Unsteady Heat Conduction

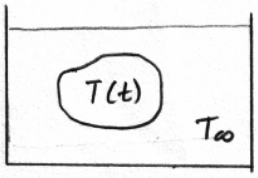
- Time dependent temperature
- Transient process (typically to a new steady state)

If temperature gradient within the solid is negligible/not important — Lumped Capacitance Method

If temperature gradient is 1D, Exact solution may be obtained

If complex geometry, and temperature gradient in the solid is of interest. Numerical solution may be obtained

Lumped Capacitance Method

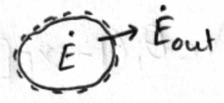


- Temperature of the solid is assumed to be spatially uniform $\Rightarrow$ negligible temperature gradients in the solid

Consider a solid at an initial uniform temperature T_i. At $t=0$, the solid is immersed in a liquid of temperature T_∞ such that $T_\infty < T_i$. The temperature of the solid will decrease from $t=0$ until it reaches T_∞ (assuming that the liquid is maintained at a constant temperature T_∞). During this transient process ($T_i \rightarrow T_\infty$), if it is reasonable to assume that the temperature of the solid is uniform at every instant t, then lumped capacitance method can be employed

Qualitatively, large thermal conductivity $\Rightarrow$ weak thermal gradient for the same rate of heat transfer

$$R_{th,\,conduction} << R_{th,\,other}$$ other = convection



Volume V
Surface area A_s

Defining control volume as the solid, and applying energy balance,

$$\dot{E} = -\dot{E}_{out} \qquad \text{(p18)}$$

$$\rho V c \frac{dT}{dt} = -hA_s\,(T(t) - T_\infty)$$

or, in terms of the excess temperature $\theta = T - T_\infty$

$$\frac{\rho V c}{hA_s}\frac{d\theta}{dt} = -\theta$$

We are primarily interested in $\theta(t)$, and need an initial condition to solve for $\theta(t)$ using the above equation.

Let

$$\theta = \theta_i \quad \text{at} \quad t = 0 \qquad \text{i.e.} \quad \theta_i = T_i - T_\infty$$

Then,

$$\int_{\theta_i}^{\theta} \frac{\rho V c}{hA_s}\frac{1}{\theta}\,d\theta = -\int_0^t dt$$

$$\frac{\rho V c}{hA_s}\ln\theta = -t$$

or

$$\frac{\theta}{\theta_i} = \frac{\overbrace{T}^{\text{time dependent}} - T_\infty}{T_i - T_\infty} = \exp\left[-\left(\frac{hA_s}{\rho V c}\right)t\right]$$

— "Exponentially decaying temperature difference"

Thermal resistance

$$R_{th,conv} = \frac{1}{hA_s}$$

We can also define,

Thermal capacitance

$$C_{th} = \rho V c$$

and hence,

Thermal time constant

$$\tau_{th} = R_{th}C_{th} \quad \Rightarrow \quad \frac{\theta}{\theta_i} = \exp\left[-\frac{t}{\tau_{th}}\right]$$

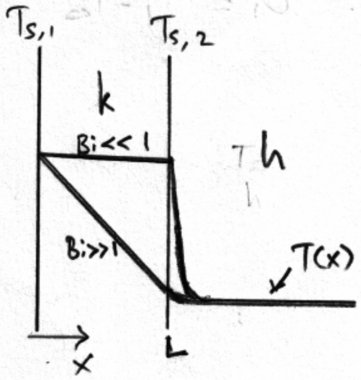
Total energy transferred upto time t $= \int_0^t Q\,dt = hA_s \int_0^t \theta\,dt = \rho V c\,\theta_i \left(1 - \exp\left[-\frac{t}{\tau_{th}}\right]\right)$

Notes:

- Simple method to solve for $T(t)$, energy
- Assumption of uniform temperature in the solid needs to be validated.

Validity of Lumped Capacitance Method

Consider heat transfer through a wall with one surface maintained at temperature $T_{s,1}$ and the other surface in contact with a liquid at temperature T_∞.



Energy balance at the surface ($x=L$)

$$-kA\frac{dT}{dx} = hA(T_{s,2} - T_\infty)$$

In absence of any generation term

$$\frac{kA}{L}(T_{s,1} - T_{s,2}) = hA(T_{s,2} - T_\infty)$$

... assuming k is uniform

Rearranging into dimensionless form

$$\frac{T_{s,1} - T_{s,2}}{T_{s,2} - T_\infty} = \frac{hL}{k} = \frac{R_{th,cond}}{R_{th,conv}} \equiv Bi$$

Bi ≡ Biot Number

Remember:

$k \to \infty \Rightarrow$ temperature in the solid $\to$ uniform

However, the "strength" of convection relative to conduction is the relevant quantity here. Thus, ratio of resistance (inverse of conductance) due to conduction to resistance due to convection decides the temperature profile.

Thumb rule: If $Bi = \frac{hL_c}{k} < 0.1$ then the error associated with lumped capacitance method is small.

