Assignment 3

Solution

1. The differential equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = 5t$$

In terms of deviation variable, the above equation is

$$\frac{d^2\hat{y}}{dt^2} + 3\frac{d\hat{y}}{dt} - \hat{y} = 5\hat{t}$$

Taking Laplace transform of the above equation

$$s^2\hat{y}(s) + 3s\hat{y}(s) - \hat{y}(s) = \frac{5}{s^2}$$

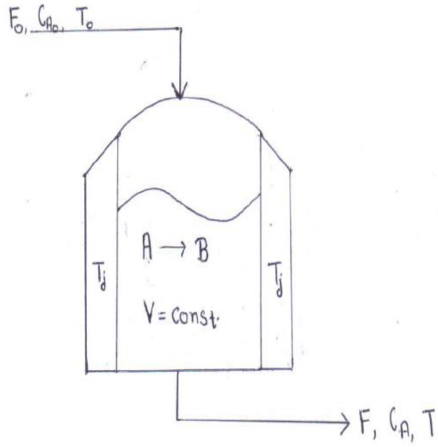
$$\hat{y}(s) = \frac{5}{s^2(s^2+3s-1)} = \frac{5}{s^2(s+3.3)(s-0.3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3.3} + \frac{D}{s-0.3}$$

On solving we get

A=-15, B=-5, C=-0.13, D=15.12

Then, $y(t) = -15 - 5t - 0.13e^{-3.3t} + 15.12e^{0.3t}$

2. V = constant i.e., $F_0 = F$



Material Balance of A and B

$$V\frac{dC_A}{dt} = F\left(C_{A_o} - C_A\right) - K_o e^{-E/RT} C_A V$$

$$\frac{dC_A}{dt} = \frac{F}{V}\left(C_{A_o} - C_A\right) - K_o e^{-E/RT} C_A$$

after Linearizing we get

$$\frac{dC_A}{dt} = \frac{1}{V}\left(\overline{F}\left(C_{A_o} - C_A\right) + \left(\overline{C}_{A_o} - \overline{C}_A\right)F\right) - \overline{K}C_A - \frac{\overline{K}\overline{C}_A E}{R\overline{T}^2}T$$

$$\frac{dC_A}{dt} + \frac{\overline{F} + \overline{K}V}{V}C_A = \frac{\overline{F}}{V}C_{A_o} + \frac{\left(\overline{C}_{A_o} - \overline{C}_A\right)F}{V} - \frac{\overline{K}\overline{C}_A E}{R\overline{T}^2}T$$

$$\frac{V}{\overline{F} + \overline{K}V}\frac{dC_A}{dt} + C_A = \frac{\overline{F}}{\overline{F} + \overline{K}V}C_{A_o} + \frac{\left(\overline{C}_{A_o} - \overline{C}_A\right)F}{\overline{F} + \overline{K}V} - \frac{\overline{K}\overline{C}_A EV}{R\overline{T}^2\left(\overline{F} + \overline{K}V\right)}T$$

$$\tau_c\frac{dC_A}{dt} + C_A = K_{c_{C_{A_o}}} C_{A_o} + K_{CF}F - K_{CT}T$$

Laplace Transforming with IC's put to 0 (because we work on deviation variables)

$$\left(\tau_c s + 1\right)C_A = K_{c_{C_{A_o}}} C_{A_o} + K_{CF}F - K_{CT}T$$

$$\therefore C_A = \frac{K_{c_{C_{A_o}}}}{\left(\tau_c s + 1\right)}C_{A_o} + \frac{K_{CF}}{\left(\tau_c s + 1\right)}F - \frac{K_{CT}}{\left(\tau_c s + 1\right)}T$$

$$C_A = G_{c_{C_{A_o}}} C_{A_o} + G_{CF}F - G_{CT}T \quad \text{.......... (a)}$$

Energy Balance

$$\text{Accumulation = flow + Generation} \pm {}^{heating}\!\big/_{cooling}$$

$$\rho V C_P \frac{dT}{dt} = \rho C_P F(T_o - T) + K_o e^{-E/RT} C_A V(-\triangle H_r) - UA\left(T - T_j\right)$$

$$\therefore \frac{dT}{dt} = \frac{F}{V}(T_o - T) + \frac{K_o e^{-E/RT} C_A V(-\triangle H_r)}{\rho C_P} - \frac{UA\left(T - T_j\right)}{\rho C_P V}$$

Linearization

$$\frac{dT}{dt} = \frac{1}{V}\left[\overline{F}(T_o - T) + \left(\overline{T_o} - \overline{T}\right)F\right] + \frac{\overline{K}(-\triangle H_r)}{\rho C_P} C_A + \frac{\overline{K}\overline{C_A}\left(-\triangle H_r\right)E}{\rho C_P R\overline{T}^2} T - \frac{UA\left(T - T_j\right)}{\rho C_P V}$$

$$\frac{dT}{dt} + \frac{\overline{F} + \frac{UA}{\rho C_P} - \frac{\overline{K}\overline{C_A}\left(-\triangle H_r\right)V\ E}{\rho C_P R\overline{T}^2}}{V} T = \frac{\overline{F}}{V} T_o + \frac{\overline{T_o} - \overline{T}}{\rho C_P V} F + \frac{UA}{\rho C_P V} T_j + \frac{\overline{K}(-\triangle H_r)}{\rho C_P} C_A$$

$$\frac{V}{\alpha}\frac{dT}{dt} + T = \frac{\overline{F}}{\alpha} T_o + \frac{\overline{T_o} - \overline{T}}{\rho C_P \alpha} F + \frac{UA}{\rho C_P \alpha} T_j + \frac{\overline{K}(-\triangle H_r)V}{\rho C_P \alpha} C_A$$

$$\text{Where, } \alpha = \overline{F} + \frac{UA}{\rho C_P} - \frac{\overline{K}\overline{C_A}\left(-\triangle H_r\right)V\ E}{\rho C_P R\overline{T}^2}$$

$$\therefore \tau_T \frac{dT}{dt} + T = K_{TT_o} T + K_{TF} F + K_{TT_j} T_j + K_{TC} C_A$$

Laplace Transforming with IC's put to 0 (because we work on deviation variables)

$$\left(\tau_T s + 1\right)T = K_{TT_o} T + K_{TF} F + K_{TT_j} T_j + K_{TC} C_A$$

$$T = \frac{K_{TT_o}}{\left(\tau_T s + 1\right)} T + \frac{K_{TF}}{\left(\tau_T s + 1\right)} F + \frac{K_{TT_j}}{\left(\tau_T s + 1\right)} T_j + \frac{K_{TC}}{\left(\tau_T s + 1\right)} C_A$$

$$T = G_{TT_o} T + G_{TF} F + G_{TT_j} T_j + G_{TC} C_A \quad \text{............(b)}$$

3.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 5u(t)$$

Assume t=0 when new products are launched

- ⇨ $y(t=0) \leq 0\ \ \&\ \ u(t=0) \leq 0$
- ⇨ $s^2 y(s) + 2sy(s) + 2y(s) = 5u(s)$
- ⇨ $\frac{y(s)}{u(s)} = \frac{5}{s^2+2s+2}$ (1)
- ⇨ $D(s) = 0\,;\quad 0.5s^2 + s + 1 = 0$
- ⇨ $s = -1 \pm \sqrt{(-1)}$

