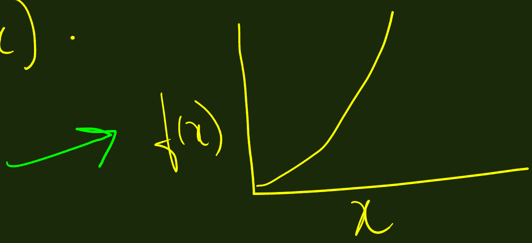
Functions of Many Variables: Partial differentials

$y = f(x)$.



$\frac{dy}{dx} \rightarrow$ local slope of the curve $y(x)$

T.P / Fluid Mechanics / Thermo :

Functions of many variables $T(x, y, z; t)$

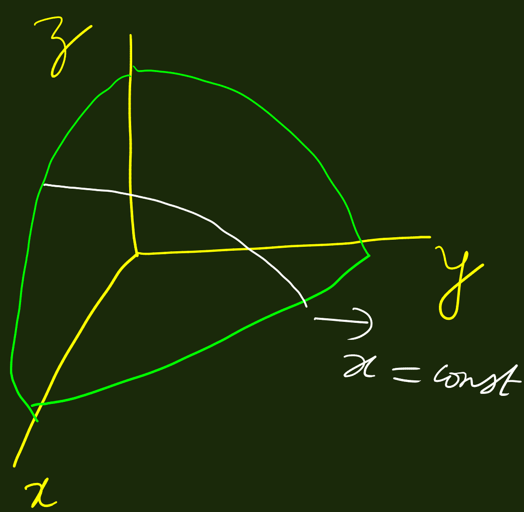
$P(V, T)$

$U(S, V) \ldots$

$z = f(x, y)$

indep. variables ...

Surface in 3D plane $\leftarrow z = f(x, y)$



$z = f(x = c, y)$

$\left.\frac{dz}{dy}\right|_{x = \text{const}}$

partial derivative : $\left.\frac{\partial z}{\partial y}\right)_{x = \text{const.}}$

$\left.\frac{\partial z}{\partial y}\right]_x \rightarrow$ a fn. of both x and y.
(in general)

$\left.\frac{\partial z}{\partial x}\right]_y \qquad \left.\frac{\partial z}{\partial y}\right]_x$

fns of both x and y (in general).

higher derivatives : $\left.\frac{\partial z}{\partial x}\right)_y \quad \left.\frac{\partial z}{\partial y}\right)_x \rightarrow g(x,y)$

further partial differentials.

$$\frac{\partial^2 z}{\partial x^2} = \left.\frac{\partial}{\partial x}\left[\left.\frac{\partial z}{\partial x}\right)_y\right]\right]_y$$

$$\left.\frac{\partial}{\partial y}\right]_x \left(\frac{\partial z}{\partial x}\right)_y = \frac{\partial^2 z}{\partial y\, \partial x}$$

$$\left.\frac{\partial}{\partial x}\right]_y \left.\frac{\partial z}{\partial y}\right)_x = \frac{\partial^2 z}{\partial x\, \partial y}$$

("Mixed derivatives")

In general : $\frac{\partial^2 z}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial y\, \partial x}$ (If the fn $z(x,y)$ and its derivatives are smooth & continuous).

$$z = f(x,y) = x^3 y - e^{xy}.$$ (two variables – x, y).

$$\left.\frac{\partial z}{\partial x}\right)_y = 3x^2 y - y\, e^{xy}$$

$$\left.\frac{\partial z}{\partial y}\right)_x = x^3 - x\, e^{xy}.$$

$$\frac{\partial z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = 3x^2 - xy\,e^{xy} - e^{xy}$$

Verify $$\frac{\partial^2 z}{\partial y \partial x} = 3x^2 - xy\,e^{xy} - e^{xy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \longrightarrow \text{verified}$$

<u>Total differential</u>: $f(x,y) \longrightarrow \left.\frac{\partial f}{\partial x}\right)_y$ (or) $\left.\frac{\partial f}{\partial y}\right)_x$

along the two coordinate axes

$$\Delta f = f(x+\Delta x,\ y+\Delta y) - f(x,y)$$

$(x, y) \dashrightarrow (x+\Delta x, y+\Delta y)$

$\Delta x, \Delta y$: small.

