Comlpex Numbers and Analysis - Part 6

ChE641, IIT Kanpur

Laurent Series Residue:

Singularities: * Removable Singularity $f(z) = \frac{\sin z}{z}$

$\lim_{z \to 0} f(z) = 1$ $f(z=0) = 1$

Simple pole. $f(z)$ near $z = a$

$$f(z) = \underbrace{\frac{C_{-1}}{(z-a)}}_{\text{"Singular part"}} + \underbrace{\sum_{n=0}^{\infty} C_n (z-a)^n}_{\text{"Regular" part (convergent)}}$$

↓ "principal part"

Residue: The coeff. of $\frac{1}{(z-a)}$ → important role.

Examples: $$\frac{\sin z}{z^2} = \underbrace{\frac{1}{z}}_{\text{singular part}} + \underbrace{\left(-\frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}_{\text{regular part}}$$

* Simple pole at $z = 0$
* Residue of $f(z)$ at $z = 0$ = 1

$\operatorname{cosec}(z) = \frac{1}{\sin z}$ $\sin z = 0$ at $z = n\pi$

Expand $\sin z$ about $z = n\pi$:

$$\sin z = (z - n\pi)\cos(n\pi) - \frac{1}{6}(z-n\pi)^3 \cos n\pi + \frac{1}{120}(z-n\pi)^6 \cos n\pi - \cdots$$

$$\sin z = (-1)^n (z - n\pi)\left[1 - \frac{1}{6}(z-n\pi)^2 + \cdots\right]$$

$$\operatorname{cosec}(z) = \frac{(-1)^n}{(z-n\pi)}\left[1 + \frac{1}{6}(z-n\pi)^2 - \cdots\right]$$

$$\operatorname{cosec}(z) = \underbrace{\frac{(-1)^n}{(z-n\pi)}}_{\text{singular}} + \underbrace{\left\{ \frac{1}{6}(-1)^n (z-n\pi) + \cdots \right\}}_{\text{regular}}$$

Simple pole at $z = n\pi$ $(n = 0, 1, 2 \ldots)$

Residue : $(-1)^n$

$$\text{Residue :} \quad \boxed{\operatorname*{Res}_{z=a} f(z) \equiv C_{-1} = \lim_{z \to a} \left[(z-a) f(z) \right]}$$

$$\frac{1}{(z-a)}$$

$$f(z) = \frac{g(z)}{h(z)} \qquad h(z) = \text{simple zero : } (z-a)$$

$$h(z) = h'(a)(z-a)$$

near $z = a$:

$$f(z) \simeq \frac{g(z)}{h'(a)(z-a)} \longrightarrow \operatorname{Res} f(z) = \lim_{z \to a} \left[\frac{\cancel{(z-a)}\, g(z)}{h'(a)\cancel{(z-a)}} \right]$$

$$\boxed{\operatorname{Res} f(z) = \frac{g(a)}{h'(a)}}$$

$$G(s) = \frac{p(s)}{q(s)}$$

Multiple or <u>Higher order poles</u> :

$$f(z) = \underbrace{\frac{C_{-m}}{(z-a)^{m}} + \frac{C_{-(m-1)}}{(z-a)^{m-1}} + \cdots + \frac{C_{-1}}{(z-a)}}_{\text{Singular}} + \underbrace{\sum_{n=0}^{\infty} C_n (z-a)^n}_{\text{Regular}}$$

<u>Residue</u>

m^{th} order pole at $z = a$

Essential singularity : The singular part does not truncate at finite m

Taylor's Series:

If $f(z)$ is analytic inside and on a circle C of radius R centered at z_0, then $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$.

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

If $f(z)$ has a singularity at z_0 inside C, then cannot use Taylor's Series.

Assume that $f(z)$ has a pole of order p within C. but is analytic otherwise.

$$g(z) = (z - z_0)^p f(z)$$

analytic inside C.

Taylor expand $g(z)$!

$$g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

$$f(z) = \frac{g(z)}{(z - z_0)^p} = \sum_{n=0}^{\infty} b_n (z - z_0)^{(n-p)}.$$

Laurent Series:

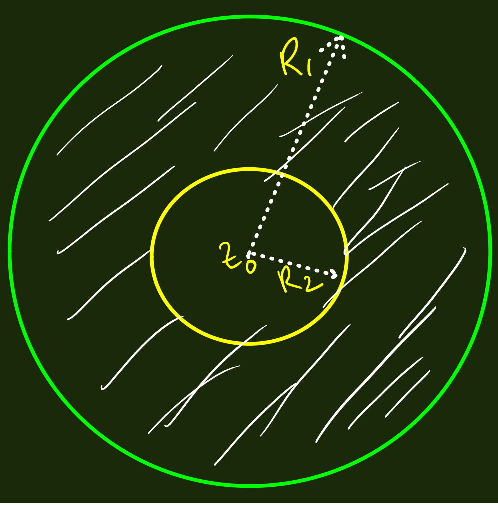
$$f(z) = \underbrace{\frac{a_{-p}}{(z - z_0)^p} + \dots + \frac{a_{-1}}{(z - z_0)}}_{\text{singular}} + \underbrace{a_0 + a_1(z - z_0) + a_2(z - z_0)^2}_{\text{regular part}} + \dots$$

$a_{-p} \neq 0$

The regular part converges for $|z| < R_1$

singular converges for $|z| > R_2$

If $R_2 > R_1 \rightarrow$ Series does not converge.



If $R_1 > R_2$, then the Laurent series will converge in an annular region.

$f(z) = \dfrac{1}{(z-z_0)^p} \longrightarrow p^{th}$ order pole

$f(z) = (z-z_0)^m \longrightarrow m^{th}$ order zero

In the Laurent Series if the singular part stops at p^{th} order... pole of order p

if the singular part does not converge $\longrightarrow$ Essential singularity.

Example : Laurent series for $\dfrac{1}{z(z-2)^3}$. about $z=0$ and $z=2$ & Find the Residue.

$$f(z) = \frac{-1}{8z\left(1-\frac{z}{2}\right)^3}$$

$$= \frac{-1}{8z}\left[1+(-3)\left(-\frac{z}{2}\right) + \frac{(-3)(-4)}{2!}\left(-\frac{z}{2}\right)^2 + \cdots\right].$$

$$= \frac{-1}{8z} - \frac{3}{16} - \frac{3z}{16} - \frac{5z^2}{32} - \cdots$$

First order pole ; at $z=0$; $\text{Res} = \dfrac{-1}{8}$

Laurent series about $z=2$: $z-2=\xi$

$$z = 2+\xi$$

$$f(z) = \frac{1}{(2+\xi)\xi^3} = \frac{1}{2\xi^3\left(1+\frac{\xi}{2}\right)}$$

$\xi = (z-2)$

$$= \frac{1}{2\xi^3}\left[1-\frac{\xi}{2}+\left(\frac{\xi}{2}\right)^2-\left(\frac{\xi}{2}\right)^3+\cdots\right]$$

Residue

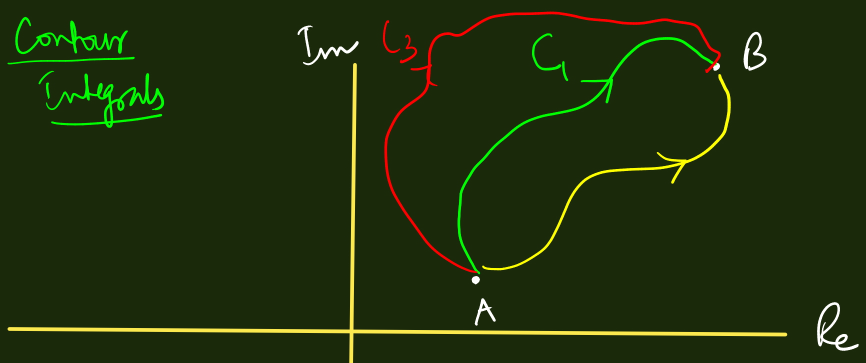
$$f(z) = \underbrace{\frac{1}{2\xi^3} - \frac{1}{4\xi^2} + \frac{1}{8\xi}}_{\text{singular part.}} \underbrace{- \frac{1}{16} + \frac{\xi}{2} - \cdots}_{\text{regular part}}$$

3rd order pole at $z=2$

Complex Integrals.

Contour Integrals

$f(z)$



A : z_1

B . z_2

$$I_1 = \int_{C_1\,A}^{B} f(z)$$

$$I_2 = \int_{C_2\,A}^{B} f(z)$$

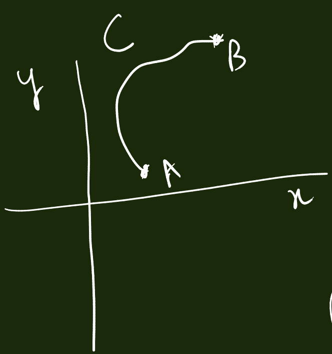
$I_1 = I_2$??

