Linear Algebra - Part II: Perfect matrices, reciprocal vectors...

ChE641, IIT Kanpur

Similarity Transforms $\longrightarrow$ Diagonalization of matrices

$\underline{v} \longrightarrow 3\hat{\underline{i}} + 4\hat{\underline{j}} - 5\hat{\underline{k}}$

in a given basis $\begin{pmatrix} 3 \\ 4 \\ -5 \end{pmatrix}$ in a diff basis $\begin{pmatrix} 5 \\ -3 \\ 2 \end{pmatrix}$

$A \longrightarrow$ N-dim basis $\longrightarrow$ $\underline{\underline{A}}$ Matrix $(N \times N)$

$$\underline{\underline{A}} \longrightarrow \boxed{\underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}} = \underline{\underline{A}}'}$$

$\underline{\underline{S}}$: transformation

$\underline{\underline{A}}\, \underline{e}_i = \lambda_i \underline{e}_i$ eig vecs

$\underline{\underline{S}} \to [\underline{e}_1 \cdots \underline{e}_N]$

$$\underline{\underline{A}}' = \underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}}$$

diagonal

$$\underline{\underline{A}}' = \underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}}$$

$$= \underline{\Lambda} \longrightarrow \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \end{pmatrix}$$

Orthogonal eig vecs. $\longrightarrow$ Hermitian matrix.

Generalization to cases where the eig vecs are not orthogonal:

$$\underline{x} \in R^n \qquad \underline{x} = \sum_{i=1}^{N} x_i \underline{e}_i \qquad \underline{e}_i = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \leftarrow i^{th} \text{ row}$$

basis vector

$$\langle \underline{e}_i | \underline{e}_j \rangle = \underline{e}_i^{+} \underline{e}_j = \delta_{ij}$$

Take another L.I basis : $\{\underline{x}_1 \cdots \underline{x}_N\}$

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i \qquad \underline{x} = \underline{\underline{X}} \cdot \underline{\alpha} \qquad \underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix}$$

coeffs

$\underline{\underline{X}} = [\underline{x}_1 \cdots \underline{x}_N]$

$\{\underline{x}_i\}$ are L.I

$\underline{\underline{X}}^{-1}$ exists.

$$\underline{\underline{X}}^{-1} \underline{x} = \underline{\alpha}$$

If $\{\underline{x}_1 \ldots \underline{x}_N\}$ are orthonormal, $\underline{x}_i^\dagger \underline{x}_j = \delta_{ij}$

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i$$

orthonormality $\longrightarrow \alpha_i$

$$x_j^\dagger \underline{x} = \sum_i \alpha_i \underbrace{x_j^\dagger x_i}_{\delta_{ji}}$$

$$= \sum_i \alpha_i \delta_{ij}$$

$$\boxed{\alpha_j = x_j^\dagger \underline{x}} = \langle x_j | \underline{x} \rangle$$

$$\underline{x} = \sum_i \alpha_i \underline{x}_i$$

$$= \sum_i \underline{x}_i \alpha_i$$

$$= \sum_i \underline{x}_i \langle \underline{x}_i | \underline{x} \rangle = \sum_i \underline{x}_i \underline{x}_i^\dagger \underline{x}$$

$$\underline{x} = \Big(\underbrace{\sum_i x_i x_i^\dagger}_{\underline{\underline{I}}}\Big) \underline{x}$$

$$\boxed{\underline{\underline{I}} = \sum_i \underline{x}_i \underline{x}_i^\dagger}$$

Resolution of identity:

$\{\underline{x}_i\}$ orthonormal basis vectors $\longrightarrow$ $\underline{\underline{I}} = \sum_{i=1}^{N} \underline{x}_i \underline{x}_i^{\dagger}$

What if $\{\underline{x}_1 \ldots \underline{x}_n\}$ is <u>not</u> orthonormal ?

