Lecture 10

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Gamma Function

For $\alpha > 0$, the **gamma function** $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$$

The most important properties of the gamma function are the following:

1. For any $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1) \cdot \Gamma(\alpha - 1)$ [via integration by parts]

2. For any positive integer, n, $\Gamma(n) = (n - 1)!$

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Gamma Function

$$f(x;\alpha) = \begin{cases} \dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

then $f(x;\alpha) \geq 0$ and $\int_0^\infty f(x;\alpha)\,dx = \Gamma(\alpha)/\Gamma(\alpha) = 1$, so $f(x;\alpha)$ satisfies the two basic properties of a pdf.

Question

Evaluate each of the following expressions, leaving the final answer in exact simplified form.

a) $\frac{3\Gamma(6)}{\Gamma(4)}$

b) $\Gamma\left(\frac{3}{2}\right)$

c) $\int_0^{\infty} x^7 e^{-x}\,dx$

d) $\frac{60\,\Gamma(5)}{\Gamma(7)}$

e) $\Gamma\left(\frac{5}{2}\right)$

f) $\int_0^{\infty} 2x^4 e^{1-x}\,dx$

a) $\dfrac{3\,\Gamma(6)}{\Gamma(4)} = \dfrac{3 \times 5!}{3!} = 3 \times 5 \times 4 = 60$

b) $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

c) $\displaystyle\int_0^\infty x^7 e^{-x}\,dx = \int_0^\infty x^{8-1} e^{-x}\,dx = \Gamma(8) = 7!$

$= 5040$

d) $\dfrac{60\,\Gamma(5)}{\Gamma(7)} = \dfrac{60 \times 4!}{6!} = \dfrac{60}{6 \times 5} = 2$

e) $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \times \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

f) $\displaystyle\int_0^\infty 2x^4 e^{1-x}\,dx = \int_0^\infty 2x^{5-1} e^{-x} . e\,dx$

$\displaystyle= 2e\int_0^\infty x^{5-1} e^{-x}\,dx$

$= 2e\,\Gamma(5)$

$= 2e(4!)$

$= 2e(4 \times 3 \times 2 \times 1)$

$= 48e$

Gamma Distribution

A continuous random variable X is said to have a **gamma distribution** if the pdf of X is

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where the parameters α and β satisfy $\alpha > 0$, $\beta > 0$.

Standard Gamma Distribution

$$\beta = 1$$

$$f(x;\alpha) = \begin{cases} \dfrac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Exponential Distribution

The exponential distribution results from taking $\alpha = 1$ and $\beta = 1/\lambda$.

$$f(x;\alpha,\beta) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}$$

$$\alpha = 1 \qquad \beta = \frac{1}{\lambda}$$

$$f\left(x; 1, \frac{1}{\lambda}\right) = \frac{1}{\frac{1}{\lambda}\Gamma(1)} x^{1-1} e^{-x\lambda}$$

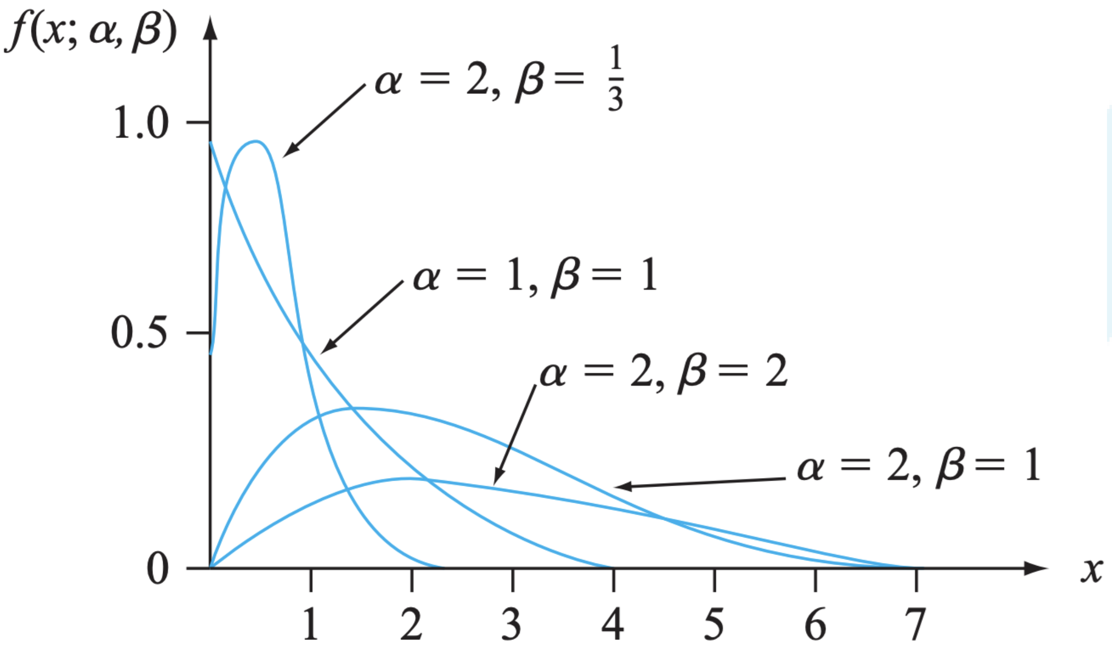
$$= \frac{\lambda x^{0} e^{-\lambda x}}{\Gamma(1)}$$

