Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

[label=1.4.0.,ref=1.4.0]The equation of the perpendicular bisector of BC is

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.1}$$

Substitute numerical values and find the equations of the perpendicular bisectors of AB, BC and CA.

Solution:

1. (a) $\mathbf{BC}$: given equation for the perpendicular bisector of $\mathbf{BC}$:

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{1.2}$$

On substituting the values,

$$\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}} = \begin{pmatrix} \frac{-3}{2} \\ 2 \end{pmatrix} \tag{1.3}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.4}$$

$$\tag{1.5}$$

solving using matrix multiplication

$$(\mathbf{B}-\mathbf{C})^{\top}\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = 0 \tag{1.6}$$

$$(\mathbf{B}-\mathbf{C})^{\top} = \begin{pmatrix} -3 & -4 \end{pmatrix} \tag{1.7}$$

$$(\mathbf{B}-\mathbf{C})^{\top}\left(\frac{\mathbf{B}+\mathbf{C}}{2}\right) = \begin{pmatrix} -3 & -4 \end{pmatrix}\begin{pmatrix} \frac{-3}{2} \\ 2 \end{pmatrix} \tag{1.8}$$

$$= \frac{-7}{2} \tag{1.9}$$

Therefore perpendicular bisector of **BC** is

$$\begin{pmatrix} -3 & -4 \end{pmatrix} \mathbf{x} = \frac{-7}{2} \tag{1.10}$$

(b) **AB**: similarly the equation for the perpendicular bisector of **AB**:

$$\left(\mathbf{x} - \frac{\mathbf{A}+\mathbf{B}}{2}\right)(\mathbf{A}-\mathbf{B}) = 0 \tag{1.11}$$

On substituting the values,

$$\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} = \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} \tag{1.12}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.13}$$

$$\tag{1.14}$$

solving using matrix multiplication

$$(\mathbf{A}-\mathbf{B})^\top \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = 0 \tag{1.15}$$

$$(\mathbf{A}-\mathbf{B})^\top = \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{1.16}$$

$$(\mathbf{A}-\mathbf{B})^\top \left(\frac{\mathbf{A}+\mathbf{B}}{2}\right) = \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} \tag{1.17}$$

$$= \frac{1}{2} \tag{1.18}$$

Therefore perpendicular bisector of **AB** is

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = \frac{1}{2} \tag{1.19}$$

(c) **CA**: similarly the equation for the perpendicular bisector of **CA**:

$$\left(\mathbf{x} - \frac{\mathbf{C} + \mathbf{A}}{2} \right) (\mathbf{C} - \mathbf{A}) = 0 \tag{1.20}$$

On substituting the values,

$$\frac{\mathbf{C} + \mathbf{A}}{\mathbf{2}} = \begin{pmatrix} \frac{1}{2} \\ \frac{7}{2} \end{pmatrix} \tag{1.21}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.22}$$

$$\tag{1.23}$$

solving using matrix multiplication

$$(\mathbf{C} - \mathbf{A})^{\top} \left(\frac{\mathbf{C} + \mathbf{A}}{2} \right) = 0 \tag{1.24}$$

$$(\mathbf{C} - \mathbf{A})^{\top} = \begin{pmatrix} -1 & 1 \end{pmatrix} \tag{1.25}$$

$$(\mathbf{C} - \mathbf{A})^{\top} \left(\frac{\mathbf{C} + \mathbf{A}}{2} \right) = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{7}{2} \end{pmatrix} \tag{1.26}$$

