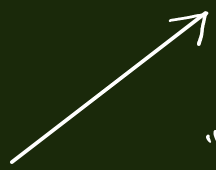
Linear Algebra: Vector Spaces, Matrices

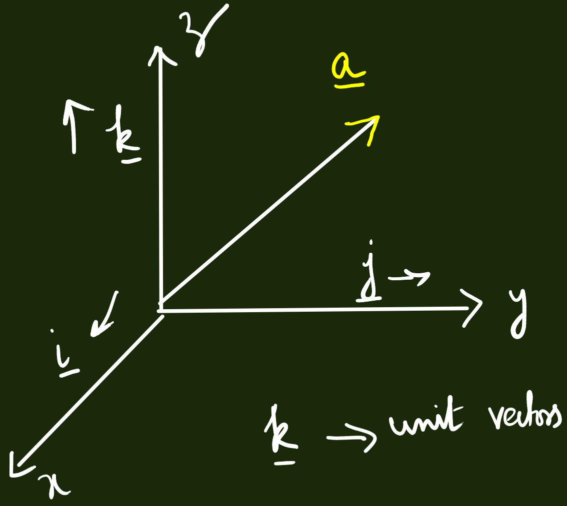
ChE641, IIT Kanpur



3D geometry: A vector is a geometrical object with a magnitude and a direction.

↓

indep. of any coordinate system



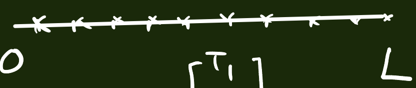
bold face font → vectors

$\underline{a}$ ← vector

$$\underline{a} = a_x \underline{i} + a_y \underline{j} + a_z \underline{k}$$

$$\to a_x' \underline{i}' + a_y' \underline{j}' + a_z' \underline{k}'$$

3D → N-dimensions → ??



$$\begin{bmatrix} T_1 \\ \vdots \\ T_{10} \end{bmatrix} \to \text{10-dimensional vector}$$

$$\begin{bmatrix} T_1 \\ \vdots \\ T_{100} \end{bmatrix} \to \text{100-dimensional vector}$$

<u>Vector space</u>: <u>Axioms</u>

A collection of objects $\underline{a}, \underline{b}, \underline{c}$ .... called vectors form a linear vector space V if

(i) closed under addition: if $a \in V$, $b \in V$

then $a + b \in V$ also

$\in$ → "element of" or "belongs to"

(ii) Commutative and associative:

$$\underline{a} + \underline{b} = \underline{b} + \underline{a}$$

$$(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$$

(iii) The set is closed under multiplication by a scalar

scalar ← real/complex number

if $\underline{a} \in V$ $\quad \lambda \underline{a} \in V$

$\lambda \to$ scalar

(iv) $\lambda(\underline{a}+\underline{b}) = \lambda\underline{a} + \lambda\underline{b}$

$(\lambda+\mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$

$\lambda(\mu\underline{a}) = (\lambda\mu)\underline{a}$

λ, μ : scalars

Distributive associative.

(v) $\underline{a} + \underline{0} = \underline{a}$ for all $\underline{a}$

$\underline{0}$: null vector

(vi) $1 \times \underline{a} = \underline{a}$

$1 \to$ unit scalar

(vii) Negative of $\underline{a}$: $-\underline{a}$ s.t $\underline{a} + (-\underline{a}) = \underline{0}$

If scalars are real #s : Real vector space. (3D geometric space)

If scalars are allowed to be complex : "Complex vector space"

No reference to magnitude/ direction

e.g : The set of all 2×2 matrices.

Vectors are neither real nor complex

Span of a set of vectors:

$\underline{a}, \underline{b}, \underline{c} \dots \underline{s}$ } original set

Set of all vectors that may be written as a linear sum of the original set:

$$\underline{x} = \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} + \dots + \sigma \underline{s}$$

$\alpha, \beta, \dots$ scalars.

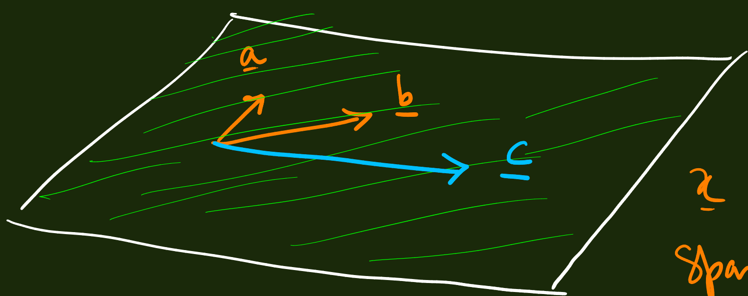
If $\underline{x} = 0$ for some choice of $\alpha, \beta, \gamma \dots \sigma$: then $\underline{a} \dots \underline{s}$ are <u>linearly dependent</u>.
(not $\alpha, \beta, \gamma \dots \sigma = 0$)

<u>1-D space</u>:

x axis

$\underline{a} = \alpha \underline{b} \longrightarrow \underline{a} - \alpha \underline{b} = 0$ $\quad \underline{a}$ & $\underline{b}$ <u>lin. dep</u>.

