Application of Contour Integration: Laplace and Fourier Inversions

ChE641, IIT Kanpur

Laplace Transform → used to solve ordinary differential eqns. (ODE's)

$f(t)$: defined s.t $\left.\begin{array}{ll} f(t) = 0 & t \leq 0 \\ f(t) \neq 0 & t > 0 \end{array}\right\}$ "Initial value problems"

t : time (real variable).

s : Laplace variable – complex in general.

Laplace transform of $f(t) \rightarrow F(s)$.

$$\rightarrow F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

integration variable

fixed parameter.

known

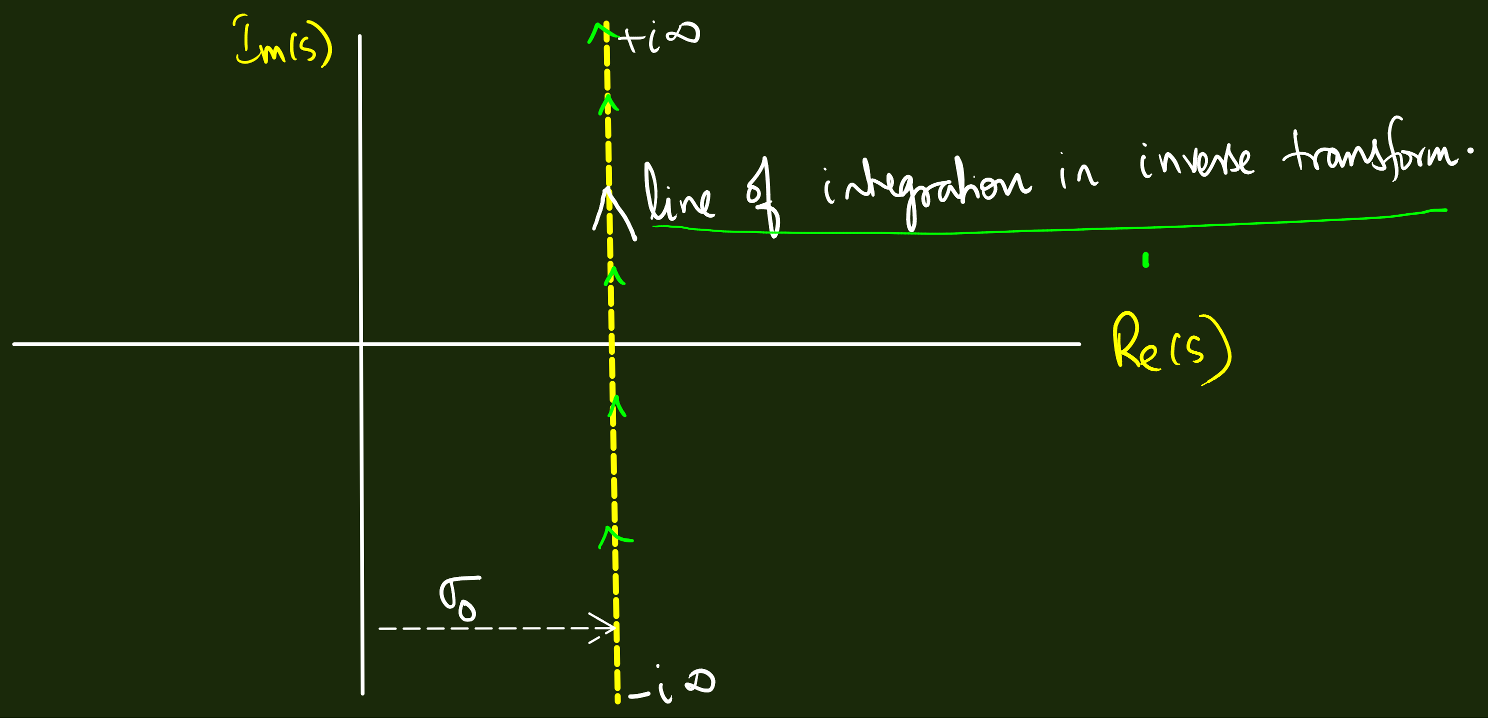
Inverse Laplace transform:

$$\rightarrow f(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\sigma_0 - i\omega}^{\sigma_0 + i\omega} e^{st} F(s)\, ds$$

fixed parameter

integration variable.

