Energy Equation

- Steady flow process in a control volume
- Energy content of the control volume is a constant
  $\Rightarrow$ Energy entering in all forms = Energy leaving in all forms
- Energy can only be transferred as heat, work or mass.

$$(\dot{E}_{in}-\dot{E}_{out})_{by\,heat}+(\dot{E}_{in}-\dot{E}_{out})_{by\,work}+(\dot{E}_{in}-\dot{E}_{out})_{by\,mass}=0$$

- For a flowing fluid, total energy per unit mass

$$e_{stream}=\underset{\substack{\uparrow\\ \text{internal}\\ \text{+ flow}}}{h}+pe+ke=h+\underbrace{gz+\frac{\vec{V}\cdot\vec{V}}{2}}_{\text{typically small}}$$

$$=c_pT$$

$$(\dot{E}_{in}-\dot{E}_{out})_{by\,mass,x}=(\dot{m}e_{stream})_x-\left[(\dot{m}e_{stream})_x+\frac{\partial(\dot{m}e_{stream})_x}{\partial x}dx\right]$$

$$=-\frac{\partial[(\rho u\,dy\,w)c_pT]}{\partial x}dx=-\rho c_p\left(u\frac{\partial T}{\partial x}+T\frac{\partial u}{\partial x}\right)dx\,dy\times w$$

Adding similar expression for y-direction, the net rate

$$(\dot{E}_{in}-\dot{E}_{out})_{by\,mass}=-\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+T\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{0\ \text{(continuity equation)}}\right)dx\,dy\,w$$

$$(\dot{E}_{in}-\dot{E}_{out})_{by\,heat}=Q_x-\left(Q_x+\frac{\partial Q_x}{\partial x}dx\right)+Q_y-\left(Q_y+\frac{\partial Q_y}{\partial y}dy\right)$$

$$=k\left(\frac{\partial^2T}{\partial x^2}+\frac{\partial^2T}{\partial y^2}\right)dx\,dy\,w \qquad \cdots\ Q_x=k(dy\,w)\frac{\partial T}{\partial x}$$

Work done = force × velocity. Negligible unless gravitational, or any other field is strong relative to rest of the terms.

Ignoring any contribution due to work (in addition to the pV term included in the enthalpy),

$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

$$\rho C_p (\vec{V} \cdot \vec{\nabla}) T = k \nabla^2 T$$

If viscous heating is significant, flow induces a source of heat. Thus,

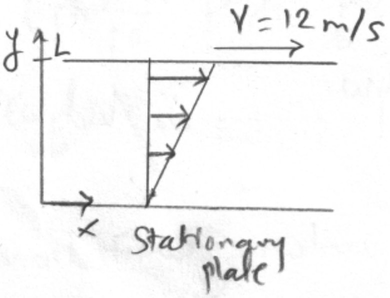
$$\rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu \Phi$$

when

$$\Phi = 2 \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2$$

- Reduces to 2D conduction equation in absence of flow
- Viscous heating is significant when shear stress is large & when viscous fluid is present.

Example: The flow of oil in a journal bearing can be approximated as parallel flow between two large plates with one plate moving and the other stationary. Both plates are maintained at 20°C. Determine the maximum temperature in the oil and the heat flux from the oil to each plate.



Given:
- top plate moving with V = 12 m/s
- bottom plate stationary
- isothermal plates separated by a distance of 2 mm

Oil:

$k = 0.145$ W/m.K $\quad \mu = 0.8374$ Pa.s @ 20°C

Assumptions:

- Steady state
- Oil is incompressible
- Negligible gravity
- No variation in z-direction
- Unidirectional flow along x-axis (velocity u)

Continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{but } v = 0 \Rightarrow u = u(y) \text{ only}$$

x-momentum balance

$$\rho\Big(\underbrace{u\frac{\partial u}{\partial x}}_{=0} + \underbrace{v\frac{\partial u}{\partial y}}_{=0}\Big) = \mu\frac{\partial^2 u}{\partial y^2} - \underbrace{\frac{\partial P}{\partial x}}_{=0}$$

