Differential Equations - Part 2

Che641, IIT Kanpur

Linear differential equations:

linearity: $\mathcal{L}[\alpha_1 y_1 + \alpha_2 y_2] = \alpha_1 \mathcal{L} y_1 + \alpha_2 \mathcal{L} y_2$

linear operator

$\alpha_1 y_1 + \alpha_2 y_2$ is also a soln. $\rightarrow$ principle of superposition.

$$\mathcal{L} y = \underbrace{f_1 + f_2 + \cdots f_k}_{\text{known fns}}$$

unknown fn $y(x)$

known fns $\rightarrow$ inhomogeneous terms

$\rightarrow$ "forcing fns"

Analogy: $\underline{A} \cdot \underline{x} = \underline{b}$

matrix, unknown vector, known vector

$\mathcal{L}(y_h) = 0$ homogeneous problem

$\rightarrow$ Homogeneous solns.

$$\mathcal{L} y_{p1} = f_1$$
$$\mathcal{L} y_{p2} = f_2$$
$$\vdots$$
$$\mathcal{L} y_{pk} = f_k$$

Add them:

$$\mathcal{L}(\underbrace{y_{p1} + y_{p2} + \cdots y_{pk}}_{\text{principle of superposition}}) = f_1 + f_2 + \cdots f_k$$

$\mathcal{L} y = f_1 + \cdots f_k$

$$y(x) = y_h(x) + y_{p1}(x) + \cdots + y_{pk}(x)$$

$$\frac{d^2 T}{dx^2} = \underset{\text{homog.}}{0}$$

B.C: $T(x=0) = 0$

$T(x=L) = T_0$

forcing in the B.C's

Solution of ODE's using the Laplace Transform method :

$\hookrightarrow$ Initial value problems $\rightarrow$ IVPs

* Linear ODE's with const coeffs.
  - $\hookrightarrow$ coeffs should be indep. of the independent variable t

$$\frac{d^2y}{dt^2} + (\underset{\downarrow\, 5}{\ }) \frac{dy}{dt} = 0.$$

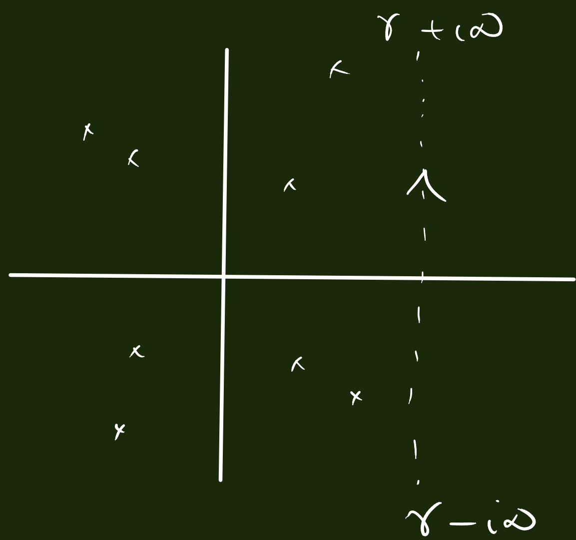
not be a fn of t

* ODE for $y(t)$ $\longrightarrow$ algebraic in s $\hookrightarrow$ Laplace variable
* Solve the algebraic eqn for $\bar{y}(s)$
* Invert $\bar{y}(s)$ to obtain $y(t)$

<u>Recall</u> : LT pair :

$$\bar{f}(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \bar{f}(s)\, e^{st}\, ds$$



L.T of $f(t)$

$\uparrow$

$$\bar{f}(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$\text{L.T of } \frac{df}{dt} = f'(t) = \overline{f'}(s)$$

$$\underset{\text{L.T of } \frac{df}{dt}}{\overline{f'}(s)} = \int_0^{\infty} \frac{df}{dt}\, e^{-st}\, dt$$

