

AMATH 481 / 581 Fall 2021  
**Homework 1 - Initial Value Problems**

**Submission open until 11:59:59pm Thursday October 21, 2021**

1. Consider the ODE

$$\frac{dy(t)}{dt} = -3y(t) \sin t, \quad y(t=0) = \frac{\pi}{\sqrt{2}},$$

which has the exact solution  $y(t) = \pi e^{3(\cos t - 1)}/\sqrt{2}$  (you can verify that). Implement the methods forward Euler and Heun's for this ODE to test the error as a function of  $\Delta t$ . In particular:

(a) Solve the ODE numerically using the forward Euler method:

$$y(t_{n+1}) = y(t_n) + \Delta t f(t_n, y(t_n))$$

with  $t = [0 : \Delta t : 5]$ , where  $\Delta t = 2^{-2}, 2^{-3}, 2^{-4}, \dots, 2^{-8}$ . For each of these  $\Delta t$  values calculate the error  $E = \text{mean}(\text{abs}(y_{\text{true}} - y_{\text{num}}))$  of the numerical method. Plot  $\log(\Delta t)$  on the  $x$  axis and  $\log(E)$  on the  $y$  axis. Using `polyfit`, find the slope of the best fit line through this data. This is the order of the forward Euler method.

**ANSWER:** Save your last numerical solution ( $\Delta t = 2^{-8}$ ) as a column vector in A1. Save the error values in a row vector with seven components in A2. Save the slope of the line in A3.

(b) Solve the ODE numerically using Heun's method:

$$y(t_{n+1}) = y(t_n) + \frac{\Delta t}{2} [f(t_n, y(t_n)) + f(t_n + \Delta t, y(t_n) + \Delta t f(t_n, y(t_n)))]$$

with  $t = [0 : \Delta t : 5]$ , where  $\Delta t = 2^{-2}, 2^{-3}, 2^{-4}, \dots, 2^{-8}$ . For each of these  $\Delta t$  values, calculate the error  $E = \text{mean}(\text{abs}(y_{\text{true}} - y_{\text{num}}))$  of the numerical method. Plot  $\log(\Delta t)$  on the  $x$  axis and  $\log(E)$  on the  $y$  axis. Using `polyfit`, find the slope of the best fit line through this data. This is the order of the Heun's method.

**ANSWER:** Save your last numerical solution ( $\Delta t = 2^{-8}$ ) as a column vector in A4. Save the error values in a row vector with seven components in A5. Save the slope of the line in A6.

2. Consider the van der Pol oscillator

$$\frac{d^2 y(t)}{dt^2} + \epsilon[y^2(t) - 1] \frac{dy(t)}{dt} + y(t) = 0$$

with  $\epsilon$  being a parameter.

(a) With  $\epsilon = 0.1$ , solve the equation for  $t = [0 : 0.5 : 32]$  using `ode45`. The initial conditions are  $y(t=0) = \sqrt{3}$  and  $dy(t=0)/dt = 1$ . Repeat this for  $\epsilon = 1$  and  $\epsilon = 20$ .

**ANSWER:** Save the solutions  $y(t)$  for different  $\epsilon$  as a matrix of 3 columns in A7.

(b) Using the time span  $t = [0, 32]$  (the step size for displaying the result is not specified), solve the van der Pol's equation with `ode45`. Use  $\epsilon = 1$  and the initial conditions  $y(t=0) = 2$  and

$$dy(t=0)/dt = \pi^2.$$

Below is an example on how to control the error tolerance TOL in `ode45`:

```
TOL = 1e-4;
options = odeset('AbsTol',TOL,'RelTol',TOL);
[T,Y] = ode45('rhs',tspan,y0,options);
```

Using the `diff` and `mean` commands on the vector `T` shown above, calculate the average step-size  $t$  needed to solve the problem for each of the following tolerance values:  $10^{-4}, 10^{-5}, \dots, 10^{-10}$ . Plot  $\log(\Delta t)$  on the  $x$  axis and  $\log(\text{TOL})$  on the  $y$  axis. Using `polyfit`, find the slope of the best fit line through this data. This is the order of the local truncation error of `ode45`. Repeat this with `ode23` and `ode113`.

**ANSWER:** The slopes should be saved in variables A8 - A10 for `ode45`, `ode23`, and `ode113` respectively.

3. To explore interaction between neurons, implement two Fitzhugh neurons coupled via linear coupling:

$$\begin{aligned}\frac{dv_1}{dt} &= -v_1^3 + (1 + a_1)v_1^2 - a_1v_1 - w_1 + I + \mathbf{d}_{12}\mathbf{v}_2 \\ \frac{dw_1}{dt} &= bv_1 - cw_1 \\ \frac{dv_2}{dt} &= -v_2^3 + (1 + a_2)v_2^2 - a_2v_2 - w_2 + I + \mathbf{d}_{21}\mathbf{v}_1 \\ \frac{dw_2}{dt} &= bv_2 - cw_2\end{aligned}$$

with parameters  $a_1 = 0.05, a_2 = 0.25$ ,  $b = c = 0.01$  and,  $I = 0.1$ . Start the simulations with the initial condition of  $(v_1(0), v_2(0)) = (0.1, 0.1)$  and  $(w_1(0), w_2(0)) = (0, 0)$  and use the `ode15s` solver. Set the interaction parameters such that  $d_{12}$  is negative and  $d_{21}$  is positive. What do you observe from the different graphical representations of the solutions?

**ANSWERS:** Set the interaction parameters to 5 different values

$(d_{12}, d_{21})$ :  $(0,0)$ ,  $(0,0.2)$ ,  $(-0.1,0.2)$ ,  $(-0.3,0.2)$ ,  $(-0.5,0.2)$ . For each interaction value solve the system for  $t = [0 : 0.5 : 100]$  and save the computed solution,  $(v_1, v_2, w_1, w_2)$ , in a  $201 \times 4$  matrix. Write out the 5 different solutions, each of which corresponds to an interaction parameter, in A11 - A15.