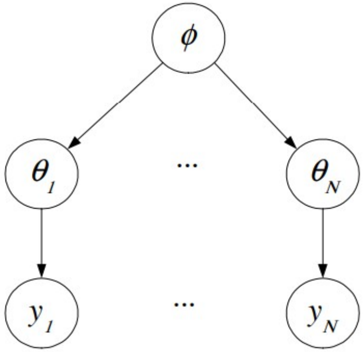
Normal Hierarchical Models



Statistics Foundations

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Motivation – Introduction

- Many stat applications involve hierarchical data, so hierarchical models are more appropriate, as it’s possible to structure some dependence.
- Having insufficient parameters, they tend to overfit.
- Can be used for “meta-analysis”: used for research in order to understand a relationship between different related experiments.

Hierarchical Models

- Some authors coin the term *Empirical Bayes* to the analysis using the data to estimate prior parameters.
- Exchangeability: if no information is given to distinguish any of the θj. Then, no order or grouping of the parameters can be made. Ignorance of this info implies exchangeability.

$$\mathrm{p}(\theta|\phi) = \prod_{j=1}^{J} p(\theta_j|\phi)$$

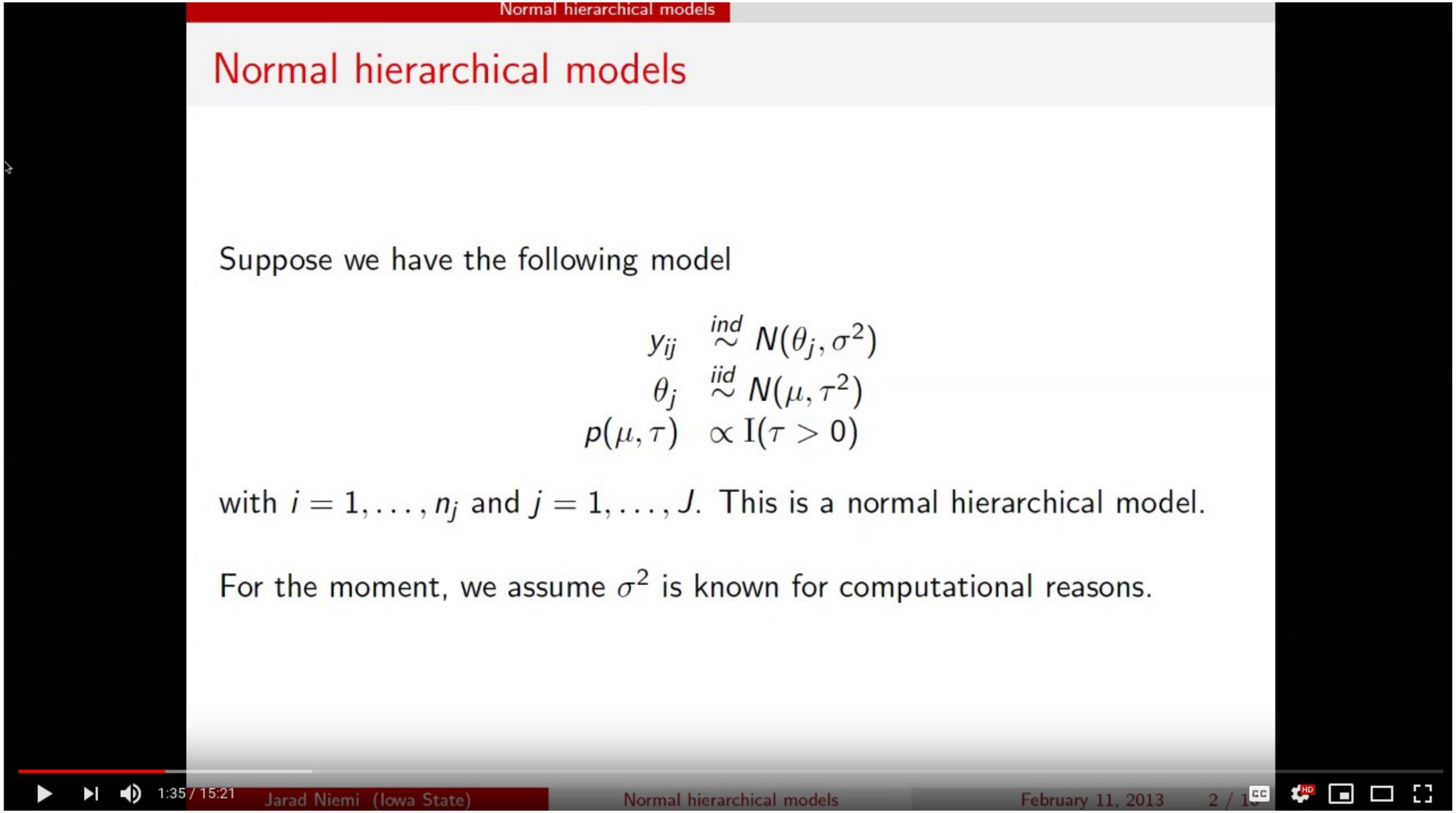
$$p(\theta) = \int [\prod_{j=1}^{J} p(\theta_j|\phi)] p(\theta_j|\phi) d\phi$$

