Data Structures and Algorithms

(ESO207)

Lecture 27

- Quick revision of Depth First Search (DFS) Traversal
- An O($m + n$):algorithm for biconnected components of a graph

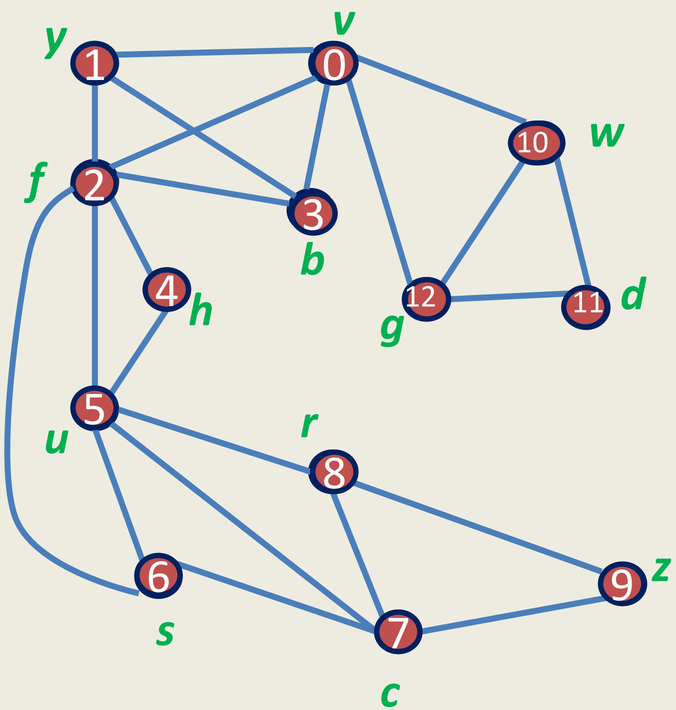
Quick revision of
Depth First Search (DFS) Traversal

DFS traversal of *G*

```
DFS(v)
{  Visited(v) ← true;  DFN[v] ← dfn ++;
   For each neighbor w of v
   {        if (Visited(w)  = false)
            {   DFS(w) ;
                 ........;
            }
            ........;
   }
}
```

```
DFS-traversal(G)
{  dfn ← 0;
   For each vertex vϵ V  {      Visited(v)← false                }
   For each vertex v ϵ V {      If (Visited(v ) = false)   DFS(v)   }
}
```

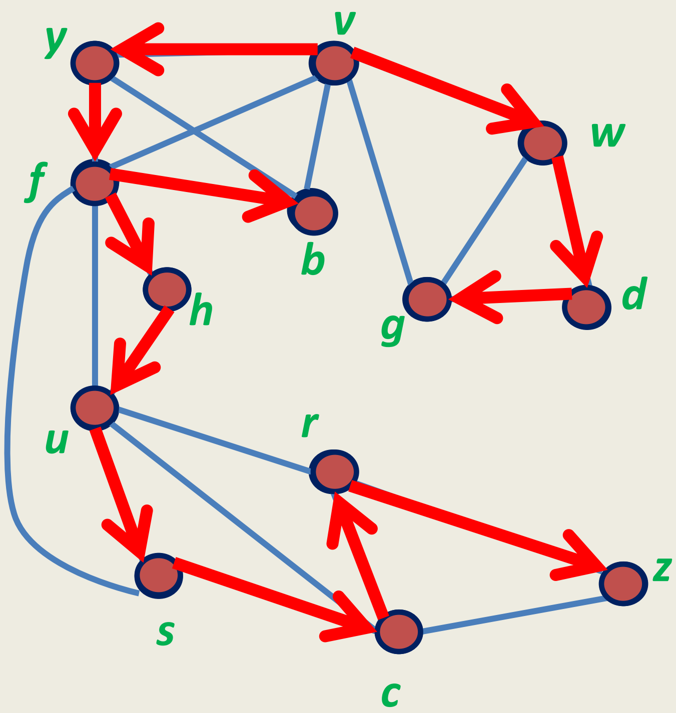
DFN number



DFN[***x***] :

The number at which ***x*** gets visited during DFS traversal.

DFS(v) computes a tree rooted at v



A **DFS** tree rooted at ***v***

If **x** is ancestor of **y** then

DFN[*x*] $\boxed{<}$ DFN[*y*]

Question: Is a **DFS** tree unique ?

