Data Structures and Algorithms

(ESO207)

Lecture 11:

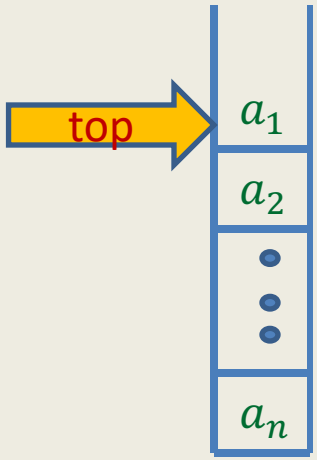
- Arithmetic expression evaluation: Complete algorithm using stack
- Two interesting problems

Quick Recap of last lecture

Stack: a new data structure

A special kind of list

where all operations (insertion, deletion, query) take place at one end only, called the **top**.



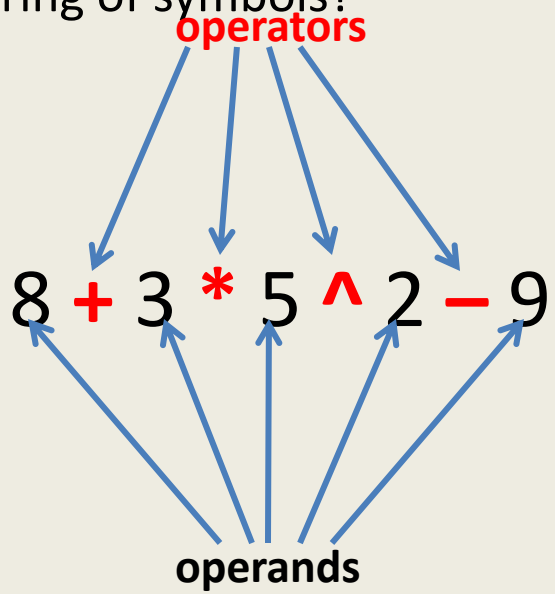
Evaluation of an arithmetic expression

Question: How does a computer/calculator evaluate an arithmetic expression given in the form of a string of symbols ?

8 + 3 * 5 ^ 2 – 9

Evaluation of an arithmetic expression

Question: How does a computer/calculator evaluate an arithmetic expression given in the form of a string of symbols?



- What about expressions involving parentheses: 3+4*(5-6/(8+9^2)+33) ?
- What about associativity of the operators ?

Overview of our solution

1. **Focusing on a simpler version of the problem:**
    1. Expressions without parentheses
    2. Every operator is left associative

2. **Solving the simpler version**

3. **Transforming the solution of simpler version to generic**

Incorporating precedence of operators through priority number

Operator	Priority
+ , -	1
* , /	2
^	3

Insight into the problem

Let o_i : the operator at position *i* in the expression.

Aim: To determine an order in which to execute the operators.

8 + 3 * 5 ^ 2 – 9 * 67

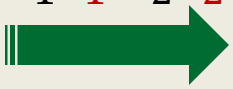
Position of an operator **does** matter

Question: Under what conditions can we execute operator o_i immediately?

Answer: if

- priority(o_i) > priority(o_{i-1})
- priority(o_i) ≥ priority(o_{i+1})

Expression: $n_1 o_1\ n_2 o_2\ n_3\ o_3\ n_4\ o_4\ n_5\ o_5\ n_6\ o_6\ \ldots$

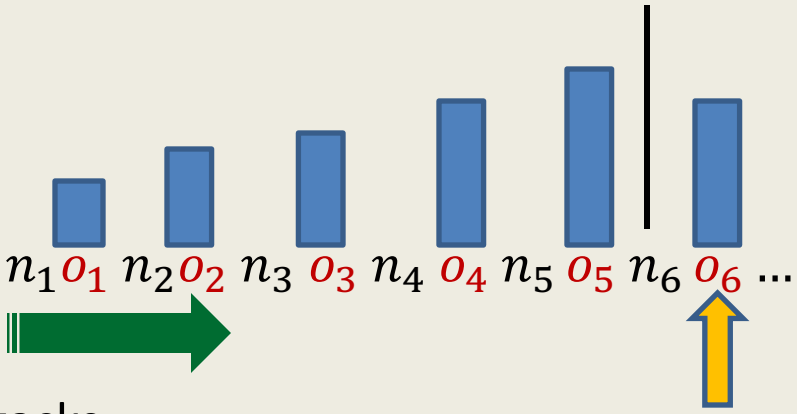


We keep two stacks:

N-stack
for operands

O-stack
for operators

Expression: $n_1 o_1\ n_2 o_2\ n_3\ o_3\ n_4\ o_4\ n_5\ o_5\ n_6\ o_6\ \ldots$



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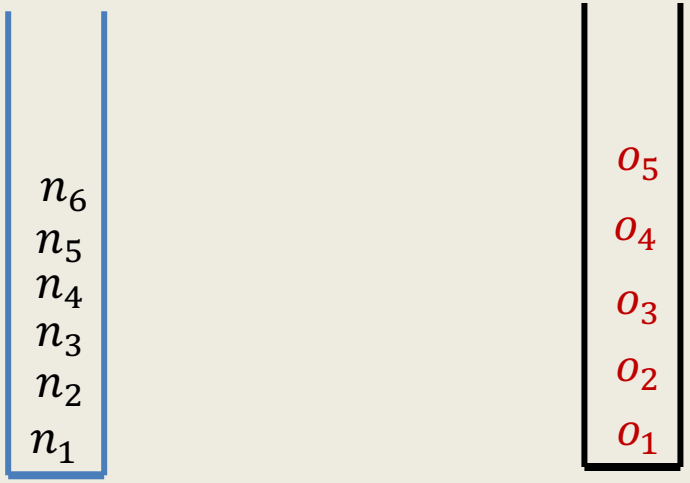
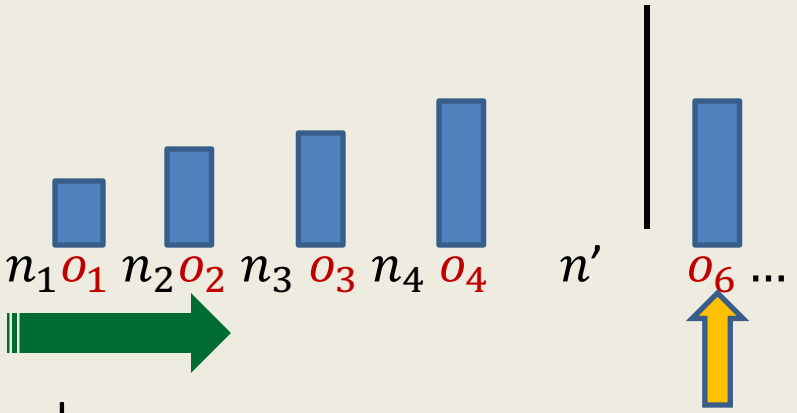


