Complex Numbers and Analysis – Part 5
ChE641, IIT Kanpur

$W = f(z)$ — complex variable

$f(z) = (x, y)$ — real, imag $\rightarrow$ z, z^*

$W = f(x, y) \rightarrow F(z, z^*)$

$\boxed{\text{"Analytic fns"}}$

$W = f(z)$ NOT z^* ; $\boxed{\dfrac{\partial W}{\partial z^*} = 0}$

$W = u + iv$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann conditions

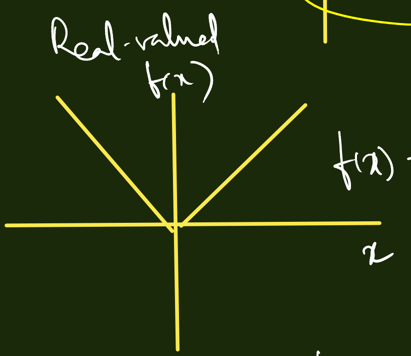
$f(z)$

$f(z)$ is analytic

↳ all the higher derivatives of $f(z)$ in the domain D are analytic.

Power Series representation is possible.

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + f''(z_0)\frac{(z-z_0)^2}{2!} + \cdots$$



$f(x) = |x|$

↳ no unique derivative at $x = 0$

$$f(x) = x|x| = \begin{cases} x^2 & x > 0 \\ -x^2 & x < 0 \end{cases}$$

$f'(x) = 2x \quad x > 0$

$= -2x \quad x < 0$

$f''(x) = 2 \quad x > 0$

$= -2 \quad x < 0$

$f(x) = x^2|x| \rightarrow$ the f', f'' will be continuous

$f(x), f'(x)$ are well-defined at $x = 0$!

$f''(x)$ is not continuous

Power Series of a Complex variable:

$$\rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z=0)$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad (z=z_0)$$

$$\downarrow z = re^{i\theta}$$

$$\rightarrow f(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta}$$

$$|f(z)| = \sum_{n=0}^{\infty} |a_n r^n| \, |e^{in\theta}| \qquad |e^{in\theta}| \leq 1$$

$$\rightarrow \sum_{n=0}^{\infty} |a_n| r^n \rightarrow \text{is this convergent ??}$$

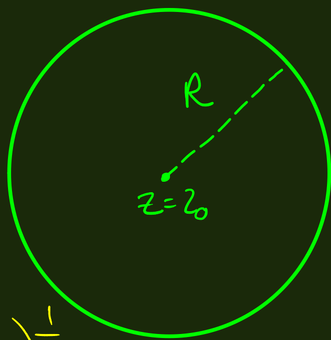
$$\boxed{\text{" Radius of Convergence " } R \qquad \frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

Convergent

$|z| < R$

Divergent

$|z| > R$



Example: (i) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

$$\frac{1}{R} = \lim_{n\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{n}}$$

$$\frac{1}{R} \rightarrow 0 \quad \text{Convergent}$$

or $\boxed{R \rightarrow \infty}$

$$\frac{1}{R} = \lim_{n\to\infty} \left(n^{-n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} n^{-1} \rightarrow 0$$

$n! = n(n-1)\cdots$

$\lim_{n\to\infty} n! \rightarrow n^n$

ii) $\sum_{n=0}^{\infty} n!\, z^n$

$$\frac{1}{R} = \lim_{n\to\infty} (n!)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} (n^n)^{\frac{1}{n}} = \lim_{n\to\infty} n \to \infty$$

not a convergent series

$$\frac{1}{R} \to \infty \Rightarrow \boxed{R \to 0}$$

iii) $\sum_{n=1}^{\infty} \frac{z^n}{n}$

$$\frac{1}{R} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$$

Series convergent if $|z| < R$

$|z| < 1$

Multivalued fns:

$$w = \ln z = \ln(re^{i\theta})$$

$$= \ln\left[r\, e^{i(\theta + 2n\pi)}\right]$$

$e^{2\pi i} = 1$

$e^{2\pi n i} = 1$

$n = 0, 1, 2, 3 \dots$

$$w = \ln r + \ln\left(e^{i(\theta + 2n\pi)}\right)$$

$$w = \ln r + i(\theta + 2n\pi) \qquad n = 0, 1, 2, 3 \dots$$

infinitely many valued.

