Module 4.6.2

Laplace Domain Analysis

Root Locus

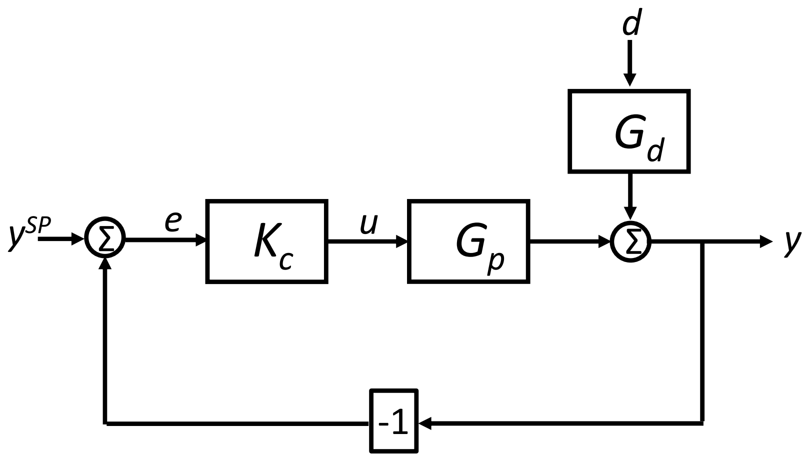
Controller Design

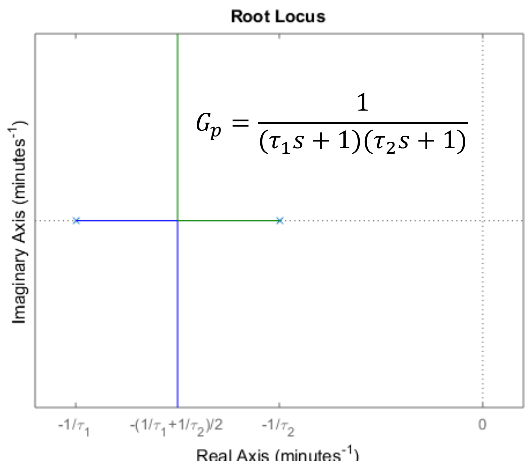
Lectures on

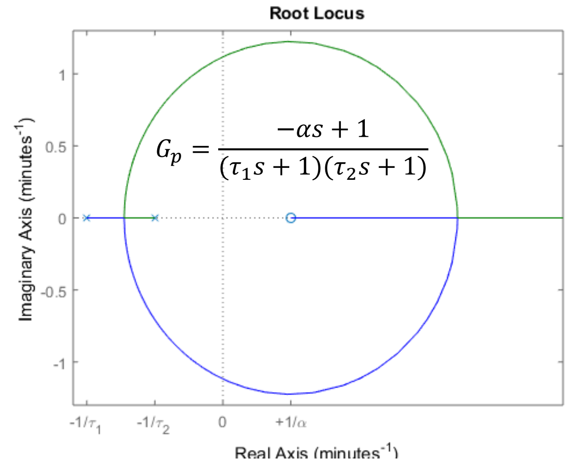
CHEMICAL PROCESS CONTROL

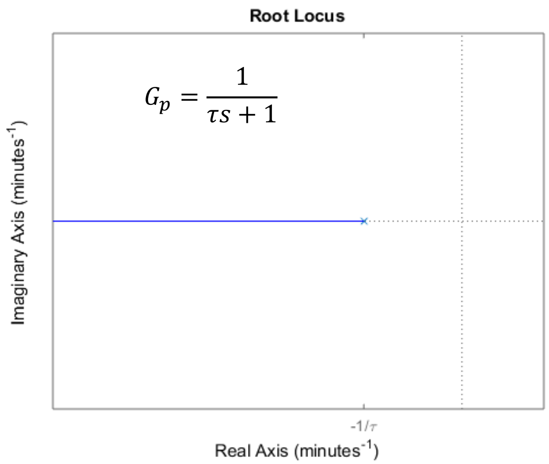
Theory and Practice

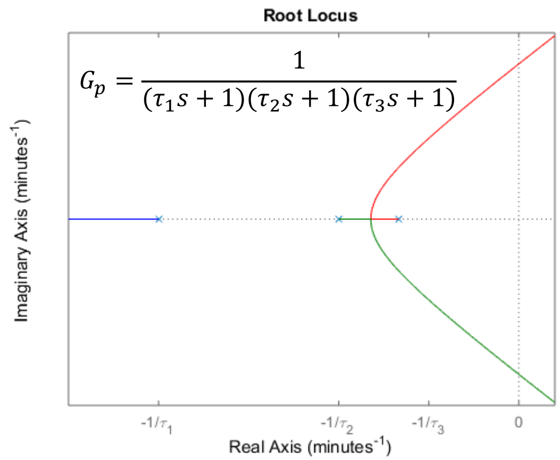
Example Root Loci

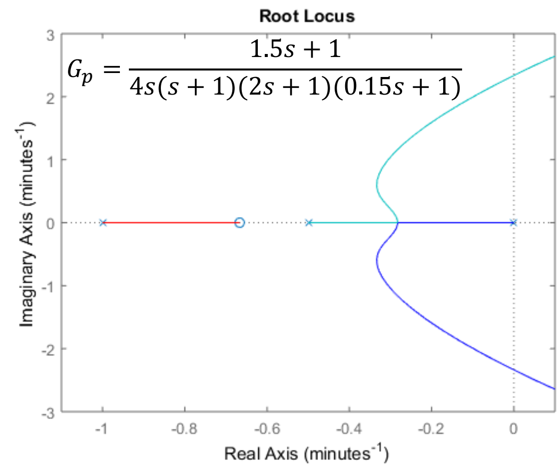












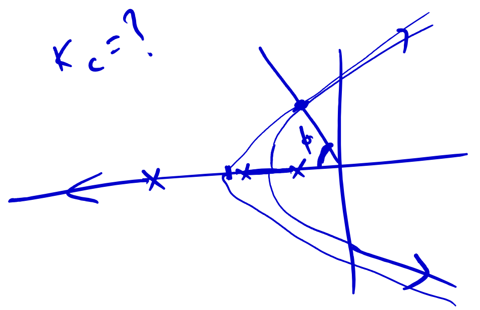
P Controller Design Using Root Locus

$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)} \qquad G_c = K_c$$

OL poles $s = -1, -\frac{1}{2}, -\frac{1}{4}$

$$\sigma = \frac{1\frac{3}{4}}{3} = \frac{7}{12}$$

$K_c = ?$



$$1 + K_c G_p = 0$$

$$\Rightarrow K_c = -\frac{1}{G_p}$$

$$K_c = \frac{1}{|G_p|_{s=\text{dom. pole}}}$$

