Complex Numbers and Analysis - Part 3

ChE641, IIT Kanpur

$z^{1/4}$ (z · complex)

4 roots.

$$z = re^{i\theta}$$

$$z^{1/4} = r^{1/4}\, e^{(i2\pi n\,\theta)/4} \quad (n = 0, 1, 2 \ldots)$$

<u>Logarithms</u>:

$$w = \ln z$$

$$z = e^{w}$$

$$z_1 z_2 = e^{w_1}\, e^{w_2} = e^{(w_1 + w_2)}$$

$$\ln(z_1 z_2) = \ln\left\{ e^{(w_1 + w_2)} \right\} = w_1 + w_2$$

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2$$

$w = \ln z$; $z = re^{i\theta}$

$$w = \ln(re^{i\theta})$$

$$= \ln r + \ln(e^{i\theta})$$

$$w = \ln z = \ln r + i\theta$$

$$z = re^{i\theta + 2\pi k i} \qquad k = 0, 1, 2, \ldots$$

$$\ln z = \ln r + i(\theta + 2\pi k)$$

infinitely many values !! $k = 0, 1, 2, \ldots \infty$

$k = 0$: $\ln z = \ln r + i\theta$

"principal value" $\rightarrow \mathrm{Ln}\, z$

$$0 \leq \theta \leq 2\pi$$

or

$$-\pi \leq \theta \leq \pi$$

<u>e.g.</u> $\ln(-1)$

$$-1 = \underset{\substack{\downarrow \\ r}}{1} \cdot e^{i\pi + 2\pi k i}$$

$$\ln(-1) = \cancel{\ln(1)}^{0} + i(\pi \pm 2\pi k)$$

$$\ln(-1) = \underline{i\pi + 2\pi k} \quad (k = 0, 1, 2 \ldots)$$

$i\pi$ $-i\pi$

$3i\pi$

$5i\pi$ . . . .

$w = z^{1/2} \rightarrow$ two valued

$z^{1/3} \rightarrow$ three values

$z^{1/N} \rightarrow N$ values

$w = \ln z \rightarrow \infty$ - valued

z^3 $\quad z^{1/4}$

Complex roots & powers ? $\quad z^w$

$\ln a^b = b \ln a \qquad$ e.g. $\quad i^{-2i}$

$a^b = e^{b \ln a}$

$a = i$
$b = -2i$ $\qquad i^{-2i}$

$\ln i = i\left(\frac{\pi}{2} + 2n\pi\right)$

$= e^{-2i \ln i}$

$= e^{-2i\left(\frac{i\pi}{2} + 2n\pi i\right)}$

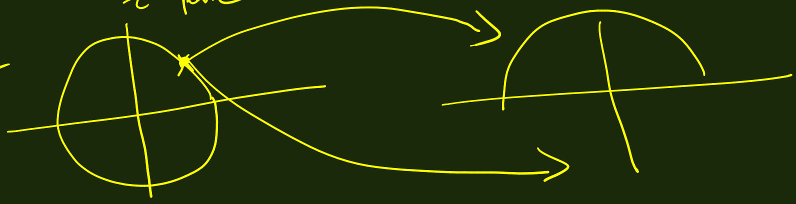
$= e^{(\pi \pm 4n\pi)}$

$i^{-2i} = e^{\pi}, \; e^{5\pi}, \; e^{-3\pi}, \; \ldots$

Branch point:

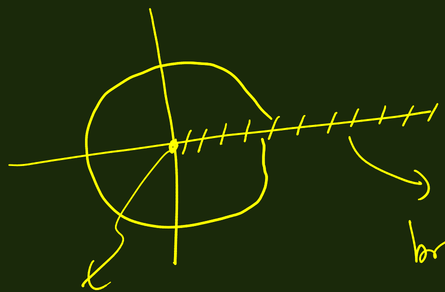
$W = f(z)$

$W = z^{1/2}$

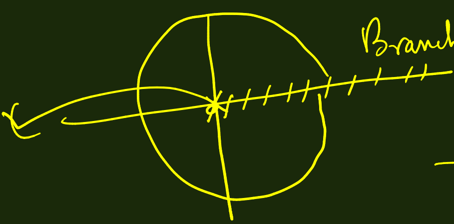


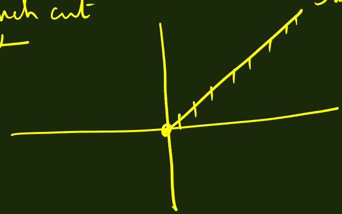
Analytic fn. fn – single-valued

$W = z^{1/2}$



$$f(z) = z^{1/2}$$
$$= r^{1/2} e^{i\theta/2}$$
$$= r^{1/2} e^{\frac{i(\theta+2\pi)}{2}}$$
$$= -r^{1/2} e^{i\theta/2}$$