Hierarchical Models

$$\text{Joint Posterior Distribution:}$$
$$\mathrm{p}(\phi, \theta) = p(\phi)p(\theta|\phi)$$
$$Hyperprior\ Distribution\ for\ \phi:$$
$$p(\phi, \theta|y) \propto p(\phi, \theta)p(y|\phi, \theta) = p(\phi, \theta)p(y|\theta)$$

- May use a diffuse distribution, if little is known.
- Should result in a posterior dist. that is proper.
- Should at least constrain the hyper params into a finite region.

Normal Hierarchical Models



https://youtu.be/MlDcNbVCGBg?t=4

Hierarchical Normal Distributions

- The marginal posterior distribution of ϕ can be computed algebraically using the conditional probability formula.
$$\mathrm{p}(\phi|y) = \frac{p(\phi,\theta|y)}{p(\theta|\phi,y)}$$
- The denominator has a normalizing factor that depends on ϕ and θ: this is the difficult part, as it depends on ϕ, y.
- Care must be taken to make sure that the proportionality constant (denominator) is actually a constant.
- Many times, a conjugate hierarchical scheme assumes normal sampling and normally distributed latent effects.

Estimating an Exchangeable Set of Params for a Normal Model

- We will show a simple normal hierarchical model: one way normal random effects model.
- Different mean for each group or experiment.
- Known variance, this assumption can be an adequate approximation at the sampling level.
- $\text{Data: } \mathrm{y}_i|\theta_j \sim N(\theta_j, \sigma^2) \ \ for \ i = 1, ..., n_j; j = 1, ...J$

$$Likelihood: \ \overline{y}_{.j}\theta_j = N(\theta_j, \sigma_j^2)$$

$$Analysis \ of \ variance: \ \overline{y_{.j}} = \frac{1}{n_j}\sum_{i=1}^{n_j} y_{ij}$$

Estimating an Exchangeable Set of Params for a Normal Model

- Pooled estimate: $\overline{y_{..}} = \dfrac{\sum_{j=1}^{J} \frac{\overline{y_{.j}}}{\sigma_j^2}}{\sum_{j=1}^{J} \frac{1}{\sigma_j^2}}$

- Traditionally, the analysis was to test differences among means. Choosing between the lesser.
- Alternatively we can use both: weighted combination: $\hat{\theta}_j = \lambda_j \overline{y_{.j}} + (1 - \lambda_j)\overline{y_{..}}$

Normal Hierarchical Model

- $$\mathrm{p}(\theta_1, ..., \theta_J|\mu, \tau) = \prod_{j=1}^{J} N(\theta_j|\mu, \tau^2)$$

  $$p(\theta_1, ..., \theta_J) = \int \prod_{j=1}^{J}[N(\theta_j|\mu, \tau^2)]p(\mu, \tau)d(\mu, \tau)$$

- We can assign a noninformative uniform hyperprior distribution to μ, given τ:

  $$p(\mu, \tau) = p(\mu|\tau)p(\tau) \propto p(\tau)$$

Joint Posterior Distribution

- Combining the sampling distribution for the observable y_{ij} we can express:

$$\mathrm{p}(\theta, \mu, \tau|y) \propto p(\mu, \tau)p(\theta|\mu, \tau)p(y|\theta)$$
$$\propto p(\mu, \tau)\prod_{j=1}^{J} N(\theta_j|\mu, \tau^2)\prod_{j=1}^{J} N(\overline{y_{.j}}|\theta_j, \sigma^2)$$

