# Lecture 7

Continuous Random Variable
Probability Density Function
Uniform Distribution
Percentile of a Distribution

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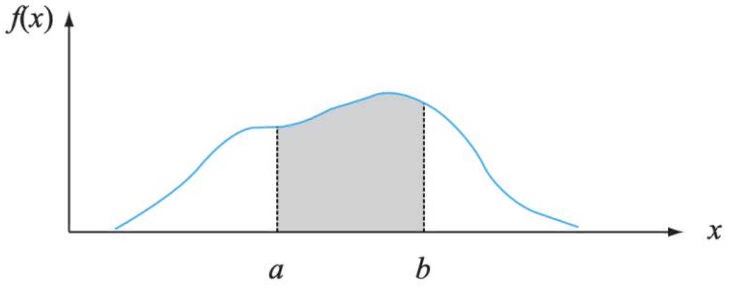
# Probability Density Function

Let $X$ be a continuous rv. Then a **probability distribution** or **probability density function** (pdf) of $X$ is a function $f(x)$ such that for any two numbers $a$ and $b$ with $a \leq b$,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$




$P(a \leq X \leq b) =$ the area under the density curve between a and b

# Probability Density Function

For $f(x)$ to be a legitimate pdf, it must satisfy the following two conditions:

**1.** $f(x) \geq 0$ for all $x$

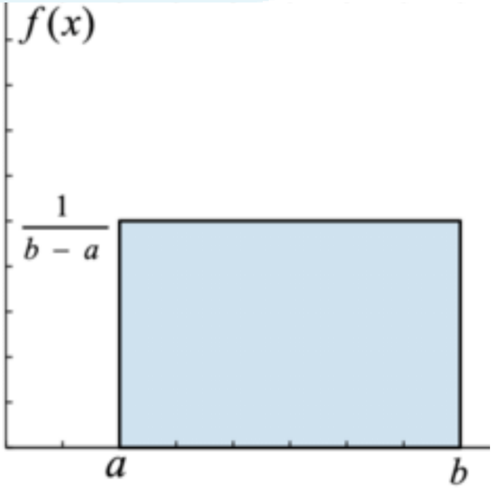
**2.** $\int_{-\infty}^{\infty} f(x)\,dx =$ area under the entire graph of $f(x)$

$= 1$

# Uniform distribution

A continuous rv $X$ is said to have a **uniform distribution** on the interval $[A, B]$ if the pdf of $X$ is

$$f(x; A, B) = \begin{cases} \dfrac{1}{B - A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$



# Mean

The **expected** or **mean value** of a continuous rv $X$ with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

If $X$ is a continuous rv with pdf $f(x)$ and $h(X)$ is any function of $X$, then

$$E[h(X)] = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)\, dx$$

# Variance

The **variance** of a continuous random variable $X$ with pdf $f(x)$ and mean value $\mu$ is

$$\sigma_X^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x)dx = E[(X - \mu)^2]$$

The **standard deviation** (SD) of $X$ is $\sigma_X = \sqrt{V(X)}$.

$$V(X) = E(X^2) - [E(X)]^2$$

$E[(X\text{-}u)^2] = E[X^2 + u^2 - 2Xu]$
$= E[X^2] + E[u^2] - 2E[Xu]$
$= E[X^2] + u^2 - 2uE[X]$
$= E[X^2] + u^2 - 2uu$
$= E[X^2] - u^2$

# Question

Calculate mean and variance for the uniform distribution (in terms of $a$ and $b$).

$$f(x; A, B) = \begin{cases} \dfrac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{A}^{B} x \left(\frac{1}{B-A}\right) dx$$

$$= \frac{1}{B-A}\left[\frac{x^2}{2}\right]_A^B$$

$$= \frac{B^2 - A^2}{2(B-A)}$$

$$= \frac{(B-A)(B+A)}{2(B-A)}$$

$$\mu = \frac{B+A}{2}$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$= E(X^2) - \mu^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{A}^{B} x^2 \left(\frac{1}{B-A}\right) dx$$