$\xi = 0.5$

$\cos\phi = 0.5$

$\Rightarrow \phi = 60^\circ$

$s = -a + \sqrt{3}aj$ lies on CLCE

$$(s+1)(2s+1)(4s+1) + K_c = 0$$

$$[2s^2 + 3s + 1][4s+1] + K_c = 0$$

CLCE $\rightarrow 8s^3 + 14s^2 + 7s + 1 + K_c = 0$

$$8 \cdot 8a^3 + 14[-2a^2 - 2\sqrt{3}a^2 j] + 7[-a + \sqrt{3}aj] + 1 + K_c = 0$$

$$64a^3 - 28a^2 - 28\sqrt{3}a^2 j - 7a + 7\sqrt{3}aj + 1 + K_c = 0$$

$$[64a^3 - 28a^2 - 7a + 1 + K_c] + aj[7\sqrt{3} - 28\sqrt{3}a] = 0$$

$$[-a + \sqrt{3}aj]^3 = -a^3 + \cancel{3\sqrt{3}a^3 j} + 3a^3 - \cancel{3\sqrt{3}a^3 j} = 8a^3$$

$$[-a + \sqrt{3}aj]^2 = a^2 - 3a^2 - 2\sqrt{3}a^2 j = -2a^2 - 2\sqrt{3}a^2 j$$

$\text{Im} = 0 \Rightarrow a = \frac{1}{4}$

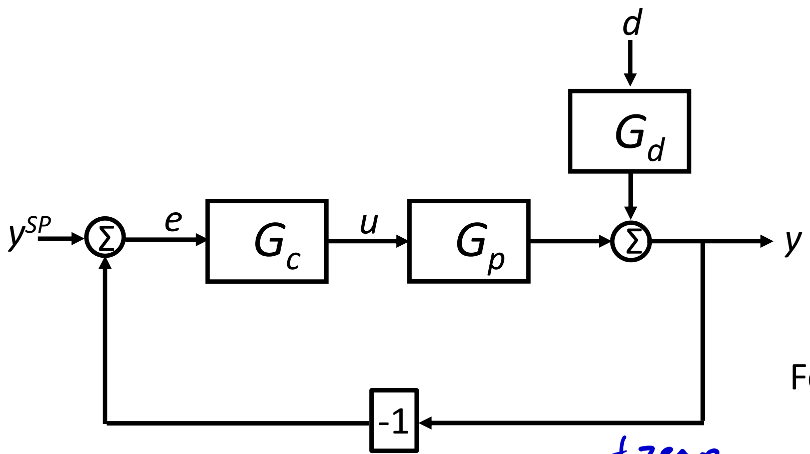
$s = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}j$ dominant pole pair

