AMATH 503: Methods for Partial Differential Equations
University of Washington Spring 2022
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Homework 1

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Problem 1

Solve the following ordinary differential equations

1.1 Part 1

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ry + b$$

where r and b are constants.

Solution.
This is a separable, first order equation. To solve, we separate the variables and integrate

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ry + b$$
$$\frac{\mathrm{d}y}{ry+b} = \mathrm{d}x$$
$$\int \frac{\mathrm{d}y}{ry+b} = \int \mathrm{d}x$$
$$\ln(ry+b) + c_1 = x + c_2 \to \ln(ry+b) = x + c$$

The coefficients c_1 and c_2 are combined into a single coefficient c. We then take the exponent of both sides and solve for y

$$\exp(\ln(ry+b)) = ry + b = \exp(x+c)$$
$$y = \frac{\exp(x+c) - b}{r}$$

This can be reduced using exponential rules to get our final solution

$$y = \frac{\exp(x+c) - b}{r} = \frac{\exp(c)\exp(x) - b}{r}$$
$$\text{Let } C = \exp(c)$$
$$\boxed{y = \frac{C\exp(x) - b}{r}}$$

1.2 Part 2

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \eta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0$$

where η and ω_0 are constants.

Solution.
This is a second order, constant coefficient, homogeneous differential equation. We assume solutions of the form ce^{rt}, where r can be found from the roots of the characteristic polynomial for the differential equation. For this differential equation, the characteristic polynomial is

$$r^2 + \eta r + \omega_0^2 = 0$$

The roots of this polynomial are given by the quadratic formula as

$$r = \frac{-\eta \pm \sqrt{\eta^2 - 4\omega_0^2}}{2}$$

There are three possibilities here depending on the value of $\eta^2 - 4\omega_0^2$. There can be two distinct real roots, a single repeated root, and two complex roots. By the principle of superposition, the solutions corresponding each root can be combined to yield a single general solution of the following form

$$x_{gen}(t) = c_1x_1(t) + c_2x_2(t)$$

For the case of real non-repeating roots, the general solution of the ordinary differential equation is given as

$$x(t) = c_1\exp(r_1t) + c_2\exp(r_2t) \tag{1}$$

For a repeated root, the general solution is given as

$$x(t) = c_1\exp(rt) + c_2t\exp(rt) \tag{2}$$

And for complex roots of the form $r = \lambda \pm \mu\mathrm{i}$, the general form is given as

$$x(t) = c_1\exp(\lambda t)\cos(\mu t) + c_2\exp(\lambda t)\sin(\mu t) \tag{3}$$

For our problem, the values will be given as $\lambda = -\eta/2$ and $\mu = \sqrt{|\eta^2 - 4\omega_0^2|}/2$. Therefore for our problem, the solution is as follows depending on the value of $\eta^2 - 4\omega_0^2$

$$\boxed{x(t) = \begin{cases} c_1\exp\left(\frac{(-\eta+\sqrt{\eta^2-4\omega_0^2})t}{2}\right) + c_2\exp\left(\frac{(-\eta-\sqrt{\eta^2-4\omega_0^2})t}{2}\right), & \eta^2 - 4\omega_0^2 > 0 \\ c_1\exp\left(-\frac{\eta t}{2}\right) + c_2t\exp\left(-\frac{\eta t}{2}\right), & \eta^2 - 4\omega_0^2 = 0 \\ c_1\exp\left(-\frac{\eta t}{2}\right)\cos\left(\frac{\sqrt{|\eta^2-4\omega_0^2|}t}{2}\right) + c_2\exp\left(-\frac{\eta t}{2}\right)\sin\left(\frac{\sqrt{|\eta^2-4\omega_0^2|}t}{2}\right), & \eta^2 - 4\omega_0^2 < 0 \end{cases}}$$

1.3 Part 3

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 4\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = 0$$