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i$$

α_i's cannot be found easily

"Reciprocal basis": Find a basis $\{\underline{z}_1 \ldots \underline{z}_n\}$ which is related to $\{\underline{x}_1 \ldots \underline{x}_n\}$ s.t

$$\boxed{\underline{z}_i^{\dagger} \underline{x}_j = \delta_{ij}}$$

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i$$

$$\underline{z}_j^{\dagger} \underline{x} = \sum_{i=1}^{N} \alpha_i \underbrace{\underline{z}_j^{\dagger} \underline{x}_i}_{\delta_{ij}}$$

$$\boxed{\alpha_j = \underline{z}_j^{\dagger} \underline{x}} \qquad \alpha_i = \underline{z}_i^{\dagger} \underline{x}$$

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i$$

$$= \sum \underline{x}_i \alpha_i$$

$$\underline{x} = \underbrace{\left(\sum_{i=1}^{N} \underline{x}_i \underline{z}_i^{\dagger}\right)}_{\underline{\underline{I}}} \underline{x}$$

Resolution of identity in a nonorthogonal basis:

$$\boxed{\sum_{i=1}^{N} \underline{x}_i \underline{z}_i^{\dagger} = \underline{\underline{I}}}$$

How to identify the Reciprocal basis:

$\{\underline{x}_i\}$ are L.I $\rightarrow$ $\underline{\underline{X}}^{-1}$ exists.

$$\underline{\underline{X}} = \{\underline{x}_1 \cdots \underline{x}_N\}$$

$$\underline{\underline{X}}^{-1}\,\underline{\underline{X}} = \underline{\underline{I}}$$

$$\downarrow$$

$$\underline{\underline{Z}}^{\dagger}\,\underline{\underline{X}} = \underline{\underline{I}}$$

$$\underline{\underline{Z}}^{\dagger} = \underline{\underline{X}}^{-1}$$

$$\underline{\underline{Z}} = \left(\underline{\underline{X}}^{-1}\right)^{\dagger}$$

$$\swarrow$$

$$[\underline{z}_1 \cdots\cdots \underline{z}_n]$$

$$\underline{\underline{Z}} = [\underline{z}_1 \cdots \underline{z}_n]$$

$$\underline{\underline{Z}}^{\dagger} = \begin{bmatrix} \underline{z}_1^{\dagger} \\ \vdots \\ \vdots \\ \underline{z}_N^{\dagger} \end{bmatrix} = \underline{\underline{X}}^{-1}$$

$$\underline{\underline{Z}}^{\dagger}\,\underline{\underline{X}} = \underline{\underline{X}}\,\underline{\underline{X}}^{-1} = \underline{\underline{I}}$$

$$\begin{bmatrix} \underline{z}_1^{\dagger} \\ \vdots \\ \underline{z}_N^{\dagger} \end{bmatrix} [\underline{x}_1 \cdots\cdots \underline{x}_N] = \underline{\underline{I}}$$

$$\langle \underline{z}_i | \underline{x}_j \rangle = \delta_{ij}$$

$$\underline{z}_i^{\dagger}\,\underline{x}_j = \delta_{ij}$$

$$\begin{bmatrix} \underline{z}_1^{\dagger}\underline{x}_1 & \underline{z}_1^{\dagger}\underline{x}_2 & \cdots & \underline{z}_1^{\dagger}\underline{x}_N \\ \underline{z}_2^{\dagger}\underline{x}_1 & \underline{z}_2^{\dagger}\underline{x}_2 & \cdots & \\ & & & \end{bmatrix} = \underline{\underline{I}}$$

"Perfect" Matrices — N lin-indep. eig vecs

$\underline{\underline{A}}\, \underline{x}_i = \lambda_i \underline{x}_i \qquad i = 1 \ldots n$

$$\underline{\underline{I}} = \sum \underline{x}_i \, \underline{z}_i^{\dagger}$$

$$\underline{\underline{A}}\,\underline{\underline{I}} = \sum \underline{\underline{A}}\, \underline{x}_i \, \underline{z}_i^{\dagger}$$

$$\boxed{\underline{\underline{A}} = \sum_{i=1}^{N} \lambda_i \underline{x}_i \, \underline{z}_i^{\dagger}}$$ Spectral resolution of a perfect matrix

