Exact Solution

- Bi is not small ⇒ spatial effects are significant.
- Exact solution can be obtained for simple geometries, in case of 1D conduction

1D Transient Conduction, (unsteady heat conduction)

- Assume that temperature gradient is nonzero only along one dimension.

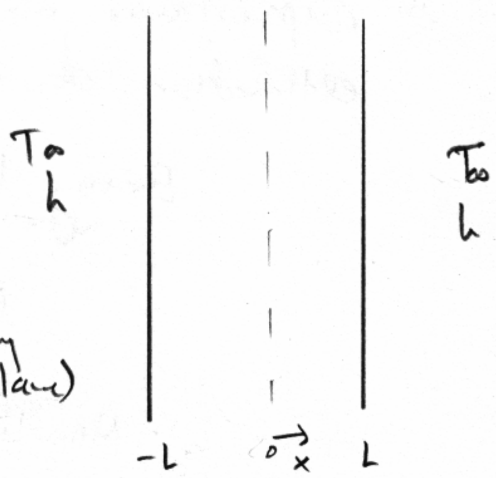
Consider a plane wall (or a problem where cartesian coordinate system can be used), with no heat source.

Heat conduction equation reduces to

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

With the initial condition

$$T(x,0) = T_i$$



and boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0,t} = 0 \quad \text{(symmetry at midplane)}$$

and

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=L,t} = h\left(T(L,t) - T_\infty\right)$$

Thus,

$$T = T(x, t; T_i, T_\infty, L, k, \alpha, h)$$

Nondimensionalization

Let us define the following dimensionless variables

temperature: $\theta^* = \dfrac{\theta}{\theta_i} = \dfrac{T - T_\infty}{T_i - T_\infty}$

x-coordinate: $x^* = \dfrac{x}{L}$

time: $t^* = \dfrac{\alpha t}{L^2} = Fo$

Substituting in the unsteady heat conduction equation and the initial and boundary conditions.

$$\frac{\partial \theta^*}{\partial t^*} = \frac{\partial \theta^*}{\partial Fo} = \frac{\partial^2 \theta^*}{\partial x^{*2}}$$

$$\theta^*(x^*, 0) = 1$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=0, t^*} = 0$$

$$\left.\frac{\partial \theta^*}{\partial x^*}\right|_{x^*=1, t^*} = -Bi\, \theta^*(1, t^*)$$

Thus, the dimensionless temperature

$$\theta^* = \theta^*(x^*, Fo, Bi)$$

Notes:

- A major advantage of nondimensionalization is reducing the number of parameters that a quantity depends on. Thus, independent values of the parameters h, L and k give the same temperature dependence as a single number, Bi.
- Useful in engineering for using existing/universal data

Separation of variables

Let $\theta^*(x,t) = F(x^*)\,G(t^*)$

Then, unsteady conduction equation in 1D reduces to

$$\underbrace{\frac{1}{F}\frac{d^2F}{dx^{*2}}}_{f^n \text{ of } x^* \text{ only}} = \underbrace{\frac{1}{G}\frac{dG}{dt^*}}_{f^n \text{ of } t^* \text{ only}} \Rightarrow \text{both equal to a constant } (= -\lambda^2)$$

$$\frac{d^2F}{dx^{*2}} + \lambda^2 F = 0 \qquad \text{and} \qquad \frac{dG}{dt^*} + \lambda^2 G = 0$$

$$F = C_1 \cos\lambda x^* + C_2 \sin\lambda x^* \qquad G = C_3 e^{-\lambda^2 t^*}$$

Thus,

$$\theta^* = (A\cos\lambda x^* + B\sin\lambda x^*)\,e^{-\lambda^2 t^*}$$

Symmetry at $x^* = 0$ gives

$$\left.\frac{\partial\theta}{\partial x^*}\right|_{x^*=0,t^*} = 0 = e^{-\lambda^2 t^*}\left(-A\lambda\sin\lambda x^* + B\lambda\cos\lambda x^*\right)_{x^*=0}$$

$$\Rightarrow B = 0$$

and $\theta^* = A\cos\lambda x^*\, e^{-\lambda^2 t^*}$

and

$$\left.\frac{\partial\theta}{\partial x^*}\right|_{x^*=1,t^*} = -Bi\,\theta^*(1,t^*) = \left(-Ae^{-\lambda^2 t^*}\lambda\sin\lambda x^*\right)_{x^*=1}$$

$$\Rightarrow \lambda\tan\lambda = Bi$$

Thus, general form of the solution is

$$\theta^* = \sum_{n=1}^{\infty} A_n e^{-\lambda^2 t^*}\cos(\lambda_n x^*)$$

with

$$\lambda_n \tan\lambda_n = Bi$$

Initial condition

$$\theta(x^*,0) = 1 \Rightarrow 1 = \sum_{n=1}^{\infty} A_n \cos(\lambda_n x^*)$$

