Partial Differential Equations: Solution by Eigenfunction Expansion
ChE641, IIT Kanpur
(Differential Equations - Part 5)

"Finite Fourier transform" $\longrightarrow$ ODE's

$\longrightarrow$ PDE $\longrightarrow$ 'Separation of variables'

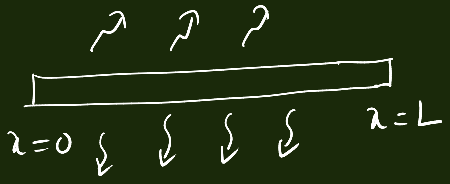
$\mathcal{L}(\phi)$

Example :

$$\frac{\partial^2 T}{\partial x^2} - \upsilon^2 T + f(x) = \frac{1}{k}\frac{\partial T}{\partial t}$$

(heat conduction)

spatial coordinate; source; heat loss to surroundings; time; tempr.



BC : $T(x=0) = 0$ all times

$T(x=L) = 0$

$T = T_0$

$(T - T_0) \to \Theta$

IC at $t=0$ $T(x, t=0) = q(x)$ $\longrightarrow$ also satisfies $q(0) = 0$, $q(L) = 0$

To find : $T(x,t)$

$$\mathcal{L}(\phi) = \frac{d^2\phi}{dx^2} - \upsilon^2 \phi$$

{Self-adj / Hermitian operator} $\to$ $\phi(0) = 0$, $\phi(L) = 0$

eig value problem.

$$\mathcal{L}\phi = -\lambda\phi \qquad \frac{d^2\phi}{dx^2} - \upsilon^2\phi = -\lambda\phi$$

given Const. parameter

$$\frac{d^2\phi}{dx^2} = -(\lambda - \upsilon^2)\phi$$

eig fns; eig value

define $\lambda - \upsilon^2 \equiv \sigma$

$$\left\{ \frac{d^2\phi}{dx^2} = -\sigma\phi \quad \phi(0) = 0, \ \phi(L) = 0 \right.$$

$$\sigma_n = \frac{n^2\pi^2}{L^2}$$

$$\sigma_n = \lambda_n - \upsilon^2 ; \quad \lambda_n = \upsilon^2 + \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3 \ldots$$

eig fns $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$; $n = 1, 2, 3 \ldots$

$\mathcal{L} = \left(\frac{\partial^2}{\partial x^2} - \nu^2\right)$.

P.D.E : $\mathcal{L}(T) + f(x) = \frac{1}{\kappa} \frac{\partial T}{\partial t}$

given

thermal diffusivity

$\langle \phi_n, \text{eqn} \rangle$:

Multiply both sides by ϕ_n and $\int_0^L dx(\quad)$:

$$\int_0^L \phi_n(x)\, \mathcal{L}(T)\, dx + \int_0^L \phi_n(x) f(x)\, dx = \frac{1}{\kappa} \int_0^L \phi_n(x) \frac{\partial T}{\partial t}\, dx \rightarrow \frac{1}{\kappa}\frac{d}{dt}\underbrace{\left(\int_0^L \phi_n(x) T(x)\right)}_{T_n(t)}$$

$\rightarrow -\lambda_n T_n(t)$ $\quad\rightarrow f_n$

$\int_0^L T \mathcal{L}\phi_n\, dx$

$$T(x,t) = \sum_{n=1}^{\infty} T_n(t)\, \phi_n(x) \quad ; \quad T_n(t) = \int_0^L T(x,t)\, \phi_n(x)\, dx$$

$\mathcal{L}$ is self-adjoint : $\int_0^L \phi_n \mathcal{L}(T)\, dx = \int_0^L T \mathcal{L} \phi_n\, dx$ $\quad \langle u, \mathcal{L}v \rangle = \langle \mathcal{L}u, v \rangle$