Here, $L_c \equiv$ characteristic length scale for conduction

typically $L_c = \frac{V}{A_s}$

or the lengthscale corresponding to the maximum temperature difference
eg: for plane wall, thickness/2

Recall that for unsteady heat conduction, lumped capacitance method gives:

$$\frac{\theta}{\theta_i} = \exp\left(-\frac{hA_s}{\rho V c}t\right) = \exp\left(-\frac{hL_c}{k}\frac{k}{\rho c}\frac{t}{L^2}\right)$$

$$\frac{\theta}{\theta_i} = \exp\left(-\frac{hL_c}{k}\frac{\alpha t}{L_c^2}\right) = \exp(-Bi \cdot F_o)$$

Where,

$$\alpha = \frac{k}{\rho c} \equiv \text{thermal diffusivity} \quad (m^2/s)$$

$\alpha \equiv$ measure of thermal inertia

$\alpha \uparrow \Rightarrow$ heat moves rapidly relative to its volumetric heat capacity

Heat conduction: 1D, no source

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} k \frac{\partial T}{\partial x} = k\frac{\partial^2 T}{\partial x^2}$$

$$\Rightarrow \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

and

$F_o \equiv$ Fourier number is the dimensionless time

Note: For the general case when convection, radiation, input/output flux and source terms could be present, energy balance is

$$q A_{s1} + \dot{E}_g - \left(h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)\right)A_s = \rho V c \frac{dT}{dt}$$

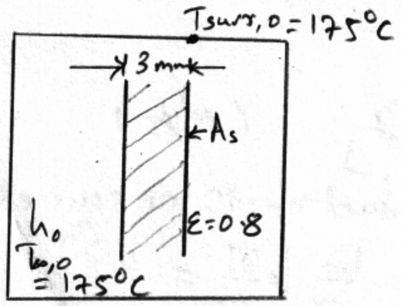
– General Lumped Capacitance Analysis

Example: convection + radiation

A 3 mm thick panel of aluminium alloy ($k = 177\ W/m \cdot K$, $c = 875\ J/kg \cdot K$, $\rho = 2770\ kg/m^3$) is finished on both sides with an epoxy coating that must be cured at or above $T_c = 150°C$ for at least 5 mins. The production line involves two steps: (1) heating in a large oven with air at $T_{\infty,o} = 175°C$ and a convection coefficient of $h_o = 40\ W/m^2 \cdot K$, and (2) cooling in a large chamber with air at $T_{\infty,c} = 25°C$ and a convection coefficient of $h_c = 10\ W/m^2 \cdot K$. The coating has an emissivity of $\varepsilon = 0.8$, and the temperatures of the oven and chamber walls are $175°C$ and $25°C$, respectively. If the panel enters the oven at an initial temperature of $25°C$ and is removed from the chamber at $37°C$, what is the total (minimum) time required for the two-step curing operation?

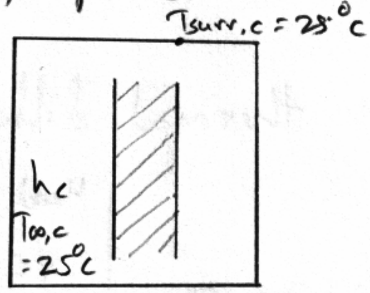
Assumptions:

- Thermal resistance due to epoxy is negligible
- Uniform properties



Step 1

$0 \leq t \leq t_e$



Step 2

$t_e < t \leq t_f$

Steps:

- check if lumped capacitance method will suffice
- proceed accordingly

Note that temperature distribution in the panel is not of interest.

Max. possible panel temperature $T_{max,o} = 175°C = T_{max,c} = 448\ K$

Thus, max. effective (convection + radiation) heat transfer coefficient during each step can be calculated to find the corresponding maximum Biot number.

Step 1: $h_{eff,max,o} = h_o + h_{r,max,o} = h_o + \varepsilon\sigma(T_{max,o} + T_{surr,o})(T_{max,o}^2 + T_{surr,o}^2)$

$$= 40 + 0.8\,\sigma\,(448 + 448)(448^2 + 448^2)$$

$$= 56 \text{ W/m}^2.\text{K}$$

$$L_c = \frac{3\times10^{-3}}{2} = 1.5\times10^{-3} \text{ m}$$

$$Bi_o = \frac{h_{eff,max,o}\,L_c}{k} = 4.8\times10^{-4}$$

Step 2: $h_{eff,max,c} = h_c + \varepsilon\sigma(T_{max,c} + T_{surr,c})(T_{max,c}^2 + T_{surr,c}^2)$

$$= 10 + 0.8\sigma(448 + 298)(448^2 + 298^2)$$

$$= 19.8 \text{ W/m}^2.\text{K}$$

$$Bi_c = \frac{h_{eff,max,c}\,L_c}{k} = 1.7\times10^{-4}$$

Thus $Bi_o < 0.1$ and $Bi_c < 0.1$, and lumped capacitance method can be used for fairly accurate estimation of the curing time

Energy balance simplifies to

$$\rho c L \frac{dT}{dt} = -\left(h(T - T_\infty) + \varepsilon\sigma(T^4 - T_{sur}^4)\right)$$

- Non-linear equation ⇒ numerical integration
- Solve for temperature in step 1 as a function of time to find t_e (≡ 5 mins after $T = 150°C$)
- Solve for temperature in step 2 as a function of time to find $t_f - t_e$ (≡ time to reach $T = 37°C$).
- The sum of these gives the total time required.