MACHAKOS UNIVERSITY

SCHOOL OF PURE & APPLIED SCIENCES

DEPARTMENT OF MATHEMATICS & STATISTICS

UNIT CODE & NAME: SCO 109 LINEAR ALGEBRA FOR COMPUTER SCIENCE

BY

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INTRODUCTION

Linear Algebra is an important part of Mathematical background required not only for Mathematicians but also for other Scientists.

This module introduces the learner to the foundations of Linear Algebra. It begins with the background information on matrices and their basic operations, determinants and inverses. Matrices are later used in solving systems of linear equations. The module also covers vectors, their basic operations and their application to equations of lines and planes in three dimensions. Other topics include Vector spaces, linear independence and Independence.

The content is divided into thirteen short lectures. Each lecture begins with a brief introduction and objectives before discussing the main content. Each concept is followed by an activity which is intended to help the student test their understanding. In addition, answers to selected self help questions are given at the ended of the module. Further reading is suggested at the end of each lecture. This is intended to help the learner get exposed to other approaches to concepts and hopefully to more challenging exercises.

The learner is strongly advised to do all the activities in each lecture before proceeding to the next lecture.

GENERAL OBJECTIVES

By the end of this module, learners should be able to:

1. Perform basic operations on matrices
2. Find the determinants of 2 X 2 and 3 X 3 matrices
3. Find the inverses of 2 X 2 and 3 X 3 matrices
4. Solve systems of linear equations using Gauss Jordan method, Cramer's rule and Inverse matrix method
5. Determine whether or not a given set of vectors is linearly dependent or linearly independent
6. Perform basic operations on vectors
7. Find the equations of planes and lines in R^3
8. Determine the basis and dimension of given of given vector spaces

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LECTURE ONE

MATRICES

INTRODUCTION

This lecture is divided into two subtopics.

In the first subtopic, the learner is introduced to the concept of matrices. Basic definitions are given including the order of a matrix, equal matrices, row matrix, column matrix e.t.c.

The second subtopic deals with operations on matrices. These include addition, subtraction, scalar multiplication, dot product and matrix multiplication.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 1.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 1.5

LECTURE OBJECTIVE

By the end of this lecture, the learner should be able to perform basic operations on matrices Including addition, subtraction, scalar multiplication and matrix multiplication.

1.1 DEFINITIONS

Matrices are used as a shorthand for keeping essential data arranged in rows and columns i.e matrices are used to summarize data in tabular form.

Definition: A matrix is an ordered rectangular array of numbers, usually enclosed in parenthesis or square brackets. Capital (Upper – case) letters are used to denote matrices.

Order of a Matrix

The size of a Matrix is specified by the number of rows (horizontal) and the number of columns (vertical).

A general matrix of order m $\times$n is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{22} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix} \quad = \lfloor a_{ij} \rfloor$$

$$i \to ith \;\; row$$

$$j \to jth \; column$$

A Square Matrix is a one with the same number of rows and columns i.e m $\times$ m matrix. Two matrices are of the same size if they have the same order.

A vector is a matrix with one row (1 $\times$ n) or one column (n $\times$ 1). A row vector is of the form 1 $\times$ n, and a column vector is of the form m $\times$ 1.

A zero matrix of order m $\times$ n is the matrix with $a_{ij} = 0 \quad i = 1, \cdots m, \quad j = 1 \ldots n$.

Similarly we talk of zero rows and column vector.

$$0 = (0,0,\cdots 0) \text{ or } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Equal Matrices:

Two matrices A & B are said to be equal if they have the same order (size) m $\times$ n and $a_{ij} = b_{ij} \ \forall i \quad \& \forall j$

1.2 OPERATIONS ON MATRICES

Addition and Subtraction of matrices
This is performed on matrices of the same order (size). Let A and B be m × n matrices.

A + B = $\left[a_{ij}\right] \pm \left[b_{ij}\right] = \left[a_{ij} \pm b_{ij}\right] = \left[c_{ij}\right] \ (m \times n)$

Scalar Multiplication
This is performed on any matrix and the resulting matrix is the same size . $c\,A = c\left[a_{ij}\right] = \left[ca_{ij}\right]$. Each entry is multiplied by same number (scalar).

Dot product: let $\vec{a}$ and $\vec{b}$ be any two vector of size n (matrices with a single column or row). $\vec{a} = \left[a_1, a_2, \cdots a_n\right], \vec{b} = \left(b_1, b_2, \cdots b_n\right)$. The dot product of $\vec{a}$ & $\vec{b}$ denoted $\vec{a} \cdot \vec{b}$, is given by $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$ .The dot product is also scalar product.

Dot product of a row & column vector of order n.

$$\left(a_1, a_2 \cdots a_n\right) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i = \vec{a} \cdot \vec{b}$$

Matrix Multiplication:
Let $A = \left[a_{ik}\right]$ be an m × n matrix, and $B = \left[b_{kj}\right]$ an n × s matrix. The matrix product AB is the m × s matrix $C = \left[c_{ij}\right]$ where c_{ij} the dot product of the i^{th} row of A and the j^{th} column of B.

i.e. $AB = C$, $\left[a_{ik}\right]\left[b_{kj}\right] = \left[c_{ij}\right]$; $C_{ij} = A_i \cdot B_j = \sum_{k=1}^{n} a_{ik} b_{kj}$

Remark:
1.Let $A(m \times n)$, $B(s \times r)$ be two matrices .
$C = AB$ exists iff $n = s$ & C is $m \times n$
$C = BA$ exists iff $r = m$ & C is $s \times n$
2. It's possible for AB to be defined while BA is not defined. i.e. matrix multiplication is not commutative.

1.2 OPERATIONS ON MATRICES (Continued)

Examples

1. Let $A=\begin{pmatrix}1 & -2 & 3\\ 4 & 5 & -6\end{pmatrix}$ and $B=\begin{pmatrix}3 & 0 & 2\\ -7 & 1 & 8\end{pmatrix}$. Then

$$A+B=\begin{pmatrix}1+3 & -2+0 & 3+2\\ 4-7 & 5+1 & -6+8\end{pmatrix}=\begin{pmatrix}4 & -2 & 5\\ -3 & 6 & 2\end{pmatrix}$$

2. $3A=\begin{pmatrix}3\cdot 1 & 3\cdot(-2) & 3\cdot 3\\ 3\cdot 4 & 3\cdot 5 & 3\cdot(-6)\end{pmatrix}=\begin{pmatrix}3 & -6 & 9\\ 12 & 15 & -18\end{pmatrix}$

3. $2A-3B=\begin{pmatrix}2 & -4 & 6\\ 8 & 10 & -12\end{pmatrix}+\begin{pmatrix}-9 & 0 & -6\\ 21 & -3 & -24\end{pmatrix}=\begin{pmatrix}-7 & -4 & 0\\ 29 & 7 & -36\end{pmatrix}$

4. $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}=\begin{pmatrix}1\cdot 1+2\cdot 0 & 1\cdot 1+2\cdot 2\\ 3\cdot 1+4\cdot 0 & 3\cdot 1+4\cdot 2\end{pmatrix}=\begin{pmatrix}1 & 5\\ 3 & 11\end{pmatrix}$

$$\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}1\cdot 1+1\cdot 3 & 1\cdot 2+1\cdot 4\\ 0\cdot 1+2\cdot 3 & 0\cdot 2+2\cdot 4\end{pmatrix}=\begin{pmatrix}4 & 6\\ 6 & 8\end{pmatrix}$$

The above example shows that matrix multiplication is not commutative, i.e.the products AB and BA of matrices need not be equal.

1.3 SUMMARY.

In this lecture we have defined a matrix, the order of a matrix, equal matrices, row matrix, column matrix and square matrix

We have also learnt how to perform addition, subtraction, scalar multiplication, dot product and multiplication of matrices.

1.4 ACTIVITY 1

5. Let $A=\begin{pmatrix} 2 & -5 & 1 \\ 3 & 0 & -4 \end{pmatrix}$, $B=\begin{pmatrix} 1 & -2 & -3 \\ 0 & -1 & 5 \end{pmatrix}$, $C=\begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & -1 \end{pmatrix}$.Find $3A+4B-2C$.

6. Find x, y, z and w if $3\begin{pmatrix} x & y \\ z & w \end{pmatrix}=\begin{pmatrix} x & 6 \\ -1 & 2w \end{pmatrix}+\begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}$

7. Find AB and BA if (a) A = $\begin{pmatrix} 1 & -2 \\ -5 & 4 \end{pmatrix}$ B=$\begin{pmatrix} -3 & -1 \\ -4 & 6 \end{pmatrix}$

   (b) $A=\begin{pmatrix} 5 & 5 & -2 \\ 4 & 1 & -3 \end{pmatrix}$ $B=\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$

1.5 FURTHER READING

1. Linear Algebra by Fraleigh & Beauregard
2. Linear Algebra: Schaum's Outline Series
3. Linear Algebra by J. N. Sharma, A.R. Vasishta

1.6 SELF-TEST QUESTIONS 1

1. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 & -4 \\ 3 & -2 & 6 \end{pmatrix}$. Find (a) AB, (b) BA

2. Given $A = (2,-1)$ and $B = \begin{pmatrix} 1 & -2 & 0 \\ 4 & 5 & -3 \end{pmatrix}$, find (a) AB, (b) BA

3. Given $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix}$, find (a) AB, (b) BA

LECTURE TWO

PROPERTIES OF MATRIX OPERATIONS

INTRODUCTION

In this lecture we deal with the transpose of a matrix ad also properties of matrix operations.

The subtopic introduces the learner to the transpose of a matrix. The definition is given, followed by a number of properties.

The second subtopic deals with properties of matrix operations. These include additive commutativity, of additive and multiplicative associativity, distributive laws etc.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 2.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 2.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to

1. Find the transpose of a given matrix.
2. Identify and verify properties of the transpose of a matrix.
3. Identify and verify properties of matrix operations like additive commutativity, additive and multiplicative associativity, distributive laws etc.

2.1 TRANSPOSE OF A MATRIX

The transpose of the matrix A is matrix $B = A^T$, such that $b_{ij} = a_{ji}$ i.e. the rows become columns and vice versa.

1. If $A = A^T$, we say A is symmetric. (only for a square matrix). If A is symmetric,

a) $A\,A^T$ and $A + A^T$ are symmetric.

b) A^k symmetric $\forall_k$

c) If A, B are symmetric, $\alpha\, A + BA$ is symmetric.

2. If $A^T = -A$, we say A is skew symmetric. In that case,

(a) $A - A^T$ is skew symmetric

(b) If A, B are skew symmetric, $\alpha 4 + B\,A$ is skew symmetric.

3. If A, S are $n \times n$ (square) matrices and A is symmetric, then $S^T A S$, $S\,A S^T$ are symmetric

4. Every square matrix can be expressed as the sum of a symmetric & skew-symmetric matrix. i.e. $A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right)$

$S(A) = \frac{1}{2}\left(A + A^T\right)$ is the symmetric part; $K(A) = \frac{1}{2}\left(A - A^T\right)$ is the skew-symmetric part.

5. If A is $n \times n$ and $f(x)$ any polynomial, then $f\left(A^T\right) = \left[f(A)\right]^T$

2.2 PROPERTIES OF MATRIX OPERATIONS

1. $A+B=B+A$ - Addition is commutative
2. $(A+B)+C=A+(B+C)$ - Addition is associative
3. $A+0=0+A=A, 0$ is the identity for addition
4. $\alpha(A+B)=\alpha A+\alpha B$, left distributive law.
5. $(\alpha+\beta)A=\alpha A+\beta A$, right distributive law
6. $(\alpha\beta)A=\alpha(\beta A)$, associativity of scalar multiplication
7. $(\alpha A)B=A(\alpha\beta)-$ scalar pull through
8. $(AB)C=A(BC)$ - associativity of matrix multiplication
9. $1_n A=A,\ AI_m=A,\ A(m\times m)$ - identity for matrix multiplication
10. $A(B+C)=AB+AC$, left distributive law
11. $(A+B)C=AC+BC$, right distributive law
12. $(A^T)^T=A$
13. $(A+B)^T=A^T+B^T$ - transpose of the sum=sum of transpose
14. $(AB)^T=B^T A^T$ - transpose of product = product of transpose

    $(AB)^*=B^*A^*$.

Proof of most of these properties involve routine computations.

Show that $(A-B)(A+B)=A^2-B^2$ iff $AB=BA$ i.e A & B commute

Solution:

$\Rightarrow$ Suppose $(A-B)(A+B)=A^2-B^2$ and show that $A \,\&\, B$ commute.

$$A^2-B^2=(A-B)(A+B)$$
$$=A(A+B)-B(A+B)$$
$$=A^2+AB-BA-B^2$$

Hence $AB-BA=0$; $AB=BA$ and so $A \,\&\, B$ commute.

$\Rightarrow$ Assume A & B commute & show that $(A-B)(A+B)=A^2-B^2$

Since $A \,\&\, B$ commute $AB=BA$

$$(A-B)(A+B)=A(A+B)-B(A+B)$$
$$=A^2+AB-BA-B^2$$

but $AB=BA;\quad =A^2+AB-AB-B^2=A^2-B^2$

2.3 SUMMARY

In this lecture, we have learnt how to find the transpose of a given matrix and to identify and verify their properties. We have also studied the properties of matrix operations including additive commutativity, additive and multiplicative associativity and distributive laws.

2.4 ACTIVITY 2

1. Given $A=\begin{pmatrix}1 & 2\\ 3 & 5\end{pmatrix}, B=\begin{pmatrix}2 & 1\\ 1 & 3\end{pmatrix}$, show that $(AB)^T = B^T A^T$.

