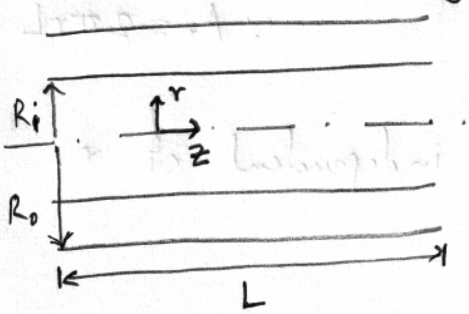
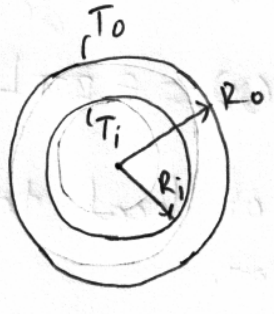
Example: Long cylinder





Assumptions
- steady state
- No gradient in temperature along z-direction or ϕ-direction
- No heat generation

1D heat conduction in cylindrical coordinates with $T = T(r)$ only

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) = \frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)$$

$$0 = \frac{d}{dr}\left(rk\frac{dT}{dr}\right)$$

$$C_1 = rk\frac{dT}{dr} \quad \Rightarrow \quad \frac{C_1}{rk} = \frac{dT}{dr}$$

$$T(r) = \frac{C_1}{k}\ln(r) + C_2$$

Boundary conditions

$T = T_i$ at $r = R_i$

$T = T_o$ at $r = R_o$

$$T_i = \frac{C_1}{k}\ln(R_i) + C_2$$

$$T_o = \frac{C_1}{k}\ln(R_o) + C_2$$

$$C_1 = \frac{k(T_i - T_o)}{\ln(R_i/R_o)} \qquad C_2 = T_i - (T_i - T_o)\frac{\ln(R_i)}{\ln(R_i/R_o)}$$

$$Q_r(r) = -kA_r\frac{dT}{dr} = -A_r\frac{C_1}{r}$$

$$Q_r(r) = -2\pi L C_1 \qquad \because A_r = 2\pi r L$$

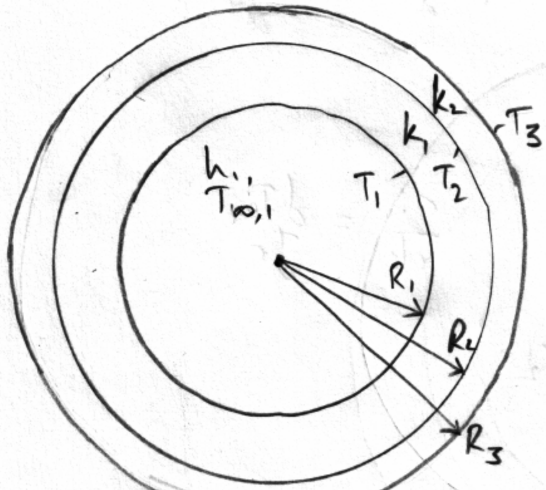
Thus, rate of heat transfer is independent of r.
For completion,

$$Q_r = 2\pi k L \frac{(T_i - T_o)}{\ln(R_o/R_i)}$$

Thus,
$$R_{th} = \frac{\ln(R_o/R_i)}{2\pi k L}$$

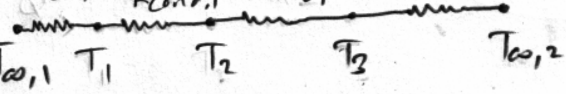
Again, a trivial result for this problem, but a powerful concept in solving complicated problems.

Example: Composite long cylinder (insulated pipe)



$$R_{tot} = R_{conv,1} + R_{cond,1} + R_{cond,2} + R_{conv,2}$$

$$= \frac{1}{h_1(2\pi R_1 L)} + \frac{\ln(R_2/R_1)}{2\pi k_1 L} + \frac{\ln(R_3/R_2)}{2\pi k_2 L} + \frac{1}{h_2(2\pi R_3 L)}$$



$$\frac{1}{h_1 2\pi R_1 L} \quad \frac{\ln(R_2/R_1)}{2\pi k_1 L} \quad \frac{\ln(R_3/R_2)}{2\pi k_2 L} \quad \frac{1}{h_2 2\pi R_3 L}$$

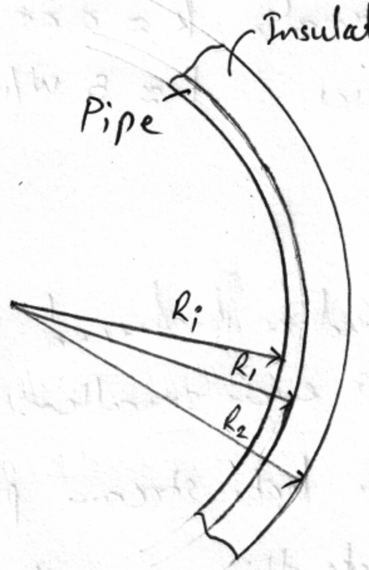
Critical Insulation Thickness

Insulation added to a wall ⟹ always results into decrease in heat transfer

However,
Insulation added to a cylindrical pipe or spherical shell ⇏ not always results into decrease in heat transfer

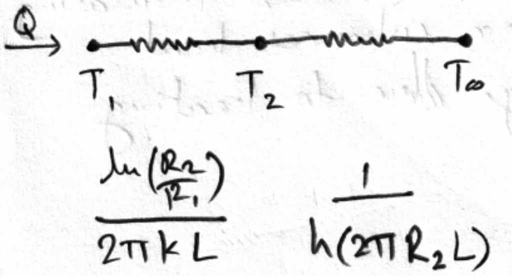
Adding insulation to a plane wall
→ increases conduction resistance
but, no change in convection resistance

However,
Adding insulation to a cylindrical pipe/spherical shell
→ increases conduction resistance
but decreases convection resistance
(due to increase in area for convection heat transfer)



Example: Insulated pipe.

