Partial differentials and Multivariable Calculus
Part - 2

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Total differential : $f = f(x, y)$

$$df = \left.\frac{\partial f}{\partial x}\right)_y dx + \left.\frac{\partial f}{\partial y}\right)_x dy$$

$$f = f(x_1, x_2 \cdots x_N)$$

$$df = \left.\frac{\partial f}{\partial x_1}\right)_{\substack{x_i = C \\ \text{except } x_1}} dx_1 + \left.\frac{\partial f}{\partial x_2}\right)_{\substack{x_i = C \\ \text{except } x_2}} dx_2 + \cdots + \left.\frac{\partial f}{\partial x_N}\right)_{\substack{\text{all } x_i = C \\ \text{except } x_N}} dx_N$$

$$\left.\frac{\partial f}{\partial x}\right)_{?} \leftarrow \boxed{\left.\frac{\partial f}{\partial x}\right)_{?}} \qquad f = f(x, y, z \ldots)$$

$$f = f(x_1, x_2 \cdots x_N)$$

$$df = \left.\frac{\partial f}{\partial x_1}\right)_{\bigcirc} dx_1 + \left.\frac{\partial f}{\partial x_2}\right)_{\bigcirc} dx_2 + \cdots + \left.\frac{\partial f}{\partial x_N}\right)_{\bigcirc} dx_N$$

In some cases $x_1, x_2 \cdots x_N$ — are not independent.

assume : $x_2, x_3, x_4 \cdots x_N \rightarrow$ fns of x_1

$$x_2 \rightarrow f_2(x_1)$$

$$x_3 \rightarrow b_3(x_1)$$

$$f = f(x_1) \Leftarrow$$

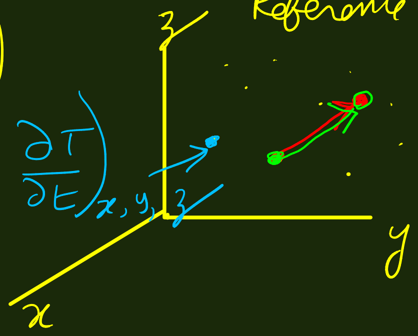
$$df = \frac{\partial f}{\partial x_1}\Big)\, dx_1 + \frac{\partial f}{\partial x_2}\Big)\, dx_2 + \cdots + \frac{\partial f}{\partial x_N}\Big)\, dx_N$$

$$\frac{df}{dx_1} = \boxed{\frac{\partial f}{\partial x_1}\Big)} + \frac{\partial f}{\partial x_2}\Big)\frac{dx_2}{dx_1} + \cdots\cdots + \frac{\partial f}{\partial x_N}\frac{dx_N}{dx_1}$$

→ partial derivative

Total derivative of f w.r.t x_1

Reference frame



Substantial derivative: (T.P / Fluid Mech.)

$$T = T(x, y, z, t)$$

$$\frac{\partial T}{\partial t}\Big)_{x,y,z} \longrightarrow \text{partial derivative}$$

$$\frac{DT}{Dt}$$

Following a particle → X, Y, Z

$$X, Y, Z \xrightarrow{t} x, y, z$$

$$x = f_x(X, Y, Z, t)$$

$$y = f_y(X, Y, Z, t)$$

$$z = f_z(X, Y, Z, t)$$

$$\frac{\partial T}{\partial t}\Big)_{X,Y,Z} \equiv \frac{DT}{Dt} \cdot \longrightarrow \frac{\partial T}{\partial t}\Big)_{x,y,z}$$

1-D example. x direction (ignore y, z)

$$T = T(x, t)$$

$$dT = \frac{\partial T}{\partial t}\Big)_x dt + \frac{\partial T}{\partial x}\Big)_t dx$$

$$x = x(X, t) \qquad \text{(ignoring } y, z)$$

$$dx = \left.\frac{\partial x}{\partial X}\right)_t dX + \left.\frac{\partial x}{\partial t}\right)_X dt$$

$$dT = \left.\frac{\partial T}{\partial t}\right)_x dt + \left.\frac{\partial T}{\partial x}\right)_t dx$$

