Linear Algebra - Part 7: The Eigenvalue problem
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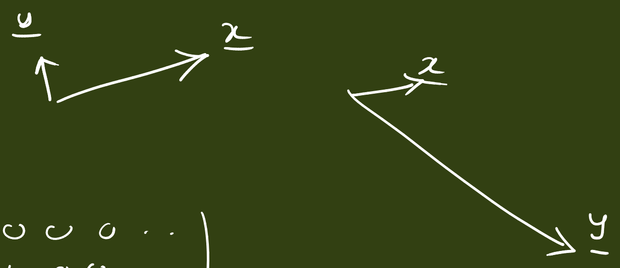
$\underline{A}\,\underline{x} \longrightarrow \underline{y}$ linear transformation

$\underline{y} = \underline{A}\,\underline{x}$



$\underline{\underline{I}} \cdot \underline{x} = \underline{x}$ ← same vector

↳ identity operator

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ & 1 & \cdots & & \end{pmatrix}$$

$\underline{A}\,\underline{x} \longrightarrow \lambda \underline{x}$ "special" vectors → "eigenvectors"

$$\underline{A}\,\underline{x} = \lambda \underline{x}$$

λ → "eigenvalue"

N-dim space: $\underline{A} \longrightarrow N$ indep. eig vectors $\underline{x}^i \longrightarrow \lambda_i$

↳ need not be distinct.

Eigen problem:

$\underline{A}\,\underline{x} = \lambda \underline{x}$ $\qquad \underline{x} = \sum_{i=1}^{N} x_i \underline{e}_i$

$$\underline{A} \sum x_i \underline{e}_i = \lambda \sum x_i \underline{e}_i$$

$$\sum_{i=1}^{N} x_i \underline{A}\,\underline{e}_i = \lambda \sum_{i=1}^{N} x_i \underline{e}_i$$

The bases $\{\underline{e}_i\}$ are orthonormal $\langle \underline{e}_i | \underline{e}_j \rangle = \delta_{ij}$

inner product w.r.t. $\underline{e}_j$

$$\sum x_i \underbrace{\langle \underline{e}_j | \underline{A}\,\underline{e}_i \rangle} = \lambda \sum_{i=1}^{N} x_i \underbrace{\langle \underline{e}_j | \underline{e}_i \rangle}_{\delta_{ij}}$$

$$\langle \underline{e}_j | A | \underline{e}_i \rangle = A_{ji}$$

$\underline{A}\,\underline{x} = \lambda \underline{x}$ $\qquad \sum_{i=1}^{N} A_{ji} x_i = \lambda x_j$

Matrix eig value prob

$\underline{\underline{A}} \cdot \underline{x} = \lambda \underline{x}$

$\underline{\underline{A}}$ ↳ $(N \times N)$ $\qquad \underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$

The Matrix eig value problem:

$$\underline{\underline{A}} \cdot \underline{x} = \lambda \underline{x}$$

$\lambda \rightarrow$ eig values

$\underline{x} \rightarrow$ eig. vectors.

$\swarrow$

do not have a specified length

$$\underline{\underline{A}}\, \underline{x} = \lambda \underline{x}$$

$$\underline{\underline{A}} (c \underline{x}) = \lambda (c \underline{x})$$

Normalize eig vectors: $\underline{x}^{\dagger} \underline{x} = 1$

$$\langle \underline{x} | \underline{x} \rangle = 1$$

$$(x_1^* \ x_2^* \ \cdots \ x_N^*) \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = 1$$

Divide by $(\underline{x}^{\dagger} \underline{x})^{1/2} \longrightarrow \langle \underline{x} | \underline{x} \rangle = 1$

$$\underline{\underline{A}}\, \underline{x} = \lambda \underline{x}$$

$$\underline{\underline{A}}\, \underline{x} = \lambda \underline{\underline{I}}\, \underline{x}$$

$$\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) \underline{x} = 0$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\underline{x} \neq \underline{0} \rightarrow$ nontrivial solutions

$\underline{x} = \underline{0} \rightarrow$ trivial soln.

For nontrivial eig vectors require $\det\left[\underline{\underline{A}} - \lambda \underline{\underline{I}}\right] = 0$

$\searrow$ N^{th} order polynomial in λ

$\downarrow$

N eig. values $\longleftarrow$ Characteristic eqn.

Some observations on the Eigen problem:

$\underline{\underline{A}}$ is a square matrix $(N \times N)$.

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

$$\underline{\underline{A}}\,\underline{\underline{A}}\,\underline{x} = \lambda\,(\underline{\underline{A}}\,\underline{x}) \rightarrow \lambda\,\underline{x}$$

$m = 1, 2, 3, \dots$

$$\underline{\underline{A}}^2\,\underline{x} = \lambda^2\,\underline{x}$$

Similarly: $\underline{\underline{A}}^m\,\underline{x} = \lambda^m\,\underline{x}$

$$\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x}$$

$$(\underline{\underline{A}}^{-1}\underline{\underline{A}})\,\underline{x} = \lambda\,\underline{\underline{A}}^{-1}\underline{x}$$

$$\underline{\underline{I}}\cdot\underline{x} = \lambda\,\underline{\underline{A}}^{-1}\cdot\underline{x}$$

$$\frac{1}{\lambda}\underline{x} = \underline{\underline{A}}^{-1}\cdot\underline{x} \longrightarrow \underline{\underline{A}}^{-1}\cdot\underline{x} = \frac{1}{\lambda}\,\underline{x}$$

$$\underline{\underline{A}}^{-1}\,\underline{x} = \lambda^{-1}\,\underline{x}$$

$$\underline{\underline{A}}^{-m}\,\underline{x} = \lambda^{-m}\,\underline{x}$$

$$\underline{\underline{A}}^j\,\underline{x} = \lambda^j\,\underline{x}$$

j: +ve or −ve integer.

