Lecture 13

Central Limit Theorem
Hypothesis Testing

Index

- Random Sample
- Sample mean
- Central Limit Theorem
- Hypothesis Testing
- Test Procedure
- Type of errors
- Level of Significance
- p value
- Lower Tail Test
- Questions

Random Sample

The rv's $X_1, X_2, \ldots, X_n$ are said to form a (simple) **random sample** of size n if

1. The X_i's are independent rv's.

2. Every X_i has the same probability distribution.

Sample mean

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with mean value μ and standard deviation σ. Then

1. $E(\overline{X}) = \mu_{\overline{X}} = \mu$

2. $V(\overline{X}) = \sigma^2_{\overline{X}} = \sigma^2/n$ and $\sigma_{\overline{X}} = \sigma/\sqrt{n}$

In addition, with $T_o = X_1 + \cdots + X_n$ (the sample total), $E(T_o) = n\mu$, $V(T_o) = n\sigma^2$, and $\sigma_{T_o} = \sqrt{n}\sigma$.

$$E(\bar{X}) = E\left(\frac{X_1 + X_2 \cdots X_n}{n}\right)$$

$$= \frac{1}{n} E(X_1 + X_2 \cdots X_n)$$

$$= \frac{1}{n}\left\{E(X_1) + E(X_2) \cdots E(X_n)\right\}$$

$$= \frac{1}{n}\left\{\mu + \mu \cdots \mu\right\}$$

$$= \frac{1}{n} n\mu = \mu$$

$$V(\bar{X}) = V\left(\frac{X_1 + X_2 \cdots X_n}{n}\right)$$

Since $X_1, X_2, X_3 \cdots X_n$ are independent

$$V(\bar{X}) = \frac{1}{n^2} V(X_1 + X_2 \cdots X_n)$$

$$= \frac{1}{n^2} V(X_1) + V(X_2) \cdots V(X_n)$$

$$= \frac{1}{n^2}(\sigma^2 + \sigma^2 \cdots \sigma^2)$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Note: $V(X_1 + X_2 \cdots X_n) = V(X_1) + V(X_2) \cdots V(X_n)$ only if $X_1, X_2 \cdots X_n$ are independent

Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be random samples from a distribution with mean μ and variance σ^2 .

Then if n is sufficiently large, $\bar{X}$ has approximately a normal distribution with mean μ and variance σ^2/n.

The larger the value of n, the better the approximation.

Rule of thumb: If $n > 30$, the Central Limit Theorem can be used.

In case $X_1, X_2, \ldots, X_n$ are normally distributed with mean μ and variance σ^2 then for any n, $\bar{X}$ has a normal distribution with mean μ and variance σ^2/n.

Question

The amount of a particular impurity in a batch of a certain chemical product is a random variable with mean value 4.0 g and standard deviation 1.5 g. If 50 batches are independently prepared, what is the (approximate) probability that the sample average amount of impurity $\bar{X}$ is between 3.5 and 3.8 g?

$$\mu_{\bar{X}} = 4.0$$

$$\sigma_{\bar{X}} = 1.5/\sqrt{50} = .2121$$

$$P(3.5 \le \bar{X} \le 3.8) \approx P\left(\frac{3.5 - 4.0}{.2121} \le Z \le \frac{3.8 - 4.0}{.2121}\right)$$
$$= \Phi(-.94) - \Phi(-2.36) = .1645$$

Hypothesis Testing

Hypothesis

A statistical hypothesis, or just hypothesis, is a claim or assertion either about the value of a single parameter (population characteristic or characteristic of a probability distribution) or about the values of several parameters, or about the form of an entire probability distribution.

The **null hypothesis,** denoted by H_0, is the claim that is initially assumed to be true (the "prior belief" claim). The **alternative hypothesis,** denoted by H_a, is the assertion that is contradictory to H_0.

The null hypothesis will be rejected in favor of the alternative hypothesis only if sample evidence suggests that H_0 is false. If the sample does not strongly contradict H_0, we will continue to believe in the plausibility of the null hypothesis. The two possible conclusions from a hypothesis-testing analysis are then *reject* H_0 or *fail to reject* H_0.