Multivalued function:

Principal value $\ln z = \mathrm{Ln}(z) = \ln r + i\theta \quad (n=0)$

$\theta : \; 0 \le \theta < 2\pi$

2) t^z both t and z are complex.

$$t^z = \exp[z \ln t]$$

multivalued multivalued

3) t : complex n real $t^{\frac{1}{n}} = \exp\left[\frac{1}{n} \ln t\right]$

Is $t^{\frac{1}{n}}$ also infinitely many valued ??

$t = r e^{i(\theta + 2\pi k)}$ $k = 0, 1, 2, 3 \dots$

$$t^{\frac{1}{n}} = \exp\left[\frac{1}{n} \ln r + \frac{i}{n}(i\theta + 2\pi k)\right]$$

$$t^{\frac{1}{n}} = r^{\frac{1}{n}} \exp\left[i\frac{\theta}{n} + \frac{2\pi k i}{n}\right]$$

$k = 0, 1, 2, 3 \dots (n-1)$

Only n distinct roots for $t^{1/n}$

If $k = n \Rightarrow \frac{2\pi \cancel{k} i}{\cancel{n}} = 2\pi i$

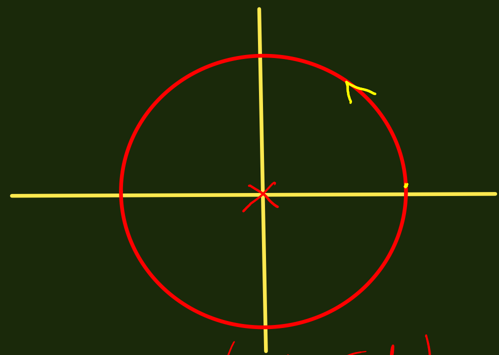
Branch Cut: / Branch point

Analytic fn $\rightarrow$ single-valued

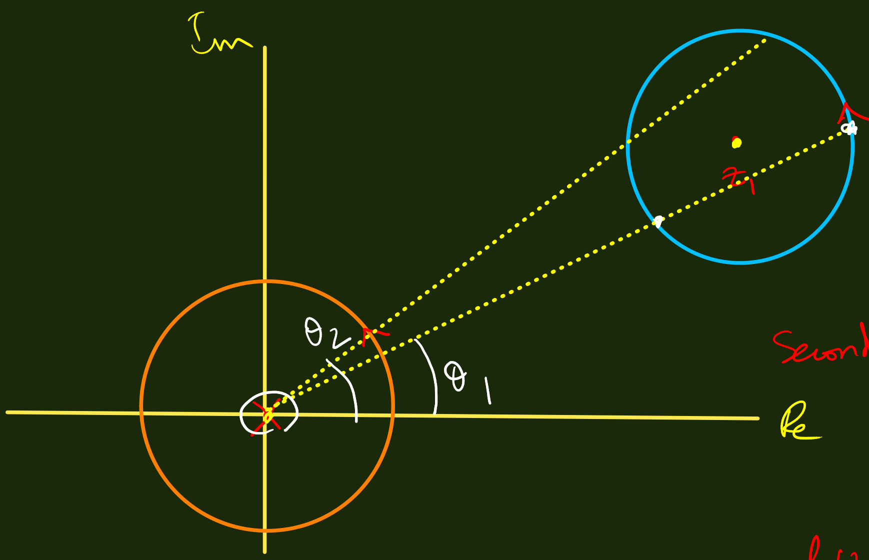
$z^{\frac{1}{n}}$, $\ln z$

multivalued

Branch point: If z is varied in the complex plane s.t. the path forms a closed curve, that encloses a branch point, then, in general, $f(z)$ will not return to its original value



