Data Structures and Algorithms

(ESO207)

Lecture 13:

- **Majority element** : an efficient and practical algorithm
- **word RAM** model of computation: further <u>refinements.</u>

Majority element

Definition: Given a **multiset** $\boldsymbol{S}$ of $\boldsymbol{n}$ elements,

$\boldsymbol{x} \in \boldsymbol{S}$ is said to be majority element **if** it appears more than $\boldsymbol{n/2}$ times in $\boldsymbol{S}$.

$$\boldsymbol{S} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}, \boldsymbol{z}, \boldsymbol{b}, \boldsymbol{w}, \boldsymbol{b}, \boldsymbol{b}, \boldsymbol{z}, \boldsymbol{b}, \boldsymbol{b}, \boldsymbol{b}, \boldsymbol{b}, \boldsymbol{c}\}$$

Majority element

Definition: Given a **multiset** S of n elements,

$x \in S$ is said to be majority element **if** it appears more than $n/2$ times in S.

8

$$S = \{a, b, d, z, b, w, b, b, z, b, b, b, b, c\}$$

Problem: Given a **multiset** S of n elements, find the majority element, if any, in S.

Majority element

Trivial algorithms:

Algorithm 1:

1. Count occurrence of each element
2. If there is any element with count > $\frac{n}{2}$, report it.

Running time: O(n^2) time

Majority element

Trivial algorithms:

Algorithm 2:

1. Sort the set S to find its median
2. Let x be the median
3. Count the occurrence of x, and
4. return x if its count is more than $\frac{n}{2}$

This assumption is **not justified** in many real life applications.

Running time: O(n **log** n) time

Critical assumption underlying Algorithm 2 :

elements of set S can be compared under some total order $(=,<,>)$

A real life application

Slots for inserting any two cards

Card-matching machine

Problem:
Given n credit cards, determine if th
using minimum no. of operations on ca

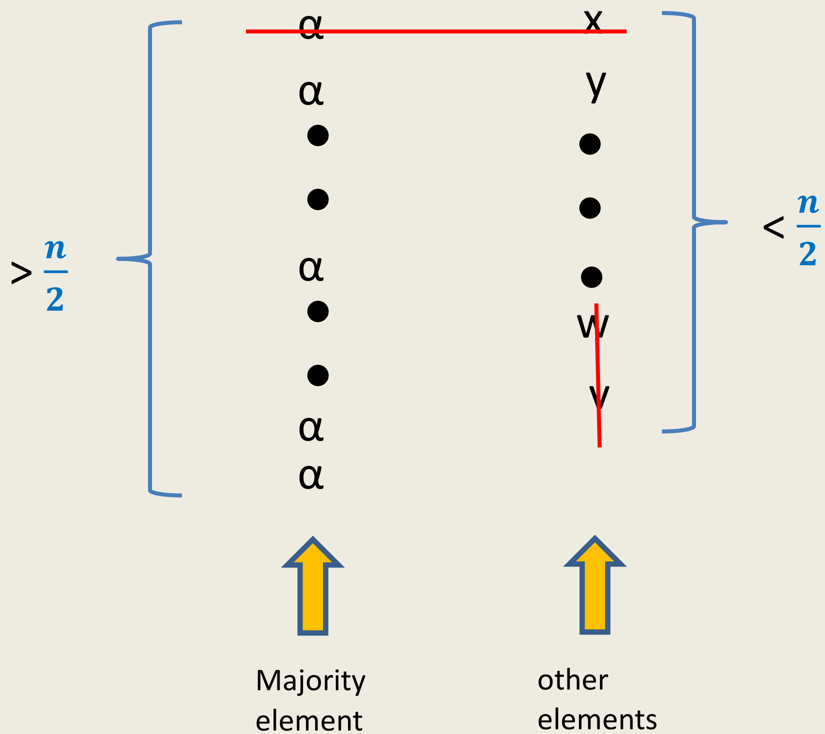
This machine takes two cards and determines whether they are identical or not.

Some observations

Problem: Given a **multiset** S of n elements,
where the only relation between any two elements is $\neq$ or $=$,
find the majority element, if any, in S.