Integrate by parts :

$$\overline{f'}(s) = \left[e^{-st} f\right]_0^{\infty} - \int_0^{\infty} f(t)\,(-s)\,e^{-st}\, dt$$

$$\overline{f'}(s) = \cancel{\left(e^{-st} f(t)\right)_{t\to\infty}}^{\,0} - \underbrace{\left(e^{-st} f(t)\right)_{t=0}}_{-f(t=0)} + s\boxed{\int_0^{\infty} f(t)\, e^{-st}\, dt}^{\,\bar{f}(s)}$$

$$\text{L.T of } \frac{df}{dt} \rightarrow \overline{f'(s)} = -f(t=0) + s\bar{f}(s)$$

Initial condition for $f(t)$

$$\text{L.T of } \frac{d^2 f}{dt^2} \rightarrow \overline{f''(s)} = -f'(t=0) - s f(t=0) + s^2 \bar{f}(s)$$

Initial conditions to complete the problem.

Example. Solve $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 2e^{-t}$ $\quad$ IC $\begin{cases} y(t=0) = 2 \\ y'(t=0) = 1 \end{cases}$

$$s^2\bar{y}(s) - s\,\overset{2}{y(t=0)} - \overset{1}{y'(t=0)} - 3\left[s\bar{y}(s) - \overset{2}{y(t=0)}\right] + 2\bar{y}(s) = 2\frac{1}{(s+1)}$$

$$\left[s^2 - 3s + 2\right]\bar{y}(s) - 2s - 1 + 6 = \frac{2}{(s+1)}$$

$$(s^2 - 3s + 2)\,\bar{y}(s) - 2s + 5 = \frac{2}{s+1}$$

$$(s^2 - 3s + 2)\,\bar{y}(s) = \frac{2}{(s+1)} + (2s - 5)$$

$$\bar{y}(s) = \frac{2 + (2s-5)(s+1)}{(s+1)(s^2 - 3s + 2)}$$

$$= \frac{2 + 2s^2 + 2s - 5s - 5}{(s+1)(s-1)(s-2)} = \frac{2s^2 - 3s - 3}{(s+1)(s-1)(s-2)}$$

$$\bar{y}(s) = \frac{1}{3(s+1)} + \frac{2}{(s-1)} - \frac{1}{3(s-2)}$$

$$\bar{y}(s) = \frac{1}{3(s+1)} + \frac{2}{(s-1)} - \frac{1}{3(s-2)}$$

Inv. Lap. transform

$$y(t) = \frac{1}{3} e^{-t} + 2 e^{t} - \frac{1}{3} e^{2t}$$

$$y(t=0) = \cancel{\frac{1}{3} \cdot 1} + 2 \cdot \cancel{-\frac{1}{3} \cdot 1} = 2$$

$$y'(t=0) = 1$$

<u>Eigenfunction expansions</u> $\rightarrow$ solution <u>of ODE's</u>

$\hookrightarrow$ * linear ODE's / PDE's.

* principle of superposition.

$$\underline{\underline{A}}\, \underline{x} = \underline{b} \qquad \underline{\underline{A}}\, \underline{e}_i = \lambda_i \underline{e}_i \quad \text{eig. val. problem for matrix } \underline{\underline{A}}.$$

$\hookrightarrow$ N lin. indep / orthogonal eig vectors

$$\underline{x} = \sum_{i=1}^{N} c_i \underline{e}_i$$

$\underline{x}$ $\downarrow$ unknown

c_i $\downarrow$ coeffs

$\underline{e}_i$ $\hookrightarrow$ eig vectors

$$\underline{\underline{A}} \left(\sum_{i=1}^{N} c_i \underline{e}_i \right) = \underline{b}$$

$\downarrow$ linear operator

$$\rightarrow \quad \sum_{i=1}^{N} c_i \underbrace{\underline{\underline{A}}\, \underline{e}_i}_{\lambda_i \underline{e}_i} = \underline{b}$$