Multiplying both sides by $\cos \lambda_m x^*$ and integrating from $x^*=0$ to $x^*=1$

$$\int_0^1 \cos \lambda_m x^* \, dx^* = \sum_{n=1}^{\infty} A_n \int_0^1 \cos(\lambda_n x^*) \cos(\lambda_m x^*) \, dx^*$$

$$\left[\frac{\sin \lambda_n x^*}{\lambda_n}\right]_0^1 = A_n \int_0^1 \cos^2(\lambda_n x^*) \, dx^*$$

$$\sin \lambda_n = A_n \frac{[\lambda_n + \sin\lambda_n \cos\lambda_n]}{2\lambda_n}$$

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin 2\lambda_n}$$

Thus, $$\theta^* = \sum_{n=1}^{\infty} A_n e^{-\lambda^2 Fo} \cos(\lambda_n x^*)$$

with

$$\lambda_n \tan \lambda_n = Bi$$

and

$$A_n = \frac{4 \sin \lambda_n}{2\lambda_n + \sin 2\lambda_n}$$

gives the temperature distribution in the wall at different times Fo.

Approximate solution, valid for $Fo > 0.2$ is given by $n=1$. Thus,

for $Fo > 0.2$ $$\theta^* \approx A_1 e^{-\lambda_1^2 Fo} \cos(\lambda_1 x^*)$$

Similarly, exact solution for an <u>infinite cylinder</u> of radius r_0, the exact solution is

$$\theta^* = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t^*} J_0(\lambda_n r^*)$$

$r^* = r/r_0$

$t^* = Fo = \frac{\alpha t}{r_0^2}$

with

$$\frac{\lambda_n J_1(\lambda_n)}{J_0(\lambda_n)} = Bi$$

and

$$A_n = \frac{2}{\lambda_n} \frac{J_1(\lambda_n)}{J_0^2(\lambda_n) + J_1^2(\lambda_n)}$$

And, for a sphere,

$$\theta^* = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t^*} \frac{\sin \lambda_n r^*}{\lambda_n r^*}$$

with

$$1 - \lambda_n \cot \lambda_n = B_i$$

and

$$A_n = \frac{4\left[\sin \lambda_n - \lambda_n \cos \lambda_n\right]}{2\lambda_n - \sin(2\lambda_n)}$$

For $F_o > 0.2$, we can approximate these using $n=1$ as

$$\theta^*_{wall} = A_1 e^{-\lambda_1^2 t^*} \cos(\lambda_1 x^*)$$

$$\theta^*_{cylinder} = A_1 e^{-\lambda_1^2 t^*} J_0(\lambda_1 r^*)$$

$$\theta^*_{sphere} = A_1 e^{-\lambda_1^2 t^*} \frac{\sin \lambda_1 r^*}{\lambda_1 r^*}$$

Note that $\cos(0) = J_0(0) = \lim_{r^* \to 0} \frac{\sin \lambda_1 r^*}{\lambda_1 r^*} = 1$

Thus, at the center of the wall/cylinder/sphere,

$$\theta^*(0,t^*) = A_1 e^{-\lambda_1^2 t^*}$$

Substituting

$$\theta^*_{wall}(x^*,t^*) = \theta^*(0,t^*) \cos(\lambda_1 x^*)$$

$$\theta^*_{cylinder}(r^*,t^*) = \theta^*(0,t^*)\ J_0(\lambda_1 r^*)$$

$$\theta^*_{sphere}(r^*,t^*) = \theta^*(0,t^*) \frac{\sin \lambda_1 r^*}{\lambda_1 r^*}$$

Notes:

- Temperature at any location is related to the temperature at the center.
- Time dependence of temperature is same at all locations