Steps involved in approximate solution

1. Calculate the Bi number
2. Calculate λ_1 using $f(\lambda_1) = Bi$ relation
3. Calculate A_1 using $A_1 = f(\lambda_1)$

$\Rightarrow$ temperature profile as a function of time.

Alternative is to find the temperature using a graphical method, with the aid of Heisler Charts

Heisler Charts

- Transient temperature charts for a large plane wall, long cylinder and sphere
  - by M.P. Heisler, Trans. ASME **69** (1947) p227.
- Supplemented with heat transfer charts for the three systems
  - by H. Grober
- Three charts for each system
  - Midplane temperature: θ_0^* vs t^* at different $1/Bi$
  - Temperature distribution: $\frac{\theta^*}{\theta_0^*}$ vs $\frac{1}{Bi}$ at different $\frac{x}{L}$
  - Total Heat transfer: $\frac{Q}{Q_{MAX}}$ vs $Bi^2\tau$ at different Bi

★ Notation for this section: $Q \equiv$ amount of heat transferred

where $Q_{max} = mc_p(T_\infty - T_i)$ ← total heat transferred

Note that using the one-term approximation for transient temperature, the amount of heat transferred

$$\frac{Q}{Q_{max}} = \frac{\int_V \rho c(T-T_i)dV}{\int_V \rho c(T_\infty - T_i)dV} = \frac{\int_V \rho c(T-T_i)dV}{\rho c V(T_\infty - T_i)}$$

$$= \frac{1}{V}\int_V (1-\theta)\,dV$$

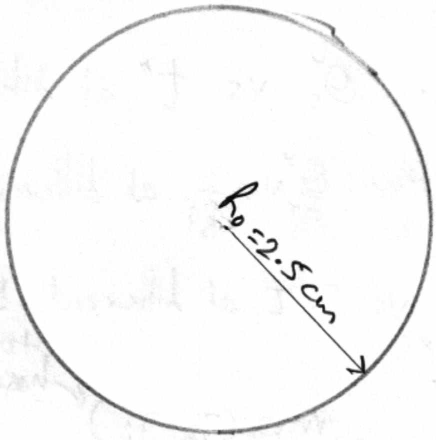
$$\left(\frac{Q}{Q_{max}}\right)_{wall} = 1 - \theta^*_{0,wall} \frac{\sin\lambda_1}{\lambda_1}$$

$$\left(\frac{Q}{Q_{max}}\right)_{cylinder} = 1 - 2\theta^*_{0,cylinder} \frac{J_1(\lambda_1)}{\lambda_1}$$

$$\left(\frac{Q}{Q_{max}}\right)_{sphere} = 1 - 3\theta^*_{0,sphere} \frac{\sin\lambda_1 - \lambda_1\cos\lambda_1}{\lambda_1^3}$$

To solve a problem quickly, either Heisler charts or these relations are used as an approximate solution.

Example: An ordinary egg, approximated as a 5 cm diameter sphere, is initially at a uniform temperature of $5°C$, and is dropped into boiling water at $95°C$. Taking the convection heat transfer coefficient to be $1200\ W/m^2 \cdot K$, determine how long it will take for the center of the egg to reach $70°C$.



Assumptions:
- Heat conduction is in 1D.
- Thermal properties are uniform and constant.
- $Fo > 0.2$ is considered $\Rightarrow$ one term approximation is valid.

