{x^2}{4Dt}) \\ &= -\frac{x}{4\sqrt{\pi}(Dt)^{3/2}} \exp(-\frac{x^2}{4Dt}) \end{aligned}$$

First, we want to show that $u(t,x) = F_x(t,x;0)$ satisfies the heat equation. We can calculate the partial derivatives using Wolfram Alpha

$$\begin{aligned} u_t &= \frac{D(6Dtx - x^3)}{16\sqrt{\pi}(Dt)^{7/2}} \exp(-\frac{x^2}{4Dt}) \\ u_{xx} &= \frac{(6Dtx - x^3)}{16\sqrt{\pi}(Dt)^{7/2}} \exp(-\frac{x^2}{4Dt}) \end{aligned}$$

From inspection, we see that $u_t = Du_{xx}$, and thus the PDE is satisfied for all x and $t > 0$. We also would like to confirm that the solution is bounded as we go to infinity. Calculating the limit

$$\begin{aligned} \lim_{x\to\pm\infty} u(t,x) &= -\frac{1}{4\sqrt{\pi}(Dt)^{3/2}} \lim_{x\to\pm\infty} x\exp(-\frac{x^2}{4Dt}) \\ &= -\frac{1}{4\sqrt{\pi}(Dt)^{3/2}} \lim_{x\to\pm\infty} \frac{x}{\exp(\frac{x^2}{4Dt})} \\ \text{L'Hospital's rule} \rightarrow &= -\frac{1}{4\sqrt{\pi}(Dt)^{3/2}} \lim_{x\to\pm\infty} \frac{1}{2x\exp(\frac{x^2}{4Dt})} \times 4Dt \\ &= 0 \end{aligned}$$

As the limit goes to 0 as $x \to \pm\infty$, we know that solution is bounded. Thus, $u(t,x)$ satisfies the heat equation.

Part 2

For each $x \in \mathbb{R}$, show that

$$\lim_{t \to 0^+} u(t, x) = 0$$

Solution.
We can rewrite our function $u(t, x)$ as follows

$$u(t, x) = -\frac{x}{4\sqrt{\pi}D^{3/2}} \frac{\exp(\frac{x^2}{4Dt})}{t^{3/2}}$$

For $x \neq 0$, the limit can be calculated by

$$\begin{aligned}
\lim_{t \to 0^+} u(t, x) &= -\frac{1}{4\sqrt{\pi}D^{3/2}} \lim_{t \to 0^+} \frac{x}{\exp(\frac{x^2}{4Dt})t^{3/2}} \\
\text{L'Hospital's rule} \rightarrow &= -\frac{1}{4\sqrt{\pi}D^{3/2}} \lim_{t \to 0^+} 0 \times \frac{1}{\frac{\exp(\frac{x^2}{4Dt})(6Dt - x^2)}{4D\sqrt{t}}} \\
&= -\frac{1}{4\sqrt{\pi}D^{3/2}} \lim_{t \to 0^+} \frac{0 \times 4D\sqrt{t}}{\exp(\frac{x^2}{4Dt})(6Dt - x^2)} \\
&= \frac{0}{-x^2 \times \infty} = 0
\end{aligned}$$

For $x = 0$, our function reduces to the following

$$u(t, 0) = -\frac{0}{4\sqrt{\pi}(Dt)^{3/2}} \exp(-\frac{0^2}{4Dt}) = 0$$

And therefore

$$\lim_{t \to 0^+} 0 = 0$$

Thus, we know for all $x \in \mathbb{R}$

$$\boxed{\lim_{t \to 0^+} u(t, x) = 0}$$

Part 3

Explain why $u(t, x)$ is not a classical solution to the heat equation with initial condition $u(0, x) = 0$? What is the classical solution to this problem? (The latter question should be very easy.)

Solution.
In addition to the conditions proved in part 1 and 2, in order for $F_x(t, x; 0)$ to be a classical solution to the heat equation it must be continuous at $t = 0$. Thus, for it to be a classical solution requires

$$\lim_{t \to 0^-} F_x(t, x; 0) = 0, \ \forall x \in \mathbb{R}$$

For $t < 0$, we can rewrite our function as follows

$$\begin{aligned}
u(t,x) &= -\frac{x}{4\sqrt{\pi}(Dt)^{3/2}}\exp(-\frac{x^2}{4Dt}) \\
&= -\frac{x}{4\sqrt{\pi}\mathrm{i}^3(D|t|)^{3/2}}\exp(-\frac{x^2}{(-1)4D|t|}) \\
&= -\frac{x}{-4\mathrm{i}\sqrt{\pi}(D|t|)^{3/2}}\exp(\frac{x^2}{4D|t|}) \\
&= \frac{-\mathrm{i}x\exp(\frac{x^2}{4D|t|})}{4\sqrt{\pi}(D|t|)^{3/2}}
\end{aligned}$$

