Linear Algebra - Part 13: System of Linear Equations - Cont.
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unknown

known

$$\underline{\underline{A}} \cdot \boxed{\underline{x}} = \underline{b}$$

known

$(m \times n) \; (n \times 1) = (m \times 1)$

$\underline{\underline{A}}$: $(m \times n)$ matrix

$\underline{b}$: $(m \times 1)$ vector

$$\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underline{\underline{A}} = [\underline{c}_1 \;\; \underline{c}_2 \; \cdots \; \underline{c}_n] \quad ; \quad \underline{\underline{A}}\,\underline{x} = \underline{b}$$

unknowns

$$\underline{b} = x_1 \underline{c}_1 + x_2 \underline{c}_2 + \cdots + x_n \underline{c}_n$$

known vector $(m \times 1)$

known vectors $(m \times 1)$

$\underline{c}_1 \cdots \underline{c}_n$

$m = n$, $\quad$ n L.I vectors in $\{\underline{c}_i\}$ then $\underline{b} = x_1 \underline{c}_1 + \cdots + x_n \underline{c}_n$

augmented matrix $\quad [\underline{\underline{A}} \mid \underline{b}] \rightarrow$ rank : # of L.I column vectors

Coeff. matrix $\quad \underline{\underline{A}}$

1) $r(\underline{\underline{A}}) = r(\underline{\underline{A}} \mid \underline{b}) = n \rightarrow$ unique soln.

2) $r(\underline{\underline{A}}) = r(\underline{\underline{A}} \mid \underline{b}) < n \rightarrow$ many solutions

3) $r(\underline{\underline{A}}) \neq r(\underline{\underline{A}} \mid \underline{b}) \rightarrow$ no soln. (inconsistent)

Example:

$$2x_2 + x_3 + 4x_4 + 3x_5 + x_6 = 2$$
$$x_1 - x_2 + x_3 + 2x_6 = 0$$
$$x_1 + x_2 + 2x_3 + 4x_4 + x_5 + 2x_6 = 3$$
$$x_1 - 3x_2 - 4x_4 - 2x_5 + x_6 = 0$$

4 eqns; 6 unknowns.

$$\underline{b} = x_1 \underset{\downarrow}{\underline{c}_1} + x_2 \underline{c}_2 + \dots + x_6 \underline{c}_6$$

4-D vectors: (4×1)

Gauss elimination:

$$\begin{cases} x_1 - 3x_2 - 4x_4 - 2x_5 + x_6 = 0 \\ x_2 - \frac{1}{2}x_3 + 2x_4 + x_5 + \frac{1}{2}x_6 = 0 \end{cases}$$

2 eqns
4 unknowns!!

$$\left.\begin{aligned} x_5 + x_6 &= -3 \\ \boxed{x_6 = -5} \end{aligned}\right\} \boxed{x_5 = 2}$$

$$\boxed{\begin{aligned} x_4 &= \alpha_1 \\ x_3 &= \alpha_2 \end{aligned}}$$

$$\boxed{\begin{aligned} x_2 &= \frac{1}{2} - 2\alpha_1 - \frac{1}{2}\alpha_2 \\ x_1 &= \frac{21}{2} - 2\alpha_1 - \frac{3}{2}\alpha_2 \end{aligned}}$$

$$\underline{x} = \begin{pmatrix} \frac{21}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 2 \\ -5 \end{pmatrix} + \alpha_1 \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -3/2 \\ -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Two-parameter family of solns

$\underline{A}\,\underline{x} = \underline{b}$

$$(\underline{A}|\underline{b}) = \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 0 & 2 & 4 \\ 1 & 0 & 3 & 3 & -1 & 6 & 3 \\ 2 & -1 & 2 & 1 & -1 & 7 & 9 \\ 1 & 0 & 5 & 8 & -1 & 7 & 1 \end{array}\right]$$

$m=4$, $n=6$ $m<n$

Gauss elimination $\longrightarrow$ Rank : 3

$$\text{3 eqns} \downarrow \text{6 unknowns} \quad \left[\begin{array}{cccccc|c} 1 & -1 & 1 & 3 & 0 & 2 & 4 \\ 0 & 1 & 2 & 0 & -1 & 4 & -1 \\ 0 & 0 & 2 & 5 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$R(\underline{A}) = R(\underline{A}|\underline{b}) = 3 < 6$

$\downarrow$ multiple solns.

$x_6 = \alpha_1, \quad x_5 = \alpha_2, \quad x_4 = \alpha_3$

$x_3 = -1 - \frac{1}{2}\alpha_1 - \frac{5}{2}\alpha_3 \;; \quad x_2 = 1 - 3\alpha_1 + \alpha_2 + 5\alpha_3$

$x_1 = 6 - \frac{9}{2}\alpha_1 + \alpha_2 + \frac{9}{2}\alpha_3$

$$\underline{x} = \begin{bmatrix} 6 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} -\frac{9}{2} \\ -3 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 9/2 \\ 5 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

3-parameter family $(\alpha_1, \alpha_2, \alpha_3) \rightarrow$ arbitrary

$$\underline{x} = \underline{x}_p + \alpha_1 \underline{x}_1^{(h)} + \alpha_2 \underline{x}_2^{(h)} + \alpha_3 \underline{x}_3^{(h)}$$

— homog. solns

$\underline{x}_p \rightarrow$ particular soln. $\quad \underline{A}\,\underline{x}_p = \underline{b}$

$\underline{A}\,\underline{x}_i^{(h)} = \underline{0}$

$r(A) = r(A|b) = n$ then there are no homog sol

$< n$

$\left.\begin{array}{l} n=6 \\ r=3 \end{array}\right\} n-r=3$

$\boxed{(n-r)} \longrightarrow$ parameter family of solns

If $\underline{\underline{A}}$ is a $(m \times n)$ matrix $\underline{\underline{A}} \cdot \underline{x} = \underline{b}$; This system has a p-parameter family of solns $(p = n - r)$

$$\underline{x} = \underline{x}_p + \alpha_1 \underbrace{\underline{x}_h^{(1)} + \alpha_2 \underline{x}_h^{(2)} + \dots + \alpha_{n-r} \underline{x}_h^{(n-r)}}_{\text{L.I homog. solns}}$$

The space spanned by

$\underline{x}_h^{(1)} \dots \underline{x}_h^{(n-r)} \rightarrow$ null space. $\dim(\text{null space}) = (n-r)$.