known



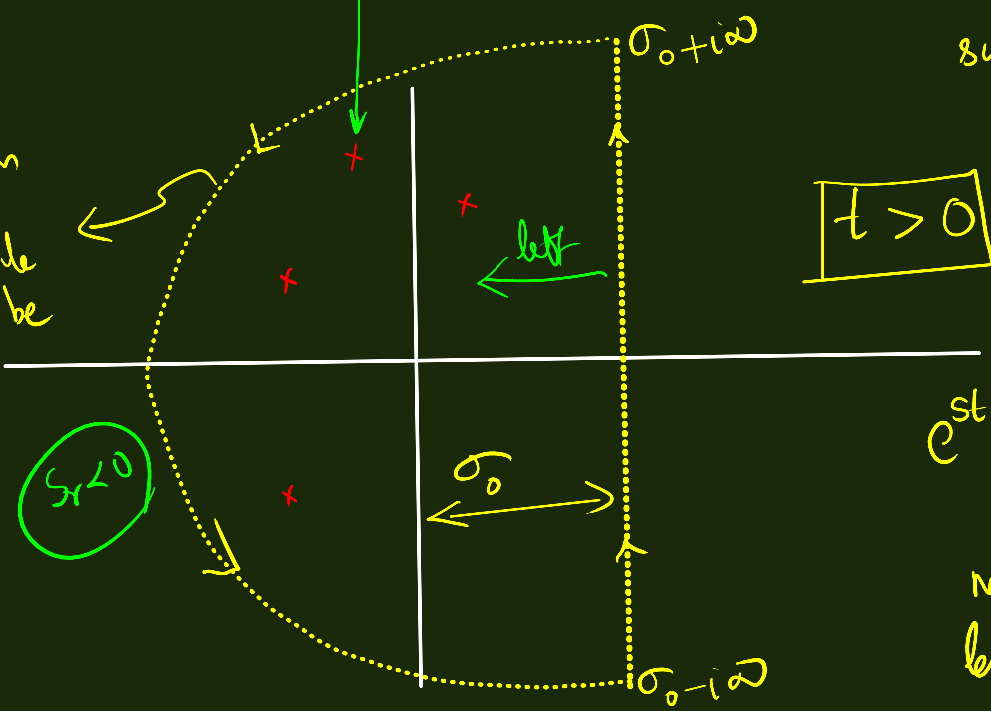
Convert $\displaystyle\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{st}\, F(s)\, ds \longrightarrow \oint_C e^{st}\, F(s)\, ds$

poles

Integral over a suitable closed contour.

Contribution from the semicircle should be zero.

$s_r<0$



$\boxed{t>0}$ $f(t)$

$s = s_r + i s_i$

$e^{st} = e^{s_r t}\, e^{i s_i t}$

negligible in the left half plane

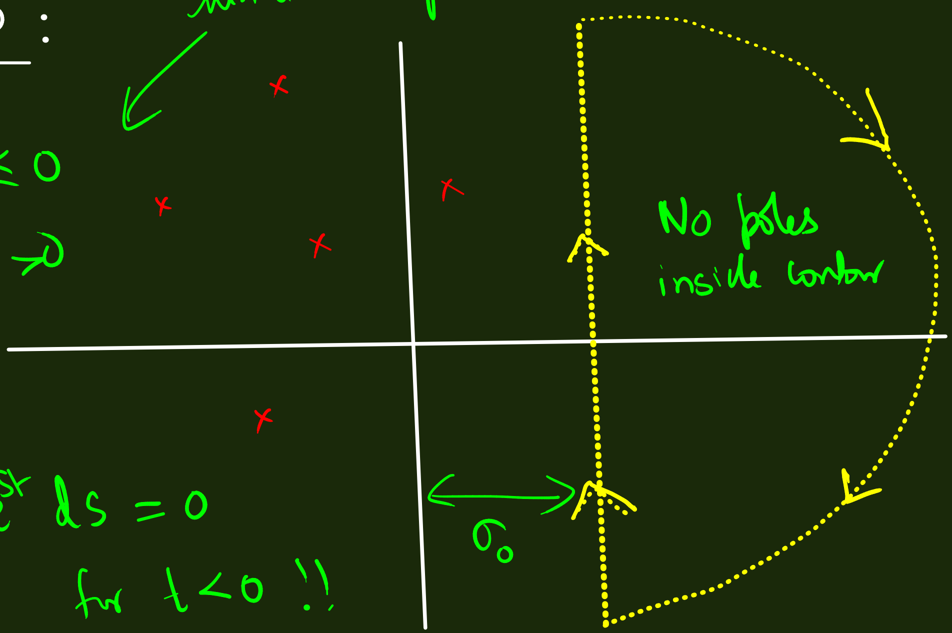
Choose σ_0 s.t all the singularities (i.e poles) are to the $\boxed{\text{left}}$ of the $s_r=\sigma_0$ line.

