Differential Equations - Part 3: Method of Eigenfunction Expansion

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linear differential equations $\rightarrow$ Eig. fn expansion (or) Finite Fourier Transform (or) "Separation of Variables" $\rightarrow$ P.D.E's

} ODE's and PDE's

Eigenfunction expansion.

$$\underset{\text{linear diff operator}}{\mathcal{L}} \overset{\text{unknown}}{u} = \overset{\text{known fn.}}{f}$$

$$u(x) = \sum c_i \phi_i(x)$$

Coffs of expansion (c_i)

eig. fns of $\mathcal{L}$ (ϕ_i)

$$\mathcal{L}\phi_i(x) = \lambda_i \phi_i$$

orthonormal eig. fns.

orthonormal eig. fns. $\phi_i(x)$

Hermitian matrices $\rightarrow$ real eig values

eig value problem

$\rightarrow$ eig vecs for distinct eig. values are orthogonal $\rightarrow$ orthonormal.

$\downarrow$

Self-adjoint matrices: $A = A^{\dagger} \neq A^*$ $\rightarrow$ adj

When is a (linear) differential operator self-adjoint ??

$$\mathcal{L} = \mathcal{L}^{\dagger}$$

The adjoint of a diff operator:

adjoint of $\mathcal{L}$

$$\langle v, \mathcal{L}u\rangle = \langle \mathcal{L}^{\dagger} v, u\rangle + \text{"Boundary terms"}$$

0

boundary conditions

For an operator $\mathcal{L}$ to be self adjoint ① $\mathcal{L} = \mathcal{L}^{\dagger}$

and Boundary terms should vanish.

② the BC's of the original and adjoint problems must be identical.

If $\mathcal{L} = \mathcal{L}^{\dagger}$ and the BC's of the original and adj problems are different $\rightarrow$ NOT self-adjoint ; If $\mathcal{L} \neq \mathcal{L}^{\dagger}$ $\rightarrow$ NOT self-adjoint

Examples: (i) $\mathcal{L} = -\dfrac{d^2u}{dx^2}$ B.C: $u(0) = 0$, $u(1) = 0$

$$\langle u, \mathcal{L}v\rangle = \langle \mathcal{L}^{\dagger} u, v\rangle + \text{Boundary terms}$$

Boundary terms made to vanish by choosing appropriate BC's of the adjoint problem.

If $\mathcal{L} = \mathcal{L}^{\dagger}$, $(BC) = (BC)^{\dagger}$ — only then $\mathcal{L}$ is self-adjoint

$$\int_0^1 u^* \left(-\frac{d^2v}{dx^2}\right) dx \longrightarrow \text{Integrate by parts}$$

$$-\left(u^* \frac{dv}{dx}\right)\Big|_0^1 + \int_0^1 \frac{dv}{dx}\frac{du^*}{dx}\, dx$$

Integrate by parts

$$\boxed{-\left(u^* \frac{dv}{dx}\right)\Big|_0^1 + \left(\frac{du^*}{dx} v\right)\Big|_0^1} - \int_0^1 v \frac{d^2u^*}{dx^2}\, dx$$

$$\int_0^1 u^* \left(-\frac{d^2v}{dx^2}\right) dx = \text{Boundary terms} + \int_0^1 \left(-\frac{d^2u^*}{dx^2}\right) v\, dx$$

$$-\left(\cancel{u^*(1)}\, \frac{dv}{dx}(1)\right) + \left(\cancel{u^*(0)}\, \frac{dv}{dx}(0)\right) + \left(\frac{du^*}{dx}(1)\, v(1)\right) - \left(\frac{du^*}{dx}(0)\, v(0)\right)$$

0 0

$v(1) = 0$ $v(0) = 0$

arbitrary

Boundary terms vanish if $\boxed{\begin{array}{l} v(0) = 0 \\ v(1) = 0 \end{array}}$

$$\int_0^1 u^* \mathcal{L}v\, dx = \int_0^1 \mathcal{L}^{\dagger}u^*\, v\, dx$$

$\boxed{\mathcal{L} = \mathcal{L}^{\dagger}}$ → self-adjoint

and BC's identical!

