Function Spaces, Eigenfunction Expansions

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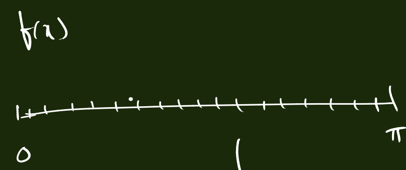
$f(x)$ in a given domain $0 \leq x \leq \pi$ → continuously

$f(x) = \sin x$ → infinite no. of values

↓

$\boxed{\text{infinite-dimensional}}$ vectors



$$N \to \infty \quad \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \Rightarrow \begin{bmatrix} \vdots \\ \vdots \\ N \end{bmatrix} \to \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix}$$

$\underline{u} = u(x)$

$\underline{v} = v(x)$

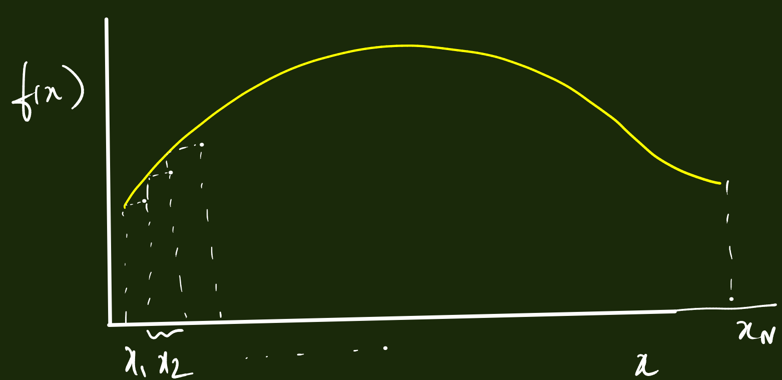
$\underline{u} + \underline{v} = u(x) + v(x)$

$\alpha\, \underline{u} = \alpha\, u(x)$ → functions satisfy the axioms of vector spaces

Inner products of two functions.

$u(x) \to (u_1, u_2, u_3 \dots u_N)$

$v(x) \to (v_1, v_2, v_3 \dots v_N)$



$$\Delta = \frac{(b-a)}{N}$$

Assume real-valued functions $\Big\}$ $\langle u, v \rangle = \cancel{\sum_{i=1}^{N} u_i v_i}$ → N-dependent! not uniquely defined

not useful.

$$\langle u, v\rangle = \sum_{i=1}^{N} u_i v_i \,\Delta \qquad \Delta = \left(\frac{b-a}{N}\right)$$

$x \in (a, b)$

$\lim N \to \infty, \ \Delta \to 0$

$$\boxed{\langle u, v\rangle = \int_a^b u(x)\, v(x)\, dx}$$

Orthogonality of two functions: $\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx = 0$

normalized $\langle f, f\rangle = 1 = \int_a^b f(x)^2\, dx$

Hillbert space

fns. that are square-integrable, & inner products are defined.

assumes that $\int_a^b f(x)^2\, dx$ exists

Square-integrable fns.

$a \to -\infty$

$b \to +\infty$

Example of orthogonal functions.

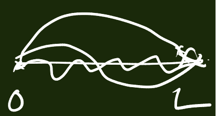
$$f_m(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi x}{L}\right) \qquad (m \to \text{integer})$$

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \qquad (n \to \text{integer})$$

$$\langle f_m, f_n\rangle = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) \cdot \sin\left(\frac{n\pi x}{L}\right) dx \quad \left.\begin{matrix} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{matrix}\right\} = \delta_{mn}$$

orthonormal functions

Boundary conditions

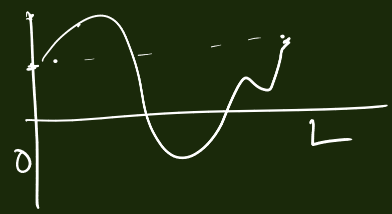


Complex-valued fns: $\quad 0 \le x \le L$

Periodic fns: $f(0) = f(L)$

$f(x) = f(x+L)$

$$\langle f, g \rangle = \int_0^L f^*(x)\, g(x)\, dx$$



$$f_m(x) = \frac{1}{\sqrt{L}}\, e^{\frac{2\pi i m x}{L}}$$

How do we get orthonormal eig functions ??

