AMATH 503: Methods for Partial Differential Equations
University of Washington Spring 2022
Instructor: Lowell Thompson

Homework 2

Philip Westphal
UW NetID: philw16@uw.edu

Problem 1

Prove that

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,\mathrm{d}x = 0$$

for all $m, n \in \mathbb{Z}^+$,

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,\mathrm{d}x = \begin{cases} \pi & m, n \in \mathbb{Z}^+ \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,\mathrm{d}x = \begin{cases} \pi & m, n \in \mathbb{Z}^+ \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

The following trigonometric identities will be helpful:

$$\begin{aligned} 2\cos\theta\cos\phi &= \cos(\theta-\phi) + \cos(\theta+\phi) \\ 2\sin\theta\sin\phi &= \cos(\theta-\phi) - \cos(\theta+\phi) \\ 2\cos\theta\sin\phi &= \sin(\theta+\phi) - \sin(\theta-\phi) \end{aligned}$$

Solution.
For the first equation, we apply the trig identity

$$2\cos(nx)\sin(mx) = \sin((m+n)x) - \sin((n-m)x)$$

Therefore

$$\begin{aligned} \int_{-\pi}^{\pi} \cos(nx)\sin(mx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} (\sin((n+m)x) - \sin((n-m)x))\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \sin((n+m)x)\ dx - \frac{1}{2}\int_{-\pi}^{\pi} \sin((n-m)x)\ dx \end{aligned}$$

This is two separate integrals of the sine function over a symmetric interval centered at $x = 0$. As sine is an odd function, these integrals will always evaluate to 0 for $n \in \mathbb{Z}^+$. Thus

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\ dx = 0 \quad \forall\ n, m \in \mathbb{Z}^+$$

For the second integral, we use the trig identity

$$2\cos(nx)\cos(mx) = \cos((n-m)x) + \cos((n+m)x)$$

Therefore

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos((n-m)x) + \cos((n+m)x))\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x)\ dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\ dx\end{aligned}$$

If $n \neq m$

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x)\Big|_{-\pi}^{\pi} + \frac{1}{n+m}\sin((n+m)x)\Big|_{-\pi}^{\pi}\right]$$

Because n, m are positive integers, $(n-m)$ and $(n+m)$ are also integers. The sine of an integer times π is zero, and therefore this integral will always evaluate to 0. Therefore

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = 0,\ \ n \neq m,\ \ n, m \in \mathbb{Z}^+$$

If $n = m$

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi} (\cos(0) + \cos(2nx))\ dx \\ &= \frac{1}{2}\left[x\Big|_{-\pi}^{\pi} + \frac{1}{2n}\sin(2nx)\Big|_{-\pi}^{\pi}\right]\end{aligned}$$

Again, as n is an integer the second term will always evaluate to 0. Therefore

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\ dx = \frac{1}{2}x\Big|_{-\pi}^{\pi} = \frac{1}{2}(\pi + \pi) = \pi,\ \ n, m \in \mathbb{Z}^+ \text{and } n = m$$

For the third equation, we can use the trig identity

$$2\sin(nx)\sin(mx) = \cos((m-n)x) - \cos((n+m)x)$$

The integral is then equivalent to

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi}(\cos((n-m)x) - \cos((n+m)x))\ dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi}\cos((n-m)x)\ dx - \frac{1}{2}\int_{-\pi}^{\pi}\cos((n+m)x)\ dx\end{aligned}$$

If $n \neq m$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx) = \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x)\bigg|_{-\pi}^{\pi} - \frac{1}{n+m}\sin((n+m)x)\bigg|_{-\pi}^{\pi}\right]$$

This again will evaluate to 0 for all $n, m \in \mathbb{Z}^+,\ n \neq m$ using the same logic as before (sum of two integers is an integer and sine of any integer times π is 0). Therefore

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx = 0,\ n \neq m,\ n, m \in \mathbb{Z}^+$$

