Inner products / Scalar products :

$\underline{a} = \sum_{i=1}^{N} a_i \underline{e}_i$

Basis : $\rightarrow$ N L.I vectors

Orthonormal Basis : $\langle \underline{e}_i | \underline{e}_j \rangle = \delta_{ij} \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

basis vectors

Norm of vector $\underline{a}$: $\| \underline{a} \|$

$\| \underline{a} \| = \langle \underline{a} | \underline{a} \rangle^{1/2}$.

abstract notation

equivalent of length

$\underline{a} \cdot \underline{a} = |\underline{a}|^2 \cos 0$ ($\cos 0 = 1$)

$|\underline{a}| = (\underline{a} \cdot \underline{a})^{1/2}$

$$\langle \underline{a} | \underline{a} \rangle = \left\langle \sum_i a_i \underline{e}_i \,\middle|\, \sum_j a_j \underline{e}_j \right\rangle$$

$$= \sum_i \sum_j a_i^* a_j \boxed{\langle \underline{e}_i | \underline{e}_j \rangle} \rightarrow \delta_{ij} \text{ if the basis is orthonormal}$$

$$= \sum_i \sum_j a_i^* a_j \delta_{ij}$$

$$\langle \underline{a} | \underline{a} \rangle = \sum_i a_i^* a_i = \sum_{i=1}^{N} |a_i|^2$$

Analogy :

$|\underline{a}| = (a_1^2 + a_2^2 + a_3^2)^{1/2}$

$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$

real #s.

Norm :

$$\| \underline{a} \| = \langle \underline{a} | \underline{a} \rangle^{1/2} = \left(\sum_{i=1}^{N} |a_i|^2 \right)^{1/2}$$

real qty :

$$\langle \underline{a} | \underline{b} \rangle = \left\langle \sum_i a_i \underline{e}_i \,\middle|\, \sum_j b_j \underline{e}_j \right\rangle$$

$$= \sum_i \sum_j a_i^* b_j \underbrace{\boxed{\langle \underline{e}_i | \underline{e}_j \rangle}}_{\delta_{ij}} \rightarrow \delta_{ij} \text{ (if orthonormal)}$$

$$\boxed{\langle \underline{a} | \underline{b} \rangle = \sum_i a_i^* b_i}$$

$\underline{a} \cdot \underline{b}$

$= a_x b_x + a_y b_y + a_z b_z$

$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$

$\langle \underline{a} | \underline{b} \rangle$: if the basis is non orthogonal.

$$\langle \underline{a} | \underline{b} \rangle = \sum_i \sum_j a_i^* b_j \boxed{\langle \underline{e}_i | \underline{e}_j \rangle}$$

G_{ij}

Linearly indep.

$\sum_i \alpha_i \underline{e}_i = 0$ iff $\alpha_i = 0$

orthogonality: $\langle \underline{e}_i | \underline{e}_j \rangle = \delta_{ij}$

$$G_{ij} = \langle \underline{e}_i | \underline{e}_j \rangle$$

$$G_{ji} = \langle \underline{e}_j | \underline{e}_i \rangle$$

Axioms

$$G_{ji}^* = \langle \underline{e}_j | \underline{e}_i \rangle^* = \langle \underline{e}_i | \underline{e}_j \rangle = G_{ij}$$

$$\boxed{G_{ij} = G_{ji}^*}$$

orthogonal: $\| \underline{a} \| = \left(\sum_i |a_i|^2 \right)^{1/2}$.

$$\langle \underline{a} | \underline{b} \rangle = \sum_i \sum_j a_i^* G_{ij} b_j$$

$\| \underline{a} \|^2 = \langle \underline{a} | \underline{a} \rangle$ Is this real even if the basis is nonorthogonal ??

$$\langle \underline{a} | \underline{a} \rangle = \sum_i \sum_j a_i^* G_{ij} a_j$$

Taking complex conjugate

$$\langle \underline{a} | \underline{a} \rangle^* = \sum_i \sum_j a_i G_{ij}^* a_j^*$$

$$= \sum_i \sum_j a_j^* G_{ij}^* a_i$$

G_{ji}

$$= \sum_i \sum_j a_j^* G_{ji} a_i \quad \text{interchange } i \text{ \& } j$$

$$\langle \underline{a} | \underline{a} \rangle^* = \sum_j \sum_i a_i^* G_{ij} a_j = \langle \underline{a} | \underline{a} \rangle \;!!$$

Even if the basis is non-orthogonal, $\langle \underline{a} | \underline{a} \rangle$ is real.

Advantageous: orthonormal basis $\langle \hat{\underline{e}}_i | \hat{\underline{e}}_j \rangle = \delta_{ij} \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$\underline{a} = \sum_{i=1}^{N} a_i \hat{\underline{e}}_i$$: How do you find a_j ??

$$\langle \hat{\underline{e}}_j | \underline{a} \rangle = \sum_{i=1}^{N} a_i \underbrace{\langle \hat{\underline{e}}_j | \hat{\underline{e}}_i \rangle}_{\delta_{ij}}$$

$$= \sum_{i=1}^{N} a_i \delta_{ij} = \boxed{a_j = \langle \hat{\underline{e}}_j | \underline{a} \rangle}$$

Example.

$$\underline{v} = \begin{bmatrix} 1+i \\ \sqrt{3}+i \end{bmatrix} = \underset{a_1}{(1+i)} \underset{\underline{e}_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \underset{a_2}{(\sqrt{3}+i)} \underset{\underline{e}_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$= \sum_{i=1}^{2} a_i \underline{e}_i = a_1 \underline{e}_1 + a_2 \underline{e}_2$$

New basis:

$$\underline{e}_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{e}_2' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\underline{v} = v_1' \underline{e}_1' + v_2' \underline{e}_2'$$

$$v_1' = \langle \underline{e}_1' | \underline{v} \rangle$$

$$= \frac{1}{\sqrt{2}} (1, 1) \begin{pmatrix} 1+i \\ \sqrt{3}+i \end{pmatrix}$$

$$v_1' = \frac{1}{\sqrt{2}} (1 + \sqrt{3} + 2i)$$

$$v_2' = \langle \underline{e}_2' | \underline{v} \rangle = \frac{1}{\sqrt{2}} (1 - \sqrt{3})$$

Orthonormal basis:

Only a set of N lin. indep. vectors $\longrightarrow$ Gram-Schmidt orthonormalisation

$\downarrow$

N orthonormal vectors.

