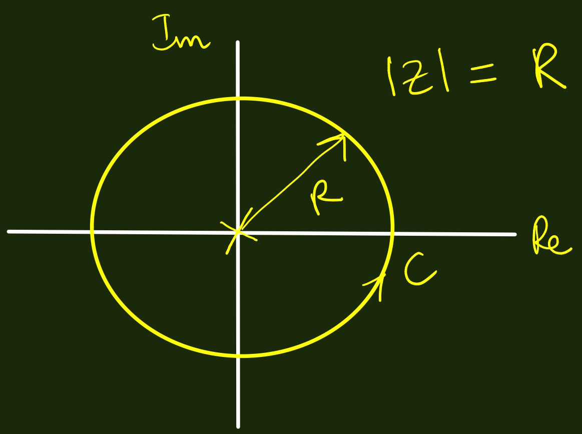
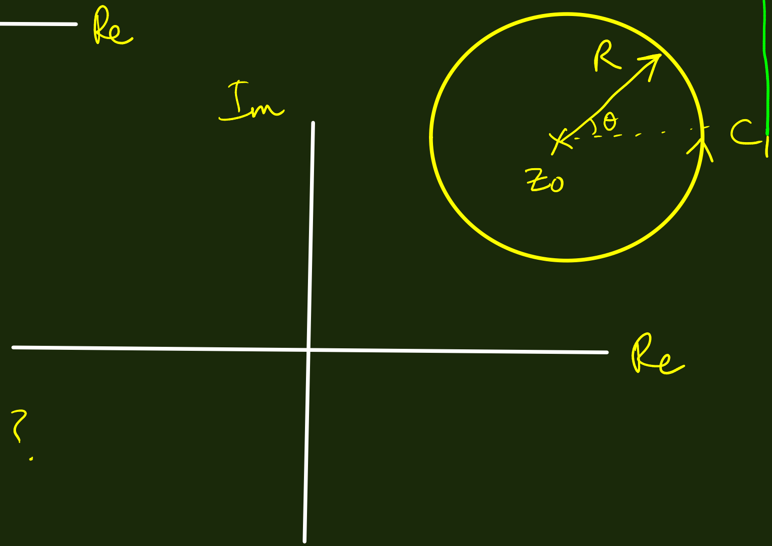
Cauchy's Integral Theorem
ChE641, IIT Kanpur



$|z| = R$

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i}$$



$$\boxed{\oint_{C_1} \frac{dz}{z} = 0 \,!!}$$

does not enclose the origin

Example : $\oint_{C_1} \frac{dz}{z} = \,?$

$$z - z_0 = R e^{i\theta}$$

$$z = z_0 + R e^{i\theta}$$

$$dz = R e^{i\theta} i \, d\theta$$

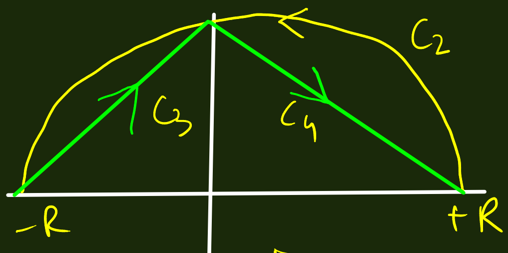
$$\oint_{C_1} \frac{dz}{z} = \int_{\theta=0}^{2\pi} \frac{R e^{i\theta} i \, d\theta}{z_0 + R e^{i\theta}} = \ln\left[R e^{i\theta} + z_0\right]_{\theta=0}^{\theta=2\pi}$$