Now, u(t)=exp(-t)

u(s)=$\frac{1}{s+1}$

put u(s) in equation (1) we get

$$y(s) = \frac{5}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}$$

A=5, B=-5, C=-5

$$y(s) = \frac{5}{s+1} - \frac{5(s+1)}{s^2+2s+2}$$

Taking Inverse of Laplace of above equation, we get

$$y(t) = 5e^{-t} - 5\cos(t)\,e^{-t} = 5e^{-t}(1-\cos(t))$$

4. The differential equation is

$$\frac{d^2h}{dt^2} + \frac{6\mu}{\rho R^2}\frac{dh}{dt} + \frac{3gh}{2L} = \frac{3}{4\rho L}\Delta P$$

Take Laplace

$$s^2H + \frac{6\mu}{\rho R^2}sH + \frac{3g}{2L}H = \frac{3}{4\rho L}\Delta P$$

$$\frac{H}{\Delta P} = G(s) = \frac{\frac{3}{4\rho L}}{s^2 + \frac{6\mu}{\rho R^2}s + \frac{3g}{2L}} = \frac{\frac{1}{2\rho g}}{\frac{2L}{3g}s^2 + \frac{4\mu L}{\rho g R^2}s + 1}$$

⇨ Gain = $G(s=0) = K = \frac{1}{2\rho g}$

$$\tau^2 = \frac{2L}{3g}\ ;\ \tau = \sqrt{\frac{2L}{3g}}$$

$$2\tau\xi = \frac{4\mu L}{\rho g R^2}\ ;\ \xi = \frac{\mu}{\rho R^2}\sqrt{\frac{6L}{g}}$$

⇨ $G(s) = \frac{K}{\tau^2 s^2 + 2\xi\tau s + 1}$

⇨ $\Delta P(t) = At$

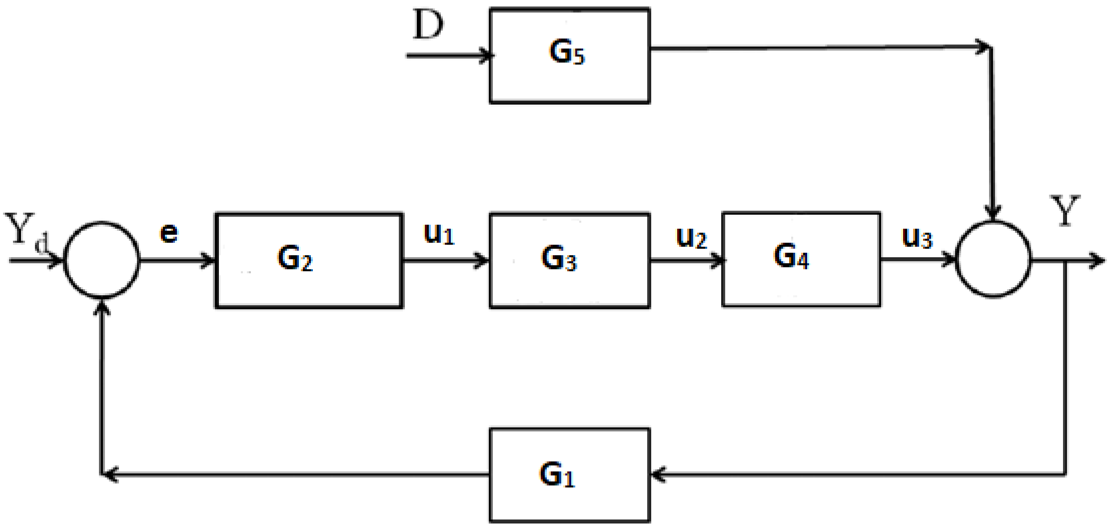
⇨ $\Delta P(s) = \frac{A}{s^2}$

⇨ $H(s) = \frac{AK}{s^2(\tau^2+2\xi\tau s+1)} = \frac{W}{s} + \frac{X}{s^2} + \frac{Ys+Z}{\tau^2 s^2+2\xi\tau s+1}$

W=-2$\tau\xi AK$, X=AK, Y=-2$\tau^3\xi AK$, Z=$\tau^2 AK(4\xi^2-1)$

Final solution will depend on the value of ξ.

5. The block diagram is



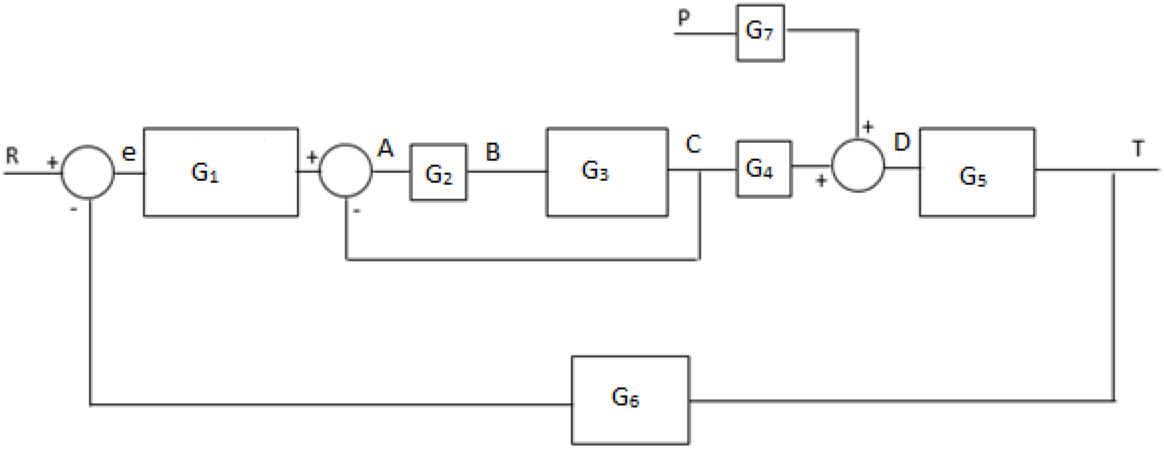
$$e = Y_d - G_1 Y$$
$$u_1 = G_2 e$$
$$u_2 = G_3 u_1$$
$$u_3 = G_4 u_2$$
$$Y = u_3 + G_5 D$$
$$Y = G_2 G_3 G_4 \left(Y_d - G_1 Y\right) + G_5 D$$
$$Y = \frac{G_2 G_3 G_4}{1 + G_1 G_2 G_3 G_4} Y_d + \frac{G_5}{1 + G_1 G_2 G_3 G_4} D$$

For servo D=0, $\frac{Y}{Y_d} = \frac{G_2 G_3 G_4}{1 + G_1 G_2 G_3 G_4}$

For regulatory Y_d=0, $\frac{Y}{D} = \frac{G_5}{1 + G_1 G_2 G_3 G_4}$

Put the values of G_1, G_2, G_3, G_4 and G_5 to get the final answer.

