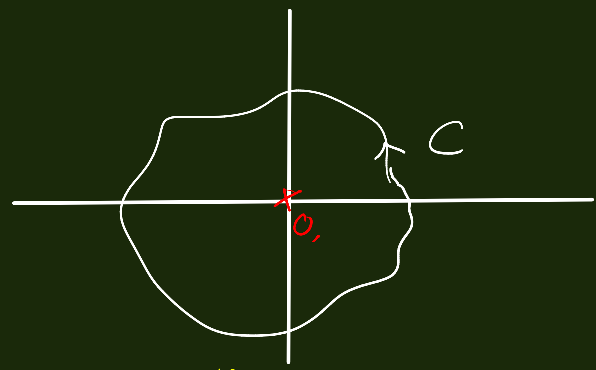
Cauchy's Residue Theorem
ChE641, IIT Kanpur

$$\boxed{\oint_C z^n \, dz = 0}$$

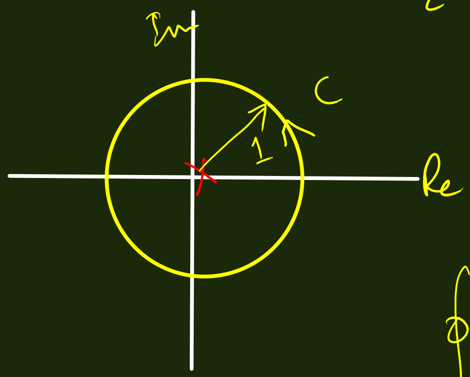
$n = 0, 1, 2, 3 \ldots$

$$\boxed{\oint_C \frac{dz}{z} = 2\pi i}$$



$\oint_C f(z)\, dz = 0$ if $f(z)$ is analytic inside C !

$z = re^{i\theta}$ $\quad r = 1$; $\quad z = e^{i\theta}$ $\quad dz = e^{i\theta} i \, d\theta$



$$\oint dz \, z^n = i \int_0^{2\pi} e^{(n+1)i\theta} \, d\theta = 0 \quad (n \geq 0)$$

$z = e^{i\theta}$ $\quad dz = ie^{i\theta} d\theta$

$$\oint \frac{dz}{z^{n+1}} = ?$$

$n = 0, 1, 2, 3 \ldots$

at the origin, the pole is of $(n+1)^{th}$ order

$$\oint \frac{dz}{z^{n+1}} = \int_{\theta=0}^{2\pi} \frac{ie^{i\theta} \, d\theta}{e^{i(n+1)\theta}} = i \int_0^{2\pi} e^{-2ni\theta} \, d\theta$$

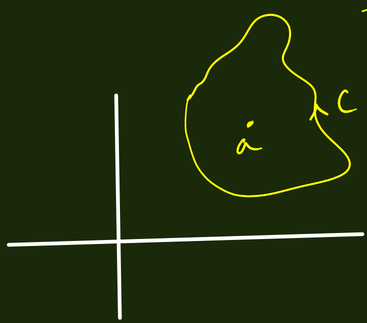
$\underline{n=0}$: $\quad i \int_0^{2\pi} d\theta = 2\pi i$

$(n > 0)$ $\quad n \neq 0$: $\quad \oint \frac{dz}{z^{n+1}} = 0$

$$\oint \frac{dz}{z^{n+1}} = 0 \quad \text{except for } n = 0.$$

Kronecker delta

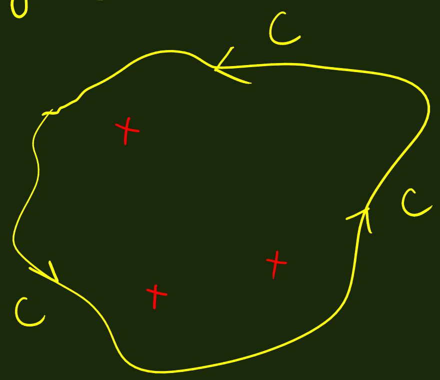
$$\frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} = \delta_{n,0} \quad \begin{cases} n = 0 \rightarrow 1 \\ n \neq 0 \rightarrow 0 \end{cases}$$



$$\boxed{\frac{1}{2\pi i} \oint \frac{dz}{(z-a)^{n+1}} = \delta_{n,0}}$$

Cauchy Residue Theorem:

$f(z)$ has finite # of poles as singularities.



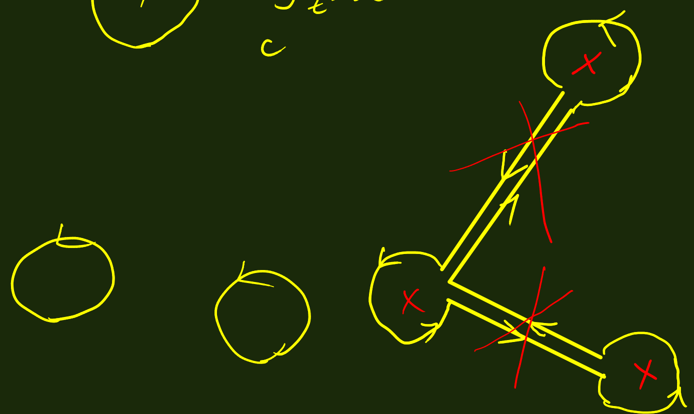
$$\boxed{\oint f(z)\, dz = 2\pi i \sum_{z_i} \mathrm{Res}(z_i)}$$

Residue Theorem.

