# Function Spaces - Part 3: The Laplace Transform

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Fourier Transform: $\longrightarrow$ generalized Fourier series

$\quad\quad \hookrightarrow$ periodic fn $-L$ to $L$


$$f(x) = \sum_{k=-\infty}^{+\infty} C_k \, e^{\frac{ik\pi x}{L}}$$

$\psi_k(x)$

Fourier series: periodic fns

eig. fn expansion

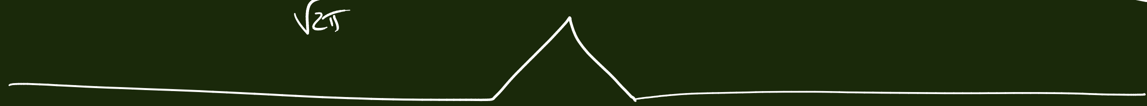
$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} C_k \, \psi_k(x)}$$

Fourier Transform: $\longrightarrow$ generalization $\rightarrow$ non-periodic fns

$T \rightarrow \infty$


product $= \frac{1}{2\pi}$

$\frac{1}{\sqrt{2\pi}}$ $\quad\quad$ $\frac{1}{\sqrt{2\pi}}$



F.T pair.

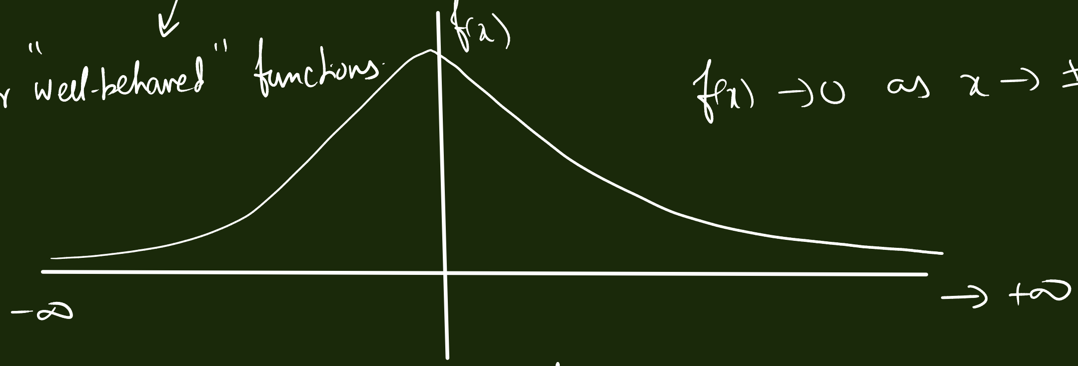
$$\hat{f}(\omega) = \frac{1}{\cdot}\int_{-\infty}^{\infty} dx \; f(x) \, e^{-i\omega x} \quad \Bigg| \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; \hat{f}(\omega) \, e^{i\omega x}$$

F.T of $f(x)$

F.T is defined for fns that decay as $x \to \pm\infty$

for "well-behaved" functions.

$f(x) \to 0$ as $x \to \pm\infty$



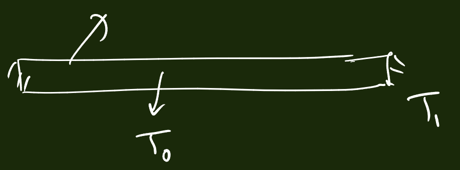

Fourier series: periodic fns → $f(t)$, $f(x)$

non-periodic fns. → Fourier transform → $f(x)$

$f(t)$ "Integral transforms" → differential eqns.

time variable → Initial condition

$(T - T_0) = T'$

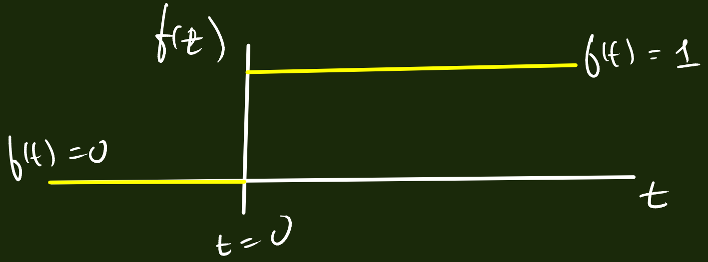



$t \geq 0 \quad f(t) = f_0$

$t < 0 \quad f(t) = 0$

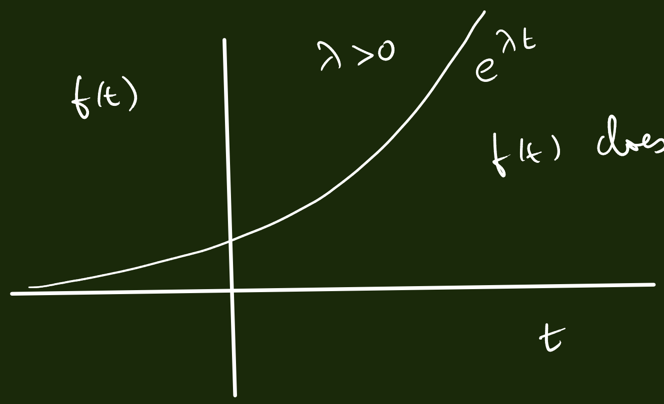
"Initial condition"

Fourier transform → $f(x, y, z)$ : spatial variables.