6. The block diagram is



$$e = R - G_6 T$$
$$A = G_1 e - C$$
$$B = G_2 A$$
$$C = G_3 B$$
$$D = G_4 C + G_7 P$$
$$T = G_5 D$$

First solve the inner loop we have,

$$A = \frac{G_1}{1 + G_3 G_2} e$$

Now use this to simplify the block diagram

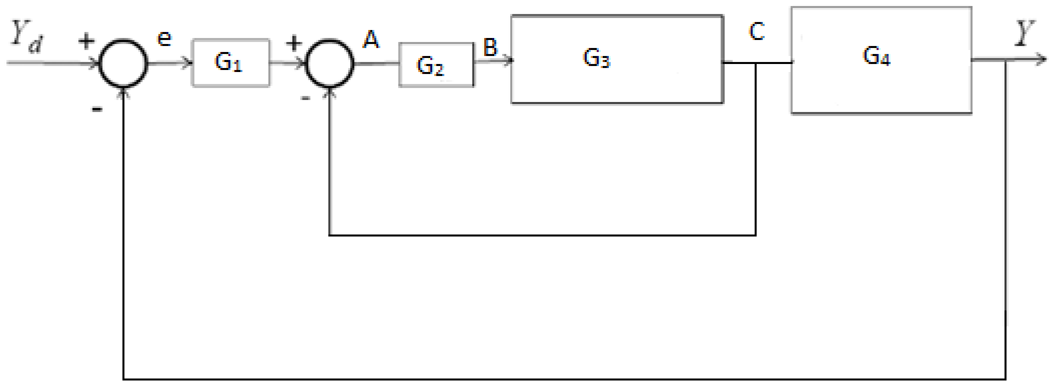
$$T = G_5 \left(G_4 G_3 G_2 \frac{G_1}{1 + G_3 G_2} \left(R - G_6 T \right) + G_7 P \right)$$
$$T = \frac{G_1 G_2 G_3 G_4 G_5}{1 + G_3 G_2} R - \frac{G_1 G_2 G_3 G_4 G_5 G_6}{1 + G_3 G_2} T + G_5 G_7 P$$
$$T = \frac{G_1 G_2 G_3 G_4 G_5}{1 + G_3 G_2 + G_1 G_2 G_3 G_4 G_5 G_6} R + \frac{G_5 G_7 \left(1 + G_3 G_2\right)}{1 + G_3 G_2 + G_1 G_2 G_3 G_4 G_5 G_6} P$$

For servo P=0, $\frac{T}{R} = \frac{G_1 G_2 G_3 G_4 G_5}{1 + G_3 G_2 + G_1 G_2 G_3 G_4 G_5 G_6}$

For regulatory R=0, $\frac{T}{P} = \frac{G_5 G_7 \left(1 + G_3 G_2\right)}{1 + G_3 G_2 + G_1 G_2 G_3 G_4 G_5 G_6}$

Put the values of G_1, G_2, G_3, G_4, G_5, G_6 and G_7 to get the final answer.

7. The block diagram is



$$e = Y_d - Y$$
$$A = G_1 e - C$$
$$B = G_2 A$$
$$C = G_3 B$$

First solve the inner loop we have,

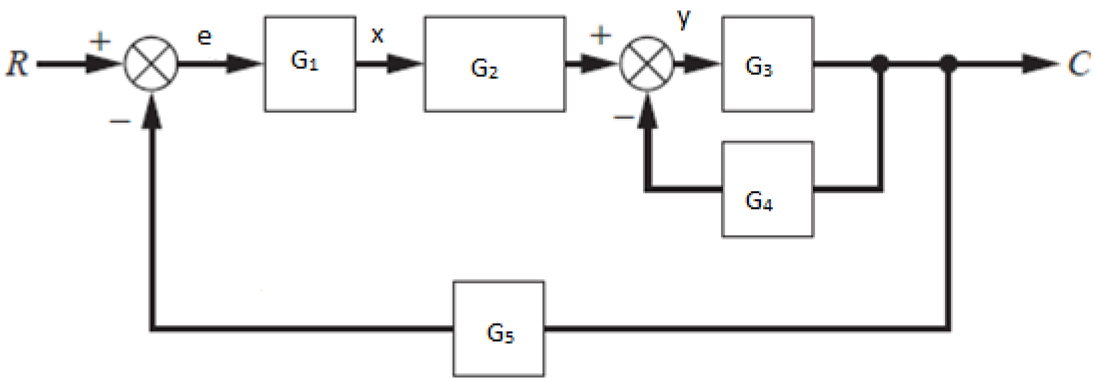
$$A = \frac{G_1}{1 + G_3 G_2} e$$

Now use this to simplify the block diagram

$$Y = G_4 C$$
$$Y = G_4 G_3 G_2 \frac{G_1}{1 + G_3 G_2} (Y_d - Y)$$
$$Y = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_2 + G_1 G_2 G_3 G_4} Y_d$$
$$\frac{Y}{Y_d} = \frac{G_1 G_2 G_3 G_4}{1 + G_3 G_2 + G_1 G_2 G_3 G_4}$$

Put the values of G_1, G_2, G_3 and G_4 to get the final answer.

8. The block diagram is



$$e = R - G_5 C$$
$$x = G_1 e$$
$$y = G_2 x - G_4 C$$
$$C = G_3 y$$

First solve the inner loop we have,

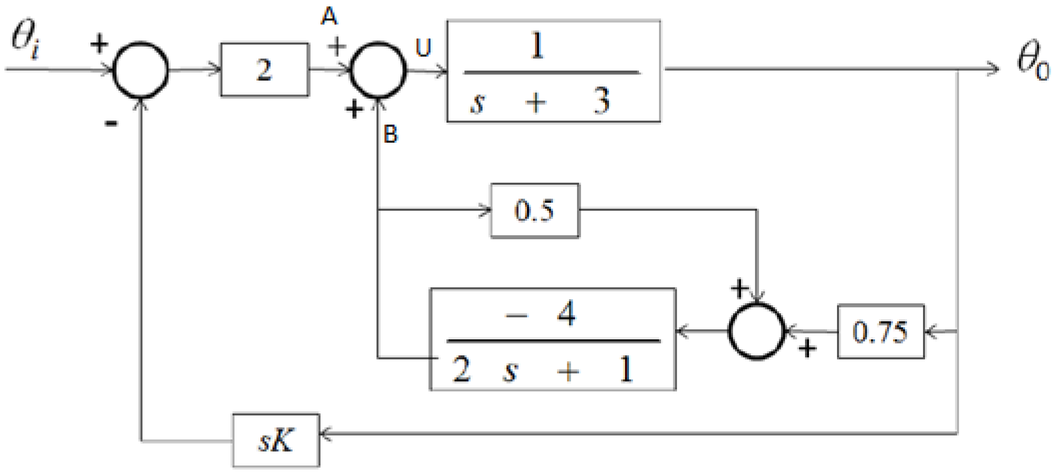
$$y = \frac{G_2}{1 + G_3 G_4} x$$

Now use this to simplify the block diagram

$$C = G_3 \frac{G_2}{1 + G_3 G_4} G_1 \left(R - G_5 C \right)$$
$$C = \frac{G_1 G_2 G_3}{1 + G_3 G_4 + G_1 G_2 G_3 G_5} R$$
$$\frac{C}{R} = \frac{G_1 G_2 G_3}{1 + G_3 G_4 + G_1 G_2 G_3 G_5}$$

Put the values of G_1, G_2, G_3, G_4 and G_5 to get the final answer.

