Lecture ~~Chapter~~ 7: Joint and Conditional Distribution & Stochastic Independence

Suppose X and Y are two random variables on the same probability space. Can we give any meaning to the following question?

$\boxed{\text{Do these two variables taken jointly can have distribution?}}$

Are these two variables independent of each other or are they related? Means we ask the question

$\boxed{\text{When is X and Y stochastically independent ?}}$

and

$\boxed{\text{If X and Y are related then how do we measure it ?}}$

We seek to answer these questions here.

Section 1: Joint distribution of a random vector

In this chapter our aim is to study a random vector and its distribution. A random vector X is the following mapping

$$X : \Omega \longrightarrow \mathbb{R}^k$$

i.e. if $\omega \in \Omega$, then $X(\omega) = (x_1 \; x_2 \ldots \ldots x_k)$

Note that each $x_1 \ldots x_k$ depend on $\omega \in \Omega$, thus can be viewed as a realization of a random variable. So we can write

$$X(\omega) = (X_1(\omega) \;\ldots\; X_k(\omega))$$

where $X_i(\omega) = x_i$, $i = 1 \ldots \ldots k$. In short

$$X = (X_1 \ldots . X_k)$$

The expression

$$X \leq x \iff X_1 \leq x_1 \;\ldots\ldots\; X_k \leq x_k$$

In fact

$$\{X \le x\} = \{X_1 \le x, \ldots X_k \le x\} = \bigcap_{i=1}^{k} \{X_i \le x_i\}$$

The distribution function of this random vector X, is given as

$$F_X(x) = F_{X_1 \ldots X_k}(x_1 \ldots x_k) = P(X_1 \le x_1 \ldots \ldots X_k \le x_k)$$

Instead of focusing on k-random variables we just focus on two random variables X and Y in order to make the discussion simpler at least for the time being. For the sake of curiosity largely we state the properties of the distribution functions of two random variables, X and Y

- Properties of $F_{X,Y}(x,y)$ (never mind if you forget these immediately after reading)

i) $\lim_{x \to -\infty} F(x,y) = 0$, for all y & $\lim_{y \to -\infty} F(x,y) = 0$, for all x.

and $\boxed{\lim_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = 1}$

ii) F(x, y) is right continuous in each variable

$$\lim_{h \to 0+} F(x+h, y) = \lim_{h \to 0+} F(x, y+h) = F(x,y).$$

Further one can relate the computing the probability using distribution functions. ~~in~~ Let $x_1 < x_2$ & $y_1 < y_2$, such that

$$P[x_1 < X \le x_2, \; y_1 < Y \le y_2]$$
$$= F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$$

- **Discrete bi-variate r.v.**

Let X and Y be discrete random variable. Then it's the joint probability mass function or joint pmf of X and Y, is given by

$$\boxed{f_{X,Y}(x,y) = P[X=x,\ Y=y]}$$

Of course one must have

- $f_{X,Y}(x,y) \geq 0, \quad \forall\ (x,y) \in Ran(X) \times Ran(Y)$
- $\cancel{\sum_x \sum_y f(x,y) = 1 = \sum_y \sum_x f(}$
- $\sum_x \sum_y f_{X,Y}(x,y) = 1 = \sum_y \sum_x f_{X,Y}(x,y).$

Example: 6.1: Consider an urn having 3 red, 4 black and 1 green ball. Two balls are drawn at random. Let X be the random variable denoting the number of ~~green~~ red balls among the two drawn balls and Y denotes the number of black balls. Can we write down the ~~pro~~ joint probability distribution

X	Y
0	0
1	1
2	2

$X \backslash Y$	0	1	2	$\sum_y f_{X,Y}(x,y)$
0	0	$\frac{4}{28}$	$\frac{6}{28}$	$\frac{10}{28}$
1	$\frac{3}{28}$	$\frac{12}{28}$	0	$\frac{15}{28}$
2	$\frac{3}{28}$	0	0	$\frac{3}{28}$
$\sum_x f_{X,Y}(x,y)$	$\frac{6}{28}$	$\frac{16}{28}$	$\frac{6}{28}$	1

$\sum_y \sum_x f_{X,Y}(x,y)$

Probability table

$\sum_x \sum_y f_{X,Y}(x,y)$

- The details of how we fill the table.

Some parts can be filled by the reader

There are 8 balls an we are choosing just two, ie in $\binom{8}{2}$ ways.

$$\binom{8}{2} = \frac{8!}{2!\,6!} = \frac{\cancel{6!} \times 7 \times 8}{2!\,\cancel{6!}}$$

$$= \frac{56}{2} = 28$$

So ~~all~~ there are 28 all possible out comes. Let us first compute

$$f_{X,Y}(0,0) = 0 \quad \text{(Think why!)}$$

$$f(0,1) = \frac{\binom{3}{0}\binom{4}{1}\binom{1}{1}}{28} = \frac{4}{28} \qquad [\,0! = 1 \text{ remember}\,]$$

$$f(0,2) = \frac{\binom{3}{0}\binom{4}{2}\binom{1}{0}}{28} = \frac{6}{28}$$

$$f(1,0) = \frac{\binom{3}{1}\binom{4}{0}\binom{1}{1}}{28} = \frac{3}{28}$$

$$f(1,1) = \frac{\binom{3}{1}\binom{4}{1}\binom{1}{0}}{28} = \frac{12}{28}$$

$$f(2,0) = \frac{\binom{3}{2}\binom{4}{0}\binom{1}{0}}{28} = \frac{3}{28}$$

$$f(2,1) = 0 \quad \& \quad f(2,2) = 0$$

I have skipped writing X,Y every time, So $f(0,1) = f_{X,Y}(0,1)$.

