AMATH 503: Methods for Partial Differential Equations

University of Washington Spring 2022

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Homework 4

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Problem 1

Solve the following boundary value problem:

$$\begin{aligned}
&u_{xx} + u_{yy} = 0 \text{ for all } 0 < x < \pi \text{ and } 0 < y < \pi \\
&u(x,0) = 0 \text{ and } u(x,\pi) = 0 \text{ for all } 0 < x < \pi \\
&u(0,y) = 0 \text{ and } u(\pi,y) = y(\pi - y) \text{ for all } 0 < y < \pi
\end{aligned}$$

Plot your solution. (Your solution will be in terms of a Fourier series - you can plot a partial sum with a reasonable number of terms.)

Solution.

We assume a solution of the form

$$u(x,y) = X(x)Y(y)$$

Which gives the following relation when substituting into the PDE. As each of the equations are only of a single variable, for the relations to be equal they must be a constant.

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \longrightarrow \frac{X''}{X} = -\frac{Y''}{Y} = K$$

As we have homogeneous boundary conditions for both values of y, the solution for $Y(y)$ should be solved for first. We have the following relation, with solutions dependent on the value of K

$$Y'' = -KY$$

If $K > 0$:

$$Y'' = -KY = -\lambda^2 Y \longrightarrow Y(y) = A\cos(\lambda y) + B\sin(\lambda y)$$

Applying the boundary conditions

$$\begin{aligned}
&Y(0) = 0 = A\cos(0) + B\sin(0) = A \\
&\therefore A = 0 \\
&Y(\pi) = 0 = B\sin(\lambda\pi) \\
&\therefore \lambda\pi = n\pi \to \lambda = n
\end{aligned}$$

If $K = 0$:

$$Y'' = 0 \longrightarrow Y(y) = Ay + B$$

As we have Dirichlet boundary conditions for y, from previous problems we know this leads to no non-trivial solutions. Finally, if $K < 0$:

$$Y'' = -KY = \lambda^2 Y \longrightarrow Y(y) = A\exp(\lambda y) + B\exp(-\lambda y)$$

Again, from previous Dirichlet boundary value problems we know this only leads to trivial solutions. Therefore, our general solution for $Y(y)$ for a given value of n is

$$Y(y) = b_n \sin(ny)$$

Using $K = n^2 > 0$, we now solve for X and apply the single homogeneous boundary condition

$$\begin{aligned}
X'' &= n^2 X \longrightarrow X(x) = A\exp(nx) + B\exp(-nx) \\
X(0) &= 0 = A + B \\
\therefore A &= -B \\
X(x) &= -B\exp(nx) + B\exp(-nx) \\
&= -2B(\frac{\exp(nx)}{2} - \frac{\exp(-nx)}{2}) \\
&= -2B\sinh(nx) \\
\therefore X_n(x) &= b_{xn}\sinh(nx)
\end{aligned}$$

Considering a linear combination of solutions for all values of n and combining the constants of $X(x)$ and $Y(y)$, the following general solution is obtained

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sinh(nx)\sin(ny)$$

Applying the final boundary condition

$$u(\pi, y) = y(\pi - y) = \sum_{n=1}^{\infty} b_n \sinh(n\pi)\sin(ny)$$

As has been previously shown for a similar problem in class, we know the coefficients b_n can be found as

$$\begin{aligned}
b_n &= \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f(y)\sin(\frac{n\pi y}{a})\ dy \\
&= \frac{2}{\pi\sinh(n\pi)}\int_0^{\pi} y(\pi - y)\sin(ny)\ dy
\end{aligned}$$

Evaluating this integral in Wolfram Alpha

$$\begin{aligned} b_n &= \frac{2(-\pi n \sin(\pi n) - 2\cos(\pi n) + 2)}{\sinh(\pi n)\pi n^3} \\ &= \frac{-4(\cos(\pi n) - 1)}{\sinh(\pi n)\pi n^3} \end{aligned}$$

The general solution for $u(x, y)$ is therefore

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} \frac{-4(\cos(\pi n) - 1)}{\sinh(\pi n)\pi n^3} \sinh(nx)\sin(ny)}$$