$\underline{b}$ $\hookrightarrow$ known

$$\langle \underline{e}_i, \underline{e}_j \rangle = \delta_{ij}$$

$$\sum_{i=1}^{N} c_i \lambda_i \underline{e}_i = \underline{b}$$

$$\underline{\underline{A}}\, \underline{x} = \underline{b}$$

$$\underline{x} = \sum_{j=1}^{N} c_j \underline{e}_j$$

eig. vectors

$$\sum_{i=1}^{N} c_i \lambda_i \underbrace{\langle \underline{e}_j, \underline{e}_i \rangle}_{\delta_{ij}} = \langle \underline{e}_j, \underline{b} \rangle$$

$$c_j \lambda_j = \langle \underline{e}_j, \underline{b} \rangle$$

$$\boxed{c_j = \frac{\langle \underline{e}_j, \underline{b} \rangle}{\lambda_j}}$$

linear differential operator $\leftarrow \mathcal{L}\, y(x) = f(x)$ with suitable B.C's

$f(x)$ → known

$\mathcal{L} \to$ self-adj operator

Eig. val. problem:

$$\boxed{\mathcal{L}\, \phi_i(x) = \lambda_i \phi_i(x)}$$

eig fns.

$\langle \phi_i, \phi_j \rangle = \delta_{ij}$

Eig. fn Expansion:

$$y(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

c_i → coeffs to be found

$$\mathcal{L}\, y(x) = \mathcal{L} \sum_{i=1}^{N} c_i \phi_i(x) = f(x)$$

$$\sum_{i=1}^{N} c_i \underbrace{\mathcal{L}\, \phi_i(x)}_{\lambda_i \phi_i(x)} = f(x)$$

$\langle \phi_i, \phi_i \rangle = \delta_{ij}$

$$\sum_{i=1}^{N} c_i \lambda_i \phi_i(x) = f(x)$$

$$\sum_{i=1}^{N} c_i \lambda_i \underbrace{\langle \phi_j, \phi_i \rangle}_{\delta_{ij}} = \langle \phi_j, f(x) \rangle$$

$$c_j \lambda_j = \langle \phi_j, f(x) \rangle$$

$$\boxed{c_j = \frac{\langle \phi_j, f(x) \rangle}{\lambda_j}}$$

$$y(x) = \sum_{j=1}^{N} c_j \phi_j(x)$$

<u>Example</u>. $\mathcal{L}\, y = -\dfrac{d^2 y}{dt^2}$ $\qquad \mathcal{L} \to -\dfrac{d^2}{dt^2}$

Eig. val. problem: $\mathcal{L}\, \phi_n = \lambda_n \phi_n$

$$-\frac{d^2 \phi_n}{dt^2} = \lambda_n \phi_n \qquad \frac{d^2 \phi_n}{dt^2} + \lambda_n \phi_n = 0$$

$$\lambda_n = \omega_n^2$$

$$\frac{d^2 \phi_n}{dt^2} + \omega_n^2 \phi_n = 0$$

$$\phi_n(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$

B.C's : periodic $\qquad \phi_n(t=0) = \phi_n(T)$

$$A = A\cos(\omega_n T) + B\sin(\omega_n T)$$

$\cos(\omega_n T) = 1$ AND $\sin(\omega_n T) = 0$

$\omega_n T = 2\pi n$

$(n = 0, 1, 2 \ldots)$

$$\boxed{\omega_n = \frac{2\pi n}{T}}$$

$$\phi_n(t) = A\cos\left(\frac{2\pi n}{T} t\right) + B\sin\left(\frac{2\pi n}{T} t\right) \qquad n = 0, 1, 2 \ldots$$

$$\frac{d^2 y}{dt^2} = f(t)$$

$\underline{\underline{A}}\, \underline{x} = \lambda \underline{x}$

$\underline{\underline{A}}\, \underline{x} = \lambda \underline{\underline{B}}\, \underline{x}$

In some cases $\qquad \mathcal{L}\,\phi_i(x) = \lambda_i\, \rho(x)\, \phi_i(x)$

$\to$ self-adj operator $\qquad \mathcal{L} = \mathcal{L}^\dagger$

Then, the eig val problem $\longrightarrow$ real eig values

eig fns corresponding to distinct eig values are orthogonal.