<u>2-D plane</u>



$\underline{a}$ and $\underline{b}$: not collinear.

$\underline{x} = \alpha \underline{a} + \beta \underline{b}$

Span: entire x-y plane

Any three vectors on a plane are <u>lin dep</u>

<u>N-dim</u>:

$$\sum_{i=1}^{N} \alpha_i \underline{a}_i = 0$$

the vectors $\underline{a}_i$ $(i = 1 \dots N)$ are <u>lin indep</u> iff all $\alpha_i = 0$

<u>Dimension of a vector space</u>: If there are N lin indep vectors, but no set of N+1 lin indep vectors, then the vector space is <u>N-dim</u>

Finite-dimensional: $i = 1, 2, 3, \dots N$

Infinite dimensional $\rightarrow$ <u>later</u>. { functions $\rightarrow$ "vectors" in infinite-dim vector space.

Basis Any set of N lin-indep vectors in an N-dim space is called a basis.

$\underline{e}_i \quad i = 1 \dots N \rightarrow$ N lin-indep vectors

$$\underline{v} = \sum_{i=1}^{N} v_i \, \underline{e}_i$$

"components" of $\underline{v}$ in the given basis.

Linear independence: A set of vectors $\underline{a}_i$ is lin indep if all α_i's $= 0$

in $$\sum_{i=1}^{N} \alpha_i \underline{a}_i = \underline{0}$$

If NOT lin indep: (say) $$\underline{a}_3 = \sum_{\substack{i=1 \\ i \neq 3}}^{N} -\frac{\alpha_i}{\alpha_3} \underline{a}_i$$

Example: $\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\begin{pmatrix}3\\2\\1\end{pmatrix}$

L.I → lin. indep.

Are these L.I ?

$$\alpha_1\begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_3\begin{pmatrix}3\\2\\1\end{pmatrix} = 0$$

lin-dep.

$\boxed{\text{Are L.D}}$

Nontrivial solns for $\alpha_1, \alpha_2, \alpha_3 \rightarrow$?

→ non-zero $\alpha_1, \alpha_2, \alpha_3$

$$\alpha_1 + \alpha_2 + 3\alpha_3 = 0 \longrightarrow -2\alpha_3 - \alpha_3 + 3\alpha_3 = 0$$

$$\alpha_1 + \quad + 2\alpha_3 = 0 \longrightarrow \alpha_1 = -2\alpha_3$$

$$\alpha_2 + \alpha_3 = 0 \longrightarrow \alpha_2 = -\alpha_3$$

Example.

$$\alpha_1\begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_3\begin{pmatrix}0\\1\\1\end{pmatrix} \overset{?}{=} \underline{0}$$

$$\alpha_1 + \alpha_2 = 0 \longrightarrow \alpha_1 = -\alpha_2$$

$$\alpha_1 + \alpha_3 = 0 \longrightarrow \alpha_1 = -\alpha_3$$

$$\alpha_2 + \alpha_3 = 0 \qquad -2\alpha_1 = 0 \quad \text{or } \alpha_1 = 0$$

$$\alpha_2 = 0$$

$$\alpha_3 = 0$$

L.I

Basis. If V is an N-dim vector space, then any set of N L.I. vectors $\underline{e}_1, \underline{e}_2 \dots \underline{e}_N$ forms a basis for V.

$$\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \dots \alpha_N \underline{e}_N = 0 \qquad \text{iff } \alpha_i = 0$$

$$\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2 + \dots \alpha_N \underline{e}_N + \beta \underline{x} = 0 \rightarrow \beta \neq 0$$

$$\underline{x} = \sum \left(\frac{-\alpha_i}{\beta}\right) \underline{e}_i \qquad x_i = -\frac{\alpha_i}{\beta}$$

$$\boxed{\underline{x} = \sum x_i \underline{e}_i}$$

If any vector $\underline{x}$ lying in the span of V can be expressed in terms of the basis vectors $\underline{e}_i$, then $\underline{e}_i$ forms a "Complete set"

Also

$\underline{x} = \sum' x_i' e_i'$

$$\underline{x} = \sum' x_i \underline{e}_i$$

$\{x_i\} \to$ unique.

$$\underline{x} = \sum' y_i \underline{e}_i$$

$$\underline{0} = \sum' (x_i - y_i)\, \underline{e}_i$$

iff $x_i - y_i = 0$

$x_i = y_i$

If any N·L·I vectors $\to$ form a basis in the N-dim vec. space.