Lecture 10: Random Sampling and Sampling Distribution

Sec 1: Sampling

Statistics concerns itself with the study of several aspects of a population, in which chance plays a role. The population can be human or otherwise. In statistics we are more concerned with target population in the study rather than whole population. Suppose we are looking for the sleeping patterns or sleeping hours of all adults in Kanpur in the age group 30-45. Then that particular segment is the target population. In most cases the target population is large enough so that collecting individual data is unviable and thus we need to seek a sub-population or a subset of a population called a "sample" on which we will carry out exhaustive studies/measurements. However to prevent bias creeping into the formation of a sample it is always advisable to randomize the process of ~~hand~~ drawing a sample.

Let there be a target population then we can consider two approaches to draw a random sample from it. The first one is sampling with replacement, where the sampled members of the population are returned back, while in the second approach they are not; since if we sample with replacement from a population of 100 a sample of ~~each~~ 10, each time, then in ten such draws we have exhausted the population

So from a theoretical framework the theory of sampling or random sampling we consider a random sampling with replacement. Thus once a value is noted it becomes eligible again to be considered.

Consider any random variable X, (say hieght, weight, etc) which represents a characteristic of a population. Let X, follows a distribution with pmf/pdf, given as $f_X(x)$. Thus by the population we shall now mean $f_X(x)$, the pmf/pdf.

A random sample is a collection of $n \in \mathbb{N}$, identically and independently ~~distribution~~ distributed (iid) random variables each having pmf/pdf $f_{X_i}(x) = f_X(x)$, $\forall i = 1, 2, \dots n$. The random sample is represented as

$$X_1, X_2, \dots \dots X_n \quad (n: \text{size of the sample})$$

Thus for each i, $X_i : \Omega \to \mathbb{R}$, a random variable. Now what is this Ω? $\omega \in \Omega$ is a collection n-individuals drawn from the population. Thus $X_i(\omega) = x_i$, is the value of characteristic of the i-th member chosen. So ω is in "some sense" the chosen physical sample. What happens is the following. Suppose $X \sim N(\mu, \sigma^2)$ but μ and σ^2 are not known though we do know that our charecteristic follows normal distribution. By the vehicle of random sampling we will estimate these parameters.

Let $X_1 \dots X_n$ be a sample of size n. The following are two important parameter or statistic associated with the random sample are the sample mean and sample variance, which are themselves, random variable given as.

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Why we have divided $(n-1)$ will be clear soon. It will dealt in more detail when we will study point estimation.

$$E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} E(X_i), \qquad \text{~~Suppose } E(X_i) = \mu\text{~~}$$

Since $X_1 \dots X_n$ are iid r.v.'s we have $E(X_i) = \mu$, $\forall i$

$$\therefore \quad E(\bar{X}) = \mu.$$

While $\mathrm{Var}(\bar{X}) = \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum \mathrm{Var}(X_i).$

Denoting $\mathrm{Var}(X_i) = \sigma^2$ we have

$$Var(\bar{x}) = \sigma_{\bar{x}}^2 = \frac{n}{n^2}\sigma^2 = \frac{1}{n}\sigma^2$$

$$\therefore \text{ Std.dev of } \bar{x} = \sigma_{\bar{x}} = \sqrt{Var(\bar{x})} = \frac{\sigma}{\sqrt{n}}$$

$\frac{\sigma}{\sqrt{n}}$ is often referred to as the standard error associated with ~~[illegible]~~ the sample

Our aim in this chapter is to know the distribution of $\bar{X}$ and S^2 depending on the density of the population ~~from~~ which the sample is drawn. The following result is important enough to be stated as a theorem.

<u>Theorem 10·1:</u> Let $X_1 \ldots X_n$ be a random sample from a population with density $f_X(\cdot)$, ~~Let~~ Then if $n>1$, we have

$$E[S^2] = \sigma^2$$

where σ^2 is the population variance. Further

$$Var(S^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right), \quad \text{for } n>1$$

where $\mu_4 = E[X^4]$, the ~~[illegible]~~ 4th population moment.

