Lecture 6

Poisson Distribution
Poisson Distribution as a limit
Poisson Process

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Question

A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.
a. Among ten randomly selected customers who want this type of racket, what is the probability that at least six want the oversize version?

b. $n = 10$

	x	0.01	0.05	0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.90	0.95	0.99
	0	.904	.599	.349	.107	.056	.028	.006	.001	.000	.000	.000	.000	.000	.000	.000
	1	.996	.914	.736	.376	.244	.149	.046	.011	.002	.000	.000	.000	.000	.000	.000
	2	1.000	.988	.930	.678	.526	.383	.167	.055	.012	.002	.000	.000	.000	.000	.000
	3	1.000	.999	.987	.879	.776	.650	.382	.172	.055	.011	.004	.001	.000	.000	.000
	4	1.000	1.000	.998	.967	.922	.850	.633	.377	.166	.047	.020	.006	.000	.000	.000
	5	1.000	1.000	1.000	.994	.980	.953	.834	.623	.367	.150	.078	.033	.002	.000	.000
	6	1.000	1.000	1.000	.999	.996	.989	.945	.828	.618	.350	.224	.121	.013	.001	.000
	7	1.000	1.000	1.000	1.000	1.000	.998	.988	.945	.833	.617	.474	.322	.070	.012	.000
	8	1.000	1.000	1.000	1.000	1.000	1.000	.998	.989	.954	.851	.756	.624	.264	.086	.004
	9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.999	.994	.972	.944	.893	.651	.401	.096

Question

A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.

a. Among ten randomly selected customers who want this type of racket, what is the probability that at least six want the oversize version?

= 1- B(5;10, 0.6)

= 1-0.367

= 0.633

Question

A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.

b. Among ten randomly selected customers, what is the probability that the number who want the oversize version is within 1 standard deviation of the mean value?

Question

A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.
b. Among ten randomly selected customers, what is the probability that the number who want the oversize version is within 1 standard deviation of the mean value?

Mean = 10*(0.6) = 6
Standard Deviation = $\sqrt{(10*0.6*0.4)}$ = 1.55
$P(4.45 <= X <= 7.55)$
$P(4 < X < 8) = B(7;10,0.6) - B(4;10,0.6) = 0.833 - 0.166 = 0.667$

Question

A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.

c. The store currently has seven rackets of each version. What is the probability that ten customers can get the version they want from current stock?

Question

A particular type of tennis racket comes in a midsize version and an oversize version. Sixty percent of all customers at a certain store want the oversize version.
c. The store currently has seven rackets of each version. What is the probability that all of the next ten customers can get the version they want from current stock?
P(3<= X <= 7) = B(7;10,0.6) - B(2;10,0.6) = 0.833 - 0.012 = 0.821

Poisson Distribution

A discrete random variable X is said to have a **Poisson distribution** with parameter μ $(\mu > 0)$ if the pmf of X is

$$p(x; \mu) = \frac{e^{-\mu} \cdot \mu^x}{x!} \qquad x = 0, 1, 2, 3, \ldots$$

Poisson Distribution as a legitimate pdf

- p(x; μ) > 0 because μ>0

$$e^{\mu} = 1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \cdots = \sum_{x=0}^{\infty} \frac{\mu^x}{x!}$$

$$1 = \sum_{x=0}^{\infty} \frac{e^{-\mu} \cdot \mu^x}{x!}$$

The Poisson Distribution as a limit

Suppose that in the binomial pmf $b(x; n, p)$, we let $n \to \infty$ and $p \to 0$ in such a way that np approaches a value $\mu > 0$. Then $b(x; n, p) \to p(x; \mu)$.

According to this proposition, *in any binomial experiment in which n is large and p is small,* $b(x; n, p) \approx p(x; \mu)$, *where* $\mu = np$. As a rule of thumb, this approximation can safely be applied if $n > 50$ and $np < 5$.

Comparing the Poisson and Three Binomial Distributions

x	$n = 30, p = .1$	$n = 100, p = .03$	$n = 300, p = .01$	Poisson, $\mu = 3$
0	0.042391	0.047553	0.049041	0.049787
1	0.141304	0.147070	0.148609	0.149361
2	0.227656	0.225153	0.224414	0.224042
3	0.236088	0.227474	0.225170	0.224042
4	0.177066	0.170606	0.168877	0.168031
5	0.102305	0.101308	0.100985	0.100819
6	0.047363	0.049610	0.050153	0.050409
7	0.018043	0.020604	0.021277	0.021604
8	0.005764	0.007408	0.007871	0.008102
9	0.001565	0.002342	0.002580	0.002701
10	0.000365	0.000659	0.000758	0.000810

