Linear Algebra: Inner Products, Linear Operators...

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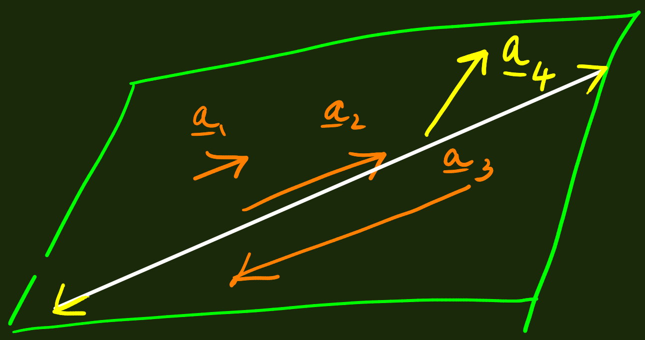
Linear indep: (LI)

$$\underline{a}_1 = \alpha_2 \, \underline{a}_2$$
$$= \alpha_3 \, \underline{a}_3$$

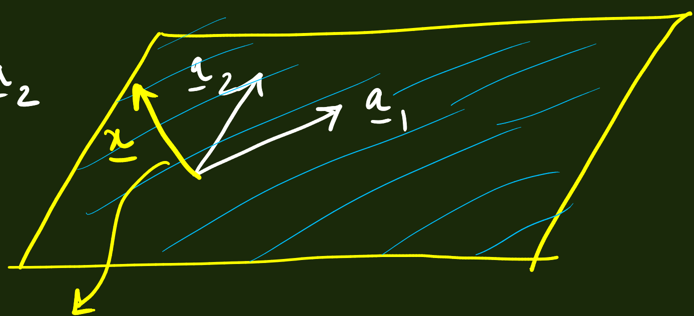


Cannot express $\underline{a}_4$ in terms of $\underline{a}_1, \underline{a}_2, \underline{a}_3$.

Span: $\underline{x} = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2 + \alpha_3 \underline{a}_3$

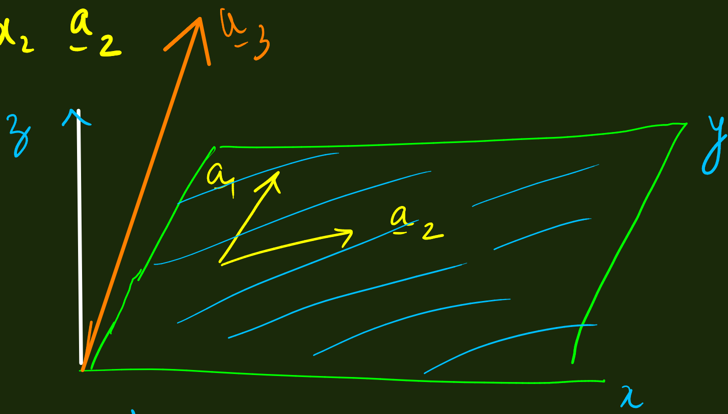
Span: $\underline{x} = \alpha_1 \underline{a}_1 + \alpha_2 \underline{a}_2$

$\underline{a}_1, \underline{a}_2$: not collinear



$\underline{a}_1, \underline{a}_2$ LI.

$$\underline{x} = x_1 \underline{a}_1 + x_2 \underline{a}_2$$



Can $\underline{a}_3$ be expressed in terms of $\underline{a}_1, \underline{a}_2$ — No !!

Linear independence, Lin dependence, Span, Basis

Inner products / Scalar products (Dot products)

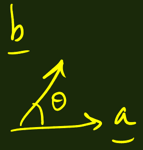
$\underline{a}, \underline{b} \xrightarrow{\text{inner pdt}}$ scalar

Some text books $(\underline{a}, \underline{b})$

$\underline{a} \cdot \underline{b}$

Inner product $\langle \underline{a} | \underline{b} \rangle$ ✓ ;
of $\underline{a}, \underline{b}$

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta$$



Axioms on inner product $\langle \underline{a} | \underline{b} \rangle$:

(i) $$\langle \underline{a} | \underline{b} \rangle = \langle \underline{b} | \underline{a} \rangle^*$$

scalar → complex no. α

complex conj α^*

(ii) $$\langle \underline{a} | (\lambda \underline{b} + \mu \underline{c}) \rangle = \lambda \langle \underline{a} | \underline{b} \rangle + \mu \langle \underline{a} | \underline{c} \rangle$$

Corollary :

$$\langle (\lambda \underline{a} + \mu \underline{b}) | c \rangle = \langle \underline{c} | (\lambda \underline{a} + \mu \underline{b}) \rangle^*$$

$$= \left[\lambda \langle \underline{c} | \underline{a} \rangle + \mu \langle \underline{c} | \underline{b} \rangle \right]^*$$

$$= \lambda^* \langle \underline{c} | \underline{a} \rangle^* + \mu^* \langle \underline{c} | \underline{b} \rangle^*$$

$$\boxed{\langle (\lambda \underline{a} + \mu \underline{b}) | \underline{c} \rangle = \lambda^* \langle \underline{a} | \underline{c} \rangle + \mu^* \langle \underline{b} | \underline{c} \rangle}$$

$$\langle \lambda \underline{a} | \mu \underline{b} \rangle = \mu \langle \lambda \underline{a} | \underline{b} \rangle$$

$$= \mu \langle \underline{b} | \lambda \underline{a} \rangle^* = \mu \lambda^* \langle \underline{b} | \underline{a} \rangle^*$$