$$= \left[\frac{x^3}{3}\right]_A^B \left(\frac{1}{B-A}\right)$$

$$= \frac{B^3 - A^3}{3(B-A)}$$

$$= \frac{(B^2 + A^2 + AB)\cancel{(B-A)}}{3\cancel{(B-A)}}$$

$$= \frac{B^2 + A^2 + AB}{3}$$

$$\sigma^2 = \frac{B^2 + A^2 + AB}{3} - \frac{(B+A)^2}{4}$$

$$= \frac{4B^2 + 4A^2 + 4AB - 3B^2 - 3A^2 - 6B}{12}$$

$$= \frac{B^2 + A^2 - 2AB}{12} = \frac{(B-A)^2}{12}$$

# Question

The time X (min) for a lab assistant to prepare the equipment for a certain experiment is believed to have a uniform distribution with A = 25 and B = 35.

**a.** Determine the pdf of $X$ and sketch the corresponding density curve.

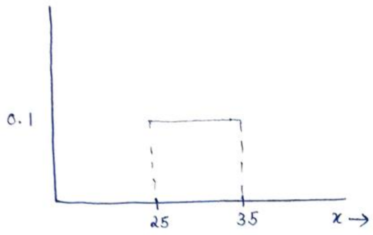
**b.** What is the probability that preparation time exceeds 33 min?

**c.** What is the probability that preparation time is within 2 min of the mean time? [*Hint*: Identify $\mu$ from the graph of $f(x)$.]

**d.** For any $a$ such that $25 < a < a + 2 < 35$, what is the probability that preparation time is between $a$ and $a + 2$ min?

a)

$$f(x) = \begin{cases} \frac{1}{35-25} = \frac{1}{10} = 0.1 & 25 \le x \le 35 \\ 0 & \text{otherwise} \end{cases}$$




b) $P(X > 33) = \int_{33}^{35} f(x)\,dx$

$$= \frac{1}{10}[x]_{33}^{35} = \frac{2}{10} = 0.2$$

c) $\mu = \frac{25+35}{2} = \frac{60}{2} = 30$

$P(28 < X < 32) = \int_{28}^{32} \frac{1}{10}\,dx$

$$= \frac{1}{10}[x]_{28}^{32} = \frac{4}{10} = 0.4$$

d) $P(a < X < a+2)$

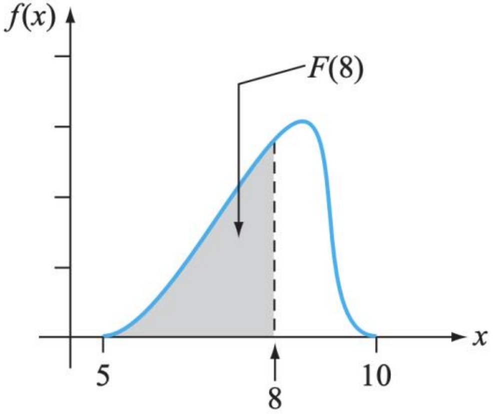
$$= \int_{a}^{a+2} \frac{1}{10}\,dx$$

$$= \frac{1}{10}[x]_{a}^{a+2} = \frac{2}{10} = 0.2$$

# Cumulative Distribution Functions

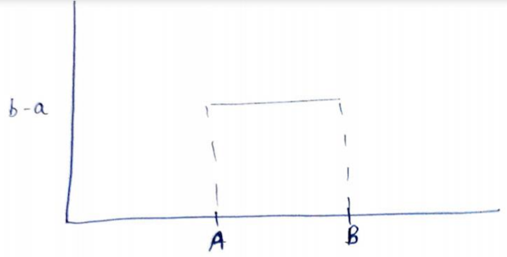
The **cumulative distribution function** $F(x)$ for a continuous rv $X$ is defined for every number $x$ by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$



# Question

Calculate cdf for uniform distribution.




$$F(x) = 0 \quad \text{for} \quad x < A$$

$$F(x) = 1 \quad \text{for} \quad x \geq B$$

How?