2. Let $A=\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}, B=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

   (a)Find A^T

   (b)Calculate $A^T BA$

   (c)Use your results in (b) above to determine whether or not $A^T BA$ is symmetric.

3. For what values of x, y and z is the following matrix symmetric?

$$\begin{bmatrix}x & y & z\\ 2 & 0 & 3\\ 4 & 3 & 3\end{bmatrix}$$

2.5 FURTHER READING

1. Linear Algebra by Michael O'nan, Herbert Enderton
2. A First Course in Linear Algebra by Daniel Zelisky
3. Elementary Linear Algebra by Bennard Kolman
4. Elementary Linear Algebra by Howard Anton

2.6 SELF-TEST QUESTION 2

1. Find the transpose A^t of the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 \end{pmatrix}$

2. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$. Find (a) AA^t , (b) $A^t A$

LECTURE THREE

DETERMINANTS OF 2×2 AND 3×3 MATRICES

INTRODUCTION

This lecture is divided into two subtopics.

The first subtopic covers determinants of 2×2 and 3×3 matrices. In each case, several worked out examples are given

The second subtopic deals with a general formula for finding the determinant of $n\times n$ matrices.

An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 3.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 3.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to:

1. Find the determinants of a 2×2 matrix
2. Find the determinants of a 3×3 matrix
3. Use the general formula for finding the determinant of $n\times n$ matrices to find the determinants of a 4 X 4 and 5 X 5 matrix

3.1 DETERMINANTS OF 2×2 AND 3×3 MATRICES

Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Det $A=\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=a_{11}\,a_{22}-a_{12}\,a_{21}$. The determinant is a scalar.

Examples:

1. Find the value of λ such that $\begin{vmatrix} \lambda & \lambda \\ 3 & \lambda-2 \end{vmatrix}=0$.

2. Let $\begin{vmatrix} 1+x & 1 \\ 2+2x & 2 \end{vmatrix}=$ find *x*.

3. Let $\begin{vmatrix} x & 3 \\ 2 & 2x+1 \end{vmatrix}=4$. Find *x*.

3×3 **Matrices**

Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\det A=\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}-a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}+a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=a_{11}\left(a_{22}\,a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}\,a_{31}\right)+a_{13}\left(a_{21}a_{32}\right)$$

$$=a_{11}\,a_{22}\,a_{33}-a_{11}\,a_{23}\,a_{32}-a_{12}\,a_{21}\,a_{33}+a_{12}\,a_{23}\,a_{31}+a_{13}\,a_{21}\,a_{32}-a_{13}\,a_{22}\,a_{31}$$

$$=a_{11}\,a_{22}\,a_{33}+a_{12}\,a_{23}\,a_{31}+a_{13}\,a_{21}\,a_{32}-a_{11}\,a_{23}\,a_{32}-a_{12}\,a_{21}\,a_{33}-a_{13}\,a_{22}\,a_{31}$$

Note: This is a sum of **6 products**, *3 positive* and *3 negative*. Each product has exactly one factor from each row and column.

3.2 DETERMINANT OF AN $n \times n$ MATRIX

For a large square matrix the determinant is a sum of products, half of which have minus signs added. Each product will have exactly one factor from each row and one factor from each column.

There are $n!$ summations.

1. Evaluate $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & 4 \\ -2 & -1 & 1 \end{vmatrix}$

Solution: $\begin{vmatrix} 3 & 1 & 2 \\ -1 & -1 & 4 \\ -2 & -1 & 1 \end{vmatrix} = 3[(-1)1+1(4)]-1[-1(1)+8]+2[1-2] = 0$

2. Evaluate $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix}$

Solution: $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = 1(4)-3(2)+1(-3) = -5$

3. Evaluate $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{vmatrix}$

Solution: $\begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 & 0 \\ -2 & -1 & 1 \\ -1 & 3 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 & 0 \\ -1 & -1 & 1 \\ 1 & 3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 & 0 \\ -1 & -2 & 1 \\ 1 & -1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 & 2 \\ -1 & -2 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 1(4)-2(-4)+3(-6)-1(-6) = 0$

3.3 SUMMARY

In this lecture, we have learnt how to find the determinants of a 2×2 matrix and a 3×3 matrix. We have also used the general formula for finding the determinant of $n \times n$ matrices to find the determinants of a 4 X 4 and 5 X 5 matrix

3.4 ACTIVITY 3

1. Find the value of c such that $\begin{vmatrix} c & -3c \\ -1 & -c \end{vmatrix} = -2$
2. Find the value of λ if the matrix $A = \begin{bmatrix} \lambda & \lambda & \\ 3 & \lambda & -2 \end{bmatrix}$ is singular.
3. Find $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & -1 \\ 4 & -1 & 2 \end{vmatrix}$
4. Determine the values of x for which the determinant of A is zero where

A= $\begin{pmatrix} x-2 & 4 & 3 \\ 1 & x+1 & -2 \\ 0 & 0 & x-4 \end{pmatrix}$

5. Evaluate $\begin{vmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{vmatrix}$

6. Compute the determinant of $A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{pmatrix}$

3.5 FURTHER READING

1. Linear Algebra by Fraleigh & Beauregard
2. Linear Algebra: Schaum's Outline Series
3. Elementary Linear Algebra by Howard Anton

3.6 SELF-TEST QUESTIONS 3

1. Evaluate the determinant of each matrix:

(a) $\begin{pmatrix} 3 & -2 \\ 4 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} a-b & a \\ b & a+b \end{pmatrix}$

(i) $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 3\cdot 5-(-2)\cdot 4 = 23$ (ii) $\begin{vmatrix} a-b & a \\ b & a+b \end{vmatrix} = (a-b)(a+b)-a\cdot a = -b^2$

2. Determine those values of k for which $\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 0$.

$\begin{vmatrix} k & k \\ 4 & 2k \end{vmatrix} = 2k^2-4k = 0$, or $2k(k-2)=0$.Hence $k=0$;and $k=2$.That is, if $k=0$ or $k=2$, the determinant is zero.

3. Compute the determinant of each matrix:

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 2 & 5 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 1 \\ 4 & 2 & -3 \\ 5 & 3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 & 1 \\ 3 & 2 & -3 \\ -1 & -3 & 5 \end{pmatrix}$

LECTURE FOUR

PROPERTIES OF DETERMINANTS

INTRODUCTION

In this lecture, the learner is introduced to the concept of properties of determinants of matrices, which can be used to find the determinant of a matrix from the determinant of another matrix.

The concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 4.5 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 4 .4

LECTURE OBJECTIVE

By the end of this lecture, the learner should be able to use properties of determinants of matrices to find the determinant of a given matrix from the determinant of another matrix.

4.1 PROPERTIES OF DETERMINANTS

1. Det (AB) = Det A . Det B

2. If one row or column of A is multiplied by scalar r to get B, det (B) = r det A

3. If row i = 0 or column j = 0, det A =0.

4. For an n $\times$n matrix, det (r A) = r^n det A

5. If two rows or columns are identical det A = 0

6. If one row or column is a scalar multiple of another, det A = 0.

7. Adding a scalar multiple of a row or a column to another row or column respectively leaves the determinant unchanged.

4.2 SUMMARY

In this lecture, we have learnt to use properties of determinants of matrices to find the determinant of a given matrix from the determinant of another matrix.

4.3 ACTIVITY 4

1. (a) Evaluate $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix}$ (b) Hence find $\begin{vmatrix} 18 & 21 & 6 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{vmatrix}$

Using properties of determinants, evaluate:

2. $\begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$ 3. $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$ 4. $\begin{vmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$ 5. $\begin{vmatrix} 4 & 1 & 3 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix}$ 6. $\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

7. $\begin{vmatrix} 4 & 2 & 3 & -4 \\ 3 & -2 & 1 & 5 \\ -2 & 0 & 1 & -3 \\ 8 & -2 & 6 & 4 \end{vmatrix}$ 8. $\begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix}$ 9. $\begin{vmatrix} 4 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix}$ 10. $\begin{vmatrix} 3 & 4 & 2 \\ 2 & 5 & 0 \\ 3 & 0 & 0 \end{vmatrix}$ 11. $\begin{vmatrix} 4 & -3 & 5 \\ 5 & 2 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

12. $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix}$ 13. $\begin{vmatrix} 2 & 0 & 1 & 4 \\ 3 & 2 & -4 & -2 \\ 2 & 3 & -1 & 0 \\ 11 & 8 & -4 & 6 \end{vmatrix}$

4.4 FURTHER READING

1. Linear Algebra by J. N. Sharma, A.R. Vasishta
2. Linear Algebra by Michael O'nan, Herbert Enderton
3. A First Course in Linear Algebra by Daniel Zelisky

4.5 SELF-TEST QUESTIONS 4

Evaluate

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$ 2. $\begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$ 3. $\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 2 & 4 & 6 \end{vmatrix}$ 4. $\begin{vmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{vmatrix}$

LECTURE FIVE

INVERSES OF 2 x 2 AND 3 x 3 MATRICES

INTRODUCTION

This lecture is divided into two subtopics.

The first subtopic deals with inverses of 2 x 2 of matrices while the second one covers the inverses of 3 x 3 matrices.

However this lecture restricts itself to the method of using the adjoint of a matrix to find the inverses. Row reduction method is covered in the following lecture.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 5.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 5.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to find the inverses of 2 x 2 and 3 x 3 matrices using the method of the adjoint of a matrix.

5.1 INVERSE OF A 2 X 2 MATRIX

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ det $A = ad - bc \neq 0$; $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -c \\ -c & a \end{bmatrix}$

Note:

$$A\,A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & -ab + ab \\ cd - cd & sd - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad ad - bc \neq 0$$

Let $A = \left[a_{ij} \right]$ be a square matrix. The classical adjoint of A is the matrix

$adj\,(A) = \left[a_{ij} \right]^T$, $a_{ij}{}' = (-1)^{i+j} \det\left(A_{ij}\right)$ where a_{ij} is the (i, j) cofactor of A.

$$A^{-1} = \frac{1}{\det(A)} \text{ adj } (A)$$

For a 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Adj(A) = $\begin{bmatrix} a_{11}{}' & a_{12} \\ a_{21}{}' & a_{22}{}' \end{bmatrix}^T = \begin{bmatrix} d & -c \\ -b & d \end{bmatrix}^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$; $A^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

5.2 INVERSE OF A 3 X 3 MATRIX

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{32} & a_{33} \end{bmatrix}$; Adj (A) = $\begin{bmatrix} a_{11}' & a_{12}' & a_{13}' \\ a_{21}' & a_{22}' & a_{23}' \\ a_{31}' & a_{32}' & a_{33}' \end{bmatrix}^T = \begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T$

$|A_{11}| = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\ a_{33} - a_{23}\ a_{32}$; $|A_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\ a_{33} - a_{23}\ a_{31}$

$|A_{13}| = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\ a_{32} - a_{22}\ a_{31}$; $|A_{21}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}\ a_{33} - a_{13} a_{32}$

$|A_{22}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = a_{11}\ a_{33} - a_{31}\ a_{13}$; $|A_{23}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}\ a_{32} - a_{12}\ a_{31}$

$|A_{31}| = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}\ a_{23} - a_{13}\ a_{22}$; $|A_{32}| = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}\ a_{23} - a_{13}\ a_{21}$

$|A_{33}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\ a_{22} - a_{12}\ a_{21}$; Det $A = a_{11}\ |A_{11}| - a_{12}\ |A_{12}| + a_{13}\ |A_{13}|$; $A^{-1} = \dfrac{1}{\det A}\ adj(A)$

Examples: Find the inverse of $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

Det A = $4\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = 4(2) - 0 + 1(2-6) = 8 - 4 = 4$; Therefore, A^{-1} exists.

$adj(A) = \begin{bmatrix} |A_{11}| & -|A_{12}| & |A_{13}| \\ -|A_{21}| & |A_{22}| & -|A_{23}| \\ |A_{31}| & -|A_{32}| & |A_{33}| \end{bmatrix}^T$; $|A_{11}| = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2$; $|A_{12}| = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} = 2$; $|A_{13}| = \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -4$;

$|A_{21}| = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$; $|A_{22}| = \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} = 1$; $|A_{23}| = \begin{vmatrix} 4 & 0 \\ 3 & 1 \end{vmatrix} = 4$; $|A_{31}| = \begin{vmatrix} 0 & 1 \\ 2 & 0 \end{vmatrix} = -2$ $|A_{32}| = \begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} = -2$

$|A_{33}| = \begin{vmatrix} 4 & 0 \\ 2 & 2 \end{vmatrix} = 8 = 8\ adj(A) = \begin{bmatrix} 2 & -2 & -4 \\ 1 & 1 & -4 \\ -2 & 2 & 8 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -2 & 1 & 2 \\ -4 & -4 & 8 \end{bmatrix}$; $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}$

5.3 SUMMARY

In this lecture we have learnt to find the inverses of 2 x 2 and 3 x 3 matrices using the method of the adjoint of a matrix.

5.4 ACTIVITY 5

1. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 2 & -3 \\ 5 & 3 & 1 \end{pmatrix}$.Find A^{-1}

2. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix}$

3. Find the value of λ if the matrix $A = \begin{bmatrix} \lambda & \lambda & \\ 3 & \lambda & -2 \end{bmatrix}$ is singular.

4. Let A= $\begin{bmatrix} 2 & -5 & 2 \\ 1 & 2 & -4 \\ 3 & -4 & -6 \end{bmatrix}$

(a)Find det A
(b)Find Adj (A)
(c)Hence find A^{-1}

5. Let A = $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \\ 2 & 0 & 7 \end{pmatrix}$ Find A^{-1}

5.5 FURTHER READING

1. Linear Algebra by Fraleigh & Beauregard
2. Elementary Linear Algebra by Bennard Kolman
3. Elementary Linear Algebra by Howard Anton

5.6 SELF-TEST QUESTIONS

Find A^{-1} if

1. $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$ 2. $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 1 \\ 4 & 5 & -2 \end{bmatrix}$

LECTURE SIX
ROW-ECHELON FORM

INTRODUCTION

This lecture is divided into three sections, each section dealing with a specific subtopic.

