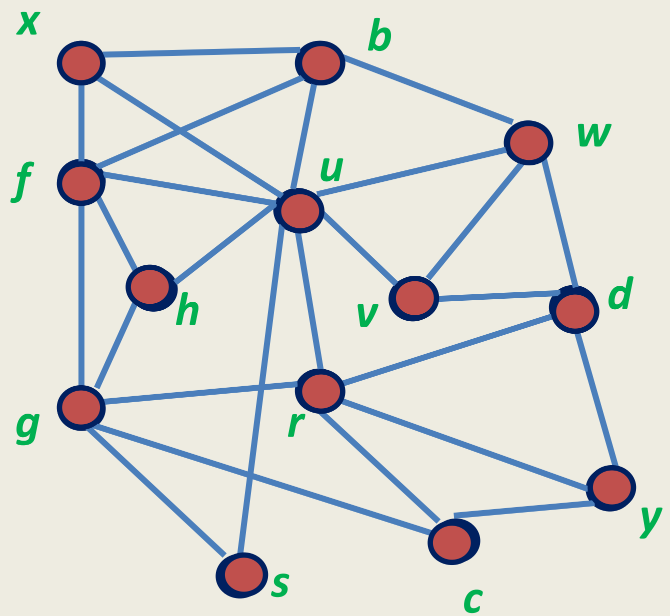
Data Structures and Algorithms

(ESO207)

Lecture 25

- A data structure problem for graphs.
- Depth First Search (DFS) Traversal
- Novel application: computing biconnected components of a graph

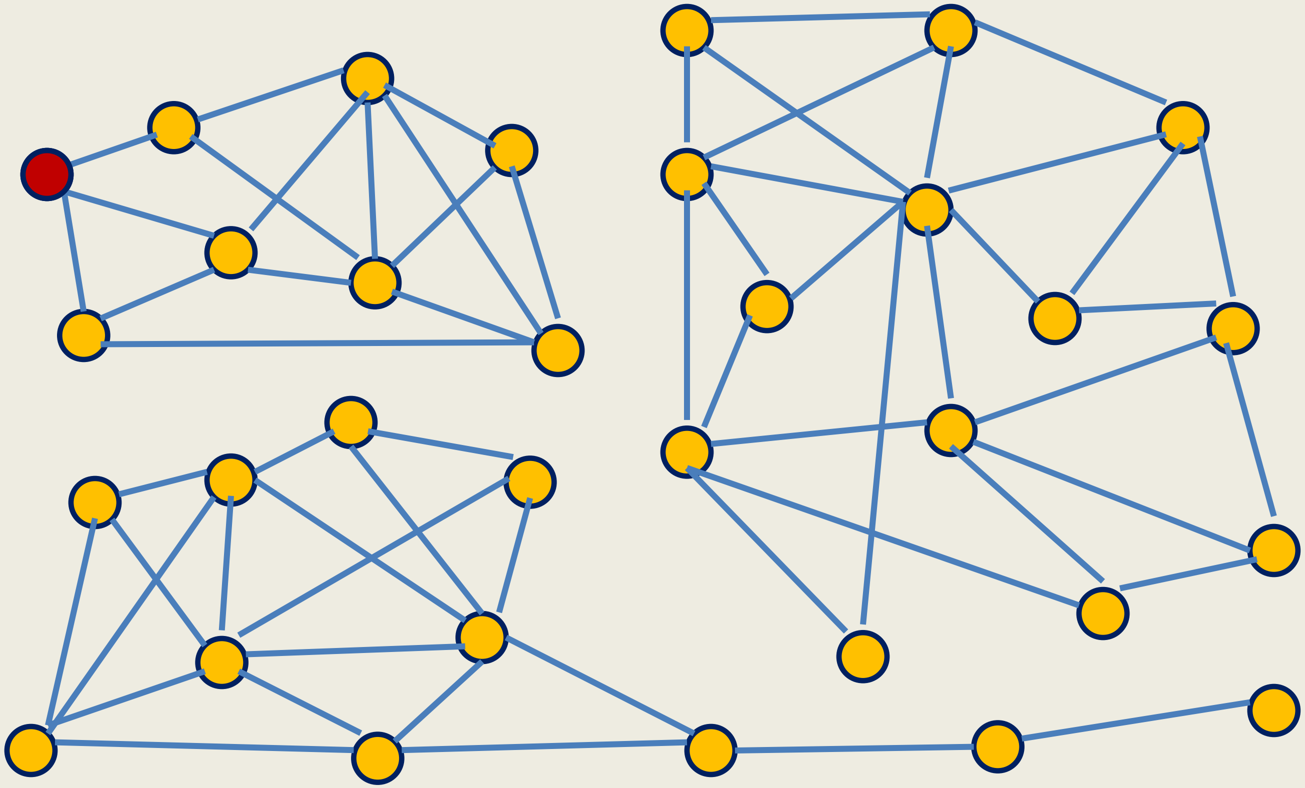
BFS Traversal
in Undirected Graphs



Theorem:

BFS Traversal from ***x*** visits all vertices reachable from ***x*** in the given graph.

