Complex Numbers and Analysis: Part 2
ChE641, IIT Kanpur

$$z = \underset{\text{Real}}{x} + i\overset{\text{imag}}{y} \qquad i = \sqrt{-1} \longrightarrow i^2 = -1 \qquad i^3 = -i$$

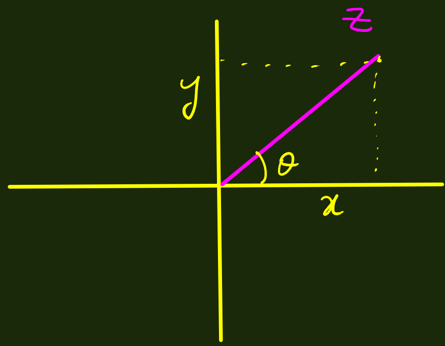
$$i^4 = +1 \ \ldots$$

$$|z| = (x^2+y^2)^{1/2} \qquad \theta \overset{\text{almost}}{=} \tan^{-1}\left(\frac{y}{x}\right)$$

quadrant

$$\theta + (2n\pi) \qquad \{n = 0, 1, 2 \ldots\}$$

$n = 0 \longrightarrow$ principal angle of z



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$z = x + iy$$

$$z = r\left[\cos\theta + i\sin\theta\right]$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \qquad \sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \qquad e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots$$

$$\frac{(i\theta)^2}{2!} = \frac{i^2\theta^2}{2!} = \frac{-\theta^2}{2!}$$

$$z = r\,\underbrace{\left[\cos\theta + i\sin\theta\right]}_{e^{i\theta}} \longrightarrow \boxed{z = re^{i\theta}}$$

"polar form"

$z : (x, y)$

$z : (r, \theta)$

DeMoivre's theorem: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

$z = 1 + i\sqrt{3}$ $\qquad r = \sqrt{1+3} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ { 1^{st} Quadrant)

$\theta = 60^\circ = \frac{\pi}{3}$ rad.

$z = re^{i\theta} = 2\,e^{i\pi/3} = 1 + i\sqrt{3}$

$= 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] = 2\left[\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right]$

$= 1 + i\sqrt{3}$

$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + x_2y_1).$

$z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$

$z_1 z_2 = r_1 r_2\, e^{i\underbrace{(\theta_1+\theta_2)}_{\text{add up}}}$

$z_1 \; z_2 \; z_3 \; .$

$= r_1 r_2 r_3 \cdots e^{i(\theta_1 + \theta_2 + \theta_3 + \cdots)}$

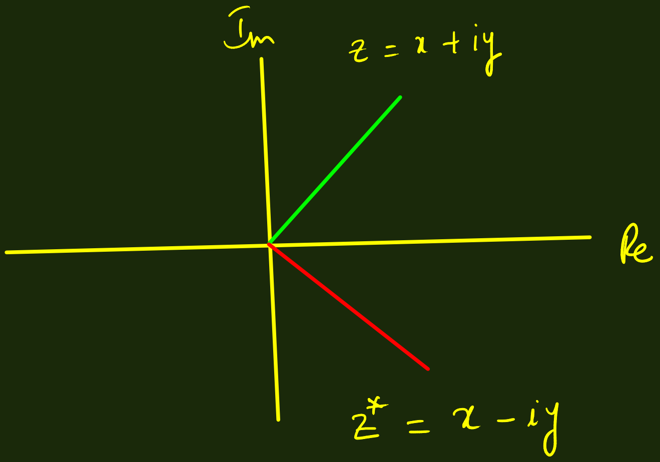
Complex conjugate

$z = x + iy$

$\downarrow$

complex conjugate

$z^* = x - iy$



$|z| = (x^2 + y^2)^{1/2}$

$$zz^* = (x+iy)(x-iy)$$
$$= x^2 - (iy)^2$$
$$= x^2 - i^2 y^2 \quad = x^2 + y^2$$

$zz^* = x^2 + y^2$

$$\boxed{zz^* = |z|^2}$$

$$z_1 . z_2 = |z_1| . e^{i\theta_1} \; |z_2| \; e^{i\theta_2}$$

$|z_1| = r_1$

$|z_2| = r_2$

$$z_1 z_2 = |z_1||z_2| \; e^{i(\theta_1 + \theta_2)}$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$|z_1 z_2| = |z_1||z_2|$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \overset{(\theta_1)}{\arg z_1} - \overset{(\theta_2)}{\arg z_2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$(e^{i\theta})^n = e^{in\theta} = (\cos\theta + i\sin\theta)^n$$

$\downarrow$

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

$n = 3$ $$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$
$$= (\cos^3\theta - 3\cos\theta\sin^2\theta)$$
$$+ i\,(3\sin\theta\cos^2\theta - \sin^3\theta)$$

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

$4^3 = 4 \cdot 4 \cdot 4 = 64 \ldots$

$z = (1+i)$ $\quad z^3$?

