Lecture 9: Transformation of Random Variables & their Distributions

Suppose we have a random variable X and another random Y is defined as $Y = g(X)$. If we assume just for the sake of it that X is a continuous random variable with a p.d.f. Sometimes specially in statistics, we need to know the density of Y. We can also have more than ~~one vari~~ random variables, $X_1, X_2, \ldots X_n$ and suppose we know their joint pmf/pdf. Then we might want to know the pmf/pdf of

$$Y = g(X_1, X_2, \ldots, X_n)$$

In this chapter we shall show how can we compute $f_Y(y)$. There are several approaches and as we will see what kind of technique to use depends on the problem at hand.

Section 1: Distribution function Approach

One of the simplest approaches ~~to~~ is to compute the distribution function of Y and then differentiate it to get the p.d.f

We have $F_Y(y) = P(Y \le y) = P(g(X_1 \ldots X_n) \le y)$

$\therefore$ Then $f_Y(y) = \dfrac{dF_Y}{dy}$

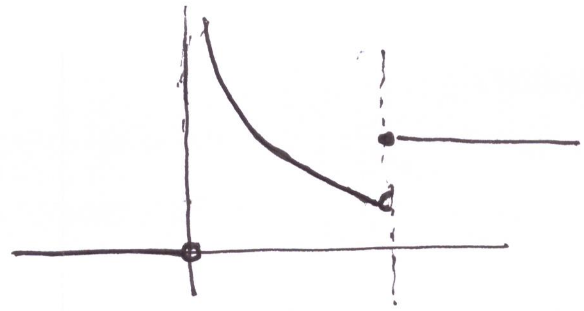
E.g. 1. Let $X \sim \text{Uniform}(0,1)$, then if $Y = g(X) = X^2$. We shall find the distribution of Y.

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = \int_{\{x:\, x^2 \le y\}} f_X(x)\,dx$$

$$= \int_0^{\sqrt{y}} dx \qquad (0 < y < 1)$$

$$= \sqrt{y}, \qquad \text{when } 0 < y < 1.$$

$$\therefore F_Y(y) = \begin{cases} 0, & y \leq 0 \\ \sqrt{y}, & 0 < y < 1 \\ 1, & y \geq 1 \end{cases}$$



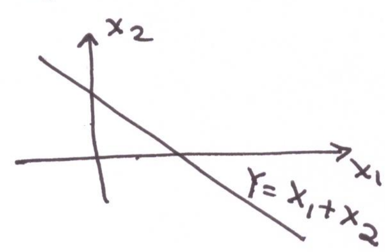
$$\therefore f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

E.g. 2: If X_1 is a continuous r.v. & X_2 is a continous r.v. whose joint density is given as

From Miller & Miller Freund's Mathematical

$$f_{X_1,X_2}(x_1, x_2) = \begin{cases} 6e^{-3x_1 - 2x_2} & ; x_1 > 0,\ x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find the distribution of $Y = X_1 + X_2$ and also its probability density



$$F_Y(y) = P(Y \leq y)$$
$$= P(X_1 + X_2 \leq y)$$
$$= P(X_1 \leq y - X_2,\ X_2 \leq y)$$
$$= \int_0^y \int_0^{y-x_2} 6e^{-3x_1 - 2x_2}\, dx_1\, dx_2$$
$$= 1 + 2e^{-3y} - 3e^{-2y}. \quad \text{when: } (y > 0)$$

Thus $f_Y(y) = 6\left(e^{-2y} - e^{-3y}\right)$, for $y > 0$

& $f_Y(y) = 0$, elsewhere.

Section 2: Transformation Approach : Single r.v.

So in this section we consider the case $Y = u(X)$, and given a probability mass / density of X, we would like to know what is the probability distribution of Y.

Consider the following simple example from Miller and Miller [ ]

Eg 1: X be the r.v. denoting the number of heads in a toss of ~~four coins~~ a fair coin four times.