Connectivity problem in a Graph

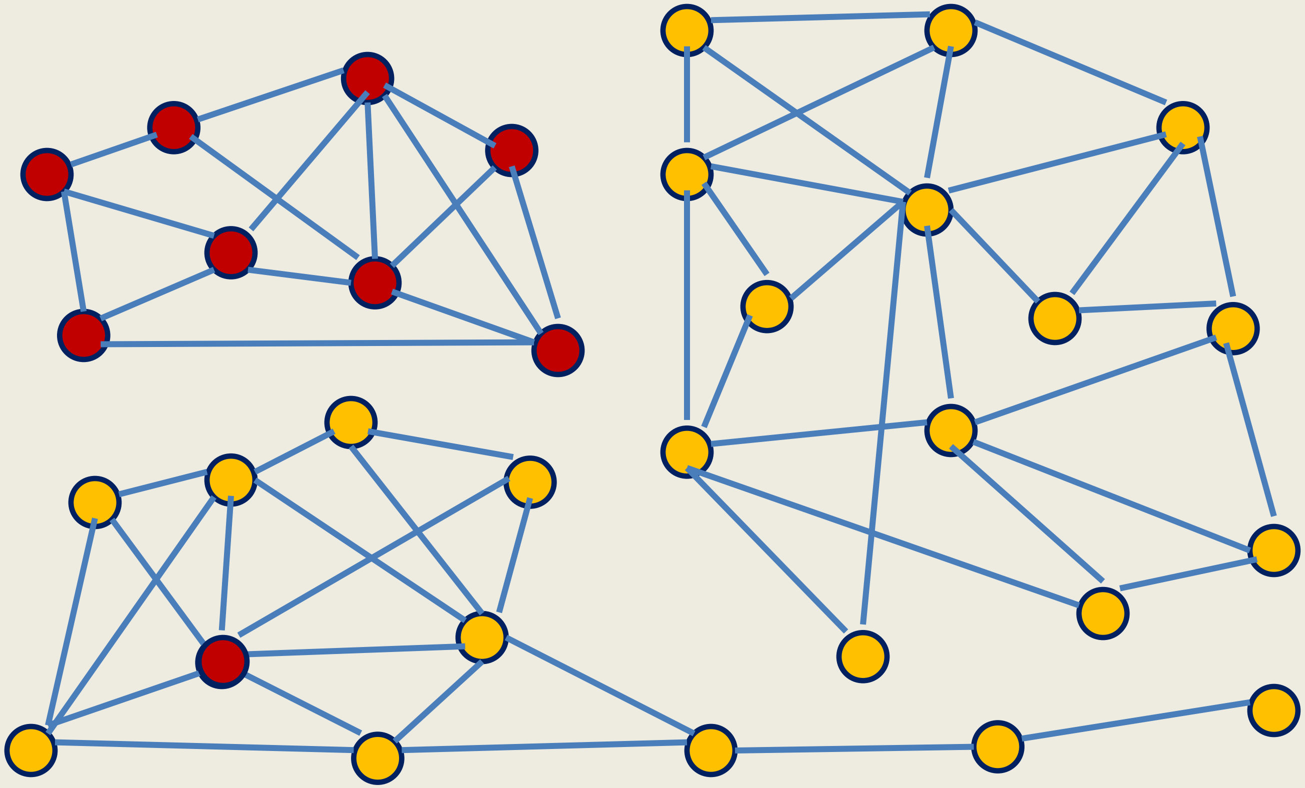


Problem:

Build an $\mathbf{O}(n)$ size data structure for a given undirected graph s.t. the following query can be answered in $\mathbf{O}(1)$ time.

Is vertex i reachable from vertex j ?

