Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.1.1. The direction vector of AB is defined as

$$\mathbf{B} - \mathbf{A} \tag{1.1.1.1}$$

Question1.1.1 :Find the Direction Vectors of AB,BC and CA.

Solution:

(a) The Direction vector of AB is

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 - (1) \\ 0 - (3) \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.1.1.2}$$

(b) The Direction vector of BC

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}0\\4\end{pmatrix}-\begin{pmatrix}-3\\0\end{pmatrix}=\begin{pmatrix}0-(-3)\\4-(0)\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix} \tag{1.1.1.3}$$

(c) The Direction vector of CA

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}1\\3\end{pmatrix}-\begin{pmatrix}0\\4\end{pmatrix}=\begin{pmatrix}1-(0)\\3-(4)\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix} \tag{1.1.1.4}$$

1.1.2. The length of side BC is

$$\|\mathbf{B}-\mathbf{A}\| \triangleq \sqrt{(\mathbf{B}-\mathbf{A})^{\top}\mathbf{B}-\mathbf{A}} \tag{1.1.2.1}$$

where

$$\mathbf{A}^{\top} \triangleq \begin{pmatrix}1 & 3\end{pmatrix} \tag{1.1.2.2}$$

Question 1.1.2 : Find the length of side AB, BC, CA.

Solution:

(a) Solving for AB

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{(\mathbf{A}-\mathbf{B})^\top (\mathbf{A}-\mathbf{B})} \tag{1.1.2.3}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.1.2.4}$$

$$(\mathbf{A}-\mathbf{B})^\top = \begin{pmatrix} 4 \\ 3 \end{pmatrix}^\top = \begin{pmatrix} 4 \ 3 \end{pmatrix} \tag{1.1.2.5}$$

$$(\mathbf{A}-\mathbf{B})^\top (\mathbf{A}-\mathbf{B}) = \begin{pmatrix} 4 & -3 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 16 + 9 = 25 \tag{1.1.2.6}$$

$$\implies \|\mathbf{A}-\mathbf{B}\| = \sqrt{(\mathbf{A}-\mathbf{B})^\top (\mathbf{A}-\mathbf{B})} = \sqrt{25} = 5 \tag{1.1.2.7}$$

(b) Solving for BC

$$\|\mathbf{B}-\mathbf{C}\| = \sqrt{(\mathbf{B}-\mathbf{C})^\top (\mathbf{B}-\mathbf{C})} \tag{1.1.2.8}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.1.2.9}$$

$$(\mathbf{B}-\mathbf{C})^\top = \begin{pmatrix} -3 \\ -4 \end{pmatrix}^\top = \begin{pmatrix} -3 & -4 \end{pmatrix} \tag{1.1.2.10}$$

$$(\mathbf{B}-\mathbf{C})^\top (\mathbf{B}-\mathbf{C}) = \begin{pmatrix} -3 & -4 \end{pmatrix} \begin{pmatrix} -3 \\ -4 \end{pmatrix} = 9 + 16 = 25 \tag{1.1.2.11}$$

$$\implies \|\mathbf{B}-\mathbf{C}\| = \sqrt{(\mathbf{B}-\mathbf{C})^\top (\mathbf{B}-\mathbf{C})} = \sqrt{25} = 5 \tag{1.1.2.12}$$

(c) Solving for CA

$$\|\mathbf{C}-\mathbf{A}\| = \sqrt{(\mathbf{C}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})} \tag{1.1.2.13}$$

$$\mathbf{C}-\mathbf{A} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.1.2.14}$$

$$(\mathbf{C}-\mathbf{A})^\top = \begin{pmatrix} -1 \\ 1 \end{pmatrix}^\top = \begin{pmatrix} -1 & 1 \end{pmatrix} \tag{1.1.2.15}$$

$$(\mathbf{C}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A}) = \begin{pmatrix} -1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1+1 = 2 \tag{1.1.2.16}$$

$$\implies \|\mathbf{C}-\mathbf{A}\| = \sqrt{(\mathbf{C}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})} = \sqrt{2} \tag{1.1.2.17}$$

1.1.3. Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are defined to be collinear if

$$\text{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 2 \tag{1.1.3.1}$$

Are the given points in (1.1) collinear?