[E.g. from Miller and Miller Freund's Mathematical Statistics]

Find the distribution of $Y = \frac{1}{1+X}$

The distribution of X is given as

X	$f_X(x)$
0	$1/16$
1	$4/16$
2	$6/16$
3	$4/16$
4	$1/16$

Here $X \sim B(4, \frac{1}{2})$

So Y takes the values, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$ corresponding to $X=0, X=1, X=2, X=3, X=4$, respectively

Y	$g_Y(y)$
1	$1/16$
$1/2$	$\frac{4}{16}$
$1/3$	$\frac{6}{16}$
$1/4$	$\frac{4}{16}$
$1/5$	$1/16$

So we have $g_Y(y) = f_X\left(\frac{1}{y} - 1\right)$. ~~But~~ Observe that

$X = \frac{1}{y} - 1.$

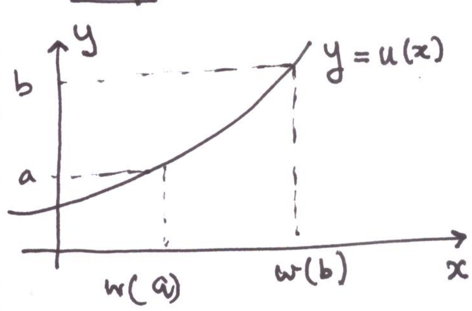
For the continuous case however we need to rely on the following result.

<u>Theorem 9.1</u>: Let X be a random variable, which is continuous one with p.d.f $f_X(x)$. Let $y = u(x)$ be a function of x, which is differentiable and monotone (either increasing or decreasing) over ~~Range (X)~~ the set $\{x \in \text{Range } X : f_X(x) \neq 0\}$. Then for these values of x, the function u has an inverse w, for which $x = w(y)$. Then the probability density ~~is given~~ of $Y = u(X)$ is given as

$$g_Y(y) = f_X[w(y)]\,|w'(y)|, \text{ provided } u'(x) \neq 0$$

and $g_Y(y) = 0$, elsewhere. We assume u and w are differentiable.

<u>Proof:</u> Let us first consider ~~the~~ the case when $y = u(x)$ is increasing



Now

$$P[a < Y < b] = P[w(a) < X < w(b)]$$

$$= \int_{w(a)}^{w(b)} f_X(x)\,dx$$

Now substitute in the integral, $x = w(y)$. ∴ ~~w(a) when~~

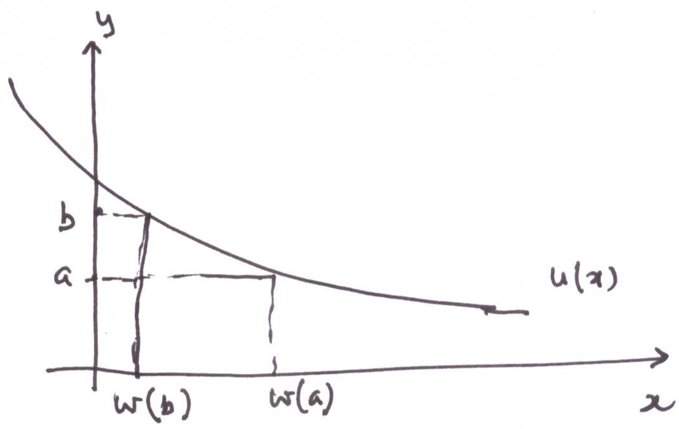
∴ When $x = w(a)$, we have $y = a$, & when $x = w(b)$, we have $y = b$ and $dx = w'(y)dy$.

$$\therefore \quad P[a < X < b] = \int_a^b f_X(w(y))\,w'(y)\,dy$$

Thus the probability density is given by the function

$$g_Y(y) = f_X(w(y))\,w'(y).$$

Let us consider now u to be decreasing



In this case we have

$$\therefore \quad P[a < Y < b] = P[w(b) < X < w(a)]$$

$$= \int_{w(b)}^{w(a)} f_X(x)\,dx$$

$$= \int_{b}^{a} f_X(w(y))\,w'(y)\,dy$$

$$= -\int_{a}^{b} f_X(w(y))\,w'(y)\,dy$$

In this case we have

$$g_Y(y) = -f_X(w(y))\,w'(y)\,dy$$

In fact. $w'(y) = \dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$ i.e. $w'(y) = \dfrac{1}{u'(x)}$

In fact when u is increasing $w'(y)$ is positive and when u is ~~decr~~ decreasing $-w'(y)$ is positive. Hence compactly we write

$$g_Y(y) = f_X[w(y)]\,|w'(y)|$$

or $g_Y(y) = f_X(u^{-1}(y))\left|\dfrac{dx}{dy}\right|$; (Here $w \equiv u^{-1}$)

$$\boxed{g_Y(y) = f_X(u^{-1}(y))\left|\frac{dx}{dy}\right|}$$

, $u'(x) \neq 0$ & $x = u^{-1}(y)$.

