Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

1.5. Angular Bisector

1.6. Matrix

The matrix of the veritices of the triangle is defined as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \tag{1.2}$$

$$= \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \tag{1.3}$$

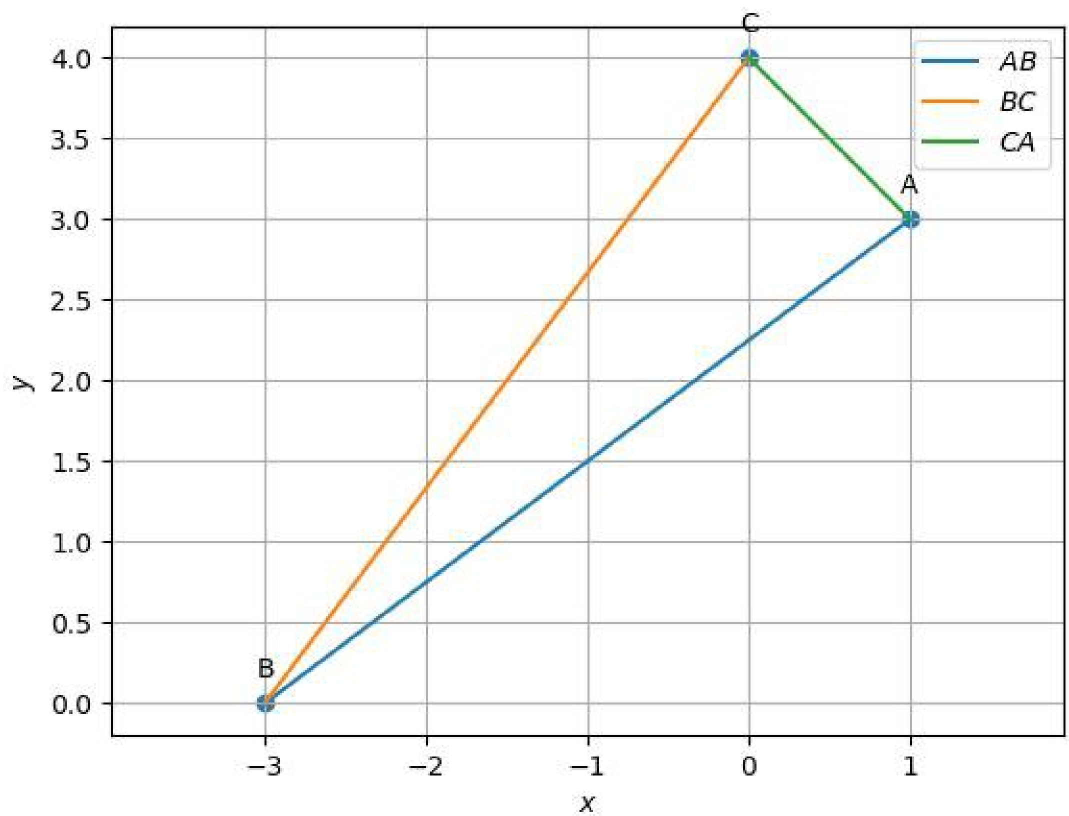

Figure 1.1: $\triangle$ ABC

1.6.1. Vectors

[label=1.6.1.0.,ref=1.6.1.0]Obtain the direction matrix of the sides of $\triangle ABC$ defined as

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} - \mathbf{B} & \mathbf{B} - \mathbf{C} & \mathbf{C} - \mathbf{A} \end{pmatrix} \tag{1.1}$$

Solution:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}-\mathbf{B} & \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} \end{pmatrix} \tag{1.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.3}$$

$$= \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \tag{1.4}$$

Using Matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 4 & -3 & -1 \\ 3 & -4 & 1 \end{pmatrix} \tag{1.5}$$

where the second matrix above is known as a circulant matrix. Note that the 2nd and 3rd row of the above matrix are circular shifts of the 1st row. Obtain the normal matrix of the sides of $\triangle ABC$

Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.1}$$

the normal matrix is obtained as

$$\mathbf{N} = \mathbf{R}\mathbf{M} \tag{2.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -3 & -1 \\ 3 & -4 & 1 \end{pmatrix} \tag{2.3}$$

Using matrix multiplication

$$\mathbf{N} = \begin{pmatrix} -3 & 4 & -1 \\ 4 & -3 & -1 \end{pmatrix} \tag{2.4}$$

Obtain a, b, c.