Question 1.1.3 : Check the collinearity of $\mathbf{A}, \mathbf{B}, \mathbf{C}$

Solution: Given that,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.1.3.2}$$

Given that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are collinear if

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} < 3 \tag{1.1.3.3}$$

Let

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \tag{1.1.3.4}$$

The matrix $\mathbf{R}$ can be row reduced as follows,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 3 & 0 & 4 \end{pmatrix} \xleftrightarrow{R_3 \leftarrow R_3 + 5R_1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ 4 & -3 & 4 \end{pmatrix} \tag{1.1.3.5}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 4 & -3 & 4 \end{pmatrix} \tag{1.1.3.6}$$

There are no zero rows. So,

$$\operatorname{rank}\begin{pmatrix} 1 & 1 & 1 \\ \mathbf{A} & \mathbf{B} & \mathbf{C} \end{pmatrix} = 3 \tag{1.1.3.7}$$

Hence, from (1.1.3.3) the points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear.

From Fig. 1.1, We can see that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are not collinear .

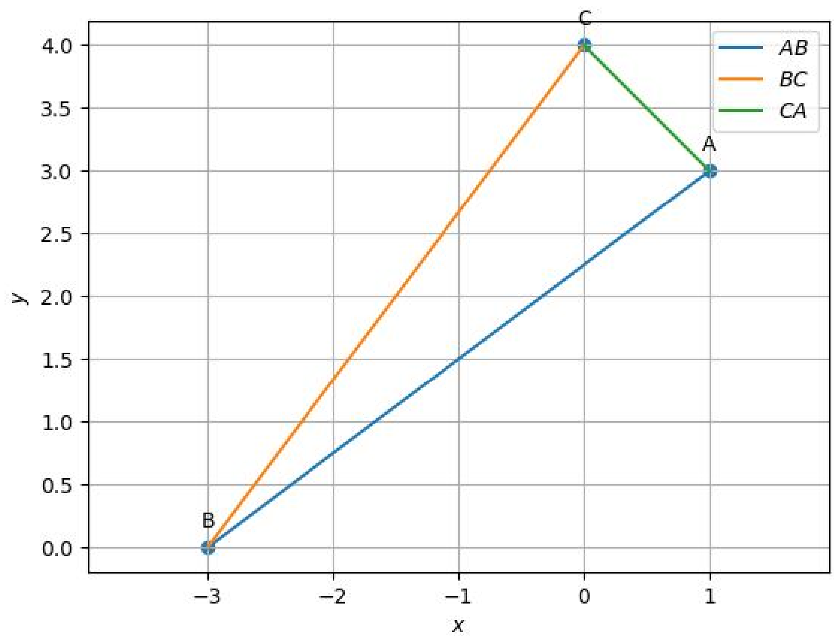

Figure 1.1: $\mathbf{A}, \mathbf{B}, \mathbf{C}$ plot

1.1.4. The parameteric form of the equation of AB is

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.1}$$

where

$$\mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.2}$$

is the direction vector of AB.

Question 1.1.4 :Find the parametric equation of AB,BC,CA.

Solution: The parametric equation for AB is given by

$$\mathbf{x} = \mathbf{A} + k\mathbf{m} \tag{1.1.4.3}$$

$$\text{where, } \mathbf{m} = \mathbf{B} - \mathbf{A} \tag{1.1.4.4}$$

$$= \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.4.5}$$

$$= \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.1.4.6}$$

Hence we get,

$$AB : \mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + k \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.1.4.7}$$

Similarly,

$$BC : \mathbf{x} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + k \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.1.4.8}$$

$$CA : \mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + k \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.4.9}$$

1.1.5. The normal form of the equation of AB is

$$\mathbf{n}^{\top} (\mathbf{x} - \mathbf{A}) = 0 \tag{1.1.5.1}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top (\mathbf{B} - \mathbf{A}) = 0 \tag{1.1.5.2}$$

$$\text{or, } \mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.3}$$

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.5.4}$$

where

$$\mathbf{n}^\top \mathbf{m} = \mathbf{n}^\top \left(\mathbf{B} - \mathbf{A}\right) = 0 \tag{1.1.5.5}$$

or,

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.6}$$

Question 1.1.5:Find the normal form of the equations of AB, BC and CA.

