Linear Algebra - Part 5: Determinant, Inverse and Rank of a Matrix

ChE641, IIT Kanpur

Linear operator in a given basis (orthonormal)

in a given basis

$$\underline{y} = \boxed{A}\, \underline{x}$$

linear operator

$$\boxed{A_{ij}}$$

$$y_i = A_{ik} x_k$$

$$\begin{pmatrix}1\\3\\1\end{pmatrix}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ \vdots & & & \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{bmatrix}}_{N \times N \text{ matrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$$

The components change if the basis is changed.

Are there some combinations of the matrix elements that remain unchanged / invariant under coordinate transformation??

Invariant

(i) Trace of a matrix

$$\sum_{i=1}^{N} A_{ii}$$

(ii) Determinant of a matrix.

Determinants: → Single scalar number → the given matrix

Both Trace and determinant → defined only for square matrices (N × N)

$\underline{a}$ $\underline{\underline{A}}$

det $\underline{\underline{A}}$ or $|\underline{\underline{A}}|$

$$|\underline{\underline{A}}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

general defn of det of $\boxed{N \times N}$ matrices

modulus:

$|-3| = 3$

$|z|$ = Absolute value

$|\underline{\underline{A}}|$ = determinant

Minors and Co-factors:

Minor

M_{ij} of the element A_{ij} is the det of the $(N-1) \times (N-1)$ matrix obtained by removing all the elements of i^{th} row & j^{th} column.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

$$M_{32} \qquad \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} \checkmark$$

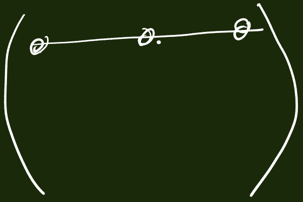
$$\boxed{M_{13}} \qquad \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \checkmark$$

Cofactor $C_{ij} = (-1)^{i+j} M_{ij}$

Cofactor of $A_{23} = (-1)^{2+3} \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$

$$C_{23} = -1 \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$$

How to find $\begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} \leftarrow$



Defn. of determinant (Laplace expansion)

= Sum of the products of the elements of <u>any</u> row <u>or</u> column and their cofactors.

$$\det A = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13}$$

$$= A_{21} C_{21} + A_{22} C_{22} + A_{23} C_{23}$$

$$= A_{13} C_{13} + A_{23} C_{23} + A_{33} C_{33}$$

$$= A_{11} (A_{22} \cancel{A_{33}} - A_{32} A_{23}) - A_{12} (A_{21} A_{33} - A_{31} A_{23}) + A_{13} (A_{21} A_{32} - A_{31} A_{22})$$

$$\begin{array}{c} (1,1) & (1,2) & (1,3) \\ \left(\begin{array}{ccc} \boxed{A_{11} & A_{12} & A_{13}} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array}\right) \end{array}$$

$$\det A = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13}$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} \; ; \; C_{12} = (-1)^{1+2} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix}$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

Consider $\begin{array}{cc} (1,1) & (1,2) \\ \left|\begin{array}{cc} A_{12} & A_{13} \\ A_{32} & A_{33} \end{array}\right| \\ (2,1) & (2,2) \end{array}$

$$\det = A_{12} (-1)^{1+1} |A_{33}| + A_{13} (-1)^{1+2} |A_{32}|.$$

$$= A_{12} A_{33} - A_{13} A_{32}$$

By defn $|A_{33}| = A_{33}$

$|A_{32}| = A_{32}$

$A_{32} = -3$

not modulus $\in |-3| = -3$

Note $\det|-2| \overset{\checkmark}{=} -2$ ~~NOT 2~~ !!

↳ not modulus.

Laplace expansion — $N \times N$ matrix $\rightarrow (N-1)\times(N-1) \rightarrow (1\times1)$

Properties of determinant:

(i) $|A^T| = |A|$ $\qquad A^\dagger = (A^T)^*$

(ii) $|A^\dagger| = |(A^*)^T| = |A^*| = |A|^*$

e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ $\det A = i$

$\underline{\underline{A}}^* = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$ $\det A^* = -i$

3) If two rows (or colums) are interchanged, then det changes its sign (but magnitude remains same).

4)
$$\det\begin{pmatrix} A_{11} & A_{12} & \cdots \\ \lambda A_{21} & \lambda A_{22} \cdots & \lambda A_{2N} \\ A_{N1} & \cdots & \cdots \end{pmatrix} = \lambda \det\begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ A_{N1} & A_{N2} & \cdots \end{pmatrix}.$$

5)
$$\det\begin{pmatrix} \lambda A_{11} & \lambda A_{22} & \lambda A_{23} \\ \vdots & & \\ \lambda A_{N1} & \lambda A_{N2} \cdots & \cdots \end{pmatrix} = \lambda^N \det\begin{pmatrix} A_{11} & A_{12} & \cdots \\ A_{21} & A_{22} & \cdots \\ \vdots & & \\ A_{N1} & A_{N2} & \cdots \end{pmatrix}$$

6) If two rows or colums are (or) multiples of one another identical $\rightarrow \boxed{\det = 0}$

7) $\det \underline{A}$ is unchanged in value if one row or colum is added to another.