Known:

$k_{egg} \approx k_{water} = 0.627\ W/m \cdot K$ (at average temperature $\frac{70+5}{2} = 37.5\ °C$)

$\alpha = \frac{k}{\rho c} = 0.151 \times 10^{-6}\ m^2/s$

$$Bi = \frac{h r_0}{k} = 47.8 \gg 0.1$$

Using one term approximation,

$$\lambda_1 = 3.0754 \quad , \quad A_1 = 1.9958$$

$$\theta^*_{o\,sphere} = A_1 e^{-\lambda_1^2 t^*}$$

$$\frac{T_f - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1^2 \tau^*} \quad \text{with } T_f = 70°C \quad T_\infty = 95°C$$
$$T_i = 5°C$$

$$\Rightarrow \tau^* = 0.209 = Fo = \frac{\alpha \tau}{R_o^2}$$

Thus,

$$\tau = \frac{R_o^2 \tau^*}{\alpha} = 865 \text{ s}.$$

('Can be compared with Heisler chart, for $\theta_o^* = 0.2778$ and $\frac{1}{Bi} = 0.021$)

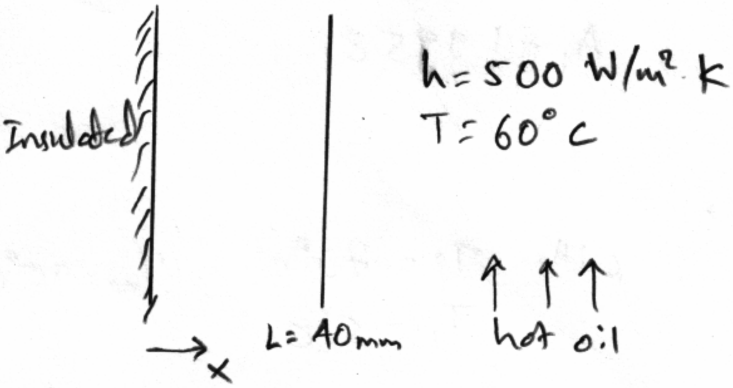
Note: Heisler charts give a quick, less accurate, estimate.

Example: Consider a steel pipe that is 1m in diameter and has a wall thickness of 40mm. The pipe is heavily insulated on the outer side. The walls of the pipe are at a uniform temperature of -20°C. With initiation of flow, a hot oil at 60°C is pumped through the pipe, creating a convective condition corresponding to h = 500 W/m²·K at the inner surface.

At t = 8 min, what are the values of Bi and Fo, temperature of the exterior pipe surface (covered by insulation), the heat flux to the pipe from the oil and energy per meter of pipe transferred from the oil?

Assumptions

- Pipe wall can be approximated as a plane wall (since thickness << diameter)
- Constant uniform properties



Known: Table A.1, Incropera

$\rho = 7832\ kg/m^3$

$c = 434\ J/kg\,K$

$k = 63.9\ W/m \cdot K$

$\alpha = 18.8 \times 10^{-6}\ m^2/s$

$$Bi = \frac{hL}{k} = 0.313$$ ($> 0.1 \Rightarrow$ lumped capacitance method cannot be used)

$$Fo = \frac{\alpha t}{L^2} = 5.64$$ at $t = 8$ mins $= 8 \times 60$ s

At the insulated surface, $q = 0 \Rightarrow \frac{dT}{dx} = 0$

Thus, the insulated wall is equivalent to a plane wall of thickness $2L$, experiencing the same surface condition. Mid plane of the latter wall is the insulated surface of the original wall (or pipe).