$$= 3 \tag{1.27}$$

Therefore perpendicular bisector of **BC** is

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = 3 \tag{1.28}$$

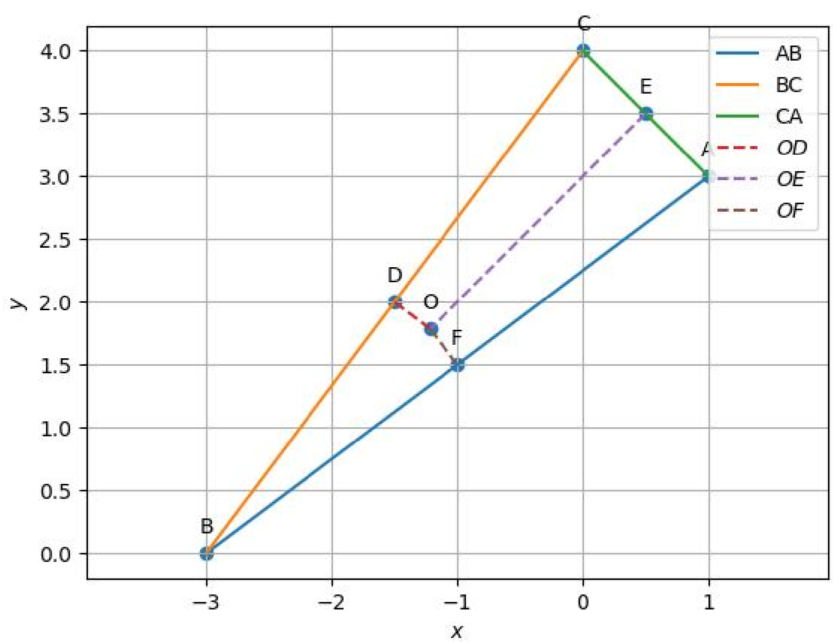

Figure 1.1: Plot of the perpendicular bisectors

2. Find the intersection **O** of the perpendicular bisectors of AB and AC.

   Solution:

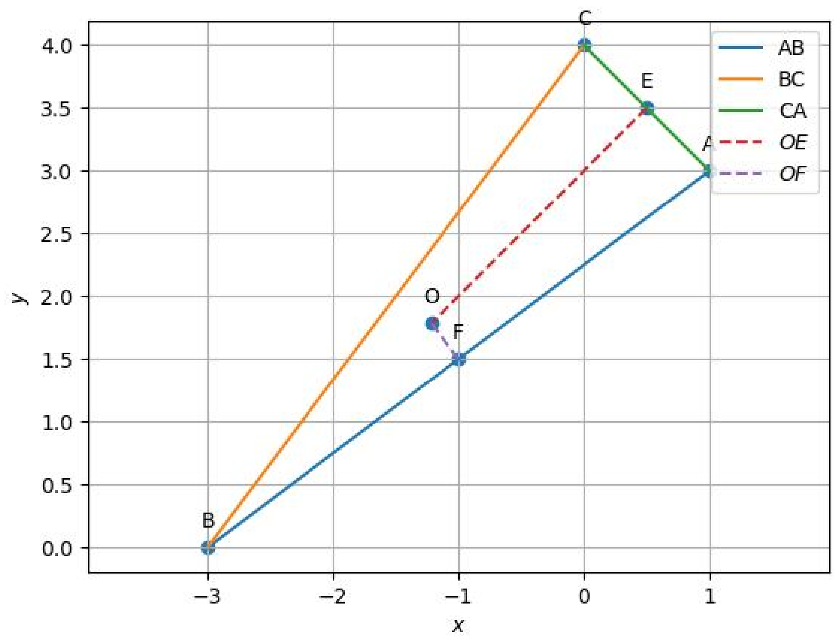

Figure 1.2: $\mathbf{O}-\mathbf{E}$ and $\mathbf{O}-\mathbf{F}$ are perpendicular bisectors of $\mathbf{A}-\mathbf{C}$ and $\mathbf{A}-\mathbf{B}$ respectively

Given,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{2.1}$$

Vector equation of perpendicular bisector of $\mathbf{A} - \mathbf{B}$ is

$$(\mathbf{A} - \mathbf{B})^\top \left(\mathbf{x} - \frac{\mathbf{A} + \mathbf{B}}{2} \right) = 0 \tag{2.2}$$

where,

$$\mathbf{A}+\mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{2.3}$$

$$= \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.4}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{2.5}$$

$$= \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{2.6}$$

$$\implies (\mathbf{A}-\mathbf{B})^\top = \begin{pmatrix} 4 & 3 \end{pmatrix} \tag{2.7}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{F}$ is