For $t<0$:

Initial value problem!

$f(t)=0 \quad t\leq 0$

$\neq 0 \quad t\to\infty$



e^{st}

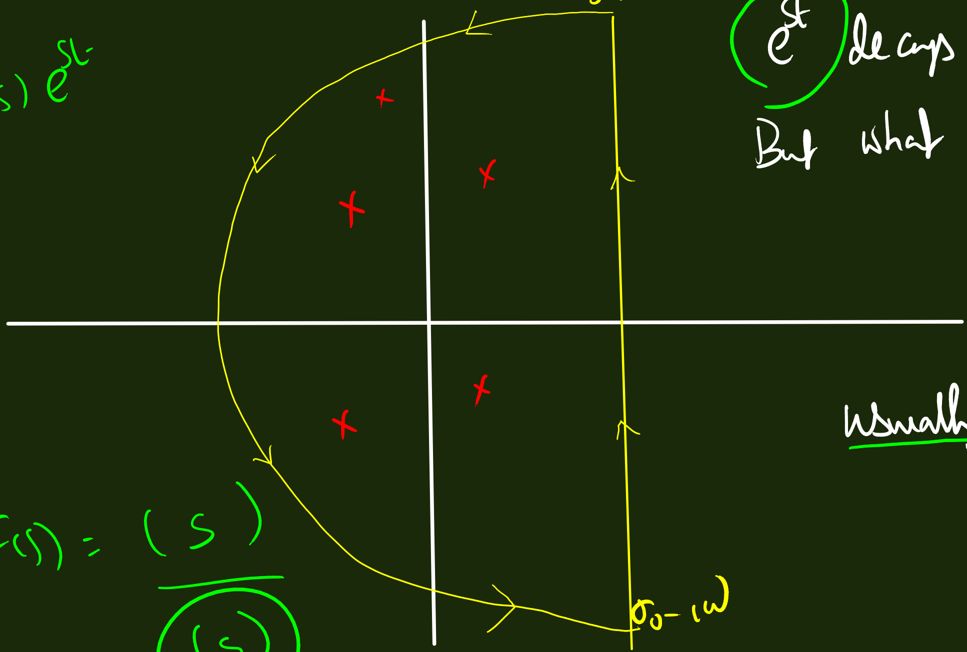
$e^{s_r t}\quad e^{i s_i t}$

decays only for $s_r>0$

$f(t) = \oint F(s)\, e^{st}\, ds = 0$

for $t<0$!!

$F(s)\,e^{st}$



e^{st} decays along the semicircle

But what about $F(s)$??

$F(s)$: Ratio of two polynomials

$F(s) = \dfrac{(s)}{(s)}$

Usually, degree of numerator $<$ degree of denominator

So $F(s) \propto \dfrac{1}{R^{(\cdot)}} \to 0$ as $R \to \infty$ if $s = Re^{i\theta}$

$\to 0$ as $R \to \infty$.

Residue Theorem.

Thus, $$\int_{\sigma_0 - i\omega}^{\sigma_0 + i\omega} e^{st} F(s)\, ds = \oint_C e^{st} F(s)\, ds = 2\pi i \sum \text{Res. of } \left(e^{st} F(s)\right) \text{ inside } C$$

"First Bromwich Contour"

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\omega}^{\sigma_0 + i\omega} e^{st} F(s)\, ds = \sum_{\text{Br. 1}} \left[\text{Res. of } e^{st} F(s) \right].$$

Example 1 : Inverse Laplace transform of $F(s) = \dfrac{1}{(s+1)(s+2)}$

Two poles at $s=-1$, $s=-2$.