E.g 2: Let $X \sim N(0,1)$. Then find the density of

$$Z = u(X) = X^2.$$

Note that the function $z = x^2$, is decreasing when $x < 0$ and increasing when $x > 0$. So conditions of ~~the~~ Theorem 9.1 are not met.

The ~~key~~ to this is to take an additional ~~step~~ and have ~~a~~ random ~~variable~~ ~~on~~ whose range the given transformation

This brings us to the case what happens if $u(x)$ behaves differently on different parts of $\{x \in Ran(x) : u(x) \neq 0\}$.

Assume that we can partition the set $\{x \in Ran(X) : u(x) \neq 0\}$ into a finite partion $\{A_n\}_{n=1}^{k}$, ie.

$$\bigcup_{n=1}^{k} A_n = \{x \in Ran(x) : u(x) \neq 0\}$$

& $A_i \cap A_j = \phi, \ \forall\, i \neq j, \quad i, j = 1, \dots, k.$

So each A_n, the function u is either increasing or decreasing.

Thus define

$$u_n(x) = \begin{cases} u(x), & x \in A_n \\ 0, & x \text{ is otherwise.} \end{cases}$$

$\therefore$ u_n has a unique inverse in A_n. Thus

$y = u_n(x)$ ~~$y = g_n(x)$~~, $x = \cancel{g_n}\, u_n^{-1}(y)$, for $x \in A_n$

Then we can get the pdf on the individual A_n and then sum

i.e.

$$\boxed{g_Y(y) = \sum_{n=1}^{k} f_X(u_n^{-1}(y)) \left| \frac{d}{dy} u_n^{-1}(y) \right|}$$

~~$g_Y(y) = \sum_{n=1}^{k} f.$~~

~~$g_Y(y) = \sum_{n=1}^{k} f_X(u_n^{-1}(y)) \left| \frac{d}{dy} u_n^{-1}(y) \right|$~~

So in our particular case we have

$$\{x \in \mathbb{R} : x^2 \neq 0\} = \underbrace{\{x \in \mathbb{R}, x < 0\}}_{A_1} \cup \underbrace{\{x \in \mathbb{R} : x > 0\}}_{A_2}$$

So on $(-\infty, 0)$ we have $x = -\sqrt{y}$ & on $(0, \infty)$ we have $x = +\sqrt{y}$. Here $y > 0$. as $y = x^2 > 0$.

$$\therefore g_Y(y) = f_X(-\sqrt{y}) \frac{1}{2\sqrt{y}} + f_X(\sqrt{y}) \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-y/2} \frac{1}{2\sqrt{y}} + \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2\pi y}} e^{-y/2}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{2\pi y}} e^{-y/2}\right)$$

$$\therefore \boxed{g_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}, \quad y > 0}$$

obsv: $Y \sim \text{Gamma}(\frac{1}{2}, \frac{1}{2})$ is called χ^2-with one deg of freedom

Of course $g_Y(y) = 0$, if $y \leq 0$.

In fact g_Y is often said to be the pdf of a χ^2 - ~~distri~~ random variable with degrees of freedom 1. (more on this later).

Section 3: Distribution of the Sum and difference of two r.v.

In this section we will be concerned with the distribution of the sum and differences of two random variables. These notions would play an important role in sampling theory. So we begin by presenting the analysis in forms of theorems, so that the reader appreciates the importance of these results. We present these results from "Introduction to the Theory of Statistics" by Mood, Graybill, Boyes.

