Linear Algebra - Part 8: Unitary Matrices, Similarity Transformations
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Normal matrices $\quad AA^\dagger = A^\dagger A \qquad A^\dagger$: Hermitian conjugate of A

A is normal matrix → eigen vecs. are orthogonal.

$$A \underline{x}^i = \lambda_i \underline{x}^i \qquad \underline{y} = \sum_i a_i \underline{x}^i$$

Unitary Matrices: $\quad \boxed{A^\dagger = A^{-1}}$

$$\xrightarrow{\text{normal}} A A^\dagger = A A^{-1} = \underline{\underline{I}} = \underline{\underline{A}}^{-1} \underline{\underline{A}} = \underline{\underline{I}}$$

$$Ax = \lambda x$$

A is unitary → $A^\dagger = A^{-1}$

$$x^\dagger x = x^\dagger \underline{\underline{I}} x$$

$$= x^\dagger A^{-1} A x$$

$$= x^\dagger A^\dagger A x$$

$$Ax = \lambda x$$

$$(Ax)^\dagger = \lambda^* x^\dagger$$

$$x^\dagger x = (Ax)^\dagger (Ax)$$

$$= \lambda^* x^\dagger \lambda x$$

$$\cancel{x^\dagger x} = \lambda^* \lambda \cancel{x^\dagger x}$$

$$\lambda^* \lambda = |\lambda|^2 = 1$$

The eigenvalues of a unitary matrix have unit modulus

(Arbitrary)
$N \times N$ square matrix (arbitrary) $\longrightarrow$ no general properties for the eigen problem.

Back to normal matrices. A, B are normal.

Question: When do two different normal matrices have the common set of eig vectors??

Answer: If $AB = BA$ (i.e. they commute)

$$A x^i = \lambda_i x^i \qquad (\lambda_i\text{s are distinct})$$

$$A B x^i = B A x^i$$

$$A B x^i = B \lambda_i x^i \quad \longrightarrow \quad Bx^i \text{ is also an eig vector of } A \text{ corresponding to eig value } \lambda_i$$

$$A(Bx^i) = \lambda_i (B x^i) \quad \longleftrightarrow \quad A x^i = \lambda_i x^i$$

$$\boxed{Bx^i = \mu_i x^i}$$

eig values of B are the same as A

Adjoint operator: $\boxed{A^*}$

$$\langle v, A u \rangle \equiv \langle A^* v, u \rangle$$

definition of the adjoint operator A^* of A

in a given basis. $\underline{v}^\dagger \underline{\underline{A}}\, \underline{u} = (\underline{\underline{A}}^* \underline{v})^\dagger \underline{u}$

$$\underline{\underline{A}} = (\underline{\underline{A}}^*)^\dagger \qquad \underline{v}^\dagger \underline{\underline{A}}\, \underline{u} = \underline{v}^\dagger (\underline{\underline{A}}^*)^\dagger \underline{u}$$

$$(\underline{\underline{A}})^\dagger = ((\underline{\underline{A}}^*)^\dagger)^\dagger \longrightarrow \boxed{A^\dagger = A^*}$$

$$\langle v, Au \rangle = \langle A^* v, u \rangle.$$

$$A^* = A^\dagger$$ → Hermitian conjugate of $\underline{\underline{A}}$.

If $\underline{\underline{A}} = \underline{\underline{A}}^\dagger$ → Hermitian operator

↕

$\underline{A} = \underline{\underline{A}}^*$ → Self-adjoint operator.

Adjoint vs Adjugate.