Mean & Variance

$$E(x) = \sum_{x=0}^{\infty} \frac{x e^{-\mu} \mu^{x}}{x!}$$

Since the $x=0$ would be zero

$$E(x) = \sum_{x=1}^{\infty} \frac{x e^{-\mu} \mu^{x}}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\mu} \mu^{x}}{(x-1)!}$$

$$= e^{-\mu} \mu \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!}$$

$$= e^{-\mu} \mu \left[\frac{\mu^0}{0!} + \frac{\mu^1}{1!} + \frac{\mu^2}{2!} \cdots \right]$$

$$= e^{-\mu} \mu \left[1 + \frac{\mu}{1} + \frac{\mu^2}{2!} \cdots \right]$$

$$= e^{-\mu} \mu e^{\mu}$$

$$E(X) = \mu$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= E(X(X-1) + X) - (E(X))^2$$

$$= E(X(X-1)) + E(X) - (E(X))^2$$

$$= E(X(X-1)) + \mu - \mu^2$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} \frac{x(x-1) e^{-\mu} \mu^{x}}{x!}$$

$x=0$ & $x=1$ terms are 0

$$= \sum_{x=2}^{\infty} \frac{e^{-\mu} \mu^{x}}{x!} (x)(x-1)$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\mu} \mu^{x}}{(x-2)!}$$

$$= \mu^2 \sum_{x=2}^{\infty} \frac{e^{-\mu} \mu^{x-2}}{(x-2)!}$$

$$= \mu^2 e^{-\mu} \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!}$$

$$= \mu^2 e^{-\mu} \left[\frac{\mu^0}{0!} + \frac{\mu^1}{1!} + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} \cdots \right]$$

$$= \mu^2 e^{-\mu} e^{\mu}$$

$$= \mu^2$$

$$Var(X) = E(X(X-1)) + \mu - \mu^2$$

$$= \mu^2 + \mu - \mu^2 = \mu$$

Mean & Variance

If X has a Poisson distribution with parameter μ, then $E(X) = V(X) = \mu$.

Question

Let X denote the number of creatures of a particular type captured in a trap during a given time period. Suppose that X has a Poisson distribution with μ = 4.5 , so on average traps will contain 4.5 creatures.

a. Find probability that the trap contains exactly 5 creatures.

$$P(X = 5) = \frac{e^{-4.5}(4.5)^5}{5!} = .1708$$

Question

Let X denote the number of creatures of a particular type captured in a trap during a given time period. Suppose that X has a Poisson distribution with μ = 4.5 , so on average traps will contain 4.5 creatures.

a. Find probability that the trap contains atleast 5 creatures.

$$P(X \leq 5) = \sum_{x=0}^{5} \frac{e^{-4.5}(4.5)^x}{x!} = e^{-4.5}\left[1 + 4.5 + \frac{(4.5)^2}{2!} + \cdots + \frac{(4.5)^5}{5!}\right] = .7029$$

Question

Let X, the number of flaws on the surface of a randomly selected boiler of a certain type, have a Poisson distribution with parameter $\mu = 5$. Calculate the following:

a. $P(X \leq 8)$
b. $P(X = 8)$
c. $P(9 \leq X)$
d. $P(5 \leq X \leq 8)$
e. $P(5 < X < 8)$

x	μ = 2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	15.0	20.0
0	.135	.050	.018	.007	.002	.001	.000	.000	.000	.000	.000
1	.406	.199	.092	.040	.017	.007	.003	.001	.000	.000	.000
2	.677	.423	.238	.125	.062	.030	.014	.006	.003	.000	.000
3	.857	.647	.433	.265	.151	.082	.042	.021	.010	.000	.000
4	.947	.815	.629	.440	.285	.173	.100	.055	.029	.001	.000
5	.983	.916	.785	.616	.446	.301	.191	.116	.067	.003	.000
6	.995	.966	.889	.762	.606	.450	.313	.207	.130	.008	.000
7	.999	.988	.949	.867	.744	.599	.453	.324	.220	.018	.001
8	1.000	.996	.979	.932	.847	.729	.593	.456	.333	.037	.002
9		.999	.992	.968	.916	.830	.717	.587	.458	.070	.005
10		1.000	.997	.986	.957	.901	.816	.706	.583	.118	.011
11			.999	.995	.980	.947	.888	.803	.697	.185	.021
12			1.000	.998	.991	.973	.936	.876	.792	.268	.039
13				.999	.996	.987	.966	.926	.864	.363	.066
14				1.000	.999	.994	.983	.959	.917	.466	.105
15					.999	.998	.992	.978	.951	.568	.157
16					1.000	.999	.996	.989	.973	.664	.221
17						1.000	.998	.995	.986	.749	.297
18							.999	.998	.993	.819	.381
19							1.000	.999	.997	.875	.470
20								1.000	.998	.917	.559
21									.999	.947	.644
22									1.000	.967	.721
23										.981	.787
24										.989	.843
25										.994	.888
26										.997	.922
27										.998	.948
28										.999	.966
29										1.000	.978
30											.987
31											.992
32											.995
33											.997
34											.999
35											.999
36											1.000