$K_c = 1.5$

$\xi = 0.5$

$$\left[\; K_c = \frac{1}{|G_p|_{s=-\frac{1}{4}+\frac{\sqrt{3}}{4}j}}\right.$$

P Controller Design Using Root Locus



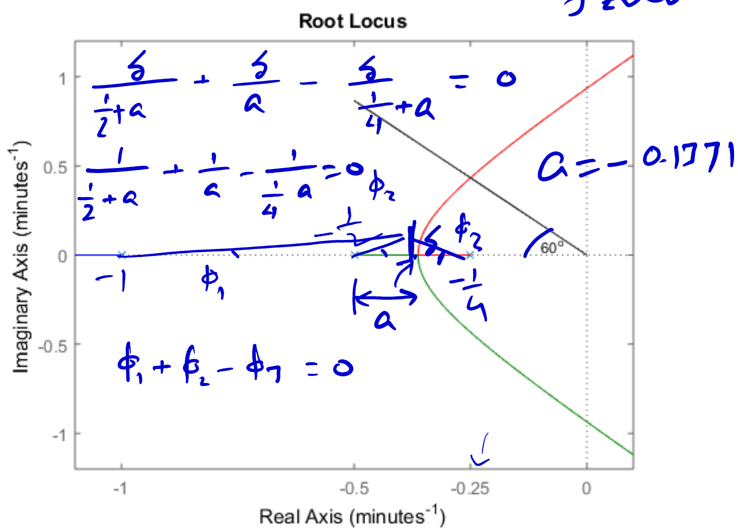
$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)} \qquad G_c = K_c$$

Obtain K_C such that $\xi = 0.5$ for CLCE dominant roots

Breakpts $s = -0.5 + 0.1371$

$s = -0.3619 \qquad K_c = \frac{1}{|G_p|_{s=-0.3619}} =$

For $\xi = 0.5$, $s = -a + \sqrt{3}aj$ satisfies CLCE ≡ $8s^3 + 14s^2 + 7s + 1 + K_c = 0$



Substituting and collecting real and imaginary terms

$$[64a^3 - 28a^2 - 7a + 1 + K_c] + aj\left[-28\sqrt{3}a + 7\sqrt{3}\right] = 0$$

$\therefore a = \frac{1}{4}$ and $s = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}j$ are CLCE dominant poles

$$K_c = \left|\frac{1}{G}\right|_{s=-\frac{1}{4}+\frac{\sqrt{3}}{4}j} = 1.5$$

PI Controller Design Using Root Locus

$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)} \qquad G_c = K_c\,\frac{\tau_I s+1}{\tau_I s} \qquad G_{OL} = \frac{K_c}{4s(s+1)(2s+1)}$$

$\tau_I = ?$ $\quad K_c \quad \xi = 0.5 \quad s = -a+\sqrt{3}\,aj$

$\phi = 60$

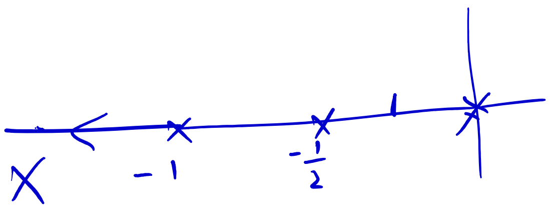
$$4s(s+1)(2s+1)+K_c=0$$

$$4s(2s^2+3s+1)+K_c=0$$

CLCE $\quad 8s^3+12s^2+4s+K_c=0$

$$64a^3+12[-2-2\sqrt{3}j]a^2+4[-a+\sqrt{3}aj]+K_c=0$$

$$[64a^3-24a^2-4a+K_c]+aj\underbrace{[4\sqrt{3}-24\sqrt{3}a]}_{=0}$$



$s = -\frac{1}{4} \pm \frac{\sqrt{3}}{4}j$

G_pG_c

$a = \frac{1}{6}$

$\tau_I = 4\,\text{min}$

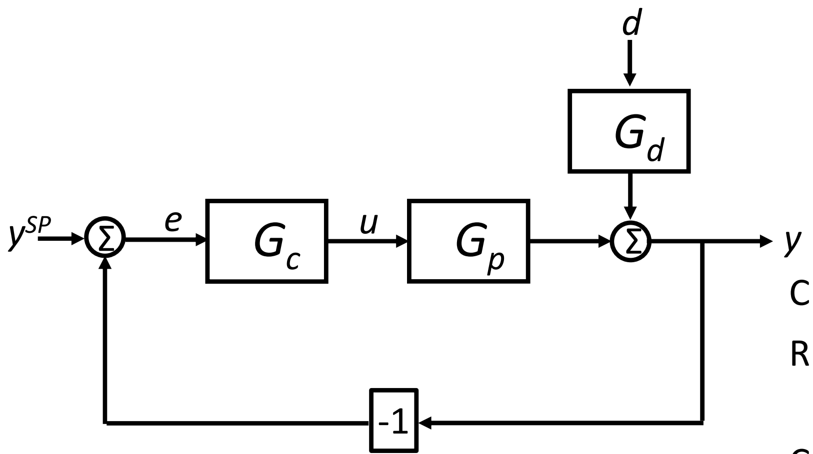
Dominant CLCE pole pair

$s = -\frac{1}{6} \pm \frac{\sqrt{3}}{6}j$

$$K_c = \frac{1}{|G_{OL}|_{s=-\frac{1}{6}+\frac{\sqrt{3}}{6}j}} = 1.037 \simeq 1.04$$