Theorem 9.1: Let X and Y be two continuous random variables, with joint pdf $f_{X,Y}(x,y)$. Let $Z = X+Y$ & $V = X-Y$, then we have

A) $$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx = \int_{-\infty}^{\infty} f_{X,Y}(z-y, y)\,dy$$

B) $$f_V(v) = \int_{-\infty}^{\infty} f_{X,Y}(x, x-v)\,dx = \int_{-\infty}^{\infty} f_{X,Y}(v+y, y)\,dy$$

We will just prove the first part of case A). The rest can be proved in an analogous way.

Proof:

$$F_Z(z) = P(Z \le z)$$
$$= P(X+Y \le z)$$
$$= \iint_{X+Y \le Z} f_X(x,y)\,dx$$
$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{Z-x} f_{X,Y}(x,y)\,dy \right] dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z} \left[f_X(x, u-x)\,du \right] dx \quad \left[\text{By substituting } y = u-x\right]$$
$$= \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} f_{XY}(x, u-x)\,dx \right] du$$

$$\therefore \quad f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{d}{dz} \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} f_{X,Y}(x, u-x)\,dx \right] du$$
$$= \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx \quad \left[\text{Use the fundamental theorem of Calculus but for the improper integrals.}\right]$$

Corollary 9.1: Let X and Y are now assumed to be independent, and $Z = X + Y$.

Then.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$$

Section 4: The distribution of products and ratio

Theorem 9.2: Let X and Y are continuous random variables with joint pdf $f_{X,Y}(x,y)$ and let $Z = XY$ and $U = \frac{X}{Y}$.

Then;

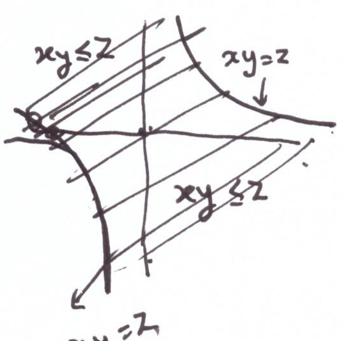
a) $$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{X,Y}\left(x, \frac{z}{x}\right) dx = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X,Y}\left(\frac{z}{y}, y\right) dy$$

b) $$f_U(u) = \int_{-\infty}^{\infty} |y|\, f_{X,Y}(uy, y)\,dy$$

We shall only prove the first part of a), the rest can be similarly worked out by the reader.

$$F_Z(z) = P(Z \le z) = \iint_{xy \le z} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{0} \left[\int_{z/x}^{\infty} f_{X,Y}(x,y)\,dy \right] dx + \int_{0}^{\infty} \left[\int_{-\infty}^{z/x} f_{X,Y}(x,y)\,dy \right] dx$$



Let us now substituting $u = xy$, we have

$$F_Z(z) = \int_{-\infty}^{0} \left[\int_{z}^{-\infty} f_{X,Y}\left(x, \frac{u}{x}\right) \frac{du}{x} \right] dx + \int_{0}^{\infty} \left[\int_{-\infty}^{z} f_{X,Y}\left(x, \frac{u}{x}\right) \frac{du}{x} \right] dx$$

$$= \int_{-\infty}^{z} \left[\int_{-\infty}^{0} \frac{1}{-x} f_{X,Y}\left(x, \frac{u}{x}\right) dx \right] du + \int_{-\infty}^{z} \left[\int_{0}^{\infty} \frac{1}{x} f_{X,Y}\left(x, \frac{u}{x}\right) dx \right] du$$

$$= \int_{-\infty}^{z} \left[\int_{-\infty}^{\infty} \frac{1}{|x|} f_{X,Y}\left(x, \frac{u}{x}\right) dx \right] du.$$

Hence

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{X,Y}\left(x, \frac{u}{x}\right) dx.$$

<u>Example:</u> Let X and Y are independent uniform random variables in $(0,1)$. Compute the distribution of $Z = XY$.

We have $f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_{X,Y}\left(x, \frac{z}{x}\right) dx$

$= \int_{-\infty}^{\infty} \frac{1}{|x|} f_X(x) f_Y\left(\frac{z}{x}\right) dx,$ by independence of X and Y.

