Linear Algebra - Part 12: Solution of Simultaneous Linear Equations

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$5x + 2y = 6$ $\quad x, y \rightarrow$ occur linearly $\rightarrow$ linear eqn

$$\left.\begin{aligned} 3x + 4y + 2z &= 7 \\ 7x - 6y - z &= 3 \\ -3x + 2y - 3z &= 4 \end{aligned}\right\}$$

e.g 3 linear eqns in 3 unknowns.

$3x^4 - 4x^3 - 6x^2 + 7 = 0 \rightarrow$ nonlinear eqn.

$3x^8 - 7y^4 = 0 \quad x, y$

non-linear - transcendental eqns $\quad e^x - 3x + 4 = 0$

$\ln(x)$
$\sinh(x)$
$\cosh(x)$

Simultaneous linear eqns:

$$\left.\begin{array}{ll} 1 - & a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ 2 - & a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots & \vdots \\ m - & a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{array}\right\}$$

m equations

n unknowns $\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

In general, $m \neq n$

$m > n$

$m = n$

$m < n$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & & & & \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{bmatrix}}_{(m \times n)} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}}_{(n \times 1)} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{bmatrix}}_{(m \times 1)} \rightarrow \underset{\text{known}}{\underset{(m \times n)}{\underline{\underline{A}}}} \cdot \underset{(n \times 1)}{\overset{\text{unknown}}{\underline{x}}} = \underset{\text{known}}{\overset{(m \times 1)}{\underline{b}}}$$

$\underset{(m\times n)}{\underline{\underline{A}}} \cdot \underset{(n\times 1)}{\underline{x}} = \underline{b}$ Inhomogeneous eqn if $\underline{b} \neq 0$

If $\underline{b} = 0$, $\underline{\underline{A}} \cdot \underline{x} = \underline{0}$ Homogeneous eqn.

$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$

(i) a unique soln.

(ii) no soln

(iii) infinitely many solns

Simplest case $m = 1, n = 1$: $\boxed{a_{11} x_1 = b_1}$

(i) if $a_{11} \neq 0$ then $x_1 = \dfrac{b_1}{a_{11}}$ $\rightarrow$ unique soln.

(ii) $a_{11} = 0$ and $b \neq 0$ $\rightarrow$ no soln. $\rightarrow$ inconsistent eqn.

(iii) if $a_{11} = 0$ and $b_1 = 0$ $\rightarrow$ any x_1 will satisfy the eqn $\rightarrow$ infinitely many solns.

$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$

$\underline{\underline{A}} = [\underline{c}_1 \ \underline{c}_2 \ \cdots \ \underline{c}_n]$

m rows
n columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

$$[\underline{c}_1 \ \underline{c}_2 \ \cdots \ \underline{c}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underline{b}$$

$\underline{\underline{A}} \, \underline{x} = \underline{b}$

$x_1 \underline{c}_1 + x_2 \underline{c}_2 + \cdots + x_n \underline{c}_n = \underline{b}$

unknown coeffs $(x_1, x_2 \ldots x_n)$

known

$\underline{b} = x_1 \underline{c}_1 + x_2 \underline{c}_2 + \cdots + x_n \underline{c}_n$

known

unknowns !!

Example: $m=2, n=3$ 2 eqn, 3 unknowns.

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} + x_3 \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$\underline{c}_1$ $\underline{c}_2$ $\underline{c}_3$

known

(2D) vectors $\rightarrow$ 2D at most 2 L.I vectors $(x_1, x_2, x_3) \rightarrow$ cannot be unique.

$(m<n)$ $\underline{A} \cdot \underline{x} = b \rightarrow$ Soln cannot be unique!

Example: $m=n=3$

$$\underline{b} = x_1 \underline{c}_1 + x_2 \underline{c}_2 + x_3 \underline{c}_3$$

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}$$

unique soln. $\leftarrow$ $\left\{ \begin{array}{l} \underline{c}_1, \underline{c}_2, \underline{c}_3 \text{ are L.I} \\ \text{and} \\ (\underline{b}, \underline{c}_1, \underline{c}_2, \underline{c}_3) \\ \quad \text{L.D} \end{array} \right.$

If $\underline{c}_1, \underline{c}_2, \underline{c}_3$ are L.D $\rightarrow$ $\underline{b} = x_1 \underline{c}_1 + x_2 \underline{c}_2 + x_3 \underline{c}_3$ cannot be unique

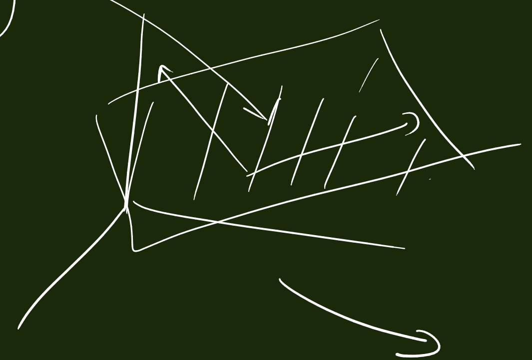
Example. $m=3, n=2$ $(m>n)$ more eqns than unknowns

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$
$$a_{31} x_1 + a_{32} x_2 = b_3$$

$\underline{b}, \underline{c}_1, \underline{c}_2 \rightarrow (3\times1)$ column vectors

$$x_1 \underline{c}_1 + x_2 \underline{c}_2 = \underline{b}$$

$\underline{c}_1$ $\underline{c}_2$



$(m\times n) \to$ lin eqns

If there exists one or more solns $\longrightarrow$ System consistent

- only one soln $\to$ unique system
- many solns $\to$ non-unique

No solns $\to$ Inconsistent.

