AMATH 503: Methods for Partial Differential Equations

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Homework 8

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Problem 1

Solve the initial-boundary value problem

$$u_{tt} = 9u_{xx} + 2\delta(x) \text{ for all } t > 0 \text{ and } -\infty < x < \infty$$
$$u(0,x) = e^{-x^2} \text{ and } u_t(0,x) = \frac{1}{1+x^2} \text{ for all } -\infty < x < \infty$$

Plot your solution at several representative times.

Solution.

Considering the following general problem

$$u_{tt} = c^2 u_{xx} + h(x) \text{ for all } t > 0 \text{ and } -\infty < x < \infty$$
$$u(0,x) = f(x) \text{ and } u_t(0,x) = g(x) \text{ for all } -\infty < x < \infty$$

we will solve three cases, one with each of $h(x), f(x)$, and $g(x)$ non-zero while the other functions are left as 0. We can then superimpose the solutions together to get the full solution. First, we consider the case where $f(x) = \exp(-x^2)$ and $g(x) = h(x) = 0$. From derivations in class, we know the general solution for a problem of this type is given by

$$u(t,x) = \frac{1}{2}[f(x+ct) + f(x-ct)]$$

Thus, for our problem we have

$$u(t,x) = \frac{1}{2}[\exp(-(x+3t)^2) + \exp(-(x-3t)^2)]$$

Next, we consider the case where $g(x) = \frac{1}{1+x^2}$ and $f(x) = h(x) = 0$. From Homework 5, we know the general solution to this problem can be found as

$$u(t,x) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\ dy$$

For our problem, we then have

$$\begin{aligned}
u(t,x) &= \frac{1}{6}\int_{x-3t}^{x+3t} \frac{1}{1+y^2}\ dy \\
&= \frac{1}{6}\arctan(y)\Big|_{x-3t}^{x+3t} \\
u(t,x) &= \frac{1}{6}[\arctan(x+3t) - \arctan(x-3t)]
\end{aligned}$$

Finally, we consider the case where $h(x) = 2\delta(x)$ and $f(x) = g(x) = 0$. From class, the general solution to this problem will be of the form

$$u(t,x) = \int_0^\infty \int_{-\infty}^\infty G(t,x;\tau,y)h(\tau,y)\ dy\ d\tau$$

We observe that as we have a delta function for $h(x)$, our integral can be simplified before considering $G(t,x;\tau,y)$.

$$\begin{aligned}
u(t,x) &= 2\int_0^\infty \int_{-\infty}^\infty G(t,x;\tau,y)\delta(y)\ dy\ d\tau \\
&= 2\int_0^\infty G(t,x;\tau,0)\ d\tau \\
&= 2\int_0^t G(t,x;\tau,0)\ d\tau \leftarrow \text{Noting integral will be 0 for } \tau \leq t
\end{aligned}$$

From class, we know $G(t,x;\tau,0)$ can be given by

$$\begin{aligned}
G(t,x;\tau,0) &= \left\{ \begin{array}{ll} \frac{1}{2c} & x > -c(t-\tau) \text{ and } x < c(t-\tau) \text{ and } \tau < t \\ 0 & \text{otherwise} \end{array} \right\} \\
&= \left\{ \begin{array}{ll} \frac{1}{2c} & \tau < t + x/c \text{ and } \tau < t - x/c \text{ and } \tau < t \\ 0 & \text{otherwise} \end{array} \right\}
\end{aligned}$$

For our problem, this will be equivalent to

$$G(t,x;\tau,0) = \left\{ \begin{array}{ll} \frac{1}{6} & \tau < t + x/3 \text{ and } \tau < t - x/3 \text{ and } \tau < t \\ 0 & \text{otherwise} \end{array} \right\}$$

We can consider this problem in two parts to find where G is non-zero. First, if $x > 0$, we note that $\tau < t + \frac{x}{3}$ will always be satisfied as the right hand side of the inequality is growing. Thus, we need to consider the more restrictive inequality $\tau < t - \frac{x}{3}$. We find the boundary where $\tau = 0$:

$$0 = t - \frac{x}{3} \rightarrow x = 3t$$

Thus, if $x \geq 3t$ than $G = 0$. For $x > 0$ & $x < 3t$, we have:

$$u(t,x) = 2\int_0^{t-\frac{x}{3}} \frac{1}{6}\ d\tau = \frac{t}{3} - \frac{x}{9}$$

We use a similar logic if $x < 0$, however in this case the more restrictive inequality will be $\tau < t + \frac{x}{3}$.