If $n = m$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx &= \frac{1}{2}\int_{-\pi}^{\pi}(\cos(0) - \cos(2nx))\ dx \\ &= \frac{1}{2}\left[x\bigg|_{-\pi}^{\pi} - \frac{1}{2n}\sin(2nx)\bigg|_{-\pi}^{\pi}\right]\end{aligned}$$

As n is an integer the second term will always evaluate to 0. Therefore

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\ dx = \frac{1}{2}x\bigg|_{-\pi}^{\pi} = \frac{1}{2}(\pi + \pi) = \pi,\ n, m \in \mathbb{Z}^+ \text{and } n = m$$

Problem 2

Find the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$ such that $f(x) = x^2$. Plot some representative partial sums of the Fourier series (i.e., plot the sum of the first n Fourier modes for several different values of n). Does this Fourier series converge to f on the interval $(-\pi, \pi)$? What about the interval $[-\pi, \pi]$? What about on the whole real line? If the series does not converge to f, what function does it converge to?

Solution.
We know the Fourier Series for $f(x)$ will be of the following form

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

The coefficients can be solved for using the following integrals, where k represents a specific value of n

$$\begin{aligned}
\frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\ dx \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\ dx
\end{aligned}$$

We first solve for the constant coefficient

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\ dx = \frac{1}{2\pi} \frac{x^3}{3} \bigg|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

Next, we solve for the a_k coefficients

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx)\ dx$$

Using WolframAlpha, this evaluates to the following

$$a_k = \frac{1}{\pi k^3} \bigg[(k^2x^2 - 2) \sin(kx) + 2kx \cos(kx) \bigg]_{-\pi}^{\pi}$$

As sine of any integer times π is 0, the first term will always evaluate to 0. Thus

$$\begin{aligned}
a_k &= \frac{1}{\pi k^3} \bigg[2kx \cos(kx) \bigg]_{-\pi}^{\pi} \\
&= \frac{1}{\pi k^3} \bigg[2k\pi \cos(kx) + 2k\pi \cos(kx) \bigg] \\
&= \frac{4}{k^2} \cos(k\pi) \\
&= \frac{4}{k^2} (-1)^k,\ \forall k \in \mathbb{Z}^+
\end{aligned}$$

We can solve for the b_n as follows

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin(kx)\ dx$$

We note that x^2 is an even function and $\sin(kx)$ is an odd function for $k \in \mathbb{Z}^+$. Therefore the product of these functions will be an odd function. The integral of an odd function over a symmetric interval will be 0, and thus

$$b_k = 0,\ \forall k \in \mathbb{Z}^+$$

Our final Fourier Series can therefore be written as follows

$$\boxed{F(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx)}$$

Some plots for the series on the interval $[-\pi, \pi]$ are shown in figure 1. The series appears to converge on the $[-\pi, \pi]$ interval as we see no Gibbs phenomenon occurring at the end points of the interval. Expanding the interval to $[-3\pi, 3\pi]$, we see that the series converges to a periodic

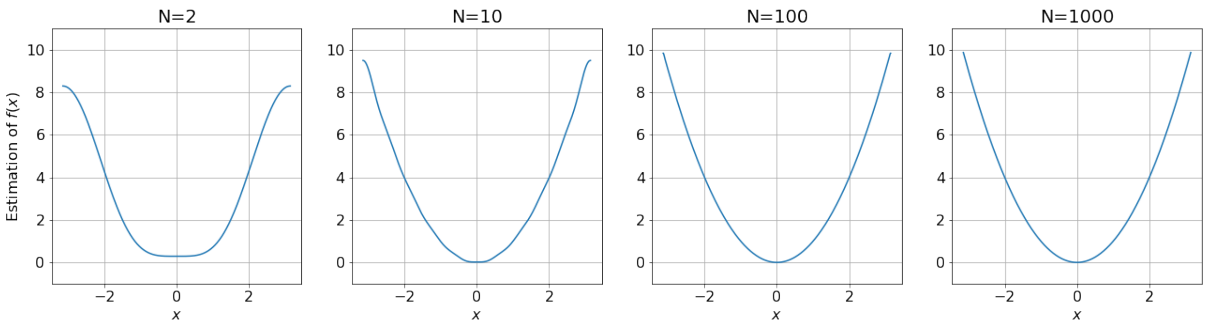

Figure 1: Fourier Series of x^2 on interval $[-\pi, \pi]$

version of x^2 with period 2π. There are no jump discontinuities in the result, with the end of one period of the function lining up perfectly with the start of the next. This can be attributed to the fact that x^2 is an even function.