Figure 1 shows the solution $u(x, y)$ using the first 100 terms

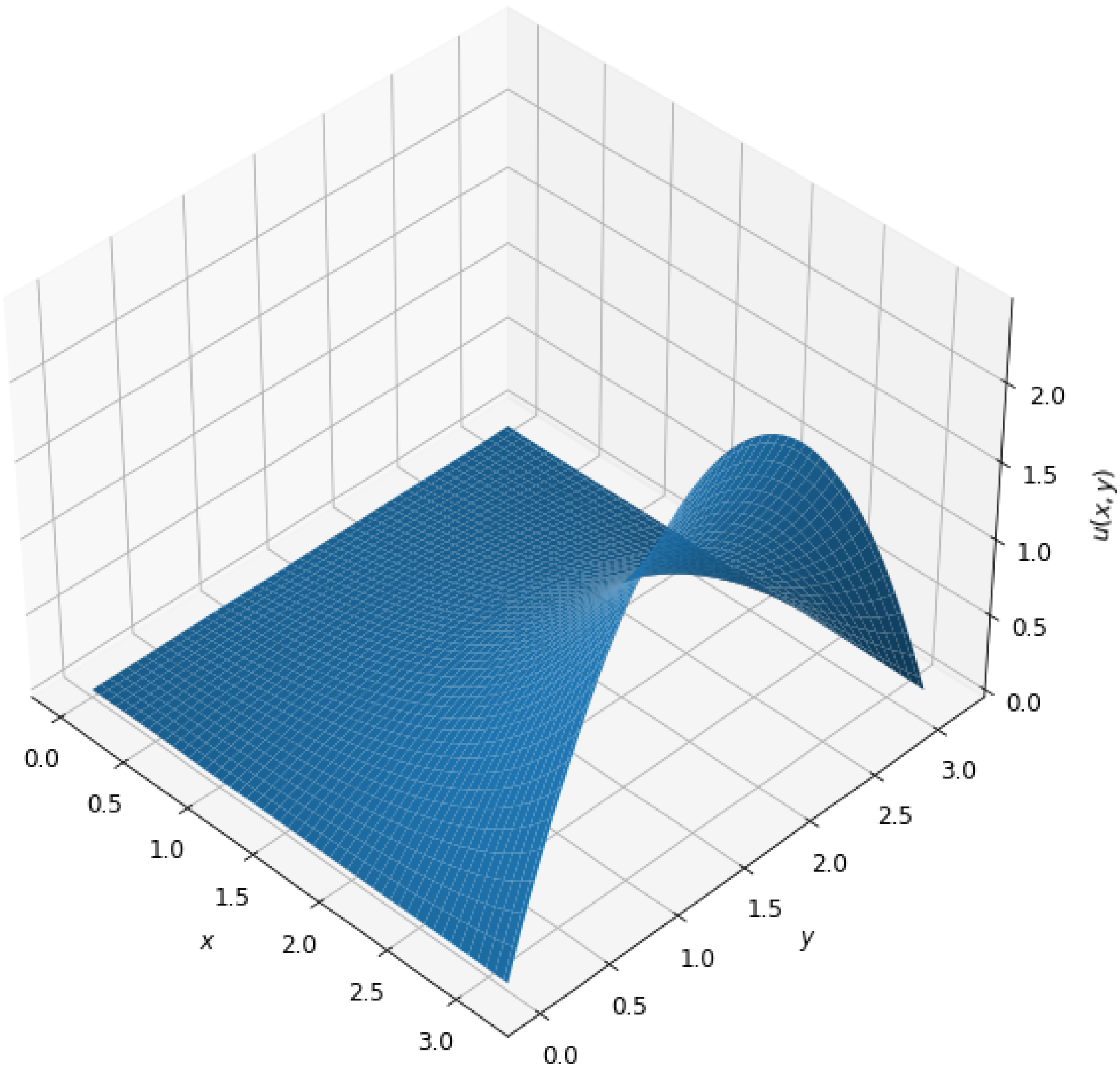

Figure 1: Solution $u(x, y)$ using 100 Series Terms

Problem 2

Consider the boundary value problem

$$u_{xx} + u_{yy} = 0 \text{ for all } x^2 + y^2 < 1$$
$$u(x, y) = f(x, y) \text{ for all } x^2 + y^2 = 1$$