The first subtopic covers the inverse of a matrix using row reduction method, while the second one covers the general concept of reducing a matrix to echelon form.

Finally, the learner is introduced to the canonical form of a matrix.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 6.7 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 6.6

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to:

1. Find the inverse of a matrix using row reduction method
2. Reduce a given matrix to echelon form.
3. Reduce a given matrix to canonical form.

6.1 INVERSE OF A MATRIX (ROW REDUCTION METHOD)

The inverse of a matrix a can be found using row reduction to echelon form of the augmented matrix (A/I) to get $\left(I / A^{-1}\right)$.

Example:

Find the inverse of $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ 2R_2 - R_1 \\ 4R_3 - 3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 4 & 1 & -3 & 0 & 4 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & -1 & -1 & 2 & 0 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right] \begin{array}{l} 2R_1 - R_3 \\ 2R_2 + R_3 \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 8 & 0 & 0 & 4 & 2 & -4 \\ 0 & 8 & 0 & -4 & 2 & 4 \\ 0 & 0 & 2 & -2 & -2 & 4 \end{array}\right] \begin{array}{l} R1/8 \\ R2/8 \\ R3/2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]; \quad \text{Inverse } \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ -1 & -1 & 2 \end{bmatrix}$$

6.2 ECHELON FORM OF A MATRIX

A $m \times n$ matrix B is said to be in row echelon form if it is of the form.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} & \\ 0 & b_{22} & b_{23} & \cdots & b_{xn} & \\ 0 & 0 & b_{33} & \cdots & b_{3n} & \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & \ldots & b_{ii} & \cdots & b_{in} \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & \cdots & & b_{mm} & b_{mn} \end{bmatrix} if \;\; m > n$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \ldots & b_{in} \\ 0 & b_{22} & b_{23} & \ldots & b_{2n} \\ 0 & 0 & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & 0 & \cdots & b_{nn} \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ldots & \ldots & 0 \end{bmatrix} if \;\; n > m$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & & b_{in} \\ 0 & b_{22} & \cdots & & b_{2n} \\ & & & & \vdots \\ & \ddots & & & \vdots \\ & & & & \cdots \\ & & & \ddots & b_{nn} \end{bmatrix} \;\; m = n$$

Note:

For a matrix in echelon form, for subsequent rows, the non-zero entries occur in later and later columns.

For a matrix in echelon form, all entries below the main diagonal are zero. Given any matrix B (not in echelon form) we perform the following **elementary row operations** to change it to echelon form:

1. Change the order of the rows (interchange some rows)
2. Multiply one row by a nonzero constant.
3. Add a multiple of one row to a nonzero multiple of another row.

6.3 REDUCED ROW-ECHELON FORM

Definition: A matrix is said to be in reduced row echelon form (canonical form) if:

1. Each nonzero row begins with a pivot entry 1. (Leading 1 of the row)
2. The rest of the columns containing the pivot entry 1 consists of 0s.
3. In subsequent rows, the pivot entries occur in later and later columns.
4. The all-zero rows are at the bottom (they are the unused rows).

Examples

1. Reduce $\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix}$ to reduced row-echelon (canonical) form.

Solution:

$$\begin{bmatrix} 3 & 4 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix} R_1 \leftrightarrow R_2 \begin{bmatrix} 1 & -1 & 3 & 1 \\ 3 & 4 & -1 & 1 \\ 4 & -3 & 11 & 2 \end{bmatrix} \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 4R_1 \end{array} \rightarrow \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 7 & -10 & -2 \\ 0 & -1 & -1 & -2 \end{bmatrix} \begin{array}{l} \\ R_2/7 \\ \\ \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 3 & 1 \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & -1 & -1 & -2 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 + R_2 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & 0 & -\frac{17}{7} & -\frac{16}{7} \end{bmatrix} \begin{array}{l} \\ \\ -\frac{7}{17} R_3 \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & \frac{11}{7} & \frac{5}{7} \\ 0 & 1 & -\frac{10}{7} & -\frac{2}{7} \\ 0 & 0 & 1 & \frac{16}{17} \end{bmatrix} \begin{array}{l} R_1 - \frac{11}{7} R_3 \\ R_2 + \frac{10}{7} R_3 \\ \\ \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{17} \\ 0 & 1 & 0 & \frac{18}{17} \\ 0 & 0 & 1 & \frac{16}{17} \end{bmatrix}$$

2. Reduce to echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & -2 & -1 & 1 \\ 1 & -1 & 3 & 4 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R1 \\ R_3 + R_1 \\ R_4 - R_1 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & -3 & 0 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - R_2 \end{array} \rightarrow$$

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 4 & 4 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4 - 2R_3 \end{array} \rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -4 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6.4 SUMMARY

In this lecture, we have learnt to find the inverse of a matrix using row reduction method, reduce a given matrix to echelon form and to canonical form.

6.5 ACTIVITY 6

1. Reduce $\begin{pmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{pmatrix}$ to (a) echelon form (b) row-reduced echelon form

2. Reduce matrix $A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$ to row-reduced echelon form.

3. Which one of the following matrices are row-reduced and which one is not.

(a) $\begin{pmatrix} 1 & 2 & -3 & 0 & 1 \\ 0 & 0 & 5 & 2 & -4 \\ 0 & 0 & 0 & 7 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 7 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 3 & 2 & 0 & 4 & 5 & -6 \\ 0 & 0 & 7 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 5 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$

6.6 FURTHER READING

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton

6.7 SELF-TEST QUESTIONS 6

1. Reduce the matrix $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ 7 & -4 & 1 \end{pmatrix}$ to echelon form .

2. Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & 2 & -3 \\ 5 & 3 & 1 \end{pmatrix}$ .Find A^{-1} using row-reduction method

2. Determine if the following matrices are in echelon canonical form or not.

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 5 & -2 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 8 & 7 & -3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

LECTURE SEVEN

SOLUTION OF SYSTEMS OF LINEAR EQUATIONS

INTRODUCTION

This lecture starts with a general introduction to the solution of a system of linear equations, followed by a more detailed section on the solution of equations using Gauss Jordan method.

Other methods are covered in the following lecture.

Several worked out examples have been included, and an activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 7.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 7.5

LECTURE OBJECTIVE

By the end of this lecture, the learner should be able to solve a system of linear equations using Gauss Jordan method.

7.1 SOLUTION OF SYSTEMS OF LINEAR EQUATIONS:

A Linear equation is an equation of the form

$$a_1\ x_1 + a_2\ x_2 + \cdots + a_n\ x_n = b$$

$a_1, a_2, \cdots, a_n, b$ are constants while $x_1, x_2, \cdots x_n$ are the variables to be determined. A solution to the equation is an *n*-tuple $(c_1, c_2, \cdots, c_n)$ such that

$$a_1\ c_1 + a_2\ c_2 + \cdots + a_n\ c_n = b$$

Example: $5x_1 - 3x_2 + 6x_3 - 4x_2 = 10$; $x_1 = 3$; $x_2 = 5$; $x_3 = 3$; $x_4 = 2$. i.e. (3,5,3,2) is a solution.

Two systems of equations are equivalent if they have the same solution set. We get equivalent system if we perform any of the following:

1. Change the order of listing the equations
2. Multiply one or more equations by a non-zero constant.
3. Add a multiple of one equation to a multiple of another equation.

There are various methods of solving a system of linear equations

Elimination Method:

The strategy is to eliminate one variable at a time:

Example:

Solve by elimination method

$x_1 + x_2 + x_3 = 5 \quad \cdots \qquad (1)$

$x_1 + 2x_2 + 3x_3 = 10 \cdots \qquad (2)$

$2x_1 + x_2 + x_3 = 0 \cdots \qquad (3)$

Solution:

Eliminate x_1

(2) – (1) $\qquad x_2 + 2x_3 = 5 \cdots$ (4)

(3)-2(1) $\qquad -x_2 - x_3 = -4 \cdots$ (5)

Eliminate x_2

(4) +(5) $\quad x_3 = 1$

Substitute in equation 5

$-x_2 - 1 = -4$, $x_2 = 3$

Substitute in equation 1

$x_1 + 3 + 1 = 5$, $x_1 = 1$

The elimination method is very tedious when there are many variables. We use a much organized elimination method in matrix form called Gauss-Jordan method.

7.2 GAUSS-JORDAN ELIMINATION METHOD

Given a linear system

$$a_{11}\ x_1 + a_{12}\ x_2 + \cdots + a_{1n} x_n = b_1$$

$$ax_1\ x_1 + a_{22}\ x_2 + \cdots + a_{2n} x_n = b_2$$

$$\vdots$$

$$a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

The system can be represented in matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Or simply $Ax = b$

Where A is the $n \times n$ is the coefficient matrix

$$x = (x_1, x_2 \cdots x_n)^T, \quad b = [b_1, b_2 \dots b_n]^T$$

An augmented matrix is the matrix $[A/b]$

The Gauss-Jordan method involves reducing the augmented matrix to reduced row echelon form.

Example:

Solve using Gauss-Jordan method

$$2x_1 - 4x_2 + 6x_3 = 20$$

$$3x_1 - 6x_2 + x_3 = 22$$

$$-2x_1 + 5x_2 - 2x_3 = -18$$

7.2 GAUSS-JORDAN ELIMINATION METHOD (continued)

Solution:

Matrix form

$$\begin{bmatrix} 2 & -4 & 6 \\ 3 & -6 & 1 \\ -2 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 20 \\ 22 \\ -18 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 20 \\ 3 & -6 & 1 & 22 \\ -2 & 5 & -2 & -18 \end{array}\right]$$

Divide row 1 by 2 i.e. $\frac{1}{2}R_1$ and use it to reduce a_{21} and a_{31} to zeros

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 3 & -6 & 1 & 22 \\ -2 & 5 & -2 & -18 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 0 & -8 & -8 \\ 0 & 1 & 4 & 2 \end{array}\right] R_2 \leftrightarrow R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 10 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -8 & -8 \end{array}\right] R_1 + 2R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & 14 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -8 & -8 \end{array}\right] \begin{array}{l} 8R_1 + 11R_3 \\ 2R_2 + R_3 \\ \end{array} \quad \left[\begin{array}{ccc|c} 8 & 0 & 0 & 24 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & -8 & -8 \end{array}\right] \begin{array}{l} R_1/8 \\ R_2/2 \\ R_3/-8 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{Solution} \quad x_1 = 3, \ x_2 = -2, \ x_3 = 1$$

7.2 GAUSS-JORDAN ELIMINATION METHOD (continued)

The general procedure to solve $Ax = b$ by Gauss-Jordan method is as follows:

1. Get the augmented matrix $[A|b]$
2. Get 1 in (1,1) position of matrix by
   a) rearranging the rows
   b) dividing row 1 by $a_{11} \neq 0$
3. Get zeros in all other positions of column 1
4. Get 1 in (2,2) position by rearranging rows or dividing all of row 2 by $a_{22} \neq 0$
5. Get zeros in all other positions of column 2.
6. Get 1 in (3,3), (4,4)… and in each case get zeros in other positions of that column.
7. Each row gives the solution

$$\left[\begin{array}{c|c} & c_i \\ I & c_2 \\ & \vdots \\ & c_n \end{array}\right] x_1 = c_1\,;\, x_2 = c_2, \ldots x_n = c_n$$

A system of linear equations may have a unique solution, many solutions or no solution.

Unique solution: $[A|b]$ reduces to $(I|c)$

No solution: The last row is a form $00\ldots a \neq 0$

Many solutions: Some variables can be written in terms of others

Previous examples have unique solutions

7.3 SUMMARY.

In this lecture, we have learnt to solve a system of linear equations using Gauss Jordan method.

7.4 ACTIVITY 7

Solve for x, y and z using Gauss-Jordan method

1. $$\begin{aligned} 2x-5y+2z&=7\\ x+2y-4z&=3\\ 3x-4y-6z&=5 \end{aligned}$$

2. $$\begin{aligned} x+y+z&=5\\ x+2y+3z&=10\\ 2x+y+z&=6 \end{aligned}$$

3. $$\begin{aligned} x_1+2x_2-x_3&=-1\\ 3x_1+8x_2+2x_3&=28\\ 4x_1+9x_2-x_3&=14 \end{aligned}$$

4. $$\begin{aligned} X_1+X_2-X_3+2X_4&=4\\ -2X_1+X_2+3X_3+X_4&=5\\ -X_1+2X_2+2X_3+3X_4&=6 \end{aligned}$$

7.5 FURTHER READING

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton

7.6 SELF-TEST QUESTIONS 7

Solve the following system by Gauss Jordan (elimination) method.

1. $$x_1 - x_2 + x_3 + 2x_4 = 1$$
$$2x_1 - x_2 + 3x_4 = 0$$
$$-x_1 + x_2 + x_3 + x_4 = -1$$
$$x_2 + x_4 = 1$$

2. $$x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1 + 2x_2 - x_3 - x_4 = 7$$
$$2x_1 - x_2 - x_3 - x_4 = 8$$
$$x_1 - x_2 + 2x_3 - 2x_4 = -7$$

3. $$x_1 + x_2 + 2x_3 + x_4 = 3$$
$$x_1 + 2x_2 + x_3 + x_4 = 2$$
$$x_1 + x_2 + x_3 + 2x_4 = 1$$
$$2x_1 + x_2 + x_3 + x_4 = 4$$

4. $$X_1 - X_2 + 2X_3 = 2$$
$$2X_1 + 3X_2 - X_3 = 14$$
$$3X_1 + 2X_2 + X_3 = 16$$
$$X_1 + 4X_2 - 3X_3 = 12$$

LECTURE EIGHT

CRAMER’S RULE AND INVERSE MATRIX METHOD

INTRODUCTION

The content of this lecture is divided into two subtopics.