Solution: : The normal equation for the side AB is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{A}\right) = 0 \tag{1.1.5.7}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{A} \tag{1.1.5.8}$$

Now our task is to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. As given in the question

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.9}$$

Here $\mathbf{m} = \mathbf{B} - \mathbf{A}$ for side $\mathbf{AB}$

$$\implies \mathbf{m} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.5.10}$$

$$= \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.1.5.11}$$

Now as we have obtained vector $\mathbf{m}$. we can use this to obtain vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.1.5.12}$$

The transpose of $\mathbf{n}$ is

$$\mathbf{n}^\top = \begin{pmatrix} -3 & 4 \end{pmatrix} \tag{1.1.5.13}$$

Hence the normal equation of side AB is

$$\begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{1.1.5.14}$$

$$\implies \begin{pmatrix} -3 & 4 \end{pmatrix} \mathbf{x} = 9 \tag{1.1.5.15}$$

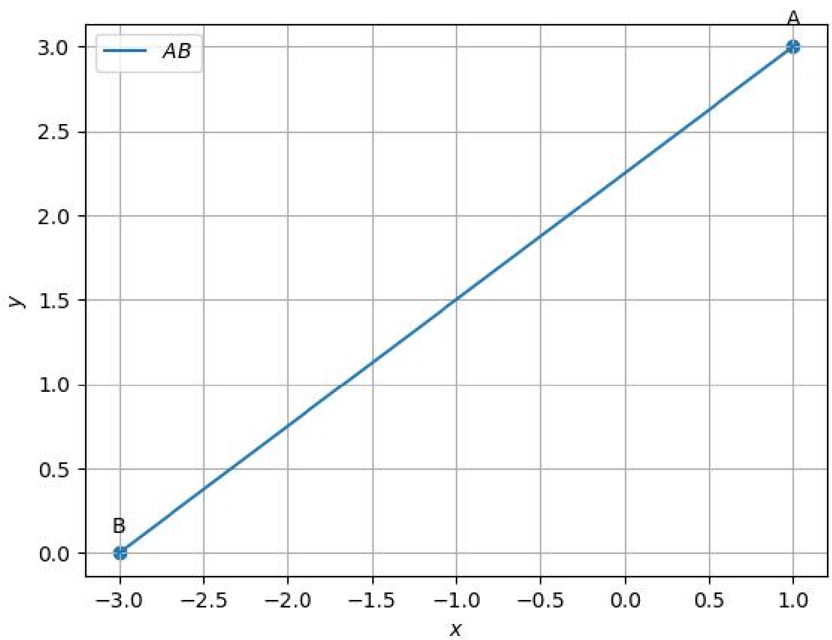

Figure 1.2: The line **AB** plotted using python

The normal equation for the side BC is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{B}\right) = 0 \tag{1.1.5.16}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{B} \tag{1.1.5.17}$$

Now our task is to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. As given in the question

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.18}$$

Here $\mathbf{m} = \mathbf{C} - \mathbf{B}$ for side $\mathbf{BC}$

$$\implies \mathbf{m} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.5.19}$$

$$= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.1.5.20}$$

Now as we have obtained vector $\mathbf{m}$. we can use this to obtain vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix} \tag{1.1.5.21}$$

The transpose of $\mathbf{n}$ is

$$\mathbf{n}^\top = \begin{pmatrix} 4 & -3 \end{pmatrix} \tag{1.1.5.22}$$

Hence the normal equation of side BC is

$$\begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 4 & -3 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \end{pmatrix} \tag{1.1.5.23}$$

$$\implies \begin{pmatrix} 4 & -3 \end{pmatrix} \mathbf{x} = -12 \tag{1.1.5.24}$$