- We can say that θ parameters are independent of the prior distribution, then:

$$\theta_j|\mu, \tau, y \sim N(\hat{\theta}_j, V_j) \ldots \ being\ proper$$
$$where, \hat{\theta}_j = \frac{\overline{y.j}\sigma^{-2} + \mu\tau^{-2}}{\sigma_j^{-2} + \tau^{-2}}, V_j = \frac{1}{\sigma_j^{-2} + \tau^{-2}}$$

Marginal Posterior Distribution (Hyperparameters)

- Prev slide is only a partial solution. As μ and τ are unknown.
- In the normal models the marginal likelihood has a simple form (which is not the case in other distributions).

$$\overline{y_{.j}}|\mu,\tau \sim N(\mu, \sigma_j^2+\tau^2)$$

$$p(\mu,\tau|y) \propto p(\mu,\tau)\prod_{j=1}^{J} N(\overline{y_{.j}}|\mu,\sigma_j^2+\tau^2)$$

$$\hat{\mu} = \frac{\sum_{j=1}^{J}\frac{\overline{y_{.j}}}{\sigma_j^2+\tau^2}}{\sum_{j=1}^{J}\frac{1}{\sigma_j^2+\tau^2}}, V_\mu^{-1} = \sum_{j=1}^{J}\frac{1}{\sigma_j^2+\tau^2}$$

Posterior distribution of τ

- We can now get it:

$$\mathrm{p}(\tau|y) = \frac{p(\mu,\tau|y)}{p(\mu|\tau,y)}$$

$$\propto \frac{p(\tau)\prod_{j=1}^{J} N(\overline{y_{.j}}|\mu\sigma_j^2+\tau^2)}{N(\mu|\hat{\mu},V_\mu)}$$

- This holds true as any value of μ, so all the factors of μ must cancel when simplification is done.

- If we set μ to hat(μ)

$$\mathrm{p}(\tau|y) \propto \frac{p(\tau)\prod_{j=1}^{J} N(\overline{y_{.j}}|\hat{\mu}\sigma_j^2+\tau^2)}{N(\hat{\mu}|\hat{\mu},V_\mu)}$$

$$\propto p(\tau)V_\mu^{\frac{1}{2}}\prod_{j=1}^{J}(\sigma_j^2+\tau^2)^{-\frac{1}{2}}exp[-\frac{(\overline{y_{.j}}-\hat{\mu})^2}{2(\sigma_j^2+\tau^2)}]$$

Prior distribution of τ

- Prior distribution for τ should be assigned: a diffuse noninf. prior will be used for convenience.
- It should be a finite integral: $p(\tau) \propto 1$
- Other priors can be defined:
  - $p(log\ \tau) \propto 1$
  - Inverse chi-squared: being a natural choice for variance parameters.