Connectivity problem in a Graph



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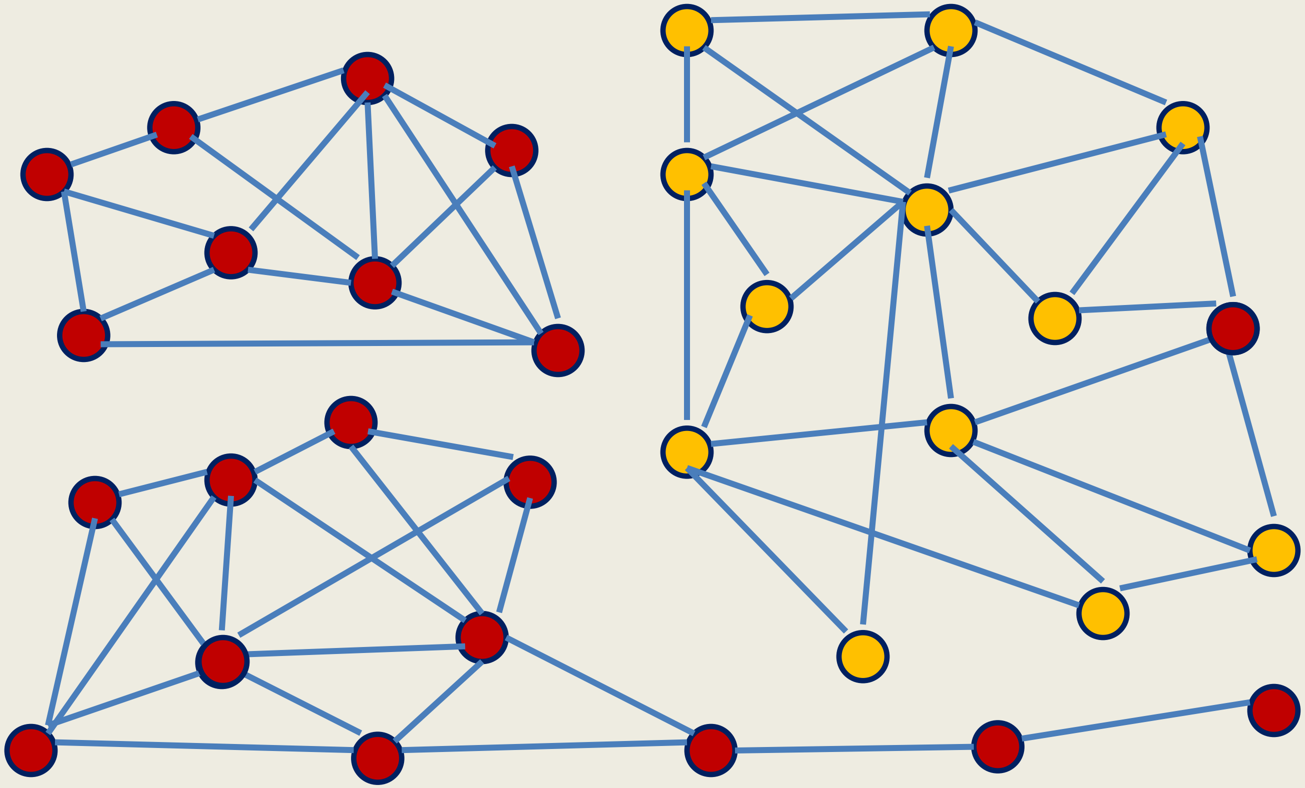

