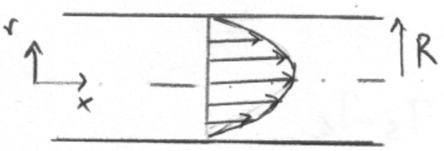
Laminar Flow in Tubes ($Re \lesssim 2300$)

Fully Developed Region

- For region beyond $L_{h,laminar}$ and $L_{t,laminar}$ from the entrance

Simplifying momentum balance in the x-direction

$$0 = -\frac{dP}{dx} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)$$



with $u = u(r)$ only and $v = w = 0$

$$\underbrace{\frac{\mu}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)}_{\text{Only a function of } r} = \underbrace{\frac{dP}{dx}}_{\text{Only a function of } x}$$

For equality to hold at any location, both LHS & RHS are equal to a constant. Thus,

$$\frac{dP}{dx} = \text{constant}$$

Integrating,

$$u(r) = \frac{1}{4\mu}\frac{dP}{dx} + C_1 \ln r + C_2$$

Boundary conditions are

$$\frac{du}{dr} = 0 \quad \text{at} \quad r = 0 \qquad \text{(symmetry)}$$

and $u = 0$ at $r = R$ (no slip)

Thus.

$$u(r) = -\frac{R^2}{4\mu}\left(\frac{dP}{dx}\right)\left(1-\left(\frac{r}{R}\right)^2\right)$$

— parabolic velocity profile

with

$$u_{avg} = \frac{\int_0^R 2\pi r\,dr\, u(r)}{\pi R^2}$$

$$= \frac{2}{R^2}\int_0^R dr \left(-\frac{R^2}{4\mu}\frac{dP}{dx}\right)\left(r - \frac{r^3}{R^2}\right)$$

$$= \frac{2}{R^2}\left(-\frac{R^2}{4\mu}\right)\frac{dP}{dx}\,\frac{R^2}{4} = -\frac{R^2}{8\mu}\left(\frac{dP}{dx}\right)$$

Thus,

$$u(r) = 2u_{avg}\left(1-\left(\frac{r}{R}\right)^2\right)$$

To determine u_{avg}, $\dot{m} = \rho u_{avg} A_c$ can be used

$$\frac{dP}{dx} = \text{constant} = \frac{P_2 - P_1}{L}$$

Thus,

$$\Delta P = P_1 - P_2 = \frac{8\mu L u_{avg}}{R^2} = \frac{32\mu L u_{avg}}{D^2}$$

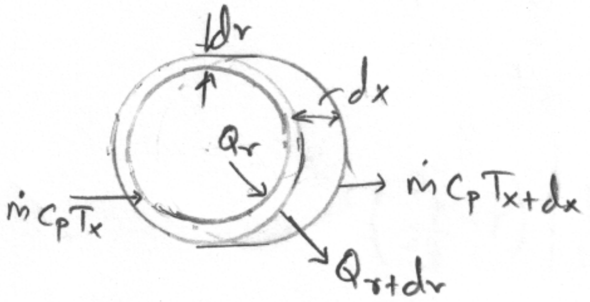
where $\Delta P \equiv$ pressure drop

Note: when $\mu \to 0$, $\Delta P \to 0$. Thus pressure drop is due to viscous effects, and is called pressure loss (ΔP_L).
Friction factor, $f = \frac{8\tau_w}{\rho u_{avg}^2}$, and hence

$$\Delta P_L = f\,\frac{L}{D}\,\underbrace{\frac{\rho u_{avg}^2}{2}}_{\text{dynamic pressure}}$$

For laminar flow through a pipe, in fully developed region. $f = 64/Re_D$

Laminar flow of an incompressible fluid with constant properties in the fully developed region results into the following energy balance over a differential element, assuming negligible viscous heating



$$\dot{m} c_p T_x - \dot{m} c_p T_{x+dx} + Q_r - Q_{r+dr} = 0$$

with

$$\dot{m} = \rho\, u(r) A_c = \rho\, u\, 2\pi r\, dr$$

Thus,

$$\rho c_p u \frac{T_{x+dx} - T_x}{dx} = -\frac{1}{2\pi r\, dx} \frac{Q_{r+dr} - Q_r}{dr} = -\frac{1}{2\pi dx} \frac{1}{r} \frac{\partial Q_r}{\partial r}$$