→ $f(t)$ does not decay to zero as $t \to \infty$

→ can't define F.T.

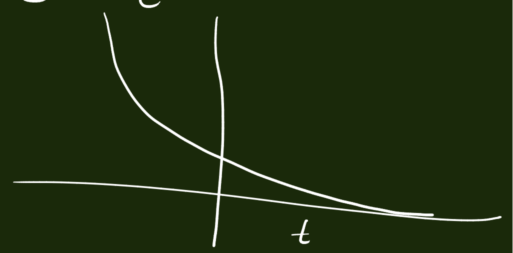



$f(t)$ does not decay as $t \to \infty$ : Can't define F.T

" not well-behaved "

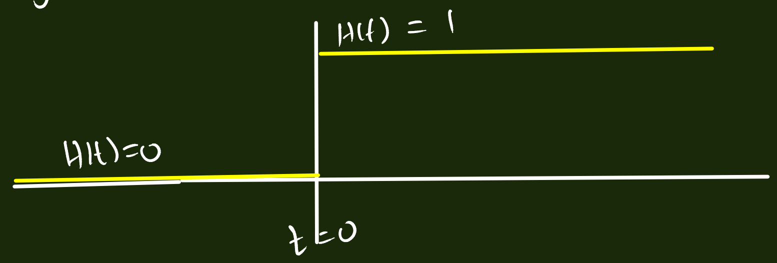
$f(t) \longrightarrow$ Not well-behaved $\longrightarrow$ No F.T exists.

(i) $f(t) \cdot e^{-\gamma t}$ s.t product $\to 0$ as $t \to +\infty$




(ii) Multiply by " Heaviside Step fn "

$H(t)$




$F(t) = f(t)\, e^{-\gamma t}\, H(t) \longrightarrow$ well-behaved as $t \to +\infty$ and $t \to -\infty$ !!

F.T of $F(t)$ $\equiv$ Laplace transform of $f(t)$

F.T of $F(t) = \hat{F}(\omega)$

$F(t) = 0 \quad t < 0$

$F(t) = f(t)\, e^{-\gamma t} \quad t \geq 0$

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} 0 \cdot dt + \int_{0}^{\infty} f(t)\, e^{-\gamma t}\, e^{-i\omega t}\, dt$$

0

$\gamma + i\omega = s$

Laplace variable.

$$\tilde{f}(s) \quad \hat{F}(\omega) = \int_{0}^{\infty} f(t)\, e^{-(\gamma + i\omega)}\, dt$$

s

The F T of F(T) is the Laplace Transform of $f(t)$

$$\boxed{\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

Defn. of Laplace transform of $f(t)$

$$f(t) = e^{\gamma t} F(t)$$

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{i\omega t}\, d\omega$$

$t > 0$:

$$f(t)\, e^{-\gamma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{i\omega t}\, d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{i\omega t + \gamma t}\, d\omega$$

$\rightarrow t(\gamma + i\omega) \rightarrow s$

$$s = \gamma + i\omega$$
$$ds = i\, d\omega$$
$$d\omega = \frac{ds}{i}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(\omega)\, e^{st}\, d\omega$$

$\hat{F}(\omega) \rightarrow \bar{f}(s)$

$$\boxed{f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)\, e^{st}\, ds}$$

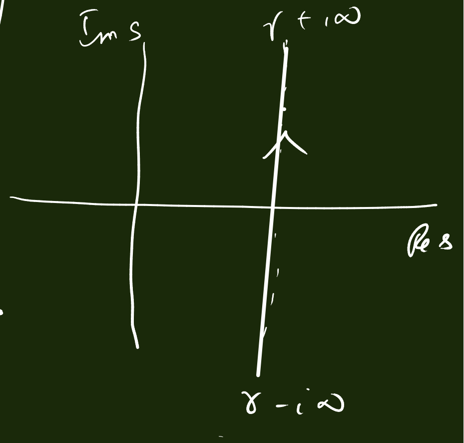
Inverse Laplace transform.

Laplace Transform $\rightarrow$ one-sided, weighted F.T of less well-behaved fns.

L.T pair:

$$\boxed{\begin{aligned} \bar{f}(s) &= \int_0^\infty f(t)\, e^{-st}\, dt \\ f(t) &= \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)\, e^{st}\, dt \end{aligned}}$$

$f(t) = 0$ for $t < 0$



Fourier and Laplace transforms.

L.T of $\frac{df}{dt} \longrightarrow s\bar{f}(s) \longrightarrow$

F.T $\frac{df}{dx} \longrightarrow (-ik)\hat{f}(k)$

$\frac{d^2}{dx^2} \longrightarrow -k^2$

$$\underline{a} = \sum_{i=1}^{N} a_i \underline{e}_i \longrightarrow f(x) = \sum_{i=1}^{N} f_i \phi_i(x)$$

For periodic fns

Fourier Series

non-periodic

Fourier Transform

Fns that do not decay as $t \to \infty$

Laplace Transform