Evaluation of Integrals Using the Residue Theorem
ChE641, IIT Kanpur

Laplace Transform, Fourier Transforms $\longrightarrow$ evaluation of integrals.

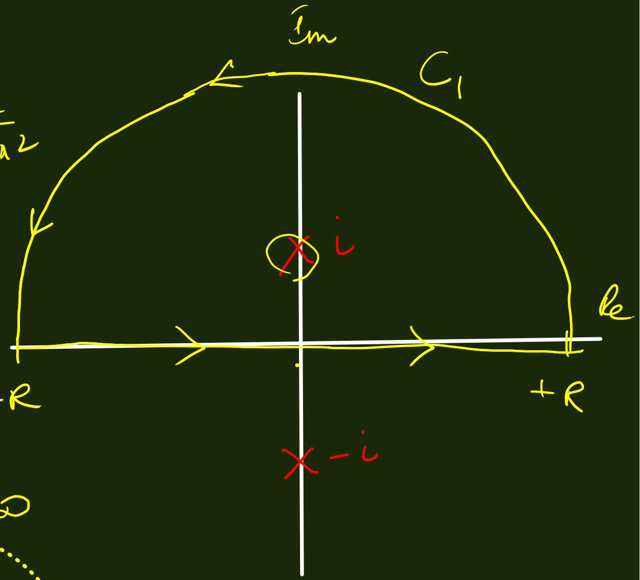
$f(x) = f(-x)$

Example 1

$$\int_0^\infty \frac{dx}{1+x^2} \; ; \quad x - \text{real !} \longrightarrow \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$\int_0^\infty \frac{dz}{1+z^2}$$

$$f(z) = \frac{1}{1+z^2}$$



$$\frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$$

$$\int_{C_1} f(z)\,dz = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz + \lim_{R\to\infty} \int f(z)\,dz$$

$$z = Re^{i\theta}$$

$$\lim_{R\to\infty} \int \frac{R\,i\,d\theta\, e^{i\theta}}{\cancel{1} + R^2 e^{2i\theta}}$$

as $R \to \infty$

$$0 \; !! \longleftarrow \lim_{R\to\infty} \int \frac{i\,d\theta}{R\,e^{i\theta}}$$

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \int_{C_1} f(z)\,dz \longrightarrow \text{Residue theorem}$$

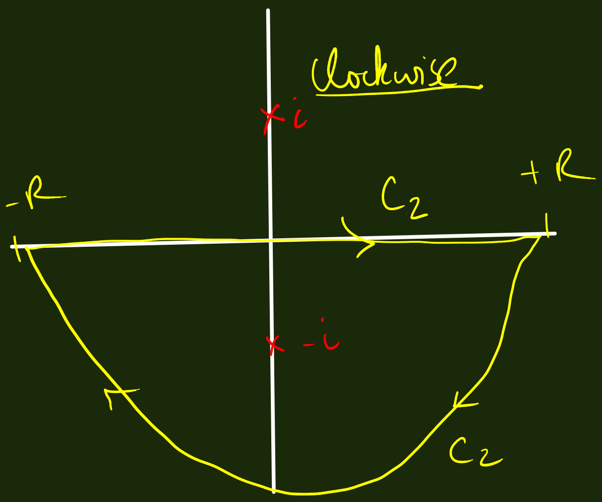
$$= 2\pi i \sum \text{Residues}$$

$$= 2\pi i \;\text{Res}[z=i]$$

$$= 2\pi i \lim_{z\to i} \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi$$



$$\oint_{C_2} f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$-2\pi i \times \text{Res}\,[z=-i]$$

$$= -2\pi i \times \lim_{z \to -i} \cancel{(z+i)}\,\frac{1}{(z-i)\,\cancel{(z+i)}}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

$$\boxed{\int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}}$$

Note regarding Residues:

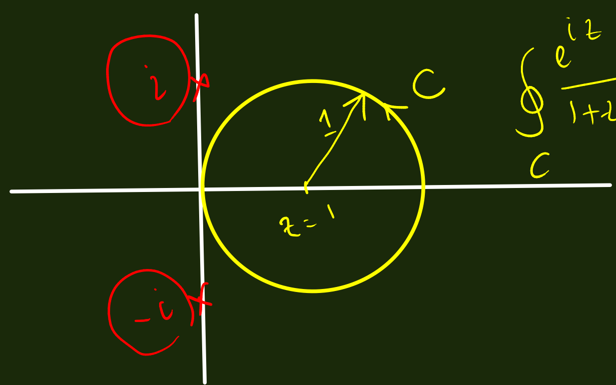
Residue is the coeff of $\frac{1}{(z-z_0)}$ in the Laurent series.

$$\frac{1}{2z-4} = \frac{1}{2\left(z-\frac{4}{2}\right)} = \frac{\frac{1}{2}}{(z-2)}$$

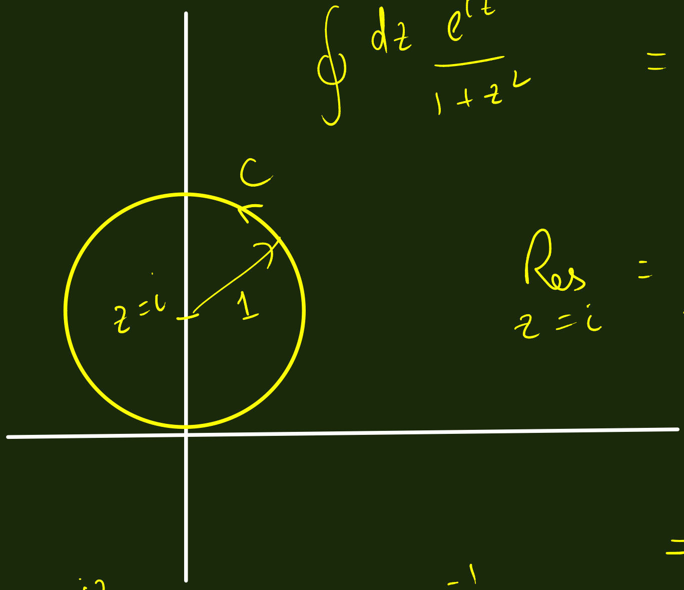
Residue

Example:

$$\oint_C \frac{e^{iz}}{1+z^2}\,dz \qquad C \text{ is a unit circle}$$



$$\oint_C \frac{e^{iz}}{1+z^2} = 0 \;!!$$

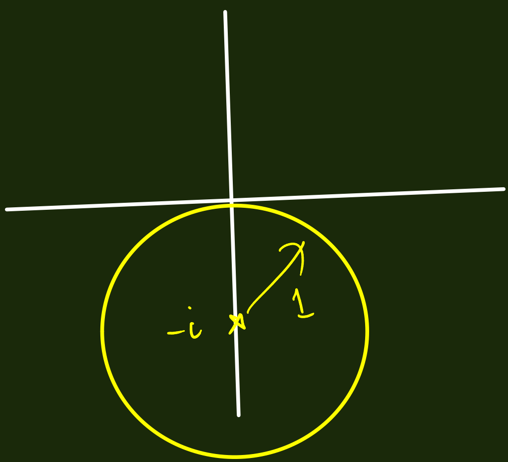


$$\oint dz \frac{e^{iz}}{1+z^2} = \oint \frac{e^{iz}\,dz}{(z+i)(z-i)}$$

$$\operatorname*{Res}_{z=i} = \lim_{z\to i}\left[\cancel{(z-i)}\frac{e^{iz}}{(z+i)\cancel{(z-i)}}\right]$$

$$= \frac{e^{-1}}{2i}$$

$$\oint dz \frac{e^{iz}}{1+z^2} = \cancel{2}\pi \cancel{i} \frac{e^{-1}}{\cancel{2i}} = \frac{\pi}{e}$$



$$\oint \frac{e^{iz}\,dz}{(z+i)(z-i)} = 2\pi i \lim_{z\to -i}\left[\cancel{(z+i)}\frac{e^{iz}}{\cancel{(z+i)}(z-i)}\right]$$