9. The block diagram is



Following relation are clearly seen from the block diagram:

$$\theta_0 = \frac{1}{s+3}U \;;\; \text{U=A+B} \;;\; A = 2(\theta_i - sk\theta_0) \text{ and } B = [0.5B + 0.75\theta_0]\left(\frac{-4}{2s+1}\right)$$

Using all these relations, we can relate θ_i and θ_0 as:

$$\frac{\theta_0}{\theta_i} = \frac{2(2s+3)}{(2+4k)s^2 + (6k+9)s + 12}$$

For critically damped response, roots of the characteristic equation

$$(2+4k)s^2 + (6k+9)s + 12 = 0 \text{ should be identical}$$

⇨ $(6k+9)^2 - 48(2+4k) = 0 \rightarrow 12k^2 - 28k - 5 = 0$

⇨ $k = 2.5, \;\; -0.17$

10. Solution of differential equation

$$24\frac{d^2y}{dt^2} + 10\frac{dy}{dt} + y = 2u$$

Taking laplace transform on both side

$$[24s^2 + 10s + 1]y_s = 2u_s$$

$$\therefore \frac{y_s}{u_s} = G_P = \frac{2}{24s^2 + 10s + 1}$$

P control Gc =K_c

Then $G^{CL} = \dfrac{K_C G_P}{1 + K_C G_P} = \dfrac{y}{y^{sp}}$

CLCE: $1 + K_C G_P$ =0

$$\Rightarrow 24s^2 + 10s + 1 + 2K_C = 0$$

$$\Rightarrow s^2 + \frac{5}{12}s + \frac{1+2K_C}{24} = 0$$

Roots are

$$s = \frac{-5}{24} \pm \frac{\sqrt{\frac{25}{144} - \frac{1+2K_C}{6}}}{2}$$

As K_c increases term under square root goes –ve & we get complex conjugate roots of CLCE

Limiting condition is

$$\frac{1+2K_C}{6} > \frac{25}{144}$$

$$K_C > \frac{1}{48}$$

For critical damping K_C = 1/48

For K_c > 1/48

$$s = \frac{-5}{24} \pm j\frac{\sqrt{\frac{1+2K_C}{6} - \frac{25}{144}}}{2}$$

Real part of the root is always –ve.

a. Closed loop system never goes unstable under P control.
b. For $K_C > \frac{1}{48}$, imaginary part appears in CLCE roots and we will have oscillations.
c. K_c =? For closed loop ε = 0.5.

   If $\varepsilon < 1$ means underdamped system means dominant closed loop poles are complex conjugate pair.

   Also

   $$\cos\varphi = \varepsilon$$

   $$\sin\varphi = \sqrt{1-\varepsilon^2}$$

   $$\tan\varphi = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}$$

   Therefore $\varepsilon = cos\varphi = \frac{1}{2}$

   So φ=60°

   Therefore s= - a+ $\sqrt{3}$ aj must satisfy CLCE which is below

   $$s^2 + \frac{5}{12}s + \frac{1+2K_C}{24} = 0$$

   $$\Rightarrow (-a+\sqrt{3}aj)^2 + \frac{5}{12}(-a+\sqrt{3}aj) + \frac{1+2K_C}{24} = 0$$

$$\frac{1+2K_C}{24}-2a^2-\frac{5}{12}a+\sqrt{3}ja(\frac{5}{12}-2a)=0$$

$$\frac{5}{12}=2a \qquad \frac{1+2K_C}{24}=\frac{5}{12}a+2a^2$$

$$\Rightarrow a=\frac{5}{24} \qquad \Rightarrow K_C=\frac{19}{12}$$

Thus, for this $K_C=\frac{19}{12}$ $s=\frac{-5}{24}\pm j\frac{5\sqrt{3}}{24}$ satisy CLCE

Which gives $\varepsilon = 0.5$

d. PI controller $G_C=K_C(1+\frac{1}{\tau_i s})=K_C\frac{(\tau_i s+1)}{\tau_i s}$

For $\tau_i=6\,\text{min}$ $G_C=K_C\frac{(6s+1)}{6s}$

$$CLCE = 1+G_C G_P=0\Rightarrow 1+K_C\frac{(6s+1)}{6s}\frac{2}{24s^2+10s+1}=0$$

$$\Rightarrow 24s^2+6s+2K_C=0$$

NOTE- This cancellation was not at all obvious in the time domain

Or $s^2+\frac{1}{4}s+\frac{K_C}{12}=0$

$$s=\frac{-1}{8}\pm\frac{\sqrt{\frac{1}{16}-\frac{K_C}{3}}}{2}$$

For $K_C>\frac{3}{16}$ we get $s=\frac{-1}{8}\pm\frac{\sqrt{\frac{1}{16}-\frac{K_C}{3}}}{2}$

Real part is always –ve.

This implies we never get instability under PI control with $\tau_i=6\,\text{min}$

(e) for $\tau_i=2\,\text{min}$

$$G_c=\frac{K_C(2s+1)}{2s}$$

$$\therefore \text{CLCE} \qquad 1+\frac{K_C(2s+1)}{2s}\frac{2}{(6s+1)(4s+1)}=0$$

Or

$$2s+(6s+1)(4s+1)+2K_C(2s+1)=0$$

We have

$$G_{OL}=\frac{2(2s+1)}{2s(6s+1)(4s+1)}=\frac{2(s+\frac{1}{2})}{24s(s+\frac{1}{6})(s+\frac{1}{4})}$$

Or

$$G_{OL} = \frac{(s+\frac{1}{2})}{12s(s+\frac{1}{6})(s+\frac{1}{4})}$$

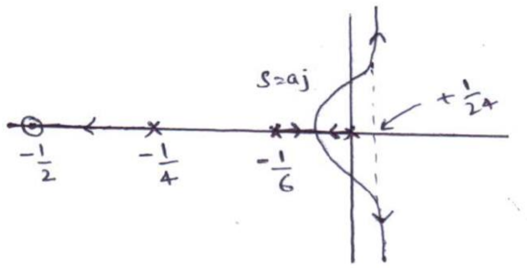
Root locus has ± 90° asymptotes

Real axis asymptotes interest = $\frac{\sum p_i - \sum z_i}{2}$

N=3, M=1 this implies 2 asymptotes.

= $\frac{-\frac{1}{4}-\frac{1}{6}-(-\frac{1}{2})}{2} = \frac{1}{24}$

Thus, root locus looks like.



From root locus, it is clear that closed loop system can go unstable as locus moves to RHP.

Let s=aj correspond to the purely imaginary (no real parts) root on the locus. It must satisfy CLCE.

$\therefore \Rightarrow 48s^3 + 20s^2 + 2s + 4K_C s + 2K_C = 0$

$48s^3 + 20s^2 + (2+4K_C)s + 2K_C = 0$

Put s=aj

$\Rightarrow -48a^3 j - 20a^2 + 2aj(2K_C + 1) + 2K_C = 0$

$(2K_C - 20a^2) + 2aj(2K_C + 1 - 24a^2) = 0$

Comparing real and imaginary parts

$2K_C = 20a^2 \quad\quad 2K_C + 1 = 24a^2$

$\Rightarrow a = \pm\frac{1}{2}$

This implies $K_C = \frac{5}{2}$

For $K_C > \frac{5}{2}$, closed loop system goes unstable.