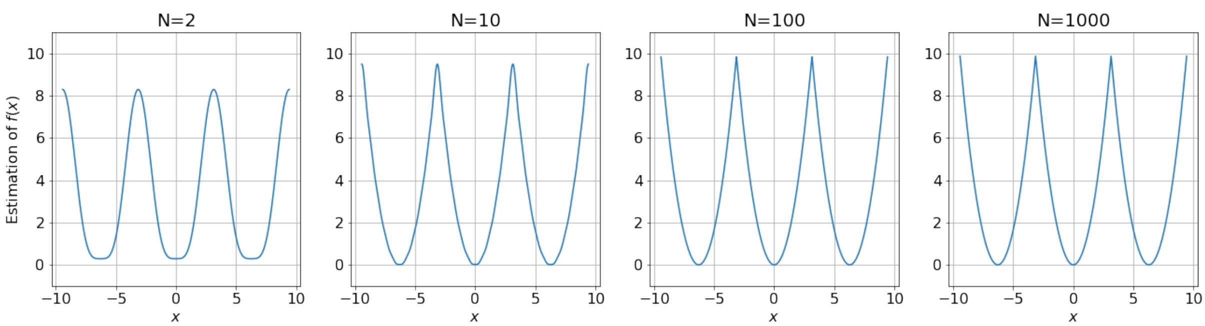

Figure 2: Fourier Series of x^2 on interval $[-3\pi, 3\pi]$

Because there is no Gibbs phenomenon observed for the series on the interval $[-\pi, \pi]$ and no jump discontinuities between successive periods on the real line, the series is converging on the interval $[-\pi, \pi]$ and thus also the interval $(-\pi, \pi)$.

Problem 3

Solve following initial-boundary value problem with separation of variables:

$$\begin{aligned} &u_t = 5u_{xx} \text{ for all } -4 < x < 4 \text{ and } t > 0 \\ &u(t,-4) = u(t,4) = 0 \text{ for all } t > 0 \\ &u(0,x) = f(x) \text{ for all } -4 < x < 4 \end{aligned}$$

where

$$f(x) = \begin{cases} x & -2 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Plot the solution at several representative times. (Your solution will be written as a Fourier series - you can plot partial sums with a reasonable number of terms.)

Solution.
Let

$$y = x + 4 \qquad dy = dx$$

The problem can then be changed to the following form, which is in a $[0, L]$ interval that we are familiar with

$$\begin{aligned} &u_t = 5u_{yy} \text{ for all } 0 < y < 8 \text{ and } t > 0 \\ &u(t,0) = u(t,8) = 0 \text{ for all } t > 0 \\ &u(0,y) = f(y-4) \text{ for all } 0 < y < 8 \end{aligned}$$

$$f(y) = \begin{cases} y-4 & 2 < y < 6 \\ 0 & \text{otherwise} \end{cases}$$

We can use separation of variables, assuming a solution of the form

$$u(t,x) = T(t)X(x)$$

Which leads to the following relation when substituting into the problem $u_t(t,x) = 5u_{yy}$

$$\frac{T'(t)}{2T(t)} = \frac{Y''(y)}{Y(y)} = K = \lambda^2$$

First, we can solve for $T(t)$. This is a first order, separable ODE with an exponential solution

$$T'(t) = 5\lambda^2 T(t) \quad \longrightarrow \quad T(t) = T_0 \exp(5\lambda^2 t)$$

