Lecture 14

Hypothesis Testing

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Hypothesis

A statistical hypothesis, or just hypothesis, is a claim or assertion either about the value of a single parameter (population characteristic or characteristic of a probability distribution), about the values of several parameters, or about the form of an entire probability distribution.

The **null hypothesis,** denoted by H_0, is the claim that is initially assumed to be true (the "prior belief" claim). The **alternative hypothesis,** denoted by H_a, is the assertion that is contradictory to H_0.

The null hypothesis will be rejected in favor of the alternative hypothesis only if sample evidence suggests that H_0 is false. If the sample does not strongly contradict H_0, we will continue to believe in the plausibility of the null hypothesis. The two possible conclusions from a hypothesis-testing analysis are then *reject* H_0 or *fail to reject* H_0.

Test Procedure

A test procedure is specified by the following:

1. A **test statistic,** a function of the sample data on which the decision (reject H_0 or do not reject H_0) is to be based
2. A **rejection region,** the set of all test statistic values for which H_0 will be rejected

The null hypothesis will then be rejected if and only if the observed or computed test statistic value falls in the rejection region.

Type of errors

		Population Condition	
		H_0 True	H_a True
Conclusion	Accept H_0	Correct Conclusion	Type II Error
	Reject H_0	Type I Error	Correct Conclusion

Level of Significance

The level of significance is the probability of making a type I error when the null hypothesis is true as an equality.
Type I error: Reject null hypothesis when it is actually true.
The greek symbol α (alpha) is used to denote the level of significance, and common choices for α are 0.05 and 0.01.
In practice, the person responsible for the hypothesis test specifies the level of significance.
In simple terms, level of significance will define the rejection region of the graph.

Tests for Population Mean when σ known: Z test

σ known case corresponds to applications in which historical data and/or other information are available that enable us to obtain a good estimate of the population standard deviation prior to significance tests.

Assumption: Population is normally distributed.

In cases where it is not reasonable to assume the population is normally distributed, these methods are still applicable if the sample size is large enough.

One tailed tests

Lower Tail Test

$$H_0\colon \mu \geq \mu_0$$
$$H_a\colon \mu < \mu_0$$

Upper Tail Test

$$H_0\colon \mu \leq \mu_0$$
$$H_a\colon \mu > \mu_0$$

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

p value

A p-value is a probability that provides a measure of the evidence against the null hypothesis provided by the sample. Smaller p-values indicate more evidence against H_0.

The value of the test statistic is used to compute the p-value.

Question

Consider the following hypothesis test:

$$H_0\colon \mu \geq 20$$
$$H_a\colon \mu < 20$$

A sample of 50 provided a sample mean of 19.4. The population standard deviation is 2.

a. Compute the value of the test statistic.

b. What is the *p*-value?

c. Using $\alpha = .05$, what is your conclusion?

d. What is the rejection rule using the critical value? What is your conclusion?

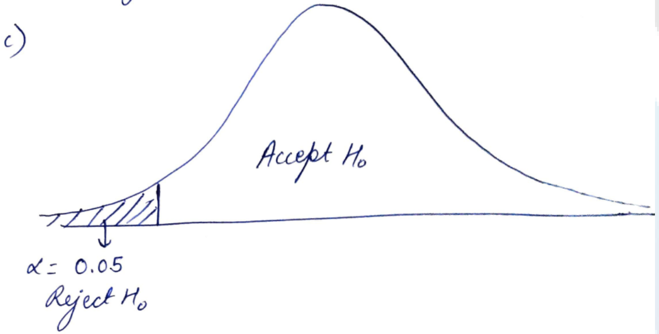
$\mu = 20$ $\qquad n = 50$

$\bar{x} = 19.4$ $\qquad \sigma = 2$

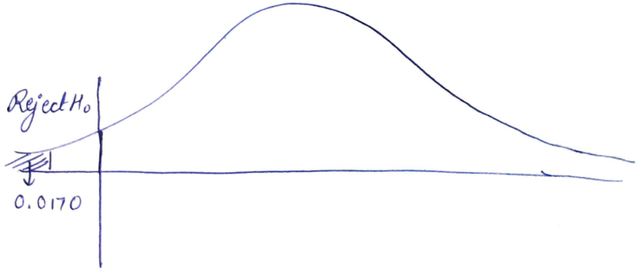
a) $$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} = \frac{19.4 - 20}{2/\sqrt{50}} = -2.12$$

b) Looking at the table $p = 0.0170$

c)