8) $\det(A \cdot B) = \det A \cdot \det B = \det(B\ A)$

$AB \neq BA$ But $\det(AB) = \det(BA)$

Inverse of a matrix

$$\bigcirc\!\!\!\!P = A \, \bigcirc\!\!\!\!B$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & \ddots \end{pmatrix}$$

$p = a \cdot b$

$b = \dfrac{p}{a}$

$$B = \frac{P}{A} \quad ✗$$

identity matrix

$$\boxed{AA^{-1} = \underline{\underline{I}} = A^{-1}A}$$

$b = a^{-1} p$

a^{-1} : multiplicative invee of a

$3 \times \frac{1}{3} = 1$

$$AB = P$$

$$A^{-1}AB = A^{-1}P$$

$$\underline{\underline{I}} \cdot B = \underline{\underline{A}}^{-1} P \longrightarrow \boxed{\underline{\underline{B}} = \underline{\underline{A}}^{-1} \cdot \underline{P}}$$

Inverse is defined only for square matrices

$$\left(A^{-1}\right)_{ik} = \frac{\left(C^T\right)_{ik}}{|A|} = \frac{C_{ki}}{|A|}$$

Cofactor of A_{ki}

'Proof'.

$$\left(A^{-1}A\right)_{ij} = \sum_k \left(A^{-1}\right)_{ik} A_{kj}$$

$$= \sum_k \frac{C_{ki}}{|A|} A_{kj}$$

$$\left(A^{-1}A\right)_{ij} = \frac{\cancel{|A|}}{\cancel{|A|}} \delta_{ij}$$

If $|A| = 0$, invaxe is not defined $\longrightarrow$ Matrix is "Singular"

$$A^{-1} = \frac{C^T}{|A|}$$

1) $(A^{-1})^{-1} = A$

2) $(A^{T})^{-1} = (A^{-1})^{T}$

3) $(A^{\dagger})^{-1} = (A^{-1})^{\dagger}$

4) $(AB)^{-1} = B^{-1}A^{-1}$

5) $(A\cdot B\cdot C\cdots G)^{-1} = G^{-1}\cdots\cdots B^{-1}A^{-1}$

$$AA^{-1} = \underline{\underline{I}} \qquad \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & \cdot & \cdots \\ 0 & 0 & 1 & \cdots \end{pmatrix}$$

$$\det(AA^{-1}) = \det \underline{\underline{I}} = 1$$

$$\det(A)\det(A^{-1}) = 1$$

$$\det(A^{-1}) = \frac{1}{\det A}.$$

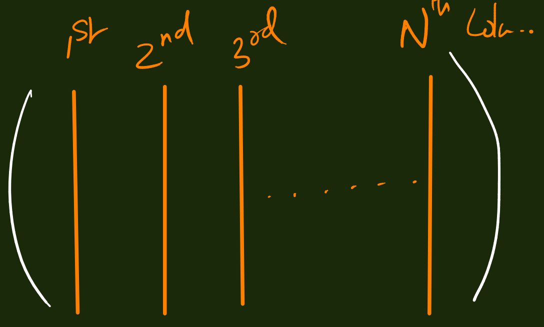
Rank of a matrix: → a single number

→ non-square matrices.

Defn. 1

$M\times N$. M rows
N colums

Rank = # of L I vectors in the set $\{\underline{v}_1, \ldots \underline{v}_N\}$



$(\underline{v}_1 \ \ \underline{v}_2 \ \cdots \ \underline{v}_N)$

$\underline{v}_1 = (M\times 1)$

$$\begin{pmatrix} \underline{W}_1 \\ \underline{W}_2 \\ \vdots \\ \underline{W}_M \end{pmatrix} = \# \text{ of L I vectors in the set } \{\underline{W}_1 \cdots \underline{W}_N\}$$

2nd Defn : Submatrix of $\underline{\underline{A}}$ by ignoring one or more rows or columns.

Rank of an $M \times N$ matrix is the size of the largest square matrix whose determinant is not zero

$$(M \times N) \longrightarrow (\overset{\overset{\text{rank}}{\downarrow}}{r} \times r) \text{ submatrix } \quad \det \neq 0$$

but $(r+1) \times (r+1)$ det $= 0$

Rank A $= r$

Rank of an $m \times n$ matrix $r \leq$ smaller of (m, n)

Square matrix : $(N \times N)$ det$|A| = 0 \rightarrow$ unless $r = N$.

$R(\underline{\underline{A}}) \rightarrow$ rank of $\underline{\underline{A}}$