Differential Equations - Part 8: Green's function for ODEs

ChE641, IIT Kanpur

Greens' functions – ODE's .

NOT PDE's.

$$\alpha^2 \frac{\partial u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$-\infty < x < \infty$

$u(x, t=0) = \delta(x)$

$u(x \to \pm\infty) \to 0$

$\to$ Green's fn. $\to K(x-z, t)$

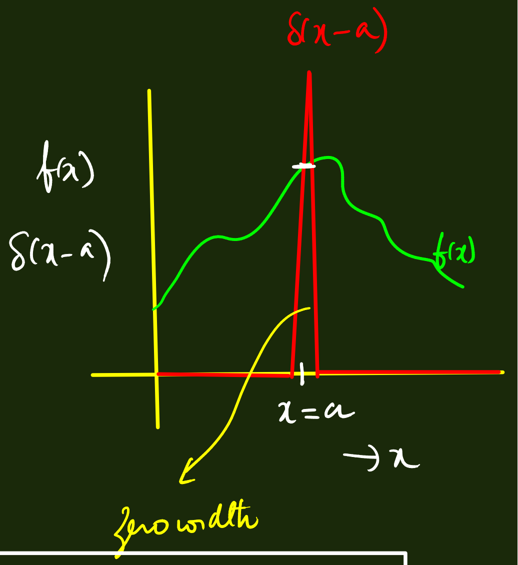
$u(x, t=0) = f(x)$

$$u(x,t) = \int_{-\infty}^{\infty} K(x-z,t)\, f(z)\, dz$$

Dirac δ-fn: $\delta(t)$

via an integral.

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$$



Suppose integration domain does not include the δ-fn peak.

$$\int_{y_1}^{y_2} f(x)\,\delta(x-a)\,dx = 0$$

if

$a \notin (y_1, y_2)$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

$$\boxed{\int_{-\infty}^{\infty} \delta(x-a)\,dx = 1}$$

$$\delta(t) = \delta(-t)$$

Green's fn. for linear ODE's.

$$a_n \frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0\, y(x) = f(x)$$

"forcing fn"

$$\mathcal{L}\, y(x) = f(x)$$

with suitable BC's

$$\boxed{\mathcal{L}\, y(x) = f(x)}$$

$G(x,z)$: Green's fn. s.t.

would like to express the soln.

$$y(x) = \int_a^b G(x,z)\, f(z)\, dz$$

dummy variable of integral.

$$\mathcal{L}\, y(x) = \mathcal{L} \int_a^b G(x,z)\, f(z)\, dz$$

$\frac{d}{dx}$

$$f(x) = \int_a^b \mathcal{L}\, G(x,z)\, f(z)\, dz$$

$$f(x) = \int_a^b \left(\delta(x-z) \right) f(z)\, dz$$

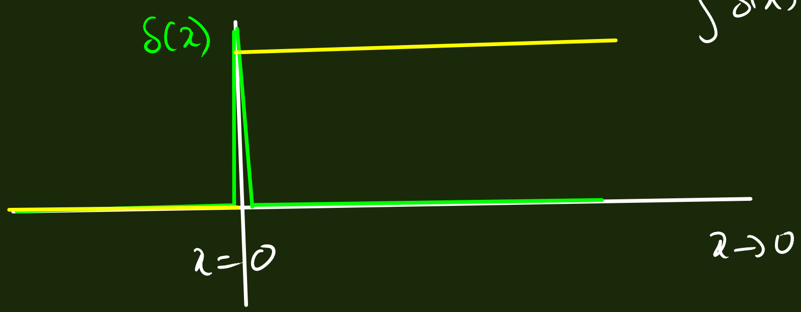
The Green's fn is obtained by solving

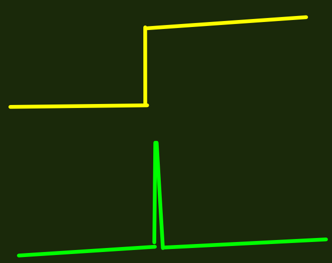
impulse forcing

$$\mathcal{L}\, G(x,z) = \delta(x-z) \longrightarrow \text{BC's at } x=a,\ x=b$$

If $y(x=a)=0$, $y(x=b)=0$ $\longrightarrow$ $G(x=a, z) = 0$, $G(x=b, z) = 0$

$$\int \delta(x)\, dx = 1$$



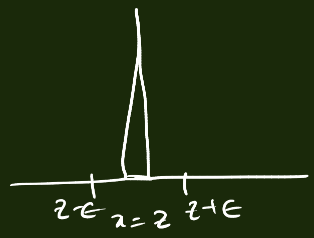


Continuity conditions for $G(x,z)$ at $x=z$:

$$\mathcal{L} G = \sum_{m=0}^{n} a_m(x) \frac{d^m G}{dx^m}$$

$$\mathcal{L}\, G(x,z) = \delta(x-z)$$

$$\lim_{\epsilon\to 0} \sum_{m=0}^{n} \int_{z-\epsilon}^{z+\epsilon} a_m(x) \frac{d^m G(x,z)}{dx^m}\, dx = \lim_{\epsilon\to 0} \int_{z-\epsilon}^{z+\epsilon} \delta(x-z)\, dx$$



$$= 1$$

$$\int \frac{d^m G}{dx^m}\, dx = \frac{d^{m-1} G}{dx^{m-1}}$$

n^{th} order $\longrightarrow$ $$\lim_{\epsilon\to 0} \left[a_n(x) \frac{d^{n-1}}{dx^{n-1}} G(x,z) \right]_{z-\epsilon}^{z+\epsilon} = 1$$

$$\left[a_{n-1} \frac{d^{n-2}}{dx^{n-2}} G(x,z) \right]_{z-\epsilon}^{z+\epsilon} = 0$$

$$\vdots$$

Continuous

$$\left.\frac{d^{n-2} G}{dx^{n-2}}\right|_{z=+\epsilon} = \int_{z=-\epsilon} \frac{d^{n-1} G}{dx^{n-1}}$$

$$a_n(z) \left[\left.\frac{d^{n-1} G}{dx^{n-1}}\right|_{z+\epsilon} - \left.\frac{d^{n-1} G}{dx^{n-1}}\right|_{z-\epsilon} \right] = 1$$

Jump cond:

$$\left.\frac{d^{n-1} G}{dx^{n-1}}\right|_{z+\epsilon} - \frac{d^{n-1} G}{dx^{n-1}} = \frac{1}{a_n(z)}$$

$$\left.\frac{d^{n-2} G}{dx^{n-2}}\right|_{z+\epsilon} - \frac{d^{n-2} G}{dx^{n-2}} = 0$$

$G(x,z)$: $\longrightarrow \mathcal{L}\, G(x,z) = \delta(x-z)$ $\qquad \mathcal{L}u = f$ $\downarrow$

$\int_{z-\epsilon}^{z+\epsilon}$

$\underline{\text{B.C's}}$ $\quad G(x=a, z) = ?$

$\qquad G(x=b, z) = ?$

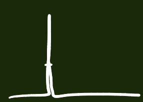
at $x = z \rightarrow$ jump or continuity conditions

$$(\underline{1}) \quad \frac{d^2 G}{dx^2} + G = \delta(x-z)$$

To jump/continuity conditions:

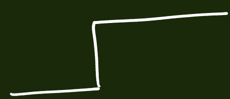
$x = z$

$$\lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} \frac{d^2 G}{dx^2}\, dx + \lim_{\epsilon \to 0} \int_{z-\epsilon}^{z+\epsilon} G(x,z)\, dx = \underbrace{\int_{z-\epsilon}^{z+\epsilon} \delta(x-z)\, dx}$$



$$\lim_{\epsilon \to 0} \left.\frac{dG}{dx}\right]_{z-\epsilon}^{z+\epsilon} + \lim_{\epsilon \to 0} G(x,z) \underbrace{\left[z+\epsilon-(z-\epsilon)\right]} = 1$$

$$\lim_{\epsilon \to 0} \cancel{G(x,z)(2\epsilon)} \to 0$$

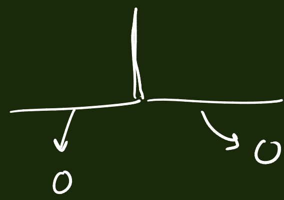


$$\lim_{\epsilon \to 0} \left.\frac{dG}{dx}\right)_{z+\epsilon} - \left.\frac{dG}{dx}\right)_{z-\epsilon} = 1$$

$$G(x, z+\epsilon) = G(x, z-\epsilon)$$

Example: $\dfrac{d^2y}{dx^2} + y = \operatorname{cosec} \cdot x \leftarrow f(x)$ BC: $y(0) = y(\pi/2) = 0$

Green's fn: $\dfrac{d^2 G(x,z)}{dx^2} + G(x,z) = \delta(x-z)$



$x > z, \quad x < z \qquad \dfrac{d^2 G(x,z)}{dx^2} + G(x,z) = 0$

$$G(x,z) = \begin{array}{ll} A(z)\sin x + B(z)\cos x & x < z \\ C(z)\sin x + D(z)\cos x & x > z \end{array}$$

BC's $G(x=0, z) = 0 \longrightarrow B(z) = 0$

$G(x=\pi/2, z) = 0 \longrightarrow C(z) = 0$

$$G(x,z) = \begin{cases} A(z)\sin x & (x < z) \\ D(z)\cos x & (x > z) \end{cases}$$