$K_c^{PI} = 1.04 \qquad K_c^{P} = 1.5$

PI Controller Design Using Root Locus



$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)} \qquad G_c = K_c \frac{\tau_I s + 1}{\tau_I s}$$

Obtain K_C such that $\xi = 0.5$ for CLCE dominant roots

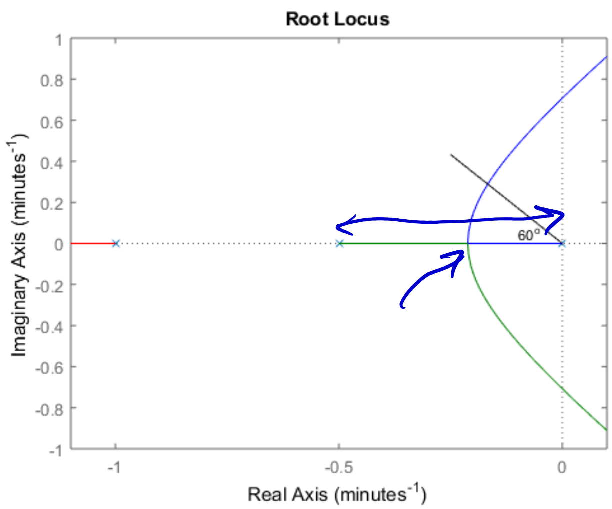
Choose reasonable value for τ_I

RULE OF THUMB: Set τ_I to largest open loop time constant. Controller zero then cancels open loop pole closest to RHP.

CLCE then becomes $8s^3 + 12s^2 + 4s + K_c = 0$ $\rightarrow s^3 + \frac{3}{2}s^2 + \frac{1}{2}s + \frac{K_c}{8} = 0$

For $\xi = 0.5$, $s = -a + \sqrt{3}aj$ satisfies CLCE $\quad (s+a)^2(s+b) = 0$

$3s^2 + 3s + \frac{1}{2} = 0 \qquad s = -\frac{3}{6} \pm \frac{\sqrt{9-6}}{6} \qquad s = -\frac{1}{2} \pm \frac{\sqrt{3}}{6}$



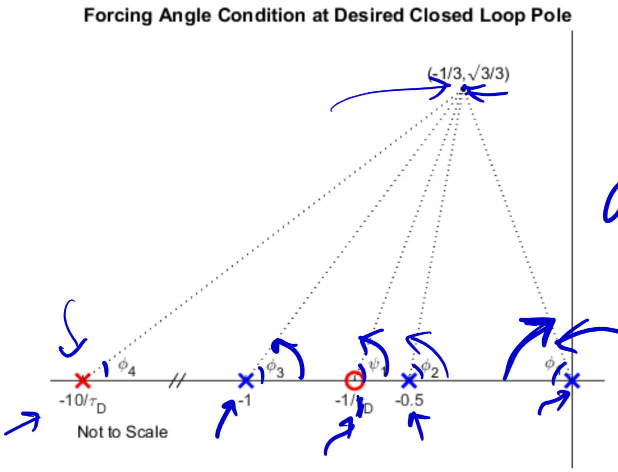
Substituting and collecting real and imaginary terms

$$[64a^3 - 24a^2 - 4a + K_c] + aj\left[-24\sqrt{3}a + 4\sqrt{3}\right] = 0$$

$\therefore a = \frac{1}{6}$ and $s = -\frac{1}{6} \pm \frac{\sqrt{3}}{6}j$ are CLCE dominant poles

$$K_c = \left|\frac{1}{G}\right|_{s=-\frac{1}{6}+\frac{\sqrt{3}}{6}j} = 1.037$$

PID Controller Design Using Root Locus



$\boxed{\tau_I = 4\,\text{min}}$

$$G_{OL} = \frac{K_c(\tau_D s+1)}{4s(s+1)(2s+1)(0.1\tau_D s+1)}$$

-120°

$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)} \qquad G_c = K_c\frac{(\tau_D s+1)}{4s(0.1\tau_D s+1)}$$

$\tau_D = ?$ $\quad K_c \quad \zeta = 0.5$

$s = 2 \times$ PI dominant pole must lie on root locus

$s = -\frac{1}{3} + \frac{\sqrt{3}}{3}j$ must lie on root locus

$\angle$ condition must be satisfied

$$\underbrace{-\phi_1 - \phi_2 - \phi_3}\; \overline{-\phi_4 + \psi_1} = -180^\circ$$

$$+\tan^{-1}\frac{\sqrt{3}/3}{\frac{1}{\tau_D}-\frac{1}{3}} - \tan^{-1}\frac{\sqrt{3}/3}{\frac{10}{\tau_D}-\frac{1}{3}}$$

$$|\phi_1| = \tan^{-1}\frac{\sqrt{3}/3}{1/3} = 60^\circ \sim \underline{\underline{-120^\circ}}$$