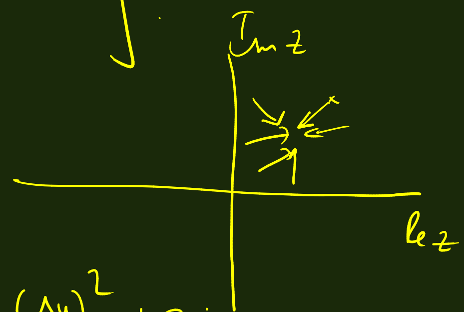




Analyticity of a Complex fn :- $z = x + iy$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f'(z) = \frac{df}{dz} = \lim_{\Delta z \to 0}\left[\frac{f(z+\Delta z) - f(z)}{\Delta z}\right]$$



$$f(z) = (x^2-y^2) + i\,2xy$$

$$\Delta z = \Delta x + i\Delta y.$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = 2x + i2y + \cancel{\frac{(\Delta x)^2 - (\Delta y)^2 + 2i\Delta x\,\Delta y}{\Delta x + i\Delta y}}$$

small

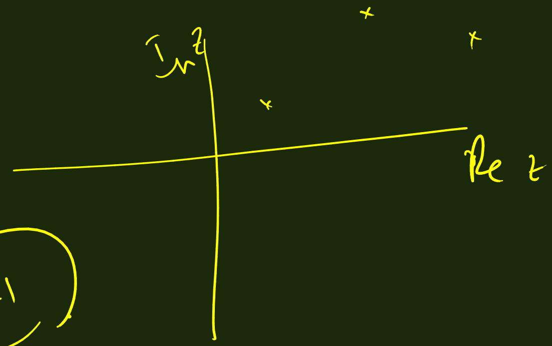
$$f(z) = z^2 \qquad f'(z) = 2z$$

$$f(z) = 2y + ix$$

$$\lim_{\Delta z \to 0}\frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{2\Delta y + i\Delta x}{\Delta x + i\Delta y}$$

$$\Delta y = m\Delta x \qquad = \frac{2m + i}{1 + im}$$

A fn. that is single-valued and differentiable at all pts in a domain R, then it is said to be analytic (regular) in R.



$$f(z) = \frac{1}{1-z}$$

$$f'(z) = \frac{1}{(1-z)^2}$$

$z \neq 1$

Cauchy-Riemann eqns:

$$L = \lim_{\Delta z \to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} \right]$$

$f = u + iv$

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$$

$$L = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\Delta y} \right]$$

Δz: purely real $\Delta z = \Delta x$ $(\Delta y = 0)$

$$L = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\Delta z = i\Delta y$ $(\Delta x = 0)$

$$L = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

$$(x, y) \longrightarrow (z, z^*)$$

$$x = \frac{1}{2}(z + z^*) \qquad y = \frac{1}{2i}(z - z^*)$$

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(\frac{-1}{2i}\right)$$

$$\frac{\partial f}{\partial z^*} = \frac{1}{2}\underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}_{\to 0} + \frac{i}{2}\underbrace{\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)}_{\to 0}$$

If Cauchy-Rieme is satisfied...

If $f(z)$ is analytic

then $\frac{\partial f}{\partial z^*} = 0$

$x, y \longrightarrow x + iy$

not as $(x - iy)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

"Harmonic fns"

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Singularities of Analytic fns:

f : C-R in $\mathcal{D}$ in complex plane.

$\hookrightarrow$ obey C-R in the entire plane ??

$$f(z) = \frac{1}{z} = \frac{x - iy}{(x^2+y^2)}$$

$$u(x,y) = \frac{x}{x^2+y^2}$$

$$v(x,y) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^{1/2}}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C-R are satisfied

$\frac{1}{z}$ has a singularity at $z = 0$

"Simple pole"

Simple pole : $\frac{c_1}{z}$

at $z = z_0$ $\quad f(z) \to \frac{R(z_0)}{(z - z_0)} + \ldots$

$R(z_0) \to$ residue.

e.g. $\frac{1}{(z+i)} \frac{1}{(z-i)}$

as $z \to i$ $\quad f(z) \to \frac{1}{2i} \frac{1}{(z-i)}$

$$\lim_{z \to i} (z-i) f(z) = \frac{1}{2i}$$

Residue at $z = i$

$\frac{1}{(z-z_0)}$. Simple poles. first order pole

If $f(z) \rightarrow \frac{R(z_0)}{(z-z_0)^n} \rightarrow n^{th}$ order pole.

$\frac{1}{(z-z_0)}$ regular singularities

Essential singularity. $f(z) = e^{1/z}$

$= \sum_{n=0}^{\infty} \frac{1}{z^n n!}$

poles, branch points, essential singularity.