MATH-COMPUTING

January 3, 2024

1. **Question(MATH-12.10.5.17):** Let **a** and **b** be two unit vectors and θ is the angle between them. Then **a** + **b** is a unit vector.

   (A) $\theta = \frac{\pi}{4}$
   (B) $\theta = \frac{\pi}{3}$
   (C) $\theta = \frac{\pi}{2}$
   (D) $\theta = \frac{2\pi}{3}$

   solution:

   Given,

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \tag{1}$$

$$\|\mathbf{a} + \mathbf{b}\| = 1 \tag{2}$$

   and, its magnitude will be as:

$$magnitude = \sqrt{\mathbf{a}^2 + \mathbf{b}^2} \tag{3}$$

$$magnitude = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{4}$$

   Squaring on both sides of (2), we get

$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \tag{5}$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top\mathbf{b} = 1 \tag{6}$$

Substituting (6) in (1), we get

$$\implies 1 + 1 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{7}$$

$$\implies 2 + 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = 1 \tag{8}$$

$$\implies 2(\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = -1 \tag{9}$$

$$\implies (\|\mathbf{a}\|\|\mathbf{b}\| \cos\theta) = \frac{-1}{2} \tag{10}$$

Substituting (1) in (10), we get

$$\implies \cos\theta = \frac{-1}{2} \tag{11}$$

$$\implies \theta = \frac{2\pi}{3} \tag{12}$$

The point f "a" is:

$$\|\mathbf{a}\| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{13}$$

The equation of the magnitude of a 2-dimensional vector is:

$$a_1^2 + a_2^2 = 1 \tag{14}$$

Assume value of a1 is given and let a1=a1

$$\implies a_2^2 = 1 - a_1^2 \tag{15}$$

$$\implies a_2 = \sqrt{1 - a_1^2} \tag{16}$$

And, The point of "b" is:

$$\|\mathbf{b}\| = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{17}$$

The equation of the magnitude of a 2-dimensional vector is:

$$b_1^2 + b_2^2 = 1 \tag{18}$$

$$\implies b_2^2 = 1 - b_1^2 \tag{19}$$

$$\implies b_2 = \sqrt{1 - b_1^2} \tag{20}$$

Since,

$$a_1 b_1 + a_2 b_2 = \frac{-1}{2} \tag{21}$$

$$\implies a_1 b_1 + \sqrt{1 - a_1{}^2}\sqrt{1 - b_1{}^2} = \frac{-1}{2} \tag{22}$$

$$\implies \sqrt{1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2} = \frac{-1}{2} - a_1 b_1 \tag{23}$$

Squaring on both the sides:

$$\implies 1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2 = \left(\frac{-1}{2} - a_1 b_1\right)^2 \tag{24}$$

$$\implies 1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2 = \left(\frac{-1}{2}\right)^2 + (a_1 b_1)^2 - 2\left(\frac{-1}{2}\right)(a_1 b_1) \tag{25}$$

$$\implies 1 - b_1{}^2 - a_1{}^2 + a_1{}^2 b_1{}^2 = \frac{1}{4} + (a_1 b_1)^2 + a_1 b_1 \tag{26}$$

$$\implies -b_1{}^2 - a_1{}^2 + a_1 b_1 = \frac{1}{4} - 1 \tag{27}$$

$$\implies {b_1}^2 + {a_1}^2 + a_1 b_1 = \frac{3}{4} \tag{28}$$

$$\implies {b_1}^2 + a_1 b_1 + \left({a_1}^2 - \frac{3}{4}\right) = 0 \tag{29}$$

We know the formula of Quadratic equation to find roots:

$$X = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{30}$$

$$\implies b_1 = \frac{(-a_1) \pm \sqrt{a_1^2 - 4(a_1^2 - \frac{3}{4})}}{2} \tag{31}$$

$$\implies b_1 = \frac{(-a_1) \pm \sqrt{-3a_1^2 + 3}}{2} \tag{32}$$

Substituting (32) in (20), we get

$$b_2 = \sqrt{1 - b_1^2} \tag{33}$$

$$\implies b_2 = \sqrt{1 - \left(\frac{-a_1 \pm \sqrt{3 - a_1^2}}{2}\right)} \tag{34}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(-a_1 \pm \sqrt{3 - a_1^2}\right)^2} \tag{35}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(-a_1^2 + 2a_1\sqrt{3 - 3a_1^2} + 3 - 3a_1^2\right)} \tag{36}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(3 - 2a_1^2 \pm 2a_1\sqrt{3 - 3a_1^2}\right)} \tag{37}$$

$$\implies b_2 = \sqrt{1 - \frac{1}{4}\left(3 - 2a_1\left(2a_1 + \sqrt{3 - 3a_1^2}\right)\right)} \tag{38}$$

$$\implies b_2 = \sqrt{1 - \frac{3}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} \tag{39}$$

$$\implies b_2 = \sqrt{\frac{1}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} \tag{40}$$

Therefore, equations (16),(32) ,(40) gives the values of a2,b1 and b2

$$\implies a_2 = \sqrt{1 - a_1^2} \tag{41}$$

$$\implies b_1 = \frac{(-a_1) \pm \sqrt{-3a_1^2 + 3}}{2} \tag{42}$$

$$\implies b_2 = \sqrt{\frac{1}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} \tag{43}$$

The angle of the two unit vectors will be:

$$\theta = \cos^{-1}\left(\frac{\mathbf{ab}}{|\mathbf{a}|\,|\mathbf{b}|}\right) \tag{44}$$

Where,

$$\mathbf{ab} = a_1b_1 + a_2b_2 \tag{45}$$

$$\left|\mathbf{a}\right| = \sqrt{a_1^2 + a_2^2} \tag{46}$$

$$\left|\mathbf{b}\right| = \sqrt{b_1^2 + b_2^2} \tag{47}$$

Let's assume the input value a1 =1, then:

$$a_2 = \sqrt{1 - a_1} = \sqrt{1 - 1} = 0 \tag{48}$$

$$b_1 = \frac{(-a_1) \pm \sqrt{-3a_1^2 + 3}}{2} = \frac{(-1) \pm \sqrt{-3(1)^2 + 3}}{2} = \frac{-1}{2} \tag{49}$$

$$b_2 = \sqrt{\frac{1}{4} - \frac{a_1}{2}\left(2a_1 + \sqrt{3 - 3a_1^2}\right)} = \sqrt{\frac{1}{4} - \frac{1}{2}\left(2(1) + \sqrt{3 - 3(1)^2}\right)} = \sqrt{\frac{-3}{4}} \tag{50}$$