$$-234.8^\circ = -180^\circ$$

$$K_c = \frac{1}{|G_{OL}|_{s=-\frac{1}{3}+\frac{\sqrt{3}}{3}j}}$$

$$\phi_2 = \tan^{-1}\frac{\sqrt{3}/3}{1/6} = 73.9^\circ$$

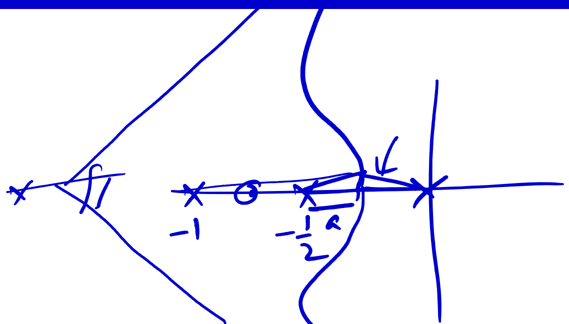
$\boxed{K_c \simeq 2.70}$

$$\tan^{-1}\frac{\sqrt{3}/3}{\frac{1}{\tau_D}-\frac{1}{3}} - \tan^{-1}\frac{\sqrt{3}/3}{\frac{10}{\tau_D}-\frac{1}{3}} = 54.8^\circ$$

$$\phi_3 = \tan^{-1}\sqrt{3}/3/2/3 = 40.9$$

$\boxed{\tau_D = 1.5\,\text{min}}$

PI~~D~~ Controller Design Using Root Locus



$$\frac{s}{a} - \frac{s}{\frac{1}{2}-a} - \frac{s}{\frac{1}{6}+a} + \frac{s}{\frac{1}{2}+a} + \frac{s}{\frac{20}{3}-\frac{1}{2}+a} = 0$$

$$\frac{1}{a} - \frac{1}{\frac{1}{2}-a} - \frac{1}{\frac{1}{6}+a} + \frac{1}{\frac{1}{2}+a} + \frac{1}{\frac{37}{6}+a} = 0$$

$$\sigma = \frac{-\frac{1}{2} - 1 - \frac{20}{3} + 2/3}{3}$$

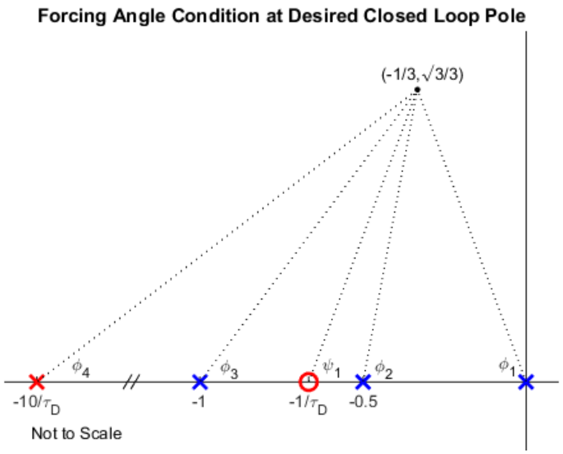
$$\sigma = -2.5$$

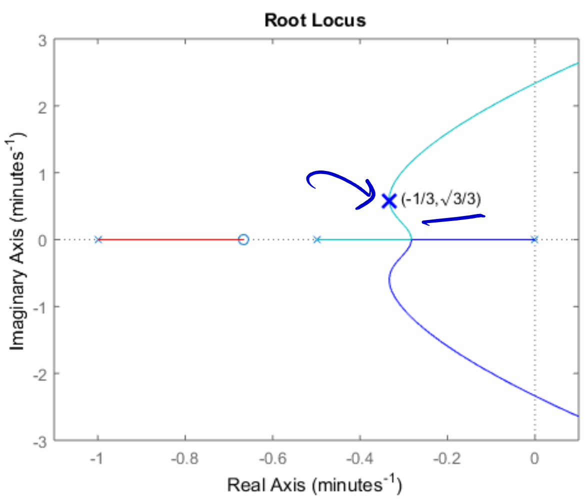
$$a = 0.2177 \qquad s = -0.5 + 0.2177$$

$$s = -0.2823 \text{ break pt}$$

$$G = \frac{1}{(s+1)(2s+1)4s(0.15s+1)} \qquad (1.5s+1)\Big| K_c^{breakpt} = \frac{1}{|G|_{s=-0.2823}} \Rightarrow K_c^{bp} = 0.5861$$

PID Controller Design Using Root Locus





$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)} \qquad G_c = K_c \frac{(\tau_I s+1)}{\tau_I s}\frac{(\tau_D s+1)}{(0.1\tau_D s+1)}$$

Want CLCE $\xi = 0.5$ and response speed twice that of PI controller

Set $\tau_I = 4$ mins as before. For twice the response speed of a PI controller, the dominant pole pair should be twice in magnitude of PI dominant pair.