We again find the boundary where $\tau = 0$

$$0 = t + \frac{x}{3} \to x = -3t$$

Therefore if $x \le -3t$ than $G = 0$. For $x < 0$ & $x > -3t$, we have:

$$u(t,x) = 2\int_0^{t+\frac{x}{3}} \frac{1}{6}\, d\tau = \frac{t}{3} + \frac{x}{9}$$

If $x = 0$, we have

$$u(t,x) = 2\int_0^{t} \frac{1}{6}\, d\tau = \frac{t}{3}$$

Combining all our solutions, we have the following full solution:

$$\boxed{u(t,x) = \frac{1}{2}[\exp(-(x+3t)^2) + \exp(-(x-3t)^2)] + \frac{1}{6}[\arctan(x+3t) - \arctan(x-3t)] + \left\{ \begin{array}{ll} \frac{t}{3} - \frac{x}{9} & 0 \le x < 3t \\ \frac{t}{3} + \frac{x}{9} & -3t < x \le 0 \end{array} \right\}}$$

Figure Some plots of $u(t,x)$ are shown in figure 1

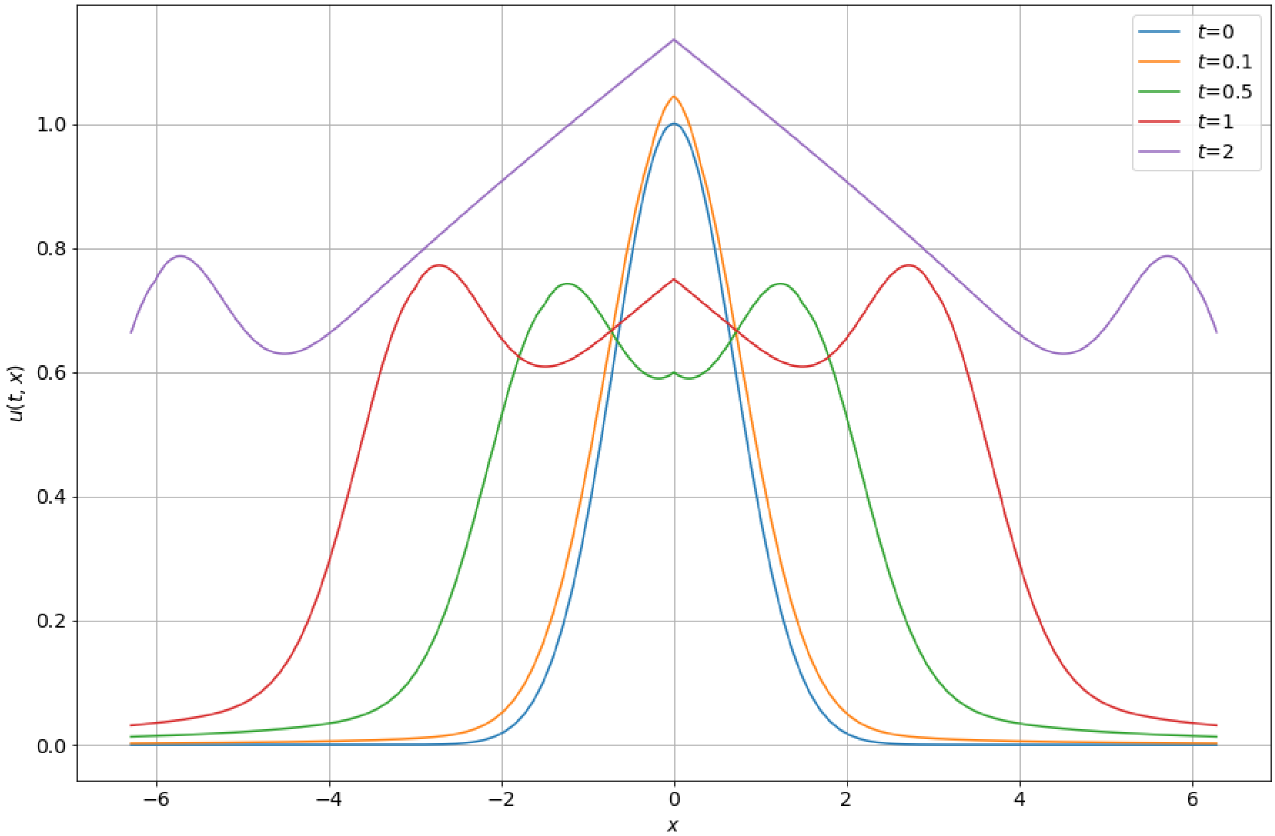

Figure 1: Solution $u(t,x)$ at various t for $x \in [-2\pi, 2\pi]$

Problem 2

Define the operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$L[\mathbf{x}] = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} \mathbf{x}$$

Find the adjoint L^* for each of the following inner products. (In each case, the same inner product is used on both the domain and range.)

Part 1

The standard Euclidean inner product.

Solution.
We can write the inner product of two vectors in the following form, where A is a matrix

$$< x, y >= x^T Ay$$

For the standard Euclidean inner product, A is equal to the identity matrix. We know an adjoint operator can be found using the following definition

$$< L[u], v >=< u, L^*[v] >$$

As the transform given in the problem is a matrix multiplication, we can define the transformation as Lx, where L is the matrix

$$L = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

The corresponding adjoint operator L^* will also be a matrix multiplication. Substituting our inner product definition from above, we can rewrite the adjoint equality in the following form.

