Contents

Chapter 1

Triangle

Consider a triangle with vertices

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.1}$$

1.1. Vectors

1.2. Median

1.3. Altitude

1.4. Perpendicular Bisector

1.5. Angle Bisector

1.5.1. Let $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$, be points on AB, BC and CA respectively such that

$$AE_3 = AF_3 = m, BD_3 = BF_3 = n, CD_3 = CE_3 = p. \tag{1.5.1.1}$$

Obtain m, n, p in terms of a, b, c obtained in Question 1.1.2.

Solution: From Question 1.1.2

$$a = 5 \tag{1.5.1.2}$$

$$b = 5 \tag{1.5.1.3}$$

$$c = \sqrt{2} \tag{1.5.1.4}$$

From the given information,

$$a = m + n, \tag{1.5.1.5}$$

$$b = n + p, \tag{1.5.1.6}$$

$$c = m + p \tag{1.5.1.7}$$

which can be expressed as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.8}$$

$$\implies \begin{pmatrix} m \\ n \\ p \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \tag{1.5.1.9}$$

Using row reduction,

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \tag{1.5.1.10}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - R_2]{R_3 \leftarrow R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.11}$$

$$\xleftrightarrow[R_1 \leftarrow 2R_1 + R_3]{R_2 \leftarrow 2R_2 - R_3} \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \tag{1.5.1.12}$$

yielding

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix} \tag{1.5.1.13}$$

Therefore,

$$\begin{aligned} p &= \frac{c+b-a}{2} = \frac{\sqrt{2}+\sqrt{25}-\sqrt{25}}{2} \\ m &= \frac{a+c-b}{2} = \frac{\sqrt{25}+\sqrt{2}-\sqrt{25}}{2} \\ n &= \frac{a+b-c}{2} = \frac{\sqrt{25}+\sqrt{25}-\sqrt{2}}{2} \end{aligned} \tag{1.5.1.14}$$

on solving above equations we get

$$p = 0.707 \tag{1.5.1.15}$$

$$m = 0.707 \tag{1.5.1.16}$$

$$n = 4.293 \tag{1.5.1.17}$$

1.5.2. Using section formula, find $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$.

Solution: Given

$$\mathbf{D}_3 = \frac{m\mathbf{C}+n\mathbf{B}}{m+n}, \mathbf{E}_3 = \frac{n\mathbf{A}+p\mathbf{C}}{n+p}, \mathbf{F}_3 = \frac{p\mathbf{B}+m\mathbf{A}}{p+m} \tag{1.5.2.1}$$

Here

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.5.2.2}$$

$$p = 0.707, m = 0.707, n = 4.293, \tag{1.5.2.3}$$

On substituting (**??**) and (**??**) in (**??**) We get

$$\mathbf{D}_3 = \frac{5\begin{pmatrix} 0 \\ 4 \end{pmatrix} + 5\begin{pmatrix} -3 \\ 0 \end{pmatrix}}{5+5} \tag{1.5.2.4}$$

$$\mathbf{E}_3 = \frac{5\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 0.707\begin{pmatrix} 0 \\ 4 \end{pmatrix}}{5+0.707} \tag{1.5.2.5}$$

$$\mathbf{F}_3 = \frac{0.707\begin{pmatrix} -3 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 1 \\ 3 \end{pmatrix}}{0.707+5} \tag{1.5.2.6}$$