Dominant pole pair is $s = 2\left[-\frac{1}{6} \pm \frac{\sqrt{3}}{6}j\right] = -\frac{1}{3} \pm \frac{\sqrt{3}}{3}j$

$s = -\frac{1}{3} \pm \frac{\sqrt{3}}{3}j$ must satisfy angle condition to be on root locus

Choose τ_D to ensure angle condition is indeed satisfied.

From figure,

$$\phi_1 = \tan^{-1}\frac{\sqrt{3}/3}{1/3} = 60^\circ \quad \phi_2 = \tan^{-1}\frac{\sqrt{3}/3}{1/6} = 73.90^\circ \quad \phi_3 = \tan^{-1}\frac{\sqrt{3}/3}{2/3} = 40.89^\circ$$

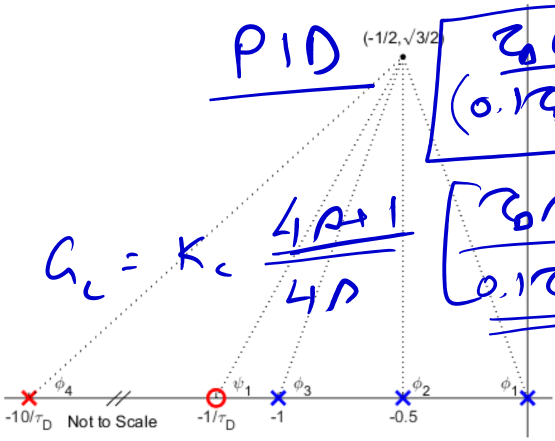
$$\psi_1 + \phi_1 - \phi_2 - \phi_3 - \phi_4 = 180^\circ$$

$$\Rightarrow \tan^{-1}\frac{\sqrt{3}/3}{1/\tau_D - 1/3} - \tan^{-1}\frac{\sqrt{3}/3}{10/\tau_D - 1/3} = 54.79^\circ \Rightarrow \tau_D = 1.5\text{min}$$

magnitude crudn. → $$K_c = \left|\frac{1}{G}\right|_{s=-\frac{1}{3}+\frac{\sqrt{3}}{3}j} = 2.70$$

PIDD Controller Design Using Root Locus

PID $\boxed{\frac{\tau_D s+1}{(0.1\tau_D s+1)}}$



$$G_c = K_c \frac{4s+1}{4s}\left[\frac{\tau_D s+1}{0.1\tau_D s+1}\right]^2$$

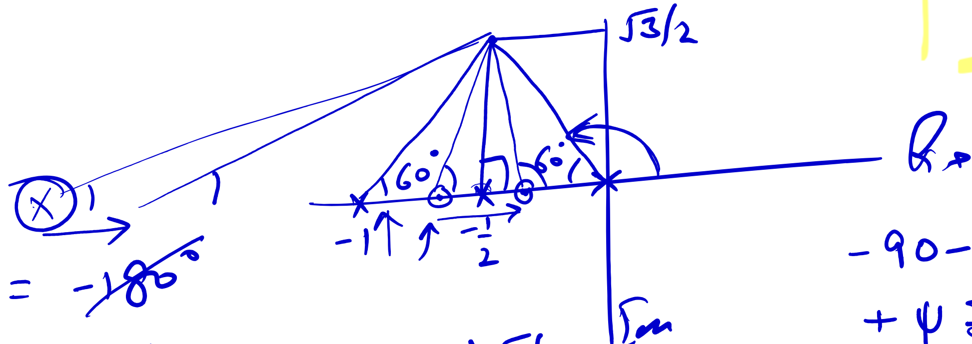
Twice as fast as P controller

$$s = -\frac{1}{4} + \frac{\sqrt{3}}{4}j$$

$\rightarrow s = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$ to lie on root locus

$\zeta = 0.5$

$$\boxed{\tau_I = 4 \text{ min}}$$



$$-60 - 120^\circ - 90^\circ + 2\psi - 2\phi = -180^\circ$$

$$\psi - \phi = 45^\circ \qquad \psi = \tan^{-1}\frac{\sqrt{3}/2}{\frac{1}{\tau_D} - \frac{1}{2}} \qquad \phi = \tan^{-1}\frac{\sqrt{3}/2}{\frac{10}{\tau_D} - \frac{1}{2}}$$

