Complex Variables and Analysis - 4

ChE641, IIT Kanpur

Complex numbers : $z = a + ib = x + iy$

a → real, ib → imag. → real qty

$i^2 = -1$ $\quad \boxed{i = \sqrt{-1}}$

Euler's formula : $e^{i\theta} = \cos\theta + i\sin\theta$

angle or argument

$a = r\cos\theta$

$b = r\sin\theta$

$\boxed{z = re^{i\theta}}$ $\quad \boxed{z = |z|\, e^{i\theta}}$

$r = |z|$

modulus, absolute value, Magnitude.

$\theta : \quad 0 \le \theta < 2\pi$ ← choose

$\theta : \quad -\pi < \theta \le 2\pi$

$z = x + iy, \quad z^* = x - iy$ ←

$z = re^{i\theta} \quad z^* = re^{-i\theta}$ ←

$i \to -i$

complex conjugate

Functions of complex variables : $z = x + iy$

$W = f(z) = u(x,y) + i\, v(x,y)$

u → real, v → imag

$z = x + iy$

$z^* = x - iy$

$W = f(x,y)$ → z, z^* $\quad W = f(z, z^*)$

$x = \frac{1}{2}(z + z^*)$

$y = \frac{1}{2i}(z - z^*)$

$W = f(x,y) \longrightarrow dw = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy$

$$\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}\underbrace{\frac{\partial x}{\partial z}}_{\frac{1}{2}} + \frac{\partial w}{\partial y}\underbrace{\frac{\partial y}{\partial z}}_{\frac{1}{2i}}$$

$$\frac{\partial w}{\partial z} = \frac{1}{2}\frac{\partial w}{\partial x} + \frac{1}{2i}\frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial z^*} = \frac{1}{2}\frac{\partial w}{\partial x} - \frac{1}{2i}\frac{\partial w}{\partial y} \rightarrow$$

Analytic functions: w is a fn. only of z and NOT of z^*

$$\boxed{\frac{\partial w}{\partial z^*} = 0}$$

$$\frac{\partial w}{\partial x} = \frac{i}{i}\frac{1}{i}\frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

$w = u + iv$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

equate real & imag parts

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy - Riemann eqns.

$w = u + iv$ if this fn is analytic then u & v have to obey the C-R eqns

If $w = f(z)$ is analytic then $\frac{\partial w}{\partial z} = \frac{dw}{dz}$ is well-defined

fn only z $\rightarrow$ $\frac{dw}{dz}$ is analytic.

$\therefore \frac{d^n w}{dz^n}$ are analytic !!

Analytic fns: A fn $w(z)$ is analytic in a given domain if the fn is single-valued, continuous & obeys the C-R eqns in the domain

$$f(z) = z^2 = \overset{u}{(x^2 - y^2)} + \overset{v}{2ixy}$$

only of z ! $f(z)$ is analytic !!

$f(x,y) = \cos y - i \sin y$ $\qquad y = \frac{z - z^*}{2i}$

is a fn of <u>both</u> z and z^* $\longrightarrow$ $f(x,y)$ is <u>not</u> analytic !

$f(x,y) = x^2 + y^2$? analytic ?

$= z z^*$ not analytic !!

$z = x + iy$

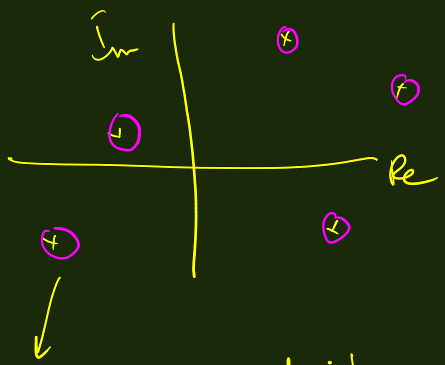
$z^* = x - iy$

$z z^* = x^2 + y^2$

If a fn $f(z)$ is analytic in D, then all its $\frac{d^n f}{dz^n}$ are also analytic

Taylor-expand $f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots$

A fn $f(z)$ obeys C-R eqns in D : will it obey C-R in the entire complex plane ??



"<u>Singularities</u>" of $f(z)$

At these singularities, the continuity of partial derivatives breaks down !

$\frac{1}{z} = \frac{z^*}{z z^*} = \frac{x - iy}{x^2 + y^2}$

$f(z) = \frac{1}{z} \longrightarrow$ analytic !

$u(x,y) = \frac{x}{x^2 + y^2}$; $v(x,y) = \frac{-y}{x^2 + y^2}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

at the origin $(0,0)$

$\frac{\partial u}{\partial x} \sim -\frac{1}{x^2}$ as you approach origin along x-axis

$\frac{\partial v}{\partial y} \sim \frac{1}{y^2}$

The C-R eqns break down at a singularity because the fn or its derivatives become ill defined as we approach the singularity.

Singularities of analytic fns:

1) Simple pole: $f(z) = \frac{1}{z}$ or $f(z) = \frac{1}{(z-z_0)}$

Residue of the fn at z_0

$$f(z) = \frac{C}{(z-z_0)}$$

as $z \to i$

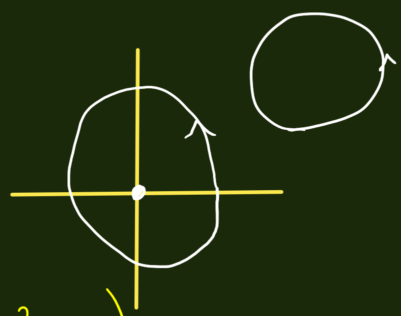
$\frac{1}{(z+i)(z-i)}$ two poles $z = i$, $z = -i$ $\qquad \frac{1}{2i} \quad \frac{1}{(z-i)}$

$R(z=-i) = \frac{-1}{2i}$ $\qquad R(z=+i) = \frac{1}{2i}$

If a fn has a simple pole at $z = z_0$, then

$$\boxed{\text{Residue} \quad R(z_0) = \lim_{z \to z_0} (z-z_0) f(z)}$$

n^{th} order pole: $\quad f(z) \to \frac{R(z_0)}{(z-z_0)^n}$



Essential singularity (at z_0) $\qquad (z-z_0)^n$

$$f(z) = \sum_0^\infty \frac{1}{z^n n!} = e^{\frac{1}{z}} \quad (\text{at } z=0).$$

Branch points: $\quad w = f(z) = z^{1/2}$

Branch Cut