$$= \frac{\lambda e^{-\lambda x}}{\Gamma(1)}$$

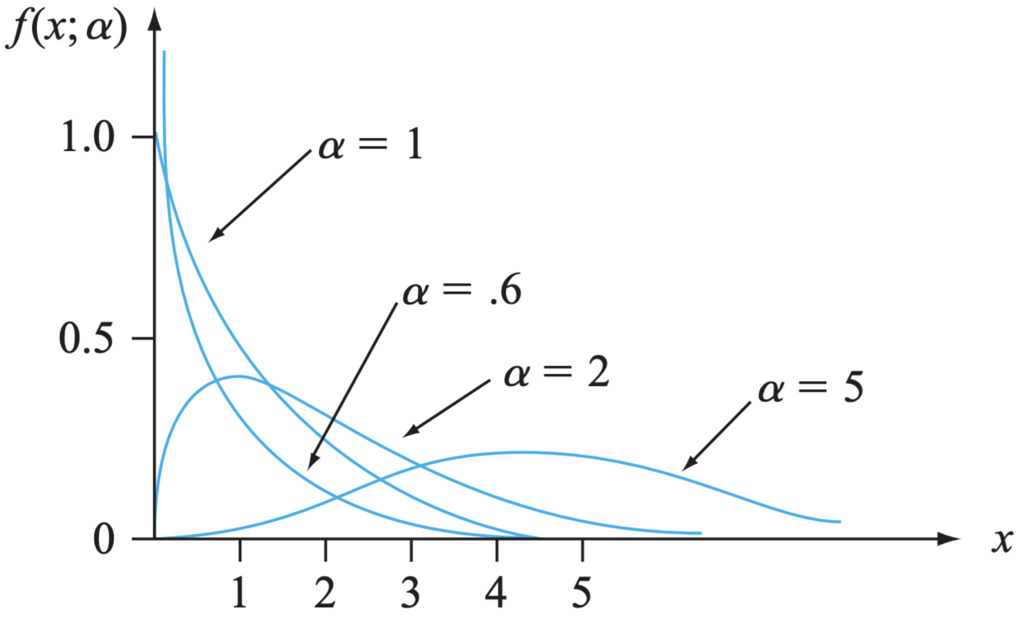
Since $\Gamma(1) = 1$

$$= \lambda e^{-\lambda x}$$

Gamma Density Curves



Standard Gamma Density Curves



Properties

- For the standard pdf, when $\alpha \leq 1$ then $f(x;\alpha)$ is strictly decreasing as x increases from 0.
- For standard pdf, when $\alpha > 1$ then $f(x;\alpha)$ rises from 0 at $x = 0$ to a maximum and then decreases.
- β is called the scale parameter because values other than 1 either stretch or compress the pdf in the x direction.

Mean & Variance

$$E(X) = \mu = \alpha\beta \qquad V(X) = \sigma^2 = \alpha\beta^2$$

Incomplete Gamma Distribution

When X is a standard gamma rv, the cdf of X,

$$F(x;\alpha) = \int_0^x \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)}\,dy \qquad x > 0$$

Question

Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with α = 2. Calculate the following:

a. $P(3 \le X \le 5)$

x \ α	1	2	3	4	5	6	7	8	9	10
1	.632	.264	.080	.019	.004	.001	.000	.000	.000	.000
2	.865	.594	.323	.143	.053	.017	.005	.001	.000	.000
3	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001
4	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008
5	.993	.960	.875	.735	.560	.384	.238	.133	.068	.032
6	.998	.983	.938	.849	.715	.554	.394	.256	.153	.084
7	.999	.993	.970	.918	.827	.699	.550	.401	.271	.170
8	1.000	.997	.986	.958	.900	.809	.687	.547	.407	.283
9		.999	.994	.979	.945	.884	.793	.676	.544	.413
10		1.000	.997	.990	.971	.933	.870	.780	.667	.542
11			.999	.995	.985	.962	.921	.857	.768	.659
12			1.000	.998	.992	.980	.954	.911	.845	.758
13				.999	.996	.989	.974	.946	.900	.834
14				1.000	.998	.994	.986	.968	.938	.891
15					.999	.997	.992	.982	.963	.930