If the function is analytic, then the integral is path-independent !!

$$\int_C f(z)\, dz = \int_C (u + iv)\,(dx + i\,dy)$$

$f(z) = u + iv$

$$= \int_C u\,dx - \int_C v\,dy + i\int_C u\,dy + i\int_C v\,dx$$

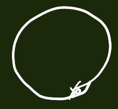


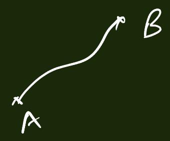
$x = x(t)$

$y = y(t)$

$$\int_C f(z)\,dz = \int_{\alpha}^{\beta} u \frac{dx}{dt}\,dt - \int_C v \frac{dy}{dt}\,dt + i\int u \frac{dy}{dt}\,dt + i\int v \frac{dx}{dt}\,dt$$

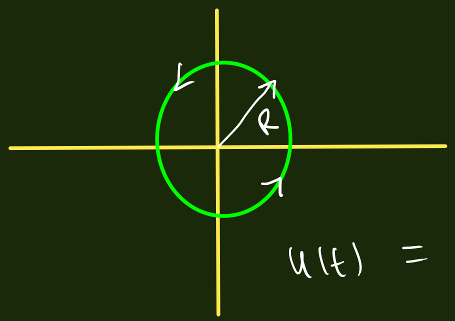
$f(z) = \frac{1}{z}$

 closed contours $\to \oint$



$\oint f(z)$ along $|z| = R$

$z(t) = R\cos t + i \sin t \qquad 0 \le t \le 2\pi$



$f(z) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$u = Re(f) = \frac{x}{x^2+y^2}$

$v = Im(f) = \frac{-y}{x^2+y^2}$

$u(t) = \frac{R\cos t}{R^2} = \frac{\cos t}{R}$

$v(t) = -\frac{\sin t}{R}$

$$\oint \frac{dz}{z} = \cancel{\int_0^{2\pi} \frac{\cos t}{R}(-R\sin t)\,dt - \int_0^{2\pi}\left(-\frac{\sin t}{R}\right) R\cos t\, dt}$$

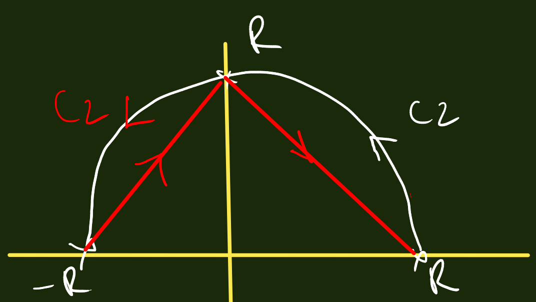
$$+ i \int_0^{2\pi} \frac{\cos t}{\cancel{R}} \cancel{R} \cos t\, dt + i \int_0^{2\pi} -\frac{\sin t}{\cancel{R}} (-\cancel{R}\sin t)\, dt$$

$$\oint \frac{dz}{z} = i \int_0^{2\pi} (\cos^2 t + \sin^2 t)\, dt = 2\pi i$$

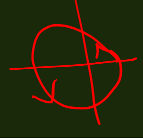
$$\boxed{\oint_{|z|=R} \frac{dz}{z} = 2\pi i}$$

$\to$ independent of R !!

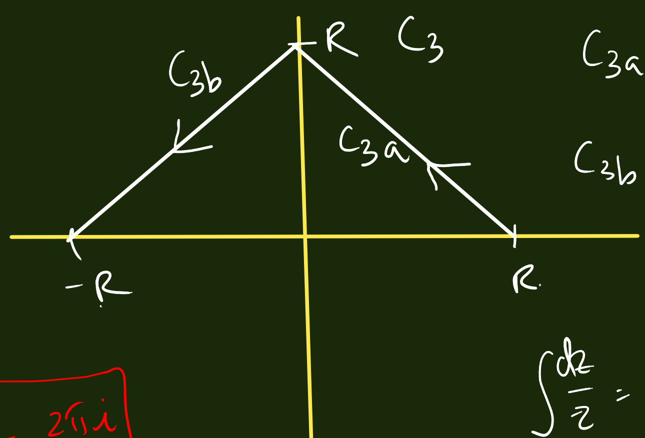
$0 \le t \le \pi \qquad \int_{C_2} \frac{dz}{z} = \pi i$



$$\boxed{\oint \frac{dz}{z} = 0 \text{ !!}}$$



$$\boxed{\oint \frac{dz}{z} = 2\pi i}$$



C_{3a}: $z = (1-t)R + itR$
$0 \le t \le 1$

C_{3b}: $-sR + i(1-s)R$
$0 \le s \le 1$

$$\int_{C_3} \frac{dz}{z} = \int_{C_{3a}} \frac{dz}{z} + \int_{C_{3b}} \frac{dz}{z} = \pi i$$