$$\theta_0^* = A_1 e^{-\lambda_1^2 Fo}$$

Matlab code:

```
fsolve(@(l) l*tan(l)-0.313, 0.1)
ans = 0.5319
```

$Bi = 0.313 \Rightarrow A_1 = 1.047$; $\lambda_1 = 0.531$

Thus,

$$\theta_0^* = 0.214$$

$$\frac{T - T_\infty}{T_i - T_\infty} = 0.214 \Rightarrow T = 42.9°C$$

with $T_\infty = 60°C$ and $T_i = -20°C$

$$\theta^* = \theta_0^* \cos(\lambda_1 x^*)$$

Thus,

$$\theta^*(x^* = 1, t^* = 8\,min) = 0.214 \cos(0.531) = 0.1845$$

$$\Rightarrow T(L, 8\,min) = 45.2°C$$

Flux, $q = h(T(L, 8\,min) - T_\infty) = -7400\ W/m^2$

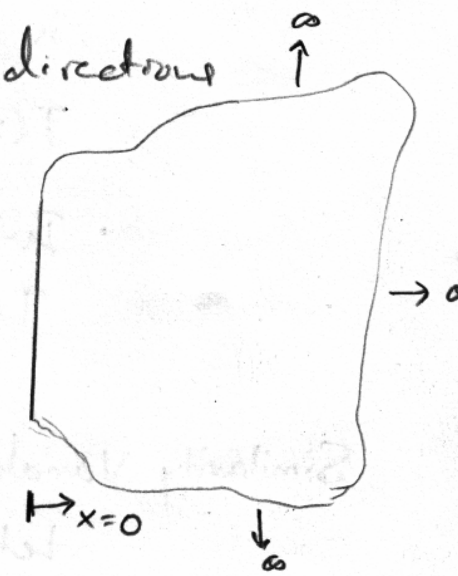
$$\frac{Q}{Q_{max}} = 1 - \frac{\sin\lambda_1}{\lambda_1}\,\theta_o^* = 0.8$$

$$Q = 0.8\,\rho C V (T_i - T_\infty) \qquad \text{with} \quad V = \pi D L$$

$$\frac{Q}{L} = -2.73 \times 10^7 \text{ J/m}$$

1D Transient Conduction: Semi-infinite solids

- Single plane surface ($x=0$)
- Extends to infinity in all other directions
- Examples
  - most bodies at short times
  - Thick slab "
  - lengthscale does not enter the heat transfer analysis at these short timescales



- Series solution (exact solution in previous section) might not converge within a few terms.
- Alternative: Similarity solution

  PDE is converted to ODE using a similarity variable $\eta(x,t)$

Consider a semi-infinite solid with
- no heat source/sink
- Constant properties
- Initially at a uniform temperature
- Heat transfer only in x-direction

Governing equation: Transient heat conduction in 1D

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Boundary Conditions:

- At the single plane surface, the temperature is uniformly T_s

  $T(x,t) = T_s$ at $x = 0$

- At the interior boundary $(x \to \infty)$, the temperature is unchanged

  $T(x,t) = T_i$ at $x \to \infty$

- Initial condition

  $T(x,t) = T_i$ at $t = 0$

Similarity Variable:

Let $\eta = \frac{x}{\sqrt{4\alpha t}}$ ... definition

then,

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{1}{\sqrt{4\alpha t}}\frac{dT}{d\eta} \quad \text{... assuming } T = T(\eta)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial \eta}\left[\frac{\partial T}{\partial x}\right]\frac{\partial \eta}{\partial x} = \frac{1}{4\alpha t}\frac{\partial^2 T}{\partial \eta^2}$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = -\frac{x}{2t\sqrt{4\alpha t}}\frac{dT}{d\eta}$$

Substituting,

$$\frac{dT}{d\eta} = -\frac{1}{2\eta}\frac{d^2 T}{d\eta^2}$$

At $x=0$, $\eta=0$. Thus, one boundary condition (at $x=0$) is

$$T(\eta=0)=T_s$$

At $x\to\infty$ or at $t=0$, $\eta\to\infty$. Thus, the other boundary condition and the initial condition give

$$T(\eta\to\infty)=T_i$$

Thus, we have the two boundary conditions needed to solve the ODE, and hence, the problem is fully defined.