Proof: We shall only compute the mean. We know that

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{x})^2.$$

We shall first show that

$$\sum_{i=1}^{n}(X_i-\mu)^2 = \sum_{i=1}^{n}(X_i-\bar{x})^2 + n(\bar{x}-\mu)^2$$

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\mu)^2 &= \sum_{i=1}^{n}(X_i-\bar{x}+\bar{x}-\mu)^2 \\ &= \sum_{i=1}^{n}(X_i-\bar{x})^2 + 2(\bar{x}-\mu)\sum_{i=1}^{n}(X_i-\bar{x}) + \sum_{i=1}^{n}(\bar{x}-\mu)^2 \\ &= \sum_{i=1}^{n}(X_i-\bar{x})^2 + n(\bar{x}-\mu)^2. \quad \left(\because \sum_{i=1}^{n}(X_i-\bar{x}) = \sum_{i=1}^{n}X_i - n\bar{X} = n\bar{X} - n\bar{x} = 0\right)\end{aligned}$$

$$E[S^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n}(x_i-\mu)^2 - n(\bar{x}-\mu)^2\right]$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} E\left[(x_i-\mu)^2\right] - \frac{n}{n-1} E\left[(\bar{x}-\mu)^2\right]$$

$$= \frac{n\sigma^2}{n-1} - \frac{n}{n-1} Var(\bar{x})$$

$$= \frac{n\sigma^2}{n-1} - \frac{n}{n-1}\frac{\sigma^2}{n}$$

$$= \frac{n\sigma^2}{n-1} - \frac{\sigma^2}{n-1}$$

$$= \frac{(n-1)\sigma^2}{(n-1)} = \sigma^2$$

$$\therefore \boxed{E[S^2] = \sigma^2}$$

So the sample mean and sample variance acts as a kind of estimator of the ~~aia~~ population mean and variance. However to compute $E(\bar{x})$ or $E(S^2)$, we need to know the distribution of $\bar{x}$ and S^2 when, the population distribution. ~~The probability distributions~~ is known. The probability distribution of $\bar{X}$ and S^2 is known as sampling distributions. This is what we discuss next.

Sec 2: Sampling Distributions

To begin with we will consider a population which obeys the normal distribution. Then we will seek in such a case the distribution of $\bar{X}$ and S^2.

Theorem 10.2 : Let $X_1 \dots X_n$ be a random sample of n iid random variables, with mean $\bar{X}$. Let each $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2 \dots n$

Then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

Proof: We will use the mgf technique

$$m_{\bar{X}}(t) = E[e^{t\bar{X}}]$$

$$= E\left[e^{t \frac{\sum_{i=1}^{n} X_i}{n}}\right]$$

$$= E\left[e^{\sum_{i=1}^{n} \frac{tX_i}{n}}\right]$$

$$= E\left[\prod_{i=1}^{n} e^{\frac{tX_i}{n}}\right]$$

$$= \prod_{i=1}^{n} E\left[e^{\frac{tX_i}{n}}\right], \text{ by independence}$$

$$= \prod_{i=1}^{n} m_{X_i}\left(\frac{t}{n}\right)$$

$$= \prod_{i=1}^{n} e^{\left(\frac{\mu t}{n} + \frac{1}{2}\frac{\sigma^2 t^2}{n^2}\right)}$$

$$= e^{\left(\mu t + \frac{1}{2}\sigma^2 \frac{t^2}{n}\right)}$$

$$\therefore \quad \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) \qquad \square$$

Now we shall focus on how to find the distribution of

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$$

when we draw a random sample of size n from a normal population.

A very special type of probability distribution that will play a crucial role in computing the probability distribution of S^2 is the so called chi-square distribution or χ^2-distribution.

A random variable or rather a continuous random variable X is said to be a χ^2 random variable (chi-square r.v.) if X has the density

$$f_X(x) = \frac{1}{\Gamma(k/2)} \left(\frac{1}{2}\right)^{k/2} x^{k/2 - 1} e^{-1/2\, x}, \quad x \in (0, \infty) \quad \cancel{0 < x < \infty}$$

$$f_X(x) = 0, \quad \text{otherwise.}$$

This a Gamma distribution with $r = k/2$ & $\lambda = 1/2$ ~~in (0,∞)~~. k is called the degrees of freedom of the χ^2-random variable X.