11. $G_P = \left(\dfrac{8}{(4s+1)(6s+1)(12s+1)} \right)$

P controller design for GM=2

$$G_{OL} = K_c G_P \Rightarrow \quad G_{OL} = \frac{8K_c}{(4s+1)(6s+1)(12s+1)}$$

$$G_{jw}^{OL} = \frac{8K_c}{(4\omega j+1)(6\omega j+1)(12\omega j+1)}$$

$$|G| = \frac{8K_c}{\sqrt{16\omega^2+1}\sqrt{36\omega^2+1}\sqrt{144\omega^2+1}} \quad \text{........(A)}$$

$$\angle G_{OL} = -\tan^{-1} 6\omega - \tan^{-1} 4\omega - \tan^{-1} 12\omega$$

At ω_c

$$\angle G_{OL} = -180^{\circ}$$

$$\therefore -\tan^{-1} 6\omega_c - \tan^{-1} 4\omega_c - \tan^{-1} 12\omega_c = -180^{\circ} \quad \text{....... (B)}$$

Solve For ω_c (iteration calculation) $\Rightarrow -\omega_c = 0.2764 \text{rad/min}$

Guess ω_c (rad/min)	ϕ
0.25	-172.9
0.28	-180.9
0.275	-179.7
0.277	-180.15
0.2764	-180.0

For GM=2

$$\left|G^{OL}\right|_{\omega=\omega_c} = \frac{1}{GM} = \frac{1}{2}$$

$$\mathrm{K}_c = \frac{\sqrt{16\omega_c^2+1}\sqrt{36\omega_c^2+1}\sqrt{144\omega_c^2+1}}{16} \qquad \Rightarrow \mathrm{K}_c = 0.625$$

P controller design

For PM = 45^0

$\phi_{\omega_{gco}} = -180 + PM = -135^{\circ}$ $\quad \omega_{gco}$ - Frequency of gain crossover

From (B) $\phi = -\tan^{-1} 6\omega_{gc0} - \tan^{-1} 4\omega_{gco} - \tan^{-1} 12\omega_{gco} = -\dfrac{3\pi}{4}$

Solve iteratively for $\omega_{gc0} \Rightarrow \omega_{gco} = 0.1521 \text{ rad/min}$

$$\left|G^{OL}\right|_{\omega=\omega_{gc0}} = 1 \qquad \Rightarrow \mathrm{K}_c = \frac{\sqrt{16\omega_{gco}^2+1}\sqrt{36\omega_{gco}^2+1}\sqrt{144\omega_{gco}^2+1}}{8} \qquad \Rightarrow \mathrm{K}_c = 0.4123$$

Ultimate gain $K_u = 2*K_c^{GM=2}$ ($K_u = GM*K_c^{GM}$)

= 1.25 , $\omega_u = \omega_c$

$$\mathrm{P}_u = \frac{2\pi}{\omega_c} = 22.73 \text{ mins}$$

For PI/PID controller design, we set $\tau_I \approx P_u$ =20 mins.

PI controller design:

For Gm=2 , τ_I = 20 mins.

$$G_c = K_C \frac{20s+1}{20s} \quad , \quad G^{OL} = 8K_c \frac{(20s+1)}{20s(4s+1)(6s+1)(12s+1)}$$

$$G_{jw}^{OL} = \frac{8K_c(20j\omega+1)}{(4\omega j+1)(6\omega j+1)(12\omega j+1)(20\omega j)}$$

$$\left|G^{OL}\right| = \frac{8K_c\sqrt{(400\omega^2+1)}}{(16\omega^2+1)(36\omega^2+1)(144\omega^2+1)(20\omega)}$$

$$\phi = \angle G_{OL} = -90^o - \tan^{-1}6\omega - \tan^{-1}4\omega - \tan^{-1}12\omega + \tan^{-1}20\omega \text{ ....(A) .}$$

$$\angle G_{OL} = -180^o \quad \Rightarrow \omega_c = 0.2336 \text{ rad/min}$$

$$\text{for GM=2, } \left|G^{OL}\right|_{\omega_c} = 1 = \frac{1}{GM} = \frac{1}{2} \quad \Rightarrow K_c = 0.4290$$

PI controller for PM = 45^0

At ω_{gco} , $\angle G^{OL} = -135^o \Rightarrow \omega_{gco} = 0.1111$ rad/min

$$\left|G^{OL}\right|_{\omega_{gco}} = 1 \qquad \Rightarrow K_c = 0.25$$

PID controller Design:

GM = 3 PM > 30^0 , τ_I = 20 mins

T_D must be chosen and add +ve phase near ω_c .

$$G_c = K_C \frac{20s+1}{20s}\left(\frac{\tau_D s+1}{0.1\tau_D s+1}\right) \quad \Rightarrow G^{OL} = 8K_c \frac{(20s+1)(\tau_D s+1)}{20s(4s+1)(6s+1)(12s+1)(0.1\tau_D s+1)}$$

$$\phi = \angle G_{OL} = -90^o - \tan^{-1}6\omega - \tan^{-1}4\omega - \tan^{-1}12\omega - \tan^{-1}0.1\tau_D\omega + \tan^{-1}20\omega + \tan^{-1}\tau_D\omega \quad \text{....... (A)}$$

Previously we have found that $\omega_c^P = 0.2764$ rad/min

$$\omega_c^{PI} = 0.25 \text{ rad/min}$$

We expect $\omega_c^{PID} > \omega_c^{PI}$ and ω_c^P

Iteration 1:

Assume ω_c^{PID} ; 0.4 rad/min.

Set $\tau_D = \frac{1}{\omega_c^{PID}} = 2.5$ mins.

From (A),

$$\omega_c\left(\phi = -180^o\right) = 0.573 \text{ rad/min}$$

$$\therefore \left|G^{OL}\right|_{\omega_c} = \frac{1}{GM} = \frac{1}{3} \qquad \Rightarrow K_c = 1.143$$

$\omega_{\phi=-150^o} = 0.217$ rad/min (PM = 30^0)

If PM condition is satisfies, $\left|G^{OL}\right|_{\omega_{\phi=-150^o}} < 1$

$\left|G^{OL}\right|_{\omega_{\phi=-150^o}} = 1.96 \ > \ 1$ $\therefore$ PM condition is not satisfied.

Must adjust τ_D appropriately

Iteration 2:

Set τ_D ; $\dfrac{1}{\omega_{\phi=-180^o}} = \dfrac{1}{0.573} = 1.7 \text{ mins}$

From eqn. (A) $\omega_{\phi=-180^o} = 0.394 \text{ rad/min}$

$\therefore \left|G^{OL}\right|_{\omega_{\phi=-180^o}} = \dfrac{1}{GM} = \dfrac{1}{3} \quad \Rightarrow \text{K}_c = 0.555$

From eqn. (A),

$$\omega_{\phi=-150^o} = 0.186 \text{ rad/min}$$

$$\left|G^{OL}\right|_{\omega_{\phi=-150^o}} = 1.21 > 1$$

PM condition still not satisfied.