Take adjoint

$$\underline{\underline{A}}^{\dagger} = \sum_i \lambda_i^{*} \underline{z}_i \, \underline{x}_i^{\dagger}$$

$$\underline{\underline{A}}^{\dagger} \underline{z}_j = \sum_i \lambda_i^{*} \underline{z}_i \, \underbrace{\underline{x}_i^{\dagger} \underline{z}_j}_{\delta_{ij}}$$

$$\underline{\underline{A}}^{\dagger} \underline{z}_j = \lambda_j^{*} \underline{z}_j \longrightarrow \text{Eig.val. prob. for } \underline{\underline{A}}^{\dagger}$$

If $\underline{\underline{A}}$ is a perfect matrix, with eig values λ_i's, then the eig values of $\underline{\underline{A}}^{\dagger}$ are complex conjugates of e.v.'s of $\underline{\underline{A}}$

The eig vecs of $\underline{\underline{A}}^{\dagger}$ are the reciprocal vectors of the eig vecs of $\underline{\underline{A}}$.

$\underline{\underline{A}}$ perfect matrix $\longrightarrow$ N lin. indep. eig vecs

$\underline{\underline{A}}^{\dagger} \underline{z}_i = \lambda_i^{*} \underline{z}_i \qquad \{\underline{z}_1 \ldots \underline{z}_N\} \longrightarrow$ Reciprocal basis.

Resol. of identity $\underline{\underline{I}} = \sum_i x_i \underline{z}_i^{\dagger}$

Spectral Resol. $\underline{\underline{A}} = \sum_i \lambda_i \underline{x}_i \, \underline{z}_i^{\dagger}$

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}} \cdot \underline{x} \qquad \underline{x}(t=0) = \underline{x}_0$$

$$\underline{\underline{A}} = \sum_i \lambda_i \underline{x}_i \underline{z}_i^\dagger$$

$$\underline{\underline{A}}^2 = \sum_i \lambda_i^2 \underline{x}_i \underline{z}_i^\dagger \quad \cdots \quad \underline{\underline{A}}^k = \sum_i \lambda_i^k \underline{x}_i \underline{z}_i^\dagger$$

$$\exp[t\underline{\underline{A}}] = \sum_{k=0}^{\infty} \frac{(t\underline{\underline{A}})^k}{k!} = \sum_{k=0}^{\infty} \sum_{i=1}^{N} \frac{t^k \lambda_i^k}{k!} \underline{x}_i \underline{z}_i^\dagger$$

$$\exp(t\underline{\underline{A}}) = \sum_{i=1}^{N} \underbrace{\left(\sum_{k=0}^{\infty} \frac{t^k \lambda_i^k}{k!} \right)}_{e^{t\lambda_i}} \underline{x}_i \underline{z}_i^\dagger$$

$$\exp(t\underline{\underline{A}}) = \sum_{i=1}^{N} \exp(t\lambda_i) \underline{x}_i \underline{z}_i^\dagger$$

$$f(\underline{\underline{A}}) = \sum_{i=1}^{N} f(\lambda_i) \underline{x}_i \underline{z}_i^\dagger$$

Linear ODE's.

$$\frac{d\underline{x}}{dt} = \underline{\underline{A}} \cdot \underline{x} \qquad \underline{x}(t=0) = \underline{x}_0 \quad \text{initial cond.}$$

$\hookrightarrow$ perfect

$$\underline{x}(t) = \exp(t\underline{\underline{A}})\, \underline{x}_0$$

$$= \sum_{i=1}^{N} \exp(t\lambda_i)\, \underline{x}_i \left(\underline{z}_i^\dagger \underline{x}_0\right)$$

$\{\underline{x}_i\}$ and $\{\underline{z}_j\}$ form a Biorthogonal set.

$$\underline{\underline{X}}^{-1} \underline{\underline{A}}\, \underline{\underline{X}} = \underline{\underline{\Lambda}} \leftarrow \text{diagonal}$$

Normal matrix $\underline{\underline{A}}\,\underline{\underline{A}}^{\dagger} = \underline{\underline{A}}^{\dagger}\underline{\underline{A}}$

orthogonal
unitary
Symm
Hermitian

$\underline{x}_i^{\dagger}\underline{x}_j = \delta_{ij}$

unitary transfr.