$s-(-1)$

Res of $e^{st}F(s)$ at $s=-1$: $(s+1)\,F(s)\,e^{st}\Big|_{s=-1} = e^{-t}$

Res. of $e^{st}F(s)$ at $s=-2$: $(s+2)\,F(s)\,e^{st}\Big|_{s=-2} = -e^{-2t}$

So, $\boxed{f(t) = e^{-t} - e^{-2t}}$ ✓

Example: 2 Laplace Inverse of $F(s) = \dfrac{1}{s^2(s+1)^2}$

Two second order poles at $s=0$, $s=-1$

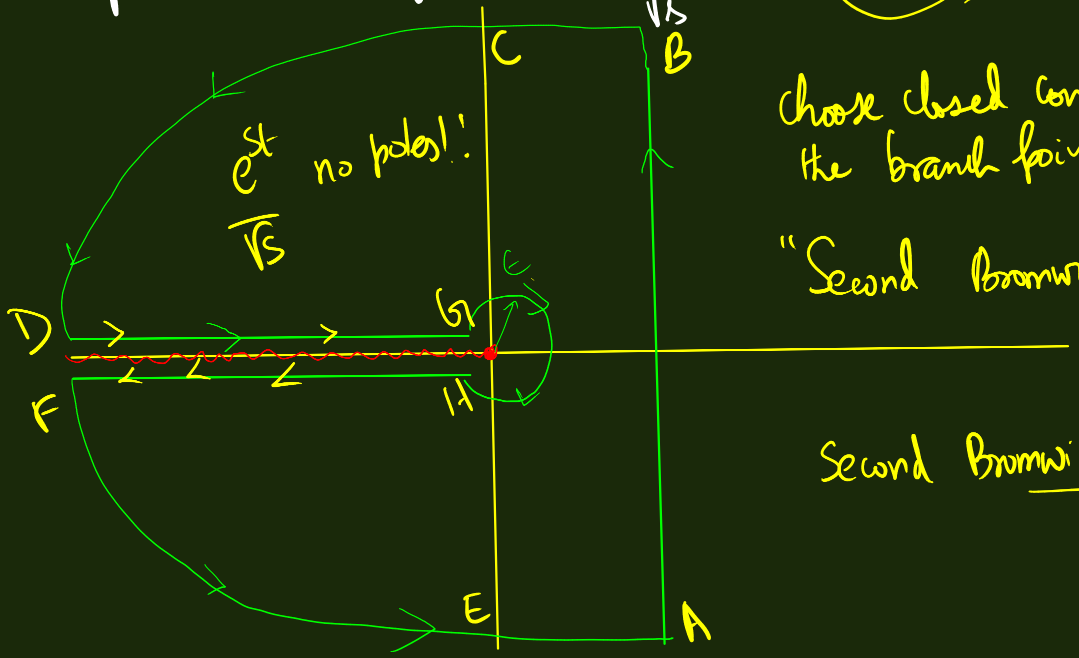
$$\text{Res}(s=0) = \frac{1}{1!}\frac{d}{ds}\left(\frac{e^{st}}{(s+1)^2}\right)_{s=0} = \frac{t\,e^{st}}{(s+1)^2}\Bigg|_{s=0} - \frac{2}{(s+1)^3}e^{st}\Bigg|_{s=0}$$

$$= \boxed{(t-2)}$$

$$\text{Res.}(s=-1) = \frac{1}{1!}\frac{d}{ds}\left(\frac{e^{st}}{s^2}\right)_{s=-1} = \boxed{t\,e^{-t} + 2e^{-t}}$$

$\boxed{f(t) = (t-2) + (2+t)\,e^{-t}.}$ Verify : $t=0$, $f(t=0) = 0.$

Inverse Laplace Transform of $F(s) = \dfrac{1}{\sqrt{s}}$. $s=0$ is a branch point.



