# MATH-COMPUTING

January 18, 2024

1. **Question(MATH-12.10.5.17):** Let **a** and **b** be two unit vectors and $\theta$ is the angle between them. Then **a** + **b** is a unit vector.

   (A) $\theta=\frac{\pi}{4}$
   (B) $\theta=\frac{\pi}{3}$
   (C) $\theta=\frac{\pi}{2}$
   (D) $\theta=\frac{2\pi}{3}$

   **solution:**

   Given,

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{1}$$

$$\|\mathbf{a} + \mathbf{b}\| = 1 \tag{2}$$

   Squaring on both sides of (2), we get

$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \tag{3}$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{4}$$

   Substituting (1) in (4), we get

$$\implies 1 + 1 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{5}$$

$$\implies 2 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{6}$$

$$\implies 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = -1 \tag{7}$$

$$\implies (\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = \frac{-1}{2} \tag{8}$$

Substituting (1) in (8), we get

$$\implies \cos\theta = \frac{-1}{2} \tag{9}$$

$$\implies \theta = \frac{2\pi}{3} \tag{10}$$

Let,

$$\mathbf{a} = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix} \tag{11}$$

and

$$\mathbf{b} = \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix} \tag{12}$$

Matrix multplication of **a**.**b** is:

$$\mathbf{a}.\mathbf{b} = \cos(\theta_1 - \theta_2) = \frac{-1}{2} \tag{13}$$

$$\theta_1 - \theta_2 = \cos^{-1}\left(\frac{-1}{2}\right) \tag{14}$$

$$\theta_1 - \theta_2 = \frac{2\pi}{3} \tag{15}$$

$$\theta_1 = \theta_2 + \frac{2\pi}{3} \tag{16}$$

Let, $\theta_2 = \frac{\pi}{3}$:

$$\theta_1 = \frac{\pi}{3} + \frac{2\pi}{3} = \pi \tag{17}$$

Therefore,

$$\mathbf{a} = \begin{pmatrix} \cos\pi \\ \sin\pi \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{18}$$

and

$$\mathbf{b} = \begin{pmatrix} \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \tag{19}$$

So, $\|\mathbf{a} + \mathbf{b}\|$ will be:

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{(\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 + \sin\theta_2)^2} \tag{20}$$

substitute (18) and (19) in (20), we will get:

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{\left(\cos\pi + \cos\frac{\pi}{3}\right)^2 + \left(\sin\pi + \sin\frac{\pi}{3}\right)^2} \tag{21}$$

$$\|\mathbf{a} + \mathbf{b}\| = \sqrt{\left(-1 + \frac{1}{2}\right)^2 + \left(0 + \frac{\sqrt{3}}{2}\right)^2} = \sqrt{0.25 + 0.74} = 1 \tag{22}$$

Hence, $\mathbf{a} + \mathbf{b}$ is also a unit vector