As we have Dirichlet conditions, we know that the non-trivial solutions of interest will result from values of $K < 0$, or $K = -\lambda^2$. The gernal solution of this problem is given by

$$Y''(y) = -\lambda^2 Y(y) \quad \longrightarrow \quad Y(y) = A\cos(\lambda y) + B\sin(\lambda y)$$

Applying the boundary conditions

$$
\begin{gathered}
Y(0) = 0 = A\cos(0) + B\sin(0) \\
\therefore A = 0 \\
Y(10) = 0 = B\sin(8\lambda)
\end{gathered}
$$

Ignoring the trivial solution $B = 0$, this relation will be satisfied when

$$8\lambda = n\pi \implies \lambda = \frac{n\pi}{8}$$

$$Y(y) = B\sin(\frac{n\pi y}{8})$$

The general solution for a specific n will then be

$$u_n = b_n \exp\big(-5(\frac{n\pi}{8})^2 t\big)\sin(\frac{n\pi y}{8}),\ n \in \mathbb{Z}^+$$

with $b_n = B \times T_0$. To match the initial condition, we assume the full general solution is an infinite combination of the individual solutions in the form

$$u(t,y) = \sum_{n=1}^{\infty} b_n \exp\big(-5(\frac{n\pi}{8})^2 t\big)\sin(\frac{n\pi y}{8})$$

Using the initial condition

$$u(0,y) = f(y) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi y}{8})$$

We solve for the coefficients b_n using the formula.

$$b_n = \frac{2}{L}\int_0^L f(y)\sin(\frac{n\pi y}{L})\ dy$$

Because $f(y)$ is piecewise, we will need to construct a piecewise integral.

$$
\begin{aligned}
b_n &= \frac{2}{8}\Big[\int_0^2 0\ dy + \int_2^6 (y-4)\sin(\frac{n\pi y}{8})\ dy + \int_6^8 0\ dy\Big] \\
&= \frac{1}{4}\int_2^6 (y-4)\sin(\frac{n\pi y}{8})\ dy
\end{aligned}
$$

We evaluate the integral using WoflramAlpha

$$\frac{1}{4}\int_2^6 (y-4)\sin(\frac{n\pi y}{8})\ dy = \frac{-8\cos(\frac{\pi n}{2})(\pi n\cos(\frac{\pi n}{4}) - 4\sin(\frac{\pi n}{4}))}{\pi^2 n^2}$$

Finally, we convert $Y(y)$ to $X(x)$ by substituting $y = x + 4$

$$X(x) = Y(x+4) = b_n\sin(\frac{n\pi(x+4)}{8})$$

The full solution to the PDE is then given as

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} \frac{-8\cos(\frac{\pi n}{2})(\pi n\cos(\frac{\pi n}{4}) - 4\sin(\frac{\pi n}{4}))}{\pi^2 n^2} \exp\Big(-5\big(\frac{n\pi}{8}\big)^2 t\Big)\sin\Big(\frac{n\pi(x+4)}{8}\Big)}$$

Plots of the solution at various t are shown in figure 3. We see the temperature decays to equilibrium fairly quickly.

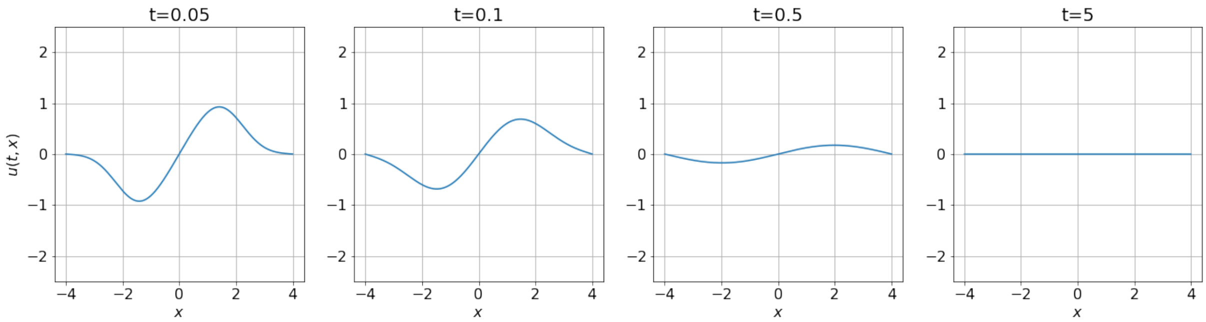

Figure 3: PDE Solution $u(t, x)$

Problem 4

Part 1

Solve the following initial-boundary value problem with separation of variables:

$$\begin{aligned}
&u_t = 2u_{xx} \;\text{ for all }\; 0 < x < 10 \;\text{ and }\; t > 0\\
&u_x(t,0) = u_x(t,10) = 0 \;\text{ for all }\; t > 0\\
&u(0,x) = 5 - x \;\text{ for all }\; 0 < x < 10
\end{aligned}$$

Plot the solution at several representative times. (Your solution will be written as Fourier series - you can plot partial sums with a reasonable number of terms.)

Part 2

You should find that your solution is identical to that of the initial-boundary value problem

$$\begin{aligned}
&u_t = 2u_{xx} \;\text{ for all }\; -10 < x < 10 \;\text{ and }\; t > 0\\
&u(t,-10) = u(t,10) \;\text{ and }\; u_x(t,-10) = u_x(t,10) \;\text{ for all }\; t > 0\\
&u(0,x) = f(x) \;\text{ for all }\; -10 < x < 10
\end{aligned}$$

for some choice of initial condition $f : [-10,10] \to \mathbb{R}$. What is the formula for this initial condition $f(x)$? (You don't have to solve this PDE, although it may be useful to check your answer.)

4.1 Part 1

Solution.
We assume a solution of the form

$$u(t,x) = T(t)X(x)$$

This leads to the following relation when substituting into the problem $u_t(t,x) = 2u_{xx}$

$$\frac{T'(t)}{2T(t)} = \frac{X''(x)}{X(x)} = K = \lambda^2$$

Each of these equations can be solved individually. First, we can solve for $T(t)$. This is a first order, separable ODE with an exponential solution

$$T'(t) = 2\lambda^2 T(t) \quad \longrightarrow \quad T(t) = T_0 \exp(2\lambda^2 t)$$

Next, we can solve the equation for $X(x)$. This is a second order, homogenous ODE and will have solutions depending on the value of λ^2. First, if $\lambda^2 > 0$

$$X''(x) = \lambda^2 X(x) \quad \longrightarrow \quad X(x) = A\exp(\lambda x) + B\exp(-\lambda x)$$

We can utilize the Neumann boundary conditions to determine the values of the constants

$$X'(0) = 0 = A\lambda - B\lambda$$
$$\therefore A = B$$
$$X'(10) = 0 = A\lambda \exp(10\lambda) - B\lambda \exp(-10\lambda)$$
$$\therefore 0 = B\lambda \exp(10\lambda) - B\lambda \exp(-10\lambda)$$

The relations are only satisfied with $A = B = 0$, which is a trivial solution we will ignore. If $\lambda = 0$

$$X''(x) = 0 \quad \longrightarrow \quad X(x) = Ax + B$$

We utilize the boundary conditions again

$$X'(0) = 0 = A$$
$$X'(10) = 0 = A$$
$$\therefore A = 0$$

This leads to $X(x) = B$, which is another trivial solution that we will ignore. Finally, if $\lambda < 0$

$$X''(x) = -\lambda^2 X(x) \quad \longrightarrow \quad X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

Applying the boundary conditions

$$X'(0) = 0 = B\lambda\cos(0) - A\lambda\sin(0)$$
$$0 = B\lambda$$
$$\therefore B = 0$$
$$X'(10) = 0 = -A\lambda\sin(10\lambda)$$