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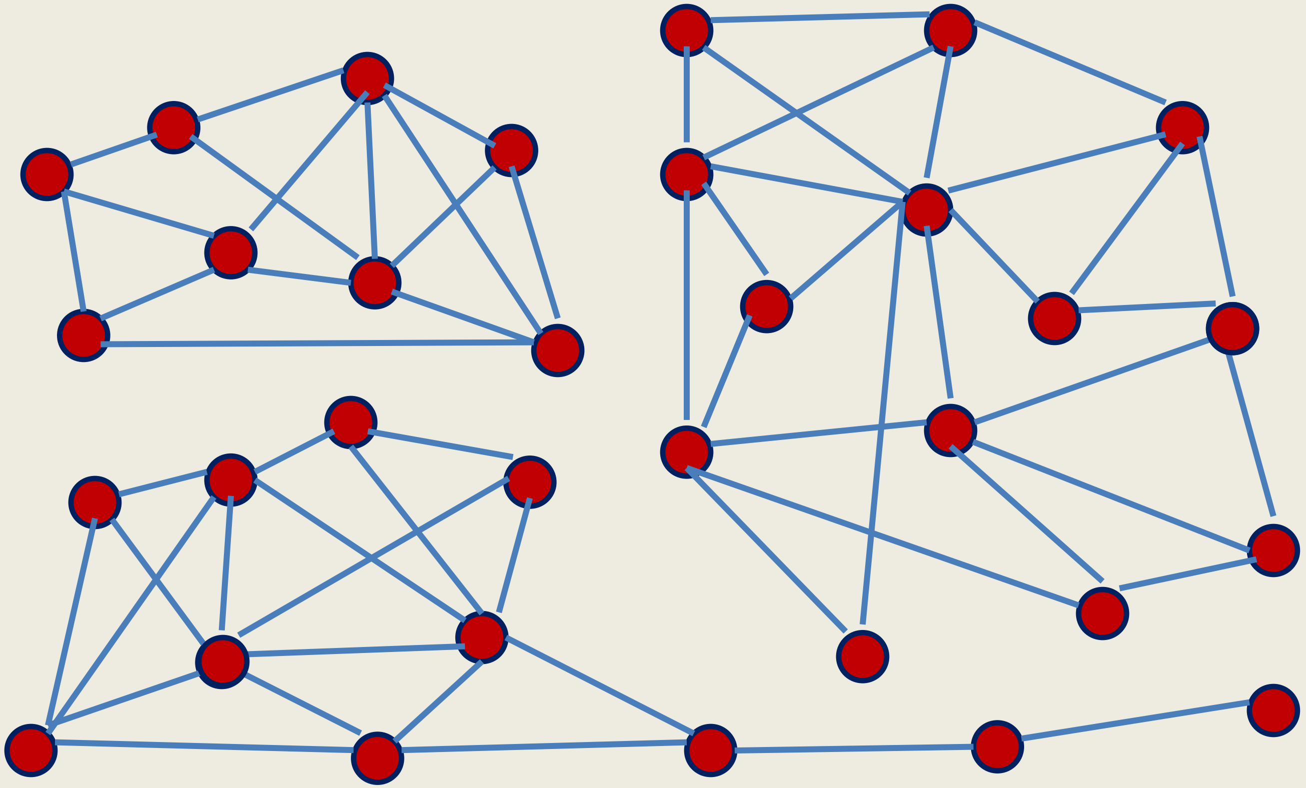


Problem:

Build an **O**(n) size data structure for a given undirected graph s.t.

the following query can be answered in **O**(1) time.

Is vertex i reachable from vertex j ?

Connectivity problem in a Graph

```
BFS(x)
    CreateEmptyQueue(Q);
    Visited(x) ← true;    Label[x] ← x;
    Enqueue(x,Q);
    While(Not IsEmptyQueue(Q))
    {        v ←Dequeue(Q);
             For each neighbor w of v
             {       if (Visited(w) = false)
                         {  Visited(w) ← true ;    Label[w] ← x ;
                            Enqueue(w, Q);
                         }
             }
    }
Connectivity(G)
{  For each vertex x Visited(x)← false;   Create an array Label;
   For each vertex v in V
          If (Visited(v) = false)    BFS(x);
   return Label;
}
```

Analysis of the algorithm

Output of the algorithm:

Array **Label[]** of size **O**(n) such that

Label[**x**]=**Label**[**y**] if and only if **x** and **y** belong to same connected component.

Running time of the algorithm :

O($n + m$)

Theorem:

An undirected graph can be processed in **O**($n + m$) time

to build an **O**(n) size data structure

which can answer any connectivity query in **O(1)** time.

Is there alternate way to traverse a graph ?

Yes

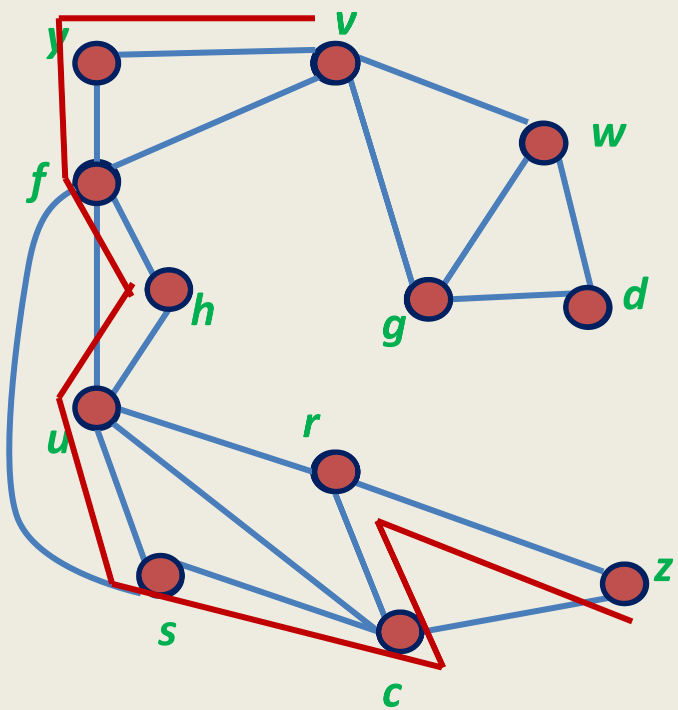
More Natural

More Powerful

Try to get inspiration from your **"<u>human executable method</u>"** to design "**a machine executable algorithm**" for traversing a graph.