Solve the eigenvalue problem of a Hermitian operator !!

$$\mathcal{L} f = \lambda f$$

$\mathcal{L}$: lin op. $\quad$ f: eig. functions $\quad$ λ: eig values

What are linear operators.

defn. of linearity

$$\mathcal{L}(\alpha f + \beta g) = \alpha \mathcal{L} f + \beta \mathcal{L} g.$$

$$\mathcal{L} f \to g$$

e.g $\mathcal{L} \equiv \frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$

$\cancel{\left(\frac{d}{dx}\right)^2} \longrightarrow \times$ not a lin. operator

Is $\mathcal{L} = \frac{d}{dx}$ Hermitian? (or) Self-adjoint ?

Defn. of adj operator: $\langle f, \mathcal{L} g \rangle = \langle \mathcal{L}^{\dagger} f, g \rangle$

$\mathcal{L}^{\dagger}$ → adj operator of $\mathcal{L}$

The operator is self-adj if

$$\mathcal{L} = \mathcal{L}^{\dagger}$$

Is $\mathcal{L} = \frac{d}{dx}$ a self-adj operator??

$$\left\langle f, \frac{dg}{dx} \right\rangle \stackrel{?}{=} \left\langle \frac{df}{dx}, g \right\rangle$$

$$\int_0^L f^* \frac{dg}{dx}\, dx \stackrel{?}{=} \int_0^L \frac{df^*}{dx} g\, dx$$

B.C's
$f(L) = f(0)$
$g(L) = g(0)$
periodic BC's

Integrate by parts:

$$\cancel{\left(f^* g\right)_0^L} - \int_0^L g \frac{df^*}{dx}\, dx \stackrel{?}{=} + \int_0^L g \frac{df^*}{dx}\, dx$$

$\cancel{\left(f^* g\right)_0^L} \to 0$

NOT equal

$\longrightarrow$ $\mathcal{L} = \frac{d}{dx}$ is NOT self-adjoint.

Can we modify $\frac{d}{dx}$ s.t it becomes self-adjoint :

$$\mathcal{L} = -i \frac{d}{dx}$$

$f(0) = f(L)$
$g(0) = g(L)$

Is $\mathcal{L}$ self-adjoint: $\langle f, \mathcal{L} g \rangle \stackrel{?}{=} \langle \mathcal{L} f, g \rangle$.

$$\int_0^L f^* \left(-i \frac{dg}{dx}\right) dx \stackrel{?}{=} \int_0^L \left(-i \frac{df}{dx}\right)^* g$$

$$-i\int_0^L f^* \frac{dg}{dx}\,dx \stackrel{?}{=} i\int_0^L \frac{df^*}{dx}\, g\, dx$$

$$-i\left[\cancel{\left(f^* g\right)_0^L}^{\,0} - \int_0^L g\,\frac{df^*}{dx}\,dx\right] \stackrel{?}{=} i\int_0^L g\,\frac{df^*}{dx}\,dx$$

$$i\int_0^L g\,\frac{df^*}{dx}\,dx \stackrel{\text{Yes}}{=} i\int_0^L g\,\frac{df^*}{dx}$$

equal !!

$\mathcal{L} = -i\dfrac{d}{dx}$ is a self-adjoint operator.

fns. that satisfy periodic BC's.

$\mathcal{L} = -i\dfrac{d}{dx}$ is self-adjoint.

