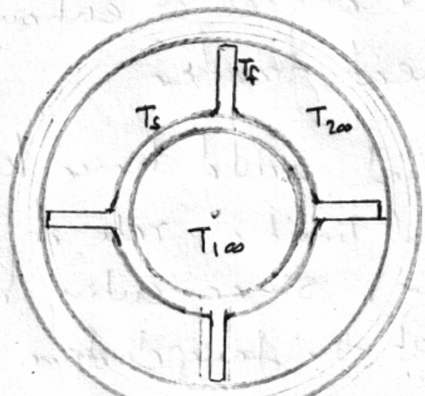
Extended Surfaces (Fins)

- Heat transfer from boundaries (by convection) is perpendicular to principal direction of heat transfer in the solid (vs walls, where the two are in the same direction)

- A typical application is in enhancing heat transfer between a solid and a surrounding fluid.

Examples: Fin oil radiators, CPU heat sink, Finned tube heat exchanger



Finned tube heat exchanger

- Assuming that the fins (metal strips) do not affect the convection heat transfer coefficient by changing the flow pattern of the annular fluid, heat transfer is enhanced due to the fins.

Concept: Rate of heat transfer,

$$Q = h A_s (T_s - T_\infty)$$

— Newton's law of cooling

Typically T_s and T_∞ are fixed by design or operating conditions. Thus, there are two ways of enhancing heat transfer, viz. enhancing

- convection heat transfer coefficient
- area of heat transfer

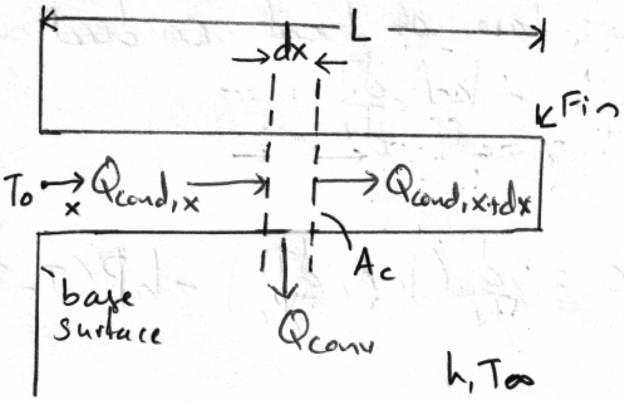
Enhancing convection heat transfer coefficient
- requires inducing flow of surrounding fluid
  - adding a pump or a fan, etc.
    or upgrading
- might not suffice

Thus,
Enhancing area of heat transfer is an alternative
- attaching fins to the surface

Fin material
- high thermal conductivity
- uniform temperature from its base to its tip $\Rightarrow$ better degree of enhancement of heat transfer
- ideal fin material would have $k \to \infty$
  $\Rightarrow$ No theoretical limit on rate of heat transfer, since entire fin would be at the temperature of the base
- practically, there is a limit on the degree to which heat transfer is enhanced by adding fins. This can be quantified once temperature distribution in the fin is determined.

Analysis: 1D steady state heat conduction

- Define a differential element
- Assume 1D heat conduction (although 2D heat conduction is more accurate).
- Assume uniform thermal conductivity



- Fin length L
- Fin crossectional area at x A_c
- Fin perimeter at x P

Steady state energy balance over the small element

Rate of heat conduction into the element (at x)	=	Rate of heat conduction from the element (x+dx)	+	Rate of heat convection from the element

$$Q_{cond,x} = Q_{cond,x+dx} + Q_{conv}$$

Area for conduction $\equiv A_c$
Area for convection $\equiv P\,dx$

Thus,

$$Q_{conv} = h(P\,dx)(T - T_\infty)$$

when

$T \equiv$ temperature of the small element located at x

Substituting and dividing by dx

$$0 = \frac{Q_{cond,x+dx} - Q_{cond,x}}{dx} + hP(T - T_\infty)$$

In the limit of $dx \to 0$.

$$0 = \frac{dQ_{cond}}{dx} + hP(T - T_\infty)$$

From Fourier's law of heat conduction

$$Q_{cond} = -kA_c\frac{dT}{dx}$$

Thus

$$0 = \frac{d}{dx}\left(kA_c\frac{dT}{dx}\right) - hP(T-T_\infty)$$

Assuming k to be uniform,

$$0 = \frac{d}{dx}\left(A_c\frac{dT}{dx}\right) - \frac{hP}{k}(T-T_\infty)$$

Analysis so far does not assume any particular geometry for the fin.

<u>Uniform Cross-sectional area</u>:

A_c is a constant
P is a constant

Energy balance equation simplifies to

$$0 = \frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T-T_\infty)$$

Defining the excess temperature

$$\theta = T - T_\infty \qquad \theta = \theta(x) \quad T = T(x)$$

since T_∞ is a constant, $\frac{dT}{dx} = \frac{d\theta}{dx}$; and energy balance is rewritten as

$$0 = \frac{d^2\theta}{dx^2} - m^2\theta$$

where

$$m^2 = \frac{hP}{kA_c} = \text{Constant}$$

The general solution of this second order, linear, homogeneous differential equation with constant coefficients is

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

Boundary Condition at the base of the fin is

$$\theta = \theta_b \quad \text{at} \quad x=0 \qquad \text{where} \quad \theta_b = T_b - T_\infty$$

One more boundary condition (at $x=L$) is required.

