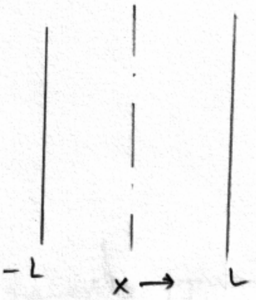
Example:

Derive an expression for the ratio of the total energy transferred from the isothermal surfaces of a plane wall to the interior of the plane wall, $Q/Q_0$, that is valid for $Fo < 0.2$. Express your results in terms of the Fourier Number, $Fo$.




Given:

Plane wall with constant surface temperature $T_s$ initially at temperature $T_i$ (uniformly)

Observation:

Heat transfer rate to the interior of the plane wall initially follows the semi-infinite solid solution, and then diverges.

Dimensionless flux (heat transfer)

$$q^* = \frac{q}{\left(\frac{k(T_s - T_i)}{L_c}\right)} \qquad \text{where } L_c \equiv \text{characteristic length scale}$$

For $Fo < 0.2$, the solution is assumed to follow the semi-infinite solid solution.

$$q = \frac{k(T_s - T_i)}{\sqrt{\pi \alpha t}}$$

Substituting,

$$q^* = \left(\frac{L_c^2}{\alpha t}\right)^{1/2} \frac{1}{\pi^{1/2}} = \frac{1}{\sqrt{\pi Fo}}$$

Heat accumulated is the heat that entered the wall upto time $t$

$$\frac{Q}{Q_0} = \frac{\int_0^t q\,dt}{\rho c L (T_s - T_i)} = \int_0^{Fo} \frac{q^* dFo \, \cancel{k(T_s - T_i)} \cancel{L}^2}{\rho \cancel{c} \cancel{L} \cancel{(T_s - T_i)} \cancel{L} \; \alpha}$$

$$= \frac{1}{\sqrt{\pi}} 2\sqrt{Fo}$$

Notes:

Exact solution would have involved infinite terms. The one-term approximation will work only for Fo > 0.2. Thus, at shorter times, the above approximation (assuming the semi-infinite solid solution to be valid) simplifies the evaluation significantly.

For $Fo = 0.2$

$$\frac{Q}{Q_o} = 0.505.$$

Thus, half of the total possible thermal energy change occurs while $Fo \leq 0.2$.

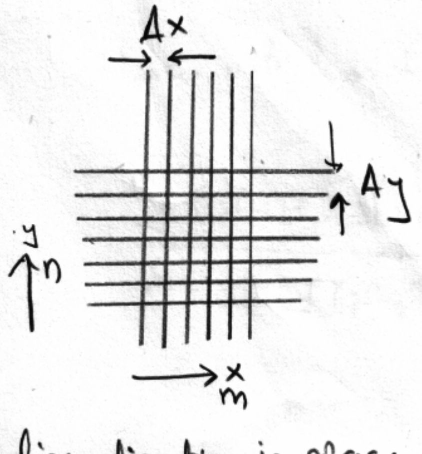
## Numerical Solution: Transient Conduction

- Analytical solution is limited to
  - simple geometries, boundary conditions
  - possible in higher dimensions, but for limited problems
- Finite difference methods
  - briefly reviewed in multidimensional steady conduction

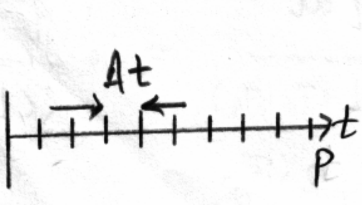
Discretization of the Heat Equation:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

$$T(x, y, t) \downarrow T(m, n, p) = T^p_{m,n}$$




discretization in space




discretization in time

Then, using a forward difference approximation for the time derivative, and centered difference for the spatial derivatives,

$$\frac{1}{\alpha}\,\frac{T_{m,n}^{p+1} - T_{m,n}^{p}}{\Delta t} = \frac{T_{m+1,n}^{p} - 2T_{m,n}^{p} + T_{m-1,n}^{p}}{\Delta x^2} + \frac{T_{m,n+1}^{p} - 2T_{m,n}^{p} + T_{m,n-1}^{p}}{\Delta y^2}$$

..... at any interior point $m,n$ at time $p$

Since the entire RHS is evaluated at time $p$ (previous time), the method is known as Explicit Method.

