Differential Equations - Part 6: Solution by Integral Transform Methods

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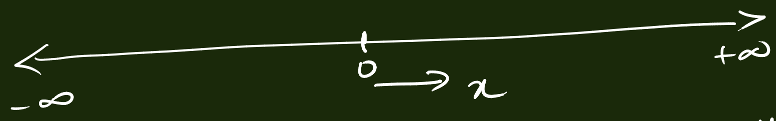
Fourier Transforms : domain $-\infty < x < +\infty$

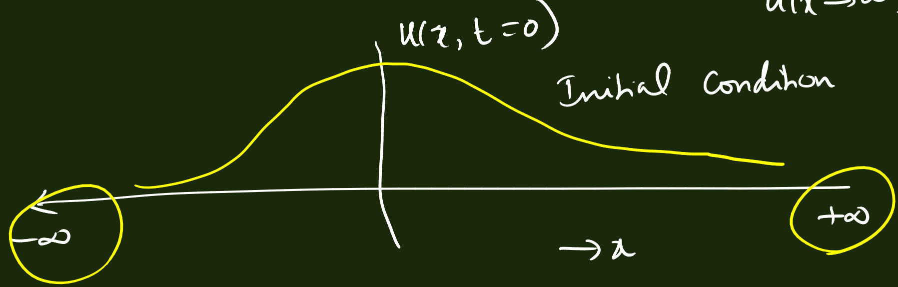
Laplace Transforms . domain $0 < x < +\infty$ (Initial value problems) → time t

$0 \le t \to \infty$

Example : unsteady heat conduction eqn.



$u(x \to \infty, t=0)$: not unbounded.



Find $u(x,t)$

domain is unbounded !! → Eig fn. expansions ↓ cannot be used.

Boundary conditions : $u(x) \to 0$ as $x \to +\infty$, $x \to -\infty$

PDE : $$\boxed{\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}}$$

$u \to$ "temp"

α^2 : thermal diffusivity

Use Fourier Transform to solve this PDE :

F T pair . $\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx$; $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k,t)\, e^{ikx}\, dk$

F.T the PDE : $\int_{-\infty}^{\infty} (\text{PDE})\, e^{-ikx}\, dx$

$$\alpha^2 \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{-ikx}\, dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-ikx}\, dx = \frac{\partial}{\partial t} \underbrace{\int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx}_{\hat{u}(k,t)}$$

$\frac{d\hat{u}}{dt}$ ↑ $\frac{\partial \hat{u}(k,t)}{\partial t}$

Integrate by parts twice

$$\left[\cancel{\frac{\partial u}{\partial x}} e^{-ikx} \right]_{-\infty}^{\infty} - \left[u\, \cancel{(-ik)}\, e^{-ikx} \right]_{-\infty}^{\infty} + \alpha^2 (-k^2) \underbrace{\int_{-\infty}^{\infty} u(x,t)\, e^{-ikx}\, dx}_{\hat{u}(k,t)}$$

0 0

$(-ik)(-ik)$

$$-\alpha^2 k^2 \hat{u}(k,t) = \frac{d\hat{u}(k,t)}{dt}$$

Original PDE in (x,t) $\longrightarrow$ ODE in t

$$\frac{\partial^2 u(x,t)}{\partial x^2} \longrightarrow -k^2 \hat{u}(k,t)$$

$$\hat{u}(k,t) = A\, e^{-\alpha^2 k^2 t}$$

$A \rightarrow$ I.C

I.C $u(x, t=0) = f(x) \rightarrow$ known

$\rightarrow$ F.T.

$$\hat{u}(k, t=0) = \int_{-\infty}^{\infty} \underbrace{u(x,t=0)}_{f(x)}\, e^{-ikx}\, dx$$

$$\hat{u}(k, t=0) = \hat{f}(k) \leftarrow \text{known}$$

$$\boxed{\hat{u}(k,t) = \hat{f}(k)\, e^{-\alpha^2 k^2 t}}$$

Inverse Fourier transfor.

$$\mathcal{F}^{-1}\left[\hat{f}(k)\cdot\hat{g}(k)\right] = \int_{-\infty}^{\infty} f(z)\, g(x-z)\, dz$$