On solving above equations We get

$$\mathbf{D}_3 = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.5.2.7}$$

$$\mathbf{E}_3 = \begin{pmatrix} 0.876 \\ 3.123 \end{pmatrix} \tag{1.5.2.8}$$

$$\mathbf{F}_3 = \begin{pmatrix} 0.504 \\ 2.628 \end{pmatrix} \tag{1.5.2.9}$$

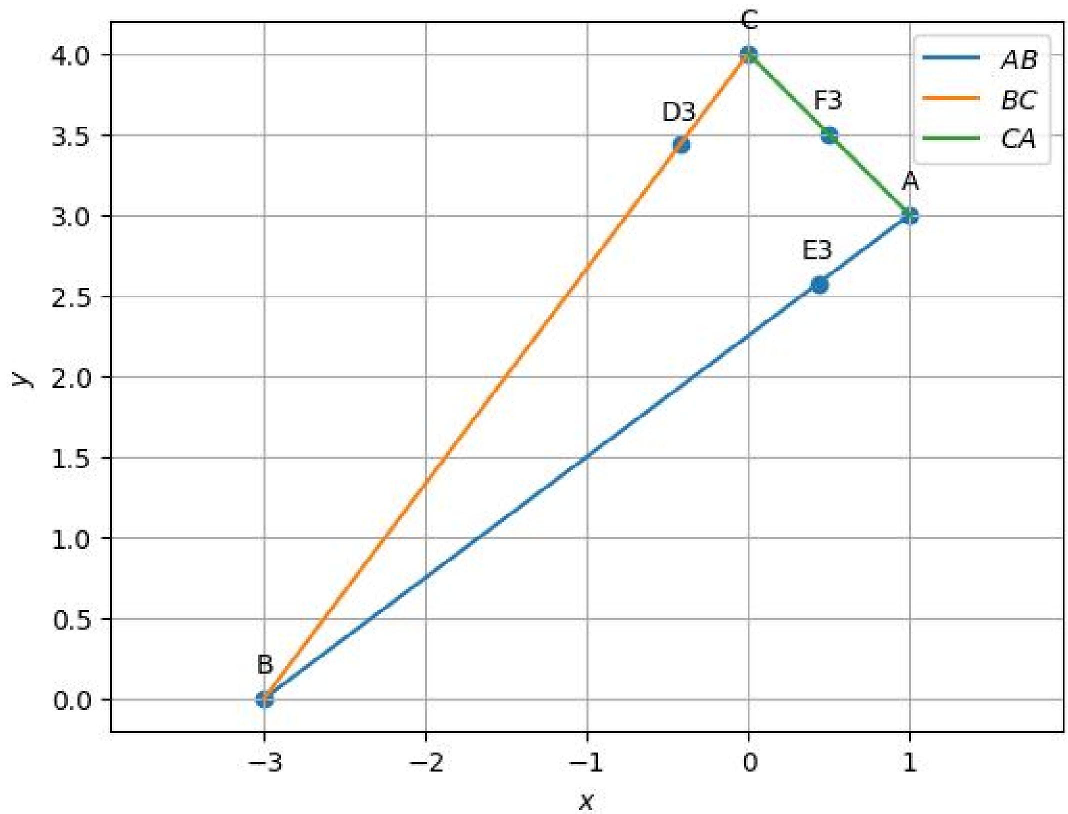

Figure 1.1: Points D3 ,E3 ,F3

1.5.3. Find the circumcentre and circumradius of $\triangle D_3E_3F_3$. These are the <u>incentre</u> and <u>inradius</u> of $\triangle ABC$.

Solution: Given

$$\mathbf{D}_3 = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.5.3.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} 0.876 \\ 3.123 \end{pmatrix} \tag{1.5.3.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} 0.504 \\ 2.628 \end{pmatrix} \tag{1.5.3.3}$$

(a) For circumcentre

Vector equation of $\mathbf{D} - \mathbf{E}$ is

$$(\mathbf{D}_3 - \mathbf{E}_3)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{E}_3}{2}\right) = 0 \tag{1.5.3.4}$$

$$(\mathbf{D}_3 - \mathbf{F}_3)^\top \left(\mathbf{x} - \frac{\mathbf{D}_3 + \mathbf{F}_3}{2}\right) = 0 \tag{1.5.3.5}$$

on Substituting the values of $\mathbf{D}_3, \mathbf{E}_3, \mathbf{F}_3$ and solving We get,

$$\begin{pmatrix} -3.876 & 0.877 \end{pmatrix} \mathbf{x} = 4.116 \tag{1.5.3.6}$$

$$\begin{pmatrix} -3.505 & 1.372 \end{pmatrix} \mathbf{x} = -4.372 \tag{1.5.3.7}$$

Thus on solving (**??**) and (**??**) using gauss elimination We get

$$\begin{pmatrix} -3.876 & 0.877 & 4.116 \\ -3.505 & 1.372 & -4.372 \end{pmatrix} \tag{1.5.3.8}$$

$$\therefore \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -4.116 \\ -4.372 \end{pmatrix} \tag{1.5.3.9}$$

$$\implies \mathbf{x} = \begin{pmatrix} -4.372 \\ 4.116 \end{pmatrix} \tag{1.5.3.10}$$