Assume a cylindrical pipe with its outer surface maintained at temperature T_1 ($r=R_1$) and an insulation of radius R_2 surrounding the pipe.



$$R_{total} = \frac{\ln\left(\frac{R_2}{R_1}\right)}{2\pi k L} + \frac{1}{h(2\pi R_2 L)}$$

To find the critical insulation thickness, we optimize (maximize) the rate of heat transfer.

$$Q = \frac{T_1 - T_\infty}{R_{total}}$$

or equivalently, minimize the total resistance by setting

$$\frac{\partial R_{total}}{\partial R_2} = 0 \quad \Rightarrow \quad \frac{\frac{1}{R_2}}{2\pi k L} - \frac{1}{h(2\pi R_2^2 L)} = 0$$

Thus, $R_{2,cr} = \frac{k}{h}$ is the critical radius of insulation.

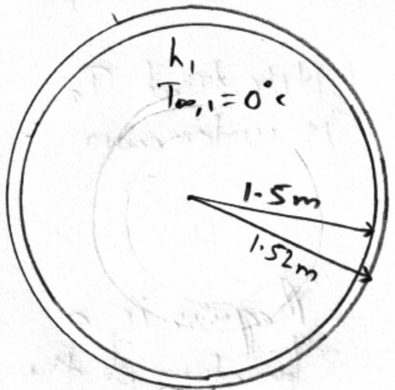
Notes:

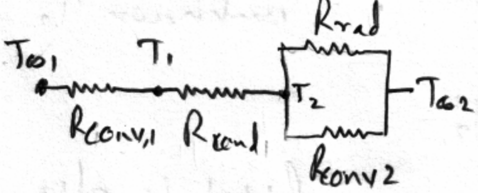
- For $R_2 < R_{2,cr}$, adding insulation increases heat losses
- For common insulating materials, $k \approx 0.05$ W/m.K
  Natural convection of gases $h \approx 5$ W/m².K

  $\Rightarrow R_{cr} \approx 1$ cm

  If radiation is considered or if forced convection is present, this value is even smaller.
- Usually safe to insulate hot stream pipes, etc
  Electric wires are much thinner, and insulating them with a thin insulator enhances heat transfer, preventing damage due to heating of the wires.

Example: Heat Transfer to a Spherical Container





A 3 m diameter spherical tank made of 2 cm thick stainless steel ($k = 15$ W/m·k) is used to store iced water at $T_{\infty 1} = 0°c$. The tank is located in a room whose temperature is $T_{\infty 2} = 22°c$. The walls of the room are also at 22°c. The outer surface of the tank is black and heat transfer between outer surface of the tank and the surroundings is by natural convection and radiation. The convection heat transfer coefficients at the inner and the outer surface of the tank are $h_1 = 80$ W/m²·k and $h_2 = 10$ W/m²·k, respectively. Determine (a) the rate of heat transfer to the iced water in the tank, and (b) the amount of ice at 0°c that melts in a 24 hour period.

Assumptions:

- steady state
- 1D heat transfer
- Constant thermal conductivity

Given:

$k = 15$ W/m.k (thermal conductivity) ; $h_{if} = 333.7$ kJ/kg (heat of fusion) ; $\epsilon = 1$ (emissivity)

Solution:

$$R_{total} = R_{conv,1} + R_{cond} + \frac{1}{\frac{1}{R_{conv,2}} + \frac{1}{R_{rad}}} = R_{conv,1} + R_{cond} + R_{equiv}$$

$$Q = \frac{T_{\infty 2} - T_{\infty 1}}{R_{total}} = \frac{T_{\infty 2} - T_2}{R_{equiv}}$$

$$R_{conv,1} = \frac{1}{h_1 \pi D_1^2} = \frac{1}{80 \times \pi \times 3^2}$$

$$R_{cond} = \frac{R_2 - R_1}{4\pi k R_1 R_2} = \frac{0.02}{4\pi \times 15 \times 1.52 \times 1.5}$$

$$R_{conv2} = \frac{1}{h_2 \pi D_2^2} = \frac{1}{10 \times \pi \times 3.04^2}$$

$$h_{rad} = \varepsilon\sigma\,[T_2^2 + T_{\infty 2}^2]\,[T_2 + T_{\infty 2}]$$

$$R_{rad} = \frac{1}{h_{rad} \pi D_2^2} = \frac{1}{h_{rad} \times \pi \times 3.04^2}$$

Note that T_2 is unknown

$$R_{equiv}(T_2) = \frac{1}{\frac{1}{R_{conv2}} + \frac{1}{R_{rad}}}$$

R_{equiv} is a function of the unknown T_2

Numerically solving the equation for T_2

$$\frac{T_{\infty 2} - T_{\infty 1}}{R_{total}(T_2)} = \frac{T_{\infty 2} - T_2}{R_{equiv}(T_2)}$$

R_{total} is also a function of the unknown T_2

$$T_2 = 3.9272\ °C$$

Thus, (a) $Q = 8037.2\ W$

(b) Heat transferred to iced water in 24 hours is equal to the mass of ice melted times latent heat of fusion of water

$$M_{ice} h_{if} = Q \Delta t$$

$$M_{ice} = \frac{8037.2 \times (24 \times 3600)}{(333.7 \times 1000)} = 2081\ kg$$

$$kg = \frac{J/s \times s}{J/kg}$$