$\underline{\underline{X}}^{\dagger} = \underline{\underline{X}}^{-1}$

$\underline{\underline{X}}^{-1}\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{\Lambda}}$

Positive definite : $\underline{x}^{\dagger}\underline{\underline{A}}\,\underline{x} = s > 0$ for any $\underline{x}$

real, scalar

$s = s^{*}$ $\quad \underline{x}^{T}\underline{\underline{A}}\,\underline{x} = \underline{x}^{\dagger}\underline{\underline{A}}^{\dagger}\underline{x}$

$\underline{\underline{A}} = \underline{\underline{A}}^{\dagger}$

Hermitian!!

Not all Hermitian matrices are positive definite

$\underline{x} = \sum \alpha_i \underline{x}_i$ eg vecs $\quad \underline{\underline{A}}$ positive definite.

$s = \underline{x}^{\dagger}\underline{\underline{A}}\,\underline{x}$

$$s = \sum_i \sum_j \alpha_i^{*}\underline{x}_i^{\dagger}\underline{\underline{A}}\,\underline{x}_j\alpha_j$$

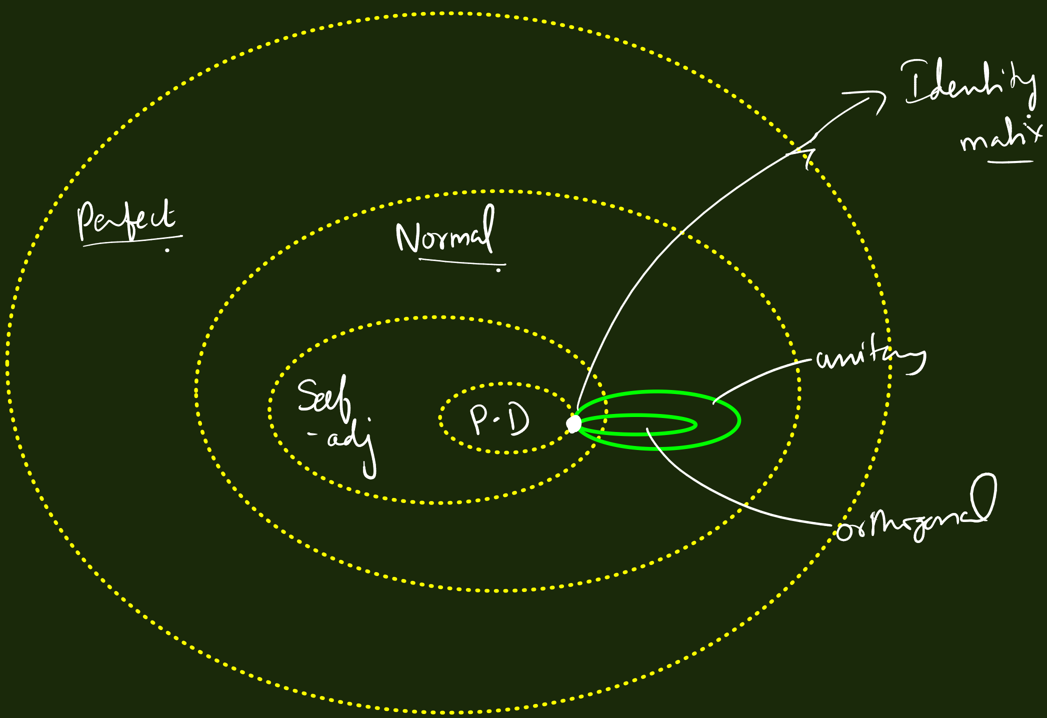
$$s = \sum_i \sum_j \alpha_i^{*}\alpha_j \lambda_j \underbrace{\underline{x}_i^{\dagger}\underline{x}_j}_{\delta_{ij}}$$

$$\boxed{s = \sum_i |\alpha_i|^2 \lambda_i}$$

Positive-definite matrices

$s > 0$ iff all λ's > 0

Imperfect



defn.