Choose closed contour avoiding the branch point at origin.

"Second Bromwich path"

Second Bromwich contour

$$\oint_{Br2} \frac{e^{st}}{\sqrt{s}}\,ds \underset{\substack{\uparrow \\ \text{By Residue theorem.}}}{=} 0 = \underbrace{\int_{BCD}}_{\to 0} + \int_{DG} + \underbrace{\int_{\epsilon}}_{\to 0} + \int_{HF} + \underbrace{\int_{FE}}_{\to 0} + \underbrace{\int_{EA}}_{\to 0} + \underbrace{\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{st}\frac{1}{\sqrt{s}}\,ds}_{\text{We want this}}$$

So we are left with:

$$\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{st}\frac{1}{\sqrt{s}}\,ds = -\int_{DG} - \int_{HF}$$

Along DG: $s = re^{i\pi} = -r$; $\sqrt{s} = \sqrt{r}\,e^{i\pi/2} = i\sqrt{r}$

$$\int_{DG} e^{st}\frac{1}{\sqrt{s}}\,ds = \int_{-\infty}^{0} \frac{e^{-rt}}{i\sqrt{r}}(-dr) = -i\int_0^\infty e^{-rt}\, r^{-1/2}\,dr$$

Along HF: $s = re^{-i\pi} = -r$, $\sqrt{s} = r^{1/2} e^{-i\pi/2} = -i\sqrt{r}$

$$\int_{HF} e^{st}\frac{1}{\sqrt{s}}\,ds = \int_0^{-\infty} \frac{e^{st}}{\sqrt{s}}\,ds = \int_0^\infty \frac{e^{-rt}}{-i\sqrt{r}}\,dr = -i\int_0^\infty e^{-rt}\, r^{-1/2}\,dr$$

So, $\displaystyle\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{st}\frac{1}{\sqrt{s}}\,ds = 2i\int_0^\infty \frac{e^{-rt}}{\sqrt{r}}\,dr$

$$f(t) = \frac{1}{\cancel{2\pi i}}\,\cancel{2i}\int_0^\infty e^{-rt}\frac{1}{\sqrt{r}}\,dr$$

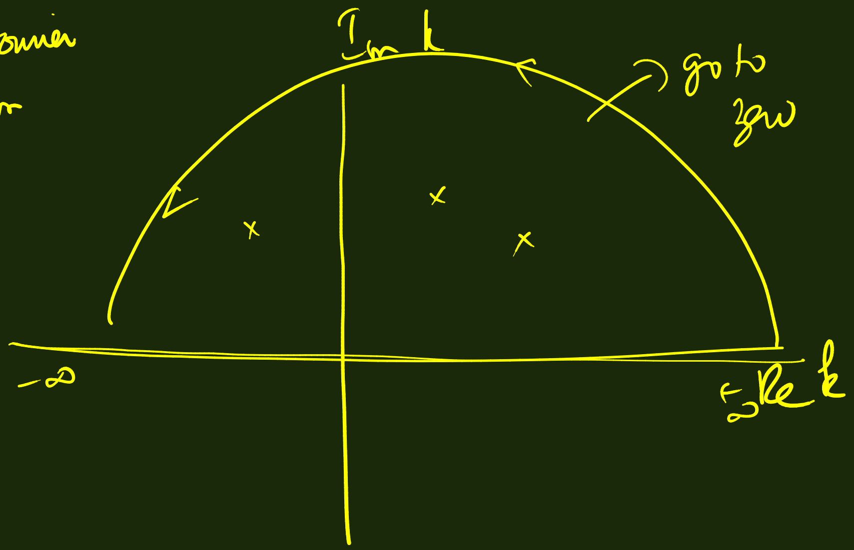
let $rt = \beta^2$

$$f(t) = \frac{1}{\pi\sqrt{t}}\int_0^\infty e^{-\beta^2}\,d\beta \;\to \sqrt{\pi}$$

$$= \frac{1}{\sqrt{\pi t}}$$

$$\frac{1}{\sqrt{s}} \to \frac{1}{\sqrt{\pi t}}$$

Inverse Fourier transform



$$k \cdot \int_{-\infty}^{\infty}$$