Partial Differentials and Multivariable Calculus
Part - 3

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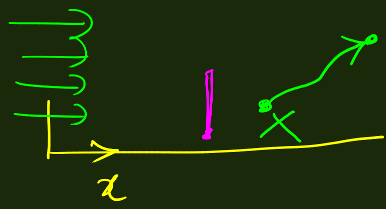
Total differential $\quad df = \left.\dfrac{\partial f}{\partial x}\right)_y dx + \left.\dfrac{\partial f}{\partial y}\right)_x dy \ldots$

$f(x,y)$

kept const.

Total derivative / substantial derivative :



$T(x,t)$ $\quad \left.\dfrac{\partial T}{\partial t}\right)_x$

$\boxed{x(t) = x(X,t)}$

$$\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_X = \left.\frac{\partial T}{\partial t}\right)_x + v_x \left.\frac{\partial T}{\partial x}\right)_t$$

$$\frac{DT}{Dt} = \left.\frac{\partial T}{\partial t}\right)_x + v_x \left.\frac{\partial T}{\partial x}\right) + v_y \left.\frac{\partial T}{\partial y}\right) + v_z \left.\frac{\partial T}{\partial z}\right)$$

$f(x,y)$ $\quad x \to r\cos\theta$, $\quad y \to r\sin\theta$

$g(r,\theta)$

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} \to$ Partial derivatives

$\nabla^2 \to$ Laplacian operator

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \longrightarrow \text{Partial diff in } (r,\theta). \qquad r = (x^2+y^2)^{1/2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\left.\frac{\partial r}{\partial x}\right)_y = \frac{x}{(x^2+y^2)^{1/2}} = \frac{x}{r} = \cos\theta$$

$$\left.\frac{\partial r}{\partial y}\right)_x = \frac{y}{r} = \sin\theta \quad \Bigg| \quad \left.\frac{\partial \theta}{\partial x}\right)_y = -\frac{\sin\theta}{r} \quad \Bigg| \quad \left.\frac{\partial \theta}{\partial y}\right)_x = \frac{\cos\theta}{r}$$

$(x,y) \longrightarrow (r,\theta)$.

$$f \to f(r,\theta) \to df = \left.\frac{\partial f}{\partial r}\right)_\theta dr + \left.\frac{\partial f}{\partial \theta}\right)_r d\theta$$

$$\left.\frac{\partial f}{\partial x}\right)_y = \left.\frac{\partial f}{\partial r}\right)_\theta \underbrace{\left.\frac{\partial r}{\partial x}\right)_y}_{\cos\theta} + \left.\frac{\partial f}{\partial \theta}\right)_r \underbrace{\left.\frac{\partial \theta}{\partial x}\right)_y}_{-\frac{\sin\theta}{r}}$$

$$\left.\frac{\partial f}{\partial x}\right)_y = \cos\theta \left.\frac{\partial f}{\partial r}\right)_\theta - \frac{\sin\theta}{r}\left.\frac{\partial f}{\partial \theta}\right)_r$$

$$\left.\frac{\partial f}{\partial y}\right)_x = \sin\theta \left.\frac{\partial f}{\partial r}\right)_\theta + \frac{\cos\theta}{r}\left.\frac{\partial f}{\partial \theta}\right)_r$$

$$\frac{\partial}{\partial x}(\ldots) = \left\{\cos\theta \left.\frac{\partial}{\partial r}\right)_\theta - \frac{\sin\theta}{r}\left.\frac{\partial}{\partial \theta}\right)_r\right\}(\ldots)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left\{\cos\theta \left.\frac{\partial}{\partial r}\right)_\theta - \frac{\sin\theta}{r}\left.\frac{\partial}{\partial \theta}\right)_r\right\}^2 f$$

$$\frac{\partial^2 f}{\partial x^2} = \left\{\cos^2\theta \frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \cancel{\frac{2\sin\theta\cos\theta}{r^2}}\frac{\partial}{\partial \theta} - \cancel{\frac{2\sin\theta\cos\theta}{r}}\frac{\partial^2}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r}\right\} f$$

$+$

$$\frac{\partial^2 f}{\partial y^2} = \left\{\sin^2\theta \frac{\partial^2 f}{\partial r^2} - \cancel{\frac{2\cos\theta\sin\theta}{r^2}}\frac{\partial f}{\partial r} + \cancel{\frac{2\cos\theta\sin\theta}{r}}\frac{\partial^2 f}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right\}$$

