Differential Equations - Part 1
ChE641, IIT Kanpur

Navier-Stokes eqns,

unsteady heat eqn.

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + \sin x = 0$$

$y(x)$ → dependent variable

x : independent variable.

lin. eqn.

$x^3 - 3 = 0$

$x + 3 = 0$

$x = -3$

(i) O.D.E vs. PDE :

only one indep. variable $y(x)$

P.D.E

$T(x,t) \longrightarrow \frac{\partial T}{\partial t} = \alpha_T \frac{\partial^2 T}{\partial x^2}$ 1-D

3-D

$$\underbrace{T(x,y,z,t)}\quad \frac{\partial T}{\partial t} = \alpha_T \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

(ii) Linear vs. nonlinear

defn. of linearity

$$\mathcal{L}(\alpha_1\phi_1 + \alpha_2\phi_2) = \alpha_1 \mathcal{L}\phi_1 + \alpha_2 \mathcal{L}\phi_2$$

$\mathcal{L} = \frac{d^2}{dx^2}$ → linear operator

$\mathcal{L} = \left(\frac{d}{dx}\right)^2$

not a linear operator

$$\frac{d^4y}{dx^4} + \frac{d^2y}{dx^2} - y(x) = e^{-x}$$

indep variable

linear

nonlinear DE's : $\left(\frac{dy}{dx}\right)^4 + \frac{d^2y}{dx^2} = 0$

$$\frac{d^4y}{dx^4} + \sin(y) = 0$$ nonlinear ODE

(iii) Dependent variable vs. independent variable.

$T(\underbrace{x,y,z,t})$

Order of an ODE : The order of the highest derivative the D.E. contains

$$\frac{d^3 y}{dx^3} + \left(\frac{dy}{dx}\right)^4 = \sin x$$

not the order!!

order 3

3rd order, 1st degree

$$\left(\frac{d^2 y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 = 0$$

2nd order

2nd degree

$$\frac{d^2 y}{dx^2} = 0 \longrightarrow \frac{dy}{dx} = C_1 \longrightarrow y(x) = C_1 x + C_2$$

2 consts of integration.

fixed used physical constraints called I.C's (or) B.C's

General solution $\longrightarrow$ Constants of integration

B.C's or I.C's

Initial value problem (I.V.P)	vs	Boundary value problem (B.V.P)
$m\frac{d^2 x}{dt^2} = F$		$\frac{d^2 T}{dx^2} = f(x)$
Initial conditions $x(t=0) = x_0$, $\frac{dx}{dt}(t=0) = v_0$		$T(0)$ $\rightarrow x$ $T(L)$
		Boundary conditions

The general soln of an n^{th} order ODE $\longrightarrow$ n constants of integration

fixed BC's (or) IC's

"particular solution"

Linear ODE's of 1st order:

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

$P(x)$, $Q(x)$: known fns. of x

"Integrating factor" $\mu(x)$

$$\underbrace{\mu(x)\frac{dy}{dx} + \mu(x) P(x)\, y}_{\downarrow} = Q(x)\,\mu(x)$$

$$\frac{d}{dx}\left\{\mu(x)\, y\right\} = Q(x)\,\mu(x)$$

$$\frac{d\mu}{dx} = \mu P$$

$$\rightarrow \mu(x) = e^{\int P(x)\, dx}$$

$$\mu \frac{dy}{dx} + y\frac{d\mu}{dx} = \mu \frac{dy}{dx} + \mu P y$$

Integrate once:

$$\mu(x)\, y = \int_{x'=a}^{x'=x} \mu(x')\, Q(x')\, dx'$$

$$e^{\int P(x)\, dx} \leftarrow \mu(x)$$

<u>Example:</u>

$$\frac{dy}{dx} + 2x\, y = 4x$$

$P(x) = 2x$, $Q(x) = 4x$

$$\mu(x) = e^{\int 2x\, dx} = e^{x^2}$$

$$y\, e^{x^2} = \int^{x} 4x'\, e^{x'^2}\, dx + C$$

$$e^{x^2} y(x) = 2e^{x^2} + C$$

$$\boxed{y(x) = 2 + C\cdot e^{-x^2}}$$

Higher order linear ODE's:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x) = f(x)$$

$f(x)$ → known.

$f(x) = 0 \longrightarrow$ Homog. DE

$f(x) \neq 0 \longrightarrow$ Inhomog. DE.

$$\mathcal{L}y = f$$

(i) $\mathcal{L}y = 0 \longrightarrow$ Complementary fn. $y_c(x) = C_1 y(x) + C_2 y_2(x) + \dots C_n y_n(x)$

$y_1(x) \dots y_n(x)$ are L.I. ??

$C_1 y_1 + C_2 y_2 + \dots + C_n y_n = 0$ iff $C_i = 0$ $i = 1 \dots N$.

$C_1 y_1' + C_2 y_2' + \dots C_n y_n' = 0$

$\vdots$

$C_1 y_1^{n-1} + C_2 y_2^{n-1} + \dots C_n y_n^{n-1} = 0$

$$\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ \vdots & & & \\ y_1^{n-1} & & \dots & y_n^{n-1} \end{vmatrix} = 0$$

general soln to $\mathcal{L}y = f$

$\mathcal{L}$ → Linear operator

$$\boxed{y(x) = y_c(x) + y_p(x)}$$

y_p → any soln to $\mathcal{L}y_p = f$

$a_n \dots a_0 \rightarrow$ Const coeff. $f(x) = 0 \rightarrow$ homog. problem.

$$a_n \frac{d^n y_c}{dx^n} + \dots + a_0 y_c = 0$$

$y = A e^{\lambda x}$ $a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0 = 0$

→ n roots for λ.

(i) all distinct roots. $e^{\lambda_m x}$ $m = 1 \dots n$

$\hookrightarrow$ L.I

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \dots + c_n e^{\lambda_n x}$$

(ii) Some roots complex —— $\alpha + i\beta$

$\alpha - i\beta$

$$c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} \longrightarrow A e^{\alpha x} \sin(\beta x + \phi)$$

$$A e^{\alpha x} \cos(\beta x + \phi)$$

(iii) Some repeated roots: λ_1 occurs k times ($k > 1$)

$$x e^{\lambda_1 x}, \ x^2 e^{\lambda_1 x} \dots x^{k-1} e^{\lambda_1 x} \text{ are also soln.}$$

Example: $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$$

R.H.S $= 0$ $$\frac{d^2y_c}{dx^2} - 2\frac{dy_c}{dx} + y_c = 0$$

$y = e^{\lambda x}$

$\lambda^2 - 2\lambda + 1 = 0$ $(\lambda - 1)^2 = 0$

$\lambda = 1$ (repeated twice)

$e^x, \ xe^x$

$$y_c = (c_1 + c_2 x) e^x$$

particular soln: $y_p = b e^x$