How will you do it without any **<u>map</u>** or asking any one for **<u>directions</u>** ?

A recursive way to traverse a graph



We need a **mechanism** to

- **Avoid** visiting a vertex multiple times

We can solve it by keeping a label "**Visited**" for each vertex like in BFS traversal.

- **Trace back** in case we reach a dead end.

Recursion takes care of it ☺

DFS traversal of *G*

```
DFS(v)
{  Visited(v) ← true;
   For each neighbor w of v
   {        if (Visited(w)  = false)
            {   DFS(w) ;
                 ……..;
            }
           ……..;
   }
}
```

Add a few extra statements here
to get an efficient algorithm
for a **new problem** ☺

```
DFS-traversal(G)
{  For each vertex vϵ V  {      Visited(v)← false;       }
    For each vertex v ϵ V {
                                   If (Visited(v ) = false)    DFS(v);
                              }
}
```

DFS traversal

a milestone in the area of graph algorithms

Invented by **Robert Endre Tarjan in 1972**

- One of the **pioneers** in the field of data structures and algorithms.
- Got the **Turing award**
  (equivalent to **Nobel prize**)
  for his fundamental contribution to
  data structures and algorithms.
- **DFS traversal** has proved to be a very powerful tool for graph algorithms.

DFS traversal

a milestone in the area of graph algorithms

Applications:

- **Connected** components of a graph.
- **Biconnected** components of a graph.

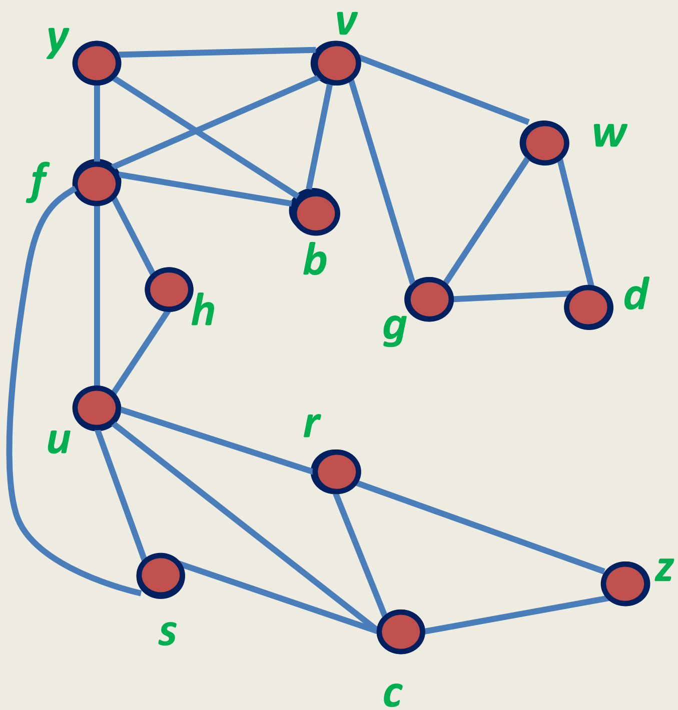
  (Is the connectivity of a graph robust to failure of any node ?)
- Finding **bridges** in a graph.

  (Is the connectivity of a graph robust to failure of any edge)
- **Planarity testing** of a graph

  (Can a given graph be embedded on a plane so that no two edges intersect ?)
- **Strongly connected** components of a directed graph.

  (the extension of connectivity in case of directed graphs)

Insight into DFS through an example



DFS(*v*) begins

v visits *y*

DFS(*y*) begins

y visits *f*

DFS(*f*) begins

f visits *b*

DFS(*b*) begins

all neighbors of *b* are already visited

DFS(*b*) ends

control returns to DFS(*f*)

f visits *h*

DFS(*h*) begins

…. and so on ….