where

$$f(x) = \begin{cases} 1 & y > 0 \\ 0 & y < 0 \end{cases}$$

Part 1

Solve this boundary value problem using separation of variables. Plot your solution. (Your solution will be in terms of a Fourier series - you can plot a partial sum with a reasonable number of terms.)

Part 2

Confirm that

$$u(r, \theta) = \begin{cases} 1 - \frac{1}{\pi} \arctan\left(\frac{1-r^2}{2r\sin\theta}\right) & 0 < \theta < \pi \\ \frac{1}{2} & \theta = 0, \pi, -\pi \\ -\frac{1}{\pi} \arctan\left(\frac{1-r^2}{2r\sin\theta}\right) & -\pi < \theta < 0 \end{cases}$$

is a solution to the given boundary value problem. (This function is defined for $0 \le r \le 1$.) Plot this function and confirm that it is the same as your Fourier series solution.

2.1 Part 1

Solution.
From class, we know that this problem can be rewritten in polar coordinates as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ for all } 0 < r < 1 \text{ and } -\pi < \theta < \pi$$
$$u(1, \theta) = f(\theta) \text{ for all } -\pi < \theta < \pi$$
$$u(r, -\pi) = u(r, \pi) \text{ and } u_\theta(r, -\pi) = u_\theta(r, \pi) \text{ for all } 0 < r < 1$$
$$u(0, \theta) \text{ is a finite constant, independent of } \theta$$

We assume a solution of the form

$$u(r, \theta) = R(r)\Theta(\theta)$$

Plugging into the PDE and simplifying, we obtain the following relations. We know for both equations to be equal they must equal a constant $K = \lambda^2$

$$\frac{\Theta''}{\Theta} = -\frac{r^2R'' + rR'}{R} = K$$

As we have periodic boundary conditions for Θ, we would like to solve this problem first. The equation

$$\Theta'' = K\Theta$$

Is a second order ODE we have solved multiple times that will have different solutions depending on the value of K and the boundary conditions. With periodic boundary conditions, from the problems we have solved in class we know the following solutions can be obtained

$$\begin{aligned} &K = 0 \longrightarrow \Theta(\theta) = B \\ &K = -\lambda^2 < 0 \longrightarrow \Theta(\theta) = A\cos(n\pi) + B\sin(n\pi), \ \lambda = n \end{aligned}$$

The values obtained for K can then be used to solve for the equation $R(r)$. We rewrite the problem for $R(r)$ as follows

$$-\frac{r^2R'' + rR'}{R} = K \longrightarrow r^2R'' + rR' + KR = 0$$

Depending on the value of K, we will get different solutions. We solved this problem in class, and the solutions are given by

$$\begin{aligned} &K = 0 \longrightarrow R(r) = A + B\ln(r) \\ &K = -n^2 < 0 \longrightarrow R(r) = Ar^n + Br^{-n} \end{aligned}$$

As we require the problem to be bounded as $r \to 0$, we throw out the solutions corresponding to r^{-n} and $\ln(r)$. Thus, the individual solutions for u (after combining coefficients) in polar coordinates are given as

$$\begin{aligned} &u_0(r,\theta) = \frac{a_0}{2} \\ &u_n(r,\theta) = a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \end{aligned}$$

And the general solution

$$u_n(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

which matches the result from class. For a unit disk in Cartesian coordinates, the relation $y > 0$ holds true in the upper two quadrants. In polar coordinates, this will correspond to $0 < \theta < \pi$. Similarly, $y < 0$ for $-\pi < \theta < 0$. Therefore, $f(x)$ can be rewritten as

$$f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & -\pi < \theta < 0 \end{cases}$$