In the first subtopic, the learner is introduced to the solution of a system of linear equations using Cramer’s rule, or the method of determinants.

The second subtopic deals with the inverse matrix method of solving linear equations. Gauss Jordan method is covered in the previous lecture.

Each method is illustrated using several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 8.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 8.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to:

1. Solve a system of linear equations using Cramer’s rule, or the method of determinants.
2. Solve a system of linear equations using inverse matrix method

8.1 CRAMERS RULE

This method uses determinants to solve a linear system $Ax = b$ provided det A (coefficient matrix) is nonzero.

Suppose $\det A \neq 0$, A square matrix $x = A^{-1}b$, $A^{-1} = \dfrac{1}{\det A}\, adj(A)$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} |A_{11}| & -|A_{21}| \\ -|A_{12}| & |A_{22}| \\ \vdots & \vdots \\ & \end{bmatrix}^T \text{; Let } \overline{a_{ij}} = (-1)^{i+j}\,|A_{ij}|;\ A^{-1} = \frac{1}{\det A}\begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & & \overline{a_{n2}} \\ \overline{a_{1n}} & \overline{a_{2n}} & & \overline{a_{nn}} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A}\begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{n1}} \\ \overline{a_{12}} & \overline{a_{22}} & & \overline{a_{n2}} \\ \vdots & \vdots & & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & & \overline{a_{nn}} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$x_j = \frac{1}{\det A}\left[b_1\,\overline{a_{1j}} + b_2\,\overline{a_{2j}} + \ldots + b_n\,\overline{a_{nj}}\right]$$

But $b_1\,\overline{a_{ij}} + b_2\,\overline{a_{2j}} + \ldots + b_n\,\overline{a_{nj}}$ is the determinant of the matrix

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{ij-1} & b_1 & a_{ij+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{1j-1} & b_2 & a_{2j+1} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nj-1} & b_n & b_{nj+1} & \ldots & a_{nn} \end{bmatrix}$$

i.e. matrix A with column j replaced with $(b_1, b_2 \ldots, b_n)$. The expansion is done along column j. Note that the determinant of A expanded along column j is

$a_{1j}\ \overline{a_{ij}} + a_{2j}\ \overline{a_{2j}} + \ldots + a_{nj}\ \overline{a_{nj}}$

Thus by replacing columns j by $[b_1, b_2 \cdots b_n]$ we have the determinants of the new matrix.

8.1 CRAMERS RULE (Continued)

For 3 $\times 3$ matrix $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad X_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & b_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \qquad X_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}}$$

Example

Solve the linear system using Crammer's rule

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3 \end{aligned}$$

Solution

$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$ .Then $x_1 = \dfrac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \dfrac{-4}{-2} = 2$, $x_2 = \dfrac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \dfrac{-6}{-2} = 3$,

And $x_3 = \dfrac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \dfrac{-8}{-2} = 4$

8.2 INVERSE MATRIX METHOD

Consider the system of linear equations represented in matrix form $A\,x = b$. Where A is an m × n matrix.

If det $A \neq 0$, then A^{-1} exists and $A^{-1}(A\,x) = A^{-1}b$; $Ix = x = A^{-1}b$. This method works only if det $A \neq 0$, and there is a unique solution $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-}b = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

Examples:

1. Solve the linear system using the inverse matrix

$x + 2y = 6$

$4x + 3y = 3$

Solution:

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad \text{det } A = 3 - 8 = -5; \quad A^{-1} = -\frac{1}{5}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\,b = -\frac{1}{5}\begin{bmatrix} 3 & -2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} 12 \\ -21 \end{bmatrix} = \begin{bmatrix} -12/5 \\ 21/5 \end{bmatrix};$$ **Solution**: $x = -12/5$, $y = 21/5$

2. Solve the linear system using the inverse matrix

$x_1 + 3x_2 + 2x_3 = 3$

$2x_1 + 4x_2 + 2x_3 = 8$

$x_1 + 2x_2 - x_3 = 10$

Solution:

$$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix}; \; A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix}; \text{ det A} = 4; \; A^{-1} = \begin{bmatrix} -2 & 7/4 & -1/2 \\ 1 & -3/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 & 7/4 & -1/2 \\ 1 & -3/4 & 1/2 \\ 0 & 1/4 & -1/2 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}; \; x_1 = 3, \; x_2 = 2, \; x_3 = -3$$

8.3 SUMMARY.

In this lecture, we have learnt to solve a system of linear equations using Cramer's rule, or the method of determinants and also by using inverse matrix method

8.4 ACTIVITY 8

Solve by Cramers rule

1. $2X_1 - 4X_2 + 6X_3 = 20$
   $3X_1 - 6X_2 + X_3 = 22$
   $-2X_1 + 5X_2 - 2X_3 = -18$

2. $X_1 + X_2 + X_3 = 5$
   $X_1 + 2X_2 + 3X_3 = 10$
   $2X_1 + X_2 + X_3 = 6$

3. $2x - 5y + 2z = 7$
   $x + 2y - 4z = 3$
   $3x - 4y - 6z = 5$

Solve by inverse matrix method

4. 2x+3y+4z=1
   x+y-z=0
   4x-y+2z=0

5. 2x-5y=-2z
   x-4z=b-2y
   -4y-6z=c-3x

6. y+x+z=5
   z+y+2x=6
   x+2y+3z=10

7. $x_1 + 3x_2 + 2x_3 = 3$
   $2x_1 + 4x_2 + 2x_3 = 8$
   $x_1 + 2x_2 - x_3 = 10$

8.5 FURTHER READING

1. A First Course in Linear Algebra by Daniel Zelisky
2. Elementary Linear Algebra by Bennard Kolman
3. Elementary Linear Algebra by Howard Anton

8.6 SELF-TEST QUESTION 8

Solve using Crammer's rule.

1.
$$2x-5y+2z=7$$
$$x+2y-4z=3$$
$$3x-4y-6z=5$$

2.
$$x_1+3x_2+2x_3=3$$
$$2x_1+4x_2+2x_3=8$$
$$x_1+2x_2-x_3=10$$

Solve the following three equations using inverse matrix method

3.
$$x+y+2z=1$$
$$x+2y-z=-2$$
$$x+3y+z=5$$

4.
$$x+2y+z=4$$
$$3x-4y-2z=2$$
$$5x+3y+5z=-1$$

LECTURE NINE

VECTORS

INTRODUCTION

This lecture covers the concept of vectors, their dot product and cross product. The application of dot product in looking for the angle between vectors is included. Application of cross product in the equations of lines and planes is covered in the last lecture in the module.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 9.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 9.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to

1. Perform addition subtraction and scalar multiplication of vectors.
2. Find the magnitude of a given vector.
3. Find dot product and cross product of vectors.
4. Use dot product to find the angle between vectors .

9.1 VECTORS (CO-ORDINATE SYSTEMS)

Any point on the Cartesian *x*-*y* plane can be represented as a pair (x_0, y_0) where x_0 is the x coordinate and y_0 is *y*-coordinate. The Cartesian plane is simply $\Re^2$. $\Re^3$ consists of three planes, *x*-*y* plane, *x*-*z* plane & *y*-*z* planes. The *x*-axis, *y*-axis and *z* – axes all meet at 90° (are perpendicular) at the origin (0,0,0). The *x*-axis consists of points $(x_0, 0, 0)$, *y*-axis $(0, y_0, 0)$ and *z*-axis $(0, 0, z_0)$. The *x*-*y* plane $(x_0, y_0, 0)$ the *x*-*z* plane $(x_0, 0, z_0)$ and the *y*-*z* plane $(0, y_0, z_0)$.

$\Re^n$ consists of all n-tuples consisting of real entries i.e.

$$\Re^n = \left\{ (a_1, a_2, \ldots, a_n) : a_i \in \Re \; \forall_i = 1, 2, \ldots n \right\}$$

VECTORS: (Quantities defined by magnitude & direction)

A **column vector is** an n-tuple of numbers written vertically $\begin{bmatrix} a_i \\ \vdots \\ a_n \end{bmatrix}$ (or n × 1 matrix)

If the numbers $a_1, a_2 \ldots, a_n$ are real we have a real column vector. If they are complex we have a complex column vector. The number a_i in the i^{th} (row) slot is the i^{th} component.

A row *n*-vector is an *n*-tuple $[a_1, a_2, \ldots, a_n]$ of numbers written horizontally.

Equality of vectors

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \Rightarrow m = n \quad a_i = b_i \quad \forall_i = 1, \ldots n \qquad i.e.\; a_1 = b_1, \; a_2 = b_2, \ldots, a_n = b_n$$

9.1 VECTORS (CO-ORDINATE SYSTEMS) (continued)

Addition & Subtraction

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \pm \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 \pm b_1 \\ a_2 \pm b_2 \\ \vdots \\ a_n \pm b_n \end{bmatrix}$$

Scalar multiplication

$$\alpha \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \alpha\, a_1 \\ \alpha\, a_2 \\ \vdots \\ \alpha\, a_n \end{bmatrix}$$ Where α is real scalar or complex scalar.

Properties:

Let a, b, c be *n*-column vectors , λ, μ scalars

1. $a+b=b+a$ - **Commutative law**
2. $(a+b)+c=a+(b+c)$ - **Associative law**
3. $0+a=a+0=a$ - **Additive identity**
4. $a+(-a)=(-a)+a=0$ **Additive inverse**
5. $\lambda(a+b)=\lambda\, a+\lambda\, b$
6. $(\lambda+\mu)\, a=\lambda\, a+\mu\, a$; $\quad (\lambda\mu)\, a=\lambda(\mu\, a)$

Length/Magnitude of Vector:

Let $a=[a_1, a_2, \ldots, a_n]$

$$|a|=\sqrt{a_1^2+a_2^2+\ldots+a_n{}^2}$$

$$|a+b| \le |a|+|b| \quad |r\, a|=|r||a|$$

9.2 DOT PRODUCTAND CROSS PRODUCT

Let $\vec{a} = [a_1, a_2, \ldots, a_n]$, $\vec{b} = [b_1, b_2, \ldots, b_n]$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$

$\vec{a} \cdot \vec{b} = a^T\, b$ if a & b are column vectors

$= a\, b^T$ if a & b are row vectors.

Properties of the dot product:

1. $u \cdot v = v \cdot u$ Commutative property
2. $u \cdot (v + w) = u \cdot v + u \cdot w$ Distributive Property
3. $(ru) \cdot v = u \cdot rv = r(u \cdot v)$ Homogeneous property
4. $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$

Note: $|a| = \sqrt{a \cdot a}$

Definition: A unit vector is a vector of length (magnitude) 1 i.e $|a| = \sqrt{a \cdot a}$ =1

Angle between vectors

Let *u* and *v* be nonzero vectors and let θ be the angle between them.
Then

$$u \cdot v = |u|\;|v|\;\cos\;\theta$$

$$\cos\;\theta = \frac{u \cdot v}{|u|\quad|v|}$$

$$\theta = \cos^{-1}\frac{u \cdot v}{|u|\;|v|}$$

Example:

Find the angle between $u = i + j + k$ and $v = i + j - k$

$(i + j + k) \cdot (i + j - k) = |i + j + k|\;|i + j - k| \cos\theta$

$1 + 1 - 1 = (\sqrt{3})(\sqrt{3})\;\cos\;\theta$

Cos $\theta = 1/3$, $\theta = \cos^{-1}\left(\frac{1}{3}\right) = 70.5^0$

9.2 DOT PRODUCT AND CROSS PRODUCT (Continued)

T Orthogonal /Perpendicular vectors:

The vectors *u* and *v* are orthogonal/perpendicular if the angle between then is 90^0.
$\theta = 90^0$, cos $90^0 = 0$
and so $u \cdot v = |u|\ |v|\ \cos\ 90^\circ$ and therefore $u \cdot v = |u|\ |v|$ (0)=0 $\Rightarrow u.v = 0$ (where $u \neq 0, v \neq 0$).

Exercise: Show that the vectors $u = [\sin\theta, \cos\theta]$ and $v = [\cos\theta - \sin\theta]$ are orthogonal

Cross/vector product:
The cross product of $\vec{a} = a_1 i + a_2 j + a_3 k$ and $\vec{b} = b_1 i + b_2 j + b_3 k$, denoted by $\vec{a} \times \vec{b}$ is a vector orthogonal to both $\vec{a}$ and $\vec{b}$ defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Example: Let $\overrightarrow{a} = (3i - j + k)$; $\overrightarrow{b} = (i + 2j - k)$

(a) Find $\vec{a} \times \vec{b}$
(b) Show that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$
(c) Show that $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$

Solution:

(a) $$(3i - j + k) \times (i + 2j - k) = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = i(1-2) - j(-3-1) + k(6 - -1) = -i + 4j + 7k$$

(b) $\overrightarrow{a} \cdot (\vec{a} \times \vec{b}) = (3i - j + k) \cdot (-i + 4j + 7k) = -3 - 4 + 7 = 0$ and therefore $\vec{a} \times \vec{b}$ is orthogonal to $\vec{a}$

(c) $\overrightarrow{b} \cdot (\vec{a} \times \vec{b}) (i + 2j - k) \cdot (-i + 4j + 7k) = -1 + 8 - 7 = 0$ and therefore $\vec{a} \times \vec{b}$ is orthogonal to $\vec{b}$

9.2 DOT PRODUCT AND CROSS PRODUCT (continued)

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (a_1 i + a_2 j + a_3 k)\cdot \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (a_1 i + a_2 j + a_3 k)\cdot \left[i\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - j\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + k\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \right]$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of the cross product:

Let a, b, c be vectors, α, β and y scalars

1. $a\times b = -b\times a$
2. $a\times(b+c) = (a\times b)+(a\times c)$

   $(a+b)\times c = a\times c + b\times c$
3. $\alpha\,(a\times b) = (\alpha\, a\times b) = a\times\alpha\, b$
4. $a\times a = -(a\times a)$

Two vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a} = k\vec{b}$ for some scalar *k*.