Posterior Predictive Distributions

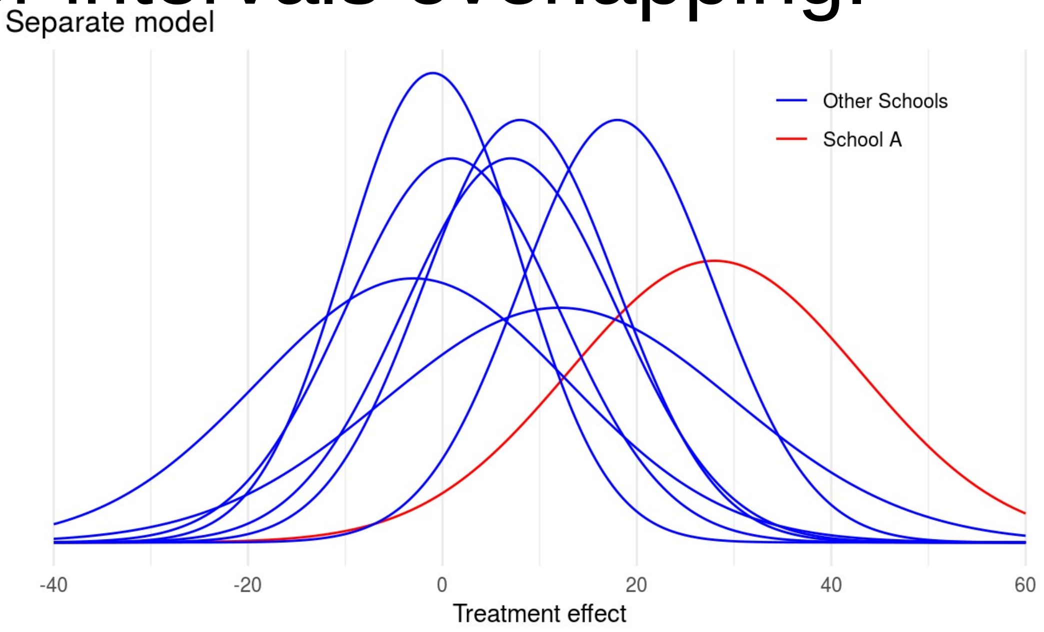
- There are 2 scenarios
  - Taking the future data from current batches
    $\theta = (\theta_1, ..., \theta_j)$
  - Future data from future batches (y_{new})
    $\tilde{\theta} = (\tilde{\theta}_1, ..., \tilde{\theta}_{\tilde{J}})$
    - Steps to follow in this scenario:
      (i) Draw (μ, τ) from the posterior, (ii) draw Ĵ new params from the population distribution $p(\tilde{\theta}_J|\mu, \tau)$, (iii) draw ŷ given $\tilde{\theta}$ from the distribution

Example

- SAT-Verbal is applied to 8 different schools with a coaching program.
  - Variable of interest: score, with values 200-800, mean 500, standard deviation 100

Example

- Separate estimates: It’s statistically difficult to distinguish between experiments: yielding 95% posterior intervals overlapping.

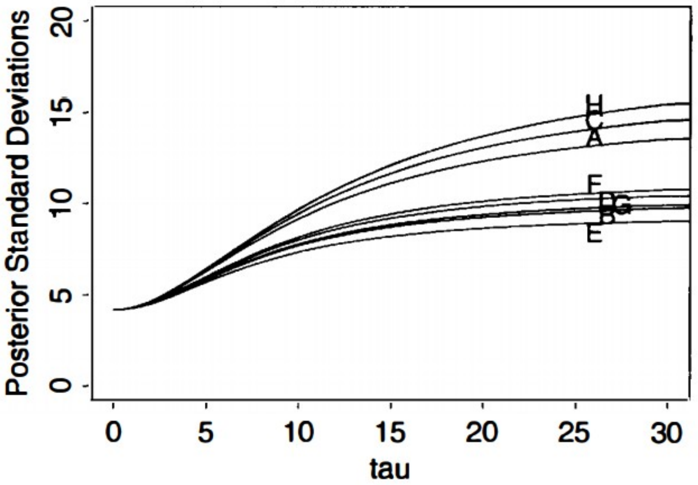


Example:

- Overlapping in the normal non pooled simple model suggests that we’re estimating the same quantity.
- Pooled estimate, in contrast, hypothesize that <u>all experiments have the same effect</u> and <u>produce independent estimates</u> of this <u>common effect</u>.

Example

- Anyway, the posterior mode of τ is on the boundary of its parameter space.
- The same for the joint posterior modal estimate.



Meta-Analysis

- It’s increasingly popular to summarize and integrate the findings of research studies.
- It’s a method for combining several parallel data sources.
- We introduce 0 subscripts for control groups, and 1 in treatment groups.