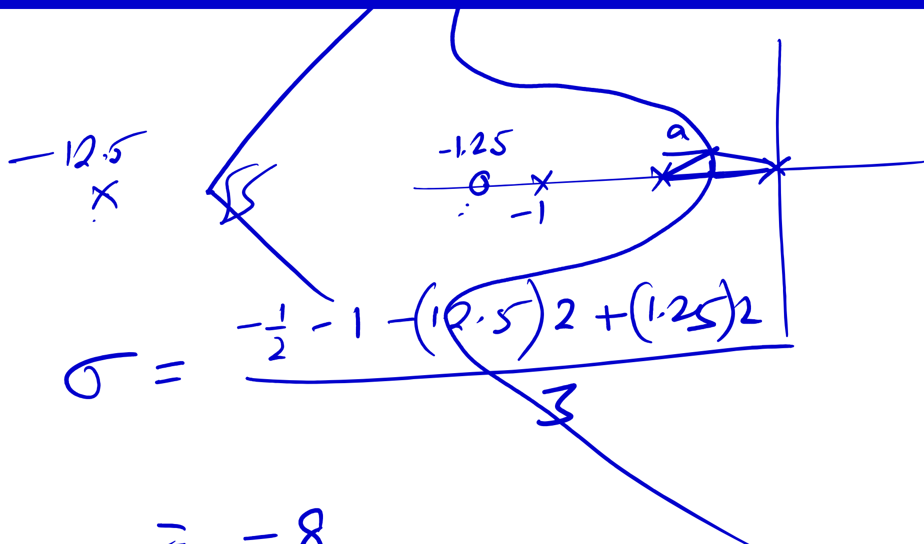
$$-90 - 60 - 120 + \psi - \phi = -180$$

$$\boxed{\psi - \phi = 90^\circ} \leftarrow X$$

$$\tan^{-1}\frac{\sqrt{3}/2}{\frac{1}{\tau_D} - \frac{1}{2}} - \tan^{-1}\frac{\sqrt{3}/2}{\frac{10}{\tau_D} - \frac{1}{2}} = 45^\circ$$

$$\boxed{\tau_D = 0.8 \text{ min}} \qquad \boxed{K_c = \frac{1}{|G_{OL}|_{s=-\frac{1}{2}+\frac{\sqrt{3}}{2}j}} = 7.64}$$

PIDD Controller Design Using Root Locus



$$\sigma = \frac{-\frac{1}{2} - 1 - (12.5)2 + (1.25)2}{3}$$

$$= -8$$

$$\frac{5}{a} - \frac{5}{\frac{1}{2} - a} + \frac{5}{\frac{1}{2} + a} - \frac{25}{0.75 + a} + \frac{25}{12 + a} = 0$$

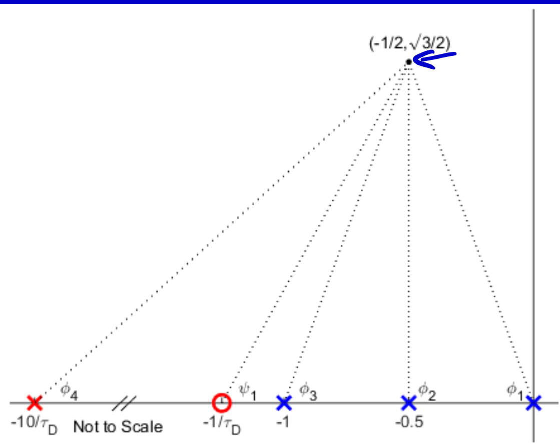
Solve for $a = 0.2342$

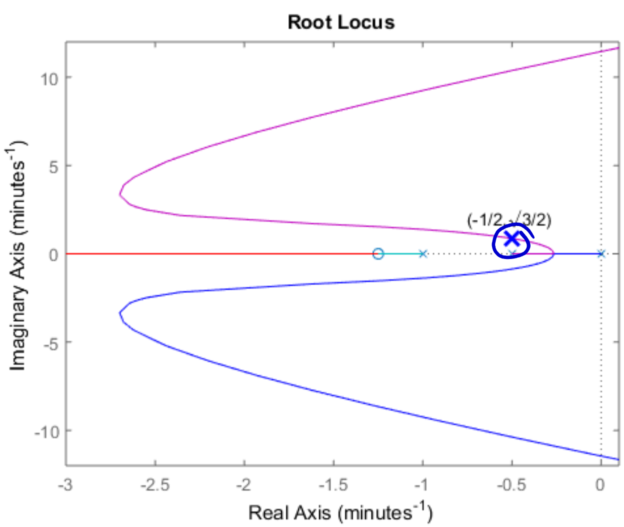
$$s = -\frac{1}{2} + 0.2342$$

$$s = -0.2658 \quad \text{break pt}$$

$$K_c^{bp} = \frac{1}{|G_{OL}|_{s=-0.2658}} = 0.5650$$

PIDD Controller Design Using Root Locus





$$G_p = \frac{1}{(s+1)(2s+1)(4s+1)}$$

Want CLCE $\xi = 0.5$ and response speed twice that of P controller

Set $\tau_I = 4$ mins. Dominant pole pair should be twice P dominant pair.

Dominant pole pair is $s = 2\left[-\frac{1}{4} \pm \frac{\sqrt{3}}{4}j\right] = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$

$s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$ must satisfy angle condition to be on root locus

Single τ_D lead-lag $\frac{(\tau_D s+1)}{(0.1\tau_D s+1)}$ cannot cause angle condition to be satisfied.