Eig values and Eig vectors ?

$$\mathcal{L} f = \lambda f$$

$$-i\frac{df}{dx} = \lambda f \longrightarrow f(x) = A e^{i\lambda x}$$

$$\frac{df}{dx} = \frac{-1}{i}\lambda f$$

$$\frac{df}{dx} = i\lambda f \longrightarrow f(x) = A e^{i\lambda x}$$

BC's $\rightarrow$ $f(0) = f(L)$ {periodic BC's}

$f(x) = A e^{i\lambda x}$

$f(0) = A$

$f(L) = A e^{i\lambda L}$

$f(0) = f(L) \longrightarrow e^{i\lambda L} = 1$

$\mathcal{L} = -i \frac{d}{dx}$ eigvalues $\boxed{\lambda = \frac{2\pi m}{L}}$; $m = 0, \pm 1, \pm 2, \pm 3 \ldots\ldots$

arbitrary const.

$f_n(x) = A e^{\frac{2\pi i m x}{L}} \rightarrow$ eig. fns

normalized. $\langle f_n, f_n \rangle = 1$

eigenfunctions $\boxed{f_n(x) = \frac{1}{\sqrt{L}} \exp\left[\frac{2\pi i m x}{L}\right]}$

$f_n \rightarrow$ infinitely many $\rightarrow n = 0, \pm 1, \pm 2, \ldots\ldots$

How do we know if this is complete :

$m = \pm 67$

$e^{\frac{\pm 2\pi i 67 x}{L}} \rightarrow$

Any periodic fn $g(x)$ in the domain $0 \le x \le L$ can be expanded as

$$g(x) = \sum_m \underset{\text{coeffs}}{g_m} \underbrace{\frac{1}{\sqrt{L}} \exp\left[\frac{2\pi i m x}{L}\right]}_{\text{eig. fns}}$$

exponential form of the Fourier series.

$\underline{a} = \sum_{i=1}^{N} a_i \underline{e}_i$

coeffs

$$\boxed{g(x) = \sum_{m=1}^{\infty} g_m e_m(x)}$$

Eigenfn. expansion.

Fourier Series

How to find the coeffs of expansion:

$$g_m = \langle e_m(x) | g(x) \rangle$$

$$\boxed{g_m = \int_0^L \frac{1}{\sqrt{L}} \exp\left[-i \frac{2\pi m x}{L}\right] g(x)\, dx}$$

Example:

$$\mathcal{L} = -\frac{d^2}{dx^2} \longrightarrow \text{self-adj ??}$$

$$\langle f, -\frac{d^2 g}{dx^2} \rangle \stackrel{?}{=} \langle -\frac{d^2 f}{dx^2}, g \rangle$$

$$\int_0^L f^* \left(\cancel{-}\frac{d^2 g}{dx^2}\right) dx \stackrel{?}{=} \int_0^L \cancel{-} \frac{d^2 f^*}{dx^2}\, g\, dx$$

$$\left(f^* \frac{dg}{dx}\right)_0^L - \int_0^L \frac{dg}{dx} \frac{df^*}{dx}\, dx$$

$$\underline{\left(f^* \frac{dg}{dx}\right)_0^L - \left(\frac{df^*}{dx} g\right)_0^L} + \int_0^L g \frac{d^2 f^*}{dx^2}\, dx \stackrel{?}{=} \int_0^L g \frac{d^2 f^*}{dx^2}$$

Yes

$\mathcal{L} = -\frac{d^2}{dx^2}$ is self-adj

When can boundary terms be zero:

$$\left.\frac{df}{dx}\right|_0 = \left.\frac{df}{dx}\right|_L = 0$$

$$\left.\frac{dg}{dx}\right|_0 = \left.\frac{dg}{dx}\right|_L = 0$$

Neuman BC's

(or)

$$\left.\begin{aligned} g(L) &= g(0) = 0 \\ f(L) &= f(0) = 0 \end{aligned}\right\} \text{Dirichlet BC}$$

(or) periodic

$$f(L) = f(0)$$

$$\left.\frac{dg}{dx}\right|_L = \left.\frac{dg}{dx}\right|_0$$

$\mathcal{L} = -\frac{d^2}{dx^2}$ is self-adj. $\rightarrow$ eig values and eig fns ?