So using our information about ~~Γ-dis~~ Gamma ~~distribut~~ distribution we can write. If $X \sim \chi^2(k)$, (i.e χ^2 with k-degrees of freedom)

$$E(X) = \frac{r}{\lambda} = \frac{k/2}{1/2} = k$$

$$Var(X) = \frac{r}{\lambda^2} = \frac{\left(\frac{k}{2}\right)}{\left(\frac{1}{2}\right)^2} = 2k$$

$$m_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

$$= \left(\frac{\frac{1}{2}}{\frac{1}{2} - t}\right)^{k/2}; \quad t < \frac{1}{2}, \ \cancel{\text{for}}$$

$$= \left[\frac{1}{1 - 2t}\right]^{k/2}; \quad t < 1/2$$

We now state the following important theorem

Theorem 10.3: Let X_i, $i = 1, 2, \ldots, k$ are continuous random variables and $X_i \sim N(\mu_i, \sigma_i^2)$ and are independent. Then

$$U = \sum_{i=1}^{k} \left(\frac{X_i - \mu_i}{\sigma_i} \right)^2$$

follows χ^2 distribution with k-degrees of freedom.

Proof: We will again use the mgf technique which validates its powers. Let us write

$$Z_i = \left(\frac{X_i - \mu_i}{\sigma_i} \right)$$

then $Z_i \sim N(0,1)$, i.e. $Z_1 \ldots Z_k$ are iid r.v.s

$$\therefore \quad U = \sum_{i=1}^{k} Z_i^2 .$$

Hence

$$\begin{aligned} m_U(t) &= E\left[e^{tU} \right], \\ &= E\left[e^{t \sum_{i=1}^{k} Z_i^2} \right] \\ &= E\left[\prod_{i=1}^{k} e^{t Z_i^2} \right] \\ &= \prod_{i=1}^{k} E\left[e^{t Z_i^2} \right] \quad \text{(by independence)} \end{aligned}$$

Now

$$\begin{aligned} E\left[e^{t Z_i^2} \right] &= \int_{-\infty}^{\infty} e^{t z^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \, dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1-2t) z^2} \, dz \end{aligned}$$

Set $\left(\sqrt{1-2t}\right) z = v \Rightarrow v^2 = (1-2t) z^2$, for $t < \frac{1}{2}$

& $dv = \sqrt{1-2t} \, dz \Rightarrow dz = \cancel{\frac{dv}{\sqrt{\ }}} \ \frac{dv}{\sqrt{1-2t}}$.

$$\therefore \quad E\left[e^{tz_i^2}\right] = \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\, dv$$

$$= \frac{1}{\sqrt{1-2t}}$$

$$\therefore \quad m_U(t) = \prod_{i=1}^{k} E\left[e^{tz_i^2}\right] = \left(\frac{1}{1-2t}\right)^{k/2}, \quad t < \tfrac{1}{2}.$$

Thus U follows χ^2 with k-degrees of freedom.

Thus if $X_1 \dots X_n$ is a random sample of size n, where for each i, $X_i \sim N(\mu, \sigma^2)$, then

$$U = \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \longrightarrow \text{A}$$

follows a χ^2 distribution with n-degrees of freedom.

<u>Question !!</u> In the expression of U as given in A. If either μ or σ^2 or both are unknown, will you call U a ~~sta~~ statistic?

We shall now state the following Theorem without proof

<u>Theorem 10.4</u> Let $X_1, \dots X_n$, be a random sample drawn from a standard normal distribution.

i) Then $\bar{X} \sim N\left(0, \frac{1}{n}\right)$

ii) $\bar{X}$ and $\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2$ are independent

iii) $\sum_{i=1}^{n} (x_i - \bar{x})^2$ has a χ^2-distribution with $(n-1)$ degrees of freedom.

Proof: Part i) is already proved in Theorem 10·2. We shall just prove part iii) here as needed for our purpose

To prove iii) observe that

$$\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 + n\bar{X}^2 \quad \text{(The reader should prove it)}$$

$$\therefore \quad m_{\sum_{i=1}^{n} x_i^2}(t) = m_{\sum_{i=1}^{n}(x_i-\bar{x})^2}(t)\, m_{n\bar{x}^2}(t) \quad \text{(Again the reader should prove it)}$$

$$\therefore \quad m_{\sum(x_i-\bar{x})^2}(t) = \frac{m_{\sum x_i^2}(t)}{m_{n\bar{x}^2}(t)}$$

Now as $X_i \sim N(0,1)$, by using Theorem 10·3 we know that $\sum x_i^2 \sim \chi^2$ ~~(n)~~ with n degrees of freedm

$$\therefore \quad m_{\sum x_i^2}(t) = \left[\frac{1}{(1-2t)}\right]^{n/2} ; \quad t < 1/2$$