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \left(\mathbf{x} - \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} \right) = 0 \tag{2.8}$$

$$\implies \begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ \frac{3}{2} \end{pmatrix} \tag{2.9}$$

Performing matrix multiplication yields

$$\begin{pmatrix} 4 & 3 \end{pmatrix} \mathbf{x} = \frac{1}{2} \tag{2.10}$$

Vector equation of perpendicular bisector of $\mathbf{A}-\mathbf{C}$ is

$$(\mathbf{A}-\mathbf{C})^\top \left(\mathbf{x}-\frac{\mathbf{A}+\mathbf{C}}{2}\right)=0 \tag{2.11}$$

where,

$$\mathbf{A}+\mathbf{C}=\begin{pmatrix}1\\3\end{pmatrix}+\begin{pmatrix}0\\4\end{pmatrix} \tag{2.12}$$

$$=\begin{pmatrix}1\\7\end{pmatrix} \tag{2.13}$$

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}0\\4\end{pmatrix} \tag{2.14}$$

$$=\begin{pmatrix}1\\-1\end{pmatrix} \tag{2.15}$$

$$\implies (\mathbf{A}-\mathbf{C})^\top=\begin{pmatrix}1 & -1\end{pmatrix} \tag{2.16}$$

$\therefore$ The vector equation of $\mathbf{O}-\mathbf{E}$ is

$$\begin{pmatrix}1 & -1\end{pmatrix}\left(\mathbf{x}-\frac{1}{2}\begin{pmatrix}1\\7\end{pmatrix}\right)=0 \tag{2.17}$$

$$\implies \begin{pmatrix}1 & -1\end{pmatrix}\mathbf{x}=\frac{1}{2}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1\\7\end{pmatrix} \tag{2.18}$$

Performing matrix multiplication yields

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{x} = -3 \tag{2.19}$$

Thus,

$$\begin{pmatrix} 4 & 3 & \frac{3}{2} \\ 1 & -1 & \frac{9}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow \frac{-1}{6} R_1} \begin{pmatrix} \frac{-2}{3} & \frac{-1}{2} & -1 \\ 1 & -1 & -3 \end{pmatrix} \tag{2.20}$$

$$\begin{pmatrix} \frac{-2}{3} & \frac{-1}{2} & -1 \\ 1 & -1 & -3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \begin{pmatrix} \frac{-2}{3} & \frac{-1}{2} & -1 \\ \frac{-5}{2} & \frac{1}{4} & 1 \end{pmatrix} \tag{2.21}$$

$$\begin{pmatrix} \frac{-2}{3} & \frac{-1}{2} & -1 \\ \frac{-5}{2} & \frac{1}{4} & 1 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow \frac{-1}{2} R_2} \begin{pmatrix} \frac{-2}{3} & \frac{-1}{2} & -1 \\ \frac{-5}{4} & \frac{1}{4} & 1 \end{pmatrix} \tag{2.22}$$

$$\therefore \begin{pmatrix} \frac{-2}{3} & \frac{-1}{2} \\ \frac{-5}{4} & \frac{1}{4} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.23}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{5}{4} \\ \frac{-5}{4} \end{pmatrix} \tag{2.24}$$

Therefore, the point of intersection of perpendicular bisectors of $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$ is $\mathbf{O} = \begin{pmatrix} \frac{5}{4} \\ \frac{-5}{4} \end{pmatrix}$

