Lecture 9

Exponential Distribution

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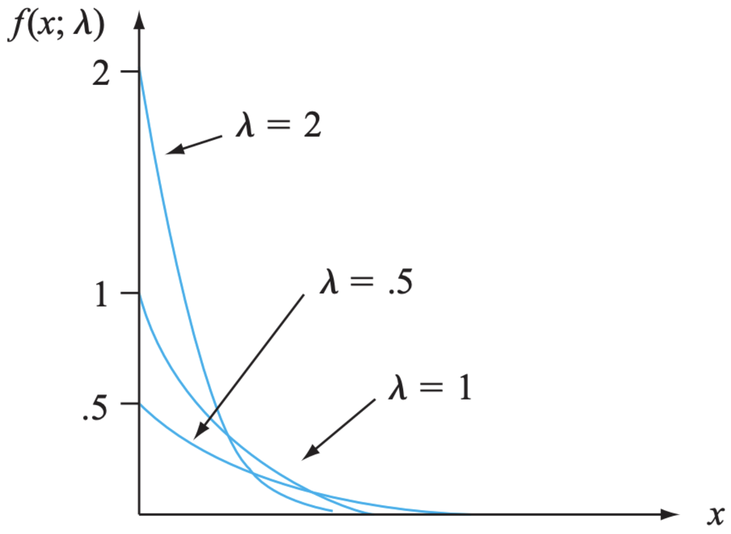
Exponential & Gamma Distribution

- The density curve corresponding to any normal distribution is bell-shaped and therefore symmetric.
- There are many practical situations in which the variable of interest to an investigator might have a skewed distribution.
- One family of distributions that has this property is the Gamma family.
- The exponential distribution is a special case of Gamma distribution.

Exponential Distribution

X is said to have an **exponential distribution** with parameter λ ($\lambda > 0$) if the pdf of X is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \ge 0 \\ 0 & \text{otherwise} \end{cases}$$



$$\int f(x)\, g(x) = f(x) \int g(x) - \int \left(f'(x) \int g(x)\, dx \right) dx$$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x\, \lambda e^{-\lambda x}\, dx$$

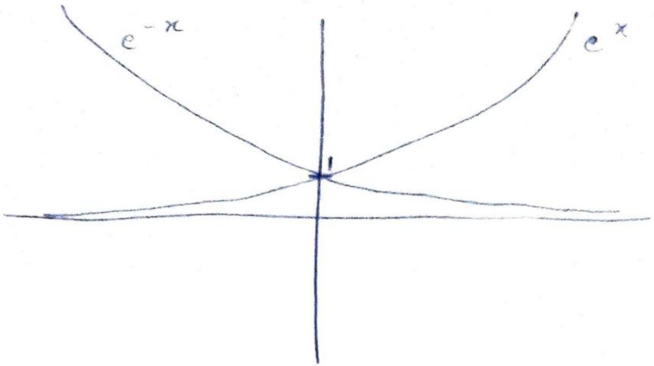
$$= \left[\lambda \frac{x\, e^{-\lambda x}}{-\lambda} \right]_0^{\infty} - \int_0^{\infty} \left(\frac{d(x)}{dx} \right) \left(\int \lambda e^{-\lambda x}\, dx \right) dx$$

$$= \left[-x e^{-\lambda x} \right]_0^{\infty} - \int_0^{\infty} \frac{\lambda e^{-\lambda x}}{-\lambda}\, dx$$

$$= \left[-x e^{-\lambda x} \right]_0^{\infty} - \int_0^{\infty} - e^{-\lambda x}\, dx$$

$$= \left[-x e^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\, dx$$

$$= \left[-x e^{-\lambda x} \right]_0^{\infty} + \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^{\infty}$$



$$= \left[-\infty\, e^{-\infty} \right] - \left[-0\, e^{-0} \right] + \left[\frac{e^{-\infty}}{-\lambda} - \frac{e^{0}}{-\lambda} \right]$$