Ignoring the trivial solution of $A = 0$, the following relation holds when

$$10\lambda = n\pi \longrightarrow \lambda = \frac{n\pi}{10}$$

Therefore

$$X(x) = A\cos(\frac{n\pi x}{10})$$

Our general solution for a specific value of n will then be of the form

$$u_n = a_n \exp\big(-2(\frac{n\pi}{10})^2 t\big)\cos(\frac{n\pi x}{10}),\ n \in \mathbb{Z}^+$$

with $a_n = A \times T_0$ representing the constant for the solution. In order to match our initial condition, we assume our solution is an infinite combination of our individual solutions in the form

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp\big(-2(\frac{n\pi}{10})^2 t\big)\cos(\frac{n\pi x}{10})$$

Using our initial condition

$$u(0,x) = 5 - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{10})$$

We can find the constants as follows

$$\begin{aligned}\frac{a_0}{2} &= \frac{1}{L}\int_0^L f(x)\ dx \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{10})\ dx\end{aligned}$$

First, we solve for $\frac{a_0}{2}$ noting that $L = 10$ in our problem

$$\frac{a_0}{2} = \frac{1}{10}\int_0^{10} (5 - x)\ dx = \frac{1}{10}\left[5x - \frac{x^2}{2}\right]_0^{10} = 0$$

Next, we can solve for the coefficients a_n

$$a_n = \frac{2}{10}\int_0^{10} (5 - x)\cos(\frac{n\pi x}{10})\ dx$$

Evaluating this integral in Wolfram Alpha

$$a_k = -\frac{10(n\pi\sin(\pi n) + 2\cos(\pi n) - 2)}{\pi^2 n^2}$$

For any integer n, the sine term will evaluate to 0. Thus

$$\begin{aligned}a_k &= -\frac{10(2\cos(\pi n) - 2)}{\pi^2 n^2} \\ &= -\frac{20(\cos(\pi n) - 1)}{\pi^2 n^2}\end{aligned}$$

The solution for the initial condition is thus

$$u(0,x) = f(x) = \sum_{n=1}^{\infty} -\frac{20(\cos(\pi n) - 1)}{\pi^2 n^2}\cos\big(\frac{n\pi x}{10}\big)$$

and the solution to the PDE is given by

$$\boxed{u(t,x) = \sum_{n=1}^{\infty} -\frac{20(\cos(\pi n) - 1)}{\pi^2 n^2}\exp\big(-2(\frac{n\pi}{10})^2 t\big)\cos\big(\frac{n\pi x}{10}\big)}$$

Solutions of $u(t,x)$ at various time t are shown in figure 4. Compared to the solution for Problem 3, the temperature $u(t,x)$ decays to equilibrium at a much slower rate.

4.2 Part 2

Solution.
We observe the initial condition $f(x)$ is represented as a Fourier Cosine series consisting only of