Rearranging, and assuming $\Delta x = \Delta y$,

$$T_{m,n}^{p+1} = Fo\left(T_{m+1,n}^{p} + T_{m-1,n}^{p} + T_{m,n+1}^{p} + T_{m,n-1}^{p}\right) + (1 - 4Fo)\,T_{m,n}^{p}$$

Where

$$Fo = \frac{\alpha \Delta t}{\Delta x^2}$$ is the discrete Fourier number

Notes:

- Explicit form is easy for marching in time.
- RHS depends on previous step's temperature distribution
- LHS explicitly gives new step's temperature distribution
- Knowing initial temperature $(p=0)$, the temperature at the next time step $(p=1)$ is explicitly given by the above formula. (taking care of the boundary conditions).
- Not unconditionally stable

  Stable only for

  $Fo < \frac{1}{4}$ or $Fo < \frac{1}{2}$ for 1D unsteady conduction

## Implicit Form

$$\frac{1}{\alpha}\frac{T_{m,n}^{p+1}-T_{m,n}^{p}}{\Delta t}=\frac{T_{m+1,n}^{p+1}-2T_{m,n}^{p+1}+T_{m-1,n}^{p+1}}{\Delta x^2}+\frac{T_{m,n+1}^{p+1}-2T_{m,n}^{p+1}+T_{m,n-1}^{p+1}}{\Delta y^2}$$

Rearranging and assuming $\Delta x=\Delta y$,

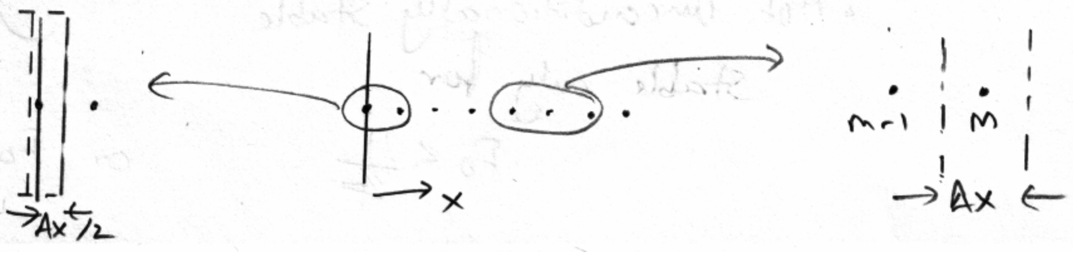
$$(1+4Fo)T_{m,n}^{p+1}-Fo\left(T_{m+1,n}^{p+1}+T_{m-1,n}^{p+1}+T_{m,n+1}^{p+1}+T_{m,n-1}^{p+1}\right)=T_{m,n}^{p}$$

Notes:

- New temperature at a node n,m depends on new temperature of adjoining nodes.
- All nodal equations must be solved simultaneously
- Unconditionally stable method

Example: 1D transient conduction in semi-infinite solids

A thick slab of copper initially at a uniform temperature of 20°C is suddenly exposed to radiation at one surface such that the net heat flux is maintained at a constant value of $3\times10^5$ W/m². Using explicit and implicit finite difference techniques with a space increment of $\Delta x = 75$ mm, determine the temperature at the irradiated surface and at an interior point that is 150 mm from the surface after 2 mins. Compare the results with corresponding analytical solution.



For the surface node ($m=0$), energy balance gives

$$q_{rad} A + kA \frac{T_1^p - T_0^p}{\Delta x} = \rho A\left(\frac{\Delta x}{2}\right) c \frac{T_0^{p+1} - T_0^p}{\Delta t}$$ — explicit form

or

$$T_0^{p+1} = 2 Fo\left(\frac{q_{rad}\Delta x}{k} + T_1^p\right) + (1 - 2Fo) T_0^p$$

For any interior node $1 \le m \le M-1$

$$T_m^{p+1} = Fo\left(T_{m+1}^p + T_{m-1}^p\right) + (1 - 2Fo) T_m^p$$

To ensure numerical stability,

$Fo \le \frac{1}{2}$. Let $Fo = 1/2$

Thus

$$\Delta t = \frac{Fo(\Delta x)^2}{\alpha} = 24\ \text{s}$$

and

$$\frac{q_{rad}\Delta x}{k} = 56.1\ ^\circ C$$

For copper, $k = 401$ W/m·K

Simplifying the finite difference equations,

$$T_0^{p+1} = 56.1 + T_1^p$$

$$T_m^{p+1} = \frac{T_{m+1}^p + T_{m-1}^p}{2}$$

$\}$ ← Solve in Matlab

After 2 mins,

$T_0 = 125.2\ ^\circ C$ $\quad$ $T_2 = 48.1\ ^\circ C$