~~Adjoint~~ : Adjugate

↓

Transpose of the matrix of cofactors

$$\langle v, Au \rangle = \langle A^* v, u \rangle$$

$$A^* = A^\dagger$$

Similarity Transformation:

$$y = \underset{=}{A}\, x$$

Matrix $(N \times N)$ in a given basis

$$\langle \underline{e}_i | \underset{=}{A} | \underline{e}_j \rangle = A_{ij}$$

$$\underline{\underline{A}} \quad \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}$$

$\underset{=}{A}$ in a basis s.t. the $\underline{\underline{A}}$ corresponding to $\underset{=}{A}$ is diagonal ?

eigen basis

$$\begin{pmatrix} \times & & & \\ & \times & & \\ & & \times & \\ & & & \times \end{pmatrix}$$

$$\langle x_j | \underset{=}{A}\, x_i \rangle = \underline{\underline{A}}_{ij} \qquad \underset{=}{A}\, x_i = \lambda_i x^i$$

$$\langle x_j | \lambda_i x_i \rangle = \lambda_i \langle x_j | x_i \rangle$$

$$= \lambda_i \delta_{ij} = A_{ij}$$

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_N \end{pmatrix}$$

Similarity Transformation

$\{\underline{e}_i\}$ in N-dim space

$i = 1 \cdots N$

$$\underline{x} = x_1 \underline{e}_1 + x_2 \underline{e}_2 + \cdots + x_N \underline{e}_N$$

$$\underline{x} \xrightarrow[\text{given basis}]{\text{in the}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$\{\underline{e}_i\}$ orthonormal

Change the basis: to $\{\underline{e}'_i\}$

$$\underline{e}'_j = \sum_{i=1}^{N} c_i^j \underline{e}_i$$

$$c_k^j = \langle \underline{e}_k | \underline{e}'_j \rangle$$

$$\boxed{\underline{e}'_j = \sum_i S_{ij} \underline{e}_i}$$

$$\langle \underline{e}_k | \underline{e}'_j \rangle = \sum_i S_{ij} \overbrace{\langle \underline{e}_k | \underline{e}_i \rangle}^{\delta_{ki}}$$

$$\boxed{\langle \underline{e}_k | \underline{e}'_j \rangle = S_{kj}}$$

$$\underline{x} = \sum x_i \underline{e}_i = \sum x'_j \underline{e}'_j \qquad (N\text{-dim space})$$

$$\sum x_i \underline{e}_i = \sum_{j=1}^{N} x'_j \sum_{i=1}^{N} S_{ij} \underline{e}_i$$

$$\sum_{i=1}^{N} \underline{e}_i x_i = \sum_{i=1}^{N} \underline{e}_i \boxed{\sum_{j=1}^{N} S_{ij} x'_j}$$

$$x_i = \sum_{j=1}^{N} S_{ij} x'_j \qquad \underline{x} = \underline{\underline{S}} \cdot \underline{x}'$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \qquad \begin{pmatrix} x'_1 \\ \vdots \\ x'_N \end{pmatrix}.$$

$$x' = S^{-1} x$$

$$\underline{e}'_j = \sum_{i=1}^{N} S_{ij} \underline{e}_i$$

$$\underline{x} = \underline{\underline{S}}\, \underline{x}' \qquad S_{ij} = \langle \underline{e}_i | \underline{e}'_j \rangle$$

Transformation matrix

$$y = A x$$

$$\boxed{\underline{y} = \underline{\underline{A}}\, \underline{x}}$$ ← original basis

$$\underline{\underline{S}}\, \underline{y}' = \underline{\underline{A}}\, \underline{\underline{S}}\, \underline{x}' \longrightarrow \underbrace{\underline{\underline{S}}^{-1} \underline{\underline{S}}}_{\underline{\underline{I}}}\, \underline{y}' = \underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}}\, \underline{x}'$$

$$\underline{y}' = \underbrace{\boxed{\underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}}}}_{\underline{\underline{A}}'}\, \underline{x}'$$

$$\underline{y}' = \underline{\underline{A}}'\, \underline{x}'$$ ← Transformed basis

$$\boxed{\underline{\underline{A}}' = \underline{\underline{S}}^{-1} \underline{\underline{A}}\, \underline{\underline{S}}}$$