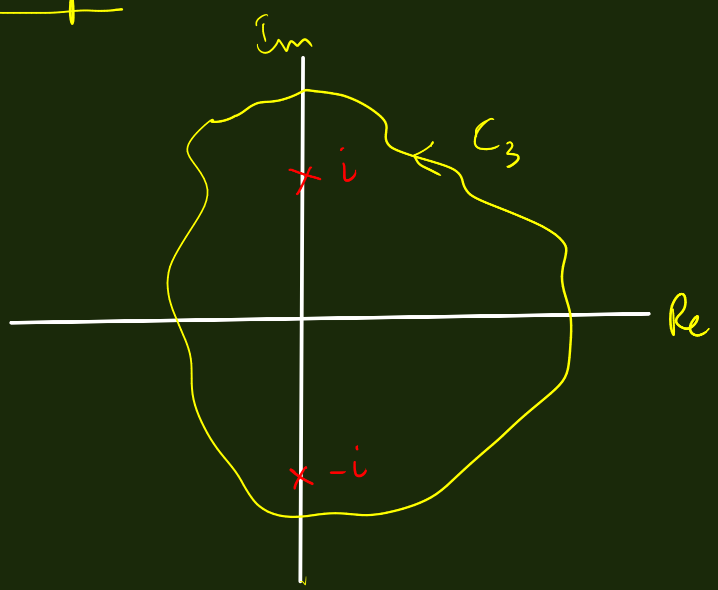
$$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{i=1}^{N} \mathrm{Res}(z_i)}$$

Residue Theorem

$$\oint_c \frac{dz}{z-a} = 2\pi i$$



Example:



$$f(z) = \frac{1}{z^2+1}$$

$$f(z) = \frac{1}{(z+i)(z-i)}$$

$$\oint_{C_1} f(z)\, dz = 2\pi i \ \mathrm{Res}(z=i)$$

$$= \lim_{z \to i} 2\pi i \left[\cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} \right]$$

$$\oint_{C_1} \frac{dz}{1+z^2} = \pi$$

$$\oint_{C_2} \frac{dz}{1+z^2} = 2\pi i \ \mathrm{Res}[z=-i]$$

$$\lim_{z \to -i} 2\pi i \left[\cancel{(z+i)} \frac{1}{\cancel{(z+i)}(z-i)} \right]$$

$$= -\pi$$

$$\oint_{C_3} \frac{dz}{1+z^2} = 2\pi i \left(\pi - \pi\right) = 0!!$$

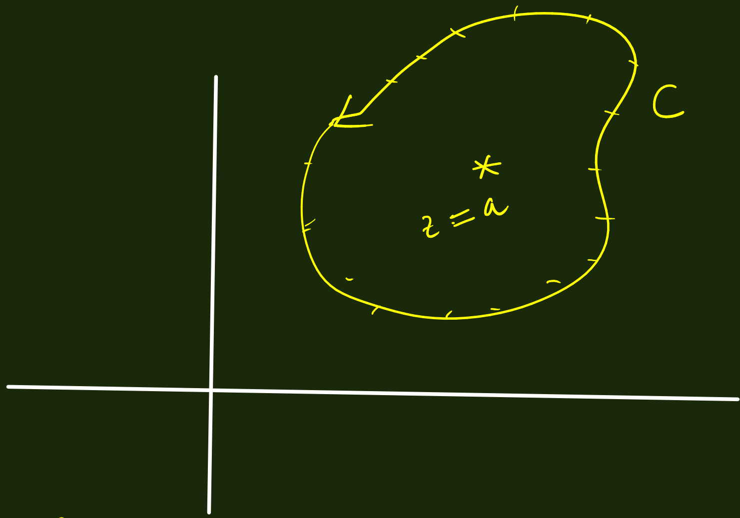
Cauchy's Integral formula:

$$F(z) = \frac{f(z)}{(z-a)}$$ ← analytic within C

$$\oint_C F(z)\,dz = 2\pi i\, f(a)$$

$$\boxed{\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-a)} = f(a)}$$

Cauchy's Integral formula.



$$\boxed{\oint_C \frac{f(z)}{z-a} = f(a)}$$

Residue

$$\frac{1}{z}$$

$$\oint_C e^{\frac{1}{z}}\,dz = 2\pi i \times 1 = 2\pi i$$

$$\oint_C e^{\frac{1}{z^2}}\,dz = 0$$

Because there is no term that is of the form $\frac{1}{z}$