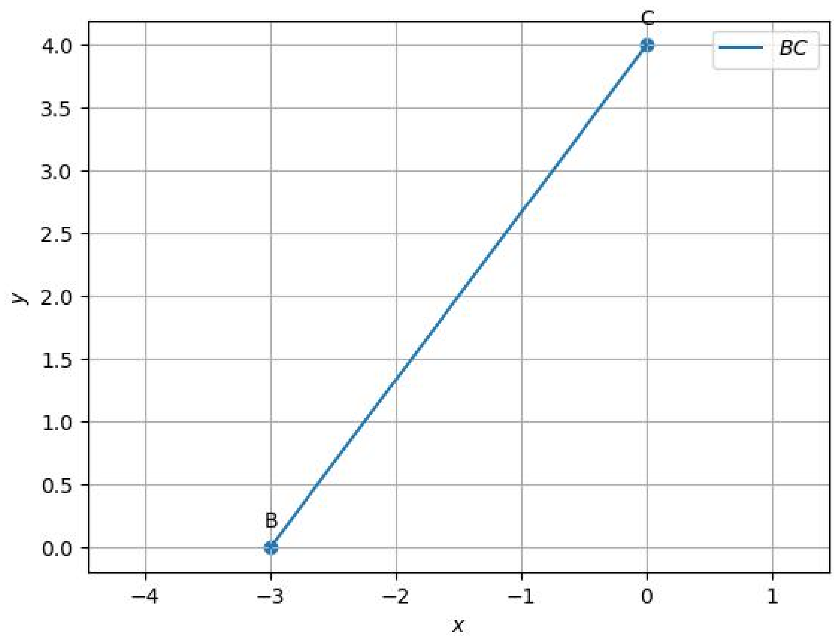

Figure 1.3: The line **BC** plotted using python

The normal equation for the side CA is

$$\mathbf{n}^\top \left(\mathbf{x} - \mathbf{C}\right) = 0 \tag{1.1.5.25}$$

$$\implies \mathbf{n}^\top \mathbf{x} = \mathbf{n}^\top \mathbf{C} \tag{1.1.5.26}$$

Now our task is to find the $\mathbf{n}$ so that we can find $\mathbf{n}^\top$. As given in the question

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{m} \tag{1.1.5.27}$$

Here $\mathbf{m} = \mathbf{A} - \mathbf{C}$ for side $\mathbf{CA}$

$$\implies \mathbf{m} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.1.5.28}$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.5.29}$$

Now as we have obtained vector $\mathbf{m}$. we can use this to obtain vector $\mathbf{n}$

$$\mathbf{n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.1.5.30}$$

The transpose of $\mathbf{n}$ is

$$\mathbf{n}^{\top} = \begin{pmatrix} -1 & 1 \end{pmatrix} \tag{1.1.5.31}$$

Hence the normal equation of side CA is

$$\begin{pmatrix} -1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.1.5.32}$$

$$\implies \begin{pmatrix} 12 & 1 \end{pmatrix} \mathbf{x} = -4 \tag{1.1.5.33}$$

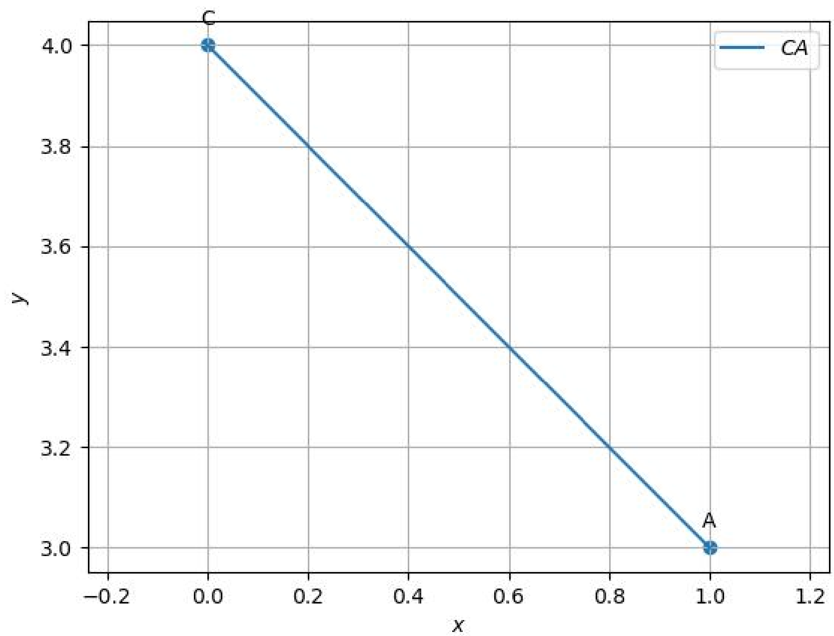

Figure 1.4: The line **CA** plotted using python

1.1.6. The area of $\triangle ABC$ is defined as

$$\frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times\mathbf{A}-\mathbf{C}\right\| \tag{1.1.6.1}$$

where

$$\mathbf{A}\times\mathbf{B} \triangleq \begin{vmatrix} 1 & -3 \\ 3 & 0 \end{vmatrix} \tag{1.1.6.2}$$

Question 1.1.6:Find the area of $\triangle$ ABC.