We solve for the coefficients as

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \; d\theta + \int_{0}^{\pi} 1\cos(n\theta) \; d\theta\right] \\
&= \frac{1}{\pi}\left[\frac{1}{n}\sin(n\theta)\right]_0^{\pi} = 0 \\
\frac{a_0}{2} &= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0 \; d\theta + \int_{0}^{\pi} d\theta\right] \\
&= \frac{1}{2\pi}\left[\theta\right]_0^{\pi} = \frac{1}{2} \\
b_n &= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \; d\theta + \int_{0}^{\pi} 1\sin(n\theta) \; d\theta\right] \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\cos(n\theta)\right]_0^{\pi} \\
&= \frac{(1-\cos(n\pi))}{n\pi}
\end{aligned}
$$

Our general solution in polar coordinates is therefore

$$
\boxed{u(r,\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-\cos(n\pi))}{n\pi} r^n \sin(n\theta)}
$$

Figure 2 shows the solution u in both polar and Cartesian space using the first 200 terms

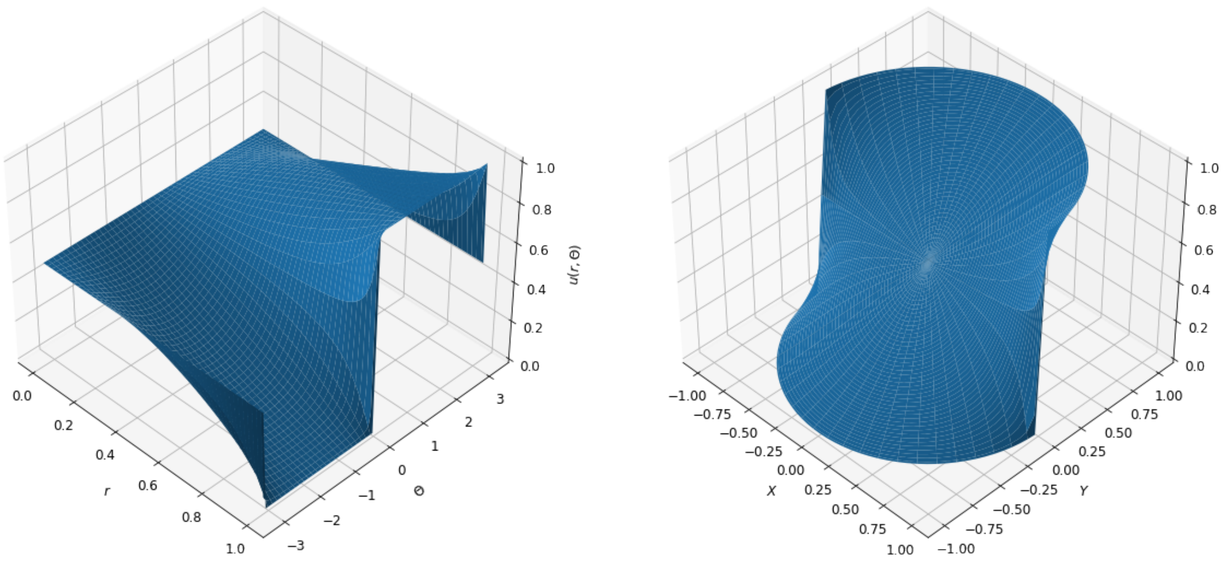

Figure 2: Solution $u(r,\theta)$ and $u(x,y)$ using 200 Series Terms

Part 2

Solution.

To show the given function is a solution, we must show that it satisfies the PDE and the boundary conditions. First, we will show the PDE is satisfied. As the constants will go to zero when any derivatives are taken, we know that the PDE is satisfied for $\theta \in 0, \pi, -\pi$. We can also ignore the constants in the other two pieces of the function as when taking any partial derivative they will go to