Solution.
This is again a second order, constant coefficient, homogeneous differential equation. The characteristic polynomial is given as

$$r^2 - 4r + 4 = 0$$

This equation has a repeated root

$$r = 2$$

Therfore, by equation 2 the general solution is given as

$$\boxed{x(t) = c_1 \exp(2t) + c_2 t \exp(2t)}$$

1.4 Part 4

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} = e^{3x}$$

Solution.
This is a second order, constant coefficient, non-homogeneous ordinary differential equation. To solve this equation, we can make use of the method of undetermined coefficients. The general solution for a problem of this type is given as

$$y = y_{homogeneous} + y_{particular}$$

The homogeneous solution is obtained by solving the base problem

$$y'' - 3y' = 0$$

Using the characteristic polynomial we can solve for the roots

$$r^2 - 3r = 0$$
$$r(r-3) = 0$$
$$r_{1,2} = 0, 3$$

This is a case of two real, non-repeating roots and thus the homogeneous solution is

$$y_{homogeneous} = c_1 \exp(0) + c_2 \exp(3x) = c_1 + c_2 \exp(3x)$$

As the equation on the right hand side of the problem is an exponential, we assume a form of $A\exp(3x)$ for our particular solution where A is a constant. The derivatives of this can be calculated and plugged into the problem to solve for a value of A

$$y'_{particular} = 3A\exp(3x) \qquad y''_{particular} = 9A\exp(3x)$$
$$9A\exp(3x) - 9A\exp(3x) = \exp(3x)$$
$$0 \neq \exp(3x)$$

This form of the particular solution is infeasible. We now try a form of $Ax\exp(3x)$

$$\begin{aligned} y'_{particular} &= 3Ax\exp(3x) + A\exp(3x) = (3Ax + A)\exp(3x) \\ y''_{particular} &= 3(3Ax + A)\exp(3x) + 3A\exp(3x) = (9Ax + 6A)\exp(3x) \end{aligned}$$
$$(9Ax + 6A)\exp(3x) - 3(3Ax + A)\exp(3x) = \exp(3x)$$
$$3A\exp(3x) = \exp(3x)$$
$$A = \frac{1}{3}$$

Using this value of A, our particular solution is given as

$$y_{particular} = \frac{x\exp(3x)}{3}$$

And our final solution is therefore

$$\boxed{y(x) = c_1 + c_2\exp(3x) + \frac{x\exp(3x)}{3}}$$

Problem 2

Show that

$$u(t, x) = \frac{1}{2\sqrt{\pi t}} e^{-x^2/(4t)}$$

is a classical solution to the heat equation

$$u_t = u_{xx}$$

Solution.

We first calculate the partial derivatives of $u(t, x)$ to show that the heat equation is satisfied

$$\begin{aligned} u_t &= \frac{1}{2\sqrt{\pi}}\partial_t\left(t^{-\frac{1}{2}}\exp(-\frac{x^2}{4t})\right) \\ &= \frac{1}{2\sqrt{\pi}}\left(t^{-\frac{1}{2}}\exp(-\frac{x^2}{4t}) \cdot -\frac{x^2}{4t} \cdot -t^{-2} + \exp(-\frac{x^2}{4t}) \cdot -\frac{t^{-\frac{3}{2}}}{2}\right) \\ &= \frac{\exp(-\frac{x^2}{4t})}{4\sqrt{\pi}}\left(\frac{t^{-\frac{5}{2}}x^2}{2} - \frac{2t^{-\frac{3}{2}}}{2}\right) \\ &= \frac{\exp(-\frac{x^2}{4t})}{8\sqrt{\pi}t^{\frac{5}{2}}}(x^2 - 2t) \end{aligned}$$

$$
\begin{aligned}
u_x &= \frac{1}{2\sqrt{\pi}t^{\frac{1}{2}}}\partial_x\left(\exp(-\frac{x^2}{4t})\right) \\
&= \frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}}x\exp(-\frac{x^2}{4t}) \\
u_{xx} &= \frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}}\left(x\cdot -\frac{x}{2t}\cdot\exp(-\frac{x^2}{4t})+\exp(-\frac{x^2}{4t})\right) \\
&= \frac{-1}{4\sqrt{\pi}t^{\frac{3}{2}}}\exp(-\frac{x^2}{4t})\left(-\frac{x^2}{2t}+1\right) \\
&= \frac{\exp(-\frac{x^2}{4t})}{8\sqrt{\pi}t^{\frac{5}{2}}}(x^2-2t)
\end{aligned}
$$