9.3 SUMMARY.

In this lecture, we have learnt performed addition, subtraction and scalar multiplication of vectors. We have also learnt to find the magnitude of a given vector and the dot product and cross product of vectors. Finally we have used the dot product to find the angle between vectors .

9.4 ACTIVITY 9

1. Find the angle between the vectors **a**=3**i**-**j**+k and **b**=-**i**+**j**+3**k**
2. Use the vectors $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}, \underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ to prove that

$\underset{\sim}{a} \times \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$

3. Find the angle between the vectors $\underset{\sim}{u} = -\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

4. Let $\begin{matrix} x = x_1 i + x_2 j + x_3 k \\ y = y_1 i + y_2 j + y_3 k \\ z = z_1 i + z_2 j + z_3 k \end{matrix}$ be vectors. Find $\underset{\sim}{x} \cdot \left(\underset{\sim}{y} \times \underset{\sim}{z}\right)$

5. Let $\begin{matrix} x = x_1 i + x_2 j + x_3 k \\ y = y_1 i + y_2 j + y_3 k \\ z = z_1 i + z_2 j + z_3 k \end{matrix}$ be vectors. Show that $\underset{\sim}{x} \cdot \left(\underset{\sim}{y} \times \underset{\sim}{z}\right) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$

6.Find the angle between the vector $\underset{\sim}{u} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and $v = -\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$

7. For any three vectors $\underset{\sim}{a} = a_1 i + a_2 j + a_3 k, \underset{\sim}{b} = b_1 i + b_2 j + b_3 k$ and $\underset{\sim}{c} = c_1 i + c_2 j + c_3 k$

Show that $\underset{\sim}{a} \times \left(\underset{\sim}{b} + \underset{\sim}{c}\right) = \underset{\sim}{a} \times \underset{\sim}{b} + \underset{\sim}{a} \times \underset{\sim}{c}$

9.5 FURTHER READING

1. A First Course in Linear Algebra by Daniel Zelisky
2. Elementary Linear Algebra by Bennard Kolman
3. Elementary Linear Algebra by Howard Anton

9.6 SELF-TEST QUESTION 9

1. Compute the length of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$.Find the angle between ***u*** and ***v***.

3. Show that the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are orthogonal

LECTURE TEN
VECTOR SPACES

INTRODUCTION

This lecture is divided into two subtopics.

In the first subtopic, the learner is introduced to the concept of a vector space.

The second subtopic deals with subspaces of vector spaces.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 10.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 10.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to

1. Define a vector space and a subspace of a vector space
2. Show that a given set is a vector space
3. Show that a given set is a sub space of a vector space

10.1 VECTOR SPACES

Definition of vector spaces

Let K be a given field and let V be a non empty set with rules of addition and scalar multiplication which assigns to any $u, v \in V$ a *sum* $u + v \in V$ and to any $u \in V, k \in K$ a *product* $ku \in V$. Then V is called a vector space over K (and the elements of V are called vectors) if the following axioms hold:

$[A_1]$: For any vectors $u, v, w \in V$, $(u+v)+w = u+(v+w)$.

$[A_2]$: There is a vector in V ,denoted by 0 and called the *zero vector*, for which $u + 0 = u$ for any vector $u \in V$

$[A_3]$: For each vector $u \in V$ there is a vector in V ,denoted by $-u$,for which $u + (-u) = 0$.

$[A_4]$: For any vectors $u, v \in V$, $u + v = v + u$.

$[M_1]$: For any scalar $k \in K$ and any vectors $u, v \in V$, $k(u+v) = ku + kv$.

$[M_2]$: For any scalars $a, b \in K$ and any vector $u \in V$, $(a+b)u = au + bu$.

$[M_3]$: For any scalars, $a, b \in K$ and any vector $u \in V$, $(ab)u = a(bu)$.

$[M_4]$: For the unit scalar $1 \in K$, $1u = u$ for any vector $u \in V$.

10.1 VECTOR SPACES (Continued)

Examples:

1. Show that the set $\Re^n = \{(a_1, a_2, \ldots, a_n) | a_n \in \Re\}$ is a vector space.
2. Show that $M_{n,m}(\Re)$, the set of all $n \times m$ matrices is a vector space.
3. Show that $p_{\Re}(x)$, the set of all polynomials in *x* with real coefficients is a vector space.
4. Show that $F(\Re)$,the set of all functions $f: \Re \to \Re$,is a vector space.
5. Define $\Re^{n+}$ to be the set of *n*-tuples $(x_1, \ldots, x_n)$ such that $x_i > 0 \quad \forall_i$
   Define addition and scalar multiplication by:
   $(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$
   $\lambda \otimes (x_1, \ldots x_n) = (x_1^{\lambda}, \ldots, x_n^{\lambda})$. Show that $\Re^{n+}$ is a vector space.

Solution:

Condition 1

$(x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n) \in \Re^{n+}$

Condition 2

$\lambda \otimes (x_1, \ldots x_n) = (x_1^{\lambda}, \ldots, x_n^{\lambda}) \in \Re^{n+}$

Condition 3

$0 \otimes (x_1, \ldots, x_n) = (x_1^0, \ldots, x_n^0) = (1, \ldots, 1)$

$(x_1, x_2, \ldots, x_n) \oplus (1, \ldots, 1) = (x_1, \ldots, x_2) \Rightarrow (1, \ldots, 1)$ is the zero vector

Condition 4

$$(x_1, \ldots, x_n) \oplus \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = \left(x_1\left(\frac{1}{x_1}\right), x_2\left(\frac{1}{x_2}\right), \ldots, x_n\left(\frac{1}{x_n}\right)\right) = (1, 1, \ldots, 1)$$

Then additive inverse of $(x_1, \ldots, x_n)$ is $\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$. Therefore $\Re^{n+}$ is a vector space.

10.2 SUBSPACE OF A VECTOR SPACE:

Definition: Let V be a vector space. A subset W of V is a subspace of V if W fulfils the requirements of a vector space, where addition and scalar multiplication in W produce the same vectors as these operations did in V.

Lemma: (Test for subspace)
A non-empty subset W of a vector space V is a subspace of V if and only if
(i) $\forall\, u, w \in W,\quad u + w \in W$ **-Closure under addition:**
(ii) $\forall\, r \in \Re, w \in W,\quad rw \in W$ **-Closure under scalar multiplication.**
i.e. W is a subspace of a vector space V *iff* its closed under vector addition and scalar multiplication.

Examples:
1. The set of diagonal n $\times$ n matrices is a subspace of M_n , the set of (n $\times$ n matrices).
2. Let $W \subset \Re^n$ with $w = (w_1 \ldots w_n)/W_i \in \Box$ with
a) $w_1 = 0$
b) w_1 even
c) w_1 is divisible by k
Show that each of them is a vector space.
3. The set D of all differentiable functions from $f : \Re \to \Re$ is a subspace of F, the set of all functions $f : \Re \to \Re$
4. (a)The set of all functions $f : \Re \to R$ such that $f(0) = 1$is not a subspace of the set $f : \Re \to \Re$.
(b) The set of all functions $f : \Re \to R$such that *f(1)*=0 is a subspace of the set $f : \Re \to \Re$.

10.3 SUMMARY

In this lecture, we have defined a vector space and a subspace of a vector space. We also have learnt to show that a given set is a vector space, or a sub space of a vector space

10.4 ACTIVITY 10

1. Show that the set $C[0,1]$ of continous functions mapping $\Re$ to $[0,1]$ is a vector space.

2. Show that $\Re^2$ with $(x_1,\ x_2)\oplus(y_1,\ y_2)=(x_1+y_1,\ x_2+y_2)$

$(x_1,x_2)\otimes(y_1,y_2)=(x_1 x_2-y_1\ y_2,\quad x_1 y_2+x_2\ y_1)$ is a vector space corresponding to C, the set of complex numbers).

$V=\Re^3$ .Show that $U=\{(x,y,0):x,y\in\Re y\}$ is a subspace of V.

3. Show whether or not the following are subspaces of $\Re^4$.
   (a) $U=\{(x,y,z)|x=2y\}$ (b) $V=\{(x,y,z)|x^2+y^2+z^2\le 1\}$

4. Let $V=\Re^3$ .Show that W is a subspace of V where $W=\{(a,b,0):a,b\in\Re\}$

5. Determine whether or not the following are subspaces of $\Re^3$.
   (a) $W=\{(x,y,z)|x+y+z=0\}$ (b) $U=\{(x,y,z)|xy=0\}$

6. Let V be a vector space over a field F and suppose A is a non-empty subset of V. State 2 necessary and sufficient conditions that A is a subspace of V

7. Let $u_1\ ,u_2,u_3,\cdots u_m\in\Re^n$, and let A be the set of all linear combinations of $u_1\ ,u_2,u_3,\cdots u_m$. Prove that A is a subspace of $\Re^n$.

10.5 FURTHER READING

1. Linear Algebra by Fraleigh & Beauregard
2. Linear Algebra: Schaum's Outline Series
3. Linear Algebra by J. N. Sharma, A.R. Vasishta

10.6 SELF-TEST QUESTIONS 10

1. Show that the line $y = 2x$ is a subspace of $\Re^2$

2. Show that the line $y = x+1$ is not a subspace of $\Re$ .i.e. $W = \{(x, y) / y = x+1\}$ $= \{(x, x+1) / x \in \Re\}$

3. Show that the set of all **invertible** n × n matrices is not a subspace of the set $M_n\,(R)$ of all n × n matrices.

LECTURE ELEVEN

LINEAR COMBINATIONS

INTRODUCTION

This lecture covers linear combinations and linear spans. Several examples have been given on how to write a given vector as a linear combination of other vectors. The relationship between linear spans, vector spaces and subspaces is also included.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 11.5 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 11.4

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to:

1. Write a given vector as a linear combination of a given set of vectors.
2. Prove theorems on the relationship between linear spans, vector spaces and subspaces.

11.1 LINEAR COMBINATIONS

Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V over scalar field K, and $a_1, a_2, \ldots a_n \in K$. A linear combination of the vectors $v_1, v_2, \ldots, v_n$ is any expression of the form

$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$

Example 1 $2(3-1, 4) + 5(1, 0, -1) - 6(0, 0, 1)$ is a linear combination of (3, -1, 4), $(1,0,-1)$ and $(0,0,1)$.

Example 2 Write the vector $v = (1,-2,5)$,as a linear combination of the vectors $e_1 = (1,1,1), e_2 = (1,2,3)$ and $e_3 = (2,-1,1)$.

Solution: We wish to express v as $v = xe_1 + ye_2 + ze_3$, with x, y and z as yet unknown scalars. Thus we require

$$\begin{aligned}(1,-2,5) &= x(1,1,1) + y(1,2,3) + z(2,-1,1)\\ &= (x, x, x) + (y, 2y, 3y) + (2z, -z, z)\\ &= (x + y + 2z, x + 2y - z, x + 3y + z)\end{aligned}$$

Form the equivalent system of equations by setting corresponding components equal to each other, and then reduce to echelon form:

$$\begin{array}{lclcl} x + y + 2z = 1 & & x + y + 2z = 1 & & x + y + 2z = 1 \\ x + 2y - z = -2 & \text{or} & y - 3z = -3 & \text{or} & y - 3z = -3 \\ x + 3y + z = 5 & & 2y - z = 4 & & 5z = 10 \end{array}$$

Note that the above system is consistent and so has a solution. Solve for the unknowns to obtain $x = -6, y = 3, z = 2$ .Hence $v = -6e_1 + 3e_2 + 2e_3$.

The span of $\{v_1, v_2, \ldots, v_n\}$ over K is the set of all linear combinations of $v_1, v_2, \ldots, v_n$.

i.e. Span $(v_1, v_2, \ldots, v_n) = \{a_1 v_1 + a_2 v_2 + \ldots + a_n\ v_n / a_1, a_2 \ldots a_n \in K\}$

For real vector spaces, Span $(v_1, v_2, \ldots, v_n) = \{a_1 v_1 + a_2 v_2 + \ldots + a_n v_n / a_i \in \Re\}$

Theorem: Span $(v_1, v_2, \ldots, v_n)$ is a subspace of the vector space V.

Proof: Show closure under addition & scalar multiplication.

Let $u, w \in$ span $(v_1, v_2, \ldots, v_n)$

Condition 1 of a subspace-**Closure under addition:**

Let $u = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$ for some $a_i \in \Re$ and $w = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n$ for some $b \in \Re$

$u + w = (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \ldots + (a_n + b_n)v_n \in$ span $(v_1, v_2, \ldots, v_n)$.

Since $a_i + b_i \in \Re$ for $i = 1, 2, \ldots, n$

i.e. a linear combination of $(v_1, v_2, \ldots, v_n)$ + a linear combination of $(v_1, v_2, \ldots, v_n)$

= a linear combination of $(v_1, v_2, \ldots, v_n)$

11.1 LINEAR COMBINATIONS (continued)

Condition 2 of a subspace-**Closure under scalar multiplication:**

$\alpha u = \alpha(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n) = (\alpha a_1) v_1 + (\alpha a_2) v_2 + \ldots + (\alpha a_n) v_n \in$ span $(v_1, v_2, \ldots, v_n)$.
since $\alpha a_i \in \Re \; \forall_i = 1, 2, \ldots, n$

i.e. a scalar multiple of a linear combination of $(v_1, v_2, \ldots, v_n)$ is also a linear combination of $(v_1, v_2, \ldots, v_n)$

$\Rightarrow$ span $(v_1, v_2, \ldots, v_n)$ is a subspace of V.

