Linear Algebra - Part 6: Special Square Matrices, Orthonormal basis

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Diagonal matrix

(Square Matrices) $\rightarrow$ $(N \times N)$

$$\underline{\underline{A}} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$$

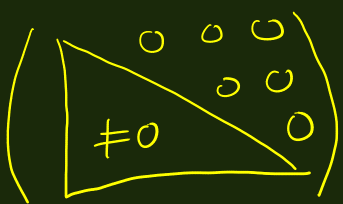
$\alpha, \beta, \gamma = 1 \rightarrow \underline{\underline{I}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\underline{\underline{A}} = \text{diag}(A_{11}, A_{22}, A_{33}, \ldots\ldots A_{NN})$$

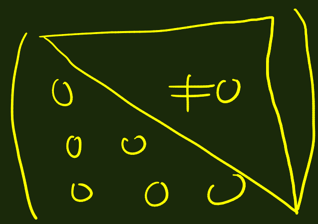
$$\underline{\underline{A}}^{-1} = \text{diag}\left(\frac{1}{A_{11}}, \frac{1}{A_{22}}, \ldots\ldots \frac{1}{A_{NN}}\right)$$

If A and B are diagonal matrices, then $AB = BA$ (Commute)

Lower and upper triangular matrices:



Lower triangular matrix



upper triangular matrix

If $\underline{\underline{A}}$ is lower/upper triangular, then $\boxed{\det \underline{\underline{A}} = A_{11} A_{22} A_{33} \ldots\ldots A_{NN}}$

Symmetric:

$\underline{\underline{A}} \rightarrow A_{ij}$ $\qquad \underline{\underline{A}}^T = A_{ji}$

$\boxed{\underline{\underline{A}} = \underline{\underline{A}}^T}$ $\qquad \boxed{A_{ij} = A_{ji}}$

$$\begin{array}{cc} & \begin{array}{ccc} & A_{12} & \end{array} \\ A_{21} & \begin{pmatrix} 2 & 1 & 4 \\ 1 & 3 & 5 \\ 4 & 5 & 6 \end{pmatrix} \end{array}$$

$A_{12} = A_{21}$

$A_{13} = A_{31}$

$A_{23} = A_{32}$

Skew (or) Anti Symmetric:

$\boxed{\underline{\underline{A}} = -\underline{\underline{A}}^T}$ or $\boxed{A_{ij} = -A_{ji}}$

$\rightarrow$ zeros in the diagonal

$$\begin{array}{cc} & \begin{array}{ccc} & A_{12} & A_{13} \end{array} \\ \begin{array}{c} \\ A_{21} \\ A_{31} \end{array} & \begin{pmatrix} 0 & 1 & 4 \\ -1 & 0 & 5 \\ -4 & -5 & 0 \end{pmatrix} \end{array}$$

$A_{11} = -A_{11} \rightarrow A_{11} = 0$

$A_{22} = -A_{22} \rightarrow A_{22} = 0$

$\vdots$

$A_{NN} = -A_{NN} \rightarrow A_{NN} = 0$

If $A = \pm A^T$

$$(A^{-1})^T = (A^T)^{-1} = \pm A^{-1}$$

$A = A^T$

$(A^T)^{-1} = A^{-1}$

antisymmetric

Any $N \times N$ matrix $\underline{\underline{A}} = \frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^T) + \frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^T)$

$A = -A^T$

Symmetric

$$\underline{\underline{A}} = \frac{\underline{\underline{A}}}{2} + \frac{\underline{\underline{A}}^T}{2} + \frac{\underline{\underline{A}}}{2} - \frac{\underline{\underline{A}}^T}{2}$$

$(A^T)^{-1} = -A^{-1}$

<u>Orthogonal Matrices.</u>

$\underline{\underline{A}}$ is orthogonal: $\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

$$(\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^T)^{-1} = (\underline{\underline{A}}^{-1})^{-1}$$

If $\underline{\underline{A}}$ is orthogonal, so is its inverse!!