N-stack
for operands

O-stack
for operators

Expression: $n_1 o_1\ n_2 o_2\ n_3\ o_3\ n_4\ o_4\quad n'\quad o_6\ \ldots$



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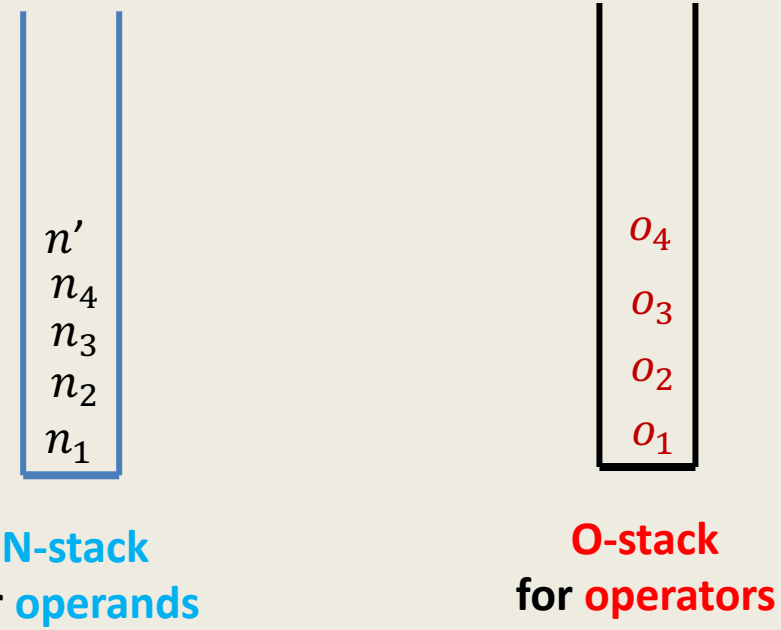
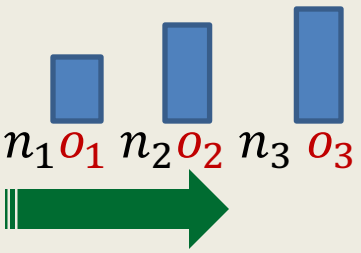
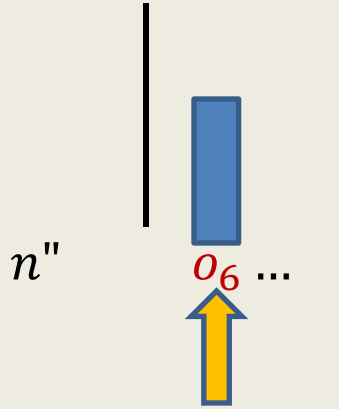


N-stack
for operands

O-stack
for operators

Expression: $n_1 o_1\ n_2 o_2\ n_3\ o_3$ n'' o_6 ...





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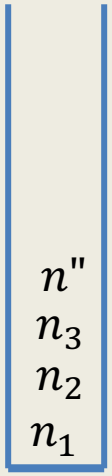
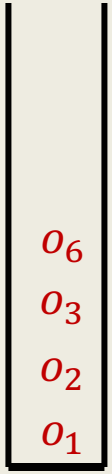




N-stack
for operands

O-stack
for operators

A simple algorithm

Priority of $: **Least**

```
push($,O-stack);
While ( ? ) do
{   x ← next_token();
    Two cases:
        x is number   :  push(x,N-stack);
        x is operator :
                         while( PRIORITY(TOP(O-stack) >= PRIORITY(x) )
                         { o ← POP(O-stack);
                           Execute(o);
                         }
                          push(x,O-stack);
}
```

- **POP** two numbers from N-stack
- apply operator **o** on them
- place the result back into N-stack

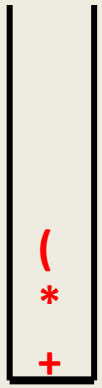
Next step

Transforming the solution to Solve the most general case

How to handle parentheses ?

3+4*(5 - 6/2)

What should we do whenever
we encounter (in the expression ?



O-stack

Answer:

Evaluate the expression enclosed by this parenthesis
before any other operator currently present in the O-stack.

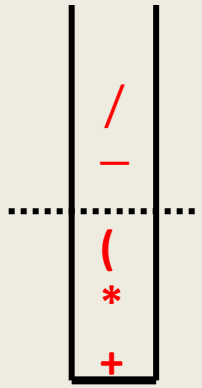
➔ So we must push (into the **O-stack**.

Observation 1: While (is the **current operator** encountered in the expression,
it must have **higher priority** than every other operator in the stack.

How to handle parentheses ?

3+4*(5 - 6/2)

What needs to be done when
(is at the top of the **O-stack** ?



O-stack

Answer:

The (should act as an *artificial bottom* of the O-stack .

➔ every other operator that follows (
should be allowed to sit on the top of (in the stack .

Observation 2 : while (**is inside the stack**,
it must have **less priority** than every other operator that follows.