Solution: Given,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}; \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \tag{1.1.6.3}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.1.6.4}$$

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.1.6.5}$$

$$\therefore (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) = \begin{vmatrix} 4 & 1 \\ 3 & -1 \end{vmatrix} \tag{1.1.6.6}$$

$$= (4) \times (-1) - 3 \times -1 \tag{1.1.6.7}$$

$$= -4 - 3 \tag{1.1.6.8}$$

$$= -7 \tag{1.1.6.9}$$

$$\implies \frac{1}{2} \left\| (\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C}) \right\| = \frac{-7}{2} = -3.5 \tag{1.1.6.10}$$

1.1.7. Question 1.1.7:Find the angles $\mathbf{A}, \mathbf{B}, \mathbf{C}$, given that

$$\cos A \triangleq \frac{(\mathbf{B} - \mathbf{A})\top(\mathbf{C} - \mathbf{A})}{\left\| \mathbf{B} - \mathbf{A} \right\| \left\| \mathbf{C} - \mathbf{A} \right\|} \tag{1.1.7.1}$$

Solution:

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$,

(a) Finding the value of angle A

$$\mathbf{B}-\mathbf{A}=\begin{pmatrix}-4\\-3\end{pmatrix} \tag{1.1.7.2}$$

and

$$\mathbf{C}-\mathbf{A}=\begin{pmatrix}-1\\1\end{pmatrix} \tag{1.1.7.3}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{A}\|=\sqrt{25} \qquad =5 \tag{1.1.7.4}$$

$$\|\mathbf{C}-\mathbf{A}\|=\sqrt{2} \tag{1.1.7.5}$$

and by doing matrix multiplication we get,

$$(\mathbf{B}-\mathbf{A})^\top(\mathbf{C}-\mathbf{A})=\begin{pmatrix}-4 & -3\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} \tag{1.1.7.6}$$

$$=1$$

so

$$\cos A=\frac{-8}{5\sqrt{2}} \tag{1.1.7.7}$$

$$\implies A=\cos^{-1}\frac{-8}{5\sqrt{2}} \tag{1.1.7.8}$$

(b) Finding the value of angle B

$$\mathbf{C}-\mathbf{B}=\begin{pmatrix}3\\-4\end{pmatrix} \tag{1.1.7.9}$$

and

$$\mathbf{A}-\mathbf{B}=\begin{pmatrix}4\\3\end{pmatrix} \tag{1.1.7.10}$$

also calculating the values of norms

$$\|\mathbf{C}-\mathbf{B}\|=\sqrt{25}=5 \tag{1.1.7.11}$$

$$\|\mathbf{A}-\mathbf{B}\|=\sqrt{25}=5 \tag{1.1.7.12}$$

and by doing matrix multiplication we get,

$$(\mathbf{C}-\mathbf{B})^\top(\mathbf{A}-\mathbf{B})=\begin{pmatrix}3 & -4\end{pmatrix}\begin{pmatrix}3\\-4\end{pmatrix} \tag{1.1.7.13}$$
$$=25$$

so

$$\cos B=\frac{25}{\sqrt{25}\sqrt{25}} \tag{1.1.7.14}$$

$$=\frac{25}{25}=1 \tag{1.1.7.15}$$

$$\implies B=\cos^{1}=0 \tag{1.1.7.16}$$

(c) Finding the value of angle C

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}1\\-1\end{pmatrix} \tag{1.1.7.17}$$

and

$$\mathbf{B}-\mathbf{C}=\begin{pmatrix}-3\\-4\end{pmatrix} \tag{1.1.7.18}$$

also calculating the values of norms

$$\|\mathbf{A}-\mathbf{C}\|=\sqrt{2} \tag{1.1.7.19}$$

$$\|\mathbf{B}-\mathbf{C}\|=\sqrt{25}=5 \tag{1.1.7.20}$$

and by doing matrix multiplication we get,

$$\begin{aligned}(\mathbf{A}-\mathbf{C})^{\top}(\mathbf{B}-\mathbf{C}) &= \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}-3\\-4\end{pmatrix}\\ &=1\end{aligned} \tag{1.1.7.21}$$

so

$$\cos C=\frac{1}{5\sqrt{2}} \tag{1.1.7.22}$$

$$\implies C=\cos^{-1}\frac{1}{5\sqrt{2}} \tag{1.1.7.23}$$