Example:

1. Let $P_{\Re}(x)$ be the set of all polynomials in x with real coefficients over $\Re$.

Span $(1, x, x^2) = \{a + bx + cx^2 / a, b, c \in \Re\}$ = $P_2(x)$, set of all polynomials in $P_{\Re}(x)$ of degree ≤ 2.

Observe that

1. Span $(1, x, x^2, 3 + 5x) \subset$ Span $(1\, x, x^2)$
2. Span $(1, 3 + 5x, x^2) \subset$ Span $(1, x, x^2)$

Theorem: Let V be a vector space and W_1 and W_2 be subspaces of V with W_1 = Span $(v_1, v_2, \ldots, v_n)$, $W_2 =$ Span $(u_1, u_2, \ldots, u_m)$. If each u_i is a linear combination of the v_i's, Then Span $(u_1, u_2, \ldots, u_m) \subset$ Span $(v_1, v_2, \ldots, v_n)$. $W_2 \subset W_1$

Proof: Since each u_i is a linear combination of v_i's, $u_i \in W_i \; \forall_i = 1, 2, \ldots, n$.

By definition of subspace, any linear combination in W_2 is in W_2

$\Rightarrow$ Span $(u_1, \ldots, u_n) = W_2 \subset W_1$

Corollary: Let W_1 = Span $(v_1, v_2, \ldots, v_n)$ and W_2 = Span $(u_1, u_2, \ldots, u_n)$. If each v_i is a linear combination of u_i's and each u_i is a linear combination of v_i's. Then $W_1 = W_2$

11.1 LINEAR COMBINATIONS (continued)

Proof: $W_1 \subset W_2$ and $W_2 \subset W_1$ implies that $W_1 = W_2$

If W = Span$\left(v_1, v_2, \ldots, v_n\right)$, we say W is a subspace **generated** or **spanned** by $v_1, v_2, \ldots, v_n$ and that $\left\{v_1, v_2, \ldots, v_n\right\}$ is a generating/spanning set for W. If a vector space V has a finite generating set, we say V is **finitely generated.**

Claim:

1. Span $\left(v_1, v_2, \ldots v_n\right)$ = span $\left(v_1, v_2, \ldots v_{i-1}, \lambda v_i, v_{i+1}, ..., v_n\right)$ for any $\lambda \neq 0$ .Thus, replacing a generator by a nonzero scalar multiple of itself leaves the subspaces unchanged.
2. Span $\left(v_1, v_2, \ldots, v_i, \ldots, v_j + \lambda v_i, \ldots, v_n\right)$= Span $\left(v_1, \ldots, v_n\right)$ $i \neq j$.

Replacing a generator by the sum of itself and a scalar multiple of another generator leaves the space unchanged.

Definition: Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V. The set spanned by $W_1, W_2, \ldots, W_n$ is the sum of W_1, $W_2, \ldots, W_n$, denoted $W_1 + W_2 + \ldots + W_2$ and defined by

$W_1 + W_2 + \ldots + W_2 = \left\{u_1 + u_2 + \ldots u_n \,/\, u_i \in W_i\right\}$

Lemma: If $W_1, W_2, \ldots, W_n$ are subspaces of V, then $W_1 + W_2 + \ldots + W_n$ is a subspace of V

Examples:

1. Let $V = P_{\Re}^5\left(x\right)$, the set of all polynomials with real coefficients with degree ≤ 5. Then,

$W_2 = P_{\Re}^2\left(x\right)$, the set of all polynomials with real coefficients with degree ≤ 2 is a subspace of V.

$W_2 = P_{\Re}^3\left(x\right)$, the set of all polynomials with real coefficients with degree ≤ 3 is a subspace of V.

$W_4 = P_{\Re}^4\left(x\right)$, the set of all polynomials with real coefficients with degree ≤ 4 is a subspace of V.

Now, $W_2 + W_3 = W_3$ and $W_2 + W_1 + W_4 = W_4$

2. Let $V = \Re^3, e_1 = \left(1, 0, 0\right), e_2 = \left(0, 1, 0\right), e_3 = \left(0, 0, 1\right)$

Let also W_1 = Span $\left(e_1\right)$, W_2= Span$\left(e_2\right)$, $W_3 =$ Span $\left(e_3\right)$ and W_4= Span $\left(e_2, e_3\right)$

Then $V = W_1 + W_2 + W_3; = W_1 + W_4 \;\; = W_1 + W_2 + W_4$ Also, $W_4 = W_2 + W_3$

11.2 SUMMARY.

In this lecture we have learnt to write a given vector as a linear combination of a given set of vectors and prove theorems on the relationship between linear spans, vector spaces and subspaces.

11.3 ACTIVITY 11

1. Show that $(11,3,-8)$ is a linear combination of $(1,1,0)$ and $(2,1,-1)$

2. If W, W_1, W_2 are subspaces of *V*, prove that

(a) $W + W = W, W + \{0\} = W, W + V = V$

(b) *If* $W_1 \subset W_2$, then $W_1 + W_2 = W_2$

(c) $W_1 \subset W_2 \Leftrightarrow W_1 + W_2 = W_2$

(d) $W_1 \bigcup W_2$ is a subspace of *V* iff $W_1 \subset W_2$ or $W_2 \subset W_1$

3. For which value of *m* will the vector $u = (1,-2,m)$ in $\Re^3$ be a linear combination of $v = (3,0,-2)$ and $w = (2,-1,-5)$

11.4 FURTHER READING

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton

11.5 SELF-TEST QUESTIONS 11

1. Write the vector $v=(2,-5,3)$ in $\Re^3$ as a linear combination of the vectors $e_1=(1,-3,2), e_2=(2,-4,-1)$ and $e_3=(1,-5,7)$.

2. For which value of k will the vector $u=(1,-2,k)$ in $\Re^3$ be a linear combination of the vectors $u=(3,0,-2)$ and $w=(2,-1,-5)$?

3. Show that the vectors $u=(1,2,3), v=(0,1,2)$ and $w=(0,0,1)$ generate $\Re^3$.

4. Find conditions on a,b and c so that $(a,b,c)\in\Re^3$ belongs to the space generated by $u=(2,1,0), v=(1,-1,2)$ and $w=(0,3,-4)$.

LECTURE TWELVE
LINEAR DEPENDENCE AND INDEPENDENCE

INTRODUCTION

The content of this lecture is divided into two subtopics.

In the first subtopic, the learner is introduced to the concept of linear dependence and independence.

The second subtopic deals with the application of linear dependence and independence in finding the basis and dimension of a given vector space.

Each concept is illustrated by several worked out examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section12.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 12.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to:

1. Determine whether a given set of vectors is linearly dependent and independent.
2. Use the concept of linear dependence and independence to find the basis and dimension of a given vector space.

12.1 LINEAR DEPENDENCE AND INDEPENDENCE

Definition: The vectors $v_1, v_2, \ldots, v_n$ of a vector space *V* are ***linearly dependent*** if there exists real numbers $a_1, a_2, \ldots, a_n$ with at least one $a_i \neq 0$ such that $a_i v_i + a_2 v_2 + \ldots + a_n v_n = 0$. The vectors are ***linearly independent*** if they are not dependent i.e. for every linear combination $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ then $a_i = 0 \ \forall \quad i = 1, 2, \ldots, n$.

Note:

1. Any set of vectors including the zero vector is linearly dependent.

2. If none of the vectors is zero and $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ with v_i's dependent, then at least two of the a_i's are non-zero.

Criterion for dependence

A finite list of nonzero vectors $v_1, \ldots, v_n$ in a vector space *V* is ***linearly dependent*** iff some vector is a linear combination of its predecessors. (OR $v_1, \ldots, v_n$ are ***linearly dependent*** iff one v_i is a linear combination of others)

Proof:

$\Rightarrow$ Suppose a vector v_k is a linear combination of $v_1, v_2, \ldots, v_{k-1}$, say $v_k = b_1 v_1 + \ldots + b_{k-1} v_{k-1}, bi \neq 0$ for some *i*. Then $b_1 v_1 + b_2 v_2 + \ldots + b_{k-1} v_{k-1} - 1 v_k + 0 v_{k+1} + \ldots + 0 v_n = 0$.

$\Leftarrow$ Suppose $a_1 v_1 + a_2 v_2 + a_n v_n = 0$ where $v_1, \ldots, v_n$ is a ***linearly dependent*** set.

Let $a_k \neq 0$ where $a_{k+1} = 0, \ a_{k+2} = 0, \ldots, a_n = 0$

Then $a_1 v_1 + \ldots + a_{k-1} v_{k-1} + a_k v_k = 0$ i.e.

$v_k = \left(-\dfrac{a_1}{a_k}\right) v_1 + \left(\dfrac{-a_2}{a_k}\right) v_2 + \ldots + \left(\dfrac{-a_{k-1}}{a_k}\right) v_{k-1}$ i.e. v_k is a linear combination of its predecessors.

Criterion for independence

A finite list of vector $v_1, v_2, \ldots, v_n$ in a vector space *V* is ***linearly independent*** iff no vector is a linear combination of the others.

Claim: Let $\{v_1, v_2, \ldots, v_n\}$ be a ***linearly independent*** set in *V*, and $u \in V$. Then $v_1, v_2, \ldots, v_n, u$ are ***linearly dependent*** iff u is a linear combination of $v_1, v_2, \ldots, v_n$.

12.1 LINEAR DEPENDENCE AND INDEPENDENCE (Continued)

Examples:

1. The vectors $e_1 = (1,0,0,...,0), e_2 = (0,1,0,\ldots,0),..., e_n = (0,0,0,\ldots,0,1)$ in $\Re^n$ are linearly independent.
2. The set $\{1, x, x^2, \ldots, x^n\}$ in $P^{n+1}(x)$ is linearly independent.
3. The set $\{1, x, 3-2x\}$ is linearly dependent since $3-2x = 3(1)-2(x)$
4. Determine whether or not $\{(1,2,3,1), (2,2,1,3), (-1,2,7-3)\}$ in $\Re^4$ is linearly dependent.

Method 1:

Find a_1, a_2, a_3 such that $a_1\,(1,2,3,1) + a_2\,(2,2,1,3) + a_3\,(-1,2,7,-3) = (0,0,0,0)$

Method 2: Simply reduce the matrix $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \\ -1 & 2 & 7 & -3 \end{pmatrix}$ to echelon form.

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \\ -1 & 2 & 7 & -3 \end{pmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_1 + R_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 4 & 10 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

OR

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 3 \\ -1 & 2 & 7 & -3 \end{pmatrix} \begin{matrix} V_1 \\ V_2 \\ V_3 \end{matrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 4 & 10 & -2 \end{bmatrix} \begin{matrix} V_1 \\ V_2 - 2V_1 \\ V_1 + V_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} V_1 \\ V_2 - 2V_1 \\ (V_1 + V_3) - 2(V_2 - 2V_1) \end{matrix}$$

$\Rightarrow (V_1 + V_3) - 2(V_2 - 2V_1) = 0 \Rightarrow -3V_1 + 2V_2 + V_3 = 0$ but the coefficients of V_1, V_2, V_3 are not zero $\Rightarrow$ V_1, V_2, V_3 are *linearly dependent.*

12.1 LINEAR DEPENDENCE AND INDEPENDENCE (Continued)

5. Show that the set $\{1, \sin^2 x, \cos^2 x\}$ is a *linearly dependent* set of functions in the vector space F of all functions mapping $\Re$ to $\Re$.

Solution: $\cos^2 x = 1 - \sin^2 x$, i.e. $\cos^2 x$ is a linear combination of $\sin^2 x$ and 1.
OR $\sin^2 x = 1 - \cos^2 x$ i.e. $\sin^2 x$ is a linear combination of $\cos^2 x$ and 1.
Since one vector can be written as a linear combination of the others, we conclude that $\{1, \sin^2 x, \cos^2 x\}$ is *linearly dependent*.

6. Any set of more than n vectors in $\Re^n$ is *linearly dependent*. In particular, $(1,-2,1)$, $(3,-5,2)$, $(2,-3,6)$ and $(1,2,1)$ in $\Re^3$ are *linearly dependent*.

Theorem: Let $v_1, v_2, \ldots, v_n$ be n vectors in $\Re^n$. The following conditions are equivalent

1. The vectors are *linearly independent*.
2. The vectors generate all of $\Re^n$
3. The matrix A having these vectors as columns (or rows) is invertible.

An infinite set S of vectors in a vector space V is *linearly independent* if there is no dependence relation involving a finite number of vectors in S.
e.g. The set $\{1, x, x^2, x^3 \ldots\}$ is *linearly independent* in the vector space $P(x)$ of all polynomials with real coefficients.

12.2 BASIS AND DIMENSION

Consider a vector space *V* generated/spanned by $v_1, v_2, \ldots, v_n$. If this set of vector is *linearly dependent*, then one vector, say v_n, is a linear combination of $v_1, v_2, \ldots, v_{n-1}$ and hence Span $(v_1, v_2, \ldots, v_n) = Span(v_1, v_2, \ldots, v_{n-1})$.

We may delete superfluous generators until we have a *linearly independent*, set say $v_1, v_2, \ldots, v_k$

Span $v_1, v_2, \ldots, v_n$ = Span $v_1, v_2, \ldots, v_k$, $k \le n$.