(P-D : positive definite)

Perfect matrix: $\longrightarrow$ Diagonalize

$\{\underline{x}_i\}$ eig vecs L.I $\quad\Big\}$ Biorthogonal set

$\{z_j\}$: reciprocal

$$\underline{x}_i^T \underline{z}_j = \delta_{ij}$$

$$\underline{\underline{X}} = [\underline{x}_1 \; \cdots \; \underline{x}_N] \qquad \underline{\underline{X}}^{-1} \underline{\underline{A}} \, \underline{\underline{X}} = \underline{\underline{\Lambda}}$$

$\hookrightarrow$ eig vecs of $\underline{\underline{A}}$

$$\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$$

If $\underline{\underline{A}}$ is perfect, it can be diagonalized by a similarity transform.

$$\underline{\underline{X}} = \{\underline{x}_1 \; \cdots \; \underline{x}_N\}$$

$$\underline{\underline{A}} \, \underline{x}_i = \lambda_i \underline{x}_i$$

$$\underline{\underline{\Lambda}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

$$\frac{d\underline{y}}{dt} = \underline{\underline{A}} \, \underline{y} \longrightarrow \quad \underline{\xi} = \underline{\underline{X}} \, \underline{y}$$

$$\frac{d\underline{\xi}}{dt} = \underbrace{\underline{\underline{X}} \, \underline{\underline{A}} \, \underline{\underline{X}}^{-1}}_{\underline{\underline{\Lambda}}} \underline{\xi}$$

Unitary and orthogonal matrices. $\rightarrow$ normal matrices $AA^{+} = A^{+}A$

$\underline{\underline{A}} \nearrow$

$$\|\underline{x}\|^2 = \|\underline{\underline{A}}\,\underline{x}\|^2$$

$$= (\underline{\underline{A}}\,\underline{x})^{+}(\underline{\underline{A}}\,\underline{x})$$

$$= \underline{x}^{+}\underbrace{\underline{\underline{A}}^{+}\underline{\underline{A}}}_{\underline{\underline{I}}}\,\underline{x}$$

$$\|\underline{a}\|^2 = \|\underline{a}\|^2$$

$\underline{\underline{A}}^{+} = \underline{\underline{A}}^{-1}$

unitary

$\boxed{\underline{\underline{A}}^{+}\underline{\underline{A}} = \underline{\underline{I}}}$

$$\underline{\underline{A}} = [\underline{a}_1 \; \cdots \; \underline{a}_N]$$

$$\underline{\underline{A}}^{+} = \begin{bmatrix} a_1^{+} \\ \vdots \\ a_N^{+} \end{bmatrix}$$

$\underline{\underline{A}}^{+}\underline{\underline{A}} = \underline{\underline{I}}$

$\langle \underline{a}_i | \underline{a}_j \rangle = \delta_{ij}$

$$\begin{bmatrix} a_1^{+} \\ \vdots \\ a_N^{+} \end{bmatrix}[\underline{a}_1 \; \cdots \; \underline{a}_N] = \underline{\underline{I}}$$

The column vectors of a unitary matrix form an orthogonal set.

$$\begin{bmatrix} a_1^{+}a_1 & a_1^{+}\underline{a}_2 & \cdots & a_1^{+}a_N \\ a_2^{+}a_1 & a_2^{+}a_2 & \cdots & a_2^{+}a_N \\ & & & \end{bmatrix} = \underline{\underline{I}}$$

$$s = \underline{x}^{+}\underline{\underline{A}}\,\underline{x} > 0$$

+ve def $\leftarrow$ $\underline{\underline{A}}\,\underline{x}_i = \lambda_i \underline{x}_i$

$$s = \sum_i \sum_j \alpha_i^{*} \underline{x}_i^{+} \underline{\underline{A}}\, \underline{x}_j \alpha_j$$

$$\rho = \sum_i \sum_j \alpha_i^* \alpha_j \lambda_j \, \underline{x}_i^\dagger \underline{x}_j \quad \delta_{ij}$$

$$\rho = \sum_{i=1}^{N} |\alpha_i|^2 \lambda_i$$

If $\rho > 0 \longrightarrow \boxed{\lambda_i\text{'s} > 0}$