$\neq \int_{0}^{1} \cancel{\frac{1}{x}}$. Now $f_Y\left(\frac{z}{x}\right) = 1$, if $0 \leq \frac{z}{x} < 1$

& $f_X(x) = 1$ if ~~$0 < x \leq 1$~~ $0 < x < 1$

$$\therefore \quad f_Y\left(\frac{z}{x}\right) = I_{(0,1)}\left(\frac{z}{x}\right)$$

$$f_X(x) = I_{(0,1)}(x)$$

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Indicator function or characteristic function

Now $\quad I_{(0,1)}(x)\, I_{(0,1)}\left(\frac{z}{x}\right)$

$= I_{(0,1)}(z)\, I_{(z,1)}(x)$ (Try to establish this)

$$\therefore \quad f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|x|} I_{(0,1)}(x)\, I_{(0,1)}\left(\frac{z}{x}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{|x|} I_{(0,1)}(z)\, I_{(z,1)}(x)\, dx$$

$$\cancel{= I_{(0,1)}(x \int_{-\infty}}$$

$$= I_{(0,1)}(z) \int_{-\infty}^{\infty} \frac{1}{|x|} I_{(z,1)}(x)\, dx$$

$$= I_{(0,1)}(z) \int_{z}^{1} \frac{1}{x}\, dx$$

$$\therefore \quad \boxed{f_Z(z) = -\log z \; I_{(0,1)}(z).}$$

i.e. $\boxed{f_Z(z) = -\log z, \quad z \in (0,1)}$

& $f_Z(z) = 0$, otherwise.

Section 5: Further Transformation Techniques

Suppose now we have the following tranformation.

$$Y = u(X_1, X_2).$$

How are we going to get the pmf or pdf in this case.

Our key idea is the following. Suppose we keep x_2 fixed and vary x_1, then y becomes a function of x_1. Suppose under this fixed x_2, the function $u(x_1, x_2)$ is increasing or decreasing and thus has an inverse, i.e. x_1 can be written as a function of Y, ie. $x_1 = \omega_1(y, x_2)$. So using Theorem 9.1 we write,

$$g_Y(y, x_2) = f_{X_1,X_2}(x_1, x_2)\left|\frac{\partial x_1}{\partial y}\right|$$

or we may have that $u(x_1, x_2)$ satisfies the requirements of Theorem 9.1 if x_1 is kept fixed and thus we will have

$$g_Y(x_1, y) = f_{X_1,X_2}(x_1, x_2)\left|\frac{\partial x_2}{\partial y}\right|.$$

Whatever be the case, ~~we can~~ ~~from~~ $g_Y(x_1, y)$ or $g_Y(y, x_2)$ can be ~~ind~~ integrated ~~out~~ with respect to x_1 or x_2, to get the marginal distribution of Y.

Let us see an example

Let X_1 and X_2 be two random variables which are independent and each follow uniform distribution in $(0,1)$. Compute the pdf of Y. Here Let us ~~fix~~ set $Y = X_1 + X_2$. As X_1 and X_2 vary from 0 to 1. Y varies from 0 to 2

If X_2 is kept fixed we have

$$\therefore \quad X_1 = Y - X_2.$$

$$\therefore \quad g_Y(y, x_2) = I_{(0,1)}(y - x_2)\, I_{(0,1)}(x_2)\,|1|$$

$$= I_{(0,1)}(y - x_2)\, I_{(0,1)}(x_2)$$

$$\therefore \quad g_Y(y) = \int_{-\infty}^{\infty} g_Y(y, x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} I_{(0,1)}(y-x_2)\, I_{(0,1)}(x_2)\, dx_2$$