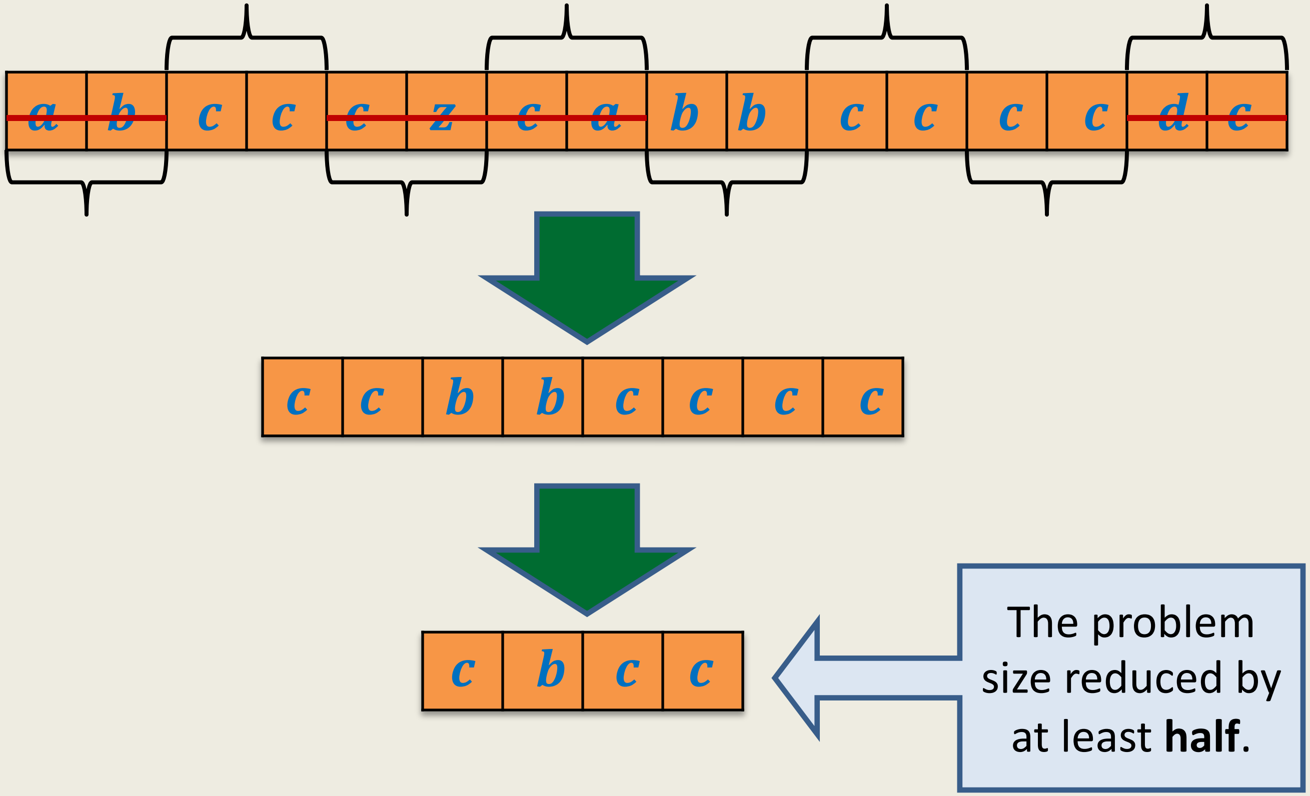
Question: How much time does it take to determine if an element $x \in S$ is majority ?
Answer: O(n) time
Observation 1: It is easy to verify whether an element is a majority

Some observations



Observation 2: whenever we cancel a pair of <u>distinct</u> elements from the array, the majority element of the array <u>remains preserved</u>.

Some observations



Observation 3: If there are m pairs of **identical elements**, then majority element is preserved even if we keep **one element** per pair.

Algorithm for 2-majority element

Repeat

1. Pair up the elements;
2. Eliminate all pairs of distinct elements;
3. Keep one element per pair of identical elements.

Take care if the no. of elements is odd

Until only one element is left.

Verify if the last element is a **majority** element.

Time complexity:

$$\mathbf{T}(n) = c\,n + c\,\frac{n}{2} + c\,\frac{n}{4} + \ldots$$

$O(n)$ time

Extra/working space requirement (assuming input is **"read only"**)

$O(n)$

Further restrictions on the problem

Restrictions:

- We are allowed to make single scan.
- We have very limited extra space.

Our current algorithm doesn't work for this real life example.

Real life example:

There are 10^{12} numbers stored on hard disk.

RAM can't provide **O**($\boldsymbol{n}$) extra (working) space in this case.