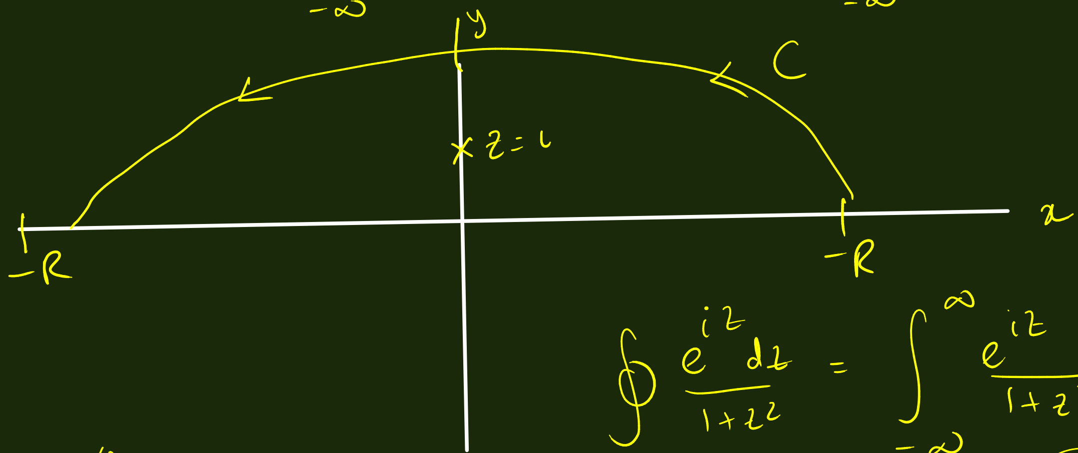
$$= -\pi e$$

$$\cos x = Re(e^{ix})$$

$$\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2} = \int_{-\infty}^{\infty}\frac{e^{ix}}{1+x^2}dx \rightarrow \int_{-\infty}^{\infty}\frac{e^{iz}}{(1+z^2)}dz$$

$$e^{iz} = e^{ix}\cancel{e^{-y}}$$

0



$$\oint \frac{e^{iz}\,dz}{1+z^2} = \int_{-\infty}^{\infty}\frac{e^{iz}}{1+z^2}dz = \frac{\pi}{e}$$

$$\int_{-\infty}^{\infty}\frac{\cos z}{1+z^2}dz + i\int_{-\infty}^{\infty}\frac{\sin z}{1+z^2}dz = \frac{\pi}{e}$$

$$\boxed{\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos x}{1+x^2} &= \frac{\pi}{e}\\ \int_{-\infty}^{\infty}\frac{\sin x}{1+x^2} &= 0\end{aligned}}$$

Cauchy's Integral formula:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0} = f(z_0)$$

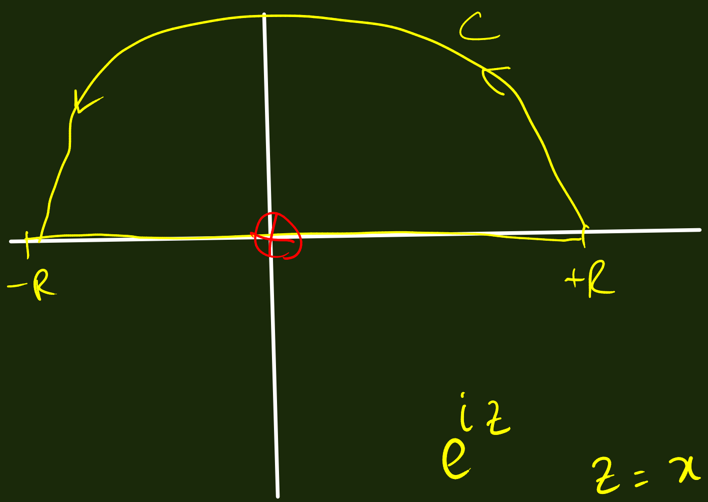
$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2} = f'(z_0) \longrightarrow \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} = f^n(z_0).$$