3. Verify that $\mathbf{O}$ satisfies (1.1). $\mathbf{O}$ is known as the circumcentre.

Solution: From the previous question we get,

$$\mathbf{O} = \begin{pmatrix} \frac{5}{4} \\ \frac{-5}{4} \end{pmatrix} \tag{3.1}$$

$$\left(\mathbf{x} - \frac{\mathbf{B}+\mathbf{C}}{2}\right)(\mathbf{B}-\mathbf{C}) = 0 \tag{3.2}$$

when substituted in the above equation,

$$= \left(\mathbf{O} - \frac{\mathbf{B}+\mathbf{C}}{2}\right).(\mathbf{B}-\mathbf{C}) \tag{3.3}$$

$$= \left(\begin{pmatrix} \frac{5}{4} \\ \frac{-5}{4} \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -3 \\ 4 \end{pmatrix}\right)^{\top} \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{3.4}$$

$$= \begin{pmatrix} \frac{-1}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{3.5}$$

$$= \frac{-9}{4} \tag{3.6}$$

It is hence proved that $\mathbf{O}$ satisfies the equation (1.1)

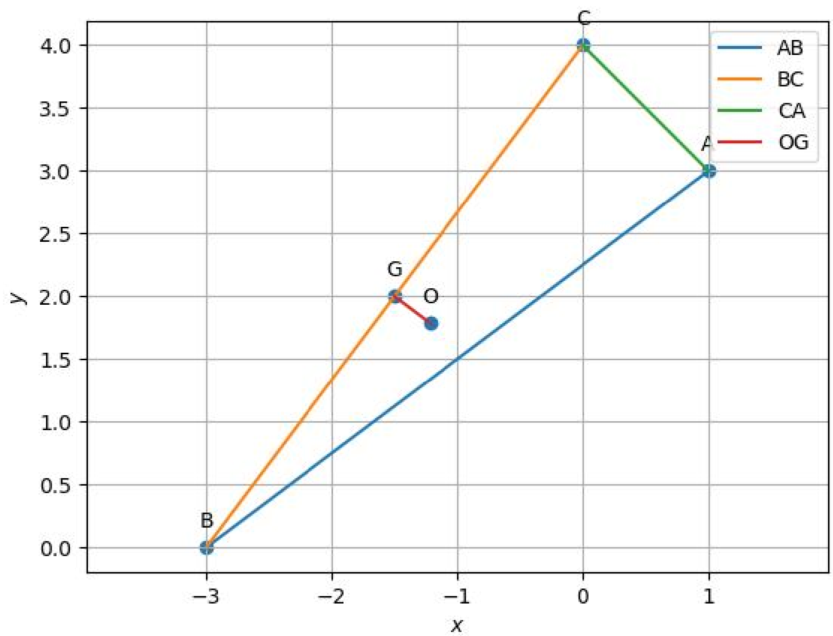

Figure 1.3: Circumcenter plotted using python

4. Verify that

$$OA = OB = OC \tag{4.1}$$

Solution: Given

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{4.2}$$

$$\mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{4.3}$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{4.4}$$

From problem-1.4.2 :

$$O = \begin{pmatrix} \frac{-5}{4} \\ \frac{5}{4} \end{pmatrix} \tag{4.5}$$

$$= \begin{pmatrix} -1.25 \\ 1.25 \end{pmatrix} \tag{4.6}$$

(a)

$$OA = \sqrt{(\mathbf{O} - \mathbf{A})^\top (\mathbf{O} - \mathbf{A})} \tag{4.7}$$

$$= \sqrt{\begin{pmatrix} \frac{1}{4} & \frac{-17}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{-17}{4} \end{pmatrix}} \tag{4.8}$$

$$= \sqrt{\frac{145}{4}} \tag{4.9}$$

$$= \frac{\sqrt{145}}{2} \tag{4.10}$$

(b)

$$OB = \sqrt{(\mathbf{O}-\mathbf{B})^\top(\mathbf{O}-\mathbf{B})} \tag{4.11}$$

$$= \sqrt{\begin{pmatrix} \frac{17}{4} & \frac{-5}{4} \end{pmatrix} \begin{pmatrix} \frac{17}{4} \\ \frac{-5}{4} \end{pmatrix}} \tag{4.12}$$

$$= \sqrt{\frac{314}{4}} \tag{4.13}$$

$$= \frac{\sqrt{314}}{2} \tag{4.14}$$

(c)

$$OC = \sqrt{(\mathbf{O}-\mathbf{C})^\top(\mathbf{O}-\mathbf{C})} \tag{4.15}$$

$$= \sqrt{\begin{pmatrix} \frac{5}{4} & \frac{-21}{4} \end{pmatrix} \begin{pmatrix} \frac{5}{4} \\ \frac{-21}{4} \end{pmatrix}} \tag{4.16}$$

$$= \sqrt{\frac{466}{4}} \tag{4.17}$$

$$= \sqrt{\frac{233}{2}} \tag{4.18}$$

$$\tag{4.19}$$

From above,

$$OA = OB = OC \tag{4.20}$$

Hence verified.