Test Procedure

A test procedure is specified by the following:

1. A **test statistic,** a function of the sample data on which the decision (reject H_0 or do not reject H_0) is to be based
2. A **rejection region,** the set of all test statistic values for which H_0 will be rejected

The null hypothesis will then be rejected if and only if the observed or computed test statistic value falls in the rejection region.

Type of errors

		Population Condition	
		H_0 True	H_a True
Conclusion	Accept H_0	Correct Conclusion	Type II Error
	Reject H_0	Type I Error	Correct Conclusion

Null and Alternate Hypothesis

For hypothesis tests involving a population mean, we let μ_0 denote the hypothesized value and we must choose one of the following three forms for the hypothesis test.
Equality part always comes in the H_0

$$H_0: \mu \geq \mu_0 \qquad H_0: \mu \leq \mu_0 \qquad H_0: \mu = \mu_0$$
$$H_a: \mu < \mu_0 \qquad H_a: \mu > \mu_0 \qquad H_a: \mu \neq \mu_0$$

Question

The manager of an automobile dealership is considering a new bonus plan designed to increase sales volume. Currently, the mean sales volume is 14 automobiles per month. The manager wants to conduct a research study to see whether the new bonus plan increases sales volume. To collect data on the plan, a sample of sales personnel will be allowed to sell under the new bonus plan for a one-month period.

a. Develop the null and alternative hypotheses most appropriate for this situation.

b. Comment on the conclusion when H_0 cannot be rejected.

c. Comment on the conclusion when H_0 can be rejected.

Question

H_0 is the default assumption that nothing has changed. So if μ becomes greater than 14, then it is a change which will be part of H_a

H_o : $\mu \leq 14$

H_a : $\mu > 14$

H_o cannot be rejected when there is no evidence that new plan increases sales.

H_o can be rejected when there is evidence that new plan increases sales.

Question

Because of high production-changeover time and costs, a director of manufacturing must convince management that a proposed manufacturing method reduces costs before the new method can be implemented. The current production method operates with a mean cost of $220 per hour. A research study will measure the cost of the new method over a sample production period.

a. Develop the null and alternative hypotheses most appropriate for this study.

b. Comment on the conclusion when H_0 cannot be rejected.

c. Comment on the conclusion when H_0 can be rejected.

Question

H_0 is the default assumption that nothing has changed. So if μ becomes less than 200, then it is a change which will be part of H_a

H_0 : $\mu \geq 200$

H_a : $\mu < 200$

H_0 cannot be rejected when there is no evidence that proposed manufacturing method reduces costs.

H_0 can be rejected when there is evidence that proposed manufacturing method reduces costs.

Level of Significance

- The level of significance is the probability of making a type I error when the null hypothesis is true as an equality.
- Type I error: Reject null hypothesis when it is actually true.
- The greek symbol α (alpha) is used to denote the level of significance, and common choices for α are 0.05 and 0.01.
- In practice, the level of significance is already specified before testing.
- In simple terms, level of significance will define the rejection region of the graph.

Level of Significance

- By selecting α, that person is controlling the probability of making a type I error.
- Applications of hypothesis testing that only control for the type I error are called significance tests.
- Because of the uncertainty associated with making a type II error when conducting significance tests, statisticians usually recommend that we use the statement “do not reject H_0” instead of “accept H_0.”

Tests for Population Mean when σ known: Z test

- σ known case corresponds to applications in which historical data and/or other information are available that enable us to obtain a good estimate of the population standard deviation prior to significance tests.
- Assumption: Population is normally distributed.
- In cases where it is not reasonable to assume the population is normally distributed, these methods are still applicable if the sample size is large enough.

One tailed tests

Lower Tail Test

$$H_0\colon \mu \geq \mu_0$$
$$H_a\colon \mu < \mu_0$$

Upper Tail Test

$$H_0\colon \mu \leq \mu_0$$
$$H_a\colon \mu > \mu_0$$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

p value

- A p-value is a probability that provides a measure of the evidence against the null hypothesis provided by the sample.
- Smaller p-values indicate more evidence against H_0.
- The value of the test statistic is used to compute the p-value.