Boundary condition at Fin Tip

A. Infinitely long fin ($L \to \infty$)

T at tip $\to T_\infty$ $\qquad \theta = 0$ at $x = L$

$\Rightarrow C_1 = 0$

Boundary condition at the base $\Rightarrow C_2 = \theta_b$

$$\theta(x) = \theta_b e^{-mx} = \theta_b e^{-\sqrt{\frac{hP}{kA_c}}x}$$

B. Adiabatic / Insulated fin tip

$\frac{dT}{dx}$ at tip $\to 0$

$$\frac{d\theta}{dx} = 0 \quad \text{at} \quad x = L$$

· more realistic, since area available for heat transfer is small compared to sides of the fin

MATH DETAILS

Condition at Fin tip $\Rightarrow$ ($x=L$) $\theta'|_L = \frac{d\theta}{dx}\Big|_L = mC_1 e^{mL} - mC_2 e^{-mL} = 0$

Fin base $\Rightarrow$ $\theta_b = C_1 + C_2 \Rightarrow C_2 = \theta_b - C_1 \Rightarrow -mC_2 e^{-mL} = -m(\theta_b - C_1)e^{-mL}$

$mC_1 e^{mL} = m\theta_b e^{-mL} - mC_1 e^{-mL} \Rightarrow C_1 = \frac{\theta_b e^{-mL}}{e^{mL} + e^{-mL}}$ and $C_2 = \theta_b - C_1$

$$\theta(x) = C_1 e^{mx} + (\theta_b - C_1)e^{-mx} = \theta_b\left[\frac{C_1}{\theta_b}e^{mx} + \left(1 - \frac{C_1}{\theta_b}\right)e^{-mx}\right] = \theta_b\left[\frac{e^{-m(L-x)}}{(e^{mL}+e^{-mL})} + \frac{e^{m(L-x)}}{(e^{mL}+e^{-mL})}\right]$$

$$\theta(x) = \theta_b \frac{\cosh m(L-x)}{\cosh mL}$$

C. Convection, Radiation from fin tip

$$h(T-T_\infty) = -k\frac{dT}{dx} \quad \text{at } x=L$$

i.e. $$h\,\theta(L) = -k\frac{d\theta}{dx}\Big|_L$$

$$\theta(x) = \frac{\cosh m(L-x) + \frac{h}{mk}\sinh m(L-x)}{\cosh mL + \frac{h}{mk}\sinh mL}$$

D. Known Temperature at fin tip

$$T = T_L \quad \text{at } x=L$$

i.e. $$\theta = \theta_L \quad \text{at } x=L$$

$$\theta(x) = \frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$$

Once the temperature profile is known, we can calculate other quantities of interest such as heat flux, rate of heat transfer.

For design purpose, an important question to be answered is: "What is the enhancement in heat transfer due to fins?"

We can calculate the rate of heat transferred from the entire fin.

At steady state, energy balance indicates

rate of heat transfer by convection from fin = rate of heat transfer by conduction through the base of the fin

Thus, the rate of heat transferred from the fin

$$Q_f = \int_{A_f} h\, dA_s\, (T(x) - T_\infty)$$

sum of heat transferred over all small elements

$$= \int_{A_f} h\, \theta(x)\, dA_s$$

Here, A_f is the total fin surface area (including the tip).

A simpler alternative is to use the energy balance. The heat transfer rate by conduction through the base of the fin

$$Q = -kA_c \frac{dT}{dx}\bigg|_{x=0}$$

$$\Rightarrow Q_f = Q = -kA_c \frac{d\theta}{dx}\bigg|_{x=0}$$

Either of these approaches gives the rate of heat transfer from the fin, once the temperature profile ($T(x)$ or $\theta(x)$) is known.

Table: Temperature profile and heat transfer rate for different boundary conditions at the fin tip. $m^2 = hP/kA_c$; $M = \sqrt{hPkA_c}\,\theta_b$

Case	Temperature θ/θ_b	Fin heat transfer rate Q_f
A. ($L \to \infty$)	e^{-mx}	M
B. (Adiabatic tip)	$\dfrac{\cosh m(L-x)}{\cosh mL}$	$M \tanh mL$
C. (Convection from tip)	$\dfrac{\cosh m(L-x) + \frac{h}{mk}\sinh m(L-x)}{\cosh mL + h/mk \sinh mL}$	$M\dfrac{\sinh mL + h/mk \cosh mL}{\cosh mL + h/mk \sinh mL}$
D. (Known temperature at tip)	$\dfrac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	$M\dfrac{\cosh mL - (\theta_L/\theta_b)}{\sinh mL}$