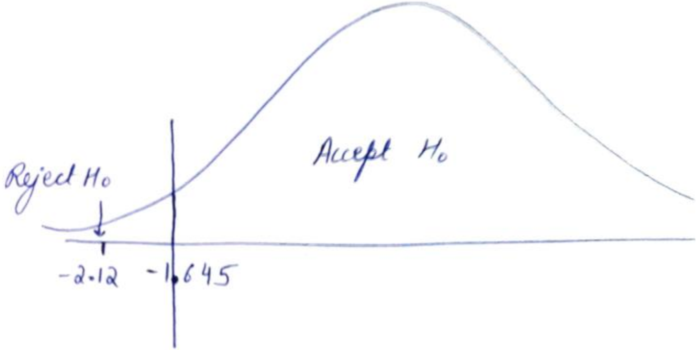
$p = 0.0170 < 0.05$



so Reject H_0

d) $\alpha = 0.05$

$-Z_{\alpha} = -1.645$

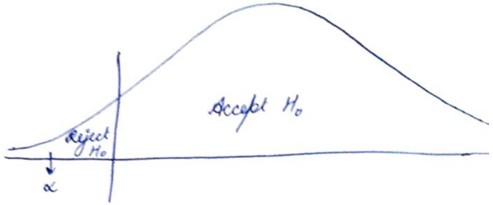


Since $-2.12 < -1.645$

Reject H_0

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379

Lower Tail Test



① Calculate z (test statistic)
z would come out to be -ve

② From z directly calculate p

③ Using p value

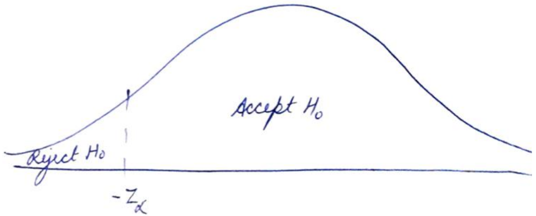
If $\alpha \geq p$ reject H_0

④ Using $-Z_\alpha$

From α directly calculate $-Z_\alpha$

Compare Z and $-Z_\alpha$

if $Z \leq -Z_\alpha$ Reject H_0



Question

Consider the following hypothesis test:

$$H_0\colon \mu \leq 25$$
$$H_a\colon \mu > 25$$

A sample of 40 provided a sample mean of 26.4. The population standard deviation is 6.

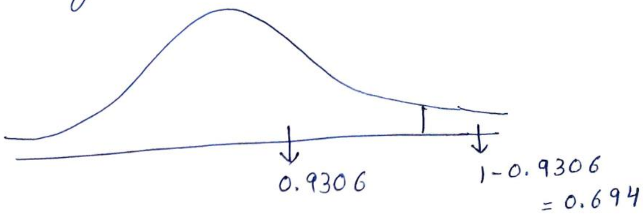
a. Compute the value of the test statistic.
b. What is the p-value?
c. At $\alpha = .01$, what is your conclusion?
d. What is the rejection rule using the critical value? What is your conclusion?

$\mu_0 = 25$ $\quad n = 40$

$\bar{x} = 26.4$ $\quad \sigma = 6$

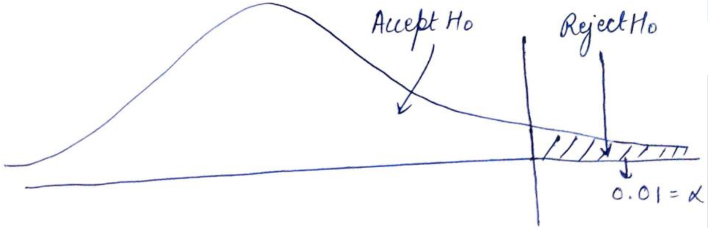
a) $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{26.4 - 25}{6/\sqrt{40}} = 1.4757 \approx 1.48$