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= f(x+\Delta x, y+\Delta y) - \underline{f(x, y+\Delta y)} + \underline{f(x, y+\Delta y)} - f(x,y)$$

$$= \left[\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}\right] \Delta x$$

$$+ \left[\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}\right] \Delta y$$

"small"

$\lim \underline{\Delta x, \Delta y \to 0}$

$$\Delta f = \left.\frac{\partial f}{\partial x}\right]_{y+\Delta y} \Delta x + \left.\frac{\partial f}{\partial y}\right]_{x} \Delta y$$

$\lim \Delta x, \Delta y \to 0$ $\quad \longrightarrow \quad \left.\frac{\partial f}{\partial x}\right)_y$

$\Delta x \to dx$

$\Delta y \to dy$

$$df = \left.\frac{\partial f}{\partial x}\right)_y dx + \left.\frac{\partial f}{\partial y}\right)_x dy$$

Total differential : $f(x,y)$

$$\boxed{df = \left.\frac{\partial f}{\partial x}\right)_y dx + \left.\frac{\partial f}{\partial y}\right)_x dy}$$

Functions of many variables $\quad f = f(x_1, x_2, x_3 \dots x_N)$

$$df = \left.\frac{\partial f}{\partial x_1}\right)_{x_2 \dots x_N = C} dx_1 + \left.\frac{\partial f}{\partial x_2}\right)_{x_1, x_3, \dots x_N = C} dx_2 + \dots\dots + \left.\frac{\partial f}{\partial x_N}\right)_{x_1 \dots x_{N-1} = C.} dx_N$$

Example:

$$\boxed{z = x^2 - y^2} \to z = z(x, y) \text{ indep. } x, y$$

$x \to r\cos\theta$

$y \to r\sin\theta$

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

$$z = x^2(\cos^2\theta - \sin^2\theta) = \boxed{z = r^2\cos 2\theta.} \quad z \to z(r, \theta)$$

$$z = r^2(\cos^2\theta - \sin^2\theta)$$

$$\left.\frac{\partial z}{\partial r}\right)_{\theta} = 2r(\cos^2\theta - \sin^2\theta)$$

$$z = x^2 - y^2 = 2x^2 - x^2 - y^2 = 2x^2 - r^2$$

$$z \to z(x, r) \qquad \left.\frac{\partial z}{\partial r}\right)_{x} = -2r$$

Not same!!

$$z = x^2 - 2y^2 + y^2$$

$$z = r^2 - 2y^2 \to \left.\frac{\partial z}{\partial r}\right)_{y} = 2r$$

(r, y)

$$x = r\cos\theta \qquad f = f(x,y) \longrightarrow \left.\frac{\partial f}{\partial x}\right)_y$$

$$y = r\sin\theta \qquad \searrow f(r,\theta)$$

$f = (r,\theta)$

$$df = \left.\frac{\partial f}{\partial r}\right)_\theta dr + \left.\frac{\partial f}{\partial \theta}\right)_r d\theta \longleftarrow$$

$$\left.\frac{\partial f}{\partial x}\right)_y = \left.\frac{\partial f}{\partial r}\right)_\theta \frac{\partial r}{\partial x}\Big)_y + \left.\frac{\partial f}{\partial \theta}\right)_r \frac{\partial \theta}{\partial x}\Big)$$

$$f(x,y) = x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = \boxed{r^2\cos 2\theta}$$

$$\left.\frac{\partial f}{\partial x}\right)_y = \left.\frac{\partial f}{\partial r}\right)_\theta \left.\frac{\partial r}{\partial x}\right)_y + \left.\frac{\partial f}{\partial \theta}\right)_r \left.\frac{\partial \theta}{\partial x}\right)_y$$

$$r = (x^2+y^2)^{1/2} \qquad \left.\frac{\partial r}{\partial x}\right)_y = \frac{1}{\cancel{2}} \frac{\cancel{2}x}{(x^2+y^2)^{1/2}} = \frac{x}{r} = \cos\theta$$

$$\left.\frac{\partial \theta}{\partial x}\right)_y = \frac{-\sin\theta}{r}$$

$$\left.\frac{\partial f}{\partial x}\right)_y = 2r\cos 2\theta\cos\theta + 2r\sin 2\theta\sin\theta$$

$$= 2r\left[\cos(2\theta - \theta)\right]$$

$$= 2r\cos\theta \qquad x = r\cos\theta$$

$$\left.\frac{\partial f}{\partial x}\right)_y = 2x \longrightarrow f(x,y) = x^2 - y^2$$

$$\left.\frac{\partial f}{\partial x}\right)_y = 2x$$

$x = r\cos\theta$ $\qquad r = \dfrac{x}{\cos\theta} \longrightarrow \left.\dfrac{\partial r}{\partial x}\right)_{\theta = \text{const}} = \dfrac{1}{\cos\theta}$

$\left.\dfrac{\partial r}{\partial x}\right)_y$ $\qquad r = \left(x^2+y^2\right)^{1/2}$

$\left.\dfrac{\partial r}{\partial x}\right)_y = \dfrac{x}{r} = \cos\theta$

$\left.\dfrac{\partial r}{\partial x}\right)_y = \cos\theta$ $\qquad \left.\dfrac{\partial r}{\partial x}\right)_\theta = \dfrac{1}{\cos\theta}$