So put 2 identical lead-lags $G_c = K_c \frac{(\tau_I s+1)}{\tau_I s}\left[\frac{(\tau_D s+1}{0.1\tau_D s+1}\right]^2$

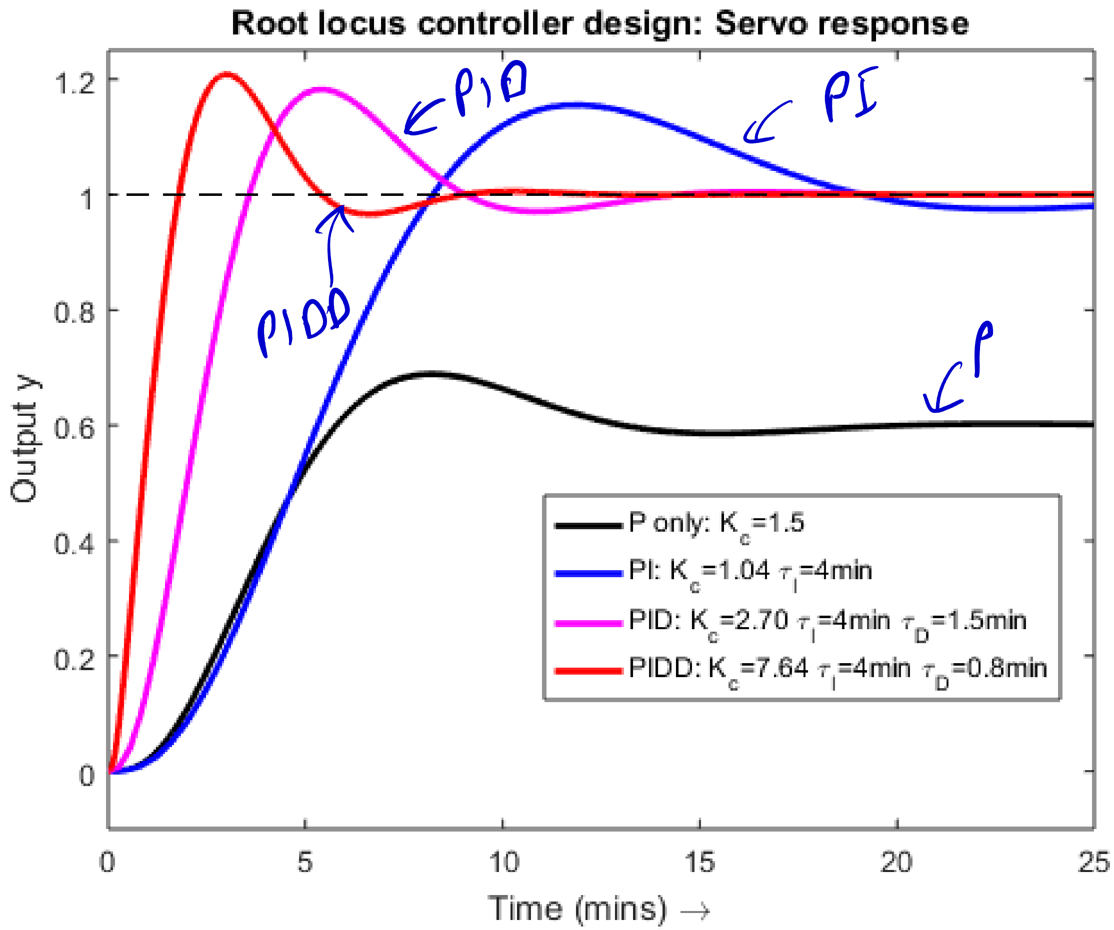
From figure, $\phi_1 = \tan^{-1}\frac{\sqrt{3}/2}{1/2} = 60^\circ \quad \phi_2 = 90^\circ \quad \phi_2 = \tan^{-1}\frac{\sqrt{3}/2}{1/2} = 60^\circ$

$$2\psi_1 + \phi_1 - \phi_2 - \phi_3 - 2\phi_4 = 180^\circ$$

$$\Rightarrow \tan^{-1}\frac{\sqrt{3}/2}{1/\tau_D - 1/2} - \tan^{-1}\frac{\sqrt{3}/2}{10/\tau_D - 1/2} = 45^\circ \Rightarrow \tau_D = 0.80\text{min}$$

$$K_c = \left|\frac{1}{G_p G_c}\right|_{s=-\frac{1}{2}+\frac{\sqrt{3}}{2}j} = 7.64$$

Closed Loop Servo Response



Root Locus Design Procedure Summary

- Obtain desired closed loop pole location (region)
  - Degree of oscillatoriness (damping coefficient ξ)
  - Speed of response
- For zero offset, set τ_I to cancel slowest (closest to RHP) open loop pole
- Choose τ_D lead-lag to force locus to pass through desired closed loop pole
  - Angle condition gives necessary τ_D
  - Magnitude condition gives necessary K_C
  - Use only if simple PI control does not give fast enough closed loop response
  - Derivative lead-lag is used to pull root locus sufficiently to the left for faster response
  - May require double derivative for very fast closed loop response
    - Amplifies noise
- Root locus construction rules allow easy visualization of reshaped root locus