$\underline{m < n}$: $\underline{\underline{A}}\, \underline{x} = \underline{b}$ — Cons $\longrightarrow$ can't have unique soln. p-parameter family $\boxed{n - m \leq p \leq n}$; — incons

$\underline{m > n}$. $\underline{\underline{A}}\, \underline{x} = \underline{b}$ — Cons. — unique ; — p-param. family of solns $1 \leq p \leq n$. — incons

Homogeneous system: $\underline{\underline{A}} \cdot \underline{x} = \underline{0}$ $\quad (\underline{b} = 0)$

$r(\underline{\underline{A}}) = r(\underline{\underline{A}} | \underline{b}) \rightarrow$ always consistent $\rightarrow$ trivial soln. $\underline{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is always a soln!!

Nontrivial solns:

(i) if $r(\underline{\underline{A}}) = n \longrightarrow \underline{x} = \underline{0}$

(ii) $r(A) = r < n$, $\longrightarrow$ $(n-r)$ parameter family of soln. (nontrivial)

$\underline{\underline{A}} = n \times n$ $\quad \boxed{\det(\underline{\underline{A}}) = 0}$

$n \times n$ system (Inhomog.) $\quad \underline{\underline{A}} \cdot \underline{x} = \underline{b}$ $\quad \underline{\underline{A}}\ (n \times n)$ $\quad \underline{b}$: $(n \times 1)$

$r = n \longrightarrow \det(\underline{\underline{A}}) \neq 0 \longrightarrow$ matrix $\underline{\underline{A}}$ is non-singular $\longrightarrow$ unique soln.

$r < n \longrightarrow \det(\underline{\underline{A}}) = 0 \longrightarrow$ multiple solns

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

Fredholm's Alternative theorem (solvability cond.)

$\underline{\underline{A}}^{\dagger}$: adjoint of $\underline{\underline{A}}$. $\qquad r(\underline{\underline{A}}) = r(\underline{\underline{A}}^{\dagger})$

$\underline{\underline{A}}$: n rows, m colns $\qquad \underline{\underline{A}}^{\dagger}$: m rows, n colns

$\underline{\underline{A}}^{\dagger} \underline{z} = 0$ (adjoint $\underline{\underline{A}}^{\dagger}$; homogeneous problem)

$\to$ $(m-r)$ L.I. solns $\quad \underline{z}_1 \ldots \underline{z}_{m-r}$

then $\underline{\underline{A}}\,\underline{x} = \underline{b}$ has a soln if and only if $\langle \underline{b} | \underline{z}_j \rangle = 0$

puts a constraint on $\underline{b}$ $\leftarrow$ $\boxed{\underline{b}^{\dagger} \underline{z}_j = 0}$

Example :

$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

What values of $\underline{b}$ will this have a soln ??

$\underline{\underline{A}}^{\dagger} \underline{z} = 0 \qquad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \underline{0} \qquad \underline{z} = \begin{bmatrix} -2/3 \\ -5/3 \\ 1 \end{bmatrix}$

Solvability (Fredholm) : $\underline{b}^{\dagger} \underline{z} = 0$

$$\boxed{-\frac{2}{3} b_1^* - \frac{5}{3} b_2^* + b_3^* = 0}$$

Solution of $\underline{\underline{A}}\,\underline{x} = \underline{b}$ via eig. vec. expansion

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b} \longrightarrow \boxed{\underline{\underline{A}}\,\underline{e}_i = \lambda_i\,\underline{e}_i}$$

Assume distinct eigvales & orthonormal eig. vecs.

$$\boxed{\underline{x} = \sum \alpha_i\,\underline{e}_i} \; ; \quad \underline{b} = \sum \beta_i\,\underline{e}_i$$

$$\underline{\underline{A}} \sum_i \alpha_i\,\underline{e}_i = \sum_k \beta_k\,\underline{e}_k \qquad \langle \underline{e}_j | \underline{e}_i \rangle = \delta_{ij}$$

$$\sum \alpha_i \overbrace{\underline{\underline{A}}\,\underline{e}_i}^{\lambda_i \underline{e}_i} = \sum_k \beta_k\,\underline{e}_k$$

Take inner pdt w.r.t. $\underline{e}_j$ on both sides.

$$\sum_i \alpha_i \lambda_i \underbrace{\langle \underline{e}_j | \underline{e}_i \rangle}_{\delta_{ij}} = \sum_k \beta_k \underbrace{\langle \underline{e}_j | \underline{e}_k \rangle}_{\delta_{kj}}$$

$$\alpha_j \lambda_j = \beta_j \qquad \text{or} \qquad \alpha_j = \frac{\beta_j}{\lambda_j}$$

$$\boxed{\alpha_j = \frac{\beta_j}{\lambda_j}} \rightarrow \text{soln to the } \underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

soln. $\rightarrow$ $\boxed{\underline{x} = \sum_{j=1}^{n} \alpha_j\,\underline{e}_j}$