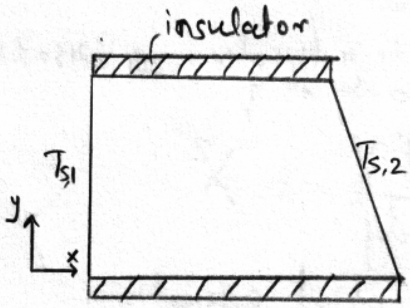
# Steady State Conduction: Multidimensional

Consider steady heat conduction through any Object (such as the trapezoidal cross section in figure), such that the temperature gradient has two (or more) components, in x and y direction.




Choice: Cartesian coordinate system

Assumptions:
- Steady state
- No heat source or sink
- Two dimensional heat conduction
- Uniform k

Simplified Heat Conduction Equation:

$$0 = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

To solve for $T(x,y)$ (and hence $Q$, $q$),

- Boundary conditions:
  - Two boundary conditions for x coordinate
  - Two boundary conditions for y coordinate

- Analytical solution
  - Separation of variables

Steps involved

Assuming Dirichlet boundary conditions

Define dimensionless temperature $\theta$

$$\theta(x,y) = \frac{T - T_1}{T_2 - T_1} \quad \begin{matrix} \leftarrow \text{shift} \\ \leftarrow \text{temperature scale} \end{matrix}$$

$$\Rightarrow \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

Assume separation of variables to be valid

$$\theta(x,y) = X(x) \cdot Y(y)$$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2}$$

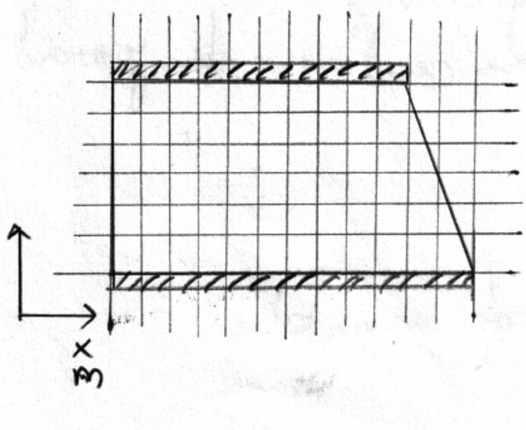
LHS is a function only of x. RHS is a function only of y $\Rightarrow$ LHS = RHS = Constant $\uparrow \lambda^2$

$$\Rightarrow \frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{d^2 Y}{dy^2} = \lambda^2$$

And solve each differential equation separately.

- Numerical Solution
  - For complex geometries or boundary conditions
  - Finite Difference, Finite Element or Boundary Element method

Finite Difference




Choose a grid with spacing $\Delta x$ along x and $\Delta y$ along y

Each discrete grid point is represented as $(m,n)$ and is at a temperature

$$T(m,n) = T_{m,n}$$

$$\left.\frac{\partial T}{\partial x}\right|_{m,n} \simeq \frac{T_{m+1,n} - T_{m-1,n}}{2\Delta x} \quad ; \quad \left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{(\Delta x)^2}$$

Central difference

$$\left.\frac{\partial T}{\partial y}\right|_{m,n} \simeq \frac{T_{m,n+1} - T_{m,n-1}}{2\Delta x} \qquad \left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2}$$

Substituting in the simplified energy balance and in boundary conditions gives algebric equations to be solved. Solution typically involves inverting a matrix.

Notes:

- Accuracy depends on $\Delta x$, $\Delta y$ and the approximation used
- Non-uniform grid can improve accuracy