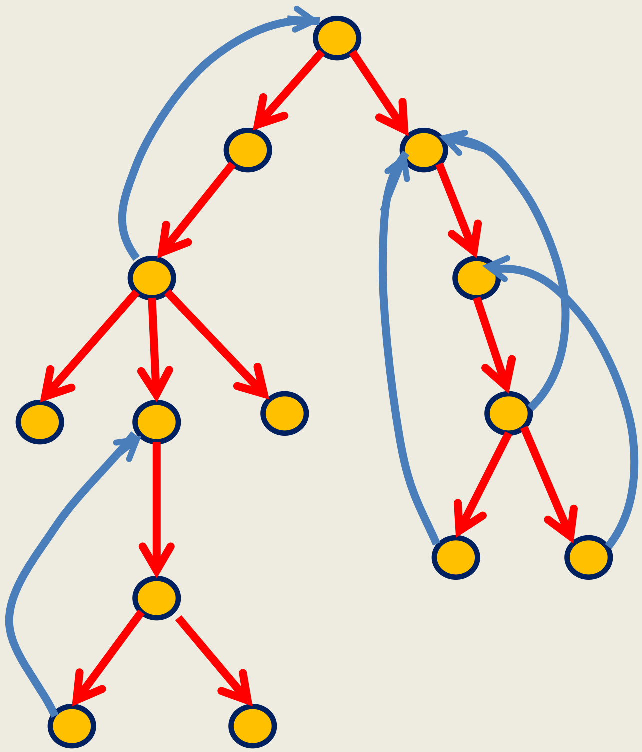
Answer: No.

Question:

Can any rooted tree be obtained through DFS ?

Answer: No.

Always remember this picture



non-tree edge ➔ **back** edge

A **DFS** representation of the graph

Verifying bi-connectivity of a graph

An **O**$(\boldsymbol{m}+\boldsymbol{n})$ time algorithm

A single **DFS** traversal

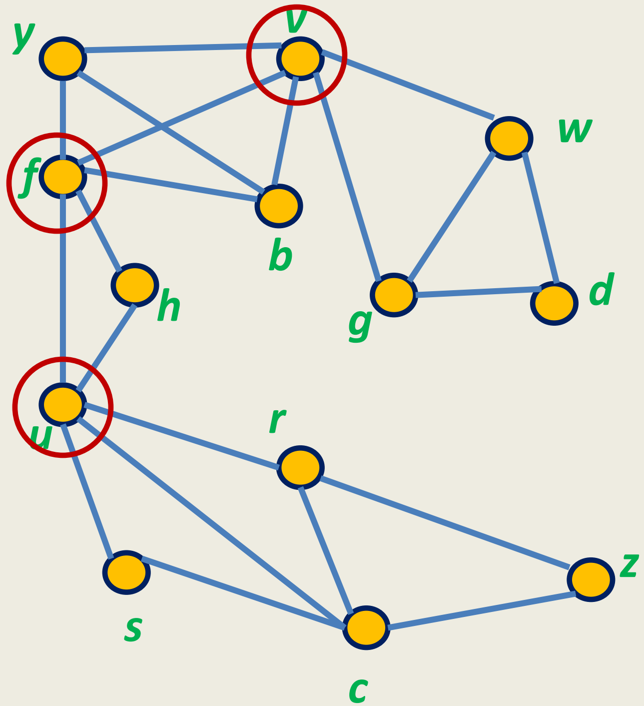
An $\mathbf{O}(m + n)$ time algorithm

- A formal **characterization** of the problem.

  (**articulation points**)

- Exploring **<u>relationship</u>** between articulation point & DFS tree.

- Using the relation **cleverly** to design an efficient algorithm.



The removal of any of {***v***,***f***,***u***} can destroy connectivity.

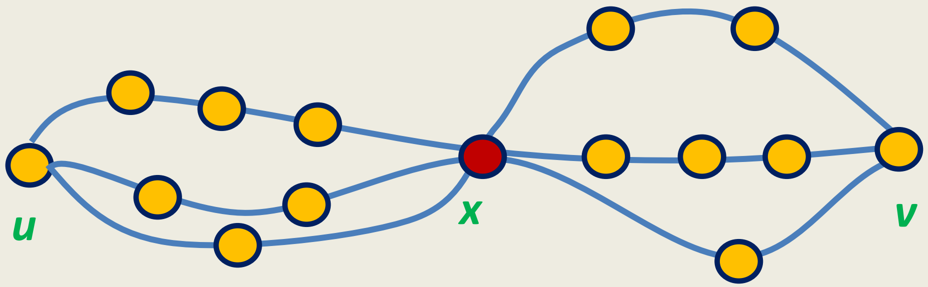
v,***f***,***u*** are called the **articulation points** of ***G***.

A formal definition of articulaton point

Definition: A vertex ***x*** is said to be **articulation point** if

∃ ***u***,***v*** different from ***x***

such that every path between ***u*** and **v** passes through ***x***.