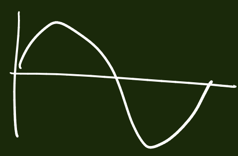
$$-\frac{d^2 f}{dx^2} = \lambda^2 f \qquad f(x) = \sin(\lambda x) \text{ or } \cos(\lambda x)$$

$$\frac{d^2 f}{dx^2} + \lambda^2 f = 0 \qquad \underline{\text{BC}}: \text{periodic} \qquad \lambda = \frac{2\pi m}{L};$$

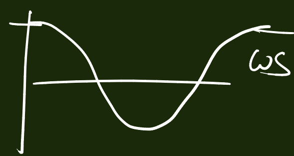
$m = 0, 1, 2, 3$

$$\cos\left(\frac{2\pi m x}{L}\right) = \cos\left(\frac{-2\pi m x}{L}\right)$$

NOT $m = -1, \cancel{-2, -3}$



$$\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi m x}{L}\right) \text{ and } \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi m x}{L}\right)$$



Any periodic fn: Fourier (sin/cos) Series:

$$\boxed{f(x) = \frac{a_0}{\sqrt{L}} + \sum_{m=1}^{\infty} a_m \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi m x}{L}\right)} + \sum_{m=1}^{\infty} b_m \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi m x}{L}\right)$$

Orthonormality conditions:

$$\int_0^L \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi m x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi n x}{L}\right) dx \begin{array}{l} = 1 \text{ if } m = n \\ = 0 \text{ if } m \neq n \end{array} \Big\} \delta_{mn}$$

$$\int_0^L \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi m x}{L}\right)\sqrt{\frac{2}{L}}\cos\left(\frac{2\pi n x}{L}\right) dx = \delta_{mn}$$

$$\int_0^L \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi m x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi n x}{L}\right) dx = 0 \text{ for all } m, n.$$

$$f(x) = \frac{a_0}{\sqrt{L}} + \sum_{m=1}^{\infty} a_m \sqrt{\frac{2}{L}}\cos\left(\frac{2\pi m x}{L}\right) + \sum_{m=1}^{\infty} b_m \sqrt{\frac{2}{L}}\sin\left(\frac{2\pi m x}{L}\right)$$

$$a_m = \int_0^L f(x)\sqrt{\frac{2}{L}}\cos\left(\frac{2\pi m x}{L}\right) dx \quad (m > 0)$$

$$b_m = \int_0^L f(x) \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi m x}{L}\right) dx$$

$$a_0 = \frac{1}{\sqrt{L}} \int_0^L f(x)\, dx.$$

$$b_m = \langle f, e_m(x) \rangle$$

$$\underline{a} = \sum_{k=1}^{3} a_k \underline{e}_k$$

$$\langle \underline{e}_i, \underline{a} \rangle = \sum_{k=1}^{3} a_k \underbrace{\langle \underline{e}_i, \underline{e}_k \rangle}_{\delta_{ik}}$$

$$\langle \underline{e}_i, \underline{a} \rangle = a_i$$

<u>Parseval theorem:</u>

$$f(x) = \sum_{k=0}^{\infty} b_k e_k(x) \qquad b_k = \langle e_k, f \rangle$$

orthonormal

$$= \int_a^b e_k^*(x) f(x)\, dx$$

$$\langle f, f \rangle = \int_a^b f^* f \, dx$$

$$= \int_a^b \left(\sum_j b_j e_j(x) \right)^* \left(\sum_k b_k e_k(x) \right)$$

$$= \sum_j \sum_k b_j^* b_k \underbrace{\boxed{\int_a^b e_j^*(x) e_k(x)\, dx}}_{\delta_{jk}} \to \langle e_j, e_k \rangle = \delta_{jk}$$

$$\langle f, f \rangle = \sum_j \sum_k b_j^* b_k \delta_{jk}$$

$$= \sum_k b_k^* b_k$$

$$\boxed{\langle f, f \rangle = \sum_k |b_k|^2}$$

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$$

$$\underline{a} \cdot \underline{a} = \sum_{i=1}^{3} a_i^2$$

$$\underline{a} \cdot \underline{a} = a_1^2 + a_2^2 + a_3^2$$