Defn of adjoint of $\mathcal{L} \longrightarrow \mathcal{L}^\dagger$

$$\langle f, \mathcal{L} g\rangle = \langle \mathcal{L}^\dagger f, g\rangle + \text{Boundary terms}$$

$$\int_a^b f^*(x)\, \mathcal{L} g(x) = \int_a^b \left[\mathcal{L}^\dagger f(x)\right]^* g(x)\, dx + \underbrace{\text{Boundary terms}}_{\text{B.C's at } x=a,\ x=b}$$

The operator is self-adj : $\quad \boxed{\mathcal{L} = \mathcal{L}^\dagger \text{ and Boundary terms vanish}}$

(or) Hermitian

$\mathcal{L} = \frac{d^2}{dt^2}$. Is $\mathcal{L} = \mathcal{L}^+$ (is $\mathcal{L}$ self-adj?)

$$\langle f, \mathcal{L} g \rangle = \langle \mathcal{L} f, g \rangle + \text{B terms}$$

$$\int_{t_0}^{t_0+T} f^* \frac{d^2 g}{dt^2} \, dt = \left[f^* \frac{dg}{dt} \right]_{t_0}^{t_0+T} - \int_{t_0}^{t_0+T} \frac{dg}{dt} \frac{df^*}{dt} \, dt$$

$$= \underbrace{\left[f^* \frac{dg}{dt} \right]_{t_0}^{t_0+T} - \left[\frac{df^*}{dt} g \right]_{t_0}^{t_0+T}}_{\text{Boundary terms.}} + \int_{t_0}^{t_0+T} g \frac{d^2 f^*}{dt^2} \, dt$$

For B. terms to vanish:

$$\left[f^* \frac{dg}{dt} \right]_{t_0}^{t_0+T} = \left[\frac{df^*}{dt} g \right]_{t_0}^{t_0+T}$$

$\mathcal{L}$ is self-adjoint.

$$\left. \begin{aligned} f(t = T_0) &= f(t_0 + T) = 0 \\ g(t = T_0) &= g(t_0 + T) = 0 \end{aligned} \right\} \text{Dirichlet}$$

$$\left. \frac{df^*}{dt} = \frac{dg}{dt} = 0 \right|_{t_0,\, t_0+T}$$

periodic

$\mathcal{L}$ is self-adj

→ real eig values

→ eig fns corresponding to distinct eig values are orthogonal.

Completeness of eig fns:

$$f(x) = \sum_{n=0}^{\infty} c_n \hat{\phi}_n(x)$$

$\hookrightarrow$ eig. fns of $\mathcal{L}$

$$c_n = \langle \phi_n, f \rangle = \int_a^b \phi_n^*(x) f(x) \, dx$$

$$f(x) = \sum_{n=0}^{\infty} \hat{\phi}_n(x) \int_a^b \phi_n^*(z) f(z) \, dz$$

$$f(x) = \int_a^b f(z) \sum_{n=0}^{\infty} \hat{\phi}_n(x)\, \phi_n^*(z)\, dz$$

$$f(x) = \int_a^b f(z)\, \delta(x-z)\, dz$$

Resolution of Identity

$$I = \sum_{i=1}^{N} z_i z_i^{\dagger}$$

Completeness.

Dirac delta fn. $\leftarrow$ $\boxed{\delta(x-z) = \sum_{n=0}^{\infty} \hat{\phi}_n(x)\, \phi_n^*(z)}$ $\rightarrow$ Complete

Sturm-Liouville equations:

$$\frac{d^2 y}{dt^2} + \omega^2 y = 0$$

$$\mathcal{L} y = \lambda \rho y$$

$$\mathcal{L} = -\left[\frac{d}{dx}\left(p(x) \frac{d}{dx} \right) + q(x) \right]$$

$\mathcal{L}$ is self-adj.

Bessel, Chebyshev, Hermite, sin/cos

$$\langle u, \mathcal{L} v \rangle = \langle \mathcal{L} u, v \rangle + \text{Boundary term}$$