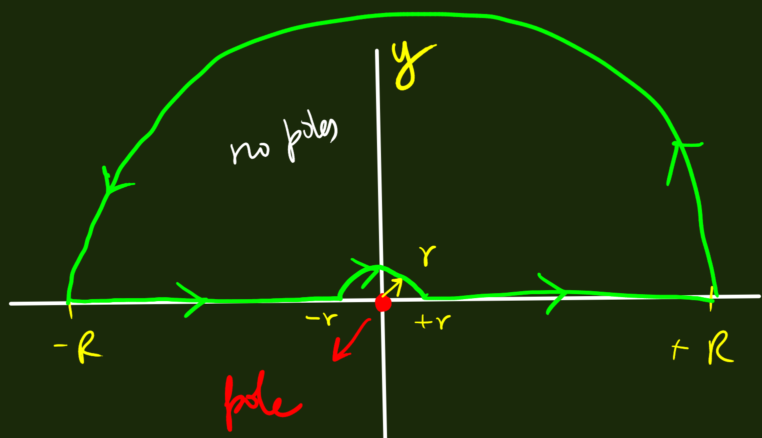
Example.

$$\oint_{|z|=1} \frac{\cos z}{z^3}\,dz = \frac{2\pi i}{2!}\,\frac{d^2}{dz^2}\cos z\Big|_{z=0} = \frac{2\pi i}{2}(-1) = -\pi i$$

Example:

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx \longrightarrow \int_{-\infty}^{\infty}\frac{\sin z}{z}\,dz \longrightarrow \int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz$$





e^{iz} $\quad z = x+iy$

$$e^{iz} = e^{ix}\,e^{-y} \longrightarrow \text{decays exponentially in the upper half}$$

$$\lim_{\substack{R\to\infty \\ r\to 0}} \oint_C \frac{e^{iz}}{z}\,dz = \int_{-R}^{-r}(\ldots) + \underbrace{\int(\ldots)}_{\text{semi circle radius } r} + \int_{+r}^{+R}(\ldots) + \underbrace{\cancel{\int(\ldots)}}_{\text{vanishes}}$$

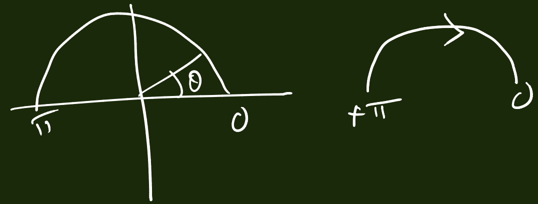
$\oint_C \to 0$

$$\int_{-\infty}^{\infty}\frac{e^{iz}}{z}\,dz + \int_{\text{semi circle } r}\frac{e^{iz}}{z}\,dz = 0$$

Integral over the smaller semicircle.

$z = re^{i\theta} \qquad dz = re^{i\theta}\, i\,d\theta$

$$\int \frac{e^{iz}}{z}\,dz = \int_{+\pi}^{0} i\,d\theta = -i\pi \qquad \frac{dz}{z} = i\,d\theta$$



as $r \to 0$, $e^{iz} \to 1$

$$\oint \frac{e^{iz}}{z}\,dz = 0 \qquad \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx - i\pi = 0$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = i\pi$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx = 0} \quad \text{and} \quad \boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi}$$

Cauchy Principal Value:

$\frac{\cos x}{x}$ as $x \to 0$, $\frac{1}{x}$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0$$

$$\left.\begin{aligned} &\int_0^3 \frac{dx}{x-3} \longrightarrow \ln(x-3)\Big|_{x=0}^{3} \\ &\int_3^5 \frac{dx}{x-3} \end{aligned}\right\} \text{divergent}$$

$$\text{P.V} \int_0^5 \frac{dx}{x-3} = \ln\frac{2}{3}$$

Cauchy principal values..

$$\int_0^{3-r} \frac{dx}{x-3} = \ln|x-3|\Big]_0^{3-r} = \cancel{\ln r} - \ln 3$$

$$\int_{3+r}^{5} \frac{dx}{x-3} = \ln 2 \cancel{- \ln r}$$