$\mathcal{L}\phi_n = -\lambda_n \phi_n$

$$\int_0^L T(x,t)\,(-\lambda_n \phi_n)\, dx = -\lambda_n \underbrace{\int_0^L T(x,t)\, \phi_n(x)\, dx}_{T_n(t)}$$

$$= -\lambda_n T_n(t)$$

$$-\lambda_n T_n + f_n = \frac{1}{\kappa} \frac{dT_n}{dt}$$

$$\frac{1}{\kappa} \frac{dT_n}{dt} + \lambda_n T_n = f_n \quad \rightarrow \text{first order ODE in } t \text{ for } T_n$$

$\frac{dy}{dx} + Py = Q$

$$T_n(t) = \frac{f_n}{\lambda_n} + \left[\overset{q_n}{T_n(0)} - \frac{f_n}{\lambda_n} \right] e^{-\lambda_n \kappa t}$$

?

$T_n(t=0)$

$T(x, t=0) = q(x)$

$T(x, t=0) = q(x) = \sum_{n=1}^{\infty} q_n \phi_n(x)$

$$q_n = T_n(t=0) = \int_0^L T(x, t=0)\, \phi_n(x)\, dx$$

known (IC)

$$T_n(t) = \frac{f_n}{\lambda_n} + \left[q_n - \frac{f_n}{\lambda_n}\right] e^{-\lambda_n k t}$$

$$T(x,t) = \sum_{n=1}^{\infty} T_n(t)\,\phi_n(x)$$

$$\boxed{T(x,t) = \sum_{n=1}^{\infty}\left[\frac{f_n}{\lambda_n} + \left(q_n - \frac{f_n}{\lambda_n}\right) e^{-\lambda_n k t}\right]\phi_n(x).}$$

$$f_n = \int_0^L f(x)\,\phi_n(x)\,dx$$

$$q_n = \int_0^L q(x)\,\phi_n(x)\,dx$$

$$\lambda_n = \upsilon^2 + \frac{n^2\pi^2}{L^2}$$

$$\phi_n(x) = \sqrt{\frac{2}{L}}\,\sin\left(\frac{n\pi x}{L}\right).$$

$t \to \infty$: Steady-limit soln :

$$T(x,t\to\infty) = \sum_{n=1}^{\infty} \frac{f_n}{\left(\upsilon^2 + \frac{n^2\pi^2}{L^2}\right)}\,\phi_n(x).$$

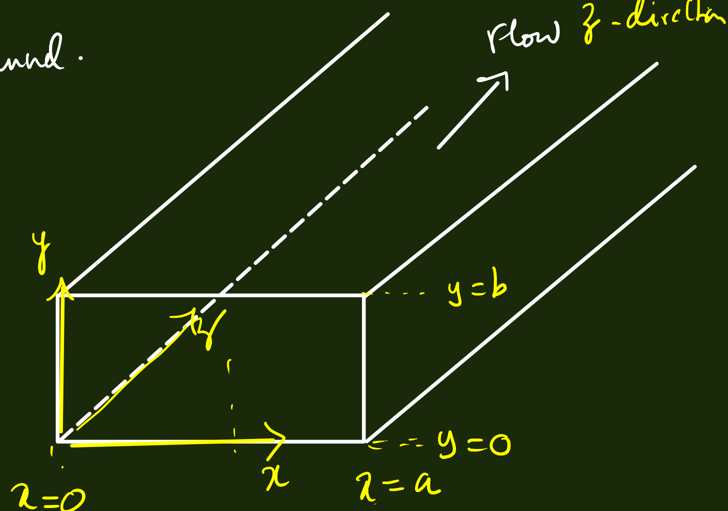
Example: PDE in two spatial coordinates

Steady Laminar flow in a rectangular channel.

no slip:

BC $u(x,y) = 0$ at $x = a$

$u(x,y) = 0$ at $x = b$



Steady-State: (unidirectional) z-comp. velocity

$$\mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial p}{\partial z} \rightarrow -\text{ve}$$

$$\frac{dp}{dz} = -\frac{\Delta P}{L} \rightarrow +\text{ve}$$

$$\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = -\frac{\Delta P}{L\mu} \rightarrow \alpha > 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\alpha$$

$$L_x(u) = \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\alpha$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2} - \alpha$$

$$\Downarrow$$

$$L_x u = -\frac{\partial^2 u}{\partial y^2} - \alpha$$

eig. val. problem:

$$L_x \phi = -\lambda \phi \qquad 0 \le x \le a$$

$$\phi(0) = 0, \quad \phi(a) = 0$$

$$\lambda_n = \frac{n^2\pi^2}{a^2}; \quad n = 1, 2 \ldots$$

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad ; n = 1, 2 \ldots$$

$$f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x)$$

$$b_n = \int_0^a f(x)\, \phi_n(x)\, dx$$

$$L_x u = -\frac{\partial^2 u}{\partial y^2} - \alpha$$

Multiply by $\phi_n(x)$ $\quad \int_0^a dx\,(\ldots)$

$$u(x,y) = \sum_{n=1}^{\infty} u_n(y)\, \phi_n(x)$$

$$\int_0^a \phi_n L_x u\, dx = -\int_0^a \frac{\partial^2 u}{\partial y^2} \phi_n\, dx - \int_0^a \alpha\, \phi_n\, dx$$

$\downarrow$ self-adj.

$$\int_0^a u \underbrace{L_x \phi_n}_{-\lambda_n \phi_n} dx = -\frac{\partial^2}{\partial y^2} \underbrace{\int_0^a u(x,y)\, \phi_n\, dx}_{u_n(y)} - \alpha \int_0^a \phi_n(x)\, dx$$

$$\int_0^a \phi_n(x)\, dx = \sqrt{\frac{2}{a}}\, \frac{a}{n\pi}\left[(-1)^n - 1\right]$$

$$\alpha \sqrt{\frac{2}{a}}\, \frac{a}{n\pi}\left[(-1)^n - 1\right] \rightarrow a_n$$

$$\underbrace{-\lambda_n \int_0^a u\, \phi_n\, dx}$$

$$-\lambda_n u_n(y) = -\frac{d^2 u_n}{dy^2} + a_n$$

$$\frac{d^2 u_n}{dy^2} - \lambda_n u_n = a_n \longrightarrow \text{ODE in } y$$

$$u_n(y=0) = 0$$

$$u_n(y=b) = 0$$

eig. val. prob.

$$L_y \psi_m = -\mu_m \psi_m$$

$$L_y = \frac{d^2}{dy^2}$$

$$\psi_m = \sqrt{\frac{2}{b}} \sin\left(\frac{m\pi y}{b}\right).$$

$$\mu_m = \frac{m^2\pi^2}{b^2}$$

$$\frac{d^2 u_n}{dy^2} - \lambda_n u_n = a_n$$

Multiply by $\psi_m(y)$ $\quad \int_0^b dy\,(\quad)$:

$$\int_0^b \underbrace{\psi_m \mathcal{L}_y u_n \, dy}_{\int_0^b u_n (\mathcal{L}_y \psi_m)\, dy} - \lambda_n \int_0^b u_n(y)\, \psi_m(y)\, dy = a_n \int_0^b \psi_m(y)\, dy$$

$\mathcal{L}_y \psi_m \to -\mu_m \psi_m$

$a_n \int_0^b \psi_m(y)\,dy \to a_n b_m$

$$-\mu_m \int_0^b u_n(y)\psi_m \, dy$$

$$-\mu_m u_{nm} - \lambda_n u_{nm} = a_n b_m$$

PDE $\to$ algebraic eqn

$$u_{nm} = -\frac{a_n b_m}{\lambda_n + \mu_m}$$

$$u_n(y) = \sum_{m=1}^{\infty} u_{nm} \psi_m(y)$$

$$u_n(y) = \sum_{m=1}^{\infty} \frac{-a_n b_m}{\lambda_n + \mu_m} \psi_m(y)$$

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} u_n \phi_n(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\left\{\frac{a_n b_m}{\lambda_n + \mu_m}\right\} \psi_m(y)\, \phi_n(x)}$$

eigvalues