Since $\bar{x} \sim N(0, \frac{1}{n})$; then $Y = n\bar{x}^2$ ~~satisfi~~ follows a χ^2 distribution with degrees of freedom 1 as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (The student /reader should be able to prove this)

$$\therefore \quad m_{n\bar{x}^2}(t) = \left[\frac{1}{(1-2t)}\right]^{1/2} ; \quad t < 1/2$$

$$\therefore \quad m_{\sum(x_i-\bar{x})^2}(t) = \left(\frac{1}{1-2t}\right)^{\frac{n-1}{2}} ; \quad t < \frac{1}{2}$$

$\therefore \quad \sum (x_i - \bar{x})^2 \sim \chi^2$ ~~(n-1)~~ distribution with $(n-1)$ degrees of freedom. □

So how do we write Theorem 10.4, if we ~~write~~ have $X_1 \ldots\ldots X_n$ a random sample ~~fro~~ from a normal distribution with mean μ and σ^2,

Note that we just set $Z_i = \dfrac{X_i - \mu}{\sigma}$

$$\therefore \quad \bar{Z} = \frac{(\bar{X} - \mu)}{\sigma}$$

and hence $\dfrac{\bar{X} - \mu}{\sigma} \sim N\left(0, \frac{1}{n}\right)$, i.e what we get from ~~(10.4) 1.)~~ ~~Theorem~~ i) of Theorem ~~2.4~~ 10.4

$$\text{Now} \quad \sum_{i=1}^{n} (Z_i - \bar{Z})^2 = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$$

$\therefore \quad \displaystyle\sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma}\right)^2 \sim \chi^2$ with $(n-1)$ degrees of freedom from iii) in Theorem 10.4

$$\text{Now} \quad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$\text{Hence} \quad \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2}$$

Thus $\dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2$ with $(n-1)$ degrees of freedom. So how do we find the p.d.f of S^2

$$\text{Set} \quad U = \frac{(n-1)S^2}{\sigma^2}$$

We have seen that $U \sim \chi^2$ with $(n-1)$ degrees of freedom. Set $S^2 = Y$.

$$\therefore \quad U = \frac{(n-1)Y}{\sigma^2} \quad \text{or} \quad Y = \frac{\sigma^2 U}{(n-1)}.$$

From standard transformation technique, (since the transformation is linear), we have for $n>1$. & $y>0$ $(\because U>0)$

$$f_{S^2}(y) = f_U\left(\frac{(n-1)y}{\sigma^2}\right)\left|\frac{dU}{dy}\right|$$

$$= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{1}{2}\right)^{\frac{n-1}{2}}\left[\frac{(n-1)y}{\sigma^2}\right]^{\frac{n-1}{2}-1} e^{-\frac{1}{2}\frac{(n-1)y}{\sigma^2}} \frac{(n-1)}{\sigma^2}$$

$$= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{1}{2}\right)^{\frac{n-1}{2}} \frac{(n-1)^{\frac{n-3}{2}}}{\sigma^{(n-3)}} \cdot y^{\frac{n-3}{2}} e^{\frac{(n-1)y}{2\sigma^2}} \cdot \frac{(n-1)}{\sigma^2}$$

$$= \frac{1}{\Gamma\left(\frac{n-1}{2}\right)}\left[\frac{(n-1)}{(2\sigma^2)}\right]^{\frac{(n-1)}{2}} y^{\frac{n-3}{2}} e^{\frac{(n-1)y}{2\sigma^2}}$$

$$\therefore \boxed{f_{S^2}(y) = \frac{1}{\Gamma\left(\frac{n-1}{2}\right)}\left(\frac{n-1}{2\sigma^2}\right)^{\frac{n-1}{2}} y^{\frac{n-3}{2}} e^{\frac{(n-1)y}{2\sigma^2}}} \quad \text{for } y>0,\ n>1$$

of course $f_{S^2}(y)=0$, for $y \leq 0$.

Our next object of focus is the F-distribution named after the celebrated statistician Ronald Fisher. This distributions plays quite an important ~~not~~ role in statistics. Let us see how an F-distribution pdf is constructed. Let $U \sim \chi^2$ with m-degrees of freedom and $V \sim \chi^2$ with ~~m~~ n degrees of freedom. Let U and V be independent, then their joint density is given as

$$f_{U,V}(u,v) = \frac{1}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)2^{\frac{(m+n)}{2}}} u^{\frac{m-2}{2}} v^{\frac{n-2}{2}} e^{-\frac{1}{2}(u+v)},$$

where ~~$0 \leq u \leq \infty$, $0 \leq v \leq \infty$~~. $0<u<\infty$, $0<v<\infty$

Let us consider transform

$$X = \frac{\frac{U}{m}}{\frac{V}{n}}$$

We shall call such a random variable X a F-variable.