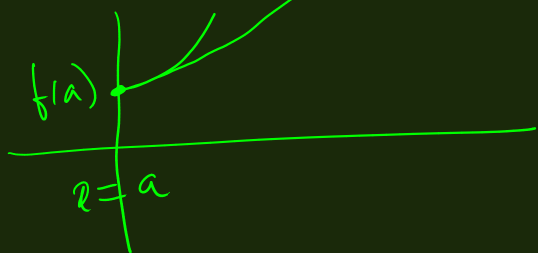
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \underbrace{\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}}_{\text{polar coordinates}} \longleftarrow \nabla^2 f$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \longrightarrow \text{Cartesian/rectangular}$$

Taylor series of a fn of multiple variables:

Taylor expansion (single variable).

$$f(x+a) = f(a) + \left.\frac{\partial f}{\partial x}\right|_a (x-a) + \frac{1}{2!}\left.\frac{\partial^2 f}{\partial x^2}\right|_a (x-a)^2 + \dots$$



$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right)_{x_0,y_0} \Delta x + \left.\frac{\partial f}{\partial y}\right)_{x_0,y_0} \Delta y$$

$\Delta x = x - x_0$

$\Delta y = y - y_0$

$$+ \frac{1}{2!}\left[\left.\frac{\partial^2 f}{\partial x^2}\right)_0 (\Delta x)^2 + 2\left.\frac{\partial^2 f}{\partial x \partial y}\right)_0 \Delta x \Delta y + \left.\frac{\partial^2 f}{\partial y^2}\right)_0 (\Delta y)^2 \right] + \dots$$

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y = \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f$$

$$\frac{1}{2!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 f(x,y)$$

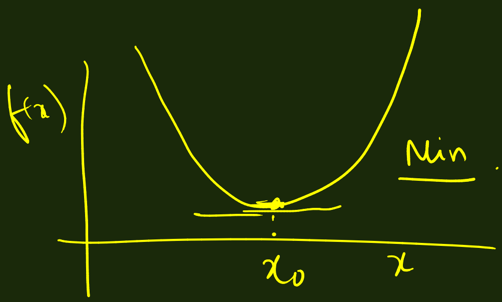
$$\boxed{f(x,y) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^n_{(x_0,y_0)} f(x,y)}$$

Stationary points:

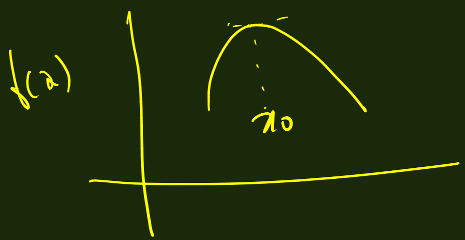
Recap. $f(x)$ $\quad \frac{df}{dx} = 0$ at x_0 $\rightarrow$ maxm

min.

pt. inflexion

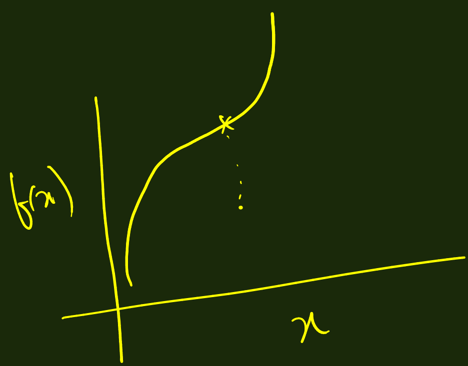


$$\left.\frac{d^2f}{dx^2}\right|_{x_0} > 0$$



$$\left.\frac{d^2f}{dx^2}\right|_{x_0} < 0$$

$$\frac{d^2f}{dx^2} = 0$$



Stationary points of a fn of two variables.

