Differential Equations - Part 7: Solution of Linear PDEs using Laplace Transform

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PDE:
$$\alpha^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t} \qquad 0 \le x < \infty, \quad 0 \le t < \infty$$

I.C: $u(x, t=0) = 0$

B.C: $u(x=0, t) = g(t)$ — To find $u(x,t)$:

$u(x \to \infty, t) = 0$

$-\infty$ to ∞ $\rightarrow$ Fourier Transform $\rightarrow$ not possible Semi infinite domains.

Laplace transform $\rightarrow$ time domain — PDE in (x,t) $\rightarrow$ ODE in x

Laplace transform of the PDE

$$\int_0^\infty (\text{PDE})\, e^{-st}\, dt$$

$$\alpha^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st}\, dt = \int_0^\infty \frac{\partial u}{\partial t} e^{-st}\, dt$$

$$\alpha^2 \frac{\partial^2}{\partial x^2} \int_0^\infty u(x,t)\, e^{-st}\, dt = s\, \bar{u}(x,s) - \underset{0}{\cancel{u(x, t=0)}}$$

$\frac{d^2}{dx^2}$ $\quad \bar{u}(x,s)$.

$$\frac{d^2 \bar{u}}{dx^2} - \frac{s}{\alpha^2} \bar{u}(x,s) = 0$$

$$\bar{u}(x,s) = \underset{0}{\cancel{A}}\, e^{\sqrt{s}\, x/\alpha} + B\, e^{-\sqrt{s}\, x/\alpha}$$

B.C $\rightarrow$ Laplace transformed.

$x \to \infty$: $\displaystyle \lim_{x\to\infty} \bar{u}(x,s) = \lim_{x\to\infty} \int_0^\infty u(x,t)\, e^{-st}\, dt$

$$= \int_0^\infty \underset{0}{\cancel{\lim_{x\to\infty} u(x,t)}}\, e^{-st}\, dt$$

$$\boxed{\bar{u}(x \to \infty, s) = 0}$$

$$\bar{u}(x,s) = B\, e^{-\sqrt{s}\,x/\alpha}$$

B.C at $\bar{u}(x=0,s) = \bar{g}(s)$ ← known.

L.T: $u(x=0,t) = g(t) \to \bar{u}(x=0,s) = \bar{g}(s)$

$$\bar{u}(x,s) = \bar{g}(s)\, e^{-\sqrt{s}\,x/\alpha}$$

↳ Inverse Laplace Transform

Convolution theorem:

$$u(x,t) = \frac{x}{2\alpha\sqrt{\pi}} \int_0^t g(t-\tau)\, \frac{e^{-x^2/4\alpha^2\tau}}{\tau^{3/2}}\, d\tau$$

Special case: $g(t) = T_0$

$$u(x,t) = \frac{T_0}{\sqrt{\pi}} \frac{x}{2\alpha} \int_0^t \frac{e^{-x^2/4\alpha^2\tau}}{\tau^{3/2}}\, d\tau$$

$$\xi^2 = \frac{x^2}{4\alpha^2\tau}$$

$$\xi = \frac{x}{2\alpha\sqrt{t}}$$

$$u(x,t) = \frac{T_0}{\sqrt{\pi}} \int_\infty^{\frac{x}{2\alpha\sqrt{t}}} e^{-\xi^2} \frac{1}{\tau^{3/2}} \left(\frac{x}{2\alpha}\right) \frac{-2x^2}{4\alpha^2} \frac{d\xi}{\xi^3}$$

$$u(x,t) = \frac{(-2)T_0}{\sqrt{\pi}} \int_\infty^{\frac{x}{2\alpha\sqrt{t}}} e^{-\xi^2} \cancel{\left(\frac{x}{2\alpha\sqrt{\tau}}\right)^3} \frac{d\xi}{\cancel{\xi^3}}$$

"Similarity transform"

$$\frac{x}{2\alpha\sqrt{t}} = \xi$$

$$u(x,t) = \frac{-2T_0}{\sqrt{\pi}} \int_\infty^{\frac{x}{2\alpha\sqrt{t}}} e^{-\xi^2}\, d\xi = T_0 \frac{2}{\sqrt{\pi}} \int_{\frac{x}{2\alpha\sqrt{t}}}^{\infty} e^{-\xi^2}\, d\xi$$

Complementary error fn

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2}\, d\xi$$

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$$

$$u(x,t) = T_0\, \mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)$$

$$\boxed{u(x,t) = T_0 \left[1 - \mathrm{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right]}$$

$g(t) = T_0$

$$\frac{x^2}{4\alpha^2\tau} = \xi^2 \qquad \xi = \frac{x}{2\alpha\sqrt{\tau}}$$

$$\tau = \frac{x^2}{4\alpha^2\xi^2} \qquad d\tau = \frac{x^2}{4\alpha^2}(-2)\frac{1}{\xi^3}\,d\xi$$