$$= \mu \lambda^* \langle \underline{a} | \underline{b} \rangle$$

Example: $\langle (\alpha \underline{a} + \beta \underline{b}) | (\gamma \underline{c} + \delta \underline{d}) \rangle$

$$= \gamma \langle (\alpha \underline{a} + \beta \underline{b}) | \underline{c} \rangle + \delta \langle (\alpha \underline{a} + \beta \underline{b}) | \underline{d} \rangle$$

$$= \gamma \langle \underline{c} | (\alpha \underline{a} + \beta \underline{b}) \rangle^* + \delta \langle \underline{d} | (\alpha \underline{a} + \beta \underline{b}) \rangle^*$$

$$= \gamma \left(\alpha \langle \underline{c} | \underline{a} \rangle + \beta \langle \underline{c} | \underline{b} \rangle \right)^* + \delta \left(\alpha \langle \underline{d} | \underline{a} \rangle + \beta \langle \underline{d} | \underline{b} \rangle \right)^*$$

$$= \alpha^* \gamma \langle \underline{c} | \underline{a} \rangle^* + \beta^* \gamma \langle \underline{c} | \underline{b} \rangle^* + \alpha^* \delta \langle \underline{d} | \underline{a} \rangle^* + \beta^* \delta \langle \underline{d} | \underline{b} \rangle^*$$

$$= \alpha^* \gamma \langle \underline{a} | \underline{c} \rangle + \beta^* \gamma \langle \underline{b} | \underline{c} \rangle + \alpha^* \delta \langle \underline{a} | \underline{d} \rangle + \beta^* \delta \langle \underline{b} | \underline{d} \rangle$$

$$\langle \alpha \underline{a} + \beta \underline{b} | \gamma \underline{c} + \delta \underline{d} \rangle$$

$$\underline{a} = \sum_{i=1}^{N} \alpha_i \underline{e}_i \qquad \underline{b} = \sum_{j=1}^{N} \beta_j \underline{e}_j$$

$\langle \underline{a} | \underline{b} \rangle$:

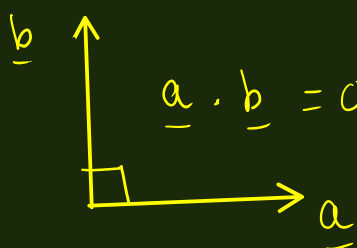
$$\left\langle \left(\sum_{i=1}^{N} \alpha_i \underline{e}_i \right) \middle| \left(\sum_{j=1}^{N} \beta_j \underline{e}_j \right) \right\rangle$$

$$\langle \underline{a} | \underline{b} \rangle = \sum_i \sum_j \alpha_i^* \beta_j \langle \underline{e}_i | \underline{e}_j \rangle$$

Orthogonality of two vectors:

$$\boxed{\langle \underline{a} | \underline{b} \rangle = 0}$$

N-dimensions



$$\underline{a} \cdot \underline{b} = 0$$

$$\underline{a} \cdot \underline{a} = |\underline{a}| |\underline{a}| = |\underline{a}|^2$$

$$|\underline{a}| = (\underline{a} \cdot \underline{a})^{1/2}$$

Norm of a vector:

$$\| \underline{a} \| = \langle \underline{a} | \underline{a} \rangle^{1/2}$$

Basis vectors $\{\underline{e}_i\}$ $i = 1 \dots N$ are orthogonal

$$\langle \underline{e}_i | \underline{e}_j \rangle = \delta_{ij} \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

orthonormal

$$\| \underline{a} \| = \langle \underline{a} | \underline{a} \rangle^{1/2}$$

$$\langle \underline{a} | \underline{a} \rangle = \sum_i \sum_j \alpha_i^* \alpha_j \underbrace{\langle \underline{e}_i | \underline{e}_j \rangle}_{\delta_{ij}}$$

$$\langle \underline{a} | \underline{a} \rangle = \sum_i \sum_j \alpha_i^* \alpha_j \delta_{ij}$$

$$\langle \underline{a} | \underline{a} \rangle = \sum_{i=1}^{N} \alpha_i^* \alpha_i = \sum_{i=1}^{N} |\alpha_i|^2$$

Norm of vector $\underline{a}$
$$\| \underline{a} \| = \langle \underline{a} | \underline{a} \rangle^{1/2} = \left(\sum_{i=1}^{N} |\alpha_i|^2 \right)^{1/2}$$

Orthonormal basis:

$$\underline{a} = \sum_{i=1}^{N} a_i \underline{e}_i \qquad \langle \underline{e}_i | \underline{e}_j \rangle = \delta_{ij}$$

What is a_j?

$$\langle \underline{e}_j | \underline{a} \rangle = \sum_{i=1}^{N} a_i \langle \underline{e}_j | \underline{e}_i \rangle$$

$$= \sum_{i=1}^{N} a_i \delta_{ij}$$

$$\underline{a} = \sum_i a_i \underline{e}_i$$

$$\underline{b} = \sum_j b_j \underline{e}_j$$

$$\boxed{\langle \underline{e}_j | \underline{a} \rangle = a_j} \qquad a_i = \langle \underline{e}_i | \underline{a} \rangle$$

$$\langle \underline{a} | \underline{b} \rangle = \sum_i \sum_j a_i^* b_j \underbrace{\langle \underline{e}_i | \underline{e}_j \rangle}_{\delta_{ij}} \rightarrow \boxed{\langle \underline{a} | \underline{b} \rangle = \sum_j a_j^* b_j}$$

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \leftarrow \text{dot}$$