Solution: The sides vector is obtained as

$$\mathbf{d} = \sqrt{\text{diag}(\mathbf{M}^\top \mathbf{M})} \tag{3.1}$$

$$\mathbf{M}^\top \mathbf{M} = \begin{pmatrix} 4 & 3 \\ -3 & -4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 & -1 \\ 3 & -4 & 1 \end{pmatrix} \tag{3.2}$$

Using matrix multiplication

$$\mathbf{M} = \begin{pmatrix} 25 & -24 & -1 \\ -24 & 25 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{3.3}$$

$$\mathbf{d} = \sqrt{\operatorname{diag}\left(\begin{pmatrix} 25 & -24 & -1 \\ -24 & 25 & -1 \\ -1 & -1 & 2 \end{pmatrix}\right)} \tag{3.4}$$

$$= \begin{pmatrix} 0 & \sqrt{3} & \sqrt{49} \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{3} & 7 \end{pmatrix} \tag{3.5}$$

Obtain the constant terms in the equations of the sides of the triangle.

Solution: The constants for the lines can be expressed in vector form as

$$\mathbf{c} = \operatorname{diag}\left\{\left(\mathbf{N}^\top\mathbf{P}\right)\right\} \tag{4.1}$$

$$\mathbf{N}^\top\mathbf{P} = \begin{pmatrix} -3 & 4 \\ 4 & -3 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \tag{4.2}$$

$$\tag{4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} 9 & 9 & 16 \\ -5 & -12 & -12 \\ 4 & 3 & -4 \end{pmatrix} \tag{4.4}$$

$$\mathbf{c} = \operatorname{diag}\left(\begin{pmatrix} 9 & 9 & 16 \\ -5 & -12 & -12 \\ -4 & 3 & -4 \end{pmatrix}\right) \tag{4.5}$$

$$= \begin{pmatrix} 21 & 5 & -1 \end{pmatrix} \tag{4.6}$$

1.6.2. Median

[label=1.6.2.0.,ref=1.6.2.0]Obtain the mid point matrix for the sides of the triangle

Solution:

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.1}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{1.2}$$

Using matrix multiplication

$$\begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} & -1 \\ 2 & \frac{7}{2} & \frac{3}{2} \end{pmatrix} \tag{1.3}$$

