ChE641 Mathematical Methods in Chemical Engineering

Assignment 5 **Due Date: 09 November 2020**
Linear Algebra

1. Use the Gram-Schmidt orthonormalization procedure to obtain three orthonormal vectors $\mathbf{x}_i$ from $[2, 1, 1]^T$, $[1, 2, 1]^T$ and $[1, 1, 2]^T$. Verify that their inner products satisfy orthonormality after orthonormalization, i.e. $\mathbf{x}_i^\dagger \mathbf{x}_j = \delta_{ij}$.

   Show by direct summation that
$$\sum_{i=1}^{3} \mathbf{x}_i \mathbf{x}_i^\dagger = \delta_{ij}$$

2. Consider the set $[2, 1, 1]^T$, $[1, 2, 1]^T$ and $[1, 1, 2]^T$. Show that this set of vectors is linearly independent, but not orthogonal. Form the reciprocal basis $\mathbf{z}_i$ from this set of vectors using the inverse of $\mathbf{X}$ which contains $\mathbf{x}_i$ as its column vectors. Show that $\mathbf{x}_i^\dagger \mathbf{z}_j = \mathbf{z}_i^\dagger \mathbf{x}_j = \delta_{ij}$. Further show that
$$\sum_{i=1}^{3} \mathbf{x}_i \mathbf{z}_i^\dagger = \mathbf{I}$$

3. Find the eigenvalues, eigenvectors, and reciprocal eigenvectors of matrix $\mathbf{A}$:
$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -2 & 2 & -1 \end{bmatrix}$$

   Verify that the reciprocal vectors $\mathbf{z}_i$ are the eigenvectors of $\mathbf{A}^\dagger$, the adjoint of $\mathbf{A}$. Also show by direct calculation that the spectral resolution of $\mathbf{A}$ is valid:
$$\sum_{i=1}^{3} \lambda_i \mathbf{x}_i \mathbf{z}_i^\dagger = \mathbf{A}$$

4. Use the spectral resolution theorem to prove the Cayley-Hamilton theorem for perfect matrices, which states that a perfect matrix $\mathbf{A}$ obeys its characteristic equation.

5. Given the nonsingular matrix $\mathbf{S}$:
$$\begin{bmatrix} -2 & 11 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

   (a) Find $\mathbf{S}^{-1}$. (b) Calculate the matrix $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$, where $\mathbf{A}$ is defined in the previous problem 3. (c) Compute the eigenvalues of $\mathbf{B}$. Are these the same as the eigenvalues of $\mathbf{A}$? (d) Find the eigenvectors of $\mathbf{B}$. Are these the same as eigenvectors of $\mathbf{A}$?

6. Compute $\exp[t\mathbf{A}]$ for the matrix $\mathbf{A}$:
$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

7. Using the spectral resolution theorem, find $\exp[\mathbf{A}t]$ for the matrix $\mathbf{A}$:

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

8. Solve the following system of ODEs by the method of eigenvalues and eigenvectors:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = 2x_1 + x_2$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -4x_1 - 3x_2$$

with the initial condition $x_1 = 1$ and $x_2 = 2$ at $t = 0$.