b) Looking at the table value = 0.9306



$p = 0.694$

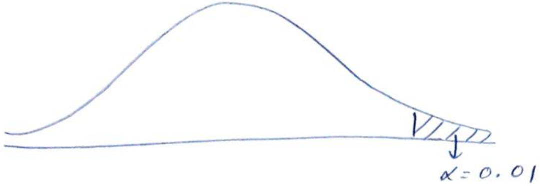
c) $\alpha = 0.01$



$0.694 > 0.01$

so do not reject H_0

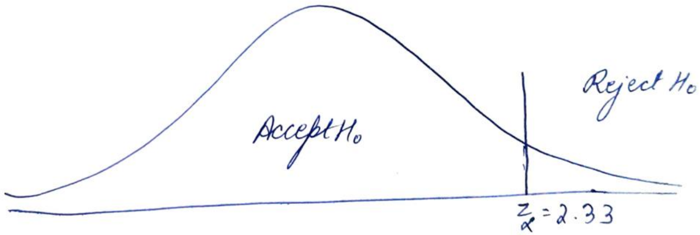
d) $\alpha = 0.01$



calculate Z_α

so area to left is $1 - 0.01 = 0.99$

Looking at the table $Z_\alpha = 2.33$

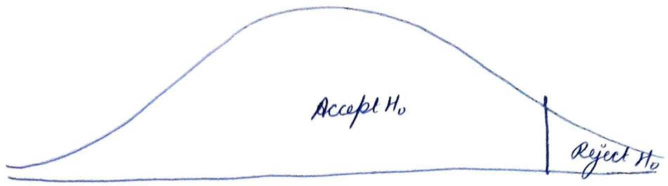


$z = 1.48 < Z_\alpha$

so accept H_0

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990

Upper tail test



① Calculate z (z would come out to be +ve)

② Calculate region to the left (Accept H_0 region) from the table

i.e. Accept H_0 region can be found directly from the table.

③ $p = 1 -$ Accept H_0 region

④ $p \leq \alpha$ Reject H_0

$p > \alpha$ Accept H_0

⑤ Calculate Z_α from $1-\alpha$ value from the table

⑥ Compare Z and Z_α

$Z \geq Z_\alpha$ Reject H_0

$Z < Z_\alpha$ Accept H_0

Question

Consider the following hypothesis test:

$$H_0\colon \mu = 15$$
$$H_a\colon \mu \neq 15$$

A sample of 50 provided a sample mean of 14.15. The population standard deviation is 3.

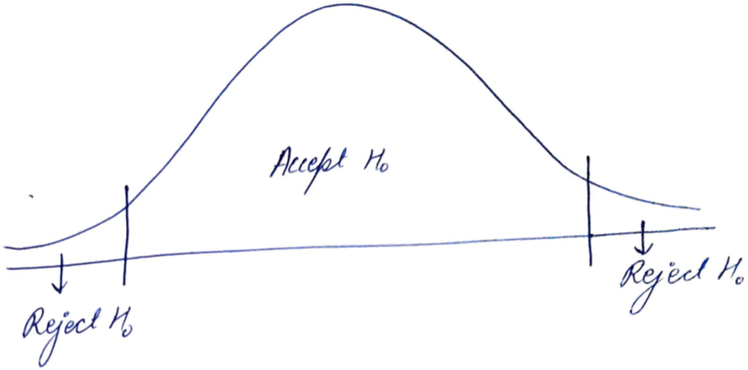
a. Compute the value of the test statistic.
b. What is the p-value?
c. At $\alpha = .05$, what is your conclusion?
d. What is the rejection rule using the critical value? What is your conclusion?

$\mu_0 = 15$ $\qquad \bar{x} = 14.15$

$n = 50$ $\qquad \sigma = 3$

a. $z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{14.15 - 15}{3/\sqrt{50}} = -2.00$