$$\begin{aligned} < L[u], v > &=< u, L^*[v] > \\ (Lu)^T Av &= u^T AL^* v \\ u^T L^T Av &= u^T AL^* v \\ 0 &= u^T L^T Av - u^T AL^* v \\ 0 &= u^T (L^T A - AL^*) v \end{aligned}$$

In order for this to always hold, we require

$$L^T A - AL^* = 0$$

Therefore, we know that

$$L^* = A^{-1} L^T A$$

For the Euclidean inner product, we know A will be the identity matrix. Thus

$$\boxed{L^* = L^T = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}}$$

Part 2

$$\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2$$

Solution.
We observe that this inner product can be written as

$$< x, y >= x^T \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} y$$

Thus for this inner product, our A matrix is given by

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

We can find A^{-1} as follows

$$A^{-1} = \frac{1}{6-0} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

And $L^T A$ as

$$L^T A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 9 \end{bmatrix}$$

Therefore

$$\boxed{L^* = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ \frac{4}{3} & 3 \end{bmatrix}}$$

Part 3

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} \mathbf{y}$$

Solution.
In this problem, we are given the matrix A in the inner product definition. We can find A^{-1} as

$$A^{-1} = \frac{1}{8-1} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

And $L^T A$ as

$$L^T A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 1 & 10 \end{bmatrix}$$

Therefore

$$\boxed{L^* = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 1 & 10 \end{bmatrix} = \begin{bmatrix} \frac{13}{7} & -\frac{10}{7} \\ \frac{5}{7} & \frac{15}{7} \end{bmatrix}}$$

Problem 3

Define the operator $L : U \to U$, where

$$U = \{C^1([0,1]) \mid u(0) = u(1) = 0\}$$

and $L[u] = u_x$. Use the weighted inner product

$$\langle u_1, u_2 \rangle = \int_0^1 u_1(x)u_2(x)e^x \, \mathrm{d}x$$

on both the domain and range.

Part 1

Find the adjoint operator L^*.