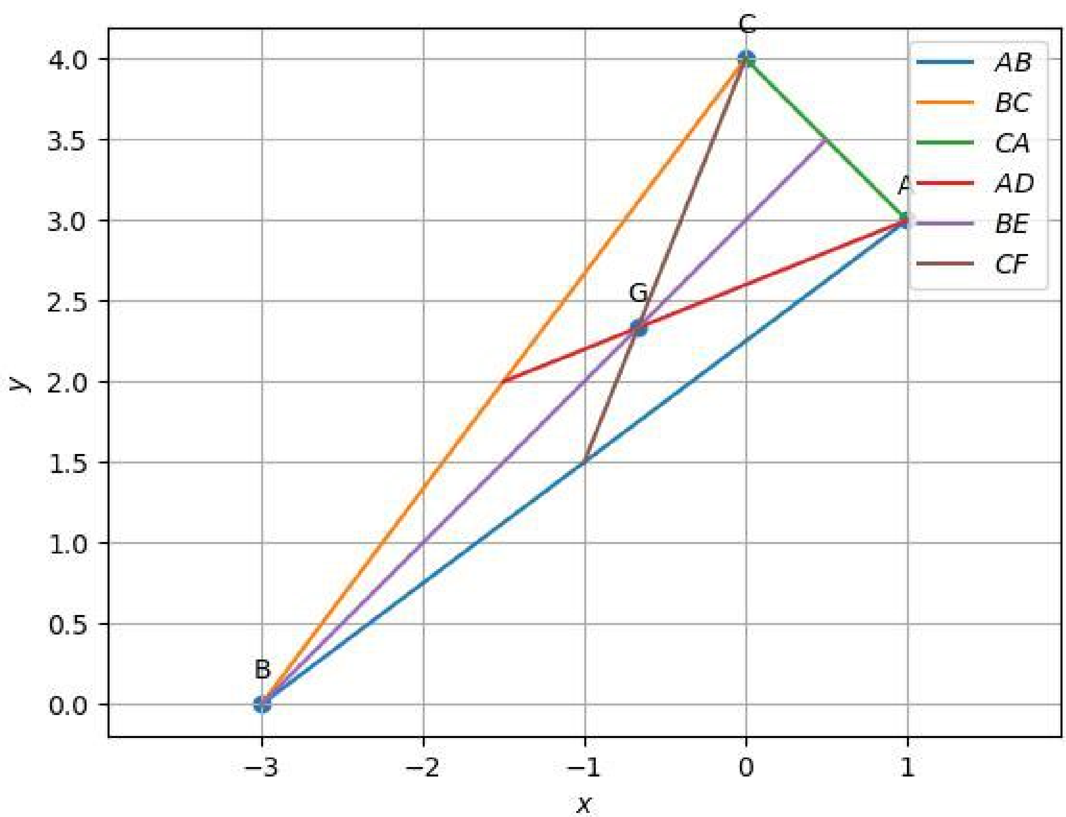

Figure 1.2: mid-points

3.

2. Obtain the median direction matrix.

   Solution: The median direction matrix is given by

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{A}-\mathbf{D} & \mathbf{B}-\mathbf{E} & \mathbf{C}-\mathbf{F} \end{pmatrix} \tag{2.1}$$

$$= \begin{pmatrix} \mathbf{A}-\frac{\mathbf{B}+\mathbf{C}}{2} & \mathbf{B}-\frac{\mathbf{C}+\mathbf{A}}{2} & \mathbf{C}-\frac{\mathbf{A}+\mathbf{B}}{2} \end{pmatrix} \tag{2.2}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{2.3}$$

$$= \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \tag{2.4}$$

   Using matrix multiplication

$$\mathbf{M}_1 = \begin{pmatrix} \frac{5}{2} & \frac{-7}{2} & 1 \\ 1 & \frac{-7}{2} & \frac{5}{2} \end{pmatrix} \tag{2.5}$$

3. Obtain the median normal matrix.

   Solution: Considering the roation matrix

$$\mathbf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

the normal matrix is obtained as

$$\mathbf{N}_1 = \mathbf{R}\mathbf{M}_1 \tag{3.2}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & \frac{-7}{2} & 1 \\ 1 & \frac{-7}{2} & \frac{5}{2} \end{pmatrix} \tag{3.3}$$

$$\mathbf{N}_1 = \begin{pmatrix} -1 & \frac{7}{2} & \frac{-5}{2} \\ \frac{5}{2} & \frac{-7}{2} & 1 \end{pmatrix} \tag{3.4}$$

4. Obtian the median equation constants.

$$\mathbf{c}_1 = \mathrm{diag}\left(\left(\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix}\right)\right) \tag{4.1}$$

$$\mathbf{N}_1^\top \begin{pmatrix} \mathbf{D} & \mathbf{E} & \mathbf{F} \end{pmatrix} = \begin{pmatrix} -1 & \frac{5}{2} \\ \frac{7}{2} & \frac{-7}{2} \\ \frac{-5}{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{-3}{2} & \frac{1}{2} & -1 \\ 2 & \frac{7}{2} & \frac{3}{2} \end{pmatrix} \tag{4.2}$$

$$\tag{4.3}$$

Using matrix multiplication

$$= \begin{pmatrix} \frac{13}{2} & \frac{33}{4} & \frac{19}{4} \\ \frac{-49}{4} & \frac{-21}{4} & \frac{-35}{4} \\ \frac{23}{4} & \frac{9}{4} & 4 \end{pmatrix} \tag{4.4}$$

$$\mathbf{c}_1 = \operatorname{diag}\left(\begin{pmatrix} \frac{13}{2} & \frac{33}{4} & \frac{19}{4} \\ \frac{-49}{4} & \frac{-21}{4} & \frac{-35}{4} \\ \frac{23}{4} & \frac{9}{4} & 4 \end{pmatrix}\right) \tag{4.5}$$

$$\mathbf{c}_1 = \begin{pmatrix} \frac{-17}{2} & \frac{49}{4} & 15 \end{pmatrix} \tag{4.6}$$

5. Obtain the centroid by finding the intersection of the medians.

   Solution:

$$\left(\mathbf{N}_1^\top \;\middle|\; \mathbf{c}^\top\right) = \left(\begin{array}{cc|c} -1 & \frac{5}{2} & \frac{-17}{2} \\ \frac{7}{2} & \frac{-7}{2} & \frac{49}{4} \\ \frac{-5}{2} & 1 & 15 \end{array}\right) \tag{5.1}$$