$$\oint_{C_1} \frac{dz}{z} = \ln\left[R + z_0\right] - \ln\left[R + z_0\right]$$

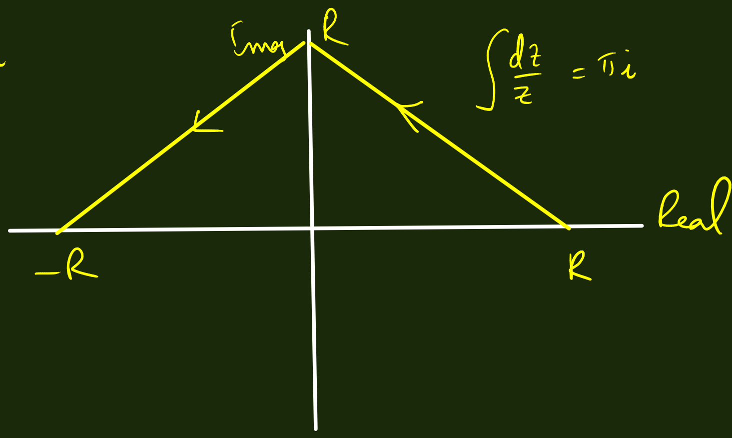
$$= 0 \,!!$$

If $z_0 = 0$, $\ln\left[R e^{i\theta}\right]_0^{2\pi} = \left(\ln R\right)_0^{2\pi} + \left(i\theta\right)_0^{2\pi}$

$$= i\left(2\pi - 0\right) = 2\pi i$$



$$\int_{C_2} \frac{dz}{z} = \pi i$$



$$\int \frac{dz}{z} = \pi i$$

$$\boxed{\int_{C_2+C_3+C_4} \frac{dz}{z} = 0 \;!!}$$

$\int \frac{dz}{z} \to 0$ if the <u>origin</u> is not included! → independent of the path.

→ nonzero otherwise

$\frac{1}{z}$ is not analytic at the origin

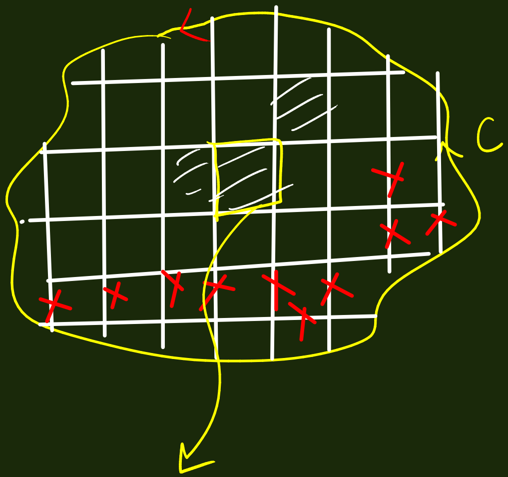
$$\oint_C f(z) = 0$$

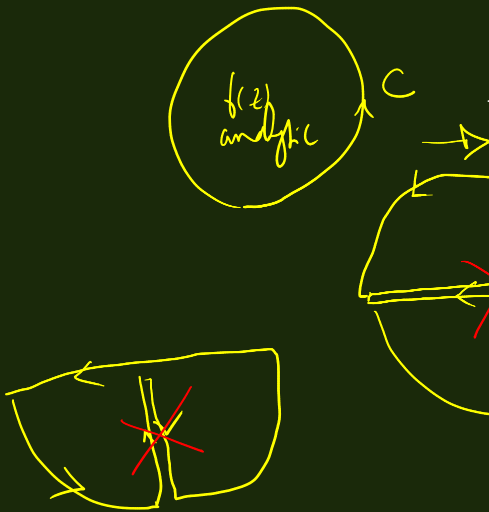
$f(z)$ is analytic within the contour C

Cauchy's Integral theorem:

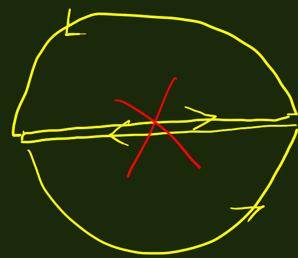
If C is a smooth contour (closed), and if $f(z)$ is analytic inside C, then

