Anologies Between Momentum and Heat Transfer

In the boundary layers, momentum balance reduces to

$$\rho\left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2}$$

and

$$0 = -\frac{\partial P}{\partial y}$$

And in absence of viscous dissipation, energy balance is

$$\rho c_p\left[u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right] = k\frac{\partial^2 T}{\partial y^2}$$

assuming $\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2}$

Nondimensionalized Form

Using $x^* = \frac{x}{L}$, $y^* = \frac{y}{L}$, $u^* = \frac{u}{u_\infty}$, $v^* = \frac{v}{u_\infty}$, $P^* = \frac{P}{\rho u_\infty^2}$

and $T^* = \frac{T - T_s}{T_\infty - T_s}$

Continuity:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

Momentum balance

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L}\frac{\partial^2 u^*}{\partial y^{*2}} - \frac{\partial P^*}{\partial x^*}$$

Energy balance

$$u^*\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = \frac{1}{Re_L Pr}\frac{\partial^2 T^*}{\partial y^{*2}}$$

with $Re_L = \frac{\rho u_\infty L}{\mu}$; $Pr = \frac{c_p \mu}{k}$

Boundary conditions can be made dimensionless using these scales. $u^*(0,y^*) = u^*(x^*,\infty) = 1$; $u^*(x^*,0) = 0 = v^*(x^*,0)$; $T^*(0,y^*) = T^*(x^*,\infty) = 1$

$T^*(x^*,0) = 0$

Advantage: reduction in number of parameters.

from $L, u_\infty, T_s, T_\infty, \nu, \alpha$ (6) $\longrightarrow$ to Re_L, Pr (2)

Thus, solutions of problems with same Re_L and Pr must be identical. In other words, there are only two parameters that are independent in the model

Note that $P^* = P^*(x^*)$ only (from y-momentum balance)

Thus, at a fixed distance x^*, the value of P^* inside the boundary layer is the same as that in the free stream. Thus, $\frac{dP^*}{dx^*}$ can be determined from free stream equations, and appears as a known quantity in these equations for the boundary layer.

For a given geometry, thus, the solution for velocity can be written as

$$u^* = f_1(x^*, y^*, Re_L)$$

from the form of dimensionless equations

Then,

$$\tau_w = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = \frac{\mu u_\infty}{L} \frac{\partial u^*}{\partial y^*}\Big|_{y^*=0} = \frac{\mu u_\infty}{L} f_2(x^*, Re_L) \quad \ldots y^*=0$$

The local friction coefficient

$$C_{f,x} = \frac{\tau_w}{\rho u_\infty^2/2} = \frac{\mu u_\infty/L}{\rho u_\infty^2/2} f_2(x^*, Re_L) = \frac{2}{Re_L} f_2(x^*, Re_L) = f_3(x^*, Re_L)$$

The friction coefficient can be expressed as a function of x^* and Re_L.

Similarly,

$$T^* = g_1(x^*, y^*, Re_L, Pr).$$

Thus, the local convection coefficient

$$h_x = \frac{-k(\partial T/\partial y)_{y=0}}{T_s - T_\infty} = \frac{-k(T_\infty - T_s)}{L(T_s - T_\infty)} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = \frac{k}{L} \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

Or in the dimensionless form,

$$Nu_x = \frac{h_x L}{k} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0} = g_2(x^*, Re_L, Pr)$$

Note: Nusselt number is equivalent to dimensionless temperature gradient at the surface, and is also interpreted as dimensionless heat transfer coefficient.

Average friction coefficient / heat transfer coefficient is found by integrating the local coefficient ($C_{f,x}$ or Nu_x) along the surface, between $x^*=0$ and $x^*=1$. Thus, the average friction coefficient / Nusselt number are given as

$$C_f = f_4(Re_L) \quad \text{and} \quad Nu = g_3(Re_L, Pr)$$

Experimental data is reasonably represented as

$$\boxed{Nu = c\, Re_L^m\, Pr^n}$$

, where m, n are constant exponents, typically between 0 & 1, and c depends on geometry.

Note:
This functional form will appear many times throughout convection.

It is useful to have a relation between C_f and Nu so that one can be estimated when the other is known. Similarity between velocity and thermal boundary layers provide a rationale for developing an analogy between the two.