Similarity transformat.

$$\boxed{\underline{\underline{A}} = \underline{\underline{S}}\, \underline{\underline{A}}'\, \underline{\underline{S}}^{-1}}$$

$$A \begin{cases} \underline{\underline{A}} \\ \underline{\underline{A}}' \end{cases}$$

Examples:

a) If $\underline{\underline{A}} = \underline{\underline{I}}$, $\underline{\underline{A}}' = \underline{\underline{S}}^{-1} \underline{\underline{I}} \, \underline{\underline{S}}$

$$= \underline{\underline{S}}^{-1} \underline{\underline{S}}$$

$$\underline{\underline{I}} = \underline{\underline{I}}$$

b) $\det \underline{\underline{A}} = \det \underline{\underline{A}}'$

$$|\underline{\underline{A}}'| = |S^{-1} A S| = |S^{-1}| \, |A| \, |S|$$

$$= |A| \, \underbrace{|S^{-1}| \, |S|}_{1}$$

$$\boxed{|A'| = |A|}$$

c) charac. eqn. of a matrix

$$|A' - \lambda I| = |S^{-1} A S - \lambda I|$$

$$= |S^{-1} (A - \lambda I) S|$$

$$= |S^{-1}| \, |(A - \lambda I)| \, |S|$$

$$|A' - \lambda I| = |A - \lambda I|$$

d) $\mathrm{Tr}\, A = \mathrm{Tr}\, A'$ — Invariants of the matrix

Unitary Transformation: S is unitary $S^{\dagger} = S^{-1}$

$$A' = S^{-1} A S = S^{\dagger} A S$$

original basis is orthogonal

$$\langle \underline{e}_i' | \underline{e}_j' \rangle = \left\langle \sum_k S_{ki} \, \underline{e}_k \,\middle|\, \sum_r S_{rj} \, \underline{e}_j \right\rangle$$

$$= \sum_k S_{ki}^{*} \sum_r S_{rj} \underbrace{\langle \underline{e}_k | \underline{e}_r \rangle}_{\delta_{kr}}$$

If the original basis is orthogonal, so is the transformed basis

$$= \sum_k S_{ki}^{*} S_{kj} = (S^{\dagger} S)_{ij} = (S^{-1} S)_{ij}$$

$$\boxed{\langle \underline{e}_i' | \underline{e}_j' \rangle = \delta_{ij}}$$

$$A' = S^{-1} A S$$

$$A x^i = \lambda_i x^i \qquad \underline{\underline{S}} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_N \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

columns of $\underline{\underline{S}}$ are eig vectors of $\underline{\underline{A}}$

S_{ij} = i^{th} comp of the j^{th} eig vec. $\underline{x}^j$ of $\underline{\underline{A}}$.

$$\boxed{S_{ij} = (\underline{x}^j)_i}$$

$$(S^{-1} A S)_{ij} = \sum_k \sum_l (S^{-1})_{ik} \, A_{kl} \, S_{lj}$$

$$= \sum_k \sum_l (S^{-1})_{ik} \, \underbrace{A_{kl} \, (\underline{x}^j)_l}_{\lambda_j \underline{x}^j{}_k}$$

$$= \sum_k \sum_l (S^{-1})_{ik} \lambda_j S_{kj} = \lambda_j \delta_{ij}$$

$$\underline{\underline{A'}} = \underline{\underline{S}}^{-1} \underline{\underline{A}} \, \underline{\underline{S}}$$

$$\underline{\underline{S}} = \begin{pmatrix} | & | & & | \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_N \\ | & | & & | \end{pmatrix}$$

$$\underline{\underline{A'}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

Diagonalization of a matrix:

$$\underline{\underline{A'}} = \underline{\underline{S}}^{-1} \underline{\underline{A}} \, \underline{\underline{S}}$$

$\underline{\underline{S}}$ → eig vectors of $\underline{A}$

$$= \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$