Taking the limit as $t \to 0^-$, we obtain the following result

$$\begin{aligned}
\lim_{t\to 0^-}\frac{-\mathrm{i}x\exp(\frac{x^2}{4D|t|})}{4\sqrt{\pi}(D|t|)^{3/2}} &= \frac{-\mathrm{i}\times\infty}{0^+} \\
&= -\mathrm{i}x\infty
\end{aligned}$$

As a two sided limit does not exist, the function F_x is not continuous at $t = 0$ and is thus not a classical solution. From class, we know any solution on the entire real line with initial condition $u(0,x) = f(x)$ will be of the form

$$u(t,x) = \int_{-\infty}^{\infty} f(y)F(t,x;y)\ dy$$

Thus, if $f(x) = 0$ than the corresponding solution will be

$$u(t,x) = \int_{-\infty}^{\infty} f(y)F(t,x;y)\ dy = 0$$

Which satisfies the PDE and is C^∞, therefore making it a classical solution.

Problem 5

It is possible to define the derivative of the Dirac delta "function" $\delta(x)$. Since δ is a distribution and not really a function, we should expect its derivative to be a distribution as well. Let $\delta'(x)$ denote the derivative of $\delta(x)$. You can assume that both δ' and δ are well-defined and that their Fourier transforms follow all of the rules we have derived in class.

Part 1

Calculate the Fourier transform of $\delta'(x)$ by integrating directly (i.e., use the definition of the Fourier transform rather than other properties). Confirm your answer from the previous homework.

Solution.
We can calculate the Fourier transform of $\delta'(x)$ as follows

$$\begin{aligned}
\mathcal{F}[\delta'(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta'(x)\exp(-\mathrm{i}\omega x)\ dx \\
&\text{Let :}u = \exp(-\mathrm{i}\omega x), \quad du = -\mathrm{i}\omega\exp(-\mathrm{i}\omega x), \quad dv = \delta'(x), \quad v = \delta(x) \\
&= \frac{1}{\sqrt{2\pi}}\left[\delta(x)\exp(-\mathrm{i}\omega x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -\mathrm{i}\omega\delta(x)\exp(-\mathrm{i}\omega x)\ dx\right] \\
&= \frac{\mathrm{i}\omega}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x)\exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{\mathrm{i}\omega}{\sqrt{2\pi}}\exp(-\mathrm{i}\omega(0))
\end{aligned}$$

Therefore

$$\boxed{\mathcal{F}[\delta'(x)] = \frac{\mathrm{i}\omega}{\sqrt{2\pi}}}$$

Which matches our result from Homework 5.

Part 2

Solve the initial-boundary value problem

$$\begin{aligned}
u_t &= Du_{xx} \ \text{ for all } \ t > 0 \ \text{ and } \ -\infty < x < \infty \\
u(0, x) &= \delta'(x) \ \text{ for all } \ -\infty < x < \infty
\end{aligned}$$

Solution.
From class, we know the solution to this problem can be expressed as

$$u(t, x) = \mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)]$$

And thus with $F(\omega) = \mathcal{F}[\delta'(x)]$ and the linearity of the inverse Fourier transform, we can obtain the following relation

$$\begin{aligned}
u(t,x) &= \mathcal{F}^{-1}[F(\omega)\exp(-D\omega^2 t)] \\
&= \mathcal{F}^{-1}[\frac{\mathrm{i}\omega}{\sqrt{2\pi}}\exp(-D\omega^2 t)] \\
&= \frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[\mathrm{i}\omega\exp(-D\omega^2 t)]
\end{aligned}$$

From the derivative property of the Fourier Transform, we know that

$$\mathcal{F}^{-1}[\mathrm{i}\omega\exp(-D\omega^2 t)] = \frac{d}{dx}\mathcal{F}^{-1}[\exp(-D\omega^2 t)]$$

From derivations in class, we know that

$$\mathcal{F}^{-1}[\exp(-D\omega^2 t)] = \frac{1}{\sqrt{2Dt}}\exp(-\frac{x^2}{4Dt})$$

Thus, our solution $u(t,x)$ can be given as

$$\begin{aligned}
u(t,x) &= \frac{1}{\sqrt{2\pi}} \times \frac{d}{dx}\mathcal{F}^{-1}[\exp(-D\omega^2 t)] \\
&= \frac{1}{\sqrt{2\pi}} \times \frac{d}{dx}\left(\frac{1}{\sqrt{2Dt}}\exp(-\frac{x^2}{4Dt})\right) \\
&= \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sqrt{2Dt}} \times -\frac{2x\exp(-\frac{x^2}{4Dt})}{4Dt}
\end{aligned}$$

$$\boxed{u(t,x) = -\frac{x\exp(-\frac{x^2}{4Dt})}{4\sqrt{\pi}(Dt)^{3/2}}}$$