$z = re^{i\theta}$

$r = \sqrt{2} \quad \theta = \frac{\pi}{4}$

$$z^3 = \left(re^{i\theta}\right)^3$$
$$= r^3 e^{i3\theta}$$
$$= 2^{3/2} e^{i3\pi/4} \longrightarrow \text{Cartesian form.}$$

$(1-i)^6$ $\quad z = 1-i$

$r = \sqrt{2} \quad \theta = \frac{7\pi}{4}$

$$z^6 = \left(2^{1/2}\right)^6 \quad e^{i\frac{7\pi}{4} \times \cancel{6}^{3}_{2}} = 8\, e^{i\,21\frac{\pi}{2}}$$
$$= 8\, e^{i\frac{(20+1)}{2}\pi}$$
$$= 8\, \underbrace{e^{i10\pi}}_{1} \cdot e^{i\pi/2}$$
$$= 8\, e^{i\pi/2} = 8i$$

$$e^{2\pi i} = \cancel{\cos 2\pi}^{1} + i\cancel{\sin 2\pi}^{0}$$
$$= 1$$
$$e^{n(2\pi i)} = 1 \quad (n = 0, 1, 2 \ldots)$$

$$(1-i)^6 = 8i$$

<u>Roots of unity:</u> $\quad x^2 = 1 \quad x = \pm 1$ two roots $\quad x = \pm(1)^{1/2}$
$= \pm 1$

$x^N = 1 \quad N$ roots

$z^N = 1 \quad z = re^{i\theta}$

$$r^N e^{iN\theta} = 1 \longrightarrow r^N e^{iN\theta} = 1 \cdot e^{i\pi 0}$$

$r^N = 1 \quad e^{iN\theta} = 1$

$r = 1 \quad N\theta = 2\pi n \quad (n = 0, 1, 2 \ldots\ldots N-1)$

$z^N = 1$ $\quad z_n = e^{\frac{i2\pi n}{N}} \quad n = 0, 1, 2, \dots N-1$

fixed number (given).

<u>N=2</u> : z_0, z_1

$z_0 = 1 \qquad z_1 = e^{\frac{\cancel{2}\pi i}{\cancel{2}}}$

$= \cos\pi + i\sin\pi$

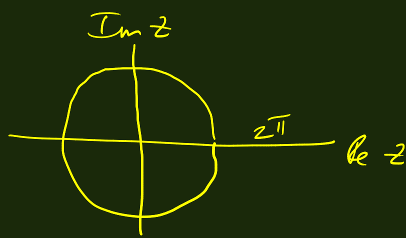
$z^2 = 1 \longrightarrow z_0 = 1 \qquad z_1 = -1$

<u>N=3</u> : z_0, z_1, z_2

$z_0 = e^0 = 1 \qquad z_1 = e^{2\pi i/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_2 = e^{4\pi i/3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$z^3 \quad z^4 \quad z^6 \quad \dots$