=0 (∵ flow purely due to relative motion of the upper plate)

$$\frac{\partial^2 u}{\partial y^2} = 0 = \frac{d^2 u}{dy^2}$$

$$\Rightarrow u = C_1 y + C_2$$

Boundary conditions

$$\left.\begin{array}{l} u = 0 \text{ at } y = 0 \\ u = V \text{ at } y = L \end{array}\right\} \Rightarrow C_2 = 0 \qquad C_1 = \frac{V}{L}$$

$$u = \frac{y}{L}V$$

Energy balance

$$0 = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{assuming } T = T(y) \text{ only}$$

$$k\frac{d^2 T}{dy^2} = -\mu\left(\frac{V}{L}\right)^2$$

$$T = \frac{-\mu}{2k}\left(\frac{V}{L}\right)^2 y^2 + C_3 y + C_4$$

Boundary conditions

$$T = T_0 \text{ at } y = 0$$
$$T = T_0 \text{ at } y = L$$

$$T = T_0 + \frac{\mu V^2}{2k}\left(\frac{y}{L} - \frac{y^2}{L^2}\right)$$

Maximum temperature is at a location corresponding to $\frac{dT}{dy}=0$

$$\frac{dT}{dy} = \frac{\mu V^2}{2kL}\left(1-\frac{2y}{L}\right) = 0 \Rightarrow y = \frac{L}{2}$$

Then, maximum temperature is $T(y=\frac{L}{2})$

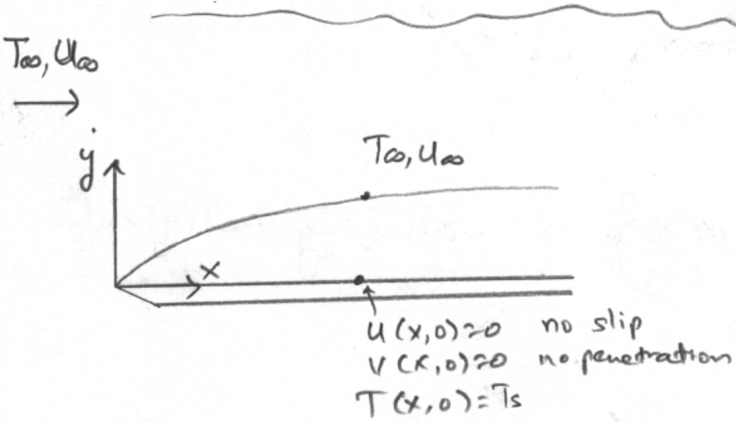
$$T_{max} = T\left(y=\frac{L}{2}\right) = T_0 + \frac{\mu V^2}{2k}\left(\frac{L/2}{L} - \frac{L^2/4}{L^2}\right) = T_0 + \frac{\mu V^2}{8k}$$

$$T_{max} = 124^\circ C$$

Notes:

- Although both plates are maintained at a temperature of 20°C, the oil reaches a maximum temperature of 124°C due to strong viscous heating.
- Viscosity was taken at 20°C, but should be taken at a higher temperature (average temperature of the oil $T_{avg} = \int_0^L T(y)\,dy$).

Convective Heat Transfer over a Flat Plate (Laminar)



- Consider a laminar flow of a fluid over a flat plate
- Free stream at velocity U_∞ and temperature T_∞.
- Flat plate surface is stationary, impenetrable and isothermal at temperature T_s

Continuity $\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x-Momentum $\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$

Energy $\quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$

Boundary Conditions

At $x=0$ $\quad u(0,y) = U_\infty \quad T(0,y) = T_\infty$

At $y=0$ $\quad u(x,0) = 0 \quad v(x,0) = 0 \quad T(x,0) = T_s$

At $y\to\infty$ $\quad u(x,\infty) = U_\infty \quad T(x,\infty) = T_\infty$

Assuming fluid properties to be independent of temperature, the continuity and x-momentum balance equations can be integrated to obtain $u(x,y)$ and $v(x,y)$. Then energy equation can be integrated to obtain $T(x,y)$.

