MATH-COMPUTING

December 28, 2023

1. **Question(MATH-12.10.5.17):** Let **a** and **b** be two unit vectors and θ is the angle between them. Then **a** + **b** is a unit vector.

   (A) $\theta=\frac{\pi}{4}$
   (B) $\theta=\frac{\pi}{3}$
   (C) $\theta=\frac{\pi}{2}$
   (D) $\theta=\frac{2\pi}{3}$

   solution:

   Given,

   $$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{1}$$

   $$\|\mathbf{a}+\mathbf{b}\| = 1 \tag{2}$$

   Squaring on both sides, we get

   $$\|\mathbf{a}+\mathbf{b}\|^2 = 1^2 \tag{3}$$
   $$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{4}$$

   Substituting (1) in (4), we get

$$\implies 1 + 1 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{5}$$

$$\implies 2 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{6}$$

$$\implies 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = -1 \tag{7}$$

$$\implies (\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = \frac{-1}{2} \tag{8}$$

Substituting (1) in (8), we get

$$\implies \cos\theta = \frac{-1}{2} \tag{9}$$

$$\implies \theta = \frac{2\pi}{3} \tag{10}$$

Assume,

$$\|\mathbf{a}\| = (\|\mathbf{a_1}\|\|\mathbf{a_2}\|) = (4, 4) \tag{11}$$

Let,

$$\|\mathbf{b_1}\|^2 + \|\mathbf{b_2}\|^2 = 1 \tag{12}$$

Since,

$$\mathbf{ab} = \mathbf{a_1 b_1} + \mathbf{a_2 b_2} \tag{13}$$

$$\implies \|\mathbf{a_1}\|\|\mathbf{b_1}\| + \|\mathbf{a_2}\|\|\mathbf{b_2}\| = \frac{-1}{2} \tag{14}$$

$$\implies 4\|\mathbf{b_1}\| + 4\|\mathbf{b_2}\| = \frac{-1}{2} \tag{15}$$

$$\implies 4(\|\mathbf{b_1}\| + \|\mathbf{b_2}\|) = \frac{-1}{2} \tag{16}$$

$$\implies (||\mathbf{b_1}|| + ||\mathbf{b_2}||) = \frac{-1}{8} \tag{17}$$

Squaring on both sides:

$$(||\mathbf{b_1}|| + ||\mathbf{b_2}||)^2 = \left(\frac{-1}{8}\right)^2 \tag{18}$$

$$\implies ||\mathbf{b_1}||^2 + ||\mathbf{b_2}||^2 + 2||\mathbf{b_1}||||\mathbf{b_2}|| = \frac{-1}{16} \tag{19}$$

$$\implies 1 + 2||\mathbf{b_1}||||\mathbf{b_2}|| = \frac{-1}{16} \tag{20}$$

$$\implies ||\mathbf{b_1}||||\mathbf{b_2}|| = \frac{-15}{32} \tag{21}$$

$$\implies \frac{||\mathbf{b_1}||}{\frac{1}{||\mathbf{b_2}||}} = \frac{-15}{32} \tag{22}$$

$$\implies (||\mathbf{b_1}||||\mathbf{b_2}||) = \left(-15, \frac{1}{32}\right) \tag{23}$$

Therefore:

$$||\mathbf{a}|| = (||\mathbf{a_1}||||\mathbf{a_2}||) = (4, 4) \tag{24}$$

$$||\mathbf{b}|| = (||\mathbf{b_1}||||\mathbf{b_2}||) = \left(-15, \frac{1}{32}\right) \tag{25}$$

To check whether the generated vector values of a and b are possible to make the requires angle:

$$\theta = \cos^{-1}\left(\frac{\mathbf{ab}}{|\mathbf{a}|\,|\mathbf{b}|}\right) \tag{26}$$

$$\implies \mathbf{ab} = \mathbf{a_1 b_1} + \mathbf{a_1 b_1} \tag{27}$$

$$\implies \mathbf{ab} = 4(-15) + 4\left(\frac{1}{32}\right) = -60.125 \tag{28}$$

$$\implies \left|\mathbf{a}\right| = \sqrt{\mathbf{a_1}^2 + \mathbf{a_2}^2} = \sqrt{4^2 + 4^2} = 5.65 \tag{29}$$

$$\implies \left|\mathbf{b}\right| = \sqrt{\mathbf{b_1}^2 + \mathbf{b_2}^2} = \sqrt{(-15)^2 + \left(\frac{1}{32}\right)^2} = \sqrt{225.0009} = 15 \tag{30}$$

Therefore:

$$\theta = \cos^{-1}\left(\frac{-60.125}{83.75}\right) = \cos^{-1}(-0.65) = 126^\circ \approx 120^\circ \tag{31}$$

Hence verified