Question

Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with α = 2. Calculate the following:

a. P(3 ≤ X ≤ 5)

$$P(3 \leq X \leq 5) = F(5; 2) - F(3; 2) = .960 - .801 = .159$$

Question

Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with α = 2. Calculate the following:

b. P(X > 4)

x \ α	1	2	3	4	5	6	7	8	9	10
1	.632	.264	.080	.019	.004	.001	.000	.000	.000	.000
2	.865	.594	.323	.143	.053	.017	.005	.001	.000	.000
3	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001
4	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008
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14				1.000	.998	.994	.986	.968	.938	.891
15					.999	.997	.992	.982	.963	.930

Question

Suppose the reaction time X of a randomly selected individual to a certain stimulus has a standard gamma distribution with α = 2. Calculate the following:

b. P(X > 4)

$$P(X > 4) = 1 - P(X \leq 4) = 1 - F(4; 2) = 1 - .908 = .092$$

Question

Let X have a standard gamma distribution with α = 7. Evaluate the following

a. $P(X \leq 5)$
b. $P(X < 5)$
c. $P(X>8)$
d. $P(3 \leq X \leq 8)$
e. $P(3 < X < 8)$
f. $P(X<4 \text{ or } X>6)$

x \ α	1	2	3	4	5	6	7	8	9	10
1	.632	.264	.080	.019	.004	.001	.000	.000	.000	.000
2	.865	.594	.323	.143	.053	.017	.005	.001	.000	.000
3	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001
4	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008
5	.993	.960	.875	.735	.560	.384	.238	.133	.068	.032
6	.998	.983	.938	.849	.715	.554	.394	.256	.153	.084
7	.999	.993	.970	.918	.827	.699	.550	.401	.271	.170
8	1.000	.997	.986	.958	.900	.809	.687	.547	.407	.283
9		.999	.994	.979	.945	.884	.793	.676	.544	.413
10		1.000	.997	.990	.971	.933	.870	.780	.667	.542
11			.999	.995	.985	.962	.921	.857	.768	.659
12			1.000	.998	.992	.980	.954	.911	.845	.758
13				.999	.996	.989	.974	.946	.900	.834
14				1.000	.998	.994	.986	.968	.938	.891
15					.999	.997	.992	.982	.963	.930

Question

Let X have a standard gamma distribution with $\alpha = 7$. Evaluate the following

a. $P(X \leq 5) = 0.238$
b. $P(X < 5) = 0.238$
c. $P(X>8) = 1 - P(X \leq 8) = 1 - 0.687 = 0.313$
d. $P(3 \leq X \leq 8) = F(8) - F(3) = 0.687 - 0.034 = 0.653$
e. $P(3 < X < 8) = F(8) - F(3) = 0.687 - 0.034 = 0.653$
f. $P(X<4 \text{ or } X>6) = F(4) + 1 - F(6) = 0.111 + 1 - 0.394 = 0.717$

Non standard gamma function to gamma function

Let X have a gamma distribution with parameters α and β. Then for any $x > 0$, the cdf of X is given by

$$P(X \leq x) = F(x; \alpha, \beta) = F\left(\frac{x}{\beta}; \alpha\right)$$

where $F(\cdot\,; \alpha)$ is the incomplete gamma function.

Question

Suppose the time spent by a randomly selected student who uses a terminal connected to a local time-sharing computer facility has a gamma distribution with mean 20 min and variance 80 min^2.

a. What are the values of α and β?

a) $\alpha\beta = 20$

$\alpha\beta^2 = 80$

$$\frac{\alpha\beta^2}{\alpha\beta} = \frac{80}{20} = 4$$

so $\beta = 4$

Now $\alpha\beta = 20$

$4\alpha = 20$

$\alpha = 5$

Question

Suppose the time spent by a randomly selected student who uses a terminal connected to a local time-sharing computer facility has a gamma distribution with mean 20 min and variance 80 min^2.

b. What is the probability that a student uses the terminal for at most 24 min?

b) $P(X \leq 24) = F(24; 5, 4)$

$= F\left(\frac{24}{4}; 5\right)$

$= F(6; 5)$

$= 0.715$

Question

Suppose the time spent by a randomly selected student who uses a terminal connected to a local time-sharing computer facility has a gamma distribution with mean 20 min and variance 80 min^2.

c. What is the probability that a student spends between 20 and 40 min using the terminal?

c) $P(20 \leq X \leq 40) = F(40; 5, 4) - F(20; 5, 4)$

$= F\left(\frac{40}{4}; 5\right) - F\left(\frac{20}{4}; 5\right)$

$= F(10; 5) - F(5; 5)$

$= 0.971 - 0.56$

$= 0.411$

References

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