Blasius' solution for velocity boundary layer

- Similarity solution
  - Noticing that the shape of velocity profile is unchanged plotting u/u_∞ vs y/δ should overlap at all x.

Defining a similarity variable

$$\eta = y\sqrt{\frac{u_\infty}{\nu x}}$$

Blasius assumed $u = \text{function}(\eta)$ only.

Defining a stream function $\psi(x,y)$,

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$ and continuity equation is satisfied.

Let $f(\eta) = \dfrac{\psi}{u_\infty\sqrt{\nu x/u_\infty}}$. Then $\psi = f(\eta)\, u_\infty \sqrt{\dfrac{\nu x}{u_\infty}}$

Then

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\infty \frac{df}{d\eta}$$

$$v = -\frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{2}\sqrt{\frac{u_\infty \nu}{x}}\left(\eta\frac{df}{d\eta} - f\right)$$

and

$$\frac{\partial u}{\partial x} = -\frac{u_\infty}{2x}\eta\frac{d^2 f}{d\eta^2} \qquad \frac{\partial u}{\partial y} = u_\infty\sqrt{\frac{u_\infty}{\nu x}}\frac{d^2 f}{d\eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_\infty^2}{\nu x}\frac{d^3 f}{d\eta^3}$$

Substituting in x-momentum balance,

$$2\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} = 0 \qquad \text{... non-linear ODE}$$

Boundary conditions

$$f(\eta=0) = 0$$

$$\frac{df}{d\eta}(\eta=0) = 0$$

$$\frac{df}{d\eta}(\eta\to\infty) = 1$$

Blasius solved this system using a series expansion. Alternatively, numerical methods can be employed to solve the system.
Table 7.1 in Incropera's book
or
Table 6.3 in Cengel's book
present results ($\eta, f, \frac{df}{d\eta} = \frac{u}{u_\infty}, \frac{d^2f}{d\eta^2}$)

The value of η is 4.91 when $u/u_\infty = 0.99$.
Thus,

$$\delta = \frac{\eta(y=\delta)}{\sqrt{u_\infty/\nu x}} = \frac{4.91}{\sqrt{u_\infty/\nu x}} = \frac{4.91x}{\sqrt{Re_x}}$$

Where
$Re_x = \frac{u_\infty x}{\nu}$ is the Reynolds number at location x, based on free stream velocity

Shear stress on the wall,

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = \mu u_\infty\sqrt{u_\infty/\nu x}\,\frac{d^2f}{d\eta^2}\bigg|_{\eta=0}$$

$$\tau_w = 0.332\,u_\infty\sqrt{\frac{\rho u_\infty \mu}{x}} \qquad \text{... from table}$$

Local friction coefficient

$$C_f = \frac{\tau_w}{\rho u_\infty^2/2} = 0.664\,Re_x^{-1/2}$$

Notes:
- Boundary layer thickness is proportional to $x^{1/2}$
- Wall shear stress and local friction coefficient are proportional to $x^{-1/2}$

Next, solve energy equation

Energy balance

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}$$

Let $\theta(x,y)=\dfrac{T-T_s}{T_\infty-T_s}$ $\Rightarrow$ $u\dfrac{\partial\theta}{\partial x}+v\dfrac{\partial\theta}{\partial y}=\alpha\dfrac{\partial^2\theta}{\partial y^2}$