(x_0, y_0) $\qquad \left.\frac{\partial f}{\partial x}\right)_{x_0, y_0} = 0 \qquad \left.\frac{\partial f}{\partial y}\right)_{x_0, y_0} = 0$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

Nature the Stationary pt :

(x_0, y_0) : $\begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases}$

$$f(x,y) - f(x_0,y_0) = \frac{1}{2!}\left[(\Delta x)^2 \frac{\partial^2 f}{\partial x^2}\bigg|_0 + 2\,\Delta x\,\Delta y\,\frac{\partial^2 f}{\partial x\,\partial y}\bigg|_0 + (\Delta y)^2 \frac{\partial^2 f}{\partial y^2}\bigg|_0\right]$$

$\Delta x = x - x_0$

$\Delta y = y - y_0$

$\dfrac{\partial^2 f}{\partial y^2}\Big)_0 , \dfrac{\partial^2 f}{\partial x^2}\Big|_0 > 0$ min

$\dfrac{\partial^2 f}{\partial x^2}\Big)_0 , \dfrac{\partial^2 f}{\partial y^2}\Big)_0 < 0$ maxm.

$$f(x,y) - f(x_0,y_0) \simeq \frac{1}{2}\left\{ f_{xx}\left(\Delta x + \frac{f_{xy}\,\Delta y}{f_{xx}}\right)^2 + (\Delta y)^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right)\right]$$

For min:

$\Delta f > 0$

$f_{xx} > 0, \quad f_{yy} > 0 \qquad f_{yy} - \dfrac{f_{xy}^2}{f_{xx}} > 0$

$$\boxed{f_{xx} > 0,\ f_{yy} > 0,\ f_{xx} f_{yy} > f_{xy}^2}$$

Criteria for Minimum.

$\begin{cases} f_{xx} = \dfrac{\partial^2 f}{\partial x^2} \\ f_{yy} = \dfrac{\partial^2 f}{\partial y^2} \end{cases}$

For max:

$f_{xx} < 0, \quad f_{yy} < 0$

$f_{xx} = -|f_{xx}|$

$f_{yy} = -|f_{yy}|$

$$\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right) = \left(-|f_{yy}| + \frac{f_{xy}^2}{|f_{xx}|}\right) < 0$$

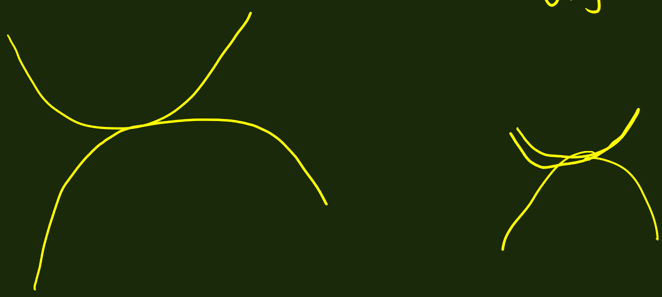
$|f_{xx}|\,|f_{yy}| > f_{xy}^2$

Maxm:

$$\boxed{f_{xx} f_{yy} > f_{xy}^2, \quad f_{xx} < 0,\ f_{yy} < 0}$$

Saddle. f_{xx} and f_{yy} have opposite signs

or $f_{xy}^2 > f_{xx} f_{yy}$.



$$\Delta f = f(x_1, x_2 \ldots x_N) - f(x_{10}, x_{20}, \ldots x_{N0})$$

$$= \frac{1}{2} \sum_i \sum_j \underbrace{\left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_0}_{M_{ij}} \Delta x_i \Delta x_j \quad \text{(to quadratic order).}$$

"Quadratic form"

$$\Delta f = \underline{\Delta x}^T \cdot \underline{\underline{M}} \cdot \underline{\Delta x} \qquad \underline{\Delta x} = \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_N \end{bmatrix}.$$

"eigenvalues"

An example from Thermo:

$$dU = T\,dS - P\,dV$$

exact diff. $U = U(S, V)$

$$dU = \left.\frac{\partial U}{\partial S}\right)_V dS + \left.\frac{\partial U}{\partial V}\right)_S dV$$

$$T = \left.\frac{\partial U}{\partial S}\right)_V \qquad P = -\left.\frac{\partial U}{\partial V}\right)_S$$

$$\left.\frac{\partial}{\partial V}\right)_S \left(\frac{\partial U}{\partial S}\right)_V = \left.\frac{\partial}{\partial S}\right)_V \left(\frac{\partial U}{\partial V}\right)_S$$

$$\boxed{\left.\frac{\partial T}{\partial V}\right)_S = -\left.\frac{\partial P}{\partial S}\right)_V}$$ "Maxwell relation"