Using Gauss-Elimination method:

$$\left(\begin{array}{cc|c} -1 & \frac{5}{2} & \frac{-17}{2} \\ \frac{7}{2} & \frac{-7}{2} & \frac{49}{4} \\ \frac{-5}{2} & 1 & 15 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - R_1} \left(\begin{array}{cc|c} -1 & \frac{5}{2} & \frac{-17}{2} \\ \frac{9}{2} & \frac{-9}{2} & \frac{83}{4} \\ \frac{-5}{2} & 1 & 15 \end{array}\right) \tag{5.2}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_1} \left(\begin{array}{cc|c} -1 & \frac{5}{2} & \frac{-17}{2} \\ \frac{9}{2} & \frac{-9}{2} & \frac{83}{4} \\ \frac{-9}{2} & 6 & -2 \end{array}\right) \tag{5.3}$$

$$\xleftrightarrow{R_2 \leftarrow \frac{1}{9} R_2} \left(\begin{array}{cc|c} -1 & \frac{5}{2} & \frac{-17}{2} \\ \frac{1}{2} & \frac{-1}{2} & \frac{83}{36} \\ \frac{-9}{2} & 6 & -2 \end{array}\right) \tag{5.4}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 + \frac{5}{2} R_2} \left(\begin{array}{cc|c} \frac{1}{4} & \frac{5}{4} & \frac{-197}{72} \\ \frac{1}{2} & \frac{-1}{2} & \frac{83}{36} \\ \frac{-9}{2} & 6 & -2 \end{array}\right) \tag{5.5}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 9R_2} \left(\begin{array}{cc|c} \frac{1}{4} & \frac{5}{4} & \frac{-197}{72} \\ \frac{1}{2} & \frac{-1}{2} & \frac{83}{36} \\ 0 & \frac{3}{2} & \frac{77}{3} \end{array}\right) \tag{5.6}$$

$$\text{Therefore } \mathbf{G} = \begin{pmatrix} \frac{-197}{72} \\ \frac{-83}{36} \\ \frac{77}{3} \end{pmatrix} \tag{5.7}$$