(b) The circiumradius is obtained from $r = \|\mathbf{I} - \mathbf{D}_3\|$

$$\mathbf{I} = \begin{pmatrix} -4.372 \\ 4.116 \end{pmatrix} \tag{1.5.3.11}$$

$$\mathbf{D}_3 = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.5.3.12}$$

$$\mathbf{I} - \mathbf{D_3} = \begin{pmatrix} -1.372 \\ 0.116 \end{pmatrix} \tag{1.5.3.13}$$

$$r = \|\mathbf{I} - \mathbf{D}_3\| = \sqrt{(\mathbf{I} - \mathbf{D}_3)^\top (\mathbf{I} - \mathbf{D}_3)} \tag{1.5.3.14}$$

$$r = 1.376 \tag{1.5.3.15}$$

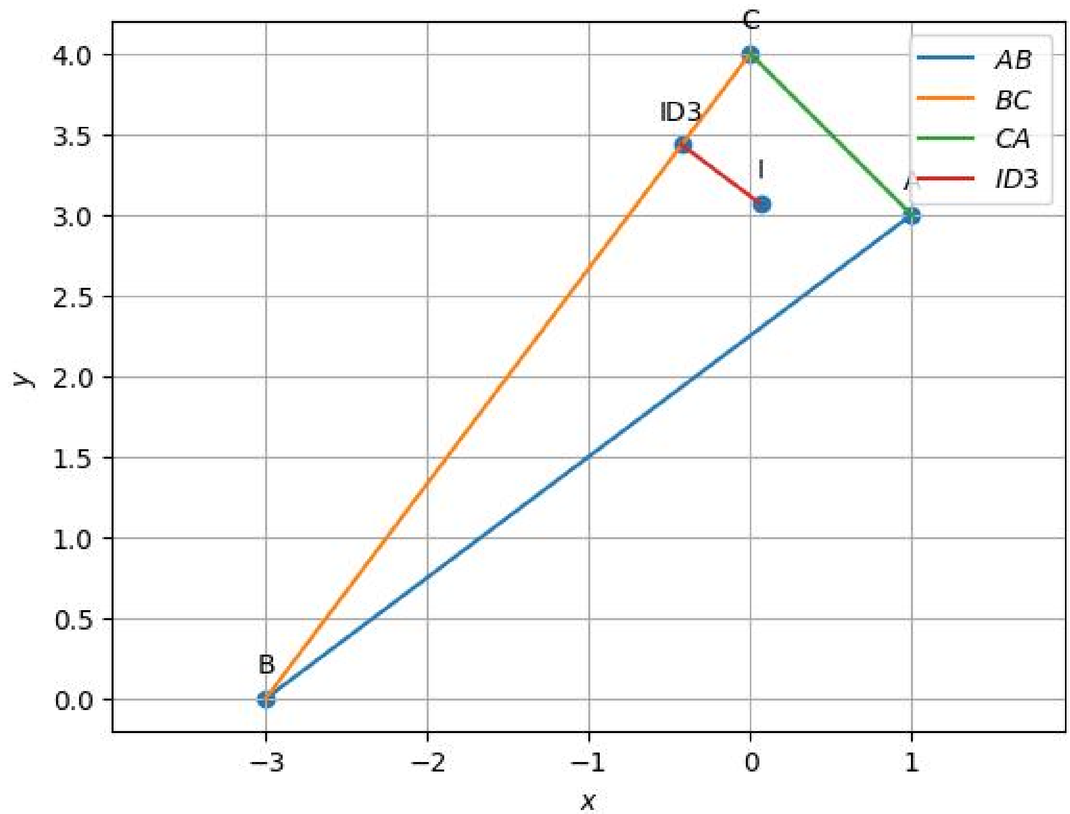

Figure 1.2: incentre and inradius of $\triangle ABC$

1.5.4. Draw the circumcircle of $\triangle D_3E_3F_3$. This is known as the incircle of $\triangle ABC$.

Solution:

$$\mathbf{D}_3 = \begin{pmatrix} -3 \\ 4 \end{pmatrix} \tag{1.5.4.1}$$

$$\mathbf{E}_3 = \begin{pmatrix} 0.876 \\ 3.123 \end{pmatrix} \tag{1.5.4.2}$$

$$\mathbf{F}_3 = \begin{pmatrix} 0.504 \\ 2.628 \end{pmatrix} \tag{1.5.4.3}$$

Incentre

$$I = \begin{pmatrix} -4.372 \\ 4.116 \end{pmatrix} \tag{1.5.4.4}$$

Radius

$$r = 1.376 \tag{1.5.4.5}$$

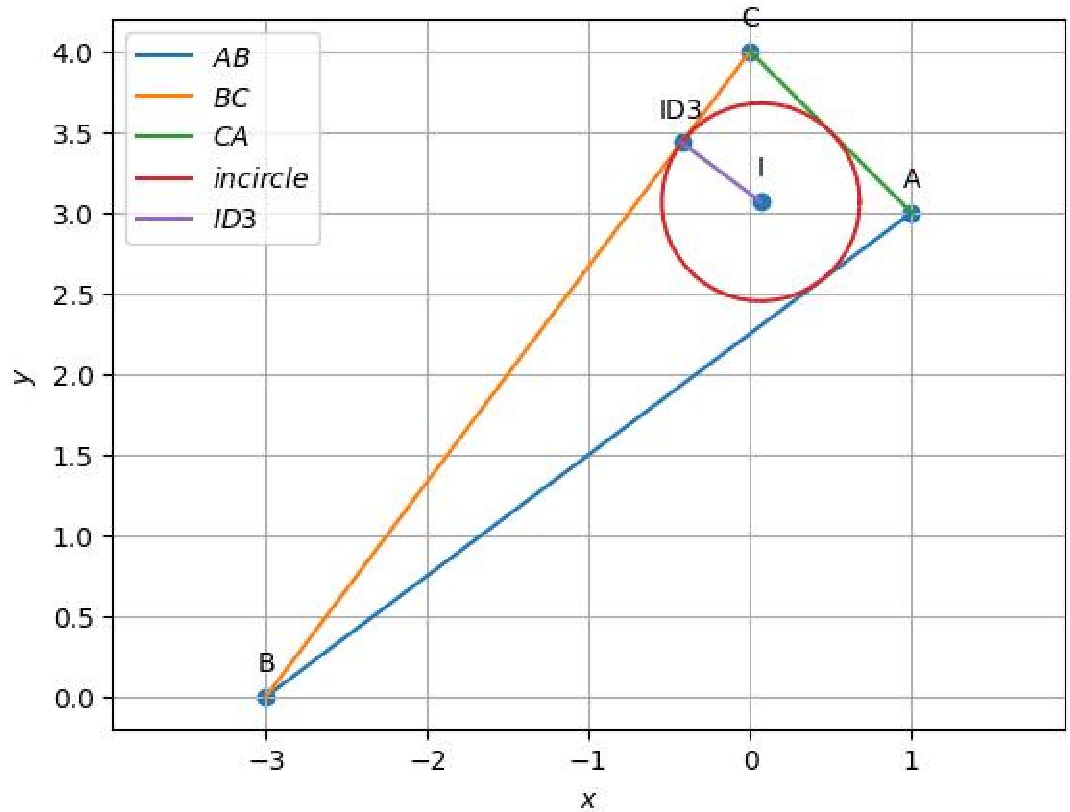

Figure 1.3: incircle of $\triangle ABC$

1.5.5. Using (1.1.7) verify that

$$\angle BAI = \angle CAI. \tag{1.5.5.1}$$

AI is the bisector of $\angle A$.

Solution:

$$\cos\angle BAI \triangleq \frac{(\mathbf{B}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{I}-\mathbf{A}\|} \tag{1.5.5.2}$$

$$\cos\angle CAI \triangleq \frac{(\mathbf{C}-\mathbf{A})\top(\mathbf{I}-\mathbf{A})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{I}-\mathbf{A}\|} \tag{1.5.5.3}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $vecI$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.5.5.4}$$

$$\mathbf{I} = \begin{pmatrix} -4.3726 \\ 4.116 \end{pmatrix} \tag{1.5.5.5}$$

$$\mathbf{B} - \mathbf{A} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \tag{1.5.5.6}$$

$$\mathbf{C} - \mathbf{A} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{1.5.5.7}$$

$$\mathbf{I} - \mathbf{A} = \begin{pmatrix} -5.372 \\ 1.116 \end{pmatrix} \tag{1.5.5.8}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{A}\| = \sqrt{25} = 5 \tag{1.5.5.9}$$

$$\|\mathbf{C}-\mathbf{A}\| = \sqrt{2} \tag{1.5.5.10}$$

$$\|\mathbf{I}-\mathbf{A}\| = 5.486 \tag{1.5.5.11}$$

$$\tag{1.5.5.12}$$

(a) for $\angle BAI$:

On substtuting the values in (**??**) ,We get

$$\cos\angle BAI \triangleq \frac{\begin{pmatrix}-4 & -3\end{pmatrix}\begin{pmatrix}-5.372\\1.116\end{pmatrix}}{5\times 5.486} \tag{1.5.5.13}$$

$$\tag{1.5.5.14}$$

On solving

$$\angle BAI = 0.6610^\circ \tag{1.5.5.15}$$

(b) for $\angle CAI$:

On substtuting the values in (**??**) ,We get

$$\cos\angle CAI \triangleq \frac{\begin{pmatrix}0 & 4\end{pmatrix}\begin{pmatrix}-5.372\\1.116\end{pmatrix}}{\sqrt{2}\times 5.486} \tag{1.5.5.16}$$

$$\tag{1.5.5.17}$$

On solving

$$\angle CAI = 0.276^\circ \tag{1.5.5.18}$$

Therefore $\angle BAI = \angle CAI$. and AI is the bisector of $\angle A$.

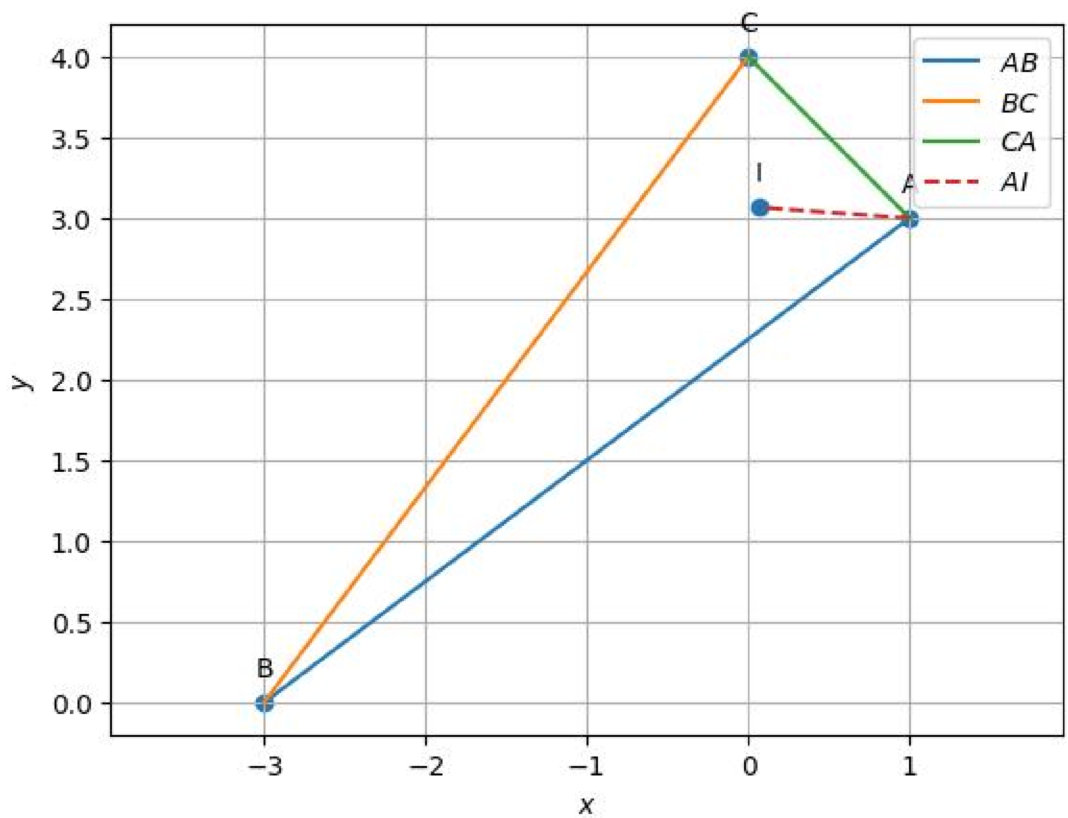

Figure 1.4: Angular bisector AI

1.5.6. Verify that BI, CI are also the angle bisectors of $\triangle ABC$.

Solution:

(a) To prove BI is an angular bisector of $\angle B$

$$\cos \angle ABI \triangleq \frac{(\mathbf{A}-\mathbf{B})\top(\mathbf{I}-\mathbf{B})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{B}\|} \tag{1.5.6.1}$$

$$\cos \angle CBI \triangleq \frac{(\mathbf{C}-\mathbf{B})\top(\mathbf{I}-\mathbf{B})}{\|\mathbf{C}-\mathbf{B}\|\,\|\mathbf{I}-\mathbf{B}\|} \tag{1.5.6.2}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.5.6.3}$$

$$\mathbf{I} = \begin{pmatrix} -4.372 \\ 4.116 \end{pmatrix} \tag{1.5.6.4}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \tag{1.5.6.5}$$

$$\mathbf{C}-\mathbf{B} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \tag{1.5.6.6}$$

$$\mathbf{I}-\mathbf{B} = \begin{pmatrix} -1.372 \\ 4.116 \end{pmatrix} \tag{1.5.6.7}$$

also calculating the values of norms

$$\|\mathbf{A}-\mathbf{B}\| = \sqrt{25} = 5 \tag{1.5.6.8}$$

$$\|\mathbf{C}-\mathbf{B}\| = \sqrt{25} = 5 \tag{1.5.6.9}$$

$$\|\mathbf{I}-\mathbf{B}\| = 18.823 \tag{1.5.6.10}$$

$$\tag{1.5.6.11}$$

i. for $\angle ABI$:

On substtuting the values in (**??**) ,We get

$$\cos\angle ABI \triangleq \frac{\begin{pmatrix} 4 & 3 \end{pmatrix}\begin{pmatrix} -1.372 \\ 4.116 \end{pmatrix}}{6 \times 18.823} \tag{1.5.6.12}$$

$$\tag{1.5.6.13}$$

On solving

$$\angle ABI = 0.060^\circ \tag{1.5.6.14}$$

ii. for $\angle CBI$:

On substtuting the values in (**??**) ,We get

$$\cos\angle CBI \triangleq \frac{\begin{pmatrix} 3 & 4 \end{pmatrix}\begin{pmatrix} -1.372 \\ 4.116 \end{pmatrix}}{\sqrt{5} \times 18.823} \tag{1.5.6.15}$$

$$\tag{1.5.6.16}$$

On solving

$$\angle CBI = 0.294^\circ \tag{1.5.6.17}$$

Therefore $\angle ABI = \angle CBI$. and BI is the bisector of $\angle B$.

(b) To prove CI is an angular bisector of $\angle C$

$$\cos \angle BCI \triangleq \frac{(\mathbf{B}-\mathbf{C})\top(\mathbf{I}-\mathbf{C})}{\|\mathbf{B}-\mathbf{C}\| \, \|\mathbf{I}-\mathbf{C}\|} \tag{1.5.6.18}$$

$$\cos \angle ACI \triangleq \frac{(\mathbf{A}-\mathbf{C})\top(\mathbf{I}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\| \, \|\mathbf{I}-\mathbf{C}\|} \tag{1.5.6.19}$$

From the given values of $\mathbf{A}, \mathbf{B}, \mathbf{C} and \mathbf{I}$,

$$\mathbf{A} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \tag{1.5.6.20}$$

$$\mathbf{I} = \begin{pmatrix} -4.372 \\ 4.116 \end{pmatrix} \tag{1.5.6.21}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix} -3 \\ -4 \end{pmatrix} \tag{1.5.6.22}$$

$$\mathbf{A}-\mathbf{C} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{1.5.6.23}$$

$$\mathbf{I}-\mathbf{C} = \begin{pmatrix} -4.372 \\ 0.116 \end{pmatrix} \tag{1.5.6.24}$$

also calculating the values of norms

$$\|\mathbf{B}-\mathbf{C}\| = \sqrt{25} = 5 \tag{1.5.6.25}$$

$$\|\mathbf{A}-\mathbf{C}\| = \sqrt{2} \tag{1.5.6.26}$$

$$\|\mathbf{I}-\mathbf{C}\| = 4.373 \tag{1.5.6.27}$$

$$\tag{1.5.6.28}$$

i. for $\angle BCI$:

   On substtuting the values in (**??**) ,We get

$$\cos\angle BCI \triangleq \frac{\begin{pmatrix}-3 & -4\end{pmatrix}\begin{pmatrix}-4.372\\0.116\end{pmatrix}}{\sqrt{25}\times 4.373} \tag{1.5.6.29}$$

$$\tag{1.5.6.30}$$

   On solving

$$\angle BCI = 0.621^\circ \tag{1.5.6.31}$$

ii. for $\angle ACI$:

   On substtuting the values in (**??**) ,We get

$$\cos\angle ACI \triangleq \frac{\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}-4.372\\0.116\end{pmatrix}}{\sqrt{2}\times 4.373} \tag{1.5.6.32}$$

$$\tag{1.5.6.33}$$

On solving

$$\angle ACI = 0.725^\circ \tag{1.5.6.34}$$

Therefore $\angle BCI = \angle ACI$. and CI is the bisector of $\angle C$.

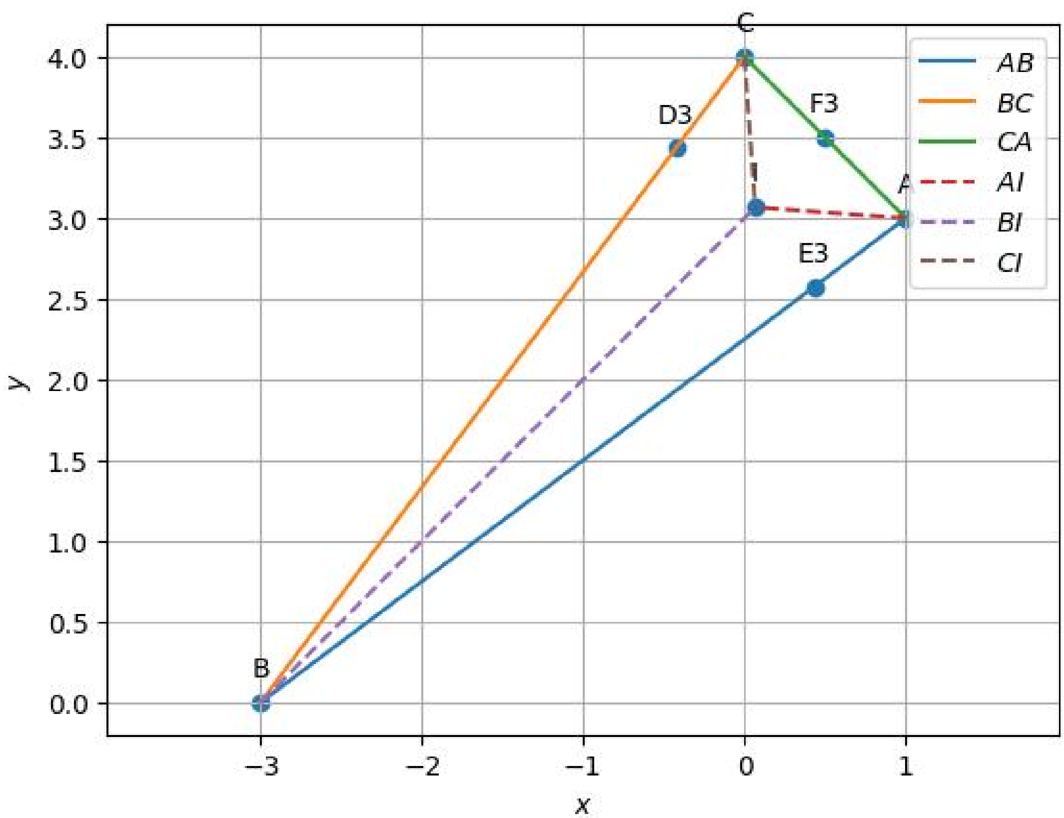

Figure 1.5: Angular bisectors BI and CI