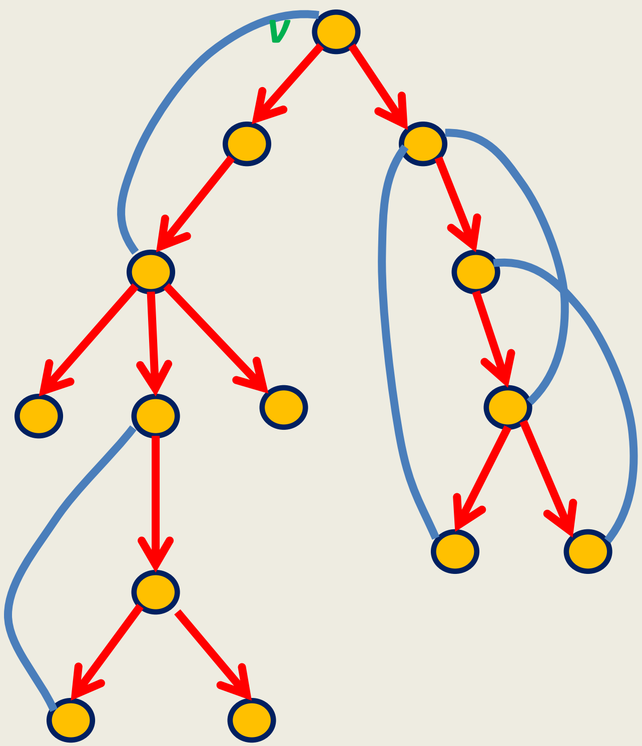


Observation: A graph is biconnected if none of its vertices is an articulation point.

AIM:

Design an **algorithm** to compute all **articulation points** in a given graph.

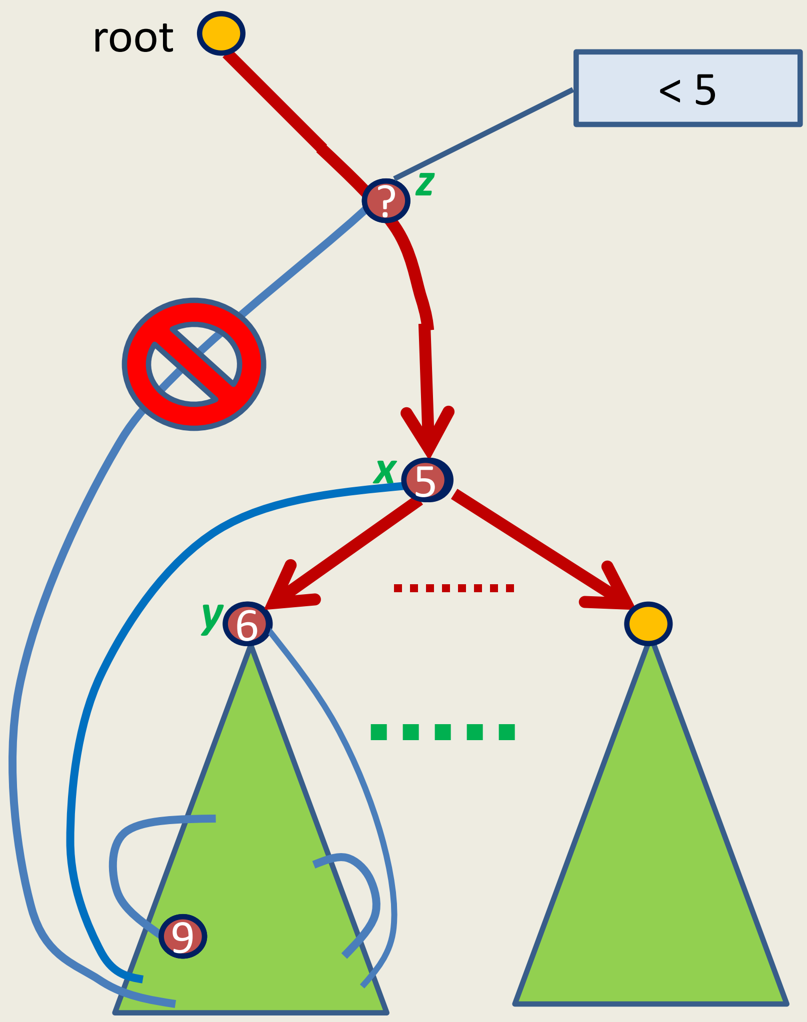
Some observations



- **A leaf node** can never be an **a.p.** ?
- **Root** is an **a.p.** iff it has two or more children.

What about an internal node ?

Necessary and Sufficient condition for *x* to be articulation point



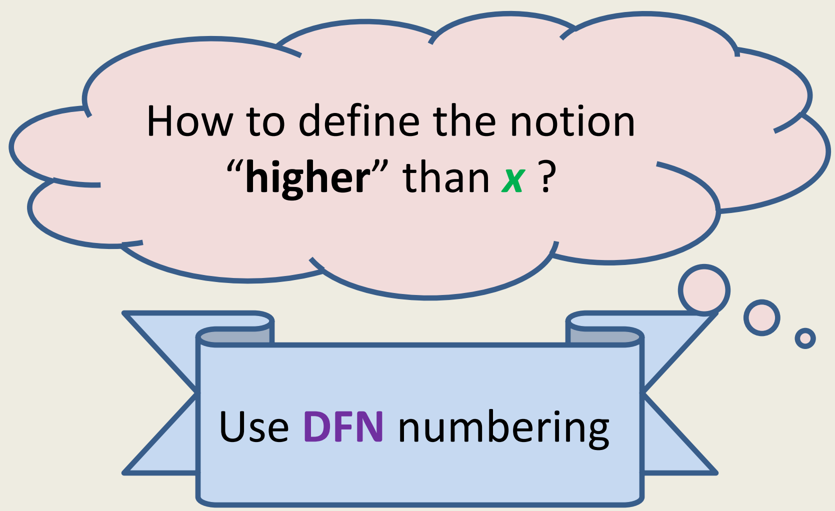
Theorem1:

An internal node ***x*** is **articulation point** iff

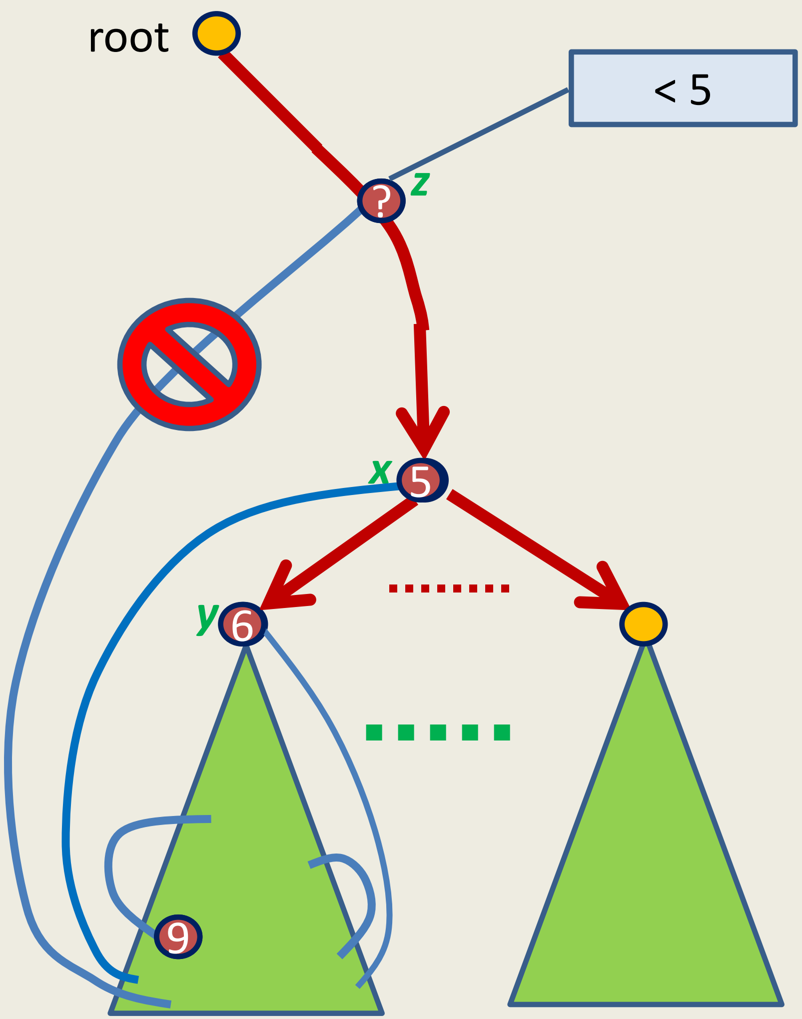
x has **at least** one child ***y*** s.t.

no back edge from **subtree**(**y**) to **ancestor** of ***x***.

➔ No back edge from **subtree**(**y**) going to a vertex "**higher**" than ***x***.



Necessary and Sufficient condition for *x* to be articulation point



Theorem1:

An internal node ***x*** is **articulation point** iff

x has **<u>at least</u>** one child ***y*** s.t.

no back edge from **subtree**(**y**) to **ancestor** of ***x***.

Invent a new function

High_pt(***v***):

DFN of the <u>*highest ancestor*</u> of ***v***

to which there is a back edge from **subtree**(***v***).

Theorem2:

An internal node ***x*** is **articulation point iff**

it has a child, say **y**, in **DFS** tree such that

High_pt(**y**) $\geq$ **DFN**(***x***).

Theorem2:

An internal node ***x*** is **articulation point** **iff**

it has a child, say **y**, in **DFS** tree such that

High_pt(**y**) ≥ **DFN**(***x***).

Good ☺!
But how to transform this Theorem
into an efficient algorithm for
articulation points ?