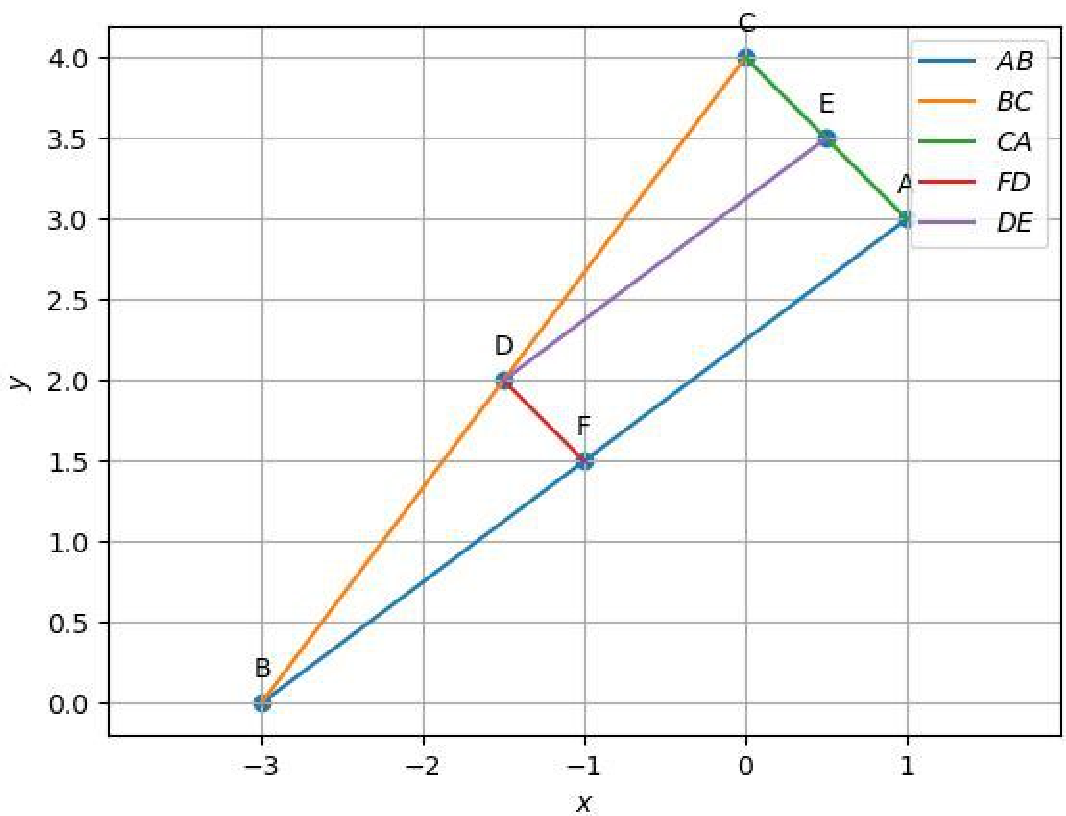

Figure 1.3: centroid of triangle ABC

1.6.3. Altitude

[label=1.6.3.0.,ref=1.6.3.0]Find the normal matrix for the altitudes

Solution: The desired matrix is

$$\mathbf{M}_2 = \begin{pmatrix} \mathbf{B}-\mathbf{C} & \mathbf{C}-\mathbf{A} & \mathbf{A}-\mathbf{B} \end{pmatrix} \tag{1.1}$$

$$= \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.2}$$

$$= \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \tag{1.3}$$

Using Matrix multiplication

$$\mathbf{M}_2 = \begin{pmatrix} -3 & -1 & 4 \\ -4 & 1 & 3 \end{pmatrix} \tag{1.4}$$

Find the constants vector for the altitudes.

Solution: The desired vector is

$$\mathbf{c}_2 = \text{diag}\left\{\left(\mathbf{M}^\top \mathbf{P}\right)\right\} \tag{2.1}$$

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} -3 & -4 \\ -1 & 1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \tag{2.2}$$

$$\tag{2.3}$$

Using matrix multiplication

$$\mathbf{M}^\top \mathbf{P} = \begin{pmatrix} -15 & 9 & -16 \\ 2 & 3 & 4 \\ 13 & -12 & 12 \end{pmatrix} \tag{2.4}$$

$$\mathbf{c}_2 = \mathrm{diag}\left(\begin{pmatrix} -15 & 9 & -16 \\ 2 & 3 & 4 \\ 13 & -12 & 12 \end{pmatrix}\right) \tag{2.5}$$