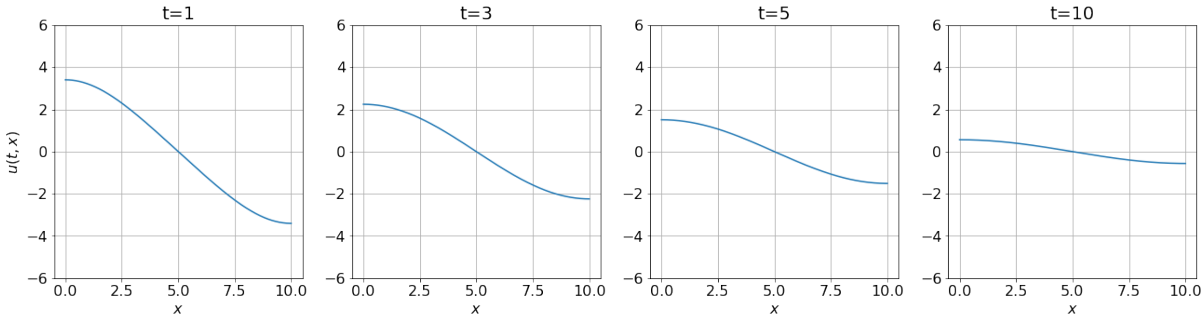

Figure 4: PDE Solution $u(t,x)$ at various t

cosine terms. This makes sense, as we have Neumann boundary conditions, which require the derivative to be zero. The derivative of cosine functions are sine functions, which will be zero for $x = 0$ and $x = L$. Because cosine is an even function, the infinite sum will also be an even function. Therefore, the result obtained in Part 1 can be extended to the periodic boundary value problem on $[-L, L]$ by considering the even extension of the function $f(x)$. The solution to this initial-boundary value problem on the interval $[-10, 10]$ will be the same as the result in Part 1 for the following initial condition $u(0, x) = f(x)$

$$\boxed{f(x) = \begin{cases} x + 5, & -10 < x < 0 \\ 5 - x, & 0 < x < 10 \end{cases}}$$

Problem 5

Suppose that $u(t,x)$ is a non-constant solution to the heat equation with homogeneous Neumann boundary conditions:

$$u_t = Du_{xx} \text{ for all } 0 < x < L \text{ and } t > 0$$
$$u_x(t,0) = u_x(t,L) = 0 \text{ for all } t > 0$$

Part 1

Prove that

$$E(t) = \int_0^L u(t,x)\,\mathrm{d}x$$

is constant.

Part 2

Prove that

$$H(t) = \int_0^L u(t,x)^2\,\mathrm{d}x$$

is strictly decreasing.

5.1 Part 1

Solution.
To show the function is constant, we want to show

$$\frac{dE(t)}{dt} = 0$$

We take the time derivative of both sides of the equation

$$\begin{aligned}
\frac{dE(t)}{dt} &= \frac{d}{dt}\int_0^L u(t,x)\ dx \\
&= \int_0^L \frac{d}{dt}u(t,x)\ dx \\
&= \int_0^L u_t(t,x)\ dx \\
&= \int_0^L Du_{xx}(t,x)\ dx \\
&= D\big[u_x(t,x)\big]_0^L
\end{aligned}$$

From the boundary conditions, we know that the derivative is zero at the boundary. Therefore

$$\boxed{\frac{dE(t)}{dt} = D[0-0] = 0}$$

As $\frac{dE(t)}{dt} = 0$, $E(t)$ must be a constant.

5.2 Part 2

Solution.
To show $H(t)$ is strictly decreasing, we want to show that

$$\frac{dH(t)}{dt} < 0$$

We take the time derivative of both sides of the equation

$$\begin{aligned}
\frac{dH(t)}{dt} &= \frac{d}{dt}\int_0^L u(t,x)^2 \ dx \\
&= \int_0^L \frac{d}{dt}u(t,x)^2 \ dx \\
&= \int_0^L 2u(t,x)u_t(t,x) \ dx \\
&= 2D\int_0^L 2u(t,x)u_{xx}(t,x) \ dx
\end{aligned}$$

This can be integrated by parts with

$$\begin{aligned}
u &= u(t,x) & v &= u_x(t,x) \\
du &= u_x(t,x) & du &= u_{xx}(t,x)dx
\end{aligned}$$

We note that the first part of the resulting equation will evaluate to 0 as $u_x(t,x) = 0$ at the boundaries

$$\begin{aligned}
\frac{dH(t)}{dt} &= 2D\left[u(t,x)u_x(t,x)\big|_0^L - \int_0^L u_x(t,x)^2\right] \\
&= -2D\int_0^L u_x(t,x)^2 \leq 0
\end{aligned}$$

We see that the derivative of $H(t) \leq 0$ for all $u(t,x)$ as the square of $u_x(t,x) \geq 0$. However if the integral over the range $[0, L]$ of $u_x(t,x)^2$ is 0, then $u_x(t,x)$ must equal zero, which requires that $u(t,x)$ is a constant. As we are looking for a non-constant solution, we know that $u_x(t,x) \neq 0$ and thus $\boxed{\frac{dH(t)}{dt} < 0 \text{ and the function is strictly decreasing}}$