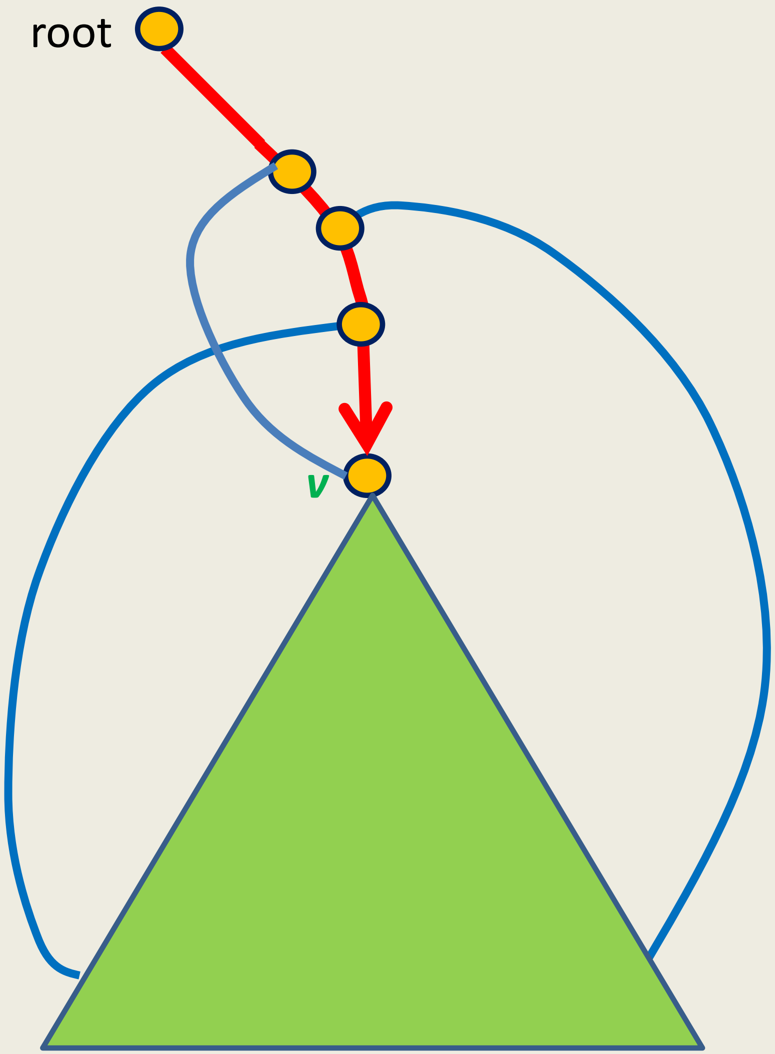
In order to compute **High_pt**(**v**) of a vertex **v**,

we have to traverse the adjacency lists of all vertices of subtree ***T***(***v***).

→ **O**(m) time in the worst case to compute **High_pt**(**v**) of a vertex **v**.

→ **O**(mn) time algorithm ☹

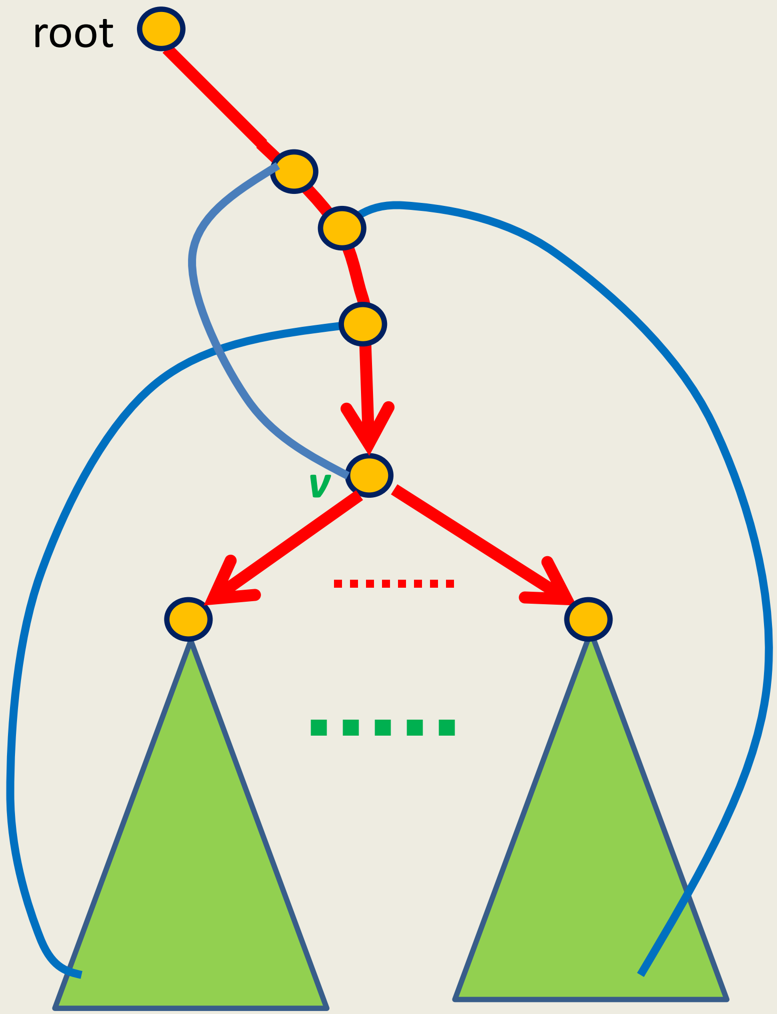
How to compute **High_pt(v)** efficiently ?