Example (ii): $\mathcal{L} = -\dfrac{d^2}{dx^2}$, B.C's $u(0) - u(1) = 0$, $u'(1) = 0$

Boundary terms:

$$-u^*(1)\frac{dv}{dx}(1) + u^*(0)\frac{dv}{dx}(0) + v(1)\cancel{\frac{du^*}{dx}(1)} - v(0)\frac{du^*}{dx}(0)$$

$$\underbrace{u(0)\left[\frac{dv}{dx}(0) - \frac{dv}{dx}(1)\right]}_{0} - v(0)\frac{du^*}{dx}(0) = 0$$

0 0

$v'(1) = v'(0)$ and $v(0) = 0 \to$ not the same as the original B.C's $\to$ NOT self-adj eventhough $\mathcal{L} = \mathcal{L}^\dagger$

general second order operator

$$\mathcal{L}u = a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)\, u$$

$$\mathcal{L}^\dagger u = \frac{d^2}{dx^2}\left(a_2^*(x)\, u(x)\right) - \frac{d}{dx}\left(a_1^*(x)\, u(x)\right) + a_0^*(x)\, u(x)$$

$$\mathcal{L}^\dagger u = \frac{d}{dx}\left[a_2^*(x)\frac{du}{dx} + u\frac{da_2^*}{dx}\right] - a_1^*\frac{du}{dx} - u\frac{da_1^*}{dx} + a_0^*(x)\, u(x)$$

$$= a_2^*(x)\frac{d^2u}{dx^2} + 2\frac{da_2^*}{dx}\frac{du}{dx} + u\frac{d^2a_2^*}{dx^2} - a_1^*\frac{du}{dx} - u\frac{da_1^*}{dx} + u\, a_0^*$$

If $\mathcal{L}^\dagger = \mathcal{L}$ then,

i) $a_2(x) = a_2^*(x)$

ii) $a_1(x) = 2\dfrac{da_2^*}{dx} - a_1^*(x)$

iii) $a_0(x) = \dfrac{d^2a_2^*}{dx^2} - \dfrac{da_1^*}{dx} + a_0^*(x)$ $\to$ trivially satisfied

Assume $a_i(x) \to$ real-valued fns.

$a_2(x) = a_2(x)$ ✓

$a_1(x) = 2\dfrac{da_2}{dx} - a_1(x) \quad \to \quad a_1(x) = \dfrac{da_2}{dx}$

$$\mathcal{L}u = a_2(x)\frac{d^2u}{dx^2} + \frac{da_2}{dx}\frac{du}{dx} + a_0(x)\, u(x)$$

$$\mathcal{L}u = \frac{d}{dx}\left[a_2(x)\frac{du}{dx}\right] + a_0(x)\, u(x) = 0$$

$\mathcal{L} = \mathcal{L}^\dagger$

Fredholm's solvability conditions for $Ax = b$

$$\mathcal{L}u = f \qquad a < x < b \qquad B_i u = \gamma_i \qquad i = 1 \cdots p$$

$$\mathcal{L}u_h = 0 \qquad a < x < b \qquad B_i u_h = 0$$

$$\mathcal{L}^{+} v = 0 \qquad a < x < b \qquad B_i^{+} v = 0$$

If $\mathcal{L}u_h = 0$ has only trivial soln, $\mathcal{L}u = f$ has unique soln

$\mathcal{L}u_h = 0$ has k. L.I soln $\quad \mathcal{L}u = f$ has soln iff

$$\langle v_h^i ; f \rangle = \sum_{j=1}^{p} \gamma_j \left(B^{+}_{2p+1-j} \, u_h^i \right)^{*}$$

Sturm-Liouville operator:

$$\mathcal{L} = - \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right] \rightarrow \text{orthogonal eig fns}$$

Self-adjoint

Simple-Harmonic
Bessel eqn
Legendre eqn
Chebyshev eqn.

$$\mathcal{L}y = \underbrace{\overbrace{\lambda}^{\text{eigvals}} s \, y}_{\text{fns}}$$

If Boundary terms in $\langle v, \mathcal{L}u \rangle - \langle u, \mathcal{L}v \rangle = 0$