$$\mathbf{c}_2 = \begin{pmatrix} -16 & 9 & 12 \end{pmatrix} \tag{2.6}$$

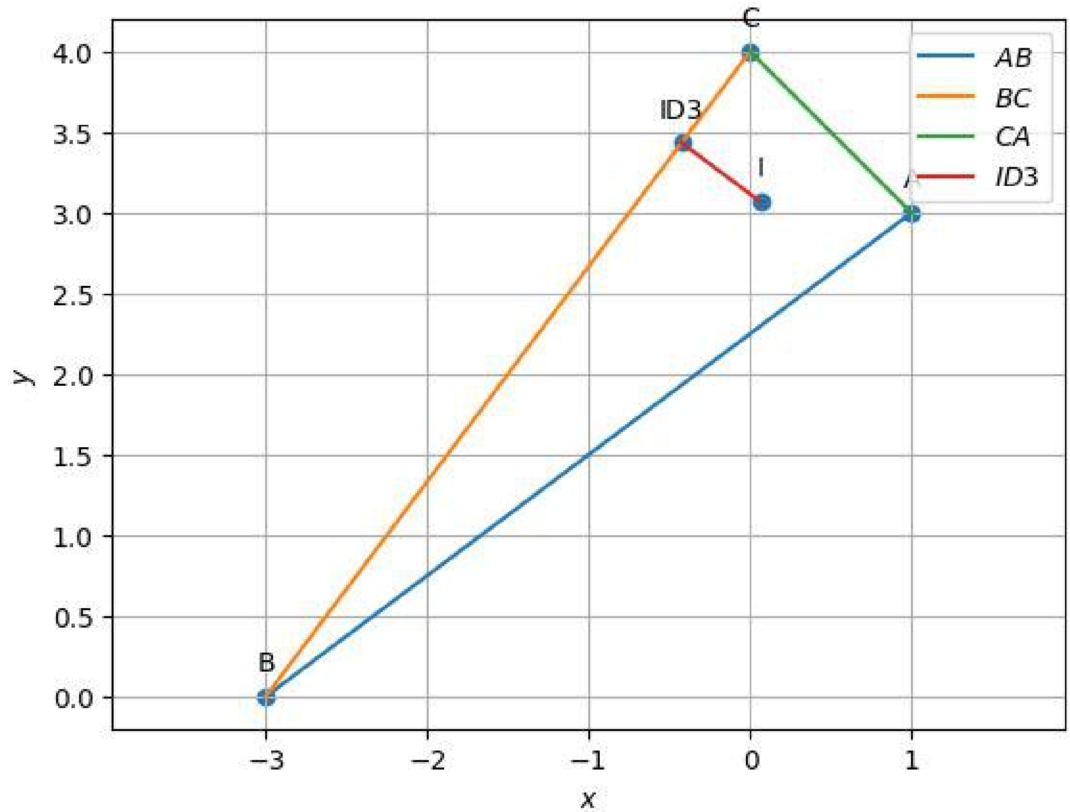

Figure 1.4: Ortho centre of △ ABC

1.6.4. Perpendicular Bisector

[label=1.6.4.0.,ref=1.6.4.0]Find the normal matrix for the perpendicular bisectors

Solution: The normal matrix is $\mathbf{M}_2$

$$\mathbf{M}_2 = \begin{pmatrix} -3 & -1 & 4 \\ -4 & 1 & 3 \end{pmatrix} \tag{1.1}$$

/sdcard/hima/geometry/figs/prep1.jpg

Figure 1.5: plot of perpendicular bisectors

2.

2. Find the constants vector for the perpendicular bisectors.

Solution: The desired vector is

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^{\top}\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{2.1}$$

Solution:

$$\mathbf{c}_3 = \text{diag}\left\{\mathbf{M}_2^{\top}\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix}\right\} \tag{2.2}$$

$$\mathbf{M}_2^{\top}\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \begin{pmatrix}-3 & -4 \\ -1 & 1 \\ 4 & 3\end{pmatrix}\begin{pmatrix}\frac{-3}{2} & \frac{1}{2} & -1 \\ 2 & \frac{7}{2} & \frac{3}{2}\end{pmatrix} \tag{2.3}$$

$$\tag{2.4}$$

Using matrix multiplication

$$\mathbf{M}_2^{\top}\begin{pmatrix}\mathbf{D} & \mathbf{E} & \mathbf{F}\end{pmatrix} = \begin{pmatrix} 0 & \frac{7}{2} \\ \frac{-7}{2} & 0 \end{pmatrix} \tag{2.5}$$

$$\mathbf{c}_3 = \operatorname{diag}\left(\begin{pmatrix} 0 & \frac{7}{2} \\ \frac{-7}{2} & 0 \end{pmatrix}\right) \tag{2.6}$$