Let $w=\frac{dT}{d\eta}$. Thus, the ODE becomes

$$\frac{dw}{d\eta}=-2\eta w \quad\Rightarrow\quad \int\frac{dw}{w}=-2\int\eta\,d\eta$$

$$\ln w=-\eta^2+\ln C_0 \quad\Rightarrow\quad w=C_0e^{-\eta^2}=\frac{dT}{d\eta}$$

$$T-T_s=C_0\int_0^{\eta}e^{-\eta_1^2}d\eta_1 \quad \text{.... using } T(\eta=0)=T_s$$

$$T_i-T_s=C_0\frac{\sqrt{\pi}}{2} \quad \text{..... using } T(\eta\to\infty)=T_i \text{ and } \int_0^{\infty}e^{-\eta^2}d\eta=\frac{\sqrt{\pi}}{2}$$

Thus,

$$\frac{T-T_s}{T_i-T_s}=\frac{2}{\sqrt{\pi}}\int_0^{\eta}e^{-u^2}du=\text{erf}(\eta)=1-\text{erfc}(\eta)$$

where

$$\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^{x}e^{-y^2}dy \quad \text{is the error function}$$

and

$$1-\text{erf}(x)=\text{erfc}(x) \quad \text{is the complementary error function}$$

Error function is tabulated (or numerically calculated) and is available in common softwares such as Matlab/Octave. It is a standard mathematical function that appears while solving a lot of problems.

Properties of error function:

$$\mathrm{erf}(0) = 0$$

$$\mathrm{erf}(\infty) = 1$$

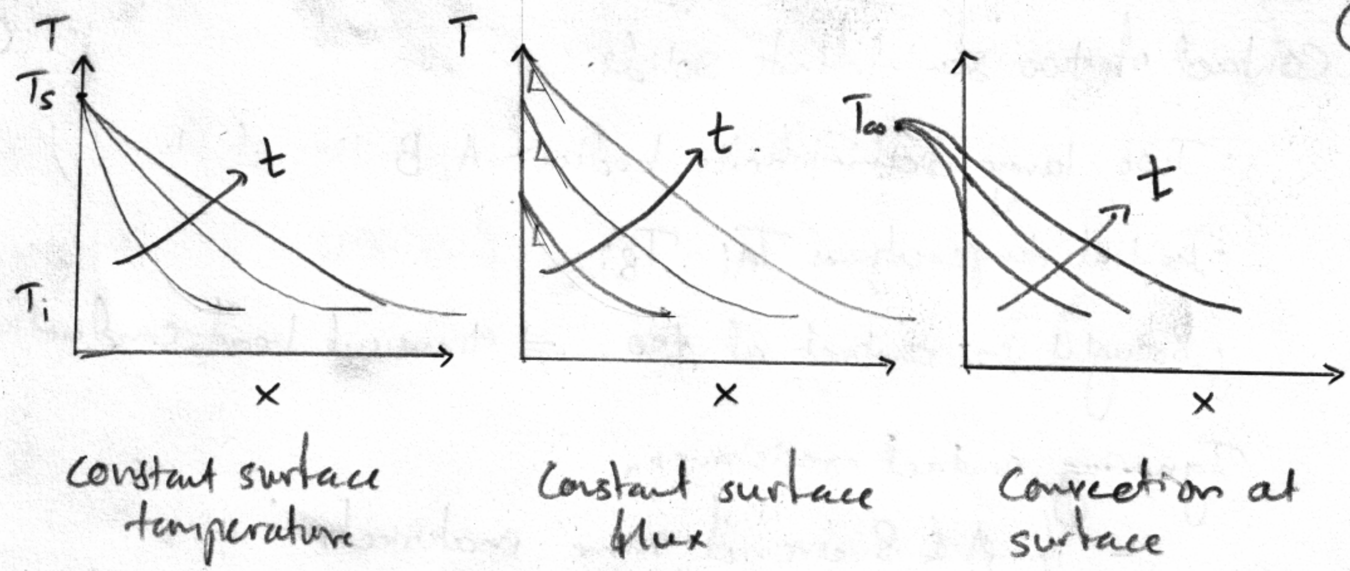
The surface flux, using Fourier's law at $x=0$, is

$$q = -k\frac{\partial T}{\partial x}\bigg|_{x=0} = -k\frac{dT}{d\eta}\frac{\partial \eta}{\partial x}\bigg|_{\eta=0} = -\frac{kC_0 e^{-\eta^2}}{\sqrt{4\alpha t}}\bigg|_{\eta=0} = \frac{-kC_0}{\sqrt{4\alpha t}}$$

$$= \frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}} = \sqrt{\frac{k\rho c}{\pi t}}(T_s-T_i)$$

Similar analysis for constant surface heat flux or convection at the surface can be obtained.