$f(z) = z^{1/2}$ $z = re^{i\theta}$



$z = re^{i(\theta + 2\pi k)}$ $k = 0, 1, 2 ..$

$k = 0$, $z = re^{i\theta}$

$f(z) = r^{1/2} e^{i\theta/2}$ $\leftarrow$

Second round: $z = r e^{i(\theta + 2\pi)}$

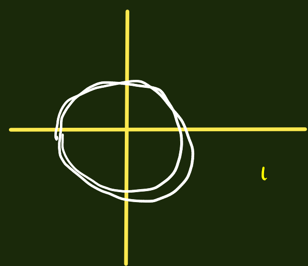
$z = re^{i\theta} e^{2\pi i}$

$f(z) = r^{1/2} e^{i\theta/2} \cancel{e^{i\pi}} \rightarrow -1$

$0 \leq \theta < 2\pi$ $f(z) = r^{1/2} e^{i\theta/2}$

$2\pi \leq \theta < 4\pi$ $f(z) = -r^{1/2} e^{i\theta/2}$

$4\pi \leq \theta < 6\pi$ $f(z) = r^{1/2} e^{i\theta/2}$



The Branch point of $f(z) = z^{1/2}$ is $z = 0$

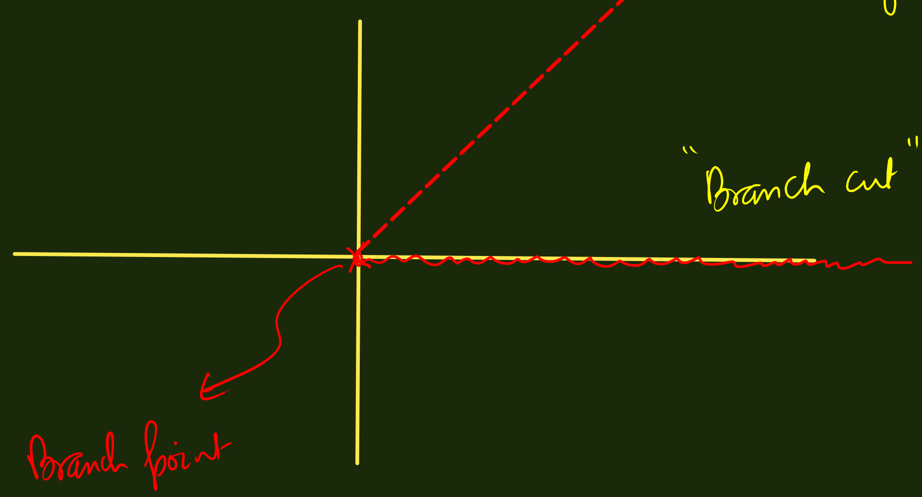
$f(z) = z^{1/n} \rightarrow$ n loops before you return to the original value of $f(z)$.

$f(z) = \ln z \longrightarrow$ never return!!

How to keep $f(z)$ single-valued ??

$f(z) = z^{1/2}$ or $z^{1/n}$

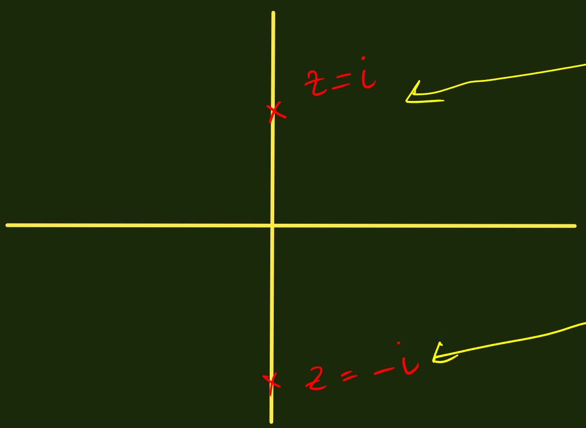
$\ln z$



$$f(z) = \sqrt{z^2+1} = \sqrt{(z+i)(z-i)}$$

$$= \sqrt{z+i}\ \sqrt{z-i}$$

$(z-z_0)^{1/2}$

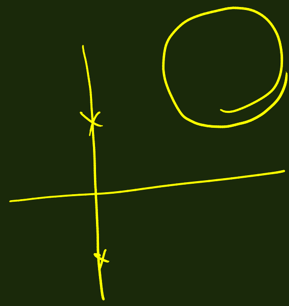


$z-i = r_1 e^{i\theta_1}$

$z+i = r_2 e^{i\theta_2}$

$$f(z) = (z+i)^{1/2}(z-i)^{1/2} \Rightarrow f(z) = \sqrt{r_1 r_2}\ e^{i\frac{\theta_1}{2}}\ e^{i\frac{\theta_2}{2}}$$

$$= (r_1 r_2)^{1/2}\ e^{i\frac{(\theta_1+\theta_2)}{2}}$$



$C \to$ closed contour.

1) if C encloses neither $z=+i$ or $z=-i$ $\longrightarrow$ $\theta_1 \to \theta_1$, $\theta_2 \to \theta_2$, $f(z) \to f(z)$.

2) C encloses $z=i$ but not $z=-i$ $\quad \theta_1 \to \theta_1 + 2\pi$, $\theta_2 \to \theta_2$

$$f(z) = -f(z)$$

3) C encloses only $z=-i$ not $z=i$ $\quad \theta_1 \to \theta_1$, $\theta_2 \to \theta_2 + 2\pi$

$$f(z) = -f(z)$$

4) C encloses both i & $-i$ $\quad f(z) \to f(z)$