b. looking at the table we get 0.0228

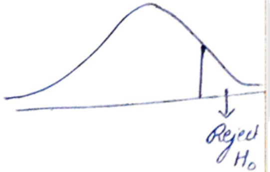


so $\beta = 0.0228 \times 2 = 0.0456$

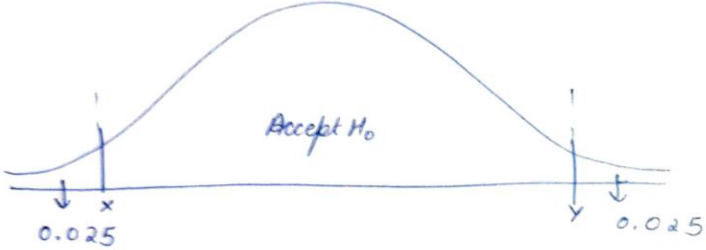
c. $\alpha = 0.05$

$\alpha > \beta$ - - - - - - ->

so reject H_0



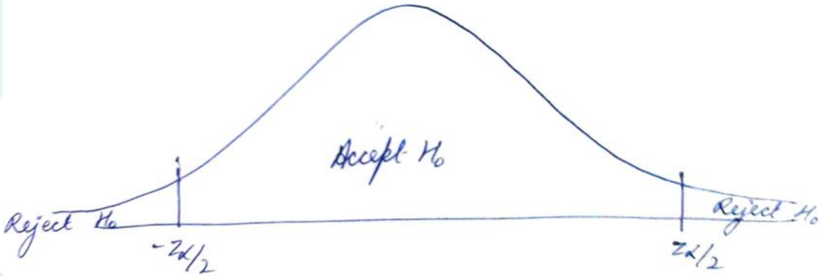
d) $\frac{\alpha}{2} = 0.025$



area to left of $X = 0.025$ so $z_{\alpha/2} = -1.96$

area to left of $Y = 1 - 0.025$

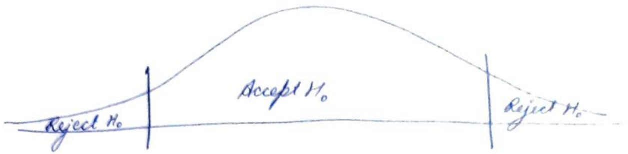
$= 0.975$ so $z_{\alpha/2} = 1.96$



$-2 < -1.96$ so reject H_0

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Two Tail Test



① Calculate z

② if z is -ve ; directly get p' from table
if z is +ve ; $p' = 1-$ value from table

③ $p = 2p'$ since reject H_0 region is on both sides

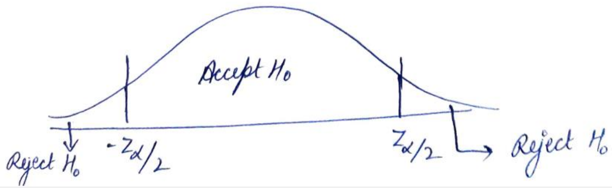
④ $p \le \alpha$ Reject H_0

⑤ To get $Z_{\alpha/2}$ look for $1-\alpha/2$ value in table
To get $-Z_{\alpha/2}$ look for $\alpha/2$ value in table

$|Z_{\alpha/2}| = |-Z_{\alpha/2}|$

so can calculate one & by symmetry get second.

⑥ $Z \le -Z_{\alpha/2}$ or $Z \ge Z_{\alpha/2}$ Reject H_0



Rules for hypothesis testing

	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0: \mu \geq \mu_0$ $H_a: \mu < \mu_0$	$H_0: \mu \leq \mu_0$ $H_a: \mu > \mu_0$	$H_0: \mu = \mu_0$ $H_a: \mu \neq \mu_0$
Test Statistic	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$	$z = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
Rejection Rule: *p*-Value Approach	Reject H_0 if p-value $\leq \alpha$	Reject H_0 if p-value $\leq \alpha$	Reject H_0 if p-value $\leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $z \leq -z_\alpha$	Reject H_0 if $z \geq z_\alpha$	Reject H_0 if $z \leq -z_{\alpha/2}$ or if $z \geq z_{\alpha/2}$