When $y - x_2 \in (0,1)$, then $0 < y - x_2 < 1$

$\therefore \quad x_2 < y$ & $y < x_2 + 1$. ~~Now if we set~~ Also

~~Set $y \in (0,1)$, then~~ if $y > 1$, & $y < 2$, then $y - x_2 < 1$

$\Rightarrow y - 1 < x_2 < 1$

Now observe that

$$I_{(0,1)}(y-x_2)\, I_{(0,1)}(x_2)$$

$$= I_{(0,y)}(x_2)\, I_{(0,1)}(y) + I_{(y-1,1)}(x_2)\, I_{[1,2)}(y)$$

$$\therefore \quad g_Y(y) = \int_{-\infty}^{\infty} \left[I_{(0,y)}(x_2)\, I_{(0,1)}(y) + I_{(y-1,1)}(x_2)\, I_{[1,2)}(y)\, dx_2 \right]$$

$$= I_{(0,1)}(y) \int_0^y I_{(0,y)}(x_2)\, dx_2 + I_{[1,2)}(y) \int_{y-1}^{1} I_{(y-1,1)}(x_2)\, dx_2$$

$$= y\, I_{(0,1)}(y) + I_{[1,2)}(y)(2-y)$$

$$\therefore \quad g_Y(y) = \begin{cases} y & \text{if } 0 < y < 1 \\ 2-y & \text{if } 1 \le y < 2 \end{cases}$$

& $g_Y(y) = 0$, elsewhere.

Now what happens if we have

$$Y_1 = u_1(X_1, X_2)$$
$$Y_2 = u_2(X_1, X_2)$$

Then if we know the joint distribution of X_1 & X_2 can we find the joint distribution of Y_1 & Y_2. The following Theorem tells us how to do it.

Theorem 9.2:

Let X_1 and X_2 are continuous random variables, with joint pdf $f_{X_1,X_2}(x_1, x_2)$. If the transformations $y_1 = u_1(x_1, x_2)$ and $y_2 = u_2(x_1, x_2)$ are bijective mappings over the set

$$\left\{ (x_1, x_2) \in \text{Range}(X_1) \times \text{Range}(X_2) : f_{X_1,X_2}(x_1, x_2) \neq 0 \right\}.$$

Then over the above set we can uniquely write

$$x_1 = w_1(y_1, y_2)$$
$$x_2 = w_2(y_1, y_2)$$

Then the joint distribution $Y_1 = u_1(X_1, X_2)$ & $Y_2 = u_2(X_1, X_2)$ is given as

$$g_{Y_1,Y_2}(y_1, y_2) = f\left[w_1(y_1, y_2), w_2(y_1, y_2)\right) \left| \det J \right|$$

where J is the Jacobian matrix of the map

$$\cancel{\phi(x_1, x_2) = \begin{pmatrix} w_1(y_1, y_2) \\ w_2(y_1, y_2) \end{pmatrix}}$$

$$\phi(y_1, y_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} w_1(y_1, y_2) \\ w_2(y_1, y_2) \end{pmatrix}$$

given as

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}.$$

Jacobian matrix is the derivative of the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$

Let us provide some interesting examples.

Eg (A): Let X_1 & X_2 are independent standard normal variables.

Let $Y_1 = X_1 + X_2$ & $Y_2 = \frac{X_1}{X_2}$

Find $g_{Y_1,Y_2}(y_1, y_2)$.

Here $y_1 = u_1(x_1, x_2) = x_1 + x_2$.

& $y_2 = u_2(x_1, x_2) = x_1/x_2$.

Then,

$$x_1 = w_1(y_1, y_2) = \frac{y_1 y_2}{1+y_2}$$

$$x_2 = w_2(y_1, y_2) = \frac{y_1}{1+y_2}$$

$$J = \begin{bmatrix} \frac{y_2}{1+y_2}, & \frac{y_1}{(1+y_2)^2} \\ \frac{1}{1+y_2}. & -\frac{y_1}{(1+y_2)^2} \end{bmatrix}$$

$\therefore$ $\det J = -\frac{y_1}{(1+y_2)^2}$ $\therefore$ $|\det J| = \frac{|y_1|}{(1+y_2)^2}$

$$\therefore g_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi}\left[e^{-\frac{1}{2}\left[\frac{(y_1 y_2)^2}{(1+y_2)^2} + \frac{y_1^2}{(1+y_2)^2}\right]}\right] \frac{|y_1|}{(1+y_2)^2}$$

$$\therefore g_{Y_1,Y_2}(y_1, y_2) = \frac{1}{2\pi} \frac{|y_1|}{(1+y_2)^2}\left[e^{-\frac{1}{2}\frac{[(1+y_2^2)\, y_1^2]}{[1+y_2]^2}}\right].$$

Question: Can you find the marginal distribution of Y_2.