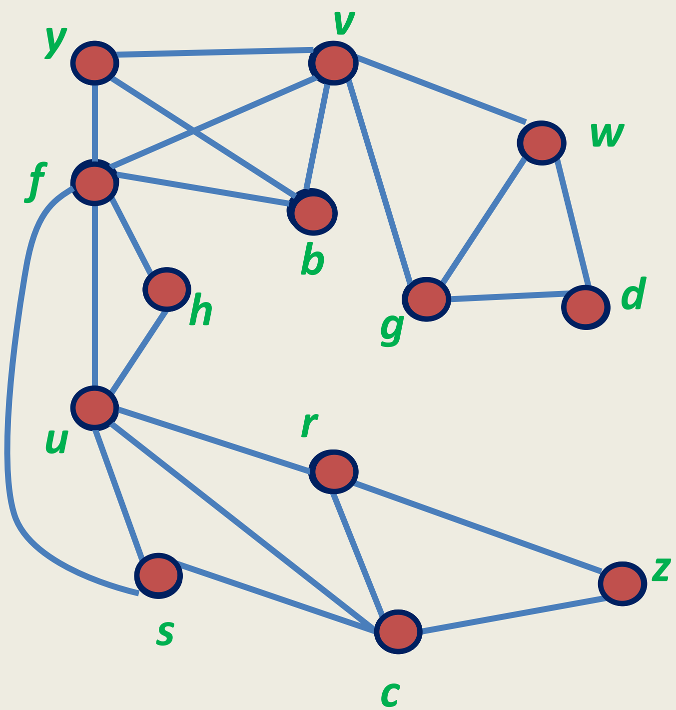
After visiting ***z*****,** control returns to ***r*→*c*→*s*→*u*→*h*→*f*→*y*→*v***

v visits *w*

DFS(*w*) begins

…. and so on ….

Insight into DFS through an example

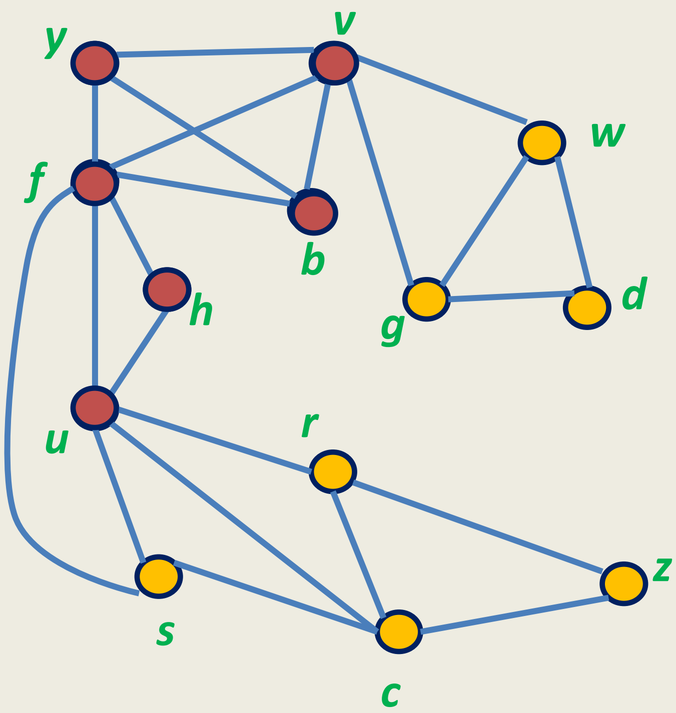


Observation1: (Recursive nature of **DFS**)

If **DFS**(*v*) invokes **DFS**(*w*), then

DFS(*w*) finishes **before** **DFS**(*v*).

Insight into **DFS** through an example

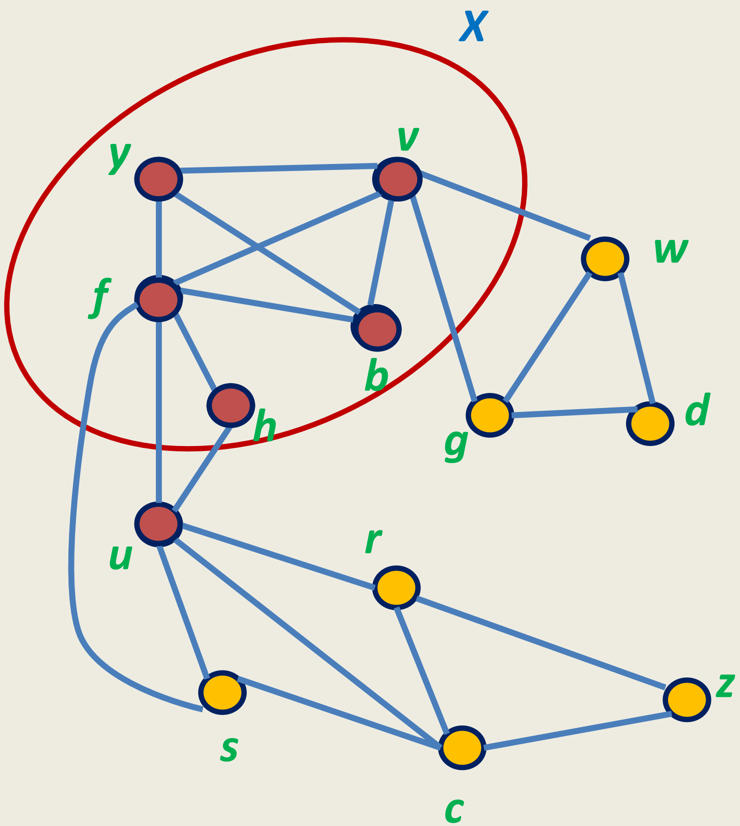


Question :

When **DFS** reaches a vertex **u**, what is the role of vertices already visited ?