5. Draw the circle with centre at **O** and radius

$$R = OA \tag{5.1}$$

This is known as the circumradius.

Solution: Given

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{5.2}$$

$$\mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{5.3}$$

$$\mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{5.4}$$

From Q1.4.2, the circumcentre is

$$\mathbf{O} = \begin{pmatrix} \frac{5}{4} \\ \frac{-5}{4} \end{pmatrix} \tag{5.5}$$

Now we will calculate the radius,

$$
\begin{aligned}
R &= OA && (5.6)\\
&= \|\mathbf{A} - \mathbf{O}\| && (5.7)\\
&= \left\| \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{5}{4} \\ \frac{-5}{4} \end{pmatrix} \right\| && (5.8)\\
&= \left\| \begin{pmatrix} \frac{1}{4} \\ \frac{-17}{4} \end{pmatrix} \right\| && (5.9)\\
&= \sqrt{\begin{pmatrix} \frac{1}{4} & \frac{-17}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{-17}{4} \end{pmatrix}} && (5.10)\\
&= \frac{\sqrt{145}}{2} && (5.11)
\end{aligned}
$$

see Fig. 1.4

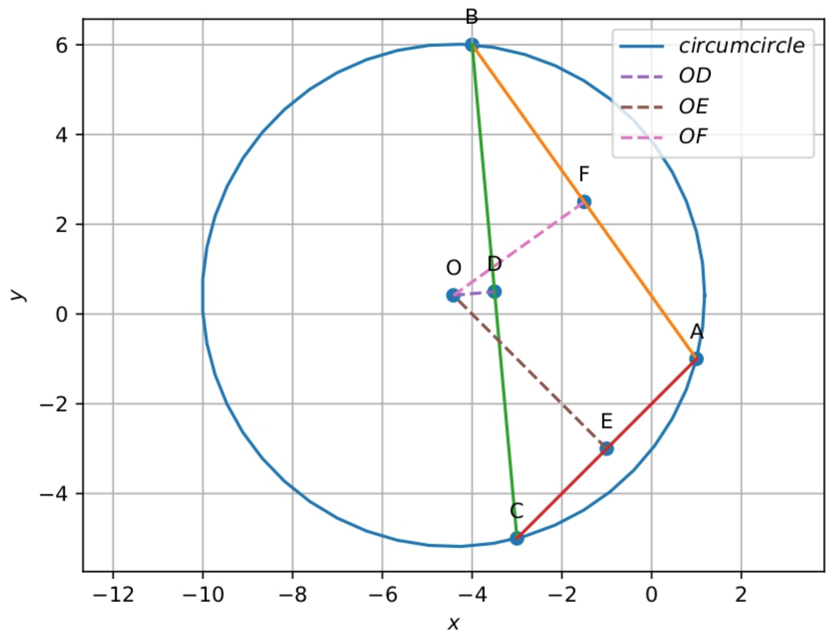

Figure 1.4: circumcircle of Triangle ABC with centre O

6. Verify that

$$\angle BOC = 2\angle BAC. \tag{6.1}$$

Solution:

(a) To find the value of $\angle BOC$:

$$\mathbf{B}-\mathbf{O}=\begin{pmatrix}\frac{-17}{4}\\ \\ \frac{5}{4}\end{pmatrix} \tag{6.2}$$

$$\mathbf{C}-\mathbf{O}=\begin{pmatrix}-\frac{-5}{4}\\ \\ \frac{21}{4}\end{pmatrix} \tag{6.3}$$

$$\implies (\mathbf{B}-\mathbf{O})^{\top}(\mathbf{C}-\mathbf{O})=\frac{5}{4} \tag{6.4}$$

$$\implies \|\mathbf{B}-\mathbf{O}\|=\sqrt{\frac{157}{8}} \tag{6.5}$$

$$\|\mathbf{C}-\mathbf{O}\|=\sqrt{\frac{233}{8}} \tag{6.6}$$

Thus,

$$\cos BOC=\frac{(\mathbf{B}-\mathbf{O})^{\top}(\mathbf{C}-\mathbf{O})}{\|\mathbf{B}-\mathbf{O}\|\,\|\mathbf{C}-\mathbf{O}\|}=\frac{10}{191} \tag{6.7}$$

$$\implies \angle BOC=\cos^{-1}\left(\frac{10}{191}\right) \tag{6.8}$$

$$=1.518 \tag{6.9}$$

Taking the reflex of above angle we get

$$\angle BOC=360-1.518 \tag{6.10}$$

$$=358.482 \tag{6.11}$$

(b) To find the value of $\angle BAC$:

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{6.12}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{6.13}$$

$$\implies (\mathbf{B} - \mathbf{A})^{\top} (\mathbf{C} - \mathbf{A}) = -6 \tag{6.14}$$

$$\|\mathbf{B} - \mathbf{A}\| = \sqrt{25} = 5 \, \|\mathbf{C} - \mathbf{A}\| = \sqrt{2} \tag{6.15}$$

Thus,

$$\cos BAC = \frac{(\mathbf{B} - \mathbf{A})^{\top} (\mathbf{C} - \mathbf{A})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{C} - \mathbf{A}\|} = \frac{1}{5\sqrt{2}} \tag{6.16}$$

$$\implies \angle BAC = \cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) \tag{6.17}$$

$$= 1.4288 \tag{6.18}$$

$$2 \times \angle BAC = 2.8576 \tag{6.19}$$

From (6.18) and (**??**),

$$2 \times \angle BAC = \angle BOC \tag{6.20}$$

Hence Verified

7. Let

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{7.1}$$

Find θ if

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{7.2}$$

Solution:

$$\mathbf{C} - \mathbf{O} = \begin{pmatrix} \frac{-5}{4} \\ \frac{21}{4} \end{pmatrix} \tag{7.3}$$

$$\mathbf{A} - \mathbf{O} = \begin{pmatrix} \frac{-1}{4} \\ \frac{17}{4} \end{pmatrix} \tag{7.4}$$

$$\mathbf{P} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{7.5}$$

$$\mathbf{C} - \mathbf{O} = \mathbf{P}\left(\mathbf{A} - \mathbf{O}\right) \tag{7.6}$$

Now from (7.6)

$$\begin{pmatrix} \frac{-5}{4} \\ \frac{21}{4} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{-1}{4} \\ \frac{17}{4} \end{pmatrix} \tag{7.7}$$

solving using matrix multiplication,we get

$$\begin{pmatrix} \frac{-5}{4} \\ \frac{21}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\cos\theta + \frac{17}{4}\sin\theta \\ \frac{-1}{4}\sin\theta + \frac{17}{4}\cos\theta \end{pmatrix} \tag{7.8}$$

Comparing on Both sides ,we get

$$\frac{1}{4}\cos\theta + \frac{17}{4}\sin\theta = \frac{-5}{4} \tag{7.9}$$

$$\frac{1}{4}\sin\theta + \frac{17}{4}\cos\theta = \frac{21}{4} \tag{7.10}$$

On solving equations (7.9) and (7.10)

$$\cos\theta = \frac{2900}{901} \tag{7.11}$$

$$\sin\theta = \frac{-53}{145} \tag{7.12}$$

$$\theta = \cos^{-1}\frac{2900}{901} \tag{7.13}$$

$$= 0.99 \tag{7.14}$$

$$\therefore \theta = 0.99 \tag{7.15}$$