$$\int_{-\infty}^{A} 0 \, dx = 0$$

$$\int_{-\infty}^{\infty} f(x) \, dx = 1$$

$$\int_{-\infty}^{B+1} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{b} f(x) \, dx + \int_{b}^{b+1} f(x) \, dx$$

$$= 0 + \frac{1}{b-a} [x]_a^b + \int_b^{b+1} 0 \, . dx$$

$$= 0 + \frac{b-a}{b-a} + 0 = 1$$

$$A \leq x < B$$

$$F(x) \quad \int_{-\infty}^{x} f(x) \, dx = \int_{-\infty}^{A} 0 . dx + \int_{A}^{x} \left(\frac{1}{B-A}\right) dx$$

$$= 0 + \frac{x-A}{B-A}$$

so

$$F(x) = \begin{cases} 0 & x < A \\ \frac{x-A}{B-A} & A \leq x < B \\ 1 & x \geq B \end{cases}$$

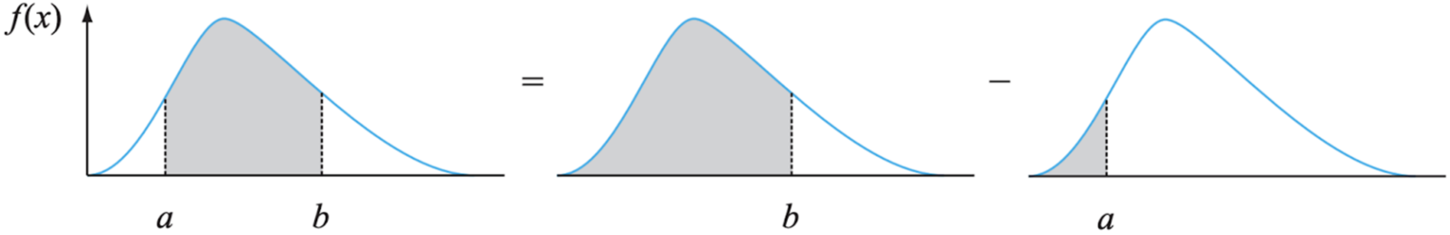
# Probability from F(x)

Let $X$ be a continuous rv with pdf $f(x)$ and cdf $F(x)$. Then for any number $a$,

$$P(X > a) = 1 - F(a)$$

and for any two numbers $a$ and $b$ with $a < b$,

$$P(a \le X \le b) = F(b) - F(a)$$



# Question

Given the pdf find out the following:

a. cdf F(x)
b. P(1<=X<=1.5)
c. P(X>1)

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

For any number $x$ between 0 and 2,

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{0}^{x}\left(\frac{1}{8} + \frac{3}{8}y\right)dy = \frac{x}{8} + \frac{3}{16}x^2$$

Thus

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{8} + \frac{3}{16}x^2 & 0 \le x \le 2 \\ 1 & 2 < x \end{cases}$$

$$\begin{aligned} P(1 \le X \le 1.5) &= F(1.5) - F(1) \\ &= \left[\frac{1}{8}(1.5) + \frac{3}{16}(1.5)^2\right] - \left[\frac{1}{8}(1) + \frac{3}{16}(1)^2\right] \\ &= \frac{19}{64} = .297 \end{aligned}$$

$$P(X > 1) = 1 - P(X \le 1) = 1 - F(1) = 1 - \left[\frac{1}{8}(1) + \frac{3}{16}(1)^2\right]$$

$$= \frac{11}{16} = .688$$

# Obtaining f(x) from F(x)

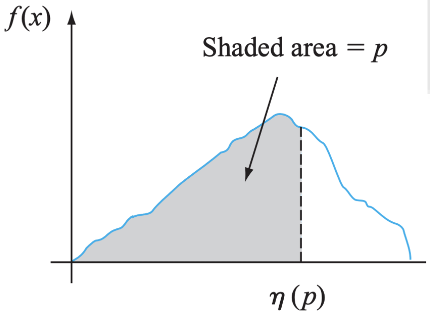
If $X$ is a continuous rv with pdf $f(x)$ and cdf $F(x)$, then at every $x$ at which the derivative $F'(x)$ exists, $F'(x) = f(x)$.