0. We thus focus on taking partial derivatives of the arc-tangent function. Let

$$A(r,\theta) = \arctan\left(\frac{1-r^2}{2r\sin(\theta)}\right)$$

We solve for the partial derivatives using Wolfram Alpha

$$\begin{aligned}
\frac{\partial^2}{\partial r^2}(1-A(r,\theta)) &= \frac{\partial^2 A}{\partial r^2} = \frac{4r\sin(\theta)(r^4+2r^2-2\cos(2\theta)-1)}{(r^4-2r^2\cos(2\theta)+1)^2} \\
\frac{1}{r}\frac{\partial A}{\partial r}(1-A(r,\theta)) &= \frac{1}{r}\frac{\partial A}{\partial r} = \frac{-2(r^2+1)\sin(\theta)}{r(r^4-2r^2\cos(2\theta)+1)} \\
\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(1-A(r,\theta)) &= \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} = \frac{-2(r^2-1)\sin(\theta)(r^4+2r^2\cos(2\theta)+4r^2+1)}{r(r^4-2r^2\cos(2\theta)+1)^2}
\end{aligned}$$

Using Wolfram Alpha to simplify, we confirm the following holds for both $A(r,\theta)$ and $1-A(r,\theta)$

$$\begin{aligned}
&-\frac{1}{\pi}\frac{\partial^2}{\partial r^2}(1-A(r,\theta)) + -\frac{1}{\pi}\frac{1}{r}\frac{\partial}{\partial r}(1-A(r,\theta)) + -\frac{1}{\pi}\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(1-A(r,\theta)) \\
&= -\frac{1}{\pi}\frac{\partial^2 A}{\partial r^2} - \frac{1}{\pi r}\frac{\partial A}{\partial r} - \frac{1}{\pi r^2}\frac{\partial^2 A}{\partial \theta^2} = -\frac{1}{\pi}\left[\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2}\right] \\
&= -\frac{1}{\pi}\left[0\right] \\
&= 0
\end{aligned}$$

Thus, the PDE is satisfied. Next, we must show the boundary conditions are satisfied. The first conditions are easily shown to be satisfied.

$$\begin{aligned}
u(r,\pi) = u(r,-\pi) &\longrightarrow \frac{1}{2} = \frac{1}{2} \\
u_\theta(r,\pi) = u_\theta(r,-\pi) &\longrightarrow 0 = 0
\end{aligned}$$

Next, we must show that $u(0,\theta)$ is finite and constant. From the piece-wise representation of the function given, we know this value must equal $\frac{1}{2}$. We again consider the arc-tangent pieces of the function, and we must take the limit as $r \to 0$. However, the value of this limit will depend on the value of $\sin(\theta)$ as the arc-tangent function has a two sided limit. Let

$$a = \sin(\theta)$$

If $a > 0$, we will be in the top two quadrants with $0 < \theta < \pi$. The limit for the corresponding segment of the piecewise function can be calculated as

$$\lim_{r\to 0^+} 1 - \frac{1}{\pi}\arctan\left(\frac{1-r^2}{2r|a|}\right) = 1 - \frac{1}{\pi}(\frac{\pi}{2}) = \frac{1}{2}$$

Conversely, if $a < 0$ we will be in the bottom two quadrants with $-\pi < \theta < 0$. The limit is now slightly different, with a negative in the denominator of the arc-tangent function. This limit is calculated as follows

$$\lim_{r\to 0^+} -\frac{1}{\pi}\arctan\left(\frac{1-r^2}{-2r|a|}\right) = -\frac{1}{\pi}(-\frac{\pi}{2}) = \frac{1}{2}$$

As all the values of the piece-wise function equal $\frac{1}{2}$ as $r \to 0$, the final boundary condition is therefore satisfied and we have confirmed $u(r,\theta)$ is a solution to the PDE.

Figure 3 shows the solution to the given u in both polar and Cartesian space. We see these plots match our results from Part 1.