We have proved $u_t = u_{xx}$ and therefore the PDE is satisfied. The second condition for the classical solution is that the function is $C^{(2)}$ on an interval D. In this case, $u(t,x)$ is only defined for values of $t > 0$ and is thus not $C^{(2)}$ for $t < 0$. Therefore,
$\boxed{u(t,x) \text{ is a classical solution for } D : \{(t,x) \in \mathbb{R}^2 \mid t > 0\}}$

Problem 3

Show that each of the following PDEs is linear and determine if the PDE is homogeneous or inhomogeneous. In addition, determine the order and dimension of the PDE.
1) $u_t = txu_{xx} - 2xu_{xy} - u_y$
2) $u_t = 5u_{xxx} + x^2u + x$

3.1 Equation 1

Solution.
The PDE can be rewritten in the form of $L[u] = f$ as

$$
\begin{gathered}
u_t - txu_{xx} + 2xu_{xy} + u_y = 0 \\
(\partial_t - tx\partial_{xx} + 2x\partial_{xy} + \partial_y)u = 0
\end{gathered}
$$

From inspection we see that $f = 0$, and therefore $\boxed{\text{the PDE is homogeneous}}$. The highest order derivatives in the equation are u_{xx} and u_{xy}, both of which are second derivatives. Therefore, $\boxed{\text{the order of the PDE is 2}}$. The partial derivatives of u contain both x and y as variables, indicating there are 2 spatial variables. Therefore, $\boxed{\text{the dimension of the PDE is 2}}$

To show the linearity of the PDE, we need to prove that the following are true

$$
\begin{gathered}
L[u+v] = L[u] + L[v] \\
L[cu] = cL[u]
\end{gathered}
$$

First, we prove that $L[u+v] = L[u] + L[v]$. We proved in class that the derivative operator is linear and thus can be distributed, which allows us to prove the first condition

$$\begin{aligned}
L[u+v] &= (\partial_t - tx\partial_{xx} + 2x\partial_{xy} + \partial_y)(u+v) \\
&= \partial_t(u+v) - tx\partial_{xx}(u+v) + 2x\partial_{xy}(u+v) + \partial_y(u+v) \\
&= u_t + v_t - tx(u_{xx} + v_{xx}) + 2x(u_{xy} + v_{xy}) + (u_y + v_y) \\
&= (u_t - txu_{xx} + 2xu_{xy} + u_y) + (v_t - txv_{xx} + 2xv_{xy} + v_y) \\
&= L[u] + L[v]
\end{aligned}$$

We now prove that $L[cu] = cL[u]$. We will again make use of the fact that the derivative operator is linear and thus can be distributed

$$\begin{aligned}
L[cu] &= (\partial_t - tx\partial_{xx} + 2x\partial_{xy} + \partial_y)cu \\
&= \partial_t(cu) - tx\partial_{xx}(cu) + 2x\partial_{xy}(cu) + \partial_y(cu) \\
&= cu_t - txcu_{xx} + 2xcu_{xy} + cu_y \\
&= c(u_t - txu_{xx} + 2xu_{xy} + u_y) \\
&= cL[u]
\end{aligned}$$

As both linearity conditions hold, $\boxed{\text{the PDE is linear}}$.

3.2 Equation 2

Solution.
We again write the PDE in the form of $L[u] = f$ as

$$\begin{aligned}
u_t - 5u_{xxx} - x^2 u &= x \\
(\partial_t - 5\partial_{xxx} - x^2)u &= x
\end{aligned}$$

In this case we see that $f \neq 0$, and therefore $\boxed{\text{the PDE is inhomogeneous}}$. The highest order partial derivative in $L[u]$ is u_{xxx} which is a third derivative. Therefore, $\boxed{\text{the order of the PDE is 3}}$. The only spatial variable in the PDE is x, thus $\boxed{\text{the dimension of the PDE is 1}}$.