Reynolds Analogy ($Pr = 1$)

- For $Pr = 1$

Reduced (non dimensional) equations for momentum and energy balance for steady, incompressible, laminar flow of a fluid with constant properties and negligible viscous dissipation, for $Pr = 1$, simplify to

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{Re_L} \frac{\partial^2 u^*}{\partial y^{*2}}$$

and

$$u^* \frac{\partial T}{\partial x^*} + v^* \frac{\partial T}{\partial y^*} = \frac{1}{Re_L\, Pr} \frac{\partial^2 T}{\partial y^{*2}}$$

with

$$u^*(0, y^*) = u^*(x^*, \infty) = 1 \quad \text{and} \quad u^*(x^*, 0) = 0$$

$$T^*(0, y^*) = T^*(x^*, \infty) = 1 \quad \text{and} \quad T^*(x^*, 0) = 0$$

and

$$v^*(x^*, 0) = 0.$$

These equations are exactly of the same form for u^* and T^*. Assuming that u^* and T^* are identical functions,

$$\left.\frac{\partial u^*}{\partial y^*}\right|_{y^*=0} = \left.\frac{\partial T^*}{\partial y^*}\right|_{y^*=0}$$

$$C_{f,x} \frac{Re_L}{2} = Nu_x \qquad \text{i.e. } f_2(x^*, Re_L) = g_2(x^*, Re_L)$$

or, in terms of another dimensionless number, Stanton number (St),

$$\frac{C_{f,x}}{2} = St_x \qquad \text{where } St_x = \frac{h_x}{\rho c_p u_\infty} = \frac{Nu_x}{Re_L\, Pr}$$

is another dimensionless heat transfer coefficient.

Note: Reynolds analogy allows estimation of heat transfer coefficient using easier measurements of friction coefficient, for $Pr = 1$ (gases)

Chilton-Colburn Analogy (modified Reynolds Analogy)

- Correction for Prandtl number.
- For a flat plate, we had determined

$$C_{f,x} = 0.664\, Re_x^{-1/2}$$

$$Nu_x = 0.332\, Re_x^{1/2}\, Pr^{1/3}$$

These imply that (dividing $C_{f,x}$ by Nu_x and rearranging)

$$C_{f,x} \frac{Re_L}{2} = Nu_x Pr^{-1/3},$$

or

$$\frac{C_{f,x}}{2} = St_x Pr^{2/3} = j_H \quad \text{for } 0.6 < Pr < 60$$

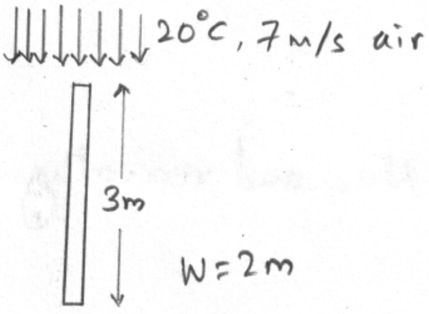
$j_H \equiv$ Colburn j-factor.

Notes:

- Developed using results for laminar flow over flat plate $\left(\frac{dP^*}{dx^*} = 0\right)$
- Experimentally also found to be applicable approximately for
  - turbulent flows over a surface
  - in presence of weak pressure gradients
- Not applicable for $\frac{dP^*}{dx^*} \neq 0$.
  - does not work for laminar flow in a pipe.
- More accurate analogies between C_f and Nu also exist (but are complicated).

Example: Estimating convection coefficient from drag force

A 2m × 3m flat plate is suspended in a room, and is subjected to air flow parallel to its surface along the 3m long side. The free stream temperature and velocity of air are 20°C and 7 m/s. Total drag force acting on the plate is measured to be 0.86 N. Determine the average convection heat transfer coefficient for the plate.



Assumptions:
- Steady state
- Negligible edge effects.

Properties: Air at 20°C and 1atm pressure

$\rho = 1.204\ \mathrm{kg/m^3}$ $\quad C_p = 1.007\ \mathrm{kJ/kg\,K}$

$Pr = 0.7309$

Lengthscale (relevant).

$L = 3\mathrm{m}$

Total surface area

$$A_s = 2WL = 12\,\mathrm{m}^2$$

Friction force on the entire surface of a flat plate

$$F_f = C_f \frac{A_s \rho u_\infty^2}{2}$$

Thus,
$$C_f = \frac{F_f}{\rho A_s u_\infty^2/2} = \frac{0.86}{1.204 \times 12 \times \frac{7\times 7}{2}} = 0.00243$$

$Pr = 0.7309$. Thus $0.6 < Pr < 60$ and Chilton-Colburn analogy can be used.

$$\frac{C_f}{2} = St\, Pr^{2/3} \quad \Rightarrow \quad h = \frac{C_f \rho u_\infty C_p}{2 Pr^{2/3}} = 12.7\ \mathrm{W/m^2\,K}$$