Constant Temperature $T(0,t)=T_s$	$\frac{T-T_s}{T_i-T_s} = \mathrm{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right)$	$q = \frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}}$
Constant flux $q(x=0)=q_0$	$T-T_i = \frac{q_0}{k}\left[\sqrt{\frac{4\alpha t}{\pi}}\exp\left(-\frac{x^2}{4\alpha t}\right) - x\,\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right)\right]$	
Convection $q(x=0)=h(T_\infty-T(0,t))$	$\frac{T-T_i}{T_\infty-T_i} = \mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}\right) - \exp\left(\frac{hx}{k}+\frac{h^2\alpha t}{k^2}\right)\mathrm{erfc}\left(\frac{x}{\sqrt{4\alpha t}}+\frac{h\sqrt{\alpha t}}{k}\right)$	



For constant surface temperature

- Temperature at any location x monotonically approaches T_s.
- Penetration depth: depth to which significant temperature effects propagate.

  e.g. x-location at which $\frac{T-T_s}{T_i-T_s} = 0.9$ (90% of the rise)

  "δ_p" ≡ penetration depth

  $\frac{T-T_s}{T_i-T_s} = \text{erf}(\eta)$ $\quad 0.9 = \text{erf}(1.16)$

  $\Rightarrow \delta_p \simeq 2.3\sqrt{\alpha t}$ $\quad \because \eta = \frac{x}{\sqrt{4\alpha t}}$

For a fixed surface flux,

- T_s increases as $t^{1/2}$

For convection at surface

- $h \to \infty$ corresponds to constant surface temperature
- for finite h, $T(x=0,t)$ approaches T_s as $t \to \infty$

Contact of two semi-infinite solids

- Two large semi-infinite bodies A, B
- Initial temperatures $T_{A,i}$, $T_{B,i}$
- Brought in contact at $t=0$ ⇒ transient heat conduction

Ignoring contact resistance,

- if A & B are of same material,

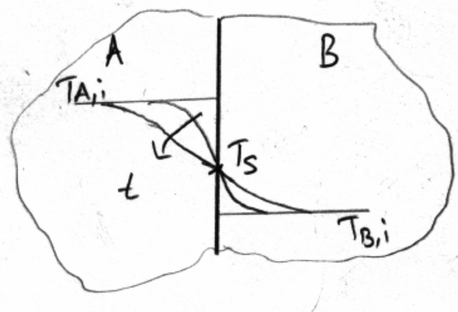
$$T_s = \frac{T_{A,i}+T_{B,i}}{2}$$ is the surface temperature

- if A and B are of different materials,

flux at the surface has to match (energy balance)

$$q_{s,A} = q_{s,B}$$

$$\frac{-k(T_s - T_{A,i})}{\sqrt{\pi \alpha_A t}} = \frac{k(T_s - T_{B,i})}{\sqrt{\pi \alpha_B t}}$$



or

$$-\sqrt{\frac{k_A \rho_A c_A}{\pi t}}(T_s - T_{A,i}) = \sqrt{\frac{k_B \rho_B c_B}{\pi t}}(T_s - T_{B,i})$$

Thus,

$$\frac{T_{A,i} - T_s}{T_s - T_{B,i}} = \sqrt{\frac{(k\rho c)_B}{(k\rho c)_A}}$$

And

$$T_s = \frac{\sqrt{(k\rho c)_A}\, T_{A,i} + \sqrt{(k\rho c)_B}\, T_{B,i}}{\sqrt{(k\rho c)_A} + \sqrt{(k\rho c)_B}}$$