We can again prove the PDE is linear by showing the linearity conditions hold for $L[u]$. Similar as to Equation 1, we will use the fact the derivative operator is linear and thus can be distributed

$$\begin{aligned}
L[u+v] &= (\partial_t - 5\partial_{xxx} - x^2)(u+v) \\
&= \partial_t(u+v) - 5\partial_{xxx}(u+v) - x^2(u+v) \\
&= (u_t + v_t) - 5(u_{xxx} + v_{xxx}) - x^2(u+v) \\
&= (u_t + u_{xxx} - x^2 u) + (v_t + v_{xxx} - x^2 v) \\
&= L[u] + L[v]
\end{aligned}$$

$$\begin{aligned}
L[cu] &= (\partial_t - 5\partial_{xxx} - x^2)cu \\
&= \partial_t(cu) - 5\partial_{xxx}(cu) - x^2(cu) \\
&= cu_t - 5cu_{xxx} - cx^2u \\
&= c(u_t - 5u_{xxx} - x^2u) \\
&= cL[u]
\end{aligned}$$

Both linearity conditions hold, showing that $\boxed{\text{the PDE is linear}}$.

Problem 4

Suppose that L is a linear operator and $f_1, \ldots, f_k$ are known functions. Show that if $u_1, \ldots, u_k$ are solutions of the inhomogeneous PDEs $L[u] = f_1, \ldots, L[u] = f_k$ and $c_1, \ldots, c_k$ are constants, then

$$u = c_1u_1 + \cdots + c_ku_k$$

is a solution of the PDE $L[u] = f$, where

$$f = c_1f_1 + \cdots + c_kf_k$$

Solution.

We first want to show the linearity conditions for an operator extend to more than two terms. Consider an odd number of equations u_1, u_2, u_3 with corresponding coefficients c_1, c_2, c_3. Since $L[u]$ is a linear operator, the following is true

$$\begin{aligned}
L[c_1u_1 + c_2u_2 + c_3u_3] &= L[(c_1u_1 + c_2u_2) + c_3u_3] \\
&= L[c_1u_1 + c_2u_2] + L[c_3u_3] \\
&= L[c_1u_1] + L[c_2u_2] + L[c_3u_3] \\
&= c_1L[u_1] + c_2L[u_2] + c_3L[u_3]
\end{aligned}$$

Considering four equations, we can show a similar result for an even number of terms

$$\begin{aligned}
L[c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4] &= L[(c_1u_1 + c_2u_2 + c_3u_3) + c_4u_4] \\
&= L[c_1u_1 + c_2u_2 + c_3u_3] + L[c_4u4] \\
&= L[c_1u_1 + c_2u_2] + L[c_3u_3] + L[c_4u4] \\
&= L[c_1u_1] + L[c_2u_2] + L[c_3u_3] + L[c_4u_4] \\
&= c_1L[u_1] + c_2L[u_2] + c_3L[u_3] + c_4L[u_4]
\end{aligned}$$

It follows that this extends to a potentially infinite number of terms when considering a single term at a time. As a result, we know that the following must hold for $u_1, \cdots, u_k$

$$L[c_1u_1 + \cdots + c_ku_k] = c_1L[u_1] + \cdots c_kL[u_k]$$

Substituting the corresponding solutions for the individual PDEs, we obtain the desired result

$$L[c_1 u_1 + \cdots + c_k u_k] = c_1 f_1 + \cdots + c_k f_k$$

Therefore, we have proved

$$\boxed{\begin{array}{c} L[u] = f \\ u = c_1 u_1 + \cdots c_k u_k \\ f = c_1 f_1 + \cdots c_k f_k \end{array}}$$

Problem 5

The PDE $u_{tt} = u_{xx}$ is known as the wave equation. Show that

$$u(t,x) = e^{x-t} \text{ and } v(t,x) = x^2 - 2xt + t^2$$

are both classical solutions to the wave equation, then find a third solution to the wave equation. (You need to show your work and justify your answer!)