ALGORITHM FOR 2-MAJORITY ELEMENT

- **Single scan and**
- **O(1) extra space**

Designing algorithm for 2-majority element
single scan and using $O(1)$ extra space

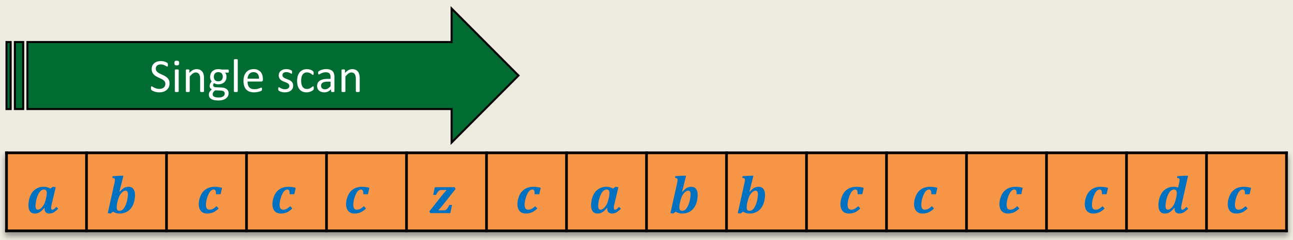
Question: Should we design algorithm from scratch to meet these constraints ?

Answer: No! We should try to adapt our current algorithm to meet these constraints.

Question: How crucial is pairing of elements in our current algorithm ?

Think over this question before going ahead.

Designing algorithm for 2-majority element
single scan and using O(1) extra space



Insightful questions:

- Do we really need to keep more than one element ?

  No. Just **cancel suitably** whenever encounter two *distinct* elements.

- Do we really need to keep multiple copies of an element **explicitly** ?

  No. Just keeping its **count** will suffice.

Ponder over these insights and make an attempt to design the algorithm before moving ahead ☺

Algorithm for 2-majority element

single scan and using O(1) extra space

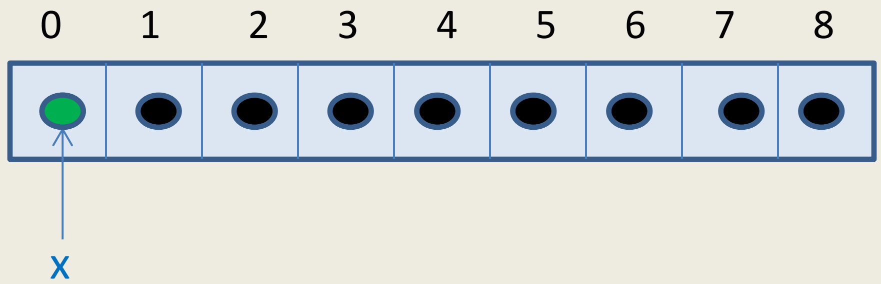
```
Algo-2-majority(A)
{   count← 0;
    for(i = 0 to n − 1)
    {     if (  count=0  ){  x← A[i];
                             count← 1;
                            }
          else if(x<> A[i])  count ← count - 1  ;
                 else        count← count + 1  ;
    }
    Count the occurrences of x in A, and if it is more than n/2, then
    print(x is 2-majority element) else print(there is no majority element in A)
}
```

Algorithm for 2-majority element
single scan and using $O(1)$ extra space

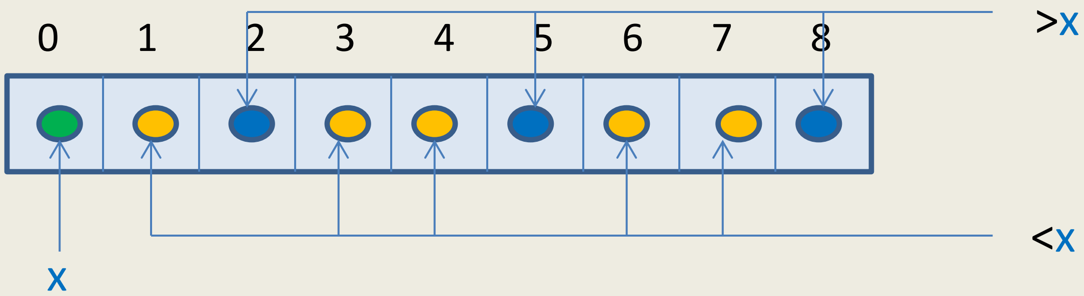
Theorem: There is an algorithm that makes just **a single scan** and uses **$O(1)$ extra space** to compute majority element for a given multi-set.