0 if $X = \text{const.}$

$$dT = \left.\frac{\partial T}{\partial t}\right)_x dt + \left.\frac{\partial T}{\partial x}\right)_t \left[\cancel{\left.\frac{\partial x}{\partial X}\right)_t dX} + \left.\frac{\partial x}{\partial t}\right)_X dt \right]$$

$$\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_X = \left.\frac{\partial T}{\partial t}\right)_x + \left.\frac{\partial T}{\partial x}\right)_t \left.\frac{\partial x}{\partial t}\right)_X$$

$$\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_X = \left.\frac{\partial T}{\partial t}\right)_x + \left.\frac{\partial T}{\partial x}\right)_t \boxed{\left.\frac{\partial x}{\partial t}\right)_X}$$

$$\boxed{\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_x + v_x \left.\frac{\partial T}{\partial x}\right)_t}$$

x

Substantial derivative
Total derivative
"Material derivative"

In 3D

$$\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_{x,y,z} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}$$

$$\boxed{\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_{x,y,z} + \underline{v} \cdot \nabla T}$$

Substantial derivative

local time variation

convected time variation

Exact & Inexact differentials:

So far: given $f \longrightarrow df$

$$f(x,y) \longrightarrow df = \left.\frac{\partial f}{\partial x}\right)_y dx + \left.\frac{\partial f}{\partial y}\right)_x dy.$$

Reverse Qn: given $df = \dots \longrightarrow$ Can we find f??

e.g. $df = x\,dy + y\,dx$

"Exact differentials"

$$f = xy + \underbrace{c}_{\text{const}}$$

eg $x\,dy + 3y\,dx$

inexact differential

$xy + c_1(x) \leftarrow$ partial integ. with y → keep $x = \text{const}$.

$3xy + c_2(y) \leftarrow$ partial integ. with x. keep $y = \text{const}$.

What are the properties of a differential - makes it exact??

$$\boxed{df = A(x,y)\,dx + B(x,y)\,dy}$$

If $df \longrightarrow f(x,y)$

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

$A(x,y)$ $B(x,y)$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} \longrightarrow \boxed{\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}}$$

$3y\,dx + x\,dy \longrightarrow$ inexact differential.

$3y \to A(x,y)$, $x \to B(x,y)$

$$\frac{\partial A}{\partial y} = 3 \neq \frac{\partial B}{\partial x} = 1 \; !$$

path fns work, heat $\longrightarrow$ inexact diff

state fns int energy, entropy $\longrightarrow$ exact diff.

<u>Useful Results</u>:

$$z = z(x,y) \longleftrightarrow x = x(y,z)$$
$$\longleftrightarrow y = y(x,z)$$

$x = x(y,z)$

$$dx = \left.\frac{\partial x}{\partial y}\right)_z dy + \left.\frac{\partial x}{\partial z}\right)_y dz$$

$y = y(x,z)$

$$dy = \left.\frac{\partial y}{\partial x}\right)_z dx + \left.\frac{\partial y}{\partial z}\right)_x dz$$

$$dx = \left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial x}\right)_z dx + \left[\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x + \left.\frac{\partial x}{\partial z}\right)_y\right] dz$$

$x = \text{const}:\ dx = 0 \longrightarrow \left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x + \left.\frac{\partial x}{\partial z}\right)_y = 0$

$$\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x = -\left.\frac{\partial x}{\partial z}\right)_y \rightarrow \frac{-1}{\left.\frac{\partial z}{\partial x}\right)_y}$$

$dz = 0$

$$\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial x}\right)_z = 1 \longrightarrow \left.\frac{\partial x}{\partial y}\right)_z = \frac{1}{\left.\frac{\partial y}{\partial x}\right)_z};$$

$$\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x = \frac{-1}{\left.\frac{\partial z}{\partial x}\right)_y} \longrightarrow \boxed{\left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x \left.\frac{\partial z}{\partial x}\right)_y = -1}$$