Using Fourier's law of conduction,

$$-Q_r = k (2\pi r\, dx) \frac{\partial T}{\partial r}$$

$$\Rightarrow \quad u \frac{\partial T}{\partial x} = \frac{\alpha}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \quad ; \quad \alpha = \frac{k}{\rho c_p}$$

Constant Heat flux

We had derived

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{\dot{q}\, p}{\dot{m} c_p} = \frac{\dot{q}\, 2\pi R}{\rho u_{avg} c_p \pi R^2} = \frac{2q}{\rho u_{avg} c_p R} = \text{constant}$$

Since $\frac{\partial T}{\partial x}$ = constant, $\frac{\partial^2 T}{\partial x^2} = 0$, consistent with the assumption that $T = T(r)$ only

Substituting $\frac{\partial T}{\partial x}$ and u,

$$2 u_{avg} \left(1 - \left(\frac{r}{R}\right)^2\right) \left(\frac{2q}{\rho u_{avg} c_p R} \right) = \frac{\alpha}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$$

$$\frac{4q}{kR} \left(1 - \left(\frac{r}{R}\right)^2\right) = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right)$$

Integrating,

$$T = \frac{q}{kR}\left(r^2 - \frac{r^4}{4R^2}\right) + C_1 \ln r + C_2$$

with boundary conditions

$$\frac{\partial T}{\partial r} = 0 \text{ at } r = 0 \quad \text{(symmetry)}$$

and $\quad T = T_s$ at $r = R$

$$T = T_s - \frac{q_s R}{k}\left(\frac{3}{4} - \left(\frac{r}{R}\right)^2 + \frac{1}{4}\left(\frac{r}{R}\right)^4\right).$$

Thus,

$$T_m = T_s - \frac{11}{24}\frac{q_s R}{k}$$

and

$$h = \frac{24}{11}\frac{k}{R} = 4.36\frac{k}{D}$$

Hence, for laminar flow in a circular tube, with constant surface heat flux, in the fully developed region,

$$Nu = \frac{hD}{k} = 4.36 \qquad \text{independent of } Re, Pr.$$

Constant Surface Temperature

Similar analysis with constant surface temperature gives,

$$Nu = 3.66$$

Notes:
- Similar relations are available for non-circular cylinders. These are reported in Cengel's book, table 8-1 or Incropera's book, table 8.1
- Properties (k) are evaluated at the bulk average fluid temperature, $(T_i + T_e)/2$

Entrance Region (Laminar flow)

- Key difference with fully developed region
  - $u(r)$ varies with x.
  - radial velocity is not zero
  - Solution to energy equation becomes complicated, since $\frac{\partial T}{\partial x}$ is no longer a constant

$\Rightarrow$ Numerical Solution to get local Nusselt number

Graetz Number

$$Gz = \frac{D}{x} Re\, Pr$$

Recall that the thermal entry length in the laminar flow regime

$$L_{t,laminar} \approx 0.05\, Re\, Pr\, D$$

$\Rightarrow (Gz)^{-1} = 0.05$ is where flow becomes fully developed, and hence $Nu = 4.36$ (constant heat flux) or 3.66 (constant surface temperature)

Thus, for $(G_z)^{-1} > 0.05$ flow is fully developed, and Nu is constant.

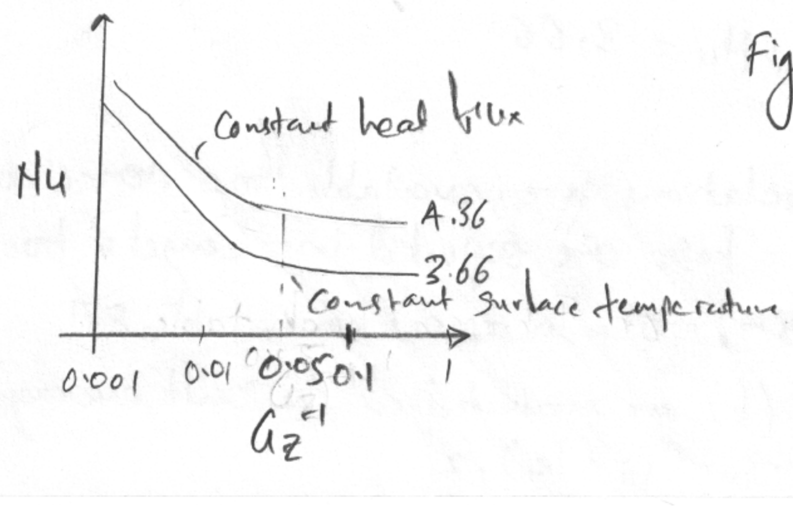


Figure 8-10 (Incropera)
8-2A (Cengel)