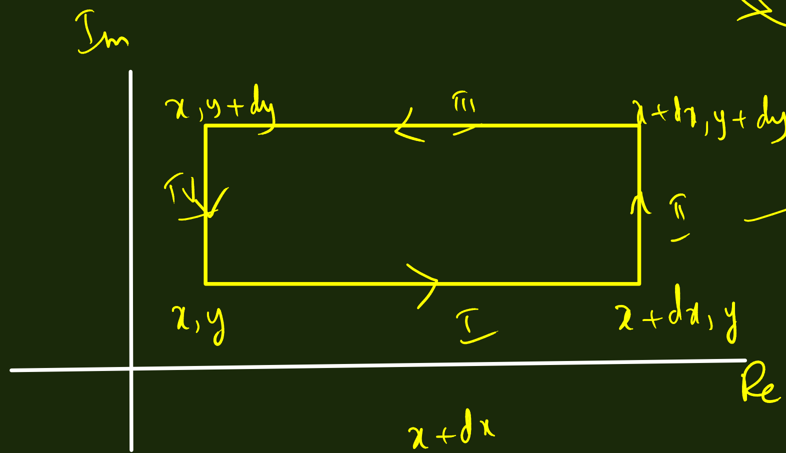
$$\boxed{\oint_C f(z)\, dz = 0}$$





$$\boxed{\oint f(z)\,dz = 0}$$





$$\boxed{\oint f(z)\,dz = 0}$$

$$\underline{I} + \underline{III} = \int_x^{x+dx} f(x,y)\,dx + \int_{x+dx}^{x} f(x,y+dy)\,dx$$

$$= \int_x^{x+dx} f(x,y)\,dx - \int_x^{x+dx} f(x,y+dy)\,dx$$

$$= -\int_x^{x+dx} \left[f(y+dy) - f(y) \right] dx$$

$$\underline{I} + \underline{III} = -\int_x^{x+dx} \frac{\partial f}{\partial y}\,dy\,dx = -\frac{\partial f}{\partial y}\,dy\,dx$$

III ly $\quad \underline{II} + \underline{IV} = i\,\frac{\partial f}{\partial x}\,dx\,dy \qquad f = u + iv$

C-R :

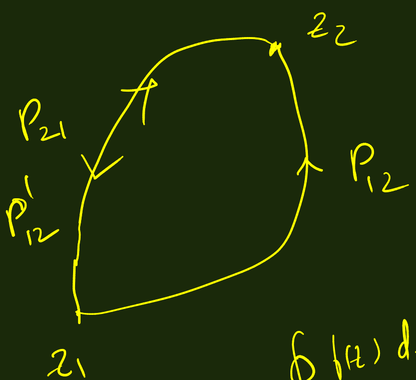
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

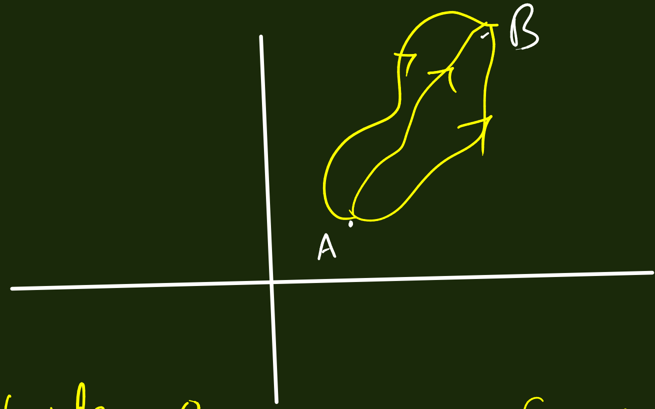
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\oint f(z)\,dz = \underline{I} + \underline{II} + \underline{III} + \underline{IV} = \left[-\frac{\partial f}{\partial y} + i\,\frac{\partial f}{\partial x} \right] dx\,dy$$

$$= \cancel{-\frac{\partial u}{\partial y}} - i\,\cancel{\frac{\partial v}{\partial y}} + i\,\cancel{\frac{\partial u}{\partial x}} \cancel{+\frac{\partial v}{\partial x}}$$

If $f(z)$ is analytic $\int_A^B f(z)\,dz$ is path-independent.





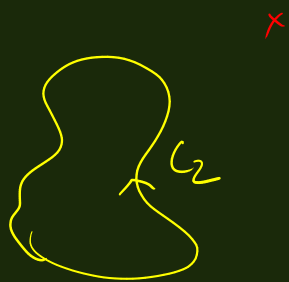
$$\oint f(z)\,dz = 0$$

$$\int_{P_{12}} f(z)\,dz + \int_{P_{21}} f(z)\,dz = 0 \Rightarrow \int_{P_{12}} f(z)\,dz - \int_{P'_{12}} f(z)\,dz = 0$$

$$\int_{P_{12}} f(z)\,dz = \int_{P'_{12}} f(z)\,dz$$

If $f(z)$ is analytic $\rightarrow \int_{z_1}^{z_2} f(z)\,dz \rightarrow$ path-independent !!

$$\oint_{C_1} f(z)\,dz = 0$$



$$\boxed{\oint_{C_2} f(z)\,dz = 0}$$