Iteration 3:

Lets try τ_D = 1.4

From eqn. (A) $\omega_{\phi=-180^o} = 0.346 \text{ rad/min}$

$$\left|G^{OL}\right|_{\omega_{\phi=-150^o}} = 1.03 \text{ almost equal to } 1$$

PM condition still not satisfied

Iteration 4:

Lets try τ_D = 1.3

From eqn. (A) $\omega_{\phi=-180^o} = 0.3323 \text{ rad/min}$

$$\left|G^{OL}\right|_{\omega_{\phi=-150^o}} = 0.9758 \Rightarrow PM > 30^o$$

PID $\boxed{\text{K}_c = 0.4046, \quad \tau_D = 1.3 \text{ min.}, \quad \tau_I = 20 \text{ min}}$

12.

$$4\frac{dx}{dt}+x=2u \qquad \Rightarrow \frac{x_s}{u_s}=\frac{2}{4s+1}$$
$$6\frac{dy}{dt}+y=2x \qquad \Rightarrow \frac{y_s}{u_s}=\frac{2}{6s+1}$$
$$12\frac{dz}{dt}+z=2y \qquad \Rightarrow \frac{z_s}{y_s}=\frac{2}{12s+1}$$

$$\therefore \frac{z_s}{y_s}.\frac{y_s}{u_s}.\frac{x_s}{u_s}=\frac{z_s}{u_s}=\frac{2}{12s+1}.\frac{2}{6s+1}.\frac{2}{4s+1}$$

$$\therefore \frac{z_s}{u_s}=G_P=\frac{8}{(12s+1)(6s+1)(4s+1)}$$

P Control

$$G_C=K_C \qquad \therefore \text{CLCE} \qquad 1+G_PG_C=0$$

Or

$$(12s+1)(6s+1)(4s+1)+8K_C=0$$
$$p(s)=288s^3+144s^2+22s+1+8K_C=0$$

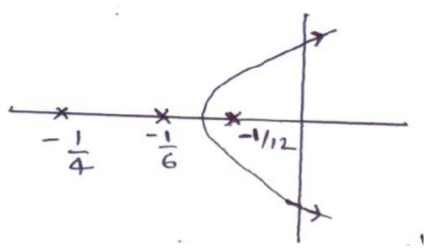
At break point

$$\frac{dp}{ds}=0 \qquad \Rightarrow 3\times 288s^2+144\times 2s+22=0$$
$$\Rightarrow 3\times 144s^2+144s+11=0$$
$$\Rightarrow s=-0.2148,-0.1186$$

Note that

$$G_{OL}=\frac{8K_C}{(12s+1)(6s+1)(4s+1)}$$
$$=\frac{8}{288(s+\frac{1}{12})(s+\frac{1}{6})(s+\frac{1}{4})}$$

Root looks like



Break point root lies between $-\frac{1}{12}$ & $-\frac{1}{6}$

Therefore s=-0.1186 is acceptable

s=-0.2148 is rejected

put s=-0.1186 in

$p(s) = 288s^3 + 144s^2 + 22s + 1 + 8K_C = 0$

To get

$K_C = 8.019 \times 10^{-2}$

break point occurs at $K_C = 8.019 \times 10^{-2}$ where s=-0.1186

for K_u, s=aj satisfies CLCE

$\therefore -288a^3 j - 144a^2 + 22aj + 1 + 8K_c = 0$

$\Rightarrow -144a^2 + 1 + 8K_c = 0$ & $-288a^3 + 22a = 0$

$\Rightarrow 144a^2 = 11$

$\Rightarrow a = 0.2764 = w_u$

$\therefore K_u = \frac{5}{4}$

$P_u = \frac{2\pi}{w_u} = 22.73 \min$

For Zn tuning

Controller type	K_c	T_i	T_d
P	5/8		
PI	2.25/4	18.9417 min	
PID	3/4	11.365 min	2.84 min

13. Differential equation is

$\frac{5d^2y}{dt^2} + 6\frac{dy}{dt} + y = u$

For P controller

$\frac{5d^2y}{dt^2} + 6\frac{dy}{dt} + y = K_C e$

$\frac{5d^2y}{dt^2} + 6\frac{dy}{dt} + (1 + K_C)y = K_C y^{SP}$

CLCE: $5s^2 + 6s + (1 + K_C) = 0$

$$s=-\frac{6}{10}\pm\frac{\sqrt{36-20\left(1+K_C\right)}}{10}$$

$$s=-\frac{6}{10}\pm\frac{2\sqrt{9-5\left(1+K_C\right)}}{10}\Rightarrow s=-\frac{3}{5}\pm\frac{\sqrt{9-5\left(1+K_C\right)}}{5}$$

At critical damping $\Delta=0\Rightarrow K_C=\frac{4}{5}$

For $\varepsilon=\frac{1}{\sqrt{2}},\quad s=-a+aj$ satisfy CLCE.

$$\frac{3}{5}=\frac{\sqrt{9-5\left(1+K_C\right)}}{5}\Rightarrow 9=\left|9-5\left(1+K_C\right)\right|$$

$$\Rightarrow 5\left(1+K_C\right)-9=9\Rightarrow K_C=\frac{13}{5}$$

Never go unstable So, $K_U=\infty$, $P_U=0$

b) PI controller

$$\frac{5d^2y}{dt^2}+6\frac{dy}{dt}+y=K_Ce+\frac{1}{\tau_i}\int edt$$

Differentiate above equation w.r.t 't'

$$\frac{5d^3y}{dt^3}+6\frac{d^2y}{dt^2}+\frac{dy}{dt}=K_C\frac{de}{dt}+\frac{1}{\tau_i}e$$

$$\frac{5d^3y}{dt^3}+6\frac{d^2y}{dt^2}+(1+K_C)\frac{dy}{dt}+\frac{K_C}{\tau_i}y=K_C\left[\frac{dy^{SP}}{dt}+\frac{1}{\tau_i}y^{SP}\right]$$

$$5s^3+6s^2+(1+K_C)s+\frac{K_C}{\tau_i}=0$$

$$5s^3+6s^2+(1+K_C)s+2K_C=0$$

For the ultimate gain and period s= -aj must satisfy CLCE

$$-5a^3j-6a^2+(1+K_C)s+2K_C=0$$

$$aj(1+K_C-5a^2)+(2K_C-6a^2)=0$$

On comparing the real part and complex part

$$a=\frac{1}{\sqrt{2}}\ rad/min$$

$$K_C=K_u=1.5$$

$$P_u=\frac{2\pi}{a}=8.89\ min$$

For $\varepsilon = \frac{1}{\sqrt{2}}, s = -a + aj$ satisfy CLCE

$$5s^3 + 6s^2 + (1+K_C)s + 2K_C = 0$$

$$10a^3(1+j) - 12a^2 j - (1+K_C)a + (1+K_C)aj + 2K_C = 0$$

$$j\left(10a^3 - 12a^2 + (1+K_C)a\right) + \left[10a^3 - (1+K_C)a + 2K_C\right] = 0$$

$$10a^3(1+j) - 12a^2 j - (1+K_C)a + (1+K_C)aj + 2K_C = 0$$
$$j\left(10a^3 - 12a^2 + (1+K_C)a\right) + \left[10a^3 - (1+K_C)a + 2K_C\right] = 0$$
$$\Rightarrow 10a^2 - 12a + (1+K_C) = 0 \Rightarrow (1+K_C) = 12a - 10a^2$$

$$\Rightarrow 10a^3 - (1+K_C)a + 2K_C = 0$$

On solving we get

$$a = 0.7527 \pm 0.7008j, 0.09455$$

For a=0.09455, $K_C = 12a - 10a^2 - 1$

$K_C = 0.04518$ for $\varepsilon = \frac{1}{\sqrt{2}}$

At critical damping, CLCE has separated roots.