$$= 0 - 0 + 0 + \frac{1}{\lambda}$$

$$E(X) = \frac{1}{\lambda}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$$\int f(x)g(x)\,dx = f(x)\int g(x) - \int\left(f'(x)\int g(x)\,dx\right)dx$$

$$= \left[x^2 \frac{\lambda e^{-\lambda x}}{-\lambda}\right]_0^{\infty} - \int\left(2x\frac{\lambda e^{-\lambda x}}{-\lambda}\right)dx$$

$$= 0 + \int_0^{\infty} 2x e^{-\lambda x}\,dx$$

$$= 2\left[\left[\frac{x e^{-\lambda x}}{-\lambda}\right]_0^{\infty} - \int_0^{\infty}\frac{e^{-\lambda x}}{-\lambda}\,dx\right]$$

$$= 2\left[0 + \int_0^{\infty}\frac{e^{-\lambda x}}{\lambda}\,dx\right]$$

$$= 2\left[\frac{e^{-\lambda x}}{\lambda(-\lambda)}\right]_0^{\infty} = \frac{2}{-\lambda^2}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{2}{-\lambda^2}[-1] = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

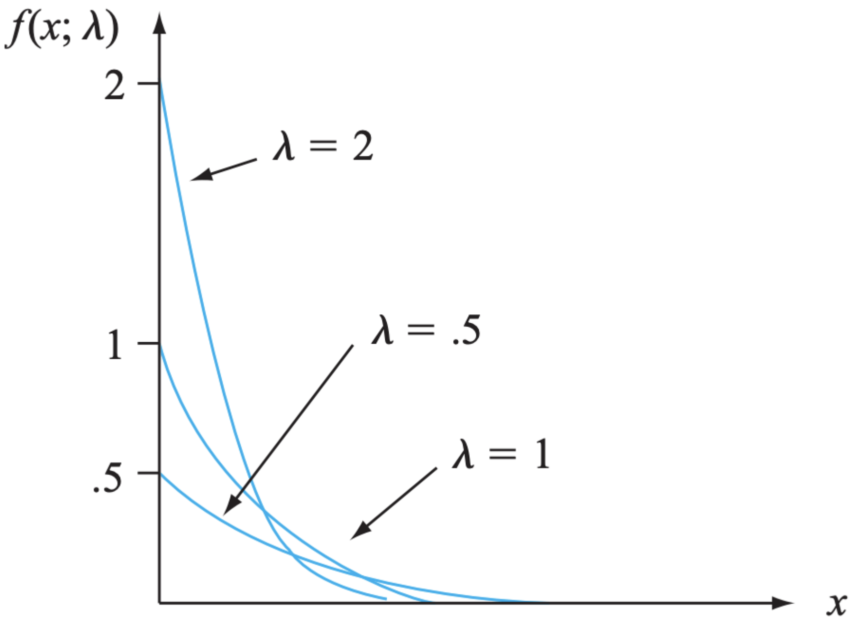
$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Expected Value and Variance

$$\mu = \frac{1}{\lambda} \qquad \sigma^2 = \frac{1}{\lambda^2}$$

Exponential Distribution



$$F(x; \lambda) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) \text{ for } x \geq 0 = \int_{-\infty}^{x} f(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} \lambda e^{-\lambda x}\, dx$$

$$= 0 + \int_{0}^{x} \lambda e^{-\lambda x} dx$$

$$= \left[\frac{\lambda e^{-\lambda x}}{-\lambda}\right]_{0}^{x} = \left[e^{-\lambda x}\right]_{x}^{0}$$

$$= 1 - e^{-\lambda x}$$

$$F(x) \text{ for } x < 0 = \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{x} 0 .\, dx = 0$$

Question

The distribution of stress range in certain bridge connections is a exponential distribution with mean value 6MPa.

a. Find the probability that stress range is at most 10 MPa.

$$E(X) = 1/\lambda = 6 \qquad \lambda = .1667$$

$$P(X \leq 10) = F(10; .1667) = 1 - e^{-(.1667)(10)} = 1 - .189 = .811$$

Question

The distribution of stress range in certain bridge connections is a exponential distribution with mean value 6 MPa.

b. Find the probability that stress range is between 5 MPa and 10 MPa.