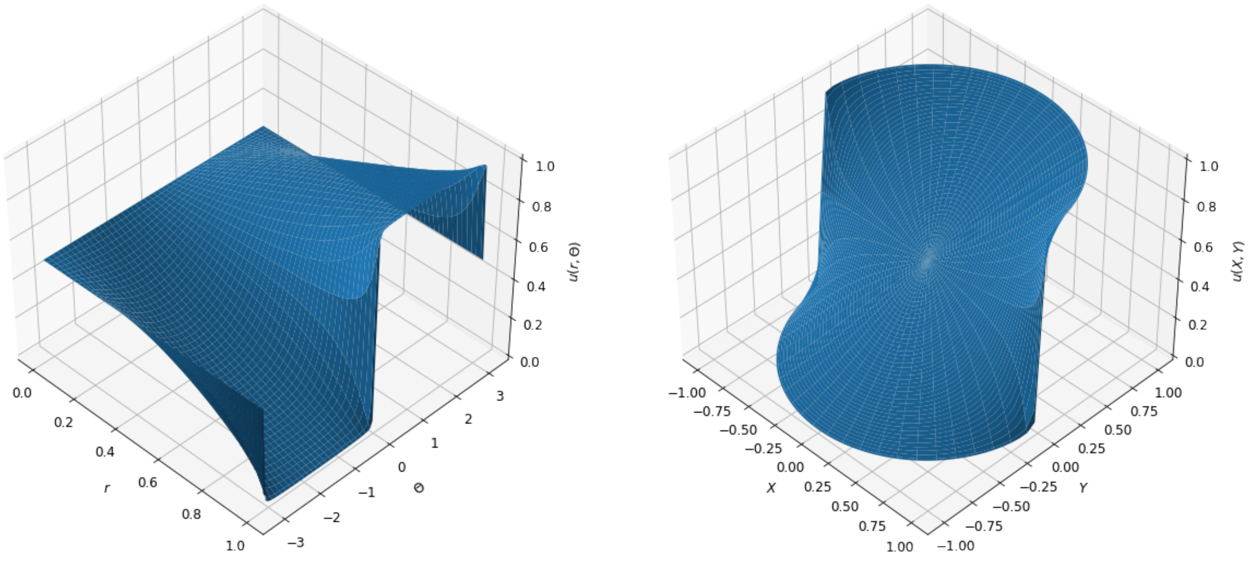

Figure 3: Solution $u(r,\theta)$ and $u(x,y)$

Problem 3

Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are functions and $c \in \mathbb{R}$ is a constant. Prove the following properties of the Fourier transform (assuming all required integrals converge):
1) $\mathcal{F}\left[f(x) + g(x)\right](\omega) = \mathcal{F}\left[f(x)\right](\omega) + \mathcal{F}\left[g(x)\right](\omega)$
2) $\mathcal{F}[cf(x)](\omega) = c\mathcal{F}[f(x)](\omega)$
3) $\mathcal{F}\left[f(cx)\right](\omega) = \frac{1}{|c|}\mathcal{F}\left[f(x)\right]\left(\frac{\omega}{c}\right)$ (assuming $c \neq 0$)
In particular, the first two properties mean that $\mathcal{F}$ is a linear operator.

3.1 Part 1

Solution.

$$\begin{aligned}
\mathcal{F}\left[f(x) + g(x)\right](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(x) + g(x)\right] \exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(f(x)\exp(-\mathrm{i}\omega x) + g(x)\exp(-\mathrm{i}\omega x)\right)\ dx
\end{aligned}$$

We separate the integral into two parts

$$\begin{aligned}
\mathcal{F}\left[f(x) + g(x)\right](\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}\omega x)\ dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)\exp(-\mathrm{i}\omega x)\ dx \\
&= \mathcal{F}\left[f(x)\right](\omega) + \mathcal{F}\left[g(x)\right](\omega)
\end{aligned}$$

$$\therefore \boxed{\mathcal{F}\left[f(x) + g(x)\right](\omega) = \mathcal{F}\left[f(x)\right](\omega) + \mathcal{F}\left[g(x)\right](\omega)}$$

3.2 Part 2

Solution.

$$\mathcal{F}\left[cf(x)\right](\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[cf(x)\right] \exp(-\mathrm{i}\omega x)\ dx$$