$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

$$(\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^{-1})^{-1} = \underline{\underline{A}}$$

$$(\underline{\underline{A}}^{-1})^T = (\underline{\underline{A}}^{-1})^{-1}$$

$$\underline{\underline{B}}^T = \underline{\underline{B}}^{-1}$$

$\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$

$$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}^{-1} \underline{\underline{A}}$$

$$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$$

$$\det(\underline{\underline{A}}^T \underline{\underline{A}}) = 1$$

$\det \underline{\underline{A}}^T = \det \underline{\underline{A}}$

$$\det(\underline{\underline{A}}^T)\det(\underline{\underline{A}}) = 1$$

$$(\det \underline{\underline{A}})^2 = 1$$

or $\det \underline{\underline{A}} = \pm 1$ $\leftarrow$ orthogonal matrices orthogonal matrix :

let $\underline{\underline{A}}$: orthogonal

$\underline{y} \in$ real vector space

$$\underline{y} = \underline{\underline{A}} \cdot \underline{x} \quad \Big| \quad \underline{y}^T = \underline{x}^T \underline{\underline{A}}^T$$

$$\langle \underline{y} | \underline{y} \rangle = \underline{y}^T \cdot \underline{y}$$

$$= \underline{x}^T \underline{\underline{A}}^T \underline{\underline{A}} \, \underline{x}$$

$$= \underline{x}^T \underbrace{\underline{\underline{A}}^{-1} \underline{\underline{A}}}_{\underline{\underline{I}}} \, \underline{x}$$

$$\underline{y} = A \, \underline{x}$$

orthogonal

norm of $|\underline{y}|$ is not necessarily $= |\underline{x}|$

$$\underline{y}^T \underline{y} = \underline{x}^T \underline{x}$$

$$\boxed{\langle \underline{y} | \underline{y} \rangle = \langle \underline{x} | \underline{x} \rangle}$$

$$\underline{y} = \underline{\underline{A}} \, \underline{x}$$

" pure rotation "

orthogonal transfor.

Hermitian Matrix : $\underline{\underline{A}} = \underline{\underline{A}}^\dagger$

Anti-Hermitian : $\underline{\underline{A}} = -\underline{\underline{A}}^\dagger$

$$\underline{\underline{A}} = \frac{1}{2}(\underline{\underline{A}} + \underline{\underline{A}}^\dagger) + \frac{1}{2}(\underline{\underline{A}} - \underline{\underline{A}}^\dagger)$$

Analogue of orthogonal matrix $\longrightarrow$ "Unitary Matrix"

$$\underline{\underline{A}}^\dagger = \underline{\underline{A}}^{-1}$$

$$\det(A^\dagger A) = \det(A^\dagger)\det(A)$$

$$= |A|^* |A|$$

$$\det(A^\dagger A) = \left(\mathrm{Abs}\left(|A^*||A|\right)\right)^2$$

$\det A$ = unit modulus

Normal Matrices : $A A^\dagger = A^\dagger A$ — A normal matrix A commutes with its Hermitian conjugate.

examples : (1) Hermitian (Symmetric)

(2) unitary (orthogonal)

$A = A^T \rightarrow$ symmetric

$\boxed{\text{"Self-adjoint"}} \leftarrow A = A^\dagger \rightarrow$ Hermitian matrix

Basis sets in a normed linear vector space

$\underline{x} \in E_N$ $\quad$ E_N: N-dim Vec. space

$$\underline{x} = \sum_{i=1}^{N} x_i \, \underline{e}_i \qquad \underline{e}_i = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \leftarrow i^{th} \text{ row}$$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \cdots$$

$\{\underline{e}_1, \underline{e}_2, \underline{e}_3 \cdots \underline{e}_N\} \rightarrow$ basis set in E_N

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} + \cdots + x_N \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\underline{e}_i^{\dagger} \, \underline{e}_j$$

$$\langle \underline{e}_i | \underline{e}_j \rangle = \delta_{ij} \rightarrow \text{orthonormal basis}$$

Can also use any set of L.I vectors as a basis in E_N.

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \, \underline{x}_i \qquad \{\underline{x}_i\} \rightarrow \text{L.I}$$

$$[\underline{x}_1 \ \underline{x}_2 \cdots \underline{x}_N]$$

$$\boxed{\underline{x} = \underline{\underline{X}} \cdot \underline{\alpha}} \qquad \underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} \qquad \begin{bmatrix} | & | & & | \\ \underline{x}_1 & x_2 & \cdots & x_N \\ | & | & & | \end{bmatrix} = \underline{\underline{X}}$$

square

$$\underline{\underline{X}}^{-1} \cdot \underline{x} = \underbrace{\underline{\underline{X}}^{-1} \cdot \underline{\underline{X}}}_{\underline{\underline{I}}} \cdot \underline{\alpha}$$

$$\boxed{\underline{\alpha} = \underline{\underline{X}}^{-1} \cdot \underline{x}}$$

$\underline{\underline{X}}^{-1}$ exists

Gram-Schmidt Orthonormalisation

$\{\underline{z}_1, \underline{z}_2, \underline{z}_3 \cdots \underline{z}_N\}$: L.I vectors $\xrightarrow[\text{we}]{\text{Can}}$ a set of N orthonormal vectors ??