The traversal will not proceed along the vertices which are **already visited**. Hence the visited vertices act as a **barrier** for the traversal from **u**.

Insight into DFS through an example



Observation 2:

Let X be the set of vertices visited before **DFS** traversal reaches vertex **u** for the first time.

The **DFS(u)** pursued now is like

fresh **DFS(u)** executed in graph **G**\\X.

NOTE:

G\\X is the graph **G** after removal of all vertices X along with their edges.

Proving that
DFS(v) visits all vertices reachable from v

By **induction** on the

size of connected component of v

Can you figure out the **inductive assertion** now?

Think over it. It is given on the following slide…

Inductive assertion

A(***i***):

If a connected component has **size** = ***i***, then **DFS** from any of its vertices **will visit** all its vertices.

PROOF:

Base case: ***i*** =1.

The component is {***v***} and the first statement of **DFS(*v*)** marks it visited.

So ***A***(***1***) holds.

Induction hypothesis:

If a connected component has **size** < ***i***, then **DFS** from any of its vertices **will visit** all its vertices.

Induction step:

We have to prove that ***A***(***i***) holds.

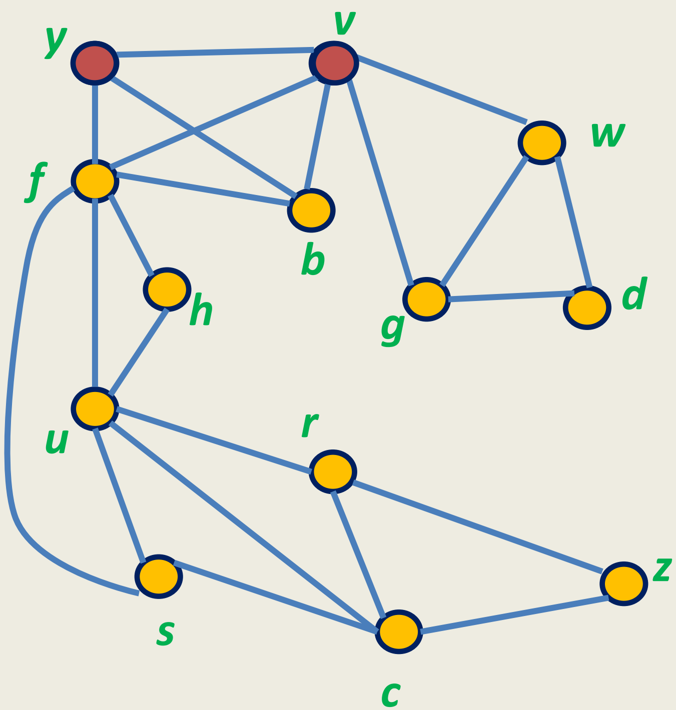
Consider any connected component of size ***i***.

Let ***V**** be the set of its vertices. |***V****|= ***i***.

Let ***v*** be any vertex in the connected component.

Watch the following slides
very slowly and **very carefully**.

DFS(v)



Let y be the first neighbor visited by v.

B= the set of vertices such that **every path** from

y to them passes through v.

C= V^*\B.

B= {v, g, w, d} |B|< i since y∉ B

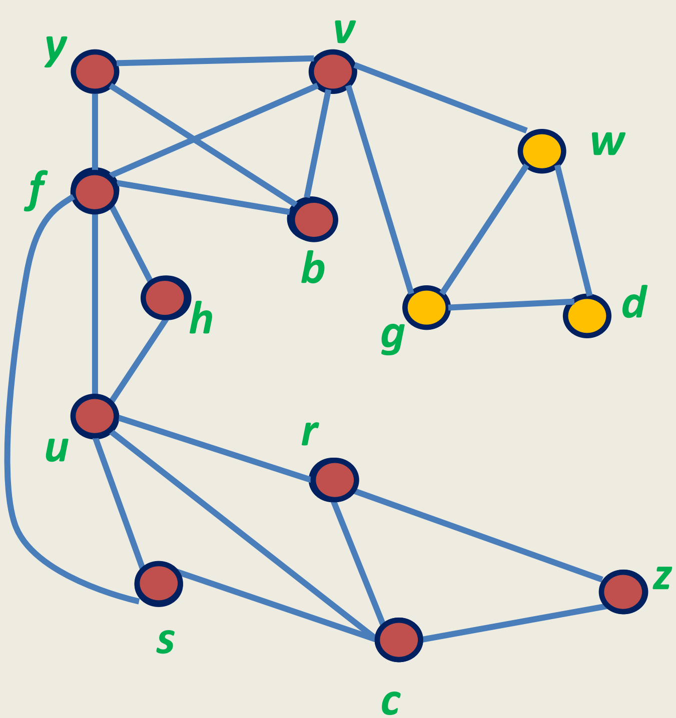
C= {y, b, f, h, u, s, c, r, z} |C|< i since v∉ C