Question: Can we express **High_pt(v)** in terms of its **children** and **proper ancestors**?

Exploit **recursive structure** of DFS tree.

How to compute High_pt(v) efficiently ?



Question: Can we express **High_pt**(**v**) in terms of its **children** and **proper ancestors**?

High_pt(*v*) =

$$\min_{(v,w)\,\in\, E} \begin{cases} \text{High\_pt}(w) & \text{If } w = \text{child}(v) \\ \text{DFN}(w) & \text{If } w = \text{proper ancestor of } v \end{cases}$$

?

The novel algorithm

Output : an array **AP[]** s.t.

AP[***v***]= **true** if and only if ***v*** is an articulation point.

Algorithm for articulation points in a graph *G*

```
DFS(v)
{  Visited(v) ← true; DFN[v] ← dfn ++;  High_pt[v]←∞ ;
   For each neighbor w of v
   {        if (Visited(w)  = false)
            {   DFS(w) ;  Parent(w) ← v;

                  ……..;

                  ……..;
            }

            ……..;
   }
}
```

```
DFS-traversal(G)
{   dfn ← 0;
   For each vertex vϵ V  {      Visited(v)← false;   AP[v]← false   }
   For each vertex v ϵ V {      If (Visited(v ) = false)    DFS(v)        }
}
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Algorithm for articulation points in a graph *G*

```
DFS(v)
{  Visited(v) ← true; DFN[v] ← dfn ++;  High_pt[v]←∞ ;
   For each neighbor w of v
   {        if (Visited(w)  = false)
            {    Parent(w)← v;   DFS(w);
                 High_pt(v) ← min(  High_pt(v)    ,  High_pt(w)  );
                 If  High_pt(w) ≥ DFN[v]        AP[v] ← true
            }
         Else if ( Parent(v) ≠  w  )
                   High_pt(v) ← min(   DFN(w)   ,   High_pt(v)    )
   }
}
```