5.1 Equation 1: $u(t,x) = e^{x-t}$

Solution.
A classical solution to the wave equation requires that the function is $C^{(2)}$ on an interval D and that it satisfies the wave equation. The function $u(t,x) = e^{x-t}$ is an exponential which is C^∞ $\forall$ $(x,t) \in \mathbb{R}^2$, and therefore the first condition is met. We can calculate the partial derivatives to show that they are equal

$$\begin{aligned} u_t &= -e^{x-t} & u_x &= e^{x-t} \\ u_{tt} &= e^{x-t} & u_{xx} &= e^{x-t} \end{aligned}$$

As $u_{tt} = u_{xx}$, the wave equation is satisfied and $u(t,x)$ is a classical solution

5.2 Equation 2: $v(t,x) = x^2 - 2xt + t^2$

Solution.
This function is a polynomial, and is therefore C^∞ $\forall$ $(x,t) \in \mathbb{R}^2$. The first condition for a classical solution is thus satisfied. We calculate the partial derivatives

$$\begin{aligned} v_t &= -2x + 2t & v_x &= 2x - 2t \\ v_{tt} &= 2 & v_{xx} &= 2 \end{aligned}$$

As $v_{tt} = v_{xx}$, the wave equation is satisfied and $v(t,x)$ is a classical solution

5.3 Custom Equation

Solution.
We define the function $w(t, x)$ as follows that we believe to be a solution to the wave equation

$$w(t, x) = \cos(x)\cos(t)$$

The function is a product of cosines, which is also $C^\infty \; \forall \; (x, t) \in \mathbb{R}^2$. Therefore, the first condition for a classical solution is satisfied. The partial derivatives can be calculated as follows

$$\begin{aligned} w_t &= -\cos(x)\sin(t) \qquad & w_x &= -\sin(x)\cos(t) \\ w_{tt} &= \cos(x)\cos(t) \qquad & w_{xx} &= \cos(x)\cos(t) \end{aligned}$$

We see that $w_{tt} = w_{xx}$, and therefore $w(t, x)$ is a classical solution of the wave equation.

Alternatively, we can write the wave equation in the form of $L[u] = f$

$$(\partial_{tt} - \partial_{xx})u = 0$$

This is a homogeneous PDE, so if we can prove L is a linear operator than the principle of superposition applies to the problem. The sum of $u(t, x)$ and $v(t, x)$ will therefore also be a solution

$$\begin{aligned} L[c_1 u + c_2 v] &= (\partial_{tt} - \partial_{xx})(c_1 u + c_2 v) \\ &= \partial_{tt}(c_1 u + c_2 v) - \partial_{xx}(c_1 u + c_2 v) \\ &= c_1 u_{tt} + c_2 v_{tt} - c_1 u_{xx} - c_2 v_{xx} \\ &= c_1(u_{tt} - u_{xx}) + c_2(v_{tt} - v_{xx}) \\ &= c_1 L[u] + c_2 L[v] \end{aligned}$$

We have proved L is a linear operator. Because the PDE is homogeneous, by the principle of superposition the following will also be a classical solution

$$w(t, x) = e^{x-t} + x^2 - 2xt + t^2$$

Problem 6

6.1 Gradient Operator

Solution.
Let $f(x, y, z)$ be a $C^{(1)}$ three dimensional function. The gradient operation is defined as follows

$$\nabla f = \langle \partial_x(f), \partial_y(f), \partial_z(f) \rangle$$

Considering another $C^{(1)}$, three dimensional function $g(x, y, z)$, we want to prove the first linearity condition. We know the derivative operator to be linear, so it can be distributed

$$\begin{aligned}
\nabla(f+g) &= \langle \partial_x(f+g), \partial_y(f+g), \partial_z(f+g) \rangle \\
&= \langle f_x + g_x, f_y + g_y, f_z + g_z \rangle \\
&= \langle f_x, f_y, f_z \rangle + \langle g_x, g_y, g_z \rangle \\
&= \nabla f + \nabla g
\end{aligned}$$

The first linearity condition is thus proved. We can then prove the second condition

$$\begin{aligned}
\nabla(cf) &= \langle \partial_x(cf), \partial_y(cf), \partial_z(cf) \rangle \\
&= \langle cf_x, cf_y, cf_z \rangle \\
&= c\langle f_x, f_y, f_z \rangle \\
&= c\nabla f
\end{aligned}$$

Both linearity conditions have been proved, and therefore the gradient is a linear operator.

