Function Spaces - Part 2: Fourier and Laplace Transforms

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$f(x)$ in a given domain (a, b) $\quad x : a \le x \le b$

$\boxed{B.C's}$ → same B.C's

$\mathcal{L} \to$ Hermitian operator

$\mathcal{L}\, e_i(x) = \lambda_i\, e_i(x)$ → same B.C's

$e_i(x)$: eg fns. $\quad \lambda_i$: eg. values

Eigenfunction expansion

$$\boxed{f(x) = \sum_{i=1}^{\infty} f_i\, e_i(x)}$$

f_i: coeffs $\quad e_i(x)$: basis fns.

analogous →

$$\underline{a} = \sum_{i=1}^{3} a_i\, \underline{e}_i$$

a_i: coeffs $\quad \underline{e}_i$: unit vectors

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$$

<u>Fourier Series</u>: $\mathcal{L} = -\dfrac{d^2}{dx^2} \longrightarrow$ sin and cos $\qquad$ <u>B.C's</u> "periodic"

<u>Domain</u>. $\quad -\pi \le x \le \pi$

Fourier Coefficients

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left\{ a_k \cos(kx) + b_k \sin(kx) \right\}$$

Fourier Series of $f(x)$ ↑ <u>periodic</u>

<u>3D vectors</u>: $\quad \underline{a} = \sum_{i=1}^{3} a_i\, \underline{e}_i$

$$\langle \underline{e}_j, \underline{a} \rangle = \sum_{i=1}^{3} a_i \underbrace{\langle \underline{e}_j, \underline{e}_i \rangle}_{\delta_{ij}}$$

$$\langle \underline{e}_j, \underline{a} \rangle = a_j$$

$$a_k = \frac{\langle \cos(kx), f(x) \rangle}{\| \cos(kx) \|^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx = \frac{\langle \sin(kx), f(x) \rangle}{\| \sin(kx) \|^2}$$

$$\underline{a} = \sum_{i=1}^{3} a_i\, \underline{e}_i \longrightarrow f(x) = \sum_{n=1}^{\infty} f_n\, e_n(x)$$

$e_n(x)$ → eg fns

∞ → <u>in practice</u>: $\sum_{n=1}^{10} \cdots$

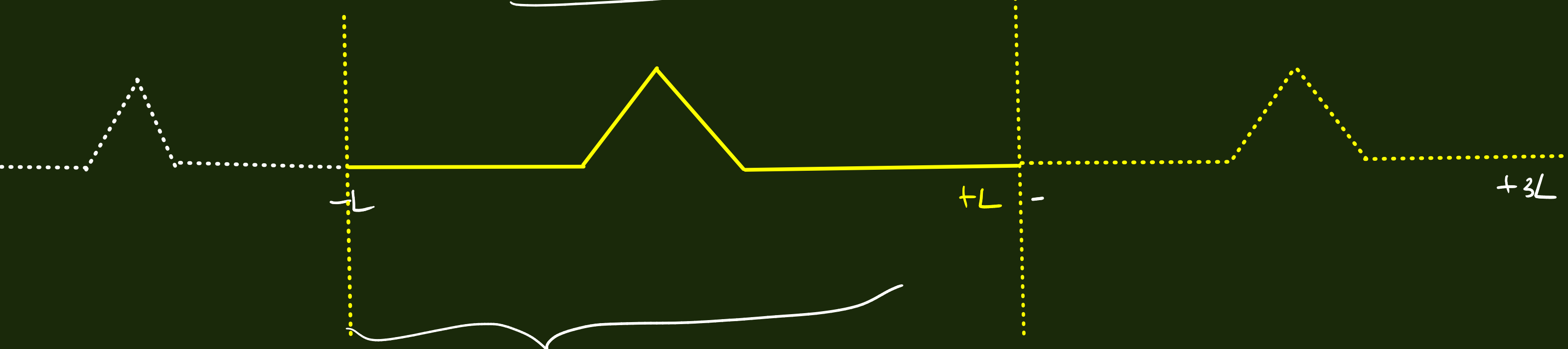
<u>truncate</u> the # of number of terms

If $f(x)$ is defined in the domain $0 \leq x \leq L$:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k x}{L}\right) + b_k \sin\left(\frac{2\pi k x}{L}\right) \right] \quad ; \; k = 1, 2, 3 \ldots$$

$$a_k = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{2\pi k x}{L}\right) dx \quad ; \quad b_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{2\pi k x}{L}\right) dx .$$

<u>periodic fns</u>



<u>Exponential form:</u>

$$f(x) = \sum_{k=-\infty}^{k=\infty} c_k \underbrace{e^{ik\pi x/L}}_{\text{eig fns}}$$

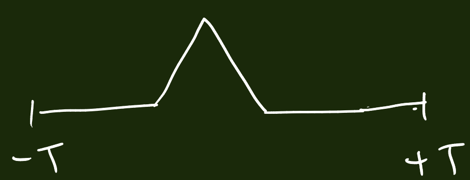
coeffs

$$\boxed{\frac{k\pi}{L} = \omega_k}$$

$$\boxed{f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k x}}$$

Generalization of Fourier Series $\xrightarrow{\text{to}}$ fns that are not periodic.