```
DFS-traversal(G)
{  dfn ← 0;
   For each vertex vϵ V  {      Visited(v)←  false;   AP[v]← false    }
   For each vertex v ϵ V {      If (Visited(v ) = false)    DFS(v)        }
}
```

Conclusion

Theorem2 : For a given graph G=(V,E), all **articulation points** can be computed in **O**($m + n$) time.

Data Structures

Range of efficient functions

Lists: (arrays, linked lists)

Binary Heap

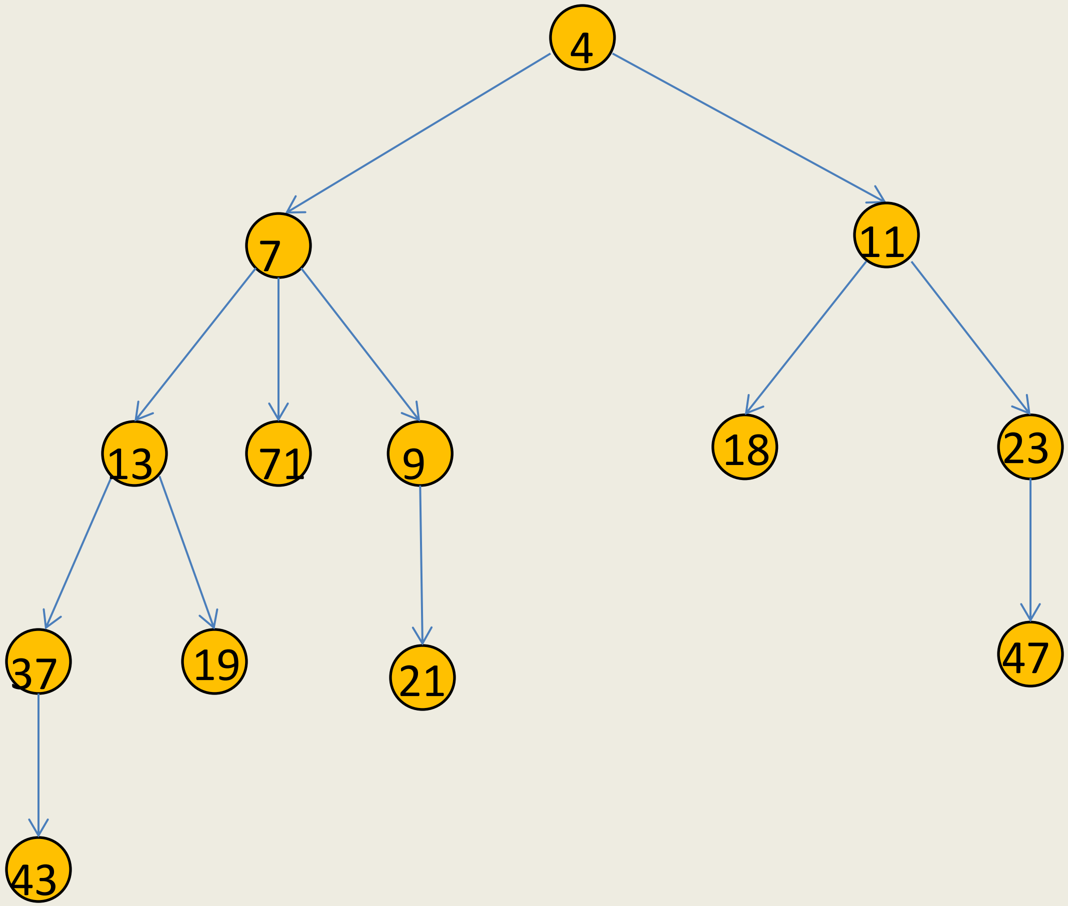
Binary Search Trees

Simplicity

Heap

Definition: a tree data structure where :

value stored in a node $<$ value stored in each of its children.



Operations on a heap

Query Operations

- **Find-min**: report the smallest key stored in the heap.

Update Operations

- **CreateHeap**(**H**) : Create an empty heap **H**.
- **Insert**(**x**,**H**) : Insert a new key with value **x** into the heap **H**.
- **Extract-min**(**H**) : delete the smallest key from **H**.