Solution.
To find the adjoint operator, we utilize the inner product equality and convert it to an integral form

$$\begin{aligned}
&< L[u], v > = < u, L^*[v] > \\
&\int_0^1 \frac{du}{dx} v(x)\exp(x)\ dx = \int_0^1 u(x)L^*[v]\exp(x)\ dx
\end{aligned}$$

Our goal is to get the left hand side of the equation into a form that matches the right hand side. We can evaluate the left hand side using integration by parts. Let

$$u = v(x)\exp(x) \quad du = \frac{d}{dx}(v(x)\exp(x)) \quad dv = \frac{du}{dx} \quad v = u(x)$$

$$\begin{aligned}
LHS &= u(x)v(x)\exp(x)\big|_0^1 - \int_0^1 u(x)\frac{d}{dx}(v(x)\exp(x))\ dx \\
&= \cancelto{0}{[u(1)v(1)e - u(0)v(0)]} + \int_0^1 u(x)(-\frac{1}{\exp(x)})\frac{d}{dx}(v(x)\exp(x))\exp(x)\ dx \\
&= \int_0^1 u(x)\big[-\frac{1}{\exp(x)}\frac{d}{dx}(v(x)\exp(x))\big]\exp(x)\ dx
\end{aligned}$$

We see this is now in a pattern that matches the right hand side of our inner product definition. Thus

$$L^*[v] = -\frac{1}{\exp(x)}\frac{d}{dx}(v(x)\exp(x)) = -\frac{1}{\exp(x)}\left(v(x)\exp(x) + \exp(x)\frac{dv}{dx}\right) = v(x) - \frac{dv}{dx}$$

Therefore, the operator L^* can be defined as

$$\boxed{L^* = -1 - \partial_x}$$

Part 2

Find the operator $S = L^* \circ L$.

Solution.
We can define $S[u]$ as

$$\begin{aligned} S[u] &= L^*[L[u]] \\ &= (-1-\partial_x)(\partial_x(u)) \\ &= (-1-\partial_x)(u_x) \\ &= -u_x - u_{xx} \end{aligned}$$

Therefore

$$\boxed{S = -\partial_x - \partial_{xx}}$$

Part 3

Solve the boundary value problem $S[u] = 2e^x$.

Solution.
Substituting for $S[u]$ from Part 2, we have the following relation

$$\begin{aligned} -u_x - u_{xx} &= 2\exp(x) \\ u_{xx} + u_x &= -2\exp(x) \end{aligned}$$

This is a second order ODE that we can solve with the the principle of superposition. First, we find the solution to the homogeneous problem using the characteristic equation

$$\begin{aligned} & u_{xx} + u_x = 0 \\ & r^2 + r = 0 \\ & r(r+1) = 0 \to r = 0, -1 \\ & \therefore u_{gen} = c_1 + c_2\exp(-x) \end{aligned}$$

Next, we look for a particular solution using the method of undetermined coefficients. We assume our particular solution is of the form

$$u_p(x) = A\exp(x)$$

Substituting into our ODE, we have

$$A\exp(x) + A\exp(x) = -2\exp(x) \to \therefore A = -1$$

By the principle of superposition, our full solution is thus

$$u(x) = c_1 + c_2\exp(-x) - \exp(x)$$

We can utilize the given boundary conditions in the problem to solve for the coefficients in the problem

$$\begin{aligned}
&u(0) = 0 = c_1 + c_2 - 1 \rightarrow c_1 = 1 - c_2 \\
&u(1) = 0 = c_1 + \frac{c_2}{e} - e \\
&0 = 1 - c_2 + \frac{c_2}{e} - e \\
&e - 1 = c_2(\frac{1}{e} - 1) \\
&\therefore c_2 = -e, \; c_1 = 1 + e
\end{aligned}$$

Our final solution can then be given as

$$\boxed{u(x) = 1 + e - \exp(1 - x) - \exp(x)}$$

Problem 4

Consider the operator $L : U \to U$ where $L[u] = -u_{xx}$ and

$$U = \{C^2([0,1]) \mid u(0) = 0 \text{ and } u'(1) + \beta u(1) = 0\}$$

for some real constant β. Use the standard L^2 inner product for both the domain and range.
Is L self-adjoint? Is L positive definite? For positive-definiteness, you should find that your answer depends on β. It is enough to find a range of β's where L is positive definite - you do not need to prove that L is not positive definite for other values of β. Show your work.