We can pull the c out of the integral as it is a constant

$$\begin{aligned}
\mathcal{F}\left[cf(x)\right](\omega) &= c\left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\exp(-\mathrm{i}\omega x)\ dx\right] \\
&= c\mathcal{F}\left[f(x)\right](\omega)
\end{aligned}$$

$$\therefore \boxed{\mathcal{F}\left[cf(x)\right](\omega) = c\mathcal{F}\left[f(x)\right](\omega)}$$

3.3 Part 3

Solution.
Let

$$g(x) = f(cx)$$

The Fourier Series of $g(x)$ is then

$$
\begin{aligned}
G(\omega) &= \mathcal{F}\left[g(x)\right](\omega) = \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[g(x)\right] \exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(cx)\right] \exp(-\mathrm{i}\omega x)\ dx
\end{aligned}
$$

We then use a change of variables, τ

$$
\tau = cx \qquad x = \frac{\tau}{c} \qquad \frac{dx}{d\tau} = \frac{1}{c}
$$

Substituting into the problem we have

$$
\begin{aligned}
G(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(cx)\right] \exp(-\mathrm{i}\omega x)\ dx \\
&= \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(\tau)\right] \exp(-\mathrm{i}\omega \frac{\tau}{c})\ d\tau
\end{aligned}
$$

Let

$$
F(\omega) = \mathcal{F}\left[f(x)\right](\omega)
$$

If $c > 0$: $\tau = cx \in [-\infty, \infty]$

$$
\begin{aligned}
G(\omega) &= \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(\tau)\right] \exp(-\mathrm{i}\frac{\omega}{c}\tau)\ d\tau \\
&= \frac{1}{c} F(\frac{\omega}{c})
\end{aligned}
$$

If $c < 0$: $\tau = cx \in [\infty, -\infty]$

$$
\begin{aligned}
G(\omega) &= \frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(\tau)\right] \exp(-\mathrm{i}\frac{\omega}{c}\tau)\ d\tau \\
&= -\frac{1}{c} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[f(\tau)\right] \exp(-\mathrm{i}\frac{\omega}{c}\tau)\ d\tau \\
&= -\frac{1}{c} F(\frac{\omega}{c})
\end{aligned}
$$

However, because $c < 0$ we know this result will end up being positive. Thus:

$$
\boxed{\mathcal{F}\left[f(cx)\right](\omega) = G(\omega) = \frac{1}{|c|} F(\frac{\omega}{c})}
$$

Problem 4

Consider the Euler equation

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \alpha t\frac{\mathrm{d}x}{\mathrm{d}t} + \beta x = 0$$

where $t > 0$.

Show that this ODE becomes a linear, constant coefficient equation for $x(\tau)$ if we make the change of variables $\tau = \ln t$.

Use this result to solve the Euler equation

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$

Solution.

If $\tau = \ln t$:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{t}\frac{dx}{d\tau} \\
\frac{d^2x}{dt^2} &= \frac{d}{dt}(\frac{1}{t}\frac{dx}{d\tau}) = -\frac{1}{t^2}\frac{dx}{d\tau} + \frac{1}{t}\frac{d^2x}{d\tau^2}\frac{d\tau}{dt} \\
&= -\frac{1}{t^2}\frac{dx}{d\tau} + \frac{1}{t^2}\frac{d^2x}{d\tau^2} \\
&= \frac{1}{t^2}(\frac{d^2x}{d\tau^2} - \frac{dx}{d\tau})
\end{aligned}$$

Substituting into the original equation

$$\begin{aligned}
0 &= t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \alpha t\frac{\mathrm{d}x}{\mathrm{d}t} + \beta x \\
&= t^2\left(\frac{1}{t^2}(\frac{d^2x}{d\tau^2} - \frac{dx}{d\tau})\right) + \alpha t\left(\frac{1}{t}\frac{dx}{d\tau}\right) + \beta x \\
&= \frac{d^2x}{d\tau^2} - \frac{dx}{d\tau} + \alpha\frac{dx}{d\tau} + \beta x \\
&= \frac{d^2x}{d\tau^2} + (\alpha - 1)\frac{dx}{d\tau} + \beta x
\end{aligned}$$

