

MATH-COMPUTING

December 28, 2023

1. **Question(MATH-12.10.5.17):** Let \mathbf{a} and \mathbf{b} be two unit vectors and θ is the angle between them. Then $\mathbf{a} + \mathbf{b}$ is a unit vector.

(A) $\theta = \frac{\pi}{4}$

(B) $\theta = \frac{\pi}{3}$

(C) $\theta = \frac{\pi}{2}$

(D) $\theta = \frac{2\pi}{3}$

solution:

Given,

$$\|\mathbf{a}\| = \|\mathbf{b}\| = 1 \quad (1)$$

$$\|\mathbf{a} + \mathbf{b}\| = 1 \quad (2)$$

and, its magnitude will be as:

$$magnitude = \sqrt{\mathbf{a}^2 + \mathbf{b}^2} \quad (3)$$

$$magnitude = \sqrt{1^2 + 1^2} = \sqrt{2} \quad (4)$$

Squaring on both sides of (2), we get

$$\|\mathbf{a} + \mathbf{b}\|^2 = 1^2 \quad (5)$$

$$\implies \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 + 2\mathbf{a}^\top \mathbf{b} = 1 \quad (6)$$

Substituting (6) in (1), we get

$$\implies 1 + 1 + 2(\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta) = 1 \quad (7)$$

$$\implies 2 + 2(\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta) = 1 \quad (8)$$

$$\implies 2(\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta) = -1 \quad (9)$$

$$\implies (\|\mathbf{a}\| \|\mathbf{b}\| \cos \theta) = \frac{-1}{2} \quad (10)$$

Substituting (1) in (10), we get

$$\implies \cos \theta = \frac{-1}{2} \quad (11)$$

$$\implies \theta = \frac{2\pi}{3} \quad (12)$$

Assume,

$$\|\mathbf{a}\| = (\|\mathbf{a}_1\| \|\mathbf{a}_2\|) = (4, 4) \quad (13)$$

Let,

$$\|\mathbf{b}_1\|^2 + \|\mathbf{b}_2\|^2 = 1 \quad (14)$$

Since,

$$\mathbf{ab} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 \quad (15)$$

$$\implies \|\mathbf{a}_1\| \|\mathbf{b}_1\| + \|\mathbf{a}_2\| \|\mathbf{b}_2\| = \frac{-1}{2} \quad (16)$$

$$\implies 4\|\mathbf{b}_1\| + 4\|\mathbf{b}_2\| = \frac{-1}{2} \quad (17)$$

$$\implies 4(\|\mathbf{b}_1\| + \|\mathbf{b}_2\|) = \frac{-1}{2} \quad (18)$$

$$\Rightarrow \langle \|\mathbf{b}_1\| + \|\mathbf{b}_2\| \rangle = \frac{-1}{8} \quad (19)$$

Squaring on both sides:

$$\langle \|\mathbf{b}_1\| + \|\mathbf{b}_2\| \rangle^2 = \left(\frac{-1}{8} \right)^2 \quad (20)$$

$$\Rightarrow \|\mathbf{b}_1\|^2 + \|\mathbf{b}_2\|^2 + 2\|\mathbf{b}_1\|\|\mathbf{b}_2\| = \frac{-1}{16} \quad (21)$$

$$\Rightarrow 1 + 2\|\mathbf{b}_1\|\|\mathbf{b}_2\| = \frac{-1}{16} \quad (22)$$

$$\Rightarrow \|\mathbf{b}_1\|\|\mathbf{b}_2\| = \frac{-15}{32} \quad (23)$$

$$\Rightarrow \frac{\|\mathbf{b}_1\|}{\frac{1}{\|\mathbf{b}_2\|}} = \frac{-15}{32} \quad (24)$$

$$\Rightarrow \langle \|\mathbf{b}_1\|\|\mathbf{b}_2\| \rangle = \left(-15, \frac{1}{32} \right) \quad (25)$$

Therefore:

$$\|\mathbf{a}\| = \langle \|\mathbf{a}_1\|\|\mathbf{a}_2\| \rangle = (4, 4) \quad (26)$$

$$\|\mathbf{b}\| = \langle \|\mathbf{b}_1\|\|\mathbf{b}_2\| \rangle = \left(-15, \frac{1}{32} \right) \quad (27)$$

To check whether the generated vector values of a and b are possible to make the requires angle:

$$\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} \right) \quad (28)$$

$$\Rightarrow \mathbf{ab} = \mathbf{a_1b_1} + \mathbf{a_1b_1} \quad (29)$$

$$\Rightarrow \mathbf{ab} = 4(-15) + 4\left(\frac{1}{32}\right) = -60.125 \quad (30)$$

$$\Rightarrow |\mathbf{a}| = \sqrt{\mathbf{a_1}^2 + \mathbf{a_2}^2} = \sqrt{4^2 + 4^2} = 5.65 \quad (31)$$

$$\Rightarrow |\mathbf{b}| = \sqrt{\mathbf{b_1}^2 + \mathbf{b_2}^2} = \sqrt{(-15)^2 + \left(\frac{1}{32}\right)^2} = \sqrt{225.0009} = 15 \quad (32)$$

Therefore:

$$\theta = \cos^{-1}\left(\frac{-60.125}{83.75}\right) = \cos^{-1}(-0.65) = 126^\circ \approx 120^\circ \quad (33)$$

Hence verified