Solution.
To determine if L is self adjoint, we use the following equality

$$< L[u], v >=< u, L^*[v] >$$

We express this in integral form as

$$\int_0^1 -u''(x)v(x)\ dx = \int_0^1 u(x)L^*[v(x)]\ dx$$

We aim to make the left hand side of the equation be in the same form of the right hand side. We can integrate by parts with the following substitiutions

$$u = v(x) \quad du = v'(x) \quad dv = -u''(x) \quad v = -u'(x)$$

$$\begin{aligned} LHS &= -v(x)u'(x)\big|_0^1 + \int_0^1 u'(x)v'(x)\ dx \\ &= [-v(1)u'(1) + \cancelto{0}{v(0)u'(0)}] + \int_0^1 u'(x)v'(x)\ dx \\ &= -v(1)u'(1) + \int_0^1 u'(x)v'(x)\ dx \end{aligned}$$

We can integrate by parts again with the following substitution

$$u = v'(x) \quad du = v''(x) \quad dv = u'(x) \quad v = u(x)$$

$$
\begin{aligned}
LHS &= -v(1)u'(1) + u(x)v'(x)\Big|_0^1 - \int_0^1 u(x)v''(x)\ dx \\
&= -v(1)u'(1) + [u(1)v'(1) + \cancelto{0}{u(0)v'(0)]} + \int_0^1 u(x)[-v''(x)]\ dx \\
&= u(1)v'(1) - u'(1)v(1) + \int_0^1 u(x)[-v''(x)]\ dx \\
&= u(1)v'(1) + \beta u(1)v(1) + \int_0^1 u(x)[-v''(x)]\ dx \leftarrow \text{Apply BCs} \\
&= u(1)\cancelto{0}{[v'(1) + \beta v(1)]} + \int_0^1 u(x)[-v''(x)]\ dx \\
&= \int_0^1 u(x)[-v''(x)]\ dx
\end{aligned}
$$

We see this matches the form of the right hand side of our initial inner product equation. Therefore, we know that

$$L^*[v(x)] = -\partial_{xx}[v(x)]$$

This is the same as the the given operator L, indicating that L is self adjoint. To find where L is positive definite, we need to indentify the conditions for which the following relation holds

$$< u, L[u] >> 0$$

Using the integral definition of the inner product, we have

$$\int_0^1 -u(x)u''(x)\ dx > 0$$

We use the following substitution to integrate by parts

$$u = u(x) \quad du = u'(x) \quad dv = -u''(x) \quad v = -u'(x)$$

$$
\begin{aligned}
&- u(x)u'(x)\Big|_0^1 + \int_0^1 [u'(x)]^2\ dx > 0 \\
&[-u(1)u'(1) + \cancelto{0}{u(0)u'(0)]} + \int_0^1 [u'(x)]^2\ dx > 0 \\
&- u(1)u'(1) + \int_0^1 [u'(x)]^2\ dx > 0
\end{aligned}
$$

Using our initial conditions, we make the substitution $u'(1) = -\beta u(1)$

$$
\begin{aligned}
&- u(1)u'(1) + \int_0^1 [u'(x)]^2\ dx > 0 \\
&\beta u(1)^2 + \int_0^1 [u'(x)]^2\ dx > 0 \\
&\beta u(1)^2 > - \int_0^1 [u'(x)]^2\ dx \\
&\beta > \frac{-\int_0^1 [u'(x)]^2\ dx}{u(1)^2}
\end{aligned}
$$

Thus, we have that L is positive definite for

$$\boxed{\beta > \frac{-\int_0^1 [u'(x)]^2 \, dx}{u(1)^2}}$$

Problem 5

Let U be the space of continuously differentiable, complex-valued, 2π-periodic functions $u : \mathbb{R} \to \mathbb{C}$. Define the linear operator $L : U \to U$ such that

$$L[u] = i\frac{\mathrm{d}u}{\mathrm{d}x}$$

Use the inner product

$$\langle u_1, u_2 \rangle = \int_{-\pi}^{\pi} u_1(x)\overline{u_2(x)}\,\mathrm{d}x$$

for both the domain and range. Show that L is self-adjoint.