E.g (B): Let $X_1 \sim \text{Gamma}(n_1, \lambda_1)$

$X_2 \sim \text{Gamma}(n_2, \lambda_2)$

and X_1 and X_2 are independent. Then find the joint distribution of Y_1 & Y_2 with

$$Y_1 = X_1 + X_2 \quad \& \quad Y_2 = \frac{X_1}{X_2}.$$

Find the marginal distribution of Y_1.

Our first ~~appro~~ job would be to compute first the joint distribution of $g_{Y_1,Y_2}(y_1, y_2)$ and then compute out the marginal;

Here

$$x_1 = w_1(y_1, y_2) = \frac{y_1 y_2}{1 + y_1 y_2}$$

$$x_2 = w_1(y_1, y_2) = \frac{y_1}{1 + y_2}$$

$$\therefore \ |\det J| = \frac{y_1}{(1+y_2)^2} \quad \left(\text{note } y_1 \ \& \ y_2 \text{ are } \cancel{\text{non-n}} \text{ positive as } x_1 \in (0,\infty) \ \& \ x_2 \in (0,\infty)\right)$$

For $y_1 \in (0,\infty)$ & $y_2 \in (0,\infty)$ we have

$$\therefore \ \cancel{f_X} \ g_{Y_1,Y_2}(y_1, y_2) = \frac{y_1}{(1+y_2)^2} \frac{1}{\Gamma(n_1)} \frac{1}{\Gamma(n_2)} \lambda^{n_1+n_2} \left(\frac{y_1 y_2}{1+y_2}\right)^{n_1-1} \left(\frac{y_1}{1+y_2}\right)^{n_2-1} e^{-\lambda(x_1+x_2)}$$

$$\therefore \ g_{Y_1,Y_2}(y_1, y_2) = \frac{y_1}{(1+y_2)^2} \frac{1}{\Gamma(n_1)} \cdot \frac{1}{\Gamma(n_2)} \lambda^{n_1+n_2} \left(\frac{y_1 y_2}{1+y_2}\right)^{n_1-1} \left(\frac{y_1}{1+y_2}\right)^{n_2-1} e^{-\lambda y_1}$$

$$= \frac{\lambda^{n_1+n_2}}{\Gamma(n_1)\Gamma(n_2)} y_1^{n_1+n_2-1} e^{-\lambda y_1} \frac{y_2^{n_1-1}}{(1+y_2)^{n_1+n_2}}$$

Now $$B(n_1, n_2) = \frac{\Gamma(n_1)\Gamma(n_2)}{\Gamma(n_1+n_2)}. \Rightarrow B(n_1,n_2)\Gamma(n_1+n_2) = \Gamma(n_1)\Gamma(n_2)$$

$$\therefore \Rightarrow g_{Y_1,Y_2}(y_1, y_2) = \underbrace{\left(\frac{\lambda^{(n_1+n_2)}}{\Gamma(n_1+n_2)} \cdot y_1^{n_1+n_2-1} e^{-\lambda y_1}\right)}_{g_{Y_1}(y_1)} \times \underbrace{\left(\frac{1}{B(n_1,n_2)} \frac{y_2^{n_1-1}}{(1+y_2)^{n_1+n_2}}\right)}_{g_{Y_2}(y_2)}$$

Thus Y_1 and Y_2 are independent random variables, with

$$\boxed{Y_1 \sim \text{Gamma}(n_1+n_2, \lambda)}$$

If $n_1 = n_2 = 1$, Compute $g_{Y_2}(y_2)$.

Section 6: The mgf technique.

When we have $Y = u(x_1, x_2)$, then, Sometimes it is useful to compute the mgf of Y and see if it matches with mgf of any distribution known to us. This is truly effective when x_1 & x_2 are independent. In fact this works pretty well even if we have a larger number of independent variables.