If we delete any vector in the *linearly independent* set, we would no longer generate *V*. A set of generators is minimal if every proper subset of them fails to span/generate *V*. Minimality and Independence are equivalent for a set of generators.

Definition: Let *V* be a vector space. A set of vectors in *V* is a **basis** for *V* if

(1) The vectors generate/span *V*

(2) The vectors are *linearly independent*

Examples:

1. The set $\{e_1, e_2, \ldots, e_n\}$ where $e_1 = (1,0,0,...,0)$, $e_2 = (0,1,0,\ldots,0)$, ..., $e_n = (0,0,0,\ldots,0,1)$ form a basis for $\Re^n$
2. The set $\{1, x, x^2, \ldots, x^{n-1}\}$ form a basis for $p_{\Re}(x)$, the set of all polynomials in *x* of degree $\le n$ with real coefficients.
3. Vectors $e_1 = (1,0)$ and $e_2 =$ (1,1) from a basis for $\Re^2$

Theorem: Let *V* be a vector space with basis $B = \{b_1, b_2, \ldots, b_n\}$.Each vector $v \in V$ can be uniquely expressed in the form $v = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n$, $\alpha_i \in \Re$ (*i.e.* there is exactly one choice for each α_i)

Proof: Suppose $v = \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n \quad \alpha_i \in \Re$

$= \beta_1 b_1 + \beta_2 b_2 + \ldots + \beta_n b_n \quad \beta_i \in \Re$ with $\alpha_i \ne \beta_i$ for some *i*. Then

$(\alpha_1 - \beta_1) b_1 + (\alpha_2 - \beta_2) b_2 + \ldots + (\alpha_i - \beta_i) b_i + \ldots + (\alpha_n - \beta_n) b_n = 0$ with $\alpha_i - \beta_i \ne 0$. This contradictions the fact that $b_1, b_2, \ldots, b_n$ are linearly independent by definition of a basis. Hence $\alpha_i = \beta_i \quad \forall i$

Example

1.Show that the vectors $(1,2,-1,0)$ (0,1,0,1), (-1,-5,2,0) and (2,3,-2,7) form a basis for $\Re^4$.

Solution: Show that the vectors are linearly independent.

Theorem: Let *V* be a vector space and $\{v_1, \ldots, v_n\}$ a basis of *V*.

(a) If $m > n$, then any set of *m* vectors of *V* is linearly dependent.

(b) Any other basis contains precisely *n* elements.

(c) *n* can be characterized as either the minimum number of generators of *V* or the maximum number of linearly independent vectors in *V*.

12.2 BASIS AND DIMENSION (Continued)

Definition: The dimension of a finitely generated vector space *V* is the number of elements in any basis of *V*, denoted *dim(V)*. We say *V* is an *n*-dimensional vector space.

Example: Find dimension of the subspace $W = S\text{pan}\,(1,-3,1),\ (-2,6,-2),\ (2,1,-4),\ (-1,10,-7)$ of $\Re^3$.

Solution:

$$\begin{bmatrix} 1 & -3 & 1 \\ -2 & 6 & -2 \\ 2 & 1 & -4 \\ -1 & 10 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & -7 & -6 \\ 0 & 7 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 1 \\ 0 & 7 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix};\ W = \text{span } \{(1,-3,1),(2,1,-4)\} \Rightarrow \ Dim(W)=2.$$

Theorem: Let *V* be an *n*-dimensional vector space and $\{v_1,\dots,v_m\}$ a linearly independent set of vectors in *V*.

(i) The set $\{v_1,\dots,v_m\}$ is a basis for *V* iff $m = n$.

(ii) Any linearly independent set of vectors of a finite dimensional space can be enlarged to a basis.

Proof:

(i) $\Rightarrow$ Suppose $\{v_1,v_2,\dots,v_m\}$ is a basis for *V*. Any two basis for a finitely generated vector space *V* have same number of elements called dim *V*. Hence $m = n$.

$\Leftarrow$ Suppose $m = n$. If $u \in V$ is not a linear combination of $v_1, v_2,\dots,v_m$. Then $\{v_1,\dots,v_m,k\} = S$ is a linearly independent set with $|S| = n+1$. But dim *V* = *n* is the maximum number of linearly independent vectors in *V*. This is a contradiction and hence $u \in \text{Span } (v_1,v_2,\dots v_m)$ and $(v_1,v_2,\dots,v_m)$ is a basis of *V*.

(ii) Let $\{v_1,v_2,\dots,v_k\}$ be a linearly independent set. If *k*=*n*, the proof follows from (i).

If $k < n$ take $V_{k+1} \notin \text{Span } (v_1,\dots,v_k)$ .Then $\{v_1,v_2,\dots,v_k,v_{k+1}\}$ is a linearly independent set. If $k+1 = n$, we have a basis from (i). If $k+1 < n$ we can repeat the process. The process stops when the number of elements in the enlarged set is *n*.

If *V* is an *n*-dimensional vector space and *W* a subspace of *V*, then *W* is finite dimensional, dim $W \subseteq$ dim *V* and any basis of *W* can be extended to a basis of *V*.

$W = V$ iff dim $W = \dim V$.

Example: Let $V = \Re^4$, W = Span *{*(1,0,0,0), (1,0,1,0)*}*. $\{(1,0,0,0),(1,0,1,0)\}$ is a basis for *W*. To enlarge this basis to a basis of $\Re^4$, we start with $U = \{(1,0,0,0),(1,0,1,0),e_1,e_2,e_3,e_4\}$ where *{* e_1,e_2,e_3,e_4 *}* is the usual basis for $\Re^4$. $\Re^4$ = span *(U)*

We delete vectors in *U* that are a linear combination of (1,0,0,0) and (1,0,1,0).

$e_1 = (1,0,0,0),\ e_3 = -(1,0,0,0)+(1,0,1,0)$

The set $\{(1,0,0,0),(1,0,1,0),e_2\}$ is linearly independent.

The set $\{(1,0,0,0),(1,0,1,0),e_2,e_3\}$ is linearly dependent since $e_3 = -(1,0,0,0)+1(1,0,1,0)$; Delete e_3

The set $\{(1,0,0,0),(1,0,1,0)\,e_2,e_4\}$ is linearly independent hence a basis of $\Re^4$.

12.3 SUMMARY.

In this lecture we have learnt to determine whether a given set of vectors is linearly dependent and independent and to use the concept of linear dependence and independence to find the basis and dimension of a given vector space.

12.4 ACTIVITY 12

1. Determine whether or not the matrices $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}, B=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $C=\begin{pmatrix}1&1\\0&0\end{pmatrix}$ are linearly independent.
2. Determine whether or not the following vectors in $\Re^4$ are linearly dependent {(1,3,-4,2),(3,8,-5,7),(2,9,4,23)}
3. Let X be the subspace $x=\{(x,y,w,z): y+w+z=0\}$. Find the dimension and the basis of X.
4. Find a basis and the dimension of the subspace **W** of $\Re^4$ spanned by $(1,-4,-2,1)(1,-3,-1,2)$ and $(3,-8,-2,7)$.Extend the basis of **W** to a basis of the whole space $\Re^4$.
5. Let **U** be the following subspace of $\Re^4 : W=\{(a,b,c,d)|a+b=0,c=2d\}$. Find the dimension and a basis of **U**.
6. Show that the vectors $(1,1,1,1),(0,1,1,1),(0,0,1,1)$ and $(0,0,0,1)$ form a basis for $\Re^4$.
7. Determine whether or not the vectors $\underset{\sim}{u}=(6,2,3,4),\underset{\sim}{v}=(0,5,-3,1)$ and $\underset{\sim}{w}=(0,0,7,-2)$ are linearly independent.
8. Let V be the vector space of polynomials of degree ≤ 3 over $\Re$ .Determine whether or not the polynomials $u=t^3-3t^2+5t+1, v=t^3-t^2+8t+2$ and $w=2t^3-4t^2+9t+5$ are linearly dependent.
9. Let W be a subspace of $\Re^5$ spanned by $(1,-2,0,0,3),(2,-5,-3,-2,6),(0,5,15,10,0)$ and $(2,6,18,8,6)$.Find the dimension and a basis of W.
10. Show that the vectors $\underset{\sim}{u}=(1,2,3), v=(0,1,2)$ and $w=(0,0,1)$ span $\Re^3$.
11. Let U and W be the following subspaces of $\Re^4$: $U=\{(a,b,c,d)|b-x+d=0\}$; $W=\{(a,b,c,d)|a=d,b=2c\}$. Find the dimension and a basis of: (a)U (b)W (c) $U\cap W$

12.5 FURTHER READING

1. Linear Algebra by Michael O'nan, Herbert Enderton
2. A First Course in Linear Algebra by Daniel Zelisky
3. Elementary Linear Algebra by Bennard Kolman
4. Elementary Linear Algebra by Howard Anton

12.6 SELF-TEST QUESTION 12

1. Determine whether or not u and v .are linearly dependent if:

(a) $u=(3,4), v=(1,-3)$ (b) $u=(2,-3), v=(6,-9)$ (c) $u=(4,3,-2), v=(2,-6,7)$

(d) $u=(-4,6,-2), v=(2,-3,1)$ (e) $u=\begin{pmatrix}1 & -2 & 4\\ 3 & 0 & -1\end{pmatrix}, v=\begin{pmatrix}2 & -4 & 8\\ 6 & 0 & -2\end{pmatrix}$

(f) $u=\begin{pmatrix}1 & 2 & -3\\ 6 & -5 & 4\end{pmatrix}, v=\begin{pmatrix}6 & -5 & 4\\ 1 & 2 & -3\end{pmatrix}$ (g) $u=2-5t+6t^2-t^3, v=3+2t-4t^2+5t^3$

2. Determine whether or not the following vectors in $\Re^3$ are linearly dependent:

(a) $(1,-2,1),(2,1,-1),(7,-4,1)$

(b) $(1,-3,7),(2,0,-6),(3,-1,-1),(2,4,-5)$

(c) $(1,2,-3),(1,-3,2),(2,-1,5)$

(d) $(2,-3,7),(0,0,0),(3,-1,-4)$

3. Determine whether or not the following form a basis for the vector space $\Re^3$:

(a) $(1,1,1)$ and $(1,-1,5)$

(b) $(1,2,3),(1,0,-1),(3,-1,0)$ and $(2,1,-2)$

(c) $(1,1,1),(1,2,3)$ and $(2,-1,1)$

(d) $(1,1,2),(1,2,5)$ and $(5,3,4)$

LECTURE THIRTEEN

PLANES AND LINES IN $\Re^3$

INTRODUCTION

This lecture is divided into two subtopics.

In the first subtopic, the learner is introduced to the vector equation of a line, while the second subtopic covers the vector equation of a plane.

Each concept is illustrated by several examples. An activity in the form of an exercise is given at the end of the lecture.

Answers to the self test questions in Section 13.6 have been provided at the end of the module.

For further understanding, the learner is encouraged to read the books recommended in Section 13.5

LECTURE OBJECTIVES

By the end of this lecture, the learner should be able to:

1. Find the parametric and symmetric equation of a line.
2. Find the equation of a plane.
3. Find the line of intersection of two planes

13.1 THE VECTOR EQUATION OF A LINE

Example: Find the equation of the line through the point A(1,2,3) and B(4,4,4), and find the co-ordinates of the point where the line meets the plane z = 0.

Solution:

$$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3i + 2j + k$$

Let R be any point on line AB, then

$\overrightarrow{OR} = \overrightarrow{OA} + t\ \overrightarrow{AB}$, where t is a scalar.

$$\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \text{OR}$$

$$\overrightarrow{OR} = \overrightarrow{OR} + (1-t)\ \overrightarrow{BA} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + (1-t)\begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \begin{pmatrix} -3+3t \\ -2+2t \\ -1+t \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

The line meets Z = 0 where the z-coordinate is 0. i.e. where 3 +t =0 $\Rightarrow$ t = -3.

At this point, $\vec{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 0 \end{pmatrix}$. Point R has coordinates $(-8, -4, 0)$

The equation of a line through a point $A(x_1, y_0, z_0)$ and parallel to vector $\begin{pmatrix} p \\ q \\ s \end{pmatrix}$ is given by

$$\vec{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} p \\ q \\ s \end{pmatrix}. \text{ i.e. } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} p \\ q \\ s \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + tp \\ y_0 + tq \\ z_0 + ts \end{pmatrix}$$

This can be written as

$$t = \frac{x - x_0}{p} = \frac{y - y_0}{q} = \frac{z - z_0}{s}$$

13.1 THE VECTOR EQUATION OF A LINE (Continued)

Example:

1. (a) Write the line $\frac{x-2}{3}=\frac{y-4}{5}=\frac{z-7}{2}$ in form $\vec{r}=\vec{a}+t\vec{u}$

(b)Show that the line passes through *(8,14,11)*

(c) Find the unit vector parallel to this line

Solution:

(a) $t=\frac{x-2}{3}=\frac{y-4}{5}=\frac{z-7}{2} \Rightarrow \quad x=2+3t,\ y=4+5t,\ z=7+2t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2+3t \\ 4+5t \\ 7+2t \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}+t\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

(b) Showing that it passes through *(8,14,11)*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2+3t \\ 4+5t \\ 7+2t \end{pmatrix}=\begin{pmatrix} 8 \\ 14 \\ 11 \end{pmatrix}\Rightarrow\begin{pmatrix} 2+3t=8 \\ 4+5t=14 \\ 7+2t=11 \end{pmatrix}\Rightarrow t=2$$

$$\begin{pmatrix} 8 \\ 14 \\ 11 \end{pmatrix}=\begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}+2\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

(c)The unit vector parallel to this line is

$$u=\frac{1}{\sqrt{3^2+5^2+2^2}}\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}=\frac{1}{\sqrt{39}}\begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix}$$

2. Show that the equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}+m\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix}+n\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ represent the same line.