6.2 Divergence Operator

Solution.
Let $F(x, y, z)$ be a three dimensional, $C^{(1)}$ vector field. The divergence operator in three dimensions is defined as

$$\nabla \cdot F = \partial_x(F_x) + \partial_y(F_y) + \partial_z(F_z)$$

We aim to show that the first linearity condition holds for F and a second three dimensional, $C^{(1)}$ vector field $G(x, y, z)$.

$$\nabla \cdot (F+G) = \partial_x(F_x + G_x) + \partial_y(F_y + G_y) + \partial_z(F_z + G_z)$$

The derivative operator can be distributed as it is linear

$$\begin{aligned}
\nabla \cdot (F+G) &= \frac{\partial F_x}{\partial x} + \frac{\partial G_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial G_y}{\partial y} + \frac{\partial F_z}{\partial z} + \frac{\partial G_z}{\partial z} \\
&= (\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) + (\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z}) \\
&= (\nabla \cdot F) + (\nabla \cdot G)
\end{aligned}$$

The second linearity condition can be proved as follows

$$\begin{aligned}
\nabla \cdot (cF) &= \partial_x(cF_x) + \partial_y(cF_y) + \partial_z(cF_z) \\
&= c\frac{\partial F_x}{\partial x} + c\frac{\partial F_y}{\partial y} + c\frac{\partial F_z}{\partial z} \\
&= c(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}) \\
&= c(\nabla \cdot F)
\end{aligned}$$

As both linearity conditions hold, the divergence has been proved to be a linear operator

6.3 Curl Operator

Solution.
Let $F(x, y, z)$ be a three dimensional, $C^{(1)}$ vector field. The curl operation is defined as

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = \langle \partial_y(F_z) - \partial_z(F_x), \partial_z(F_x) - \partial_x(F_z), \partial_x(F_y) - \partial_y(F_x) \rangle$$

With $G(x, y, z)$ as another three dimensional, $C^{(1)}$ vector field, we can prove the first linearity condition holds.

$$\begin{aligned} \nabla \times (F + G) = \langle &\partial_y(F_z + G_z) - \partial_z(F_x + G_x), \partial_z(F_x + G_x) - \partial_x(F_z + G_z), \\ &\partial_x(F_y + G_y) - \partial_y(F_x + G_x) \rangle \end{aligned}$$

We make use of the fact the derivative operation is linear to distribute it in the vector operations.

$$\begin{aligned} \nabla \times (F + G) &= \langle \frac{\partial F_z}{\partial y} + \frac{\partial G_z}{\partial y} - \frac{\partial F_x}{\partial z} - \frac{\partial G_x}{\partial z}, \frac{\partial F_x}{\partial z} + \frac{\partial G_x}{\partial z} - \frac{\partial F_z}{\partial x} - \frac{\partial G_z}{\partial x}, \\ &\quad \frac{\partial F_y}{\partial x} + \frac{\partial G_y}{\partial x} - \frac{\partial F_x}{\partial y} - \frac{\partial G_x}{\partial y} \rangle \\ &= \langle \frac{\partial F_z}{\partial y} - \frac{\partial F_x}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \rangle + \\ &\quad \langle \frac{\partial G_z}{\partial y} - \frac{\partial G_x}{\partial z}, \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x}, \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \rangle \\ &= (\nabla \times F) + (\nabla \times G) \end{aligned}$$

The second linearity condition can be proved as follows

$$\begin{aligned} \nabla \times (cF) &= \langle \partial_y(cF_z) - \partial_z(cF_x), \partial_z(cF_x) - \partial_x(cF_z), \partial_x(cF_y) - \partial_y(cF_x) \rangle \\ &= \langle c\frac{\partial F_z}{\partial y} - c\frac{\partial F_x}{\partial x}, c\frac{\partial F_x}{\partial z} - c\frac{\partial F_z}{\partial x}, c\frac{\partial F_y}{\partial x} - c\frac{\partial F_x}{\partial y} \rangle \\ &= c\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_x}{\partial x}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \rangle \\ &= c(\nabla \times F) \end{aligned}$$

The curl operation satisfies both linearity conditions and is therefore a linear operator