The probability that stress range is between 5 and 10 MPa is

$$P(5 \leq X \leq 10) = F(10; .1667) - F(5; .1667) = (1 - e^{-1.667}) - (1 - e^{-.8335})$$
$$= .246$$

Poisson Distribution & Exponential Distribution

Suppose that the number of events occurring in any time interval of length t has a Poisson distribution with parameter αt (where α, the rate of the event process, is the expected number of events occurring in 1 unit of time) and that numbers of occurrences in nonoverlapping intervals are independent of one another. Then the distribution of elapsed time between the occurrence of two successive events is exponential with parameter $\lambda = \alpha$.

Poisson Distribution & Exponential Distribution

Although a complete proof is beyond the scope of the text, the result is easily verified for the time X_1 until the first event occurs:

$$P(X_1 \le t) = 1 - P(X_1 > t) = 1 - P[\text{no events in } (0, t)]$$
$$= 1 - \frac{e^{-\alpha t} \cdot (\alpha t)^0}{0!} = 1 - e^{-\alpha t}$$

which is exactly the cdf of the exponential distribution.

Memoryless Property

Suppose component lifetime is exponentially distributed with parameter λ. After putting the component into service, we leave for a period of t_0 hours and then return to find the component still working; what now is the probability that it lasts at least an additional t hours? In symbols, we wish $P(X \geq t + t_0 | X \geq t_0)$. By the definition of conditional probability,

$$P(X \geq t + t_0 | X \geq t_0) = \frac{P[(X \geq t + t_0) \cap (X \geq t_0)]}{P(X \geq t_0)}$$

But the event $X \geq t_0$ in the numerator is redundant, since both events can occur if and only if $X \geq t + t_0$. Therefore,

$$P(X \geq t + t_0 | X \geq t_0) = \frac{P(X \geq t + t_0)}{P(X \geq t_0)} = \frac{1 - F(t + t_0; \lambda)}{1 - F(t_0; \lambda)} = e^{-\lambda t}$$

This conditional probability is identical to the original probability $P(X \geq t)$ that the component lasted t hours. Thus *the distribution of additional lifetime is exactly the same as the original distribution of lifetime,* so at each point in time the component shows no effect of wear. In other words, the distribution of remaining lifetime is independent of current age.

$$P(X \geq t + t_0 \mid X \geq t_0)$$

$$= \frac{P(X \geq t + t_0 \cap X \geq t_0)}{P(X \geq t_0)}$$

$$= \frac{P(X \geq t + t_0)}{P(X \geq t_0)}$$

$$= \frac{1 - P(X < t + t_0)}{1 - P(X < t_0)}$$

$$= \frac{1 - F(t + t_0 ; \lambda)}{1 - F(t_0 ; \lambda)}$$

$$= \frac{1 - \left(1 - e^{-\lambda(t_0 + t)}\right)}{1 - \left(1 - e^{-\lambda t_0}\right)}$$

$$= \frac{e^{-\lambda t_0 - \lambda t}}{e^{-\lambda t_0}} = e^{-\lambda t}$$

Question

Let X = the time between two successive arrivals at the drive-up window of a local bank. If X has an exponential distribution with $\lambda = 1$, compute the following:

a. The expected time between two successive arrivals

b. The standard deviation of the time between successive arrivals

c. $P(X<=4)$

d. $P(2<= X< =5)$

a) $E(X) = \frac{1}{\lambda} = 1$

b) $\sigma(X) = \frac{1}{\lambda} = 1$

c) $P(X \leq 4) = 1 - e^{-\lambda x}$

$= 1 - e^{-4} = 0.9816$

d) $P(2 \leq X \leq 5) = F(5; 1) - F(2; 1)$

$= 1 - e^{-5} - 1 + e^{-2}$

$= e^{-2} - e^{-5} = 0.12859$

References

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