Rules for hypothesis testing

	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Test Statistic	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
Rejection Rule: *p*-Value Approach	Reject H_0 if p-value $\leq \alpha$	Reject H_0 if p-value $\leq \alpha$	Reject H_0 if p-value $\leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $z \leq -z_\alpha$	Reject H_0 if $z \geq z_\alpha$	Reject H_0 if $z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$

Question

Consider the following hypothesis test:

$$H_0\text{: } \mu \geq 20$$
$$H_a\text{: } \mu < 20$$

A sample of 50 provided a sample mean of 19.4. The population standard deviation is 2.

a. Compute the value of the test statistic.

b. What is the *p*-value?

c. Using $\alpha = .05$, what is your conclusion?

d. What is the rejection rule using the critical value? What is your conclusion?

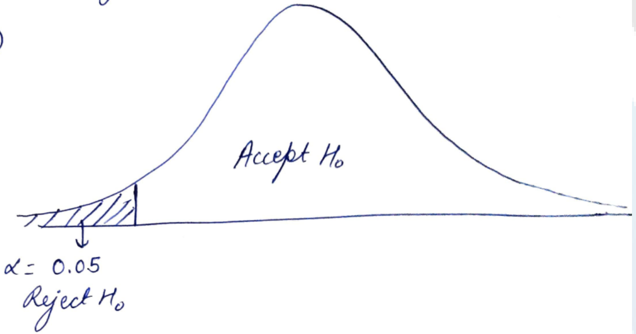
$\mu = 20$ $\quad n = 50$

$\bar{x} = 19.4$ $\quad \sigma = 2$

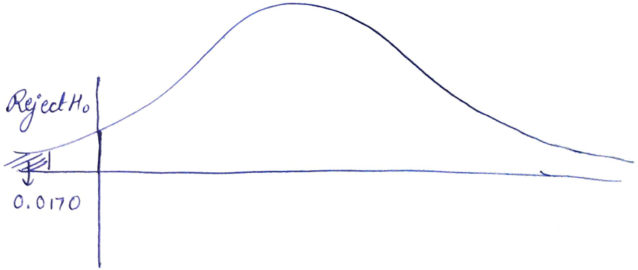
a) $z = \dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \dfrac{19.4 - 20}{2/\sqrt{50}} = -2.12$

b) Looking at the table $p = 0.0170$

c)



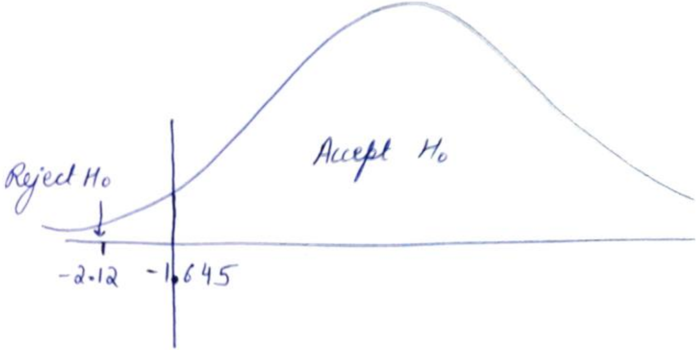
$p = 0.0170 < 0.05$



so Reject H_0

d) $\alpha = 0.05$

$-Z_{\alpha} = -1.645$

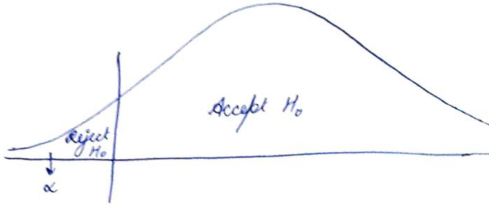


Since $-2.12 < -1.645$

Reject H_0

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379

Lower Tail Test



① Calculate z (test statistic)
z would come out to be -ve

② From z directly calculate p

③ Using p value

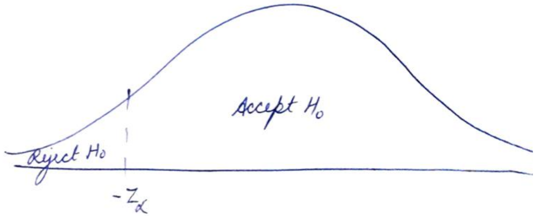
If $\alpha \geq p$ reject H_0

④ Using $-Z_\alpha$

From α directly calculate $-Z_\alpha$

Compare Z and $-Z_\alpha$

if $Z \leq -Z_\alpha$ Reject H_0



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