Solution.
With the given definition $L[u]$, the resulting operator L can be defined as

$$L = \mathrm{i}\partial_x$$

A function $f : \mathbb{R} \to \mathbb{C}$ can be written in the following form of real and complex components

$$f(x) = a(x) + \mathrm{i}b(x)$$

Thus, it follows that

$$f'(x) = a'(x) + \mathrm{i}b'(x)$$

The complex conjugate of $f'(x)$ is given as

$$\overline{f'(x)} = a'(x) - \mathrm{i}b'(x)$$

We also can calculate the derivative of the conjugate of $f(x)$

$$\frac{d}{dx}(\overline{f(x)}) = \frac{d}{dx}(a(x) - \mathrm{i}b(x)) = a'(x) - \mathrm{i}b'(x)$$

Thus, we see that

$$\overline{f'(x)} = \frac{d}{dx}(\overline{f(x)})$$

To show the operator L is self adjoint, we use the equality relation of the inner product definition

$$< L[u], v >=< u, L^*[v] >$$
$$\int_{-\pi}^{\pi} \mathrm{i}\frac{du}{dx}\overline{v(x)}\ dx = \int_{-\pi}^{\pi} u(x)\overline{L^*[v(x)]}\ dx$$

We attempt to match the left hand side of the equation to the right hand side. We can use integration by parts with the following substitutions on the left hand side

$$u = \overline{v(x)} \quad du = \overline{\frac{d}{dx}(v(x))} \quad dv = \frac{du}{dx} \quad v = u(x)$$

$$
\begin{aligned}
LHS &= \mathrm{i}\left[u(x)\overline{v(x)}\big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u(x)\overline{\frac{d}{dx}(v(x))}\ dx\right] \\
&= \mathrm{i}\left[(u(\pi)\overline{v(\pi)} - u(-\pi)\overline{v(-\pi)}) - \int_{-\pi}^{\pi} u(x)\overline{\frac{d}{dx}(v(x))}\ dx\right]
\end{aligned}
$$

As from the problem definitions the functions in U are 2π periodic, we know that $u(\pi) = u(-\pi)$. By the same logic, $v(\pi) = v(-\pi)$, and if these values are equal than it follows that $\overline{v(\pi)} = \overline{v(\pi)}$. Therefore, we know that

$$
u(\pi)\overline{v(\pi)} - u(-\pi)\overline{v(-\pi)} = 0
$$

Thus

$$
LHS = \int_{-\pi}^{\pi} u(x)[-\mathrm{i}\overline{\frac{d}{dx}(v(x))}]\ dx
$$

We know by complex conjugate rules that $\overline{zw} = \overline{z}\ \overline{w}$, and as a result the following relationship holds true

$$
\overline{\mathrm{i}\frac{d}{dx}(v(x))} = \bar{\mathrm{i}}\ \overline{\frac{d}{dx}(v(x))} = -\mathrm{i}\overline{\frac{d}{dx}(v(x))}
$$

We now have a pattern that matches the right hand side of our initial integral relation

$$
\int_{-\pi}^{\pi} u(x)\overline{\mathrm{i}\frac{d}{dx}(v(x))}\ dx = \int_{-\pi}^{\pi} u(x)\overline{L^*[v(x)]}\ dx
$$

From inspection, we have that

$$
L^*[v] = \mathrm{i}\frac{dv}{dx} \to L^* = \mathrm{i}\partial_x
$$

As $L^* = L$, we have proved that the operator is self adjoint.