Looking at the table one can see that all the properties of the joint pmf is satisfied. $\square$

Of course we can now write down the distribution function as follows

$$F_{X,Y}(x,y) = \sum_{s \le x} \sum_{t \le y} f(s,t), \qquad x \in \mathbb{R} \;\&\; y \in \mathbb{R}.$$

Exercise. Using the ~~above~~ table in the previous page compute $F_{XY}(2,2)$.

Now we can of course think of ~~independent~~ continuous random variables X, Y. Thus $f_{X,Y}(x,y)$ is a joint probability density function if

$$f_{X,Y}(x,y) \geq 0 \qquad \forall\ (x,y) \in \text{Ran}(X) \times \text{Ran}(Y)$$

and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dy\, dx = 1$$

which is same as saying

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = 1.$$

The distribution function can now be written as

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(x,y)\, dx\, dy, \qquad x \in \mathbb{R}, y \in \mathbb{R}$$

$$= P(X \leq x, Y \leq y).$$

For your happiness only.

$$\boxed{f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\, \partial y}}$$

~~I urge~~ We urge the reader to extend this definition for k random variables.

Example: Continuous case: Consider

$$f(x,y) = K(x+y), \qquad 0 < x < 1,\ 0 < y < 1$$

If $K \geq 0$, then $f(x,y) \geq 0$, $\forall\ (x,y) \in [0,1] \times [0,1]$

Now to be a joint ~~pmf~~ density we must have

$$\int_0^1\int_0^1 K(x+y)\, dx\, dy = 1$$

$$\therefore\ K\int_0^1\left[\int_0^1 (y+x)\, dx\right] dy = 1 \Rightarrow K\int_0^1\left[y + \tfrac{1}{2}\right] dy = 1$$

$$\Rightarrow K\int_0^1 y\, dy + K\int_0^1 \tfrac{1}{2}\, dy = 1$$

$$\Rightarrow \frac{K}{2} + \frac{K}{2} = 1$$

$$\Rightarrow K = 1.$$

Knowing the joint distribution of (X,Y) can we get the individual distributions of X and Y.

The marginal distribution of X : $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

The marginal distribution of Y: $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

We have to show that $f_X(x)$ and $f_Y(y)$ are actually densities.

Of course we have ~~to show that~~ $f_X(x) \geq 0$ & $f_Y(y) \geq 0$, $\forall (x,y) \in Ran(X) \times Ran(Y)$.

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\right]dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = 1$$

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\right]dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

We can also define all the above ideas for the discrete case. Following on the ideas of conditional ~~distribution~~ probability, we define, what is called conditional pmf and conditional p.d.f. Then let X and Y be two continuous random variables. the conditional density of Y given $X=x$ is given as

$$\boxed{f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$

[This is defined if $f_X(x) > 0$ and not defined $f_X(x) = 0$]

Let X and Y be discrete, then, with $P[X=x_i] > 0$, $\forall x_i \in Ran(X)$, then

$$f_{Y|X}(y_j|x_i) = P[Y=y_j \mid X=x_i] = \frac{P[X=x_i\ Y=y_j]}{P[X=x_i]}$$

Our question again remains the same. If $f_{Y|X}(y|x)$ a density if (X,Y) are jointly continuous random variable. Let us check this fact. ~~If~~ $f_{Y|X}(\cdot|x)$ is a density function of y. The fact that $f_{Y|X}(y|x) \geq 0$, $\forall y$ is clear. Then

$$\int_{-\infty}^{\infty} f_{Y|X}(y|x)\,dy = \int_{-\infty}^{\infty} \frac{\cancel{f_{Y|X}(y|x)}\, f_{X,Y}(x,y)}{f_X(x)}\,dy$$

$$= \frac{1}{f_X(x)} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \frac{1}{f_X(x)} \cdot f_X(x) = 1$$

Hence $f_{Y|X}(\cdot|x)$ is truly a density function. How can one write down the cumulative distribution function in this case. If $f_X(x)>0$ then then the cdf of Y given $X=x$ is given as

$$F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(z|x)\,dz$$

Let us now look into the important concept of independence of the random variables X and Y.

We say X and Y are two given random variables.

If X and Y are discrete

Independence $\Leftrightarrow$ Joint pmf $\cancel{f_X(x,y)}\ f_{X,Y}(x,y) = f_X(x) f_Y(y)$ marginal pmf

If X and Y are continuous

Independence $\Leftrightarrow$ Joint density = Product of marginals.

$$f_{X,Y}(x,y) = f_X(x) f_X(y),$$

$$\Rightarrow f_{Y|X}(y|x) = f_Y(y)$$

$$\& \; f_{X|Y}(x|y) = f_X(x)$$

A collection of k random variables $X_1, X_2, \ldots, X_k$ iff

$$f_{X_1 \ldots X_n}(x_1, x_2 \ldots x_n) = f_{X_1}(x_1) \cdot \ldots \cdot f_{X_n}(x_n).$$

Depends on wether you consider pmf or pdf.

E.g. $f_{X,Y}(x, y) = e^{-(x+y)}$; $x \geq 0,\ y \geq 0$

Is X and Y independent. Let us first compute the marginals

$$f_X(x) = \int_0^\infty e^{-(x+y)}\, dy = e^{-x}$$

$$f_Y(y) = \int_0^\infty e^{-(x+y)}\, dx = e^{-y}$$

$$f_{X,Y}(x+y) = e^{-(x+y)} = e^{-x} e^{-y}$$
$$= f_X(x) f_Y(x).$$

—— x ——