For example if we have $X_1, X_2, \ldots X_n$ as independent random variables, then, if

$$Y = X_1 + \cdots + X_n$$

Then $m_Y(t) = \prod_{i=1}^{n} m_{X_i}(t)$. (Prove it in the assignment)

Consider again for example that $X_1, X_2 \ldots X_n$ are independent random variables with

$$X_i \sim \text{Poisson}(\lambda_i)$$

$$\therefore \quad m_{X_i}(t) = e^{\lambda_i (e^t - 1)}.$$

Let $Y = X_1 + \cdots + X_n$.

$$\therefore \quad m_Y(t) = \cancel{\prod_{i=1}^{n} m_{X_i}(t} \prod_{i=1}^{n} m_{X_i}(t)$$

$$= \prod_{i=1}^{n} e^{\lambda_i (e^t - 1)}$$

$$= e^{(\lambda_1 + \cdots + \lambda_n)(e^t - 1)}$$

Then $Y \sim \text{Poisson}(\lambda_1 + \cdots + \lambda_n)$.

Now consider another example.

Let X_1, X_2 are standard normal variables.

Find the joint distribution of Y_1 and Y_2

where $Y_1 = X_1 + X_2$ & $Y_2 = X_2 - X_1$

$$\therefore m_{Y_1,Y_2}(t_1,t_2) = E\left[e^{t_1Y_1 + t_2Y_2}\right]$$

$$= E\left[e^{t_1(X_1+X_2) + t_2(X_2 - X_1)}\right]$$

$$= E\left[e^{(t_1-t_2)X_1 + (t_1+t_2)X_2}\right]$$

$$= E\left[e^{(t_1-t_2)X_1}\, e^{(t_1+t_2)X_2}\right]$$

$$= E\left[e^{(t_1-t_2)X_1}\right] E\left[e^{(t_1+t_2)X_2}\right]$$

($\because X_1$ & X_2 are indept.)

$$\therefore m_{Y_1,Y_2}(t_1,t_2) = m_{X_1}(t_1-t_2)\, m_{X_2}(t_1+t_2)$$

$$= e^{\frac{1}{2}(t_1-t_2)^2}\, e^{\frac{1}{2}(t_1+t_2)^2}$$

$$= e^{(t_1^2+t_2^2)}$$

$$= e^{2\frac{t_1^2}{2}}\, e^{2\frac{t_2^2}{2}}$$

$$\cancel{= m_{Y_1}(t_1)\, m_{Y_2}(t_2)}$$ (See next page)

Now $\boxed{m_{Y_1,Y_2}(t_1,t_2) = m_{Y_1}(t_1)\, m_{Y_2}(t_2)}$. We shall establish this fact when X_1 & X_2 are independent

Let $Y_1 = \cancel{g_1(x_1}\ u_1(x_1, x_2)$

$Y_2 = u_2(x_1, x_2)$.

$$\therefore m_{Y_1,Y_2}(t_1\, t_2) = E\left[e^{t_1Y_1 + t_2Y_2}\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1u_1(x_1,x_2) + t_2 u_2(x_1,x_2)}\, f_{X_1,X_2}(x_1,x_2)\, dx_1 dx_2$$

$$m_{Y_1,Y_2}(t_1, t_2) = e^{2t_1^2/2}\, e^{2t_2^2/2} \longrightarrow (*)$$

Now set $t_2 = 0$

$$m_{Y_1,Y_2}(t_1, 0) = E\left[e^{t_1 Y_1}\right] = m_{Y_1}(t_1)$$

But $m_{Y_1,Y_2}(t_1, 0) = e^{2t_1^2/2}$

$$\therefore \quad m_{Y_1}(t_1) = e^{2t_1^2/2}$$

$$\Rightarrow \quad Y_1 \sim N(0, 2)$$

Similarly setting $t_1 = 0$, we have ~~$Y_2 \sim N(0,2)$~~

i.e. $m_{Y_2}(t_2) = e^{2t_2^2/2}$

$$\therefore \quad Y_2 \sim N(0, 2).$$

$$\therefore \quad m_{Y_1 Y_2}(t_1, t_2) = m_{Y_1}(t_1)\, m_{Y_2}(t_2).$$

Thus Y_1 & Y_2 are both independent and identically distributed normal random variable with mean 0 and variance 2

— x —

$\boxed{\text{End of the Probability Part}}$