Using implicit form at the surface node ($m=0$)

$$q_{rad} + k \frac{T_1^{p+1} - T_0^{p+1}}{\Delta x} = \rho \frac{\Delta x}{2} c \frac{T_0^{p+1} - T_0^{p}}{\Delta t}$$

or

$$(1+2F_0) T_0^{p+1} - 2 F_0 T_1^{p+1} = \frac{2\alpha q_{rad} \Delta t}{k \Delta x} + T_0^{p}$$

For any interior node $1 \leq m \leq M$.

$$(1+2F_0) T_m^{p+1} - F_0 (T_{m-1}^{p+1} + T_{m+1}^{p+1}) = T_m^{p}$$

At node $M+1$ $\quad T_{M+1}^{p+1} = 20°C$ (boundary condition at $x \to \infty$)

Thus, we have to solve M equations simultaneously. Equations are linear in T_m^{p+1}.

Let

$$\underline{\underline{A}} = \begin{bmatrix} (1+2F_0) & -2F_0 & 0 & 0 & 0 & 0 \\ -F_0 & (1+2F_0) & -F_0 & 0 & 0 & 0 \\ 0 & -F_0 & (1+2F_0) & -F_0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & 0 & -F_0 & (1+2F_0) & -F_0 \\ \cdots & \cdots & 0 & 0 & -F_0 & (1+2F_0) \end{bmatrix}$$

$$\vec{C}^p = \begin{bmatrix} \frac{2\alpha q_{rad} \Delta t}{k \Delta x} + T_0^p \\ T_1^p \\ T_2^p \\ \vdots \\ T_M^p + F_0 T_{M+1}^{p+1} \end{bmatrix}$$

where $p \equiv$ time (discrete) index.

Initial condition

$$T(1:M, p=0) = 20^\circ C \qquad \text{i.e. } \vec{T}^{\,1} = \begin{bmatrix} 20 \\ 20 \\ \vdots \\ 20 \end{bmatrix}_{M\times 1}$$

Then,

$$\vec{T}^{\,p+1} = \underline{\underline{A}}^{-1} \cdot \vec{C}^{\,p} \quad \Big\} \text{ Solve in Matlab}$$

After 2 mins,

$$T_0 = 114.7^\circ C \qquad T_2 = 44.2^\circ C \quad \text{for } Fo = 1/2$$

Note: Since implicit method is unconditionally stable, Fo can exceed 1/2 with no numerical stability issues. Note that from the definition of Fo,

$$\Delta t = \frac{Fo\,\Delta x^2}{\alpha}$$. Thus,

Fo	1/2	2.5
Δt	~24s	~120s

Thus, a single timestep of 120s can be used (i.e. Fo=2.5) to find T_0 and T_2 at 2 mins.

On page 76, the analytical solution (by approximating the thick slab to be a semi-infinite solid) is

$$T - T_i = \frac{q_0}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\,\exp\left(\frac{-x^2}{4\alpha t}\right) - x\,\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right]$$

Thus,

$$T_0(x=0, t=120\,s) = 120^\circ C$$

$$T(x=0.15\,m, t=120\,s) = 45.4^\circ C$$