Differential Equations - Part 4: Separation of Variables and Finite Fourier Transform
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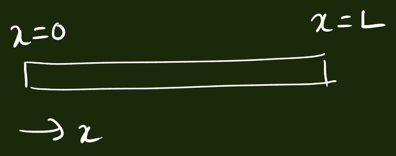
linear ODE's and PDE's "Separation of variables"

Separation of variables: $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ $0 \le x \le L$. $t > 0$



unsteady heat conduction equation.

$\dfrac{\partial^2 T}{\partial x^2}$ → 2nd order in x → 2 B.C's

$\dfrac{\partial T}{\partial t}$ → 1st order in t → 1 I.C

$T(x, t=0) = T_0(x)$

$$\left.\begin{aligned} T(x=0,t) &= T_a \\ T(x=L,t) &= T_a \end{aligned}\right\} t > 0$$

→ inhomog. B.C's

To find $T(x,t)$.

$y = T - T_a$

↓

$y(x,t)$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\alpha}\frac{\partial y}{\partial t}$$

α → thermal diffusivity

$$\left.\begin{aligned} y(x=0,t) &= 0 \\ y(x=L,t) &= 0 \end{aligned}\right\} \text{B.C's}$$

$y(x,t=0) = T_0(x) - T_a = y_0(x)$

Separation of variables: $y(x,t) = X(x)\,\Theta(t)$ (postulate / guess)

Substitute in the PDE

$$\frac{\partial^2}{\partial x^2}\left(X(x)\,\Theta(t)\right) = \frac{1}{\alpha}\frac{\partial}{\partial t}\left(X(x)\,\Theta(t)\right)$$

$$\Theta(t)\,\frac{\partial^2 X}{\partial x^2} = \frac{1}{\alpha}\,X(x)\,\frac{\partial \Theta(t)}{\partial t}$$

Divide by $\Theta(t) \cdot X(x)$

$$\frac{1}{X(x)}\frac{\partial^2 X(x)}{\partial x^2} = \frac{1}{\alpha}\,\frac{1}{\Theta(t)}\frac{\partial \Theta}{\partial t}$$

fn. only of x ←

→ fn. only of t

Both sides must equal a constant.

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{\alpha}\frac{1}{\Theta}\frac{d\Theta}{dt} = -\lambda$$

$$\frac{d\Theta}{dt} = -\alpha\lambda t$$

$$\Theta(t) = \Theta(0)\, e^{-\alpha\lambda_n t}$$

BC: $y(x=0,t) = X(x=0)\,\Theta(t) = 0$

$X(x=0) = 0$

$y(x=L,t) = 0 \longrightarrow X(x=L) = 0$

I.C.: $X(x)\,\Theta(t=0) = y_0(x)$

eig value proble!!

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda \longrightarrow \boxed{\frac{d^2X}{dx^2} + \lambda X = 0 \qquad X(0) = 0 \quad X(L) = 0}$$

$X = 0 \to$ trivial soln.

$$X(x) = C_1 \sin\sqrt{\lambda}\,x + C_2 \cos\sqrt{\lambda}\,x$$

At $x=0$, $X(x=0) = 0 \to C_1 \cdot 0 + C_2 \cdot 1 = 0$

$\to C_2 = 0$

At $x=L$, $X(x=L) = 0 \qquad \sin(\sqrt{\lambda}L) = 0$

$$\sqrt{\lambda}L = \pm n\pi \qquad n = \cancel{0}, 1, 2, \ldots$$

For nontrivial soln. $\quad \lambda_n L^2 = n^2\pi^2$ eig values!

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

$$X_n(x) = \sin\sqrt{\lambda_n}\,x = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, 4 \ldots$$

eig fns

no need to have -ve values

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = 0 \quad m \neq n$$

$$\boxed{y(x,t) = \Theta(t)\,X(x) = \sum_{n=1}^{\infty} A_n \sin\sqrt{\lambda_n}\,x\; e^{-\alpha\lambda_n t}}$$

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

I.C: $y(x, t=0) = y_0(x) = \sum_{n=1}^{\infty} A_n \sin(\sqrt{\lambda_n}\, x) \cdot 1$

known

Find A_n's $\rightarrow$ orthonormality.

Eigenfunction expansion or Finite Fourier Transform.

self-adj second order operator $\mathcal{L}$ with B.C's.

eigvalue problem: $\mathcal{L}\phi = -\lambda\, s(x)\, \phi$

$B[\phi] = 0$

infinite set real eigvalues

$\hookrightarrow$ for each distinct eigvalue $\rightarrow \phi_n(x)$

$\langle \phi_n, \phi_m \rangle = \delta_{nm}$

$$\int_a^b s(x)\, \phi_n(x)\, \phi_m(x)\, dx = \delta_{nm}$$

"weighting fn"

Any fn in that domain (a,b)

$$\boxed{f(x) = \sum_{i=1}^{\infty} c_i\, \phi_i(x)}$$

$\rightarrow$ generalized Fourier series

To find c_i's: use orthonormality

$$c_i = \int_a^b s(x)\, f(x)\, \phi_i(x)\, dx$$

$$f(x) = \sum_{i=1}^{\infty} \left[\int_a^b s(z)\, f(z)\, \phi_i(z)\, dz \right] \phi_j(x)$$

$f(x) \longleftrightarrow c_i$'s

Solve BVP's involving ODE's :

example
$$\boxed{\frac{d^2y}{dx^2} - \upsilon^2 y = -f(x)} \qquad 0 \le x \le L.$$

$y(0) = 0$

$y(L) = 0$

$$\boxed{L_1 \equiv \frac{d^2}{dx^2}}$$

self-adj

$$L_1 y - \upsilon^2 y = -f(x)$$

eig value prob of L_1 : $\quad L_1 \phi_i = -\lambda_i \phi_i$

$\phi_i(0) = 0$

$\phi_i(L) = 0$

$$\lambda_n = \frac{n^2\pi^2}{L^2} \; ; \; n = 1, 2, \ldots$$

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).$$

$\langle u, Lv \rangle = \langle Lu, v \rangle$

$f(x) = \sum b_n \phi_n$

$$L_1 y - \upsilon^2 y = -f(x)$$

Multiply by $\phi_n(x)$ and $\int_0^L dx$:

$$\int_0^L \phi_n L_1 y \, dx - \upsilon^2 \underbrace{\int_0^L \phi_n(x) y \, dx}_{y_n} = - \underbrace{\int_0^L f(x) \phi_n(x) \, dx}_{b_n}$$

$$\int_0^L y \, L_1 \phi_n \, dx \qquad L_1 \phi_n \to -\lambda_n \phi_n$$

$$\int_0^L y(-\lambda_n \phi_n) \, dx = -\lambda_n \underbrace{\int_0^L y(x) \phi_n(x) \, dx}_{y_n}$$

$$-\lambda_n y_n - \upsilon^2 y_n = -b_n \qquad y_n = \frac{b_n}{\lambda_n + \upsilon^2}$$

$$y(x) = \sum_{n=1}^{\infty} y_n \, \phi_n(x) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n + v^2} \, \phi_n(x)$$

$$y(x) = \frac{2}{L} \sum_{n=1}^{\overset{N}{\infty}} \frac{1}{v^2 + n^2 \frac{\pi^2}{L^2}} \left[\int_0^L f(z) \sin\left(\frac{n\pi z}{L}\right) dz \right] \sin \frac{n\pi x}{L} .$$