Question

Let X, the number of flaws on the surface of a randomly selected boiler of a certain type, have a Poisson distribution with parameter $\mu = 5$. Calculate the following:

a. $P(X \leq 8) = F(8; 5) = 0.932$
b. $P(X = 8) = F(8;5) - F(7;5) = 0.932 - 0.867 = 0.065$
c. $P(9 \leq X) = 1 - F(8;5) = 1 - 0.932 = 0.068$
d. $P(5 \leq X \leq 8) = F(8;5) - F(4;5) = 0.932 - 0.44 = 0.492$
e. $P(5 < X < 8) = F(7;5) - F(5;5) = 0.867 - 0.616 = 0.251$

Poisson Process

$P_K(t)$ denote the probability that k events will be observed during any particular time interval of length t. The occurrence of events over time as described is called a Poisson process; the parameter α specifies the rate for the process.

$P_k(t) = e^{-\alpha t} \cdot (\alpha t)^k/k!$ so that the number of events during a time interval of length t is a Poisson rv with parameter $\mu = \alpha t$. The expected number of events during any such time interval is then αt, so the expected number during a unit interval of time is α.

Question

The number of requests for assistance received by a towing service is a Poisson process with rate $\alpha = 4$ per hour.

a. Compute the probability that exactly ten requests are received during a particular 2-hour period.

b. If the operators of the towing service take a 30-min break for lunch, what is the probability that they do not miss any calls for assistance?

c. How many calls would you expect during their break?

		μ										
		2.0	**3.0**	**4.0**	**5.0**	**6.0**	**7.0**	**8.0**	**9.0**	**10.0**	**15.0**	**20.0**
	0	.135	.050	.018	.007	.002	.001	.000	.000	.000	.000	.000
	1	.406	.199	.092	.040	.017	.007	.003	.001	.000	.000	.000
	2	.677	.423	.238	.125	.062	.030	.014	.006	.003	.000	.000
	3	.857	.647	.433	.265	.151	.082	.042	.021	.010	.000	.000
	4	.947	.815	.629	.440	.285	.173	.100	.055	.029	.001	.000
	5	.983	.916	.785	.616	.446	.301	.191	.116	.067	.003	.000
	6	.995	.966	.889	.762	.606	.450	.313	.207	.130	.008	.000
	7	.999	.988	.949	.867	.744	.599	.453	.324	.220	.018	.001
	8	1.000	.996	.979	.932	.847	.729	.593	.456	.333	.037	.002
	9		.999	.992	.968	.916	.830	.717	.587	.458	.070	.005
	10		1.000	.997	.986	.957	.901	.816	.706	.583	.118	.011
	11			.999	.995	.980	.947	.888	.803	.697	.185	.021
	12			1.000	.998	.991	.973	.936	.876	.792	.268	.039
	13				.999	.996	.987	.966	.926	.864	.363	.066
	14				1.000	.999	.994	.983	.959	.917	.466	.105
	15					.999	.998	.992	.978	.951	.568	.157
	16					1.000	.999	.996	.989	.973	.664	.221
	17						1.000	.998	.995	.986	.749	.297
	18							.999	.998	.993	.819	.381
x	19							1.000	.999	.997	.875	.470
	20								1.000	.998	.917	.559
	21									.999	.947	.644
	22									1.000	.967	.721
	23										.981	.787
	24										.989	.843
	25										.994	.888
	26										.997	.922
	27										.998	.948
	28										.999	.966
	29										1.000	.978
	30											.987
	31											.992
	32											.995
	33											.997
	34											.999
	35											.999
	36											1.000

a) $P(X=10)$

$$\alpha t = 4(2) = 8$$

$$P(X=10;8) = F(10;8) - F(9;8)$$

$$= 0.816 - 0.717$$

$$= 0.099$$

b) $$\alpha t = 4\left(\frac{1}{2}\right) = 2$$

$$P(X=0;2) = F(0;2) = 0.135$$

c) $$\alpha t = 4\left(\frac{1}{2}\right) = 2$$

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