At separated root s=a, p =0

$\frac{dp}{ds} = 0$, p=0 is CLCE.

$$p = 5s^3 + 6s^2 + (1 + K_C)s + 2K_C = 0$$

$$\frac{dp}{ds} = 15s^2 + 12s + 1 + K_C = 0$$

$$1 + K_C = -15s^2 - 12s$$

Put the value in above

$$5s^3 + 6s^2 - (15s^2 + 12s)s + 2(-15s^2 - 12s) - 2 = 0$$

On simplifying

$$5s^3 + 18s^2 + 12s + 1 = 0$$

s=-2.7553, -0.7476, -0.09709

$$K_C = -15s^2 - 12s - 1$$

$$K_C = -81.81, -0.4125, 0.02370$$

So $K_C = 0.02370$ for critical damping

14. Controller design for inverse response process

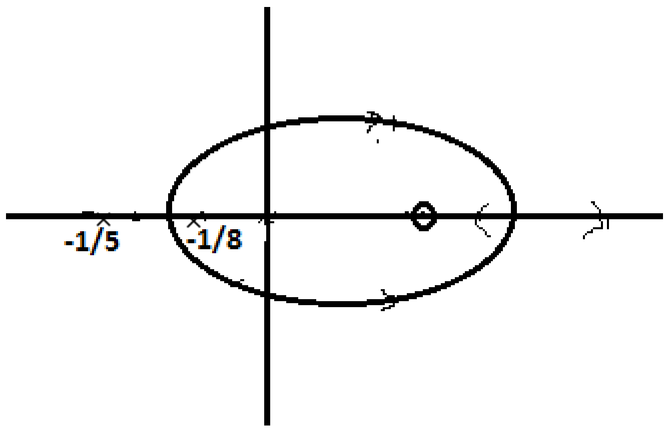
$$G_P = \frac{-2s+1}{(8s+1)(5s+1)}$$

P controller

CLCE: $1 + K_c\left(-\frac{1}{20}\right)\frac{s-0.5}{\left(s+\frac{1}{8}\right)(s+0.2)} = 0$

$$\frac{s-0.5}{\left(s+\frac{1}{8}\right)(s+0.2)} = \frac{20}{K_c}$$

$\therefore$ RHS is +ve real condition is $\angle G_{OL}^{j\omega} = 2n\pi \Rightarrow n = 0,1,2,3....$



Want CLCE root such that $\tau = \frac{1}{\sqrt{2}} \Rightarrow s = -a + aj$ satisfies CLCE

$$(8s+1)(5s+1) + K_c(-2s+1) = 0$$

$$40s^2 + 13s + 1 + K_c(-2s+1) = 0 \quad \text{or } 40s^2 + (13-2K_c)s + 1 + K_c = 0$$

Put $s = -a + aj$

$$40a^2(-1+j)^2+(13-2K_c)a(-1+j)+1+K_c=0$$
$$40a^2(-2j)+(13-2K_c)a(-1+j)+1+K_c=0$$
$$1+K_c-(13-2K_c)a+j\left[-8a^2+(13-2K_c)a\right]=0$$
$$\Rightarrow 13-2K_c=8a \qquad \& \; 1+K_c-(13-2K_c)a=0$$
$$\Rightarrow K_c=\frac{13-8a}{2} \qquad \therefore\; 1+\frac{13-8a}{2}-8a^2=0 \Rightarrow 16a^2+8a-15=0$$
$$\Rightarrow a=0.75,-1.25$$

We retain the +ve a else we get the unstable root. $\therefore a=0.75$

$K_c=3.5$ so, $s=-0.75\pm0.75j$ is the CLCE root.

PI controller design:

Set τ_I to cancel dominant G_P pole

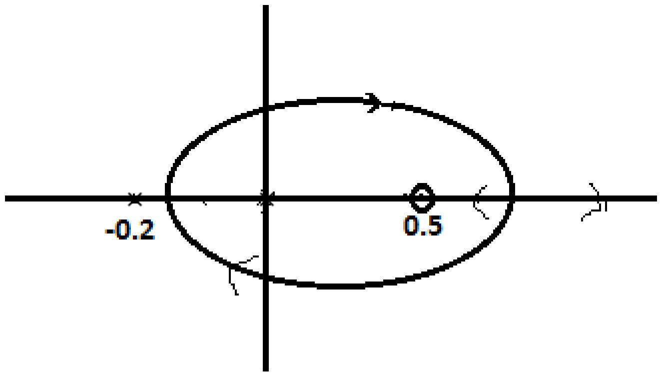
$\Rightarrow \tau_I=8$ mins.

$$\therefore G_c=\frac{8s+1}{8s}, \quad G_{OL}=\frac{-2s+1}{(5s+1)(8s+1)}\frac{8s+1}{8s}=-0.05\frac{s-0.5}{(s+0.2)s}$$

With the modified angle condition root locus looks like

CLCE: $8s(5s+1)+(-2s+1)K_c=0$

$$40s^2+(8-2K_c)s+K_c=0$$
$s=-a+aj$ must satisfy CLCE. for $\varepsilon=0.7071$
$$40a^2(-1+j)^2+(4-K_c)2a(-1+j)+K_c=0$$
$$40a^2(-2j)+(4-K_c)2a(-1+j)+K_c=0$$
$$K_c-2a(4-K_c)+2aj(4-K_c-40a)=0$$
$$\therefore 4-K_c=40a \qquad \& \quad K_c-2a(4-K_c)=0$$
$$or\; K_c=4(1-10a) \quad \text{or} \quad 4-40a-80a^2=0$$



$$\text{or } 20a^2+10a-1=0 \Rightarrow a=0.0854,-0.5854$$

Taking +ve a, we have $K_c=0.5836$ with $s=-0.0854+0.0854j$ as the CLCE root.

PID controller design

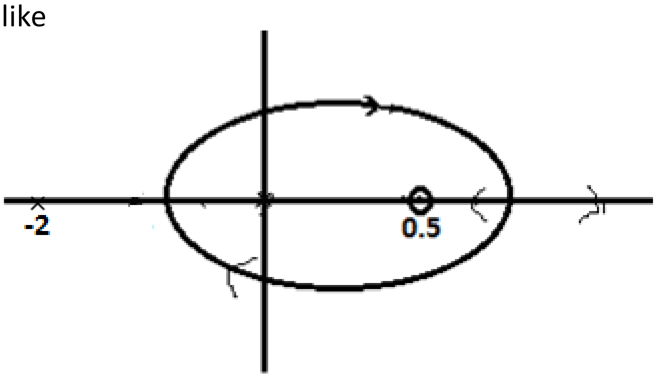
Set τ_I to cancel dominant G_p pole

Set τ_D to cancel fastest G_p pole.