Homework: Algorithm for 3-majority element

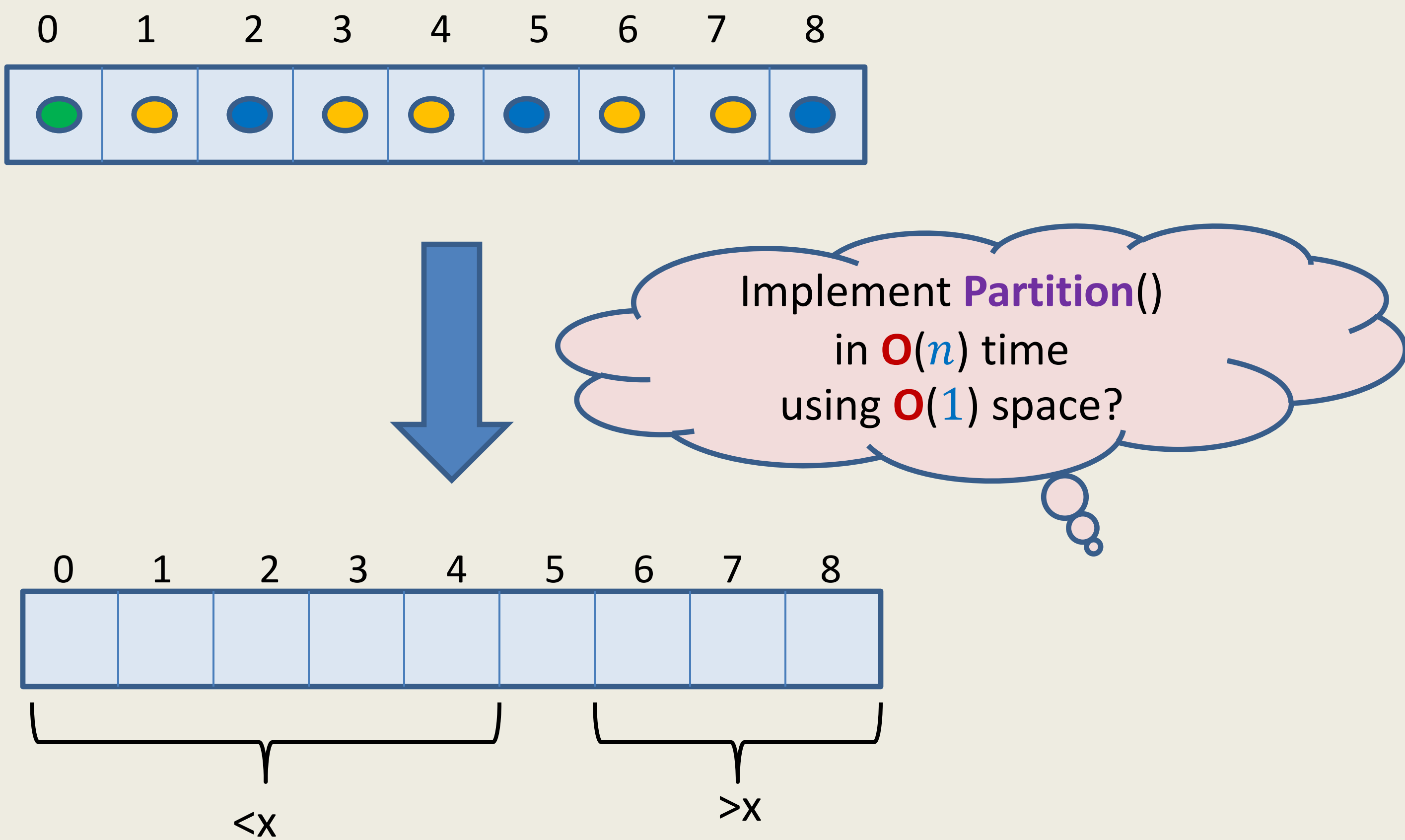
A nice programming exercise ?



A **nice** programming exercise ?



A nice programming exercise ?