$\{\underline{y}_1 \cdots \underline{y}_N\}$ orthogonal

$$\underline{y}_1 = \underline{z}_1$$

$$\underline{y}_2 = \underline{z}_2 - \frac{(\underline{y}_1^\dagger \underline{z}_2)}{\|\underline{y}_1\|^2}\underline{y}_1$$

$\underline{a}, \underline{b}, \underline{c}$

$\underline{x}_1 = \underline{a}$

$\underline{x}_2 = \underline{b} - \left(\underline{b}\cdot\frac{\underline{x}_1}{|\underline{x}_1|}\right)\frac{\underline{x}_1}{|\underline{x}_1|}$

$\underline{x}_1\cdot\underline{x}_2 = 0$

$\underline{x}_1\cdot\underline{x}_2 = \underline{b}\cdot\underline{x}_1 - \frac{(\underline{b}\cdot\underline{x}_1)\cancel{(\underline{x}_1\cdot\underline{x}_1)}^{1}}{\cancel{|\underline{x}_1|^2}}$

$= 0$

$$\underline{y}_3 = \underline{z}_3 - \frac{(\underline{y}_1^\dagger \underline{z}_3)\underline{y}_1}{\|\underline{y}_1\|^2} - \frac{(\underline{y}_2^\dagger \underline{z}_3)\underline{y}_2}{\|\underline{y}_2\|^2}$$

$$\underline{y}_N = \underline{z}_N - \sum_{j=1}^{N-1}\frac{(\underline{y}_j^\dagger \underline{z}_N)}{\|\underline{y}_j\|^2}\underline{y}_j$$

$\underline{y}_1^\dagger \underline{y}_2 = 0 \quad \underline{y}_1^\dagger \underline{y}_3 = 0 \quad \underline{y}_2^\dagger \underline{y}_3 = 0 \qquad \underline{y}_j^\dagger \underline{y}_i = 0 \quad i \neq j$

$$\underline{x}_i = \frac{\underline{y}_i}{\|\underline{y}_i\|} \qquad i = 1 \cdots N \qquad \boxed{\underline{x}_i^\dagger \underline{x}_j = \delta_{ij}}$$

ortho-normal.

If $\{\underline{x}_1 \cdots \underline{x}_N\}$ form an orthonormal basis,

$$\underline{x} = \sum_{i=1}^{N}\alpha_i \underline{x}_i$$

$$\underline{x}_j^\dagger \underline{x} = \sum_{i=1}^{N}\alpha_i (\underline{x}_j^\dagger \underline{x}_i) \quad \delta_{ij}$$

$\alpha_i = \underline{x}_i^\dagger \underline{x}$

$$\boxed{\underline{x}_j^\dagger \underline{x} = \alpha_j}$$

$\underline{x} = \sum_i \alpha_i \underline{x}_i$

$= \sum_i \underline{x}_i \alpha_i$

$\underline{x} = \underbrace{\left(\sum_i \underline{x}_i \underline{x}_i^\dagger\right)}_{\underline{\underline{I}}}\underline{x}$

$$\boxed{\sum_i \underline{x}_i \underline{x}_i^\dagger = \underline{\underline{I}}}$$

Resolution of the identity in a given orthonormal basis.

