FINANCIAL ENGINEERING
IME611A

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SESSION OBJECTIVES

- The Dividend Discount Model
- Measures of returns
- Short selling
- Portfolio return and variance

PORTFOLIO THEORY

Equity pricing and portfolio management

REQUIRED PRE-READING: DIVIDEND DISCOUNT MODEL

- A **Stock** which is expected **to pay dividends** $(D_1, D_2, \dots, D_n)$ at **different time-points** $(t = 1, 2, 3, \dots, n\ years)$ and can be sold at price P_n at the end of $n^{th}\ year$ can be priced as below, given a discount rate of r.

$$P_0 = \frac{D_1}{(1+r)^1} + \frac{D_2}{(1+r)^2} + ,,, + \frac{D_n}{(1+r)^n} + \frac{P_n}{(1+r)^n}$$

$$P_0 = \sum_{t=1}^{n} \frac{D_t}{(1+r)^t} + \frac{P_n}{(1+r)^n}$$

- Above formula is known as **Dividend Discount Model (DDM)**

UNCERTAIN CASHFLOW

- Investments where **initial cash outlay is known**, but amount to be returned is uncertain
- Uncertainty is handled using
  - Mean –variance analysis
  - Utility function analysis
  - Arbitrage (or comparison analysis)

ASSET RETURN

- Consider buying an asset at time zero (t_0), and selling the same 1 year later (t_1)

$$Total\ return\ (R) = \frac{amount\ received}{amount\ invested} = \frac{X_1}{X_0}$$

$$Rate\ of\ return\ (r) = \frac{amount\ received\ -\ amount\ invested}{amount\ invested} = \frac{(X_1\ - X_0)}{X_0}$$

$$R = 1 + r$$

$$X_1 = \ (1 + r)X_0$$

SHORT SELLING OF AN ASSET

- **Short selling (or shorting):** To sell an asset that you do not own, by borrowing from a broker
- Borrow the stock from a broker
- Sell it at X_0
- Repay the loan later by purchasing the stock at X_1
- Your net payoff = $+X_0 - X_1$
- You earn a **profit if the stock price declines**.
- Short selling is risky, potentially the loss could be unlimited.

RETURN IN SHORT SELLING

- **Short selling results** in receiving a cash inflow of X_0 today at t_0, and experiencing a cash outflow of X_1 at t_1.

$$R = \frac{-X_0}{-X_1} = \frac{X_0}{X_1}$$

$$-X_1 = -X_0 R = -X_0(1 + r)$$

Practice Example 6.1, and Exercise 1

PORTFOLIO RETURN (1/2)

- Portfolio is a combination of multiple assets.
- Suppose, there are $\boldsymbol{n}$ **assets**, we can form a **master asset or portfolio**
- $\boldsymbol{X_0}$ amount is invested across $\boldsymbol{n}$ assets.
- $\boldsymbol{X_{0i}}$ is the amount invested in $\boldsymbol{i^{th}}$ asset, where $i = 1, 2, \ldots n$.
- We have $\sum_{i=1}^{n} X_{0i} = X_0$

- Alternatively, if we consider the **weight** or **fraction** of asset i in the portfolio as w_i then
  - $X_{0i} = w_i X_0$
  - $\sum_{i=1}^{n} w_i = 1$

PORTFOLIO RETURN (2/2)

- Let R_i denote the total return of asset i.
- Amount of money generated at the end of period: $R_i X_{0i} = R_i w_i X_0$
- Total amount received at the end of the period $\sum_{i=1}^{n} R_i w_i X_0$
- So, overall **total return on portfolio**

$$R = \frac{\sum_{i=1}^{n} R_i w_i X_0}{X_0} = \sum_{i=1}^{n} w_i R_i$$

- Equivalently,

$$r = \sum_{i=1}^{n} w_i r_i$$

IMPORTANT RESULT

- **Portfolio Return:** Both the total return and rate of return of a portfolio of assets are **equal to the weighted sum of the corresponding individual asset returns**, with the weight of an asset being its relative weight (in purchase cost) in the portfolio, that is,

$$R = \sum_{i=1}^{n} w_i R_i$$

$$r = \sum_{i=1}^{n} w_i r_i$$

An example illustration

SOME PRELIMINARY FROM PROBABILITY AND STATISTICS

- **Random Variable**
- **Expected Value**
  - Properties of expected value
    1. Certainty value
    2. Linearity
    3. Nonnegativity
- **Variance**
  - Several random variables
  - Covariance and correlation
  - Covariance bound, uncorrelated, positively correlated, negatively correlated random variables
  - Properties of variance
    1. Variance of sum of two random variables

RANDOM RETURNS

- An asset, when acquired, typically has an uncertain rate of return
- To summarize the uncertainty
- Expected value: $E(r) \equiv \bar{r}$
- Variance: $E[r - \bar{r}]^2 \equiv \sigma^2$
- Covariance: $E[(r_i - \bar{r})][(r_j - \bar{r})] \equiv Cov(r_i, r_j)$

MEAN RETURN OF A PORTFOLIO

- Suppose, there are n assets with (random) rates of return $r_1, r_2, r_3, \dots, r_n$ having expected values as $E(r_1) = \overline{r_1}, E(r_2) = \overline{r_2}, \dots, E(r_n) = \overline{r_n}$.

- We form a portfolio of these n assets using the weights $w_i, i = 1, 2, \dots, n$.

- The return on portfolio is given by

$$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n$$

- Taking expectation and using linearity,

$$E(r) = w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n)$$

VARIANCE OF PORTFOLIO RETURN (1/2)

- Suppose,
- σ^2 denote the portfolio variance,
- σ_i^2 denote the variance of i^{th} stock, and
- σ_{ij} denote the covariance of return on asset i and asset j.

VARIANCE OF PORTFOLIO RETURN (2/2)

$$\sigma^2 = E[(r - \bar{r})^2]$$

$$\sigma^2 = E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)^2\right]$$

$$\sigma^2 = E\left[\left(\sum_{i=1}^{n} w_i (r_i - \bar{r}_i)\right)\left(\sum_{j=1}^{n} w_j (r_j - \bar{r}_j)\right)\right]$$

$$\sigma^2 = E\left[\left(\sum_{i,j=1}^{n} w_i w_j (r_i - \bar{r}_i)(r_j - \bar{r}_j)\right)\right]$$

$$\sigma^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

Practice Example 6.8

DISCLAIMER

- The information in this presentation has been compiled from the following textbook which has been mentioned as a reference text for this course on **Financial Engineering.**

- Reference Text:
  - **Investment Science**, 2nd Edition, Oxford University Press, David G. Luenberger