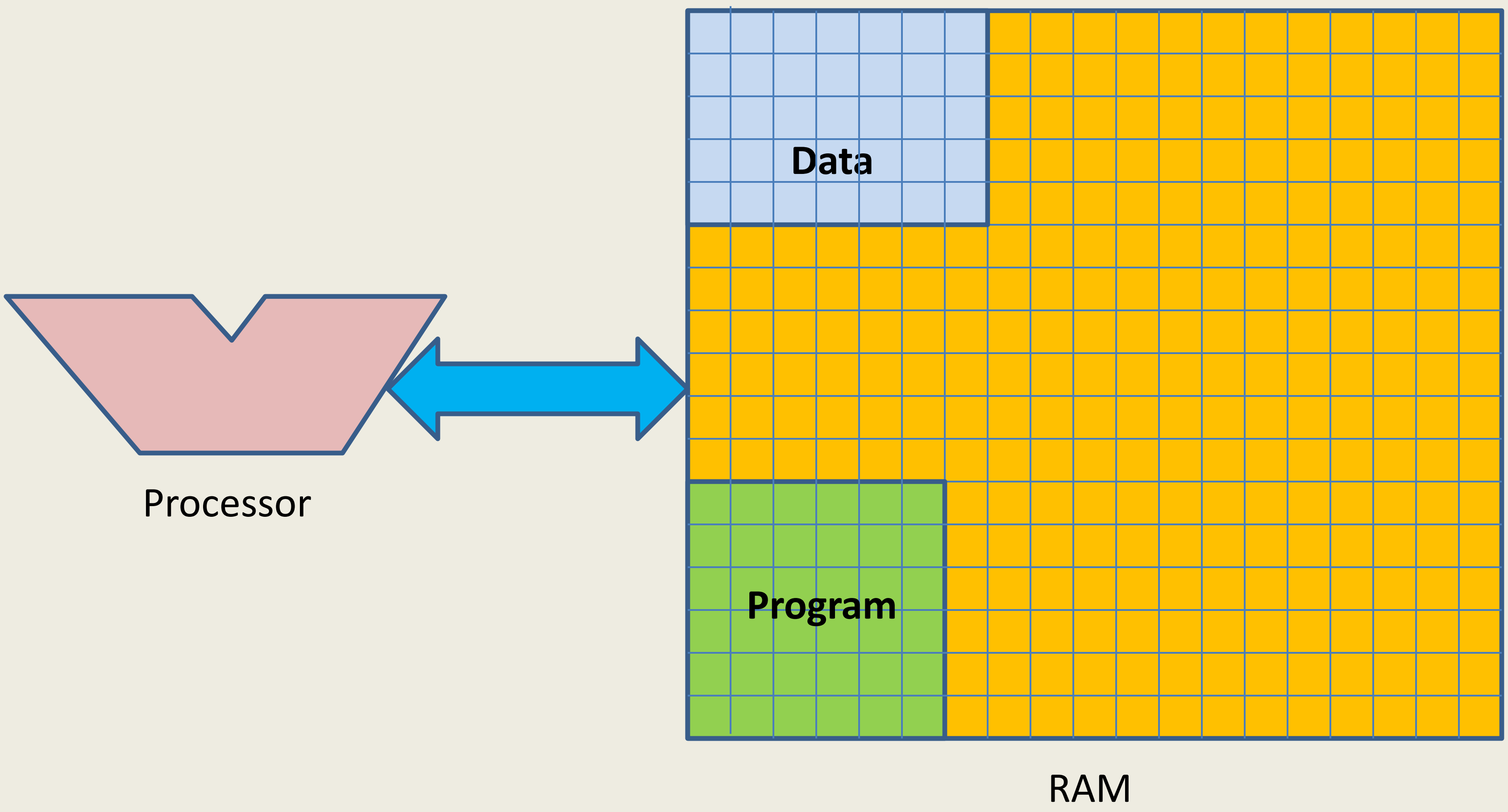
This procedure is called **Partition**.

It **rearranges** the elements so that all elements less than x appear to the left of x and all elements greater than x appear to the right of x.

