ChE641 Mathematical Methods in Chemical Engineering

Assignment 7 **Due Date: 26 November 2020**
Function spaces, Eigenfunction expansions

1. Consider the differential operator $\mathcal{L}[u] = \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2}$, in the domain $0 < x < 1$ and with a set of "*anti-periodic*" boundary conditions:

$$u(0) = -u(1)\,, \qquad\qquad \frac{\mathrm{d}u(0)}{\mathrm{d}x} = -\frac{\mathrm{d}u(1)}{\mathrm{d}x}$$

   (a) Find the set of adjoint BCs. (b) Is this a self-adjoint operator?

2. Consider the eigenvalue problem $\mathcal{L}[y] \equiv \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\lambda y$, $0 < x < 1$ along with the following types of boundary conditions (prime denotes a derivative with respect to x):

   (a) $y[0] = y[1] = 0.$
   (b) $y'[0] = y'[1] = 0.$
   (c) $y[0] = y'[1] = 0.$
   (d) $y'[0] = y[1] = 0.$

   For these boundary conditions, determine the eigenvalues and normalized eigenfunctions.

3. Find the adjoints $\mathcal{L}^\dagger$ of the operator $\mathcal{L}u \equiv \dfrac{\mathrm{d}^4 u}{\mathrm{d}x^4}$, $0 < x < 1$ with the following boundary conditions:

   (a) $u(0) = u(1) = u''(0) = u''(1) = 0.$
   (b) $u'(0) = u'(1) = u'''(0) = u'''(1) = 0.$
   (c) $u(0) = u(1) = u''(0) = u''(1) = 0$ and $u'(0) = u'(1).$
   (d) $u(0) = u'''(0) = u(1) = u'''(1) = 0.$ For which set(s) of boundary conditions is the operator self-adjoint?

4. Find the adjoint operator and boundary conditions for $\mathcal{L}u = \dfrac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \dfrac{\mathrm{d}u}{\mathrm{d}x}$ subject to $u(0) = 2u'(1) + 3$ and $u(1) = 0$.

5. Show by explicit integration (where $m, n = 0, 1, 3, \cdots, \infty$) that

$$\frac{2}{L}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \delta_{mn}$$

6. Show by explicit integration (where $m, n = 0, \pm 1, \pm 2 \cdots \infty$) that $\langle \phi_m, \phi_n \rangle = \delta_{mn}$ where $\phi_m(x) = 1/\sqrt{L}\exp[2\pi i m x/L]$ in the domain $0 < x < L$.