$\Rightarrow \tau_I = 8$ mins. & $\tau_D = 5$ mins.

$$\therefore \mathrm{G}_c = \frac{8s+1}{8s}\frac{5s+1}{0.5s+1}, \quad G_{OL} = \frac{-2s+1}{(5s+1)(8s+1)}\frac{8s+1}{8s}\frac{5s+1}{0.5s+1} = -0.5\frac{s-0.5}{(s+2)s}$$

With the modified angle condition root locus looks like



for $\varepsilon = 0.7071$

CLCE: $8s(0.5s+1)+(-2s+1)K_c = 0$

$$4s^2 + (8-2K_c)s + K_c = 0$$

$s = -a + aj$ must satisfy CLCE.

$$4a^2(-1+j)^2 + (4-K_c)2a(-1+j) + K_c = 0$$
$$4a^2(-2j) + (4-K_c)2a(-1+j) + K_c = 0$$
$$K_c - 2a(4-K_c) + 2aj(4-K_c-4a) = 0$$
$$\therefore 4-K_c = 4a \quad \& \quad \mathrm{K}_c - 2a(4-K_c) = 0$$
$$or\ K_c = 4(1-a) \quad \text{or} \quad 4-4a-8a^2 = 0$$

$$\text{or } 2a^2 + a - 1 = 0 \Rightarrow a = 0.5, -1$$

Taking +ve a, we have $K_c = 2$ with $s = -0.5 + 0.5j$ as the CLCE root.

$$K_c = 2 \;\; >> \;\; K_c^{PI} \;\; \& \;\; K_c^{P}$$

Derivative action significantly improves K_c for same ε

15. We have given the transfer function for first order level control. If we compare it with the general form of first order system, we will observe that time constant of 5 and gain of 1.

$$G(s) = \frac{k}{\tau s + 1}$$

$$G(s) = \frac{1}{5s + 1}$$

Expression for first order system in time domain

$$\tau\frac{dy}{dt} + y = k$$

Expression for PI controller in time domain

$$u = k_C e\{1 + \frac{1}{\tau_i}\int edt\ \}$$

We know

$$\tau\frac{dy}{dt}+y=kk_C(y_{sp}-y)\{1+\frac{1}{\tau_i}\int(y_{sp}-y)dt\ \}$$

$$\tau\frac{d^2y}{dt^2}+\frac{dy}{dt}(1+kk_C)+\frac{y(kk_C)}{\tau_i}=\{y_{sp}+\tau_i\frac{dy_{sp}}{dt}\}\frac{kk_C}{\tau_i}$$

Characteristic Equation

$$\tau\lambda^2+\lambda(1+kk_C)+\frac{(kk_C)}{\tau_i}=0$$

$$\lambda^2+\lambda\frac{(1+kk_C)}{\tau}+\frac{(kk_C)}{\tau\tau_i}=0$$

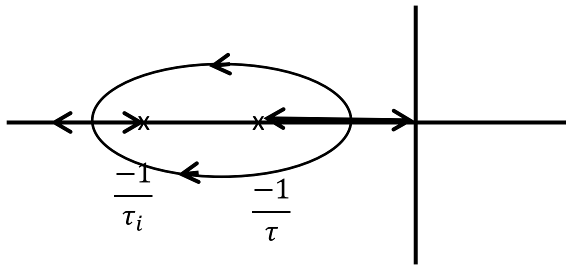
On solving this equation by quadratic formula

$$\lambda=-\frac{(1+kk_C)}{2\tau}\pm\frac{(1+kk_C)}{\tau}\sqrt{1-\frac{4k\tau}{\tau_i}\frac{(k_C)}{(1+kk_C)^2}}$$

Now for judging the nature of roots

$D=(1-kk_C)^2+4kk_C(1-r)$ $\qquad r=\frac{\tau}{\tau_i}$

Case 1 -For complex root r >1 $\qquad \tau>\tau_i$

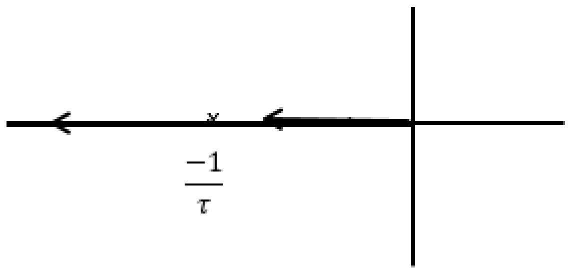


Hence a PI controller is stable.

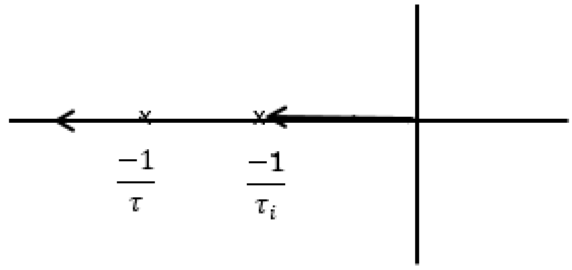
Case 2 - when r=1

$D=(1-kk_C)^2$

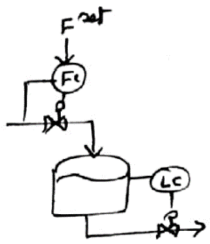
$\tau=\tau_i$



Case 3 – when r<1, $\tau < \tau_i$



16.



$G_p = -\frac{K_p}{s}$ Controller gain $= -K_c$ $K_c > 0$.

$G_c = -K_c \frac{\tau_I s+1}{\tau_I s}$ $G_{OL} = G_c G_p$

$$= K_p K_c \frac{(\tau_I s+1)}{\tau_I s^2}$$

2 poles & one zero
$s = 0$ $s = -\frac{1}{\tau_I}$

Root Locus

As $K_c \to K_c + \Delta K_c$, $\left|\begin{matrix}\text{Real part of CLCE} \\ \text{root}\end{matrix}\right|$ increases $\Rightarrow \varepsilon \downarrow$.

$\left|\begin{matrix}\text{Im part of CLCE} \\ \text{root}\end{matrix}\right|$ decreases

Thus as K_c is increased, underdamping decreases as long as $K_c < K_c$ for break-in pt.

17.

$$G_{OL} = \frac{2K_c}{(s-1)(2s+1)(\frac{1}{2}s+1)}$$

CLCE: $1 + G_{OL} = 0$

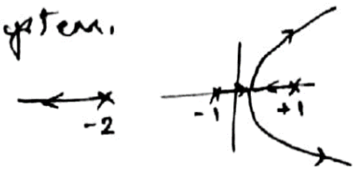
$(s-1)(2s+1)(\frac{1}{2}s+1) + 2K_c = 0$

$\equiv (s-1)(s^2 + \frac{5}{2}s + 1) + 2K_c = 0$

$\therefore$ CLCE $\equiv s^3 + \frac{3}{2}s^2 - \frac{3}{2}s + (2K_c - 1) = 0$

Since a coef of CLCE is -ve, an RHP root is guaranteed $\forall K_c$

$\therefore$ P controller cannot stabilize the open loop unstable system.



$\Rightarrow$ Always unstable $\forall K_c$

Root locus.