Word RAM model of computation

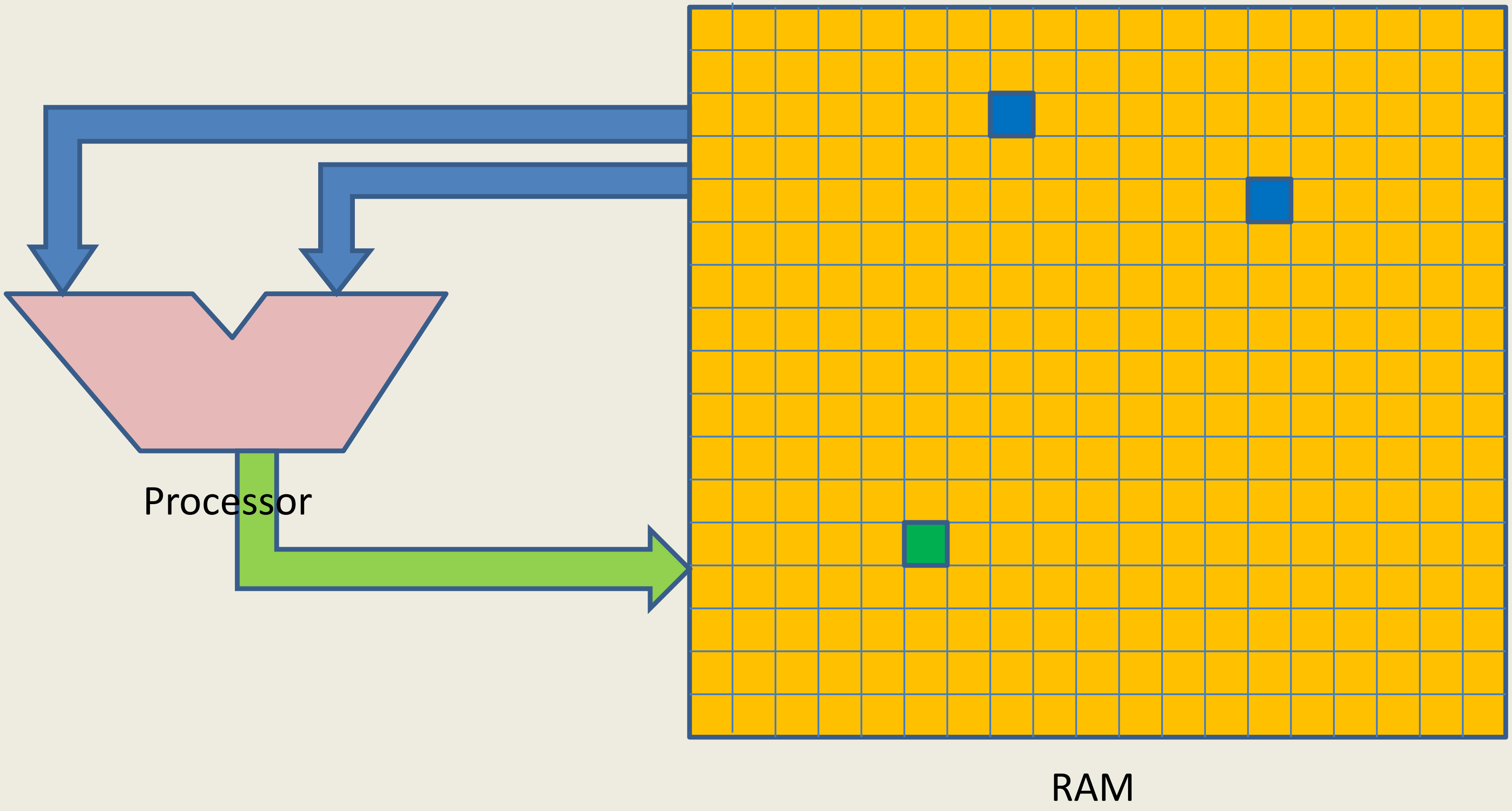
Further refinements

word RAM : a model of computation



Execution of a instruction

(fetching the operands, arithmetic/logical operation, storing the result back into RAM)



A more realistic RAM

n **: input size**

Input resides completely in **RAM**.

Question: How many bits are needed to access an input item from **RAM** ?

Answer: At least **log** n**.**

(k bits can be used to create at most 2^k different addresses)

Current-state-of-the-art computers:

- RAM of size **4GB**

  Hence 32 bits to address any item in RAM.

- Support for **64-bit arithmetic**

  Ability to perform arithmetic/logical operations on any two 64-bit numbers.

word RAM model of computation: Characteristics

- Word is the **basic storage** unit of RAM. Word is a collection of few bytes.
- Data as well as Program **reside fully** in RAM.
- Each input item (number, name) is stored in **binary format**.
- RAM can be viewed as a huge array of words. Any arbitrary location of RAM can be **accessed** in the same time **irrespective** of the location.
- Each arithmetic or logical operation (+,-,*,/,or, xor,…) involving **$O(\log n)$ bits** takes **a constant number of steps** by the CPU, where n is the number of bits of input instance.