Tests for Population Mean when σ unknown: t test

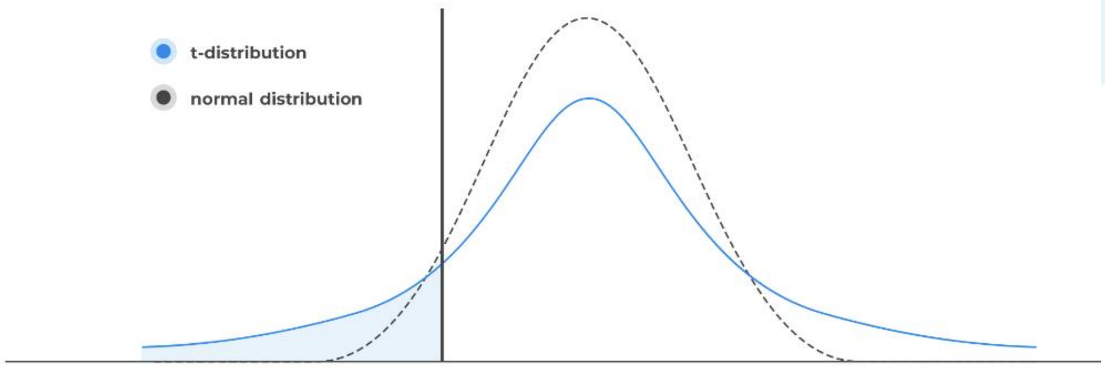
Because the σ unknown case corresponds to situations in which an estimate of the population standard deviation cannot be developed prior to sampling, the sample must be used to develop an estimate of both μ and σ.

t is the test statistic

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Degree of freedom

Degrees of Freedom refers to the maximum number of logically independent values, which are values that have the freedom to vary, in the data sample.



t Distribution

The t distribution, also known as the Student's t-distribution, is a type of probability distribution that is similar to the normal distribution with its bell shape but has heavier tails.

t distributions have a greater chance for extreme values because of the fatter tails.

The t statistics has t distribution with n-1 degrees of freedom.

Rules for hypothesis testing

	Lower Tail Test	Upper Tail Test	Two-Tailed Test
Hypotheses	$H_0\colon \mu \geq \mu_0$ $H_a\colon \mu < \mu_0$	$H_0\colon \mu \leq \mu_0$ $H_a\colon \mu > \mu_0$	$H_0\colon \mu = \mu_0$ $H_a\colon \mu \neq \mu_0$
Test Statistic	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$	$t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$
Rejection Rule: *p*-Value Approach	Reject H_0 if p-value $\leq \alpha$	Reject H_0 if p-value $\leq \alpha$	Reject H_0 if p-value $\leq \alpha$
Rejection Rule: Critical Value Approach	Reject H_0 if $t \leq -t_\alpha$	Reject H_0 if $t \geq t_\alpha$	Reject H_0 if $t \leq -t_{\alpha/2}$ or if $t \geq t_{\alpha/2}$

Question

Consider the following hypothesis test:

$$H_0: \mu = 18$$
$$H_a: \mu \neq 18$$

A sample of 48 provided a sample mean $\bar{x} = 17$ and a sample standard deviation $s = 4.5$.

a. Compute the value of the test statistic.
b. Use the t distribution table to compute a range for the p-value.
c. At $\alpha = .05$, what is your conclusion?
d. What is the rejection rule using the critical value? What is your conclusion?