Solution: $\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$ is parallel to $\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$ since $\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}=-2\begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 10 \\ 15 \\ -3 \end{pmatrix}=\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}+2\begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$; (10,15,-3) and (2,3,1) are on this line.

13.2 EQUATION OF A PLANE

It is always possible to find a plane through 3 points. A fourth point may not lie on the plane. A plane is uniquely determined by 3 points.

The general equation of a plane through points A,B and C is given by

$\overrightarrow{AP} = m\overrightarrow{AB} + n\,\overrightarrow{AC}$ i.e. $\overrightarrow{OP} = \alpha\,\overrightarrow{OA} + \beta\,\overrightarrow{OB} + \gamma\,\overrightarrow{OC}$

where $\alpha + \beta + \gamma = 1$

Since $\overrightarrow{OP} - \overrightarrow{OA} = m\left(\overrightarrow{OB} - \overrightarrow{OA}\right) = n\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$

$\overrightarrow{OP} = (1 - m - n)\,\overrightarrow{OA} + m\,\overrightarrow{OB} + n\,\overrightarrow{OC}$

$(1 - m - n) + m + n = 1$

Examples:

Find the equation of the plane through $A(1,1,1),\ B(5,0,0)\ and\ C\ (3,2,1)$

Solution:

$\overrightarrow{AP} = m\overrightarrow{AB} + n\,\overrightarrow{AC}$ where *P (x,y,z)*

$$\begin{pmatrix} x-1 \\ y-1 \\ z-1 \end{pmatrix} = m\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + n\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + m\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} + n\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$x = 1 + 4m + 2n$

$y = 1 - m + n$

$z = 1 - m$

Eliminate *n* from equation 1 and 2 to get

$x - 2y = -1 + 6m$

$z = 1 - m$

Eliminate *m* to get

$x - 2y + 6z = 5$ which is the equation of the plane

13.2 EQUATION OF A PLANE (Continued)

Intersection of two planes

Two planes always meet in a straight line.

Examples:

1. Find the equation of the line of intersection for the planes $3x-5y+z=8$ and $2x-3y+z=3$

Solution: At the line of intersection, the values of *x,y,z* satisfy both equations.

$$\begin{array}{r} 3x-5y+z = 8 \\ \underline{2x-3y+z = 3} \ - \\ x-2y = 5 \end{array}$$

i.e.eliminate *z* from equation 1and 2 to get

$x - 2y = 5$, or $x = 5+2y$. Thus the equation of line is $x - 2y - 5 = 0$

Since the line can also be obtained by eliminating x or y, we get it in the form $\vec{r} = \vec{a} + t\vec{u}$, we Let $y = t$, $\Rightarrow x = 5+2t$

Substitute in plane $3x-5y+z = 8$ to get $3(5+2t) - 5t+z = 8 \Rightarrow z = -7 - t$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5+2t \\ t \\ -7-t \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Note that the point $(5+2t, t, -7-t)$ lies in both planes. i.e.

$3x -5y+z=3(5+2t) - 5t+(-7-t) = 8$ and

$2x - 3y+ z = 2(5+2t) - 3t + (-7-t) = 3$

2. Show that the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ lies on the plane $2x +3y-5z = -7$.

Solution: Substitute the point $(1+t, 2+t, 3+t)$ on line in plane equation $2x +3y-5z = -7$ to get

$2x+3y - 5z = 2(1+t) +3(2+t)-5(3+t)=2+2t+6+3t-15-5t=-7$

3. Show that the lines $\vec{r} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + m\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\vec{s} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + n\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$ do not meet i.e. they are skew lines.

Solution: Suppose they meet. Then, $\left.\begin{array}{l} 3+m=1+2n \\ 5+2m=2+3n \\ 7+m \ \ =3+5n \end{array}\right\}$. Solving equation 1 and 2 gives $n=1$, and $m=0$. Substituting these values in equation 3 gives $7+0 \neq 3+5 \Rightarrow$ There is no

13.3 SUMMARY.

In this lecture, we have learnt to find the parametric and symmetric equation of a line, the equation of a plane and the line of intersection of two planes

13.4 ACTIVITY 13

1. Find the angle between the planes $2x + y - 2z = 1$ and $x - 2y - 2z = 2$.

2. Find the equation of the plane through the points $A(1,1,1), B(2,0,2)$ and $C(-1,1,2)$

3. Find the equation of the plane through $(-1,2,3)$ and perpendicular to the planes $2x - 3y + 4z = 1$ and $3x - 5y + 2z = 3$.

4. Write the line $\frac{x-1}{2} + \frac{y-3}{3} + \frac{z-4}{5}$ in the form $\vec{r} = \vec{a} + t\vec{u}$ and show that it passes through $\begin{pmatrix} 7 \\ 12 \\ 19 \end{pmatrix}$.

5. Find the distance between the parallel planes $x + 2y - 2z = 3$ and $2x + 4y - 4z = 7$

6. Find the equation of the plane passing through the points $P(-4,-1,-1), Q(-2,0,1)$ and $R(-1,-2,-3)$.

7.Find the equation of the line of intersection for the planes 4x+3y + z=10 and x + y + z = 6

8.Find the point of intersection of the lines $\vec{r} = (1+m)i + (2+m)i + (4+2m)k$ and $\vec{s} = (1+3n)i + 5nj + (3+7n)k$. **Ans:** (4,5,10)

13.5 FURTHER READING

1. Linear Algebra: Schaum's Outline Series
2. Linear Algebra by J. N. Sharma, A.R. Vasishta
3. Linear Algebra by Michael O'nan, Herbert Enderton
4. Elementary Linear Algebra by Bennard Kolman
5. Elementary Linear Algebra by Howard Anton

13.6 SELF-TEST QUESTIONS 13

1. Find an equation of the plane passing through the point $(3,-1,7)$ and perpendicular to the vector $\mathbf{n}=(4,2,-5)$.

2. Find the equation of the plane passing through the points $P_1(1,2,-1), P_2(2,3,1)$ and $P_3(3,-1,2)$.

3. Find The line through the point $(1,2,-3)$ and parallel to the vector $\mathbf{v}=(4,5,-7)$ has parametric equations

4. (a)Find parametric equations for the line l passing through the points $P_1(2,4,-1)$ and $P_2(5,0,7)$.

(b)Where does the line intersect the xy-plane?

5. Find parametric equations for the line of intersection of the planes $3x+2y-4z-6=0$ and $x-3y-2z-4=0$.

ANSWERS TO SELF-TEST QUESTIONS

SELF TEST QUESTIONS 1

1 (a) $AB = \begin{pmatrix} 11 & -6 & 14 \\ 1 & 2 & -14 \end{pmatrix}$ 1(b) BA is not defined. 2. (a) $AB = (6,1,-3)$ (b) BA is not defined.

3. (a) $AB = \begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$ (b) $BA = \begin{pmatrix} 15 & -21 \\ 10 & -3 \end{pmatrix}$

SELF TEST QUESTIONS 2

$A^t = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 4 \\ 1 & 4 & 4 \\ 0 & 5 & 4 \end{pmatrix}$ 2. $A^t = \begin{pmatrix} 1 & 3 \\ 2 & -1 \\ 0 & 4 \end{pmatrix}$, Then $AA^t = = \begin{pmatrix} 5 & 1 \\ 1 & 26 \end{pmatrix}$; $A^t A = \begin{pmatrix} 10 & -1 & 12 \\ -1 & 5 & -4 \\ 12 & -4 & 16 \end{pmatrix}$

SELF TEST QUESTIONS 3

1. (a) $\begin{vmatrix} 3 & -2 \\ 4 & 5 \end{vmatrix} = 23$ (b) $\begin{vmatrix} a-b & a \\ b & a+b \end{vmatrix} = -b^2$ 2. $k = 0$;and $k = 2$ .That is, if $k = 0$ or $k = 2$

3. (a) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & -2 & 3 \\ 2 & 5 & 1 \end{vmatrix} = 79$ (b) $\begin{vmatrix} 2 & 0 & 1 \\ 4 & 2 & -3 \\ 5 & 3 & 1 \end{vmatrix} = 24$ (c) $\begin{vmatrix} 2 & 0 & 1 \\ 3 & 2 & -3 \\ -1 & -3 & 5 \end{vmatrix} = -5$ (d) $\begin{vmatrix} 1 & 0 & 0 \\ 3 & 2 & -4 \\ 4 & 1 & 3 \end{vmatrix} = 10$

SELF TEST QUESTIONS 4

1. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & 0 & 9 \\ 2 & -1 & 3 \\ 1 & 0 & 1 \end{vmatrix}$,obtained by adding twice the second row to the first row Det=4.

2. -120

ANSWERS TO SELF-TEST QUESTIONS (Continued)

SELF TEST QUESTIONS 5

$$1. A^{-1} = \frac{1}{|A|}(adjA) = \begin{pmatrix} -18/-46 & -11/-46 & -10/-46 \\ 2/-46 & 14/-46 & -4/-46 \\ 4/-46 & 5/-46 & -8/-46 \end{pmatrix} = \begin{pmatrix} 9/23 & 11/46 & 5/23 \\ -1/23 & -7/23 & 2/23 \\ -2/23 & -5/46 & 4/23 \end{pmatrix}$$

2.

SELF TEST QUESTIONS 6

1. $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{pmatrix}$

2. *b*,*e*,*f* are in echelon form; *a* and *f* are in canonical form. *c* and *d* are not in echelon form.

SELF TEST QUESTIONS 7

1. $x_1 = \frac{13}{2}, x_2 = 4, x_3 = 9/2, x_4 = -3$

2. $x_1 = 4_1 \quad x_2 = 1, x_3 = -3, \; x_4 = 2$

3. $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = -1$

4. $x_3 = k, \; x_2 = 2 + k, \; x, = 4 - k$

ANSWERS TO SELF-TEST QUESTIONS (Continued)

SELF TEST QUESTIONS 8

1. $\|\mathbf{v}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ 2. $0 \le \theta \le \prod, \theta = 60^0$ 3. $\mathbf{u}.\mathbf{v} = (2)(4) + (-4)(2) = 0$

SELF TEST QUESTIONS 9

SELF TEST QUESTIONS 10

1. $L = \{(x, y) / y = 2x\} = \{(x, 2x) / x \in R\}$
2. Let $w_1, w_2 \in W$. $w_1 = (x_1, x_1 + 1)$, $w_2 = (x_2, x_2 + 1)$

$w_1 + w_2 = (x_1 + x_2, x_1 + x_2 + 2) \notin W$ .OR: $\alpha\,(x_1, x_1 + 1) = (\alpha x_1, \alpha x_1 + \alpha) \notin W$,since $\alpha \neq 1$. Therefore the line $y = x + 1$is not a subspace of $\Re$

3. For any invertible n $\times$ n matrix *A*, -*A* is also invertible.

But $A + (-A) = 0$ not invertible and therefore the set of all **invertible** n $\times$ n matrices is not a subspace of the set $M_n\,(R)$

ANSWERS TO SELF-TEST QUESTIONS (Continued)

SELF TEST QUESTIONS 11

1. Set v as a linear combination of the e_1 using the unknowns x, y and z: $v = xe_1 + ye_2 + ze_3$.
v cannot be written as a linear combination of the vectors e_1, e_2 and e_3.
2. Set $u = xv + yw$; $k = -8$.
3We need to show that an arbitrary vector $(a,b,c) \in \Re^3$ is a linear combination of u, v and w.
$x = a, y = b - 2a, z = c - 2b + a$ is a solution. Thus u, v and w generate $\Re^3$.
4. Set (a,b,c) as a linear combination of u, v and w using unknowns x, y and z:
$(a,b,c) = xu + yv + zw$; $2a - 4b - 3c = 0$.

SELF TEST QUESTIONS 12

1. Two vectors u and v are dependent if and only if one is a multiple of the other.
(a)No (b) Yes; for $v = 3u$ (c)No (d)Yes; for $u = -2v$ (e)Yes; for $v = 2u$ (f)No (g)No (h)Yes;
2. (a) Since the echelon matrix has a zero row the vectors are dependent. (The three given vectors generate a space of dimension 2.)
(b)Yes, since any four (or more) vectors in $\Re^3$ are dependent.
(c)Since the echelon matrix has no zero rows, the vectors are independent. (The three given vectors generate a space of dimension 3.)
(d)Since $0 = (0,0,0)$ is one of the vectors, the vectors are dependent
3. (a)and (b). No; for a basis of $\Re^3$ must contain exactly 3 elements, since $\Re^3$ is of dimension 3.
(c)The vectors form a basis if and only if they are independent. The echelon matrix has no zero rows; hence the three vectors are independent and so form a basis for $\Re^3$.
(d)The echelon matrix has a zero row, i.e. only two non zero rows; hence the three vectors are dependent and so do not form a basis for $\Re^3$.

SELF TEST QUESTIONS 13

1. $4x + 2y - 5z + 25 = 0$ 2. $9x + y - 5z - 16 = 0$ 3. $\begin{array}{lcl} x & = & 1+4t \\ y & = & 2+5t \\ z & = & -3-7t \end{array}$ 4.(a) $\begin{array}{lcl} x & = & 2+3t \\ y & = & 4-4t \\ z & = & -1+8t \end{array}$ *(b)*

$(x, y, z) = \left(\frac{19}{8}, \frac{7}{2}, 0\right)$ 5. $\begin{array}{lcl} x & = & \frac{26}{11} + \frac{16}{11}t \\ y & = & -\frac{6}{11} - \frac{2}{11}t \\ z & = & t \end{array}$ where $-\infty < t < +\infty$