If $\{\underline{x}_1, \underline{x}_2 \cdots \underline{x}_N\}$ is a nonorthonormal, but LI. $\longrightarrow x_i^\dagger x_j \neq \delta_{ij}$

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i$$

Find "Reciprocal basis" $\longrightarrow \{\underline{z}_1, \underline{z}_2 \cdots \underline{z}_N\}$ s.t

$$\underline{z}_i^\dagger \underline{x}_j = \delta_{ij}$$

geometry:

Three non-coplanar vectors: $\underline{a}, \underline{b}, \underline{c}$: $\longrightarrow$ $\underline{b} \times \underline{c}$

$$\hat{\underline{a}} = \frac{\underline{b} \times \underline{c}}{\underline{a} \cdot (\underline{b} \times \underline{c})} \qquad \underline{a} \cdot \hat{\underline{a}} = 1$$

$$\hat{\underline{b}} = \frac{\underline{a} \times \underline{c}}{\underline{b} \cdot (\underline{a} \times \underline{c})} \qquad \underline{b} \cdot \hat{\underline{b}} = 1$$

$$\hat{\underline{c}} = \frac{\underline{a} \times \underline{b}}{\underline{c} \cdot (\underline{a} \times \underline{b})} \qquad \underline{c} \cdot \hat{\underline{c}} = 1$$

$$\underline{a} \cdot \hat{\underline{b}} = \underline{a} \cdot \hat{\underline{c}} = \underline{c} \cdot \hat{\underline{a}} = \cdots \cdots = 0$$

If we have a LI nonorthogonal basis $\{\underline{x}_1, \underline{x}_2, \cdots \underline{x}_N)$

$$\underline{x} = \sum_i \alpha_i \underline{x}_i$$

Assume Reciprocal basis $\{\underline{z}_i\}$

$$\underline{z}_j^\dagger \underline{x} = \sum_i \alpha_i \underbrace{\underline{z}_j^\dagger \underline{x}_i}_{\delta_{ij}} \longrightarrow \boxed{\alpha_j = \underline{z}_j^\dagger \underline{x}}$$

$$\underline{x} = \sum \alpha_i \underline{x}_i$$

$$\underline{x} = \sum \underline{x}_i \alpha_i = \underbrace{\left(\sum \underline{x}_i \underline{z}_i^\dagger\right)}_{\underline{\underline{I}}} \underline{x} \qquad \boxed{\sum_i \underline{x}_i \underline{z}_i^\dagger = \underline{\underline{I}}}$$

How do find the Reciprocal basis:

$$\{\underline{x}_1,\ \underline{x}_2 \ldots \underline{x}_N\} \rightarrow \underline{\underline{X}} = \begin{bmatrix} | & | & & | \\ \underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_N \\ | & | & & | \end{bmatrix}$$

$\underline{\underline{X}}^{-1}$ exists

$$\underline{\underline{X}}^{-1}\underline{\underline{X}} = \underline{\underline{I}}$$

define $\boxed{\underline{\underline{Z}}^{\dagger} = \underline{\underline{X}}^{-1}}$ or $\underline{\underline{Z}} = \left(\underline{\underline{X}}^{-1}\right)^{\dagger}$

$$\underline{\underline{Z}}^{\dagger} = \begin{bmatrix} \underline{z}_1^{\dagger} \\ \underline{z}_2^{\dagger} \\ \vdots \\ \underline{z}_N^{\dagger} \end{bmatrix} = \underline{\underline{X}}^{-1}$$

$$\underline{\underline{Z}} = \begin{bmatrix} | & | & & | \\ \underline{z}_1 & \underline{z}_2 & \cdots & \underline{z}_N \\ | & | & & | \end{bmatrix}$$

$$\underline{\underline{Z}} = \begin{bmatrix} \underline{z}_1 & \cdots & \underline{z}_N \end{bmatrix}$$

$$\underline{\underline{Z}}^{\dagger}\underline{\underline{X}} = \begin{bmatrix} \underline{z}_1^{\dagger} \\ \underline{z}_2^{\dagger} \\ \vdots \\ \underline{z}_N^{\dagger} \end{bmatrix} \begin{bmatrix} \underline{x}_1, & \underline{x}_2 & \cdots & \underline{x}_N \end{bmatrix} = \langle \underline{z}_i | \underline{x}_j \rangle = \underline{\underline{X}}^{-1}\underline{\underline{X}} = \underline{\underline{I}}$$

$$\boxed{\underline{z}_i^{\dagger} | \underline{x}_j = \delta_{ij}}$$

$$\underline{z}_i^{\dagger}\underline{x}_j = \delta_{ij}$$

$$\underline{x} = \sum_{i=1}^{N} \alpha_i \underline{x}_i \qquad \langle \underline{z}_i | \underline{x}_j \rangle = \delta_{ij}$$