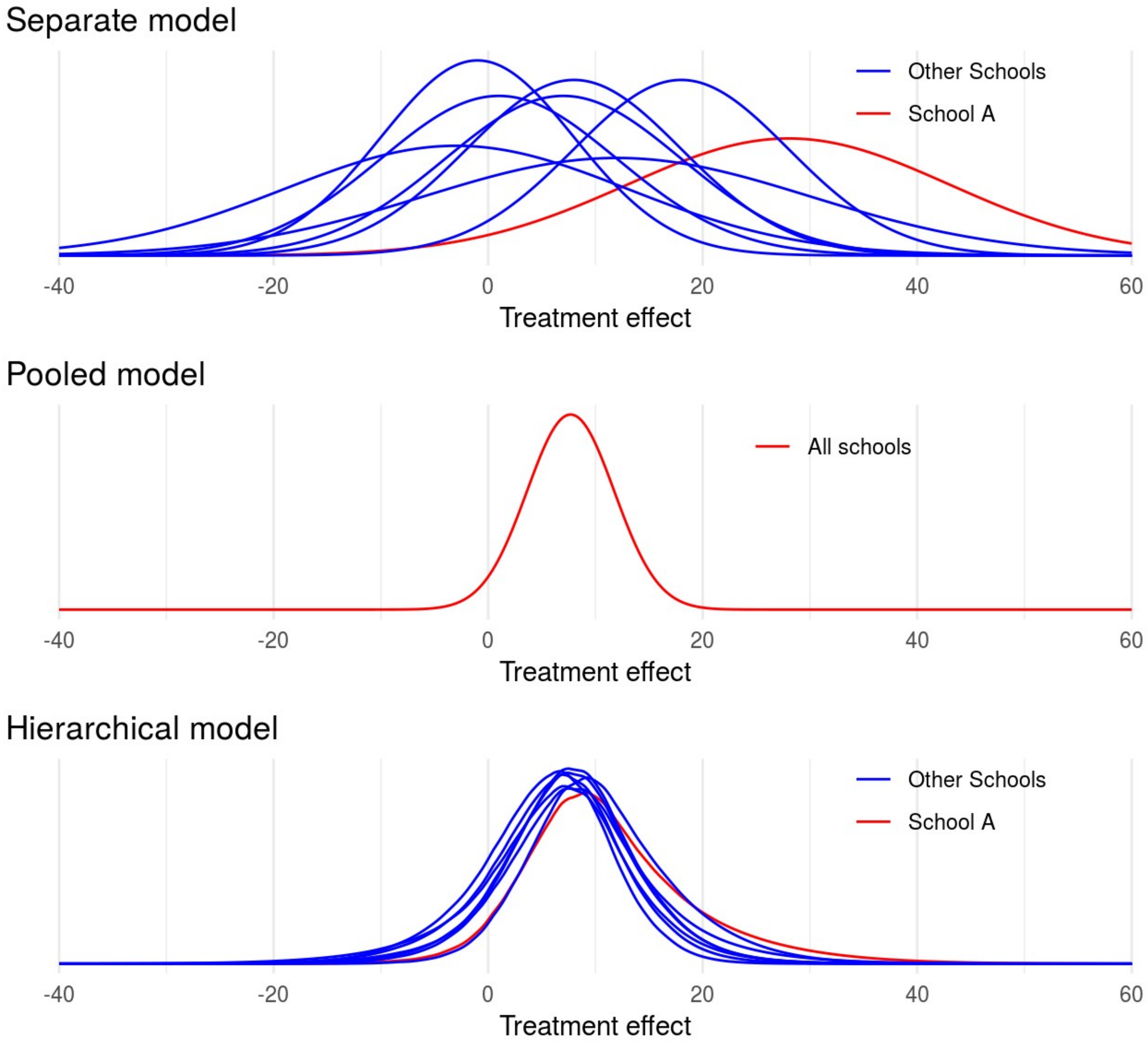
Example: Hierarchical Normal Model

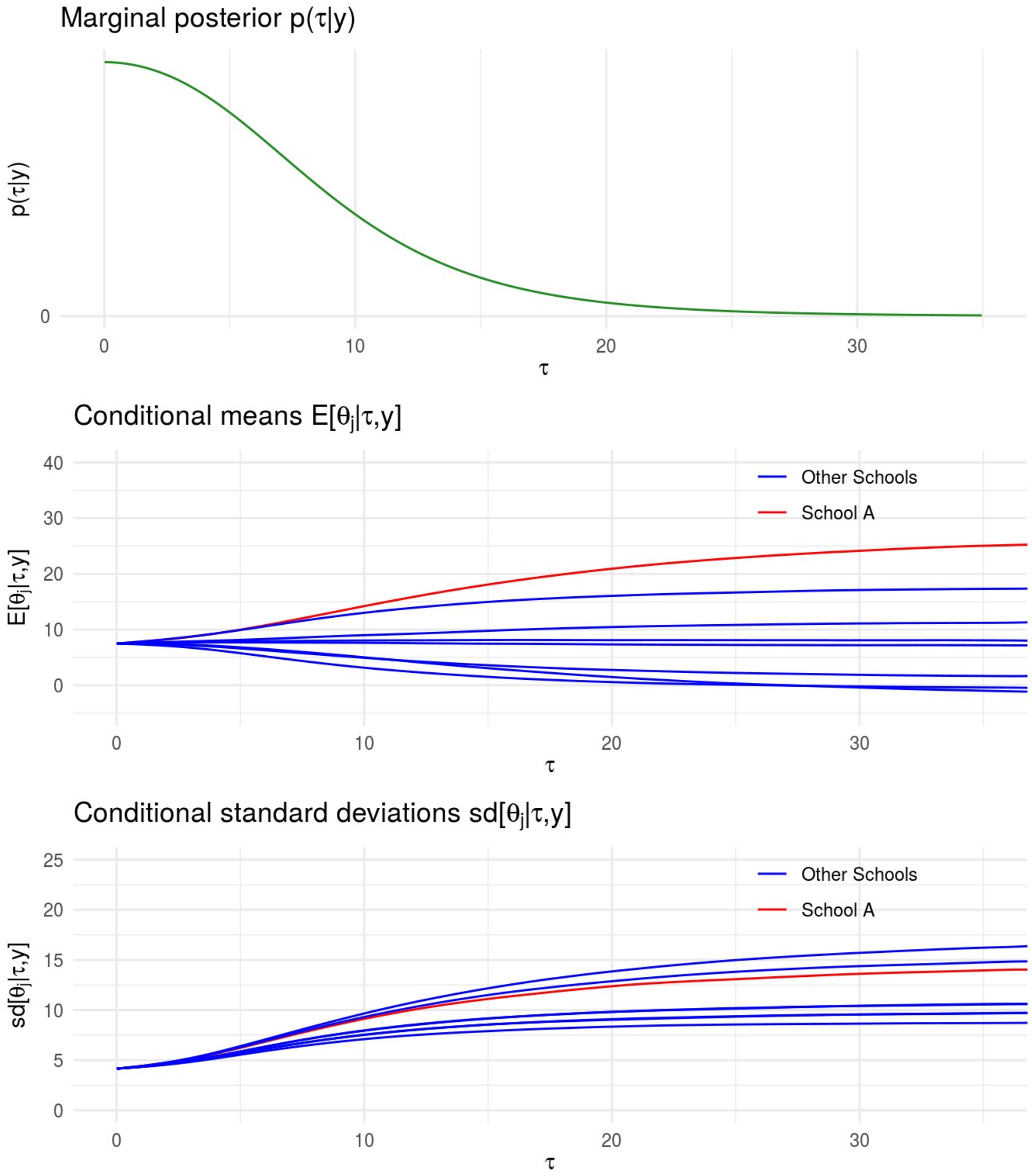
- $\sigma_j^2 = \frac{1}{y_{1j}} + \frac{1}{n_{1j}-y_{1j}} + \frac{1}{y_{0j}} + \frac{1}{n_{0j}-y_{0j}}$ $\quad Appr\ sampling\ variance$

$y_j = log(\frac{y_{ij}}{n_{1j}-y_{1j}}) - log\frac{y_{0j}}{n_{0j}-y_{0j}}$ $\quad empirical\ logits$

$y_j|\theta_j, \sigma_j^2 \sim N(\theta_j, \sigma_j^2), j = 1, \ldots, J$

- This method has a marginal posterior that peaks at nonzero value, which is plausible.
- The Expected value of μ is shrinked, compared with the non-hierarchical model. But the variance is accounted for, as it’s uncertain in the estimation.


Separate model
Other Schools
School A
-40
-20
0
20
40
60
Treatment effect
Pooled model
All schools
-40
-20
0
20
40
60
Treatment effect
Hierarchical model
Other Schools
School A
-40
-20
0
20
40
60
Treatment effect



Marginal posterior p(τ|y)
p(τ|y)
0
0
10
20
30
τ
Conditional means E[θj|τ,y]
E[θj|τ,y]
40
30
20
10
0
0
10
20
30
τ
Other Schools
School A
Conditional standard deviations sd[θj|τ,y]
sd[θj|τ,y]
25
20
15
10
5
0
0
10
20
30
τ
Other Schools
School A

BUGS Example

- https://philwebsurfer.github.io/fundstats/

References

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