

MATH-COMPUTING

December 21, 2023

1. **Question(MATH-12.10.5.17):** Let \mathbf{a} and \mathbf{b} be two unit vectors and θ is the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector.

(A) $\theta = \frac{\pi}{4}$

(B) $\theta = \frac{\pi}{3}$

(C) $\theta = \frac{\pi}{2}$

(D) $\theta = \frac{2\pi}{3}$

solution:

Given,

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \quad (1)$$

$$\|\mathbf{a} + \mathbf{b}\| = 1 \quad (2)$$

Squaring on both sides, we get

$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \quad (3)$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top \mathbf{b} = 1 \quad (4)$$

Substituting eq (4) in eq (1), we get

$$\implies 1 + 1 + 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = 1 \quad (5)$$

$$\implies 2 + 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = 1 \quad (6)$$

$$\implies 2\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = -1 \quad (7)$$

$$\implies \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta = \frac{-1}{2} \quad (8)$$

Substituting eq (1) in eq (8), we get

$$\Rightarrow \cos \theta = \frac{-1}{2} \quad (9)$$

$$\Rightarrow \theta = \frac{2\pi}{3} \quad (10)$$

Assuming the co-ordinates:

$$A = \begin{pmatrix} -2.31 \\ 3.98 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, c = \begin{pmatrix} 1.69 \\ 0 \end{pmatrix} \quad (11)$$

$$(12)$$

To find angle B in a triangle ABC:

$$\cos B \triangleq \frac{(\mathbf{A} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B})}{\|\mathbf{A} - \mathbf{B}\| \|\mathbf{C} - \mathbf{B}\|} \quad (13)$$

$$(14)$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} -2.31 \\ 3.98 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2.31 \\ 3.98 \end{pmatrix} \quad (15)$$

$$\mathbf{C} - \mathbf{B} = \begin{pmatrix} 1.69 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1.69 \\ 0 \end{pmatrix} \quad (16)$$

$$(17)$$

$$\|\mathbf{A} - \mathbf{B}\| \triangleq \sqrt{(\mathbf{A} - \mathbf{B})^\top (\mathbf{A} - \mathbf{B})} = \sqrt{(-2.31 - 3.98) \begin{pmatrix} -2.31 \\ 3.98 \end{pmatrix}} = 4.60 \quad (18)$$

$$(19)$$

$$\|\mathbf{C} - \mathbf{B}\| \triangleq \sqrt{(\mathbf{C} - \mathbf{B})^\top (\mathbf{C} - \mathbf{B})} = \sqrt{(1.69 - 0) \begin{pmatrix} 1.69 \\ 0 \end{pmatrix}} = 1.69 \quad (20)$$

(21)

Therefore:

$$\cos B = \frac{(-2.31 - 3.98) \begin{pmatrix} 1.69 \\ 0 \end{pmatrix}}{(4.60)(1.69)} = \frac{-3.903}{7.774} = 0.501 \quad (22)$$

(23)

$$B = \cos^{-1}(0.5) = 120^\circ \quad (24)$$

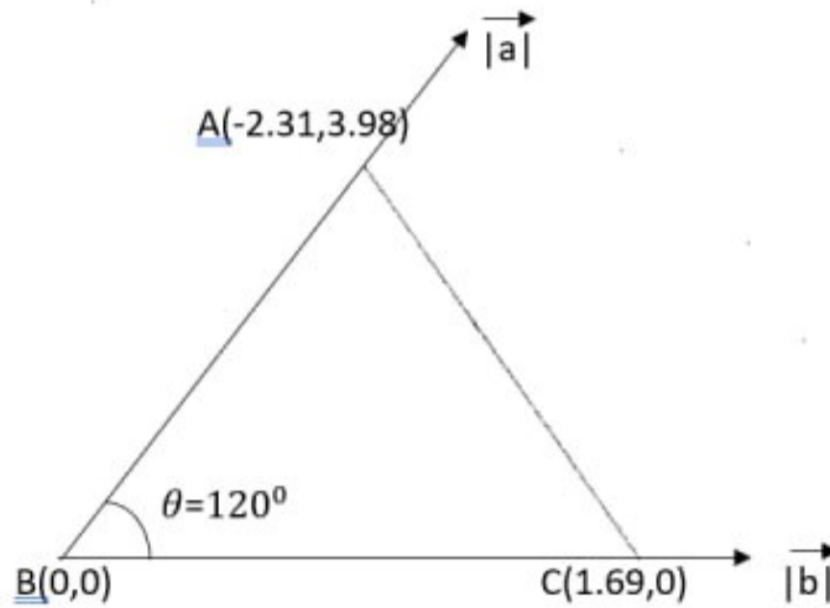


Figure 1: vectors