We see the problem has been reduced to the following constant coefficient ODE

$$\boxed{\frac{d^2x}{d\tau^2} + (\alpha - 1)\frac{dx}{d\tau} + \beta x = 0}$$

We now can use this substitution to solve the Euler equation. From inspection, we see that for the Euler equation $\alpha = 1$. We make our substitution as follows:

$$t^2\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + t\frac{\mathrm{d}x}{\mathrm{d}t} = 0 \longrightarrow \frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} = 0$$

The general solution for this equation will be a line

$$x(\tau) = c_1 + c_2\tau$$

We can now substitute for τ to obtain our final solution

$$\boxed{x(t) = c_1 + c_2 \ln(t)}$$

Problem 5

Consider the boundary value problem

$$
\begin{aligned}
&u_{xx} + u_{yy} = 0 \text{ for all } 0 < x < a \text{ and } 0 < y < b \\
&u(x, 0) = f(x) \text{ and } u(x, b) = g(x) \text{ for all } 0 < x < a \\
&u(0, y) = 0 \text{ and } u(a, y) = 0 \text{ for all } 0 < y < b
\end{aligned}
$$

Explain how you to solve this problem by splitting it into two sub-problems, each of which has homogeneous boundary conditions on three sides of the rectangle $[0, a] \times [0, b]$. You do not need to find a formula for the solution, but you should justify your explanation. (The proof should be similar in format to what we did in class when solving the heat equation with inhomogeneous boundary conditions.)

Solution.
The problem can be broken down to two sub problems with solutions $u_1(x, y)$, $u_2(x, y)$ as follows

$$
\begin{aligned}
&u_{1_{xx}} + u_{1_{yy}} = 0, \ \forall\ 0 < x < a,\ 0 < y < b \\
&u_1(x, 0) = f(x),\ u_1(x, b) = 0,\ \forall\ 0 < x < a \\
&u_1(0, y) = u_1(a, y) = 0,\ \forall\ 0 < y < b
\end{aligned}
\qquad
\begin{aligned}
&u_{2_{xx}} + u_{2_{yy}} = 0, \ \forall\ 0 < x < a,\ 0 < y < b \\
&u_2(x, 0) = 0,\ u_2(x, b) = g(x),\ \forall\ 0 < x < a \\
&u_2(0, y) = u_2(a, y) = 0,\ \forall\ 0 < y < b
\end{aligned}
$$

Each of these problems contains one of the inhomogeneous boundary conditions. We can also prove that the Laplacian is a linear operator

$$
\begin{aligned}
L[u] &= (\partial_x x + \partial_y y)u \\
L[c_1 u + c_2 v] &= (\partial_x x + \partial_y y)(c_1 u + c_2 v) \\
L[c_1 u + c_2 v] &= \partial_x x(c_1 u + c_2 v) + \partial_y y(c_1 u + c_2 v) \\
&= c_1 u_x x + c_2 v_x x + c_1 u_y y + c_2 v_y y \\
&= c_1(u_x x + u_y y) + c_2(v_x x + v_y y) \\
\therefore L[c_1 u + c_2 v] &= c_1 L[u] + c_2 L[v]
\end{aligned}
$$

As we have shown the Laplacian is a linear operator, we know that the principle of superposition applies and the sum of the solutions $u_1(x, y)$ and $u_2(x, y)$ will also be a solution. As a result, to solve the problem with two inhomogenous boundary conditions we can solve each individual case and the general solution will be given as

$$u(x, y) = u_1(x, y) + u_2(x, y)$$

As a confirmation, we can quickly show the boundary conditions are satisfied

$$
\begin{aligned}
u(x, 0) &= u_1(x, 0) + u_2(x, 0) = f(x) + 0 = f(x) \\
u(x, b) &= u_1(x, b) + u_2(x, b) = 0 + g(x) = g(x) \\
u(0, y) &= u_1(0, y) + u_2(0, y) = 0 + 0 = 0 \\
u(a, y) &= u_1(a, y) + u_2(a, y) = 0 + 0 = 0
\end{aligned}
$$