# Solution for Practice Problem : Set 5 (HSO 201 A).

1. Marginal densities:

1 $$f_X(x) = \int_0^1 4xy\,dy$$

$$= 4x\int_0^1 y\,dy = 2x$$

$$f_Y(y) = \int_0^1 4xy\,dx = 2y$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_X(y)}, \quad 0<x<1, \quad 0<y<1$$

$$= \frac{4x\cancel{y}}{2\cancel{y}} = 2x.$$

2 $$\therefore \quad f_{X|Y}(x|y) = 2x = f_X(x).$$ , when $0<x<1$ and zero elsewhere.

2. Since $f_{X,Y}$ is to be joint p.d.f. we must have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} kx(x-y)\,dx\,dy = 1$$

$$\therefore \int_0^1\int_{-x}^{x}(kx^2 - kxy)\,dy\,dx = 1$$

$$\Rightarrow \int_0^1\left[\int_{-x}^{x}(kx^2-kxy)\,dy\right]dx = 1$$

$$\Rightarrow k\int_0^1\left[\int_{-x}^{x}x^2\,dy - \int_{-x}^{x}xy\,dy\right]dx = 1$$

$$\Rightarrow k\int_0^1\left[x^2(2x) - 0\right]dx = 1$$

$$\Rightarrow k\int_0^1 2x^3\,dx = 1 \Rightarrow k\,2\times\frac{1}{4} = 1 \Rightarrow \frac{k}{2} = 1 \text{ or } k = 2.$$

Thus $kx(x-y) \geq 0, \ \forall x,y$ in $0<x<1, -x<y<x$

3.) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$.

Here $f_X(x) = \int_0^1 (x+y)\,dy$

$= \int_0^1 x\,dy + \int_0^1 y\,dy$

$= x + \frac{1}{2}$

$\therefore f_{Y|X}(y|x) = \dfrac{x+y}{x+1/2}$, $0 < x < 1$, $0 < y < 1$, & zero elsewhere

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} \frac{x+z}{x+1/2}\,dz, \quad 0<y<1$$

$$= \int_{-\infty}^{y} \frac{x}{x+1/2}\,dz + \int_{-\infty}^{y} \frac{z}{x+1/2}\,dz$$

$$= \int_0^y \frac{x}{x+1/2}\,dz + \int_0^y \frac{z\,dz}{x+1/2}$$

$$= \frac{1}{x+1/2}\left(xy + \frac{y^2}{2}\right), \quad 0<y<1$$

4) a) $E[g_1(Y) + g_2(Y) \mid X = x]$

We prove for the continuous case only

$$= \int_{-\infty}^{\infty} (g_1(Y) + g_2(Y))\, f_{Y|X}(y|x)\,dy$$

$$= \int_{-\infty}^{\infty} g_1(Y)\, f_{Y|X}(y|x)\,dy + \int_{-\infty}^{\infty} g_2(Y)\, f_{Y|X}(y|x)\,dy$$

$$= E[g_1(Y) \mid X=x] + E[g_2(Y) \mid X=x]$$

b) $E[g_1(Y)\, g_2(X) \mid X=x] = \int_{-\infty}^{\infty} g_1(y)\, g_1(x)\, f_{Y|X}(y|x)\,dy$

Since $x$ is the value of the random variable $X$, we observe that $g(x)$ is just a real number hence

$$E[g_1(Y)\, g_2(X) \mid X=x] = g_2(x) \int_{-\infty}^{\infty} g_1(y)\, f_{Y|X}(y|x)\,dy = g_2(x)\, E[g_1(Y) \mid X=x]$$

$$E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\, f_X(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^y 2xy\, dx\, dy$$

$$= 2\int_0^1 \left[ \int_0^y xy\, dx \right] dy$$

$$= 2\int_0^1 \frac{y^3}{2}\, dy = \left[\frac{y^4}{4}\right]_0^1$$

$$= \frac{1}{4}$$

$$E(X) = \int_0^1 f_X(x)\,dx$$

while $f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = 2\int_0^1 dy = 2$

$$E(X) = \int_0^1 f_X(x)\,dx. = \int_0^1 2\,dx = 2$$

$\therefore$ $E(Y) = \int_0^1 f_Y(y)\,dy$, where $f_Y(y) = \int_0^y 2\,dx$

$$= 2y$$

$$\therefore E(Y) = 2\int_0^1 y\,dy = 1.$$

$$\therefore \quad Cov(X,Y) = \frac{1}{4} - 2\times 1$$

$$= \frac{1}{4} - 2 = -\frac{7}{4}.$$

$f_{Y|X}(y|x) = \frac{2}{2} = 1$ for all $0 < y < 1$.

6) Note here that if $X_2$ is the random variable showing the number on the second ball.

For example the event

$\{X = 2, Y = 2\}$ is the event $\{X = 2, X_2 = 1\}$

or $\{X = 3, Y = 3\}$ is the event $\{X = 3, X_2 = 1\} \cup \{X = 3, X_2 = 2\}$

These are mutually exclusive

Once we understand this fact it is simple to write down the all possible configurations and their probability

$$P[X = 1, Y = 2] = P[X = 1]\, P[X_2 = 2 \mid X = 1]$$

$$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

(When the first ball drawn is having the number 1, then the second ball is picked from the remaining 2)

$$P[X = 1, Y = 3] = P[X = 1]\, P[X_2 = 3 \mid X = 1]$$

$$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$P[X = 2, Y = 2] = P[X = 2, X_2 = 1]$$

$$= P[X = 2]\, P[X_2 = 1 \mid X = 2]$$

$$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$P[X = 2, Y = 3] = P[X = 2, X_2 = 3]$$

$$= P[X = 2]\, P[X_2 = 3 \mid X = 2]$$

$$= \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$$

$$P[X = 3, Y = 3] = P[X = 3, X_2 = 1] + P[X = 3, X_2 = 2]$$

$$= \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$$

Observe that the total probability sums upto 1.

When the distribution is known the students can compute 6 a) & 6 b)

Q7). This is a straight forward application of the definition of

[Hint:] a bivariate normal distribution and the fact the marginals are also normal distribution.