# Percentile of a Distribution

Let $p$ be a number between 0 and 1. The **(100$p$)th percentile** of the distribution of a continuous rv $X$, denoted by $\eta(p)$, is defined by

$$p = F(\eta(p)) = \int_{-\infty}^{\eta(p)} f(y)\,dy$$

$\eta(p)$ is that value on the measurement axis such that 100p% of the area under the graph of f(x) lies to the left of $\eta(p)$ and 100(1-p)% lies to the right.



$$f(x) = \begin{cases} \frac{3}{2}(1-x^2) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We want to calculate 50 th percentile

$$\int_0^{\eta(p)} \frac{3}{2}(1-x^2)\,dx = 0.5$$

$$\frac{3}{2}\left[x - \frac{x^3}{3}\right]_0^{\eta(p)} = 0.5$$

$$\frac{3}{2}\left[\eta(p) - \frac{(\eta(p))^3}{3}\right] = 0.5$$

Solve to get value of $\eta(p)$

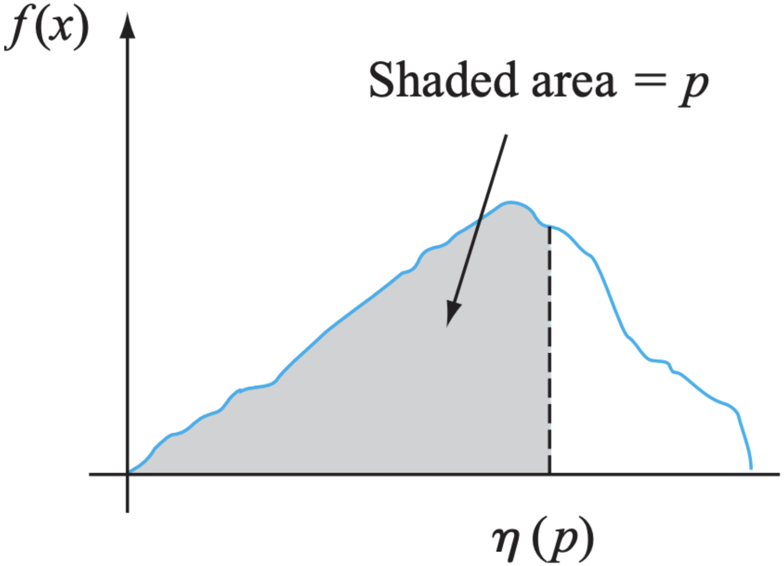
To get $\eta(p)$ given $p$

$$\int_{-\infty}^{\eta(p)} f(x)\,dx = p \rightarrow$$ integerate, apply limits & solve the eqn.

Solve the equation to get $\eta(p)$

To get $p$ given $\eta(p)$

$$\int_{-\infty}^{\eta(p)} f(x)\,dx = p \rightarrow$$ integerate in the limits $-\infty$ to $\eta(p)$ and find $p$.



# Question

Given the pdf. $f(x) = \begin{cases} \frac{1}{20} & 10 \leq x \leq 30 \\ 0 & \text{otherwise} \end{cases}$

Find out the 10 th, 25 th, 50 th, 75 th percentile

$$0.1 = \int_{-\infty}^{z} \frac{1}{20}\, dx$$

$$0.1 = \int_{-\infty}^{10} 0 . dx + \int_{10}^{z} \frac{1}{20}\, dx$$

$$0.1 = 0 + \left[\frac{x}{20}\right]_{10}^{z}$$

$$0.1 = 0 + \frac{z-10}{20}$$

$$0.1 = \frac{z-10}{20}$$

$$2 = z - 10$$

$$12 = z$$

so 10 th percentile is 12

$$0.25 = \int_{-\infty}^{z} \frac{1}{20}\,dx$$

$$0.25 = \int_{-\infty}^{10} 0\,dx + \int_{10}^{z} \frac{1}{20}\,dx$$

$$0.25 = 0 + \frac{z-10}{20}$$

$$5 = z - 10$$

$$z = 15$$

$$0.5 = \int_{10}^{z} \frac{1}{20}\,dx$$

$$0.5 = \frac{z-10}{20}$$

$$10 + 10 = z$$

$$z = 20$$

$$0.75 = \int_{10}^{z} \frac{1}{20}\,dx$$

$$0.75 = \frac{z-10}{20}$$

$$15 + 10 = z$$

$$z = 25$$

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