- **Decrease-key**(***p***, Δ, **H**) : decrease the value of the key ***p*** by amount Δ.
- **Merge**(**H_1,H_2**) : Merge two heaps **H_1 and H_2**.

Why heaps when we can use a binary search tree ?

Compared to binary search trees, a heap is usually

-- much **simpler** and

-- more **efficient**

Existing heap data structures

- **Binary heap**
- **Binomial heap**
- **Fibonacci heap**
- **Soft heap**

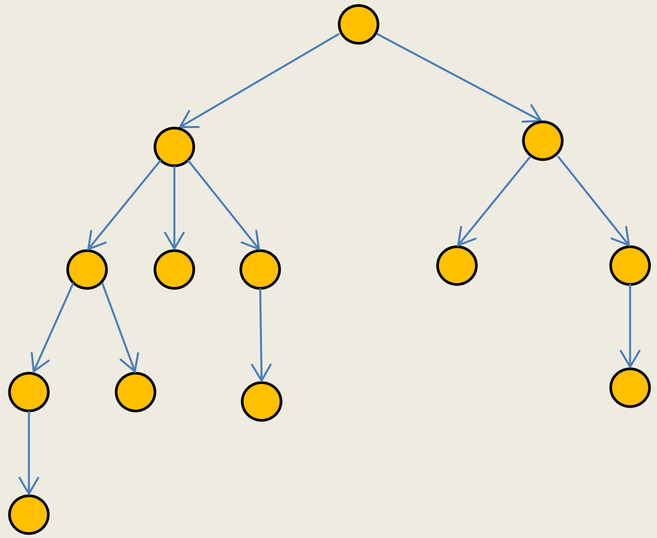
Can we implement a binary tree using an array ?

Yes.

In some special cases

fundamental question

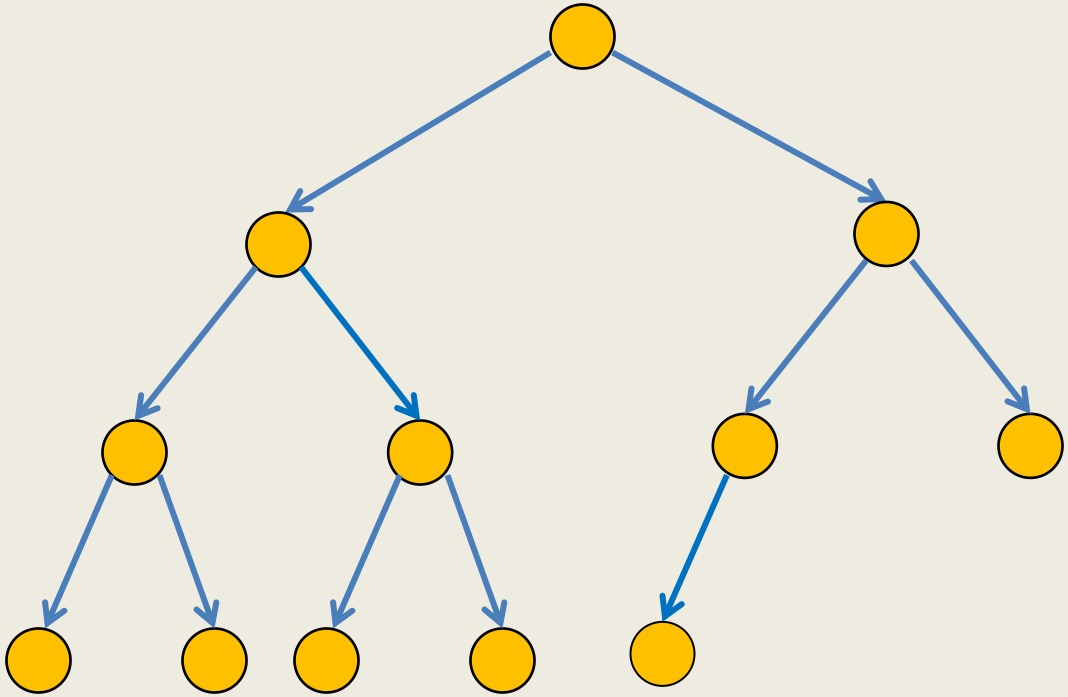
Question: What does the implementation of a tree data structure require ?



Answer: a mechanism to

- access **parent** of a node
- access **children** of a node.

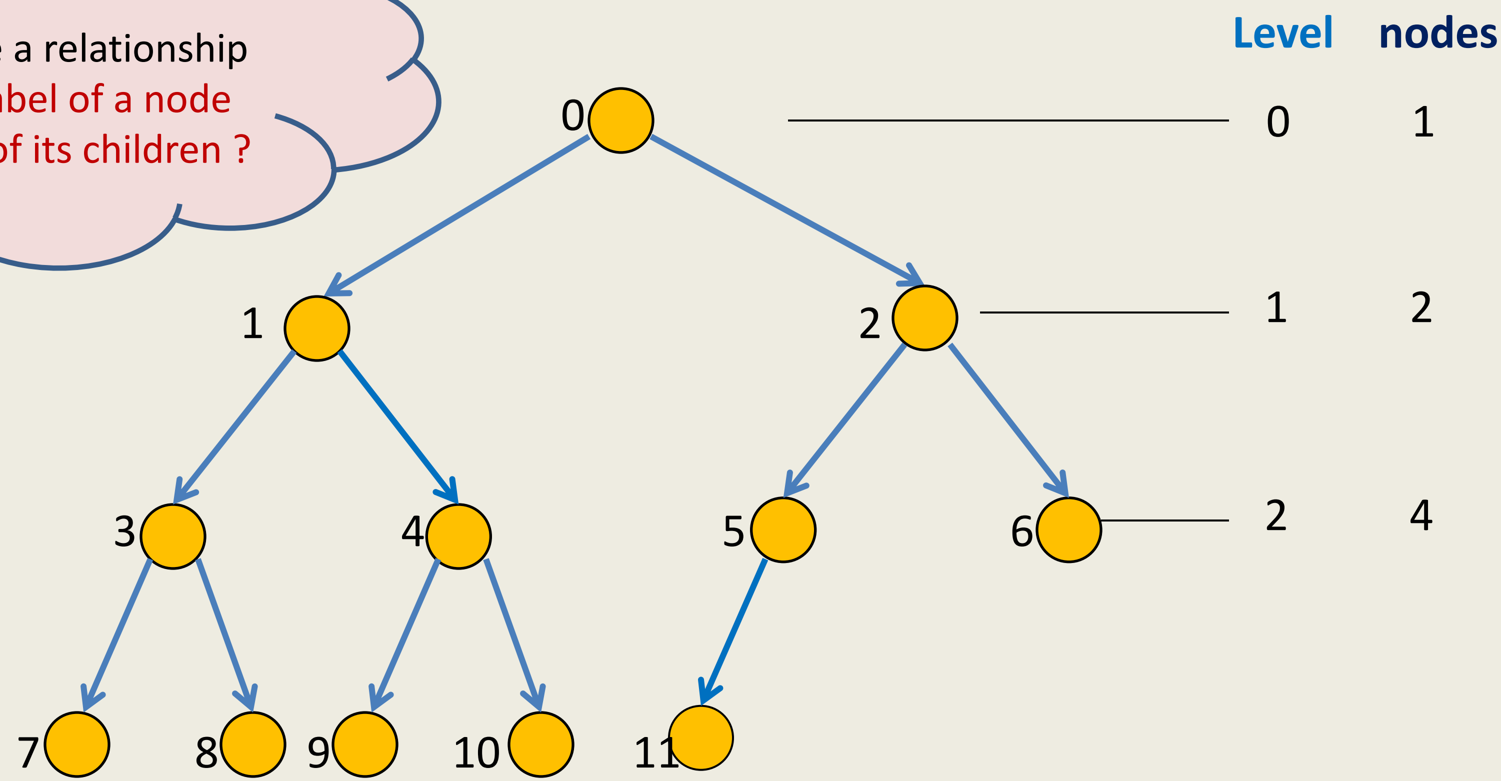
A complete binary tree



A complete binary of 12 nodes.

A complete binary tree

Can you see a relationship
between label of a node
and labels of its children ?



Think over it before next lecture.