$\rightarrow$ <u>Fourier Transform</u>



$T \rightarrow \infty \rightarrow$ non-periodic fn.

$$\omega_r = \frac{2\pi}{T} \cdot r$$

$$\omega_r = \Delta r$$

$\lim T \to \infty$: $\quad \frac{1}{T} = \frac{\Delta\omega}{2\pi}$

as $T \to \infty$: $\quad \omega_{r+1} - \omega_r = \frac{2\pi}{T}(r+1-r) = \frac{2\pi}{T}$

$$\Delta\omega = \frac{2\pi}{T} \to 0 \quad \text{as } T \to \infty$$

$$f(t) = \sum_{r=-\infty}^{\infty} c_r \, e^{\frac{2\pi i r t}{T}}$$

$$c_r = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-2\pi i r t/T}\, dt$$

$$c_r = \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega_r t}\, dt$$

$\Delta\omega \to 0$ as $T \to \infty$

$$f(t) = \sum_{r=-\infty}^{\infty} c_r \, e^{-i\omega_r t}$$

$$f(t) = \sum_{r=-\infty}^{\infty} \left(\frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} f(\xi)\, e^{-i\omega_r \xi}\, d\xi \right) e^{-i\omega_r t}$$

as $T \to \infty$, $\Delta\omega \to 0$

$$\sum_{r=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\, g(\omega_r)\, e^{i\omega_r t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, g(\omega)\, e^{i\omega t}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \left(\int_{-\infty}^{\infty} f(\xi)\, e^{-i\omega_r \xi}\, d\xi \right) e^{i\omega_r t}$$

$\hat{f}(\omega)$ or $\tilde{f}(\omega)$

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \tilde{f}(\omega)\, e^{i\omega_r t}} \qquad \boxed{\tilde{f}(\omega) = \int_{-\infty}^{\infty} du \, f(u)\, e^{-i\omega u}}$$

Fourier Transform pair:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega \, \tilde{f}(\omega)\, e^{i\omega_r t} \; ; \quad \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du \, f(u)\, e^{-i\omega u}.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; \tilde{F}(\omega) \; e^{i\omega t}$$

$$\frac{d^2F}{dt^2} = (i\omega)^2 \tilde{F}(\omega)$$

$$\text{F.T of } \frac{dF}{dt} = i\omega \tilde{f}(\omega)$$

Solving ODEs & PDE's

$$\frac{df}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; \tilde{f}(\omega) \, (i\omega) \; e^{i\omega t}$$

$$\frac{df}{dt} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \; \boxed{(i\omega) \; \tilde{f}(\omega)} \; e^{i\omega t}$$

must be the F.T of $\frac{df}{dt}$

Example: Find the FT of $f(t) = 0 \quad t < 0$

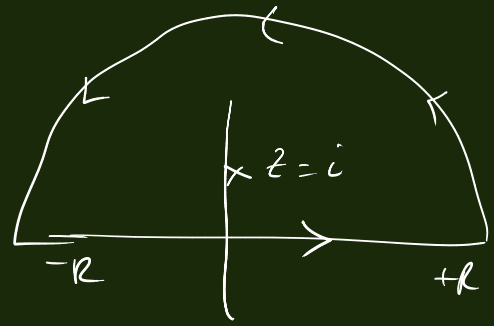
$$= A e^{-\lambda t} \quad t \geq 0 \qquad (\lambda > 0)$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \underbrace{\cancel{0 \cdot e^{-i\omega t}}}_{0} \, dt + \frac{A}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\lambda t} \; e^{-i\omega t} \, dt$$

$$\tilde{f}(\omega) = \frac{A}{\sqrt{2\pi} \; (\lambda + i\omega)}$$

Inverse:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A}{\sqrt{2\pi}} \; \frac{1}{\lambda + i\omega} \; e^{i\omega t} \, d\omega$$



$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{\lambda + i\omega} \; e^{i\omega t} \, d\omega$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{i \left(\omega + \frac{\lambda}{i}\right)} \; e^{i\omega t} \, d\omega$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{A}{(\omega - i\lambda)} \; e^{i\omega t} \, d\omega$$

Simple pole at $\omega = i\lambda$

$$f(t) = \frac{A}{2\pi i} \underbrace{\int_{-\infty}^{\infty} \frac{1}{(\omega - i\lambda)}\, e^{i\omega t}\, d\omega}_{\text{Residue theorem.}}$$

pole at $\omega = i\lambda$

$$f(t) = \frac{A}{\cancel{2\pi}} \times \cancel{2\pi i} \left[\text{Res} \right]_{\omega = i\lambda}$$

$$f(t) = A\, e^{i(i\lambda)t} = A e^{-\lambda t}$$

$$f(t) = A e^{-\lambda t}$$

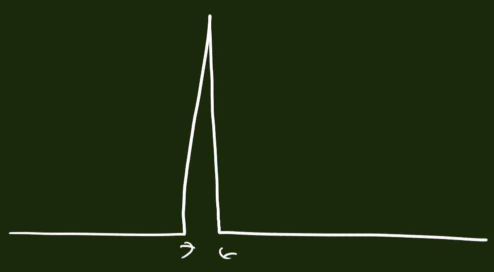
<u>Dirac Delta fn:</u>

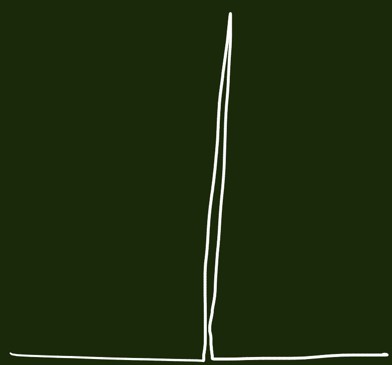
$$\int_{-L}^{L} \delta(t-a) \cdot dt = 1$$

$$\delta(t) = \delta(-t)$$

$$\delta(at) = \frac{1}{|a|}\, \delta(t)$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-a) = f(a)$$





Convert PDE's $\rightarrow$ ODE's

$$\delta(t) = \delta(-t)$$

$$\delta(at) = \frac{1}{|a|}\, \delta(t)$$