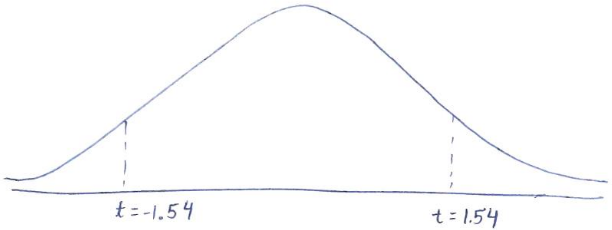
Degrees of Freedom	Area in Upper Tail .20	.10	.05	.025	.01	.005
35	.852	1.306	1.690	2.030	2.438	2.724
36	.852	1.306	1.688	2.028	2.434	2.719
37	.851	1.305	1.687	2.026	2.431	2.715
38	.851	1.304	1.686	2.024	2.429	2.712
39	.851	1.304	1.685	2.023	2.426	2.708
40	.851	1.303	1.684	2.021	2.423	2.704
41	.850	1.303	1.683	2.020	2.421	2.701
42	.850	1.302	1.682	2.018	2.418	2.698
43	.850	1.302	1.681	2.017	2.416	2.695
44	.850	1.301	1.680	2.015	2.414	2.692
45	.850	1.301	1.679	2.014	2.412	2.690
46	.850	1.300	1.679	2.013	2.410	2.687
47	.849	1.300	1.678	2.012	2.408	2.685
48	.849	1.299	1.677	2.011	2.407	2.682
49	.849	1.299	1.677	2.010	2.405	2.680
50	.849	1.299	1.676	2.009	2.403	2.678
51	.849	1.298	1.675	2.008	2.402	2.676
52	.849	1.298	1.675	2.007	2.400	2.674
53	.848	1.298	1.674	2.006	2.399	2.672
54	.848	1.297	1.674	2.005	2.397	2.670
55	.848	1.297	1.673	2.004	2.396	2.668
56	.848	1.297	1.673	2.003	2.395	2.667
57	.848	1.297	1.672	2.002	2.394	2.665
58	.848	1.296	1.672	2.002	2.392	2.663
59	.848	1.296	1.671	2.001	2.391	2.662
60	.848	1.296	1.671	2.000	2.390	2.660
61	.848	1.296	1.670	2.000	2.389	2.659
62	.847	1.295	1.670	1.999	2.388	2.657
63	.847	1.295	1.669	1.998	2.387	2.656
64	.847	1.295	1.669	1.998	2.386	2.655
65	.847	1.295	1.669	1.997	2.385	2.654
66	.847	1.295	1.668	1.997	2.384	2.652
67	.847	1.294	1.668	1.996	2.383	2.651
68	.847	1.294	1.668	1.995	2.382	2.650
69	.847	1.294	1.667	1.995	2.382	2.649
70	.847	1.294	1.667	1.994	2.381	2.648
71	.847	1.294	1.667	1.994	2.380	2.647
72	.847	1.293	1.666	1.993	2.379	2.646
73	.847	1.293	1.666	1.993	2.379	2.645
74	.847	1.293	1.666	1.993	2.378	2.644
75	.846	1.293	1.665	1.992	2.377	2.643
76	.846	1.293	1.665	1.992	2.376	2.642
77	.846	1.293	1.665	1.991	2.376	2.641
78	.846	1.292	1.665	1.991	2.375	2.640
79	.846	1.292	1.664	1.990	2.374	2.639

$\mu_0 = 18 \qquad n = 48$

$\bar{x} = 17 \qquad s = 4.5$

a) test statistic $t = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}} = \dfrac{17 - 18}{4.5/\sqrt{48}} = -1.54$

b)



Since t distribution is symmetric we can use $t = 1.54$ to look at the table

$dof = n - 1 = 48 - 1 = 47$

at $dof = 47$

α	0.10	0.05
t	1.300	1.678

1.54 in between

so p' is between 0.10 & 0.05

so $2p' = p$

$0.1 < p < 0.2$

c) $\alpha = 0.05$

$0.1 < p < 0.2$

$\alpha < p$

so we do'nt reject H_0

d) critical value $= t_{\alpha/2}$

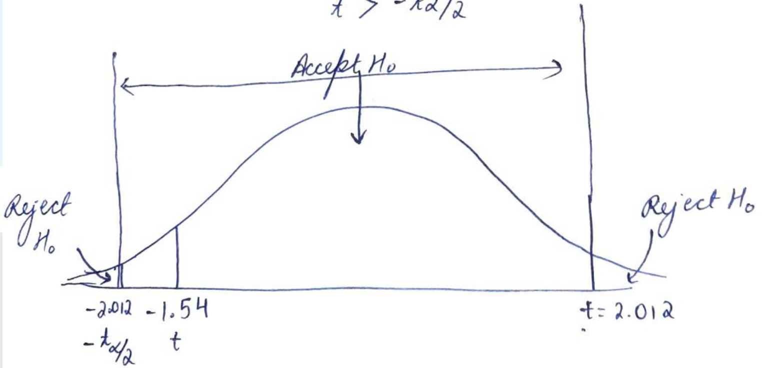
$\alpha = 0.05 \qquad \alpha/2 = 0.025$

$t_{\alpha/2} = 2.012$

Reject H_0 when $t \leq -2.012$ or $t \geq 2.012$

$t = -1.54 > -2.012$ so we do not reject H_0

$t > -t_{\alpha/2}$



so we do not reject H_0

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