$$\mathbf{c}_3 = \begin{pmatrix} \frac{-7}{2} & 0 \end{pmatrix} \tag{2.7}$$

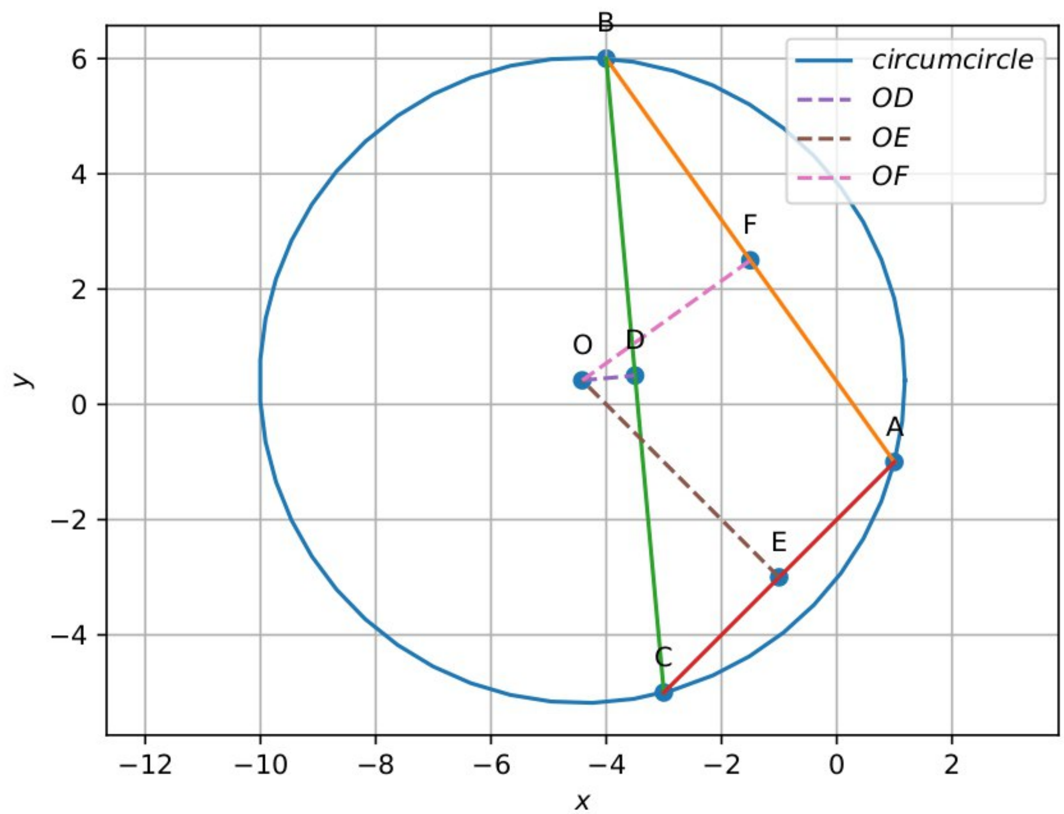

Figure 1.6: circumcentre and circumcircle of $\triangle$ ABC

1.6.5. Angle Bisector

[label=1.6.5.0.,ref=1.6.5.0]Find the points of contact.

Solution: The points of contact are given by

$$\begin{pmatrix} \frac{n\mathbf{A}+p\mathbf{C}}{n+p} & \frac{p\mathbf{B}+m\mathbf{A}}{p+m} & \frac{m\mathbf{C}+n\mathbf{B}}{m+n} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} \frac{n}{b} & \frac{m}{c} & 0 \\ 0 & \frac{p}{c} & \frac{n}{a} \\ \frac{p}{b} & 0 & \frac{m}{a} \end{pmatrix} \tag{1.1}$$

$$\begin{pmatrix} \mathbf{p} & \mathbf{m} & \mathbf{n} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.2}$$

$$= \frac{1}{2} \begin{pmatrix} 5 & 5 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.3}$$

$$= \frac{1}{2} \begin{pmatrix} 5 & 5 & 1.414 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \tag{1.4}$$

Using matrix multiplication

$$\begin{pmatrix}\mathbf{p} & \mathbf{m} & \mathbf{n}\end{pmatrix} = \begin{pmatrix}1.414 & 1.414 & 8.586\end{pmatrix} \tag{1.5}$$

$$\begin{pmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{pmatrix}\begin{pmatrix}\frac{n}{b} & \frac{m}{c} & 0\\ 0 & \frac{p}{c} & \frac{n}{a}\\ \frac{p}{b} & 0 & \frac{m}{a}\end{pmatrix} = \begin{pmatrix}1 & -3 & 0\\ 3 & 0 & 4\end{pmatrix}\begin{pmatrix}\frac{8.586}{5} & \frac{1.414}{\sqrt{2}} & 0\\ 0 & \frac{1.414}{\sqrt{2}} & \frac{8.586}{5}\\ \frac{1.414}{5} & 0 & \frac{1.414}{5}\end{pmatrix} \tag{1.6}$$