Square root of a complex no : $W = z^{1/2}$

$z = re^{i\theta}$

$W = z^{1/2} = r^{1/2} \, e^{i\theta/2}$

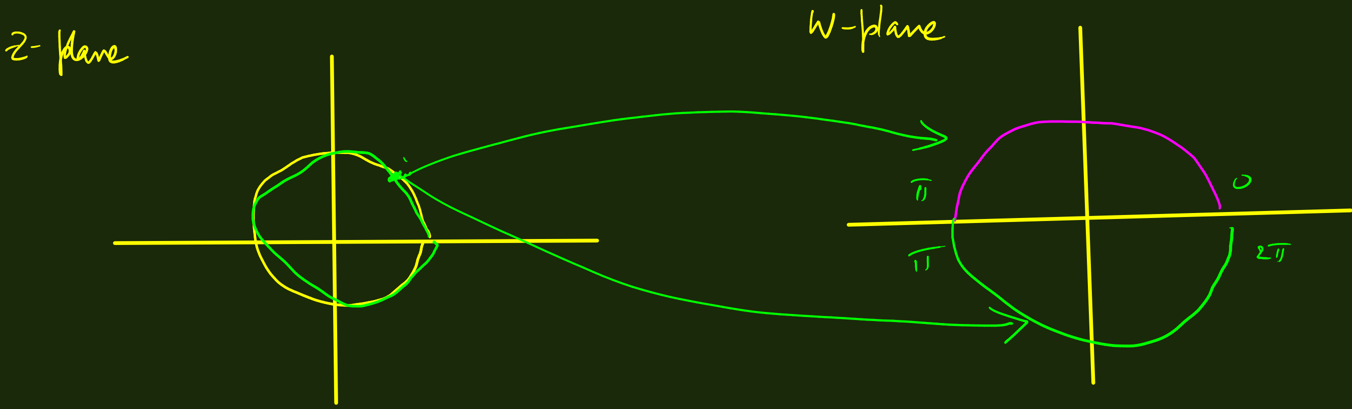
$\theta = 0 \qquad z = r$

$W = r^{1/2}$

$\theta = 2\pi$

$W = r^{1/2} \, e^{i2\pi/2} = r^{1/2} e^{i\pi}$

$= -r^{1/2}$

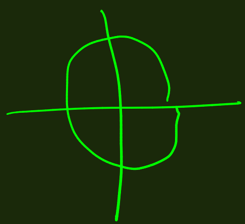


$w = z^{1/2} \longrightarrow$ double-valued fn.

two copies of the z-plane $\longrightarrow$ "Riemann sheets"

	z	w
θ :	$0 - 2\pi$	$0 - \frac{2\pi}{3}$
	$2\pi - 4\pi$	$\frac{2\pi}{3} - \frac{4\pi}{3}$
	$4\pi - 6\pi$	$\frac{4\pi}{3} - 2\pi$

$z^{1/3}$



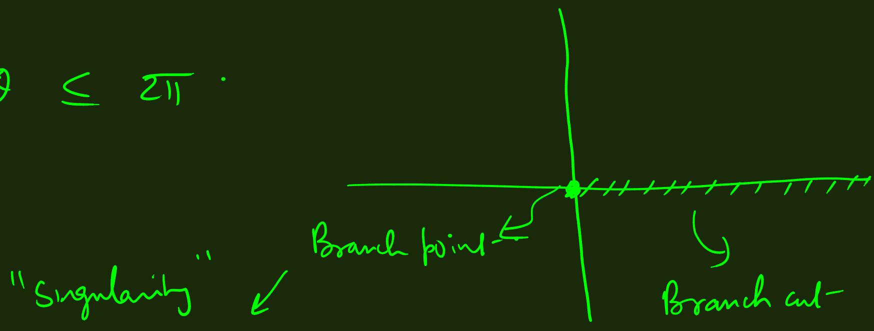
3 Riemann sheets.

$w = z^{1/3}$

$w = r^{1/3} \, e^{i\frac{\theta}{3}}$

How to make w single-valued:

$\text{Arg } z \quad \theta : \quad 0 < \theta \leq 2\pi$.



Cube root of 8.

$z = 8$

$= 8 \, e^{in\pi}$

$n = 0, 2, 4, 6 \ldots$

go back to same roots

$8 \, e^{i 2\pi i}$

$e^{i 4\pi i}$

$e^{i 2m\pi i}$

$m = 0, 1, 2 \ldots$

$2m = n$

$z^{1/3} = 8^{1/3} \, e^{i n\pi/3}$

$= 8^{1/3} \cdot e^{0}$

$= 8^{1/3} \, e^{i 2\pi/3}$

$8^{1/3} \, e^{i 4\pi/3}$

$8^{1/3} \, e^{i 6\pi/3}$

$8^{1/3}$

$8^{1/3} \, e^{i 8\pi/3}$

$\underline{1} \leftarrow e^{i\frac{6\pi}{3}} \, e^{i\frac{2\pi}{3}}$

$8^{1/3} = 2, \; 2e^{i2\pi/3}, \; 2e^{i4\pi/3}$

$2\left[\frac{-1}{\sqrt{2}} + i\frac{\sqrt{3}}{2}\right]$

$\cancel{+} - i\sqrt{3}$

$-1 + i\sqrt{3}$