Assuming similarity solution (since temperature profiles at different x locations are known to be similar)

$$\theta=\theta(\eta)$$

where $\eta=y\sqrt{\dfrac{u_\infty}{\nu x}}$ Substituting the velocity profiles u, v

$$u=u_\infty\frac{df}{d\eta}\qquad v=\frac{1}{2}\sqrt{\frac{u_\infty\nu}{x}}\left(\eta\frac{df}{d\eta}-f\right)$$

and using the equations

$$\frac{d\eta}{dy}=\sqrt{\frac{u_\infty}{\nu x}}\ ,\quad \frac{d\eta}{dx}=-\frac{1}{2}y\sqrt{\frac{u_\infty}{\nu}}\,x^{-3/2}=-\frac{1}{2}\frac{\eta}{x}$$

in the energy equation

$$u_\infty\frac{df}{d\eta}\frac{d\theta}{d\eta}\frac{d\eta}{dx}+\frac{1}{2}\sqrt{\frac{u_\infty\nu}{x}}\,\eta\frac{df}{d\eta}\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y}-\frac{1}{2}\sqrt{\frac{u_\infty\nu}{x}}\,f\frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y}=\alpha\frac{d^2\theta}{d\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2$$

$$\frac{d\theta}{d\eta}\left[\cancel{u_\infty\frac{df}{d\eta}\left(-\frac{1}{2}\right)\frac{\eta}{x}}+\cancel{\frac{1}{2}\sqrt{\frac{u_\infty\nu}{x}}\,\eta\frac{df}{d\eta}\sqrt{\frac{u_\infty}{\nu x}}}-\frac{1}{2}\sqrt{\frac{u_\infty\nu}{x}}\,f\sqrt{\frac{u_\infty}{\nu x}}\right]=\alpha\frac{d^2\theta}{d\eta^2}\frac{u_\infty}{\nu x}$$

$$-\frac{1}{2}\frac{d\theta}{d\eta}\frac{\cancel{u_\infty}}{x}f=\alpha\frac{d^2\theta}{d\eta^2}\frac{\cancel{u_\infty}}{\nu x}$$

$$\frac{d^2\theta}{d\eta^2}+\frac{Pr}{2}f\frac{d\theta}{d\eta}=0 \qquad \text{with } Pr=\frac{\nu}{\alpha}$$

Boundary Conditions

$\theta=0$ at $\eta=0$

$\theta=1$ at $\eta\to\infty$

Notes:
- ODE in $\theta(\eta)$
- Fluid flow information enters through f
- Can be solved numerically

From numerical solution at various Pr values, it is found that the surface temperature gradient

$$\frac{d\theta}{d\eta}\bigg|_{\eta=0} = 0.332\, Pr^{1/3} \qquad \text{for } Pr > 0.6$$

Thus,

$$\frac{dT}{dy}\bigg|_{y=0} = (T_\infty - T_s)\frac{d\theta}{d\eta}\bigg|_{\eta=0}\frac{\partial \eta}{\partial y}\bigg|_{y=0}$$

$$= 0.332\, Pr^{1/3}\,(T_\infty - T_s)\sqrt{\frac{u_\infty}{\nu x}} \qquad \text{for } Pr > 0.6$$

Thus, the local convection heat transfer coefficient is

$$h_x = \frac{-k\frac{\partial T}{\partial y}\big|_{y=0}}{T_s - T_\infty} = 0.332\, Pr^{1/3}\, k\sqrt{\frac{u_\infty}{\nu x}}$$

and

$$\frac{h_x x}{k} = 0.332\, Pr^{1/3}\, Re_x^{1/2} \qquad \text{for } Pr > 0.6$$

The dimensionless convection heat transfer coefficient on the left hand side is known as <u>Nusselt Number</u>.

$$Nu_x = 0.332\, Pr^{1/3}\, Re^{1/2} \quad \text{.... local Nusselt number}$$

Numerical solution also gives δ_t. It is found that

$$\frac{\delta}{\delta_t} = Pr^{1/3} \qquad \text{or} \qquad \delta_t = \frac{4.91x}{\sqrt{Re_x}\, Pr^{1/3}}$$

Notes:

- $Pr = 1 \Rightarrow \delta/\delta_t = 1$ and hence the velocity and thermal boundary layers coincide, with identical nondimentional velocity and temperature profiles.

- Nusselt number is essentially the ratio of heat flux by convection to that by conduction across a layer of fluid

$$\frac{q_{conv}}{q_{cond}} = \frac{h\Delta T}{k\Delta T/L} = \frac{hL}{k} = Nu$$