Using matrix multiplication We get the points of contact

$$= \begin{pmatrix}1.8 & -2 & -5.4\\ 6.2 & 3 & 0.8\end{pmatrix} \tag{1.7}$$

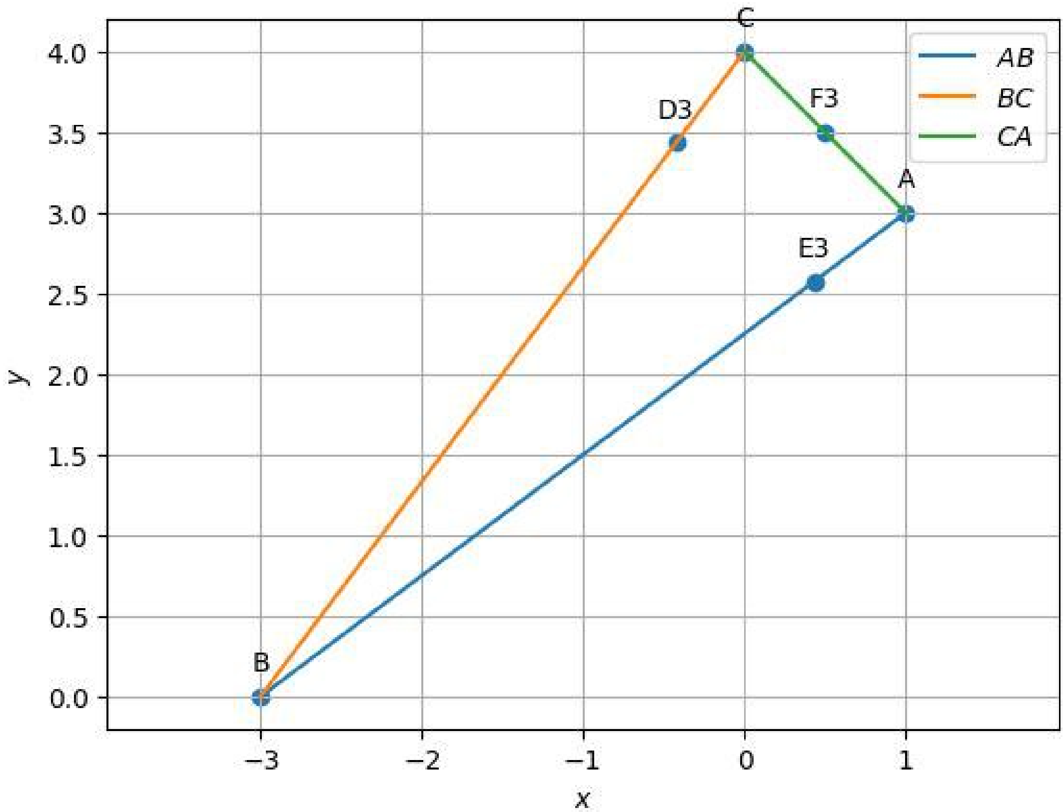

Figure 1.7: Contact points of incircle of *triangle* ABC

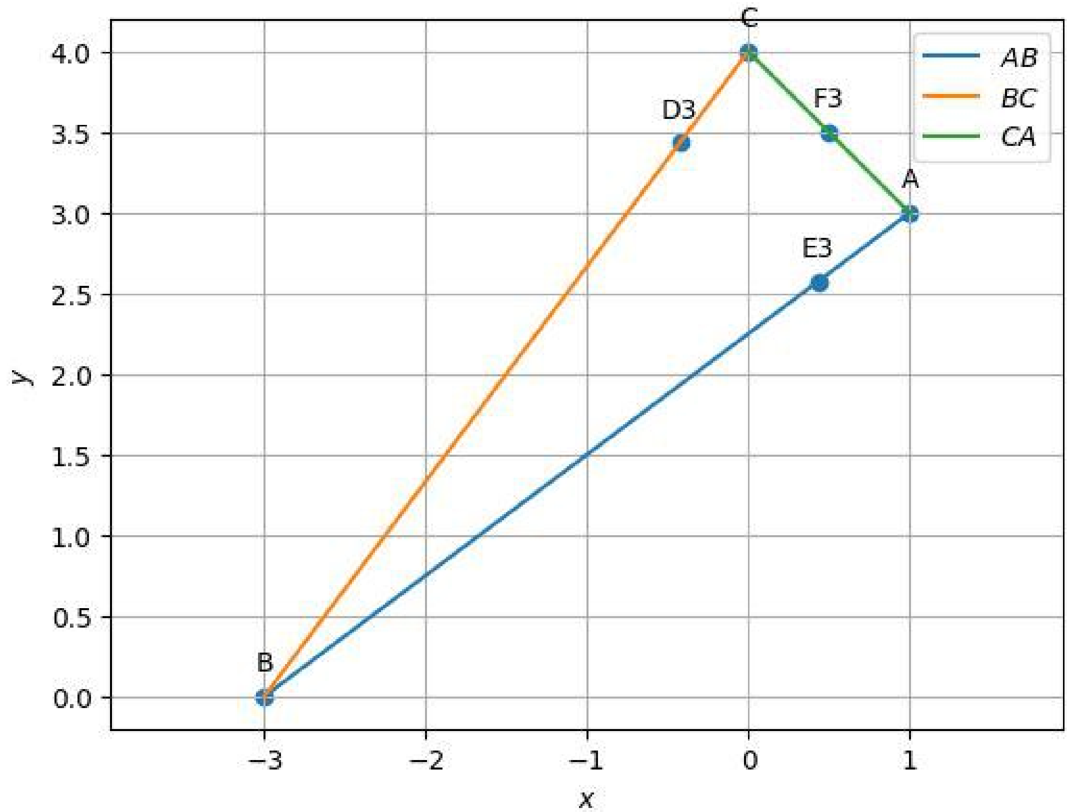

Figure 1.8: Incircle and Incentre of △ ABC

1.