A CONTRADICTION !!

Observation 1: While (is the **current operator** encountered in the expression,
it must have **higher priority** than every other operator in the stack

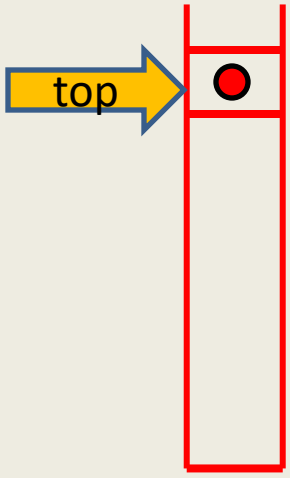
Take a pause for a few minutes to realize surprisingly that the contradicting requirements for the priority of (in fact hints at a suitable solution for handling (.

How to handle parentheses ?

Using two **types** of priorities of each operator ●.

InsideStack priority

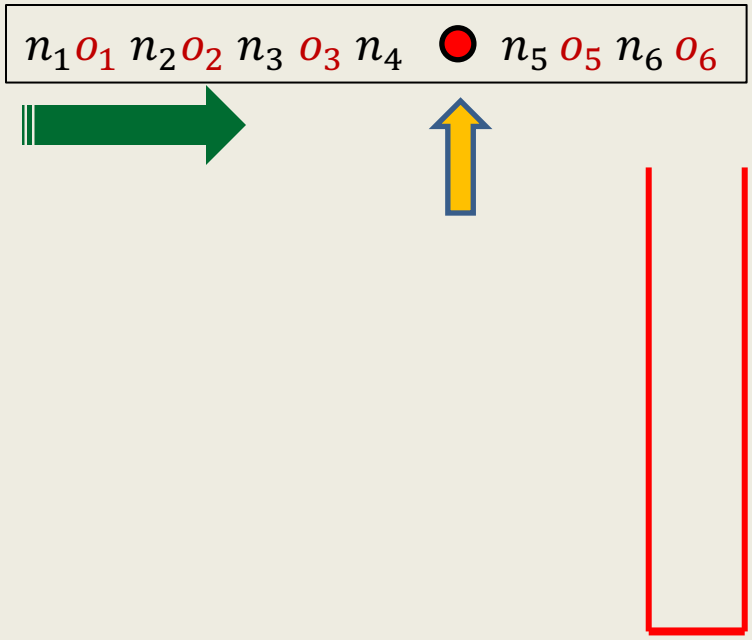
The priority of an operator ●
when it is **inside** the stack.



O-stack

OutsideStack priority

The priority of an operator ●
when it is **encountered** in the expression.



O-stack

How to handle parentheses ?

Using two **types** of priorities of each operator ●.

Operator	InsideStackPriority	OutsideStackPriority
+ , -	1	1
* , /	2	2
^	3	3
(	0	4

Does it take care of nested parentheses ? Check it yourself.

How to handle parentheses ?

3+4*(5 - 6/2)

Question: What needs to be done whenever we encounter) in the expression ?

Answer: Keep **popping** O-stack and evaluating the operators until we get its matching (.

The algorithm generalized to handle **parentheses**

```
push($,O-stack);
While (    ?    ) do
  x ← next_token();
  Cases:
    x is number :    push(x,N-stack);
    x is )          :    while(    TOP(O-stack) <> (    )
                         {    o ← Pop(O-stack);
                              Execute(o);
                         }
                         Pop(O-stack);          //popping the matching (
    otherwise   :        while( InsideStackPriority(TOP(O-stack)) >=  OutsideStackPriority(x)  )
                         {    o ← Pop(O-stack);
                              Execute(o);
                         }
                         Push(x,O-stack);
```

Practice exercise

Execute the algorithm on 3+4*((5+6*(3+4)))^2 and convince yourself through proper reasoning that the algorithm handles parentheses suitably.

How to handle associativity of operators ?

