MATH-COMPUTING

January 5, 2024

1. **Question(MATH-12.10.5.17):** Let **a** and **b** be two unit vectors and θ is the angle between them. Then **a** + **b** is a unit vector.

   (A) $\theta = \frac{\pi}{4}$

   (B) $\theta = \frac{\pi}{3}$

   (C) $\theta = \frac{\pi}{2}$

   (D) $\theta = \frac{2\pi}{3}$

   solution:

   Given,

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{1}$$

$$\|\mathbf{a} + \mathbf{b}\| = 1 \tag{2}$$

   and, its magnitude will be as:

$$magnitude = \sqrt{\mathbf{a}^2 + \mathbf{b}^2} \tag{3}$$

$$magnitude = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{4}$$

   Squaring on both sides of (2), we get

$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \tag{5}$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{6}$$

Substituting (6) in (1), we get

$$\implies 1 + 1 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{7}$$
$$\implies 2 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{8}$$
$$\implies 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = -1 \tag{9}$$
$$\implies (\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = \frac{-1}{2} \tag{10}$$

Substituting (1) in (10), we get

$$\implies \cos\theta = \frac{-1}{2} \tag{11}$$
$$\implies \theta = \frac{2\pi}{3} \tag{12}$$

The point f ”a” is:

$$\|\mathbf{a}\| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{13}$$

The equation of the magnitude of a 2-dimensional vector is:

$$a_1^2 + a_2^2 = 1 \tag{14}$$

Assume value of a1 is given and let $a_1 = a_1$

$$\implies a_2^2 = 1 - a_1^2 \tag{15}$$

$$\implies a_2 = \sqrt{1 - a_1^2} \tag{16}$$

Condition-1: $(-1 \geq a_1 \geq 1)$

$$\implies a_1 = 0; a_2 = \sqrt{1 - 0} = 1 \tag{17}$$

$$\implies a_1 = 1; a_2 = \sqrt{1 - 1} = 0 \tag{18}$$

$$\implies a_1 = -1; a_2 = \sqrt{1-1} = 0 \tag{19}$$

Condition-2: $a_1 \geq 2$

$$\implies a_1 = 2; a_2 = \sqrt{1-4} = \sqrt{-3} \tag{20}$$

Condition-3: $a_1 \leq -2$

$$\implies a_1 = -2; a_2 = \sqrt{1-4} = \sqrt{-3} \tag{21}$$

And, The point of ”b” is:

$$||\mathbf{b}|| = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{22}$$

The equation of the magnitude of a 2-dimensional vector is:

$$b_1^2 + b_2^2 = 1 \tag{23}$$

$$\implies b_2^2 = 1 - b_1^2 \tag{24}$$

$$\implies b_2 = \sqrt{1 - b_1^2} \tag{25}$$

Condition-1: $(-1 \geq b_1 \geq 1)$

$$\implies a_1 = 0; b_1 = \frac{\sqrt{3-0}}{2} = \frac{\sqrt{3}}{2} = 0.866 \tag{26}$$

$$\implies a_1 = 1; b_1 = \frac{1 \pm \sqrt{3-3}}{2} = \frac{1}{2} \tag{27}$$

$$\implies a_1 = -1; b_1 = \frac{1 \pm \sqrt{3-3}}{2} = \frac{1}{2} \tag{28}$$

Condition-2: $a_1 \geq 2$

$$\implies a_1 = 2; b_1 = \frac{-2 \pm \sqrt{-9}}{2} \tag{29}$$

Condition-3: $a_1 \leq -2$

$$\implies a_1 = -2; b_1 = \frac{2 \pm \sqrt{-9}}{2} \tag{30}$$

Since,

$$a_1 b_1 + a_2 b_2 = \frac{-1}{2} \tag{31}$$

$$\implies a_1 b_1 + \sqrt{1 - a_1{}^2}\sqrt{1 - b_1{}^2} = \frac{-1}{2} \tag{32}$$

$$\implies \sqrt{1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2} = \frac{-1}{2} - a_1 b_1 \tag{33}$$