Correlations for the average Nusselt number:

- For hydrodynamically fully developed thermal entry region, in laminar flow regime.

$$Nu = 3.66 + \frac{0.065 \frac{D}{L} Re\,Pr}{1 + 0.04\left[\frac{D}{L} Re\,Pr\right]^{2/3}}$$

<u>Edwards</u>

where $L \equiv$ length of the circular tube

also applicable when $Pr \geqslant 5$

(recall: $L_{t,laminar} = Pr\, L_{h,laminar}$)

- For hydrodynamically and thermally developing laminar flow, with viscosity varying due to large temperature differences.

$$Nu = 1.86\left(Re\,Pr\,\frac{D}{L}\right)^{1/3}\left(\frac{\mu_b}{\mu_s}\right)^{0.14}$$

$\mu_s \equiv$ viscosity at surface temperature

all other properties at bulk average fluid temperature $(T_i + T_e)/2$

<u>Sieder-Tate</u>

Recommended for

$$0.6 \leq Pr \leq 5$$

and

$$0.0044 \leq \frac{\mu_b}{\mu_s} \leq 9.75$$

Note: Gr appears in both correlations

Turbulent Flow in Tubes

- Flow is truely turbulent for $Re > 10000$ for smooth tubes, and heat transfer coefficient is relatively high

For fully developed (hydrodynamically and thermally) flow in turbulent flow regime, in a smooth circular tube,

$$Nu_D = 0.023\, Re_D^{4/5}\, Pr^n$$

with $n = 0.4$ for heating $(T_s > T_m)$
$n = 0.3$ for cooling $(T_s < T_m)$

$0.6 \leq Pr \leq 160$
$Re_D \geq 10{,}000$
$L/D \geq 10$

Dittus-Boelter Equation

Fluid properties are evaluated at the bulk average fluid temperature, assuming small temperature differences between the fluid and the surface of the tube.

For large temperature differences leading to significant property variations,

$$Nu_D = 0.027\, Re^{4/5}\, Pr^{1/3} \left(\frac{\mu_b}{\mu_s}\right)$$

with μ_s evaluated at T_s
other properties evaluated at the mean temperature

$0.6 \leq Pr \leq 16{,}700$
$Re_D \geq 10{,}000$
$L/D \geq 10$

Sieder-Tate Equation

Dittus-Boelter and Sieder-Tate equations are simpler for calculations, but can lead to large (upto 25%) errors. More complex correlations with less than 10% error are available.

E.g. for smooth tubes

$$Nu_D = \frac{f/8\, Re_D Pr}{1.07 + 12.7\left(\frac{f}{8}\right)^{0.5}(Pr^{2/3} - 1)} \qquad \begin{array}{l} 0.5 \le Pr \le 2000 \\ 10^4 \le Re_D \le 5\times10^6 \end{array}$$

second Petukhov Correlation

or

$$Nu_D = \frac{f/8\,(Re_D - 1000)\, Pr}{1 + 12.7\,(f/8)^{0.5}(Pr^{2/3} - 1)} \qquad \begin{array}{l} 0.5 \le Pr \le 2000 \\ 3000 \le Re_D \le 5\times10^6 \end{array}$$

Gnielinski Correlation

where

$f \equiv$ friction factor

$$f = (0.790 \ln Re_D - 1.64)^{-2}$$

$$3000 \le Re_D \le 5\times10^6$$

first Petukhov Correlation

Note:
- Entry lengths for turbulent flow are typically small (Recall the correlations for $L_{h,turbulent}$, $L_{t,turbulent}$). Thus the fully developed region's Nusselt number can be approximately extended to the entire tube.
- The same correlations are "approximately" applicable for noncircular tubes, with D replaced by the hydraulic diameter, D_h.