Associativity of arithmetic operators

Left associative operators : +, - , * , /

- a+b+c = (a+b)+c
- a-b-c = (a-b)-c
- a*b*c = (a*b)*c
- a/b/c = (a/b)/c

We have already handled left associativity in our algorithm.

Right associative operators: ^

- 2^3^2 = 2^(3^2) = 512.

How to handle right associativity ?

What we need is the following:

If ^ is **current operator** of the expression, and ^ is on **top of stack**,

then ^ should be evaluated before ^.

How to incorporate it ? Play with the **priorities** ☺

How to handle associativity of operators ?

Using two **types** of priorities of each **right associative** operator.

Operator	InsideStackPriority	Outside-stack priority
+ , -	1	1
* , /	2	2
^	3	4
(	0	5

The general Algorithm

It is the same as the algorithm to handle parentheses :-)

```
While ( ? ) do
  x ← next_token();
  Cases:
    x is number :   push(x,N-stack);
    x is )      :   while(  TOP(O-stack) <> (  )
                    {   o ← Pop(O-stack);
                        Execute(o);
                    }
                    Pop(O-stack);     //popping the matching (
    otherwise   :   while(InsideStackPriority(TOP(O-stack)) >= OutsideStackPriority(x))
                    {   o ← Pop(O-stack);
                        Execute(o);
                    }
                    Push(x,O-stack);
```

Homeworks

1. Execute the general algorithm on 3+4*((4+6)^2)/2 and convince yourself through proper reasoning that the algorithm handles nested parentheses suitably.

2. Execute the general algorithm on 3+4^2^2*3 and convince yourself through proper reasoning that the algorithm takes into account the right associativity of operator ^.

3. What should be the priorities of $?

4. How to take care of the end of the expression ?

Hint: Introduce a new operator symbol #

at the end of the expression so that upon seeing #,

we do very much like what we do on seeing **)**. What should be the priorities of # ?

Homeworks

- How is recursion implemented during program execution ?

Using **stack**

```
int Recur(int i)
{
    int j, k, val;
    …
    …
    val = Recur(t);
    …
    …
}
```

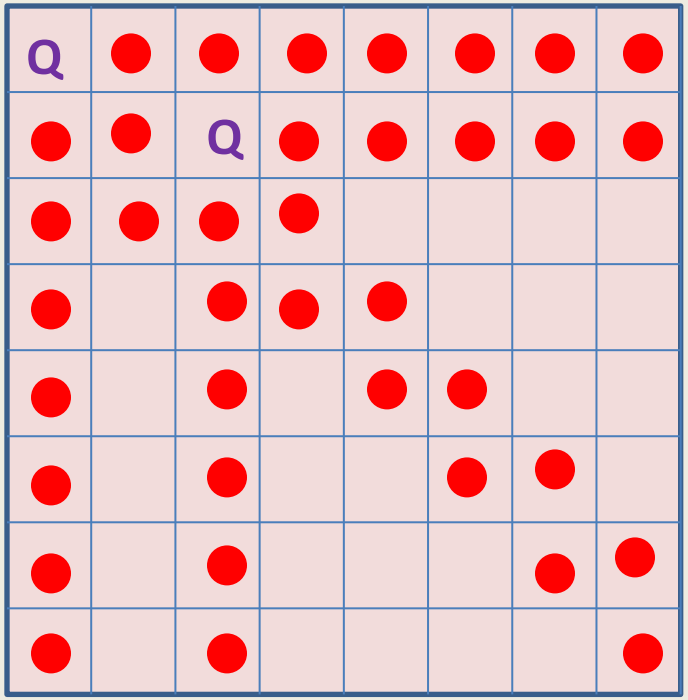
Learn about it from **wikipedia** …

Two interesting problems

Applications of simple data structures

8 queen problem

Place 8 queens on a chess board so that no two of them attack each other.



With this sketch/hint, try to design the complete algorithm using stack or otherwise.

Shortest route in a grid

From a cell in the grid, we can move to any of its neighboring cell in one step.

From top left corner, **find shortest route** to each green cell avoiding obstacles.