Question: What is DFS(y) like ?

Answer: DFS(y) in **G\{v}.**

Question: What is the connected component of y in **G\{v}** ?

DFS(v)



Let *y* be the first neighbor visited by *v*.

B= the set of vertices such that **every path** from *y* to them passes through *v*.

C= *V**\B.

B= {*v*, *g*, *w*, *d*}

|B|< i since y∉ B

C= {*y*, *b*, *f*, *h*, *u*, *s*, *c*, *r*, *z*}

|C|< i since v∉ C

Question: What is DFS(y) like ?

Answer: DFS(y) in G\{*v*}.

Question: What is the connected component of *y* in G\{v} ?

Answer: C.

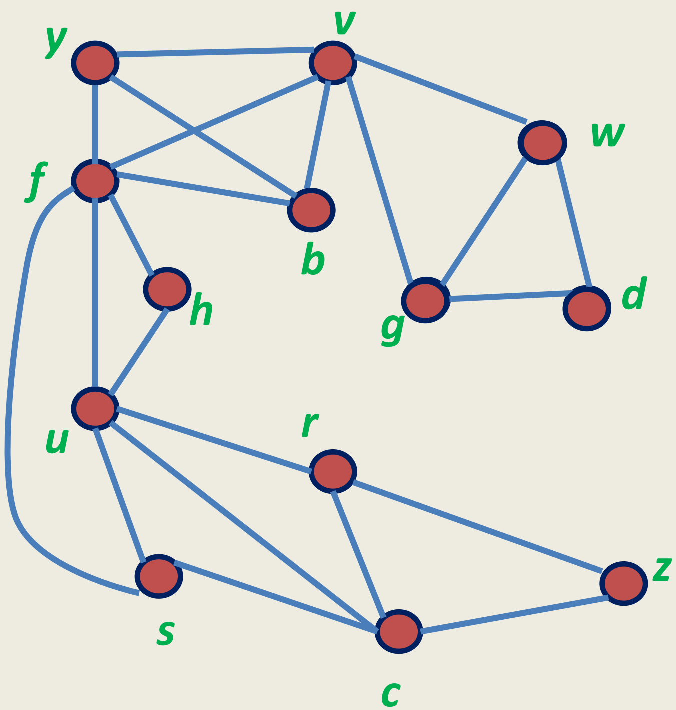
|C|< i, **so by I.H.,** DFS(*y*) visits entire set C & we return to v.

Question: What is DFS(*v*) like when DFS(*y*) finishes ?

Answer: DFS(v) in G\C.

Question: What is the connected component of *v* in G\C ?

DFS(v)



Hence entire component of *v* gets visited

Let *y* be the first neighbor visited by *v*.

B= the set of vertices such that **every path** from *y* to them passes through *v*.

C= *V**\B.

B= {*v*, *g*, *w*, *d*} $\leftarrow$ |B|< i since y∉ B

C= {*y*, *b*, *f*, *h*, *u*, *s*, *c*, *r*, *z*} $\leftarrow$ |C|< i since v∉ C

Question: What is **DFS**(**y**) like ?

Answer: DFS(**y**) in **G\{*v*}.**

Question: What is the connected component of ***y*** in **G\{v}** ?

Answer: C.

|C|< i, so by I.H., DFS(***y***) visits entire set C & we return to **v**.

Question: What is **DFS**(***v***) like when **DFS**(***y***) finishes ?

Answer: DFS(**v**) in **G\C.**

Question: What is the connected component of ***v*** in **G\C** ?

Answer: B.

|B|< i, so by I.H., DFS(***v***) pursued after finishing **DFS**(***y***) visits entire set B.

Theorem: **DFS(*v*)** visits all vertices of the connected component of ***v***.

Homework:

Use **DFS** traversal to compute all connected components of a given **G** in time **O**($\boldsymbol{m} + \boldsymbol{n}$).