"Convolution theorem"

$$\mathcal{F}^{-1}[\hat{f}(k)] = f(x) \rightarrow \text{I.C (known)}$$

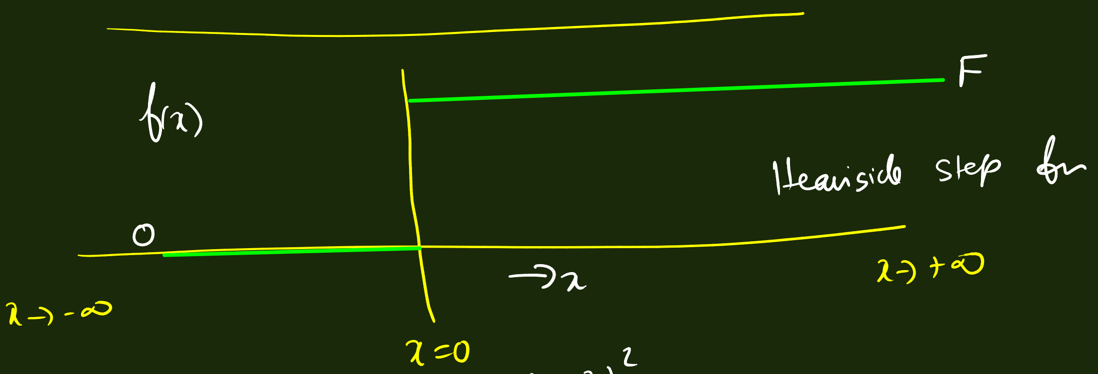
$$\mathcal{F}^{-1}\left[e^{-k^2\alpha^2 t}\right] = \frac{1}{2\alpha\sqrt{\pi t}}\, e^{\frac{-x^2}{4\alpha^2 t}}$$

$g(x) \rightarrow \mathcal{F}^{-1}(\hat{g}(k))$

$f(x) \rightarrow \mathcal{F}^{-1}(\hat{f}(k))$

$$\boxed{u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \underbrace{f(z)}_{\text{prescribe}}\, e^{\frac{-(x-z)}{4\alpha^2 t}}\, dz}$$

Special case:



$$u(x,t) = \frac{F}{2\alpha\sqrt{\pi t}} \int_{z=0}^{z=\infty} e^{-\frac{(x-z)^2}{4\alpha^2 t}} \, dz$$

let $\eta = \dfrac{x-z}{2\alpha\sqrt{t}}$ $\qquad d\eta = \dfrac{-dz}{2\alpha\sqrt{t}}$

$$u(x,t) = \frac{-F}{2\alpha\sqrt{\pi t}} \int_{\frac{x}{2\alpha\sqrt{t}}}^{-\infty} e^{-\eta^2} \, 2\alpha\sqrt{t} \, d\eta = \frac{-F}{\sqrt{\pi}} \int_{\frac{x}{2\alpha\sqrt{t}}}^{-\infty} e^{-\eta^2} \, d\eta$$

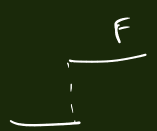
$$u(x,t) = \frac{F}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{2\alpha\sqrt{t}}} e^{-\eta^2} \, d\eta$$

$$u(x,t) = \frac{F}{\sqrt{\pi}} \left[\underbrace{\int_{-\infty}^{0} e^{-\eta^2} \, d\eta}_{\frac{\sqrt{\pi}}{2}} + \int_{0}^{\frac{x}{2\alpha\sqrt{t}}} e^{-\eta^2} \, d\eta \right]$$

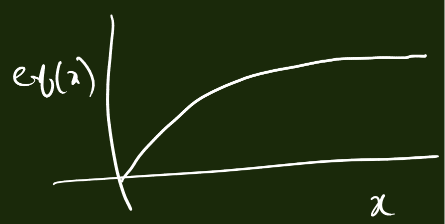
Error fn:

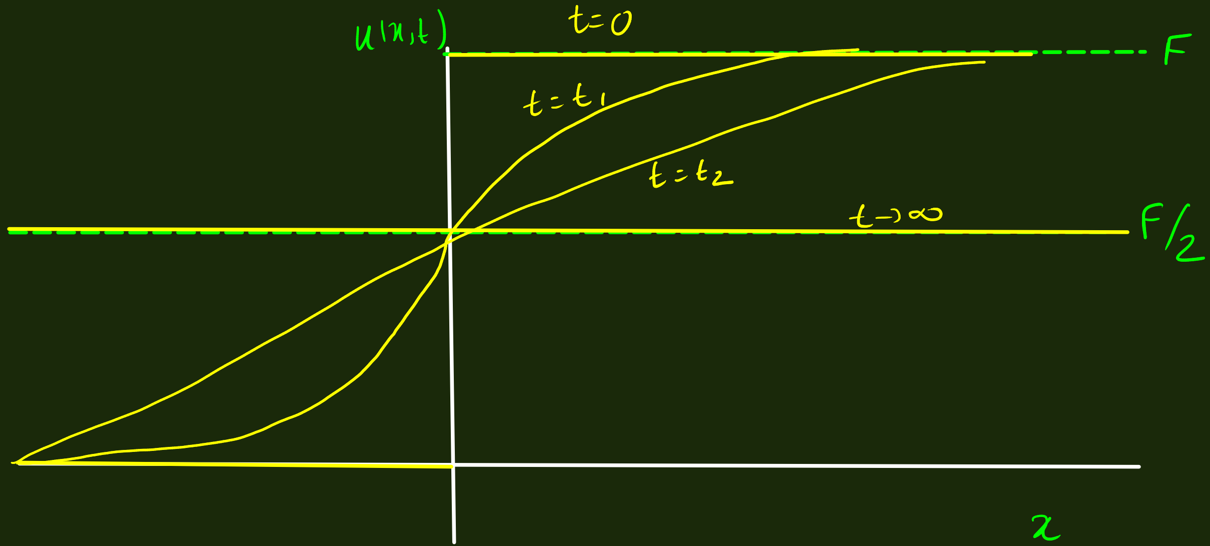
$$\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-y^2} \, dy$$

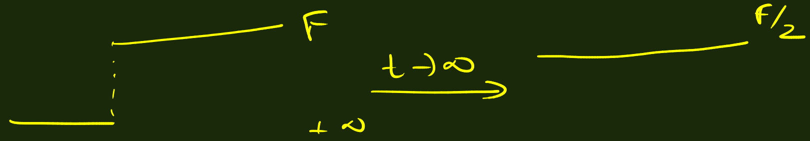
$$u(x,t) = \frac{F}{2} \left[1 + \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\alpha\sqrt{t}}} e^{-\eta^2} \, d\eta \right]$$



$$\boxed{u(x,t) = \frac{F}{2} \left[1 + \mathrm{erf}\left(\frac{x}{2\alpha\sqrt{t}} \right) \right]}$$







$$u(x,t) = \int_{-\infty}^{\infty} f(z)\, K(z-x,t)\, dz$$

↳ soln to δ fn fn y

"Green's function"

$$u(x,t) = \int_{-\infty}^{\infty} f(z) \frac{1}{2\alpha\sqrt{\pi t}} e^{-(x-z)^2/4\alpha^2 t}\, dz$$

$f(z)$ → Initial condn

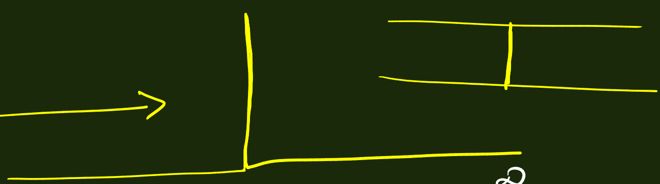
"Kernel" $K(z-x,t)$ → Soln to the PDE to $\delta(x)$ input!!

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$u(x, t=0) = f(x)$$

$$u(x,t) = \int_{-\infty}^{\infty} f(z)\, K(z-x,t)\, dz$$

Special I.C $\quad f(z) = \delta(z-z_0)$ →



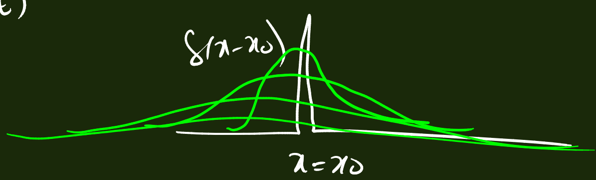
$$u(x,t) = \int_{-\infty}^{\infty} \delta(z-z_0)\, K(z-x,t)\, dz$$

$$\int_{-\infty}^{\infty} \delta(x-x_0) f(x)\, dx = f(x_0)$$

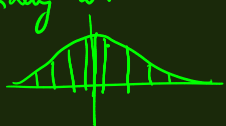
$$u(x,t) = K(z_0-x,t) = K(x-z_0,t) = \frac{e^{-\frac{(x-z_0)^2}{4\alpha^2 t}}}{2\alpha\sqrt{\pi t}}$$

even fn.

↳ soln. to a δ fn input.



spatially dist.



$K(x-z)$ → soln to δ-fn. I.C

linearity

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\, \underbrace{K(\xi-x)}\, d\xi$$