$$\alpha_j\,\underline{\underline{A}}^j\,\underline{x} = \alpha_j\,\lambda^j\,\underline{x}$$

M, N: +ve integers

$$\sum_{j=-M}^{j=N} \alpha_j\,\underline{\underline{A}}^j\,\underline{x} = \sum_{j=-M}^{N} \alpha_j\,\lambda^j\,\underline{x}.$$

$Q(\underline{\underline{A}})$

if λ is the eig value of $\underline{\underline{A}}$

then eig value of $Q(\underline{\underline{A}}) = Q(\lambda)$

Normal matrices: $\quad A^\dagger A = A A^\dagger \quad \left\{ \begin{array}{l} \text{Hermitian / Symmetric ;} \\ \text{unitary / orthogonal} \end{array} \right\}$

let A 'normal'

$$(A - \lambda I)x = 0 \qquad \text{let } B = A - \lambda I$$

$$Bx = 0 \qquad x^\dagger B^\dagger = 0$$

$$(Bx)^\dagger = 0 \qquad \boxed{x^\dagger B^\dagger B x = 0}$$

$$B^\dagger B = (A - \lambda I)^\dagger (A - \lambda I)$$

$$= (A^\dagger - \lambda^* I)(A - \lambda I)$$

$$= \underbrace{A^\dagger A}_{\to AA^\dagger} - \lambda^* A - \lambda A^\dagger + \lambda \lambda^*$$

$$B^\dagger B = A A^\dagger - \lambda^* A - \lambda A^\dagger + \lambda \lambda^*$$

$$= (A - \lambda I)(A - \lambda I)^\dagger$$

i.e if A is normal, so is $(A - \lambda I)$

$$\boxed{B^\dagger B = B B^\dagger}$$

$BB^\dagger$

$$x^\dagger B^\dagger B x = 0 \longrightarrow x^\dagger B B^\dagger x = 0$$

$$B^\dagger x = z \qquad (B^\dagger x)^\dagger (B^\dagger x) = 0$$

$$z^\dagger z = 0$$

$$\langle z | z \rangle = 0 \qquad AA^\dagger = A^\dagger A$$

$$B^\dagger x = 0$$

$$z = 0 \qquad Ax = \lambda x$$

$$A^\dagger x - \lambda^* I x = 0$$

$$\boxed{A^\dagger x = \lambda^* x} \longrightarrow \text{Complex conj.}$$

If A is a normal matrix, the eigvalues of $A^\dagger$ are the complex conjugates of the eigvalues of A

$A \rightarrow$ normal :

$$A \underline{x}^i = \lambda_i \underline{x}^i \quad \text{--- (1)}$$

$$A \underline{x}^j = \lambda_j \underline{x}^j \quad \text{--- (2)}$$

Multiply (2) by $(x^i)^\dagger$ on the left

$$\underbrace{(x^i)^\dagger A}\, x^j = \lambda_j (x^i)^\dagger x^j \qquad (A^\dagger)^\dagger = A$$

$$(A^\dagger x^i)^\dagger x^j =$$

$$(\lambda_i^* \underline{x}^i)^\dagger x^j =$$

$$\underline{\lambda_i \underline{x}_i^\dagger x_j} = \lambda_j \underline{(x^i)^\dagger x^j}$$

$$(\lambda_i - \lambda_j)\, x_i^\dagger x_j = 0 \qquad \text{If } \lambda_i \neq \lambda_j$$

$$\langle x_i | x_j \rangle = 0$$

For a normal Matrix:
The eigen vectors corresponding to distinct eigvalues are orthogonal

$$\underline{\underline{A}} \longrightarrow (\underline{x}^1 \;\ldots\; \underline{x}^N) \rightarrow \text{orthogonal eig vecs}$$

$$\underline{y} = \sum a_i \underline{x}^i$$

$$a_j = \langle \underline{x}^j | \underline{y} \rangle = \underline{x}_j^\dagger \underline{y}$$

normal matrix
$\underline{\underline{A}} \longrightarrow N$ orthog eig vecs $\longrightarrow$ basis

$x_i^\dagger \underline{y} = a_i$!!

Spectral Resolution:

normal matrices

$$A = \sum_{i=1}^{N} \lambda_i \, x^i (x^i)^\dagger \qquad\qquad y = \sum a_i x^i$$

"Proof":

$$A y = \sum_{i=1}^{N} \lambda_i \, x^i \underbrace{(x^i)^\dagger y}_{a_i}$$

$$A \sum a_i x^i = \sum a_i \lambda_i x^i$$

$$\sum a_i A x^i = \sum a_i \lambda_i x^i$$

$$\sum a_i \lambda_i x^i = \sum a_i \lambda_i x^i$$

$$A = \sum_{i=1}^{N} \lambda_i \, x^i (x^i)^\dagger$$

Spectral Resolution of a normal matrix.

Hermitian matrices:

$$A = A^\dagger$$

real eig values !!

$$A \underline{x} = \lambda \underline{x}$$

$$A^\dagger \underline{x} = \lambda^* \underline{x}$$

$$\lambda = \lambda^* \rightarrow \boxed{\lambda\text{'s are real}}$$

orthogonal eig vectors !! → Quantum Mechanics.

Anti Hermitian:

$$A^\dagger = -A \quad (\text{normal})$$

$$A A^\dagger = A(-A) = (-A) A = A^\dagger A$$

$$A x = \lambda x$$

$$A^\dagger x = \lambda^* x$$

$$\lambda = -\lambda^* \longrightarrow \lambda\text{'s are purely imag or zero.}$$