Squaring on both the sides:

$$\implies 1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2 = \left(\frac{-1}{2} - a_1 b_1\right)^2 \tag{34}$$

$$\implies 1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2 = \left(\frac{-1}{2}\right)^2 + (a_1 b_1)^2 - 2\left(\frac{-1}{2}\right)(a_1 b_1) \tag{35}$$

$$\implies 1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2 = \frac{1}{4} + (a_1 b_1)^2 + a_1 b_1 \tag{36}$$

$$\implies -b_1{}^2 - a_1{}^2 + a_1 b_1 = \frac{1}{4} - 1 \tag{37}$$

$$\implies b_1{}^2 + a_1{}^2 + a_1 b_1 = \frac{3}{4} \tag{38}$$

$$\implies {b_1}^2 + a_1 b_1 + \left({a_1}^2 - \frac{3}{4}\right) = 0 \tag{39}$$

We know the formula of Quadratic equation to find roots:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{40}$$

$$\implies b_1 = \frac{(-a_1) \pm \sqrt{a_1^2 - 4(a_1^2 - \frac{3}{4})}}{2} \tag{41}$$

$$\implies b_1 = \frac{(-a_1) \pm \sqrt{-3a_1^2 + 3}}{2} \tag{42}$$

Substituting (42) in (25), we get

$$b_2 = \sqrt{1 - b_1^2} \tag{43}$$

$$\implies b_2 = \sqrt{1 - \left(\frac{-a_1 \pm \sqrt{3 - a_1^2}}{2}\right)} \tag{44}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(-a_1 \pm \sqrt{3 - a_1^2}\right)^2} \tag{45}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(-a_1^2 + 2a_1\sqrt{3 - 3a_1^2} + 3 - 3a_1^2\right)} \tag{46}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(3 - 2a_1^2 \pm 2a_1\sqrt{3 - 3a_1^2}\right)} \tag{47}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(3 - 2a_1\left(2a_1 + \sqrt{3 - 3a_1^2}\right)\right)} \tag{48}$$

$$\implies b_2 = \sqrt{1 - \frac{3}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} \tag{49}$$

$$\implies b_2 = \sqrt{\frac{1}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} \tag{50}$$

Condition-1: $a_1 = 0$

$$\implies a_1 = 0; b_2 = \sqrt{\frac{1}{2}} = 0.5 \tag{51}$$

Condition-1: $a_1 \neq 0$

$$\implies a_1 \neq 0; b_2 = \sqrt{\frac{1}{4} - 1} = \sqrt{\frac{-3}{4}} \tag{52}$$

Therefore, equations (16),(42) ,(50) gives the values of a2,b1 and b2

$$\implies a_2 = \sqrt{1 - a_1^2} \tag{53}$$

$$\implies b_1 = \frac{(-a_1) \pm \sqrt{-3a_1^2 + 3}}{2} \tag{54}$$

$$\implies b_2 = \sqrt{\frac{1}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} \tag{55}$$

The angle of the two unit vectors will be:

$$\theta = \cos^{-1}\left(\frac{\mathbf{a}^\top \mathbf{b}}{||\mathbf{a}||||\mathbf{b}||}\right) \tag{56}$$

Where,

$$\mathbf{a}^\top \mathbf{b} = a_1 b_1 + a_2 b_2 \tag{57}$$

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2} \tag{58}$$

$$\|\mathbf{b}\| = \sqrt{b_1^2 + b_2^2} \tag{59}$$

Substituting (57),(58),(59) in (56),we get:

$$\theta = \cos^{-1}\left(\frac{a_1 b_1 + a_2 b_2}{\left(\sqrt{a_1^2 + a_2^2}\right)\left(\sqrt{b_1^2 + b_2^2}\right)}\right) \tag{60}$$