Jump + Continuity Conds: $x = z$

$G(x, z-\epsilon) = G(x, z+\epsilon)$

$$\left.\frac{dG}{dx}\right|_{z+\epsilon} - \left.\frac{dG}{dx}\right|_{z-\epsilon} = 1$$

$$\left.\begin{array}{l} D(z)\cos z - A(z)\sin(z) = 0 \\ -D(z)\sin z - A(z)\cos z = 1 \end{array}\right\} \quad \begin{array}{l} A(z) = -\cos z \\ D(z) = -\sin z \end{array}$$

$$G(x,z) = \begin{cases} -\cos z \sin x & x < z \\ -\sin z \cos x & x > z \end{cases}$$

$$\mathcal{L}y = f(x) \leftarrow \operatorname{cosec}(x)$$

$$y(x) = \int_0^{\pi/2} G(x,z)\, \overset{\operatorname{cosec}(z)}{f(z)}\, dz$$

$$= \int_0^x \frac{-\sin z \cos x}{\operatorname{cosec} z}\, dz + \int_x^{\pi/2} \frac{-\cos z \sin x}{\operatorname{cosec} z}\, dz$$

$$= \cos x \int_0^x -\sin z \operatorname{cosec} z\, dz - \sin x \int_x^{\pi/2} \cos z \operatorname{cosec} z\, dz$$

$\mathcal{L}y = \operatorname{cosec} x$

$\downarrow$

any $f(x)$

$$\boxed{y(x) = -x\cos x + \sin x \ln(\sin x)}$$

Green's fn are sensitive to B.C's:

$$\frac{d^2y}{dx^2} + y = f(x) \qquad y(0) = 0, \quad \frac{dy}{dx}(0) = 0$$

$$G(x,z) = \begin{cases} \underset{0}{A(z)} \sin x + \underset{0}{B(z)} \cos x & x < z \\ C(z) \sin x + D(z) \cos x & x > z \end{cases}$$

$$\left.\begin{aligned} G(0,z) &= 0 \\ \frac{dG}{dx}(0,z) &= 0 \end{aligned}\right\} \quad \begin{aligned} A(z) &= 0 \\ B(z) &= 0 \end{aligned}$$

Continuity/Jump conditions at $x = z$:

$$C(z)\sin z + D(z)\cos z = 0$$

$$C(z)\cos z - D(z)\sin z = 1$$

$$C(z) = \cos z$$

$$D(z) = -\sin z$$

$$G(x,z) = \begin{cases} 0 & x < z \\ \sin(x-z) & x > z \end{cases}$$

$$y(x) = \int_0^\infty G(x,z)\, f(z)\, dz$$

$$= \int_0^x \underbrace{G(x,z)}_{\sin(x-z)} f(z)\, dz + \int_x^\infty \cancel{G(x,z)} f(z)\, dz$$

$$y(x) = \int_0^x \sin(x-z)\, f(z)\, dz$$

Green's fn as a superposition of eigen fn :

$$\mathcal{L}\, y(x) = f(x)$$

$$\boxed{y(x) = \sum c_n \phi_n(x)}$$

Eig value problem

$$\mathcal{L}\, \phi_n = \lambda_n\, \rho(x)\, \phi_n(x)$$

eig val.

self-adj operator

eig fns orthonorml.

$$f(x) = \mathcal{L}\, y(x) = \mathcal{L} \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$= \sum_{n=0}^{\infty} c_n\, \mathcal{L}\, \phi_n$$

$$f(x) = \sum_{n=0}^{\infty} c_n\, \lambda_n\, \rho_n(x)\, \phi_n(x)$$

multiply both sides by $\phi_j^*(x)$ $\int_a^b$

$$\int_a^b \phi_j^*(x) f(x)\,dx = \sum_{n=0}^{\infty} c_n \lambda_n \underbrace{\int_a^b g(x)\,\phi_j^*(x)\,\phi_n(x)\,dx}_{\delta_{jn}}$$

$$\int_a^b \phi_j^*(x) f(x)\,dx = c_j \lambda_j$$

$$c_n = \frac{1}{\lambda_n} \int_a^b \phi_n^*(z) f(z)\,dz$$

$$y(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

$$y(x) = \sum_{n=0}^{\infty} \left[\frac{1}{\lambda_n} \int_a^b \phi_n^*(z) f(z)\,dz \right] \phi_n(x)$$

$$y(x) = \int_a^b \underbrace{\left[\sum_{n=0}^{\infty} \frac{1}{\lambda_n} \phi_n^*(z)\,\phi_n(x) \right]}_{G(x,z)} f(z)\,dz$$

$$\mathcal{L}u = f$$

$$y(x) = \int G(x-z) f(z)\,dz$$

$$\boxed{G(x,z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \phi_n^*(z)\,\phi_n(x)}$$

eigen fn. expansion