Rank of a matrix :

(i) A $m\times n$ matrix has a rank r if it has at least one $r\times r$ Submatrix (square) with nonzero determinant, but no square matrix larger than $r\times r$ with nonzero det.

(ii) rank $\underline{\underline{A}}$ = # of L.I column vectors = # of L.I row vectors

$$\underline{\underline{A}}\,\underline{x} = \underline{b} \longrightarrow \underline{b} = x_1\underline{c}_1 + x_2\underline{c}_2 + \dots + x_n\underline{c}_n$$

$\underline{\underline{A}}.\underline{x} = \underline{b}$

$\underline{\underline{A}} \to$ coefficients matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & & \\ & & \end{bmatrix}$$

$$\underline{b} \to \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\left[\begin{array}{c|c} \underline{\underline{A}} & \begin{matrix} b_1 \\ b_2 \\ \vdots \end{matrix} \end{array}\right] \to \text{augmented matrix}$$

$\underline{b} = x_1\underline{c}_1 + \dots x_n\underline{c}_n$

(i) $r(\underline{\underline{A}}|\underline{b}) \neq r(\underline{\underline{A}}) \longrightarrow$ No solutions

(ii) $r(\underline{\underline{A}}|\underline{b}) = r(\underline{\underline{A}}) = n \longrightarrow$ unique soln.

(iii) $r(\underline{\underline{A}}|b) = r(\underline{\underline{A}}) < n \longrightarrow$ non unique solns

Gauss elimination:

Example.

original set
$$\begin{cases} x_1 + x_2 - x_3 = 1 & \text{— (1)} \\ 3x_1 + x_2 + x_3 = 9 & \text{— (2)} \\ x_1 - x_2 + 4x_3 = 8 & \text{— (3)} \end{cases}$$

$m = 3$ eqns
$n = 3$ unknowns

$(1) \longrightarrow (4) \longrightarrow x_1 + x_2 - x_3 = 1$

$(2) \longrightarrow -3 \times (1) + (2) \longrightarrow (5) \longrightarrow -2x_2 + 4x_3 = 6$

$(3) \longrightarrow -1 \times (1) + (3) \longrightarrow (6) \longrightarrow -2x_2 + 5x_3 = 7$

$(4) \rightarrow$ same
$(5) \rightarrow$ same
(7) $-1 \times (5) + (6)$

$\longrightarrow$ upper triangular set
$$\begin{cases} x_1 + x_2 - x_3 = 1 \longrightarrow x_1 = 3 \\ -2x_2 + 4x_3 = 6 \longrightarrow x_2 = -1 \\ x_3 = 1 \end{cases}$$

upper-triangular matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right] \qquad \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix}$$

unique soln

$$r(\underline{A}) = 3 = r(\underline{A} \mid \underline{b})$$

Two systems of lin eqs with n unknowns are equivalent if their soln sets are identical.

(i) eq $j + \alpha$(eq k) $\rightarrow$ eq j

(ii) eq $j \rightarrow \alpha$(eq j)

(iii) interchange eq j and eq k.

Example   Inconsistent system.

$$2x_1 + 3x_2 - 2x_3 = 4$$
$$x_1 - 2x_2 + x_3 = 3$$
$$7x_1 - x_3 = 2$$

Gauss elimination

$$① \to ①$$
$$② \to ② - \tfrac{1}{2}①$$
$$③ \to ③ - \tfrac{7}{2}①$$

$$\left\{\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 4 \quad —④ \\ -\tfrac{7}{2}x_2 + 2x_3 &= 1 \quad —⑤ \\ -\tfrac{21}{2}x_2 + 6x_3 &= -12 \quad —⑥ \end{aligned}\right.$$

$$⑥ \to -3\times⑤ + ⑥ \to 0 = -15$$

Inconsistent

$$\cancel{+\tfrac{21}{2}x_2} \; \cancel{-6x_3} = -3$$
$$\cancel{-\tfrac{21}{2}x_2} \; \cancel{+6x_3} = -12$$

$$\left.\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 4 \\ x_1 - 2x_2 + x_3 &= 3 \\ 7x_1 - x_3 &= 17 \end{aligned}\right\} \to \text{Gauss eli.} \left\{\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 4 \\ -\tfrac{7}{2}x_2 + 2x_3 &= 1 \\ 0 &= 0 \end{aligned}\right.$$

2 eqns for 3 unkn.

$x_3 = \alpha$ (anything)

$$\boxed{\begin{aligned} x_2 &= -\tfrac{2}{7} + \tfrac{4}{7}\alpha \\ x_3 &= \tfrac{17}{7} + \tfrac{1}{7}\alpha \\ x_3 &= \alpha \end{aligned}}$$

one-parameter family of soln