Linear Algebra. Linear operators

functions:

$x \to \boxed{f(x)} \to y$

$y = f(x)$

$x = 1 \times x$

$x = 0 + x$

"operators":

$\underline{y} = \mathcal{A}\, \underline{x}$

input

operator

output

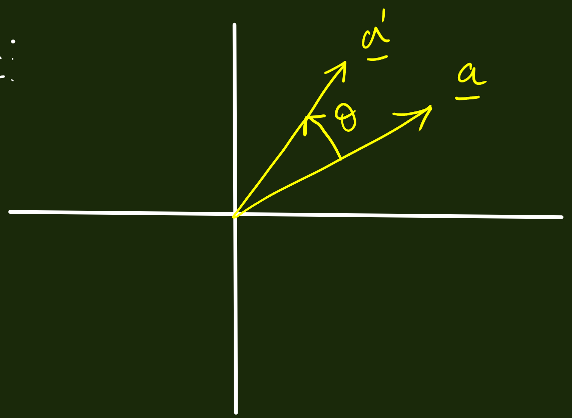
Examples: 2D geometric vectors.

$\underline{x} = \underline{\underline{I}} \cdot \underline{x}$

Identity operator

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rotation matrix:



$$\underline{\underline{R}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\underline{a}' = \underline{\underline{R}} \cdot \underline{a}$$

Linear operators.

$$\mathcal{A}(\lambda \underline{a} + \mu \underline{b}) = \lambda \mathcal{A} \underline{a} + \mu \mathcal{A} \underline{b}$$

defn. of linear operator.

$y = \mathcal{A}(\underline{x})$ "abstract relation"

How does the linear transformation $y = \mathcal{A} x$ happen in a given basis?

given basis: $\{\underline{e}_i\}$ $\quad \underline{y} = \sum_{i=1}^{N} y_i\, \underline{e}_i \qquad \underline{x} = \sum_{i=1}^{N} x_j\, \underline{e}_j$

$\underline{y} = \mathcal{A} x$

$$\sum y_i\, \underline{e}_i = \mathcal{A} \sum x_j\, \underline{e}_j \xrightarrow[\text{linear}]{\text{Because } \mathcal{A} \text{ is}} \sum_{i=1}^{N} y_i\, \underline{e}_i = \sum_{i=1}^{N} x_j\, \mathcal{A}\, \underline{e}_j$$

$\underline{y} = \mathcal{A}\, \underline{x}$ $\quad\searrow$ lin. operator

$\mathcal{A}(\lambda \underline{x} + \mu \underline{y}) = \lambda \mathcal{A}\underline{x} + \mu \mathcal{A}\underline{y}$

$\underline{a} = \sum a_i \underline{e}_i$

$\langle \underline{e}_j | \underline{a} \rangle = a_j$

$$\sum_i y_i \underline{e}_i = \sum_j x_j \mathcal{A} \underline{e}_j$$

Need k^{th} component y_k. Take inner prod with $\underline{e}_k$

$$\sum_i y_i \underbrace{\langle \underline{e}_k | \underline{e}_i \rangle}_{\delta_{ki}} = \sum_j x_j \langle \underline{e}_k | \mathcal{A} \underline{e}_j \rangle$$

$$y_k = \sum_j x_j \boxed{\langle \underline{e}_k | \mathcal{A}\, \underline{e}_j \rangle} \Rightarrow A_{kj}$$

$\underline{y} = \mathcal{A}\, \underline{x}$

$$\boxed{y_k = \sum_j x_j A_{kj}}$$

Matrices

$$\mathcal{A}\, \underline{e}_j = \underline{e}_j' \qquad A_{kj} = \langle \underline{e}_k | \underline{e}_j' \rangle$$

Components of the linear operator in the given orthonormal basis

$\underline{y} = \mathcal{A}\, \underline{x}$ linear transformation $\mathcal{A}(\lambda \underline{x} + \mu \underline{y}) = \lambda \mathcal{A}\underline{x} + \mu \mathcal{A}\underline{y}$

In a given basis

$$y_k = \sum_j A_{kj} x_j$$

$$\underline{y} \rightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

$$\underline{x} \rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

linear operator $\rightarrow \mathcal{A} \rightarrow \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ & & & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}$

$\underline{y} = A \underline{x}$ (A : Linear operator).

Matrix Representation of a linear operator in a given basis

In a given basis

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ & & & \\ \vdots & \vdots & & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$\underline{y} = A \underline{x}$

$$\boxed{y_k = \sum_j A_{kj} x_j}$$

→ represents, in a given basis the linear transformation $\underline{y} = A \underline{x}$

$$y_1 = \sum_j A_{1j} x_j = A_{11} x_1 + A_{12} x_2 + A_{13} x_3 + \cdots + A_{1N} x_N$$

$$\underline{\sigma} = \underline{\underline{T}} \cdot \underline{n}$$

Tensors