Consider the transformation

$$X = \frac{\frac{U}{m}}{\frac{V}{n}} \quad \& \quad Y = V$$

We seek to compute $f_{X,Y}(x,y)$, since we know $f_{U,V}(u,v)$. Now

~~Jacobian~~
$$U = \frac{m}{n} XY$$
$$V = Y$$

$$\therefore \text{Jacobian matrix } J = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{m}{n} y & \frac{m}{n} x \\ 0 & 1 \end{bmatrix}$$

$$\therefore \det J = \frac{m}{n} y \Rightarrow |\det J| = \frac{m}{n} y. \quad (\text{as } m>0, n>0, y>0)$$

$$\therefore f_{X,Y}(x,y) = \frac{m}{n} y \cdot \frac{1}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2}) 2^{\frac{m+n}{2}}} \left(\frac{m}{n} xy\right)^{\frac{m-2}{2}} y^{\frac{(n-2)}{2}} e^{-[\frac{m}{n} xy + y]/2}$$

where, $0 < x < \infty, 0 < y < \infty$

$$\therefore f_X(x) = \int_0^\infty f_{X,Y}(x,y)\,dy$$

$$= \frac{1}{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2}) 2^{\frac{m+n}{2}}} \left(\frac{m}{n}\right)^{\frac{m}{2}} x^{\frac{(m-2)}{2}} \int_0^\infty y^{\frac{m+n-2}{2}} e^{-\frac{1}{2}[\frac{m}{n}x+1]y}\,dy$$

$$\text{Set } \frac{1}{2}\left[\frac{m}{n}x+1\right] y = t \Rightarrow \frac{dt}{\frac{1}{2}\left[\frac{m}{n}x+1\right]} = dy$$

$$f_X(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{m}{2}\right)} \left(\frac{m}{n}\right)^{m/2} \frac{x^{\frac{(m-2)}{2}}}{\left[\frac{m}{n}x+1\right]^{\frac{(m+n)}{2}}}$$

Note that

$$\int_0^\infty y^{\frac{m+n-2}{2}} e^{-t}\, dt = \Gamma\left(\frac{m+n}{2}\right)$$

(after substitution)

$$\therefore \boxed{f_X(x) = \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{m/2} \frac{x^{(m-2)/2}}{\left[\frac{m}{n}x+1\right]^{\frac{(m+n)}{2}}}} \qquad x \in (0,\infty)$$

F-distribution (We say $X \sim F$ with ~~(m,n)~~ m & n degrees of freedom)

F-variable $X = \frac{U}{m} \Big/ \frac{V}{n}$ is often called variance ratio.

But how is this discussion relevant to sampling distributions. This is how it is.

Let $X_1 \ldots X_{m+1}$ be a sample of size $(m+1)$ from a normal distribution with mean μ_1 & variance σ^2 & $Y_1 \ldots Y_{n+1}$ is a sample of size $(n+1)$ drawn from a normal distribution with mean μ_2 and variance σ^2, then we know that

$$\frac{1}{\sigma^2}\sum_{i=1}^{m+1}(X_i-\bar{X})^2 \sim \chi^2 \text{ with } m \text{ degrees of freedom}$$

and

$$\frac{1}{\sigma^2}\sum_{i=1}^{n+1}(Y_i-\bar{Y})^2 \sim \chi^2 \text{ with } n \text{ degrees of freedom}$$

~~The statis~~

Consider the statistic $T = \dfrac{\dfrac{\frac{1}{\sigma^2}\sum_{i=1}^{m+1}(x_i-\bar{x})^2}{m}}{\dfrac{\frac{1}{\sigma^2}\sum_{i=1}^{n+1}(x_i-\bar{x})^2}{n}}$

$$\therefore \quad T = \frac{\frac{1}{m}\sum_{i=1}^{m+1}(x_i-\bar{x})^2}{\frac{1}{n}\sum_{i=1}^{n+1}(x_i-\bar{x})^2}$$

Then $T \sim F$ with m & n degrees of freedom.

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Note: Though the material here is standard, a large part of the discussion here is based on Chapter 6 of Mood, Graybill and Boes, "Introduction to the Theory of ~~Statistics~~" Statistics" 1974, Mc. Graw Hill.