Mathematical models of stock prices

Additive Models

Consider discrete time models, with time points $k = 0, 1, \dots, N$
Price at k is $S(k)$, and price at any one time is dependent to some extent on previous prices.

$$S(k+1) = aS(k) + u(k)$$

- $(a > 1)$, and $u(k)$ is r.v. that induces "shocks" on prices. (usually independent normally distributed). $u(k) \sim N(0, \sigma^2)$
- $S(0)$ could be specified or is 1.

Then the recursive expression for $S(k)$ is,

$$S(k) = a^k S(0) + \underbrace{\sum_{i=1}^{k} a^{k-i} \cdot u(i-1)}_{\text{Sum of normal r.v.}}$$

Therefore, $S(k)$ is normal with mean, $E(S(k)) = a^k S(0)$

Limitations:

1. Normal r.v. can take $-ve$ values; but real stock prices are never $-ve$.
2. If a stock with current price ₹1, and $\sigma = ₹.5$, drifts upward to ₹100. Then it is unlikely that its σ would remain ₹ 0.5. Stand dev. ~~should~~ would be proportional to price.
3. May be useful for localized analysis

Multiplicative Model

$$S(k+1) = S(k) \cdot u(k), \quad k = 0, 1, \ldots N-1$$

$$\frac{S(k+1)}{S(k)} = u(k)$$

relative change.

$$\ln S(k+1) = \ln S(k) + \ln u(k) \qquad (1)$$

Let $w(k) = \ln u(k) =$ independent normal disturbances.

$$w(k) \sim N(v, \sigma^2)$$

$$u(k) = e^{w(k)}$$

Lognormal

Note $u(k)$ is always +ve.

$$S(k) = u(k-1) \cdot u(k-2) \cdots u(0) \cdot S(0)$$

$$\ln S(k) = \ln S(0) + \sum_{i=0}^{k-1} \ln u(i)$$

$$= \underbrace{\ln(S_0)}_{\text{const}} + \underbrace{\sum_{i=0}^{k-1} w(i)}_{\text{normal}}$$

$\ln(S(k)) =$ normal

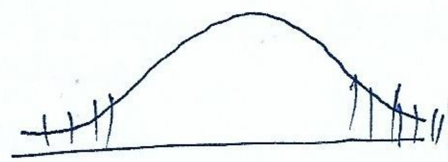
$$\left.\begin{aligned} E\left[\ln(S(k))\right] &= \ln(S_0) + k \cdot v \\ Var\left[\ln(S(k))\right] &= k \cdot \sigma^2 \end{aligned}\right\} \quad -(A)$$

Both are linearly related with k. (periods)

Properties of Real Stock prices

1. close to lognormal, $k = 1$ week
2. Tails acount for a greater share of prob. than lognormal

"fat tails"



3. Skewness (-ve): lower tail is heavier than the upper.
   - prices drop quickly but recover slowly.

4. For portfolio these observations are not serious becaused pool of many asset cancel extreme events (in general).

But prevalent in derivatives, such as stock options.

Effect of period Length

$K = 52k = Ck$.

$$\ln\left(\frac{S_1}{S_0}\right) = w(1), \ldots, \ln\left(\frac{S_{52}}{S_{51}}\right) = w(52)$$

$$\ln\left(\frac{S_{52}}{S_0}\right) = \ln\left(\frac{S_1}{S_0}\right) + \cdots + \ln\left(\frac{S_{52}}{S_{51}}\right) = w(1) + \cdots + w(52)$$

$$\cancel{= 52 \times w(1)}$$

$$\cancel{= 52 \times N(v, \sigma^2)}$$

$$= N(52v, 52\sigma^2)$$

$$= N(Kv, K\sigma^2)$$

$$= (Kv, \sqrt{K}\sigma)$$

Random Walks and Wiener Process

Suppose N periods of length Δt.

Define the process, $Z(t_{k+1}) = Z(t_k) + \epsilon(t_k)\cdot\sqrt{\Delta t}$

(Random Walk) $\qquad t_{k+1} = t_k + \Delta t, \quad k = 0, 1, \dots N$

$\epsilon(t_k)$ = normal r.v. with mean 0 and variance 1.

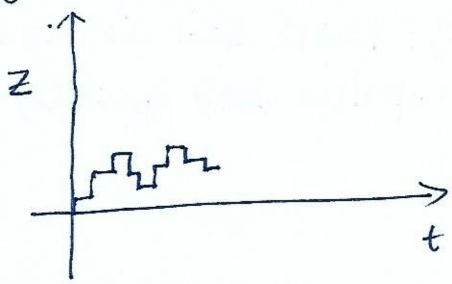
(Standard normal r.v.)

$\downarrow$

mutually uncorrelated

$$E\left[\epsilon(t_i)\cdot\epsilon(t_j)\right] = 0, \quad \forall\, i \neq j$$

$Z(t_0) = 0,$



For $(j > k)$, $Z(t_k) - Z(t_j)$

$$= \cancel{\sum_{i=j}^{k} \epsilon(t_j)\cdot\sqrt{\Delta t}} = \sum_{i=j}^{k-1} \epsilon(t_i)\cdot\sqrt{\Delta t}$$

$\downarrow$

Normal r.v.

$$E\left[Z(t_k) - Z(t_j)\right] = 0$$

and $\operatorname{var}\left(Z(t_k) - Z(t_j)\right] = \sum_{i=j}^{k-1} \Delta t = (k-j)\Delta t = t_k - t_j$

Note that for $t_{k_1} < t_{k_2} \leq t_{k_3} < t_{k_4}$, $\quad Z(t_{k_2}) - Z(t_{k_1})$ & $Z(t_{k_4}) - Z(t_{k_3})$

are uncorrelated. Why?

(Wiener process) obtained by taking the limit of the random walk, $\Delta t \to 0$

$$dZ = \epsilon(t)\sqrt{dt}, \quad \epsilon(t) \sim N(0,1)$$

and $\epsilon(t')$, $\epsilon(t'')$ are uncorrelated.

Wiener process (Brownian motion)

1. For any $s<t$, $Z(t)-Z(s) \sim N(0, \underset{\text{var}}{\underset{\downarrow}{t-s}})$

2. $0 \le t_1 < t_2 \le t_3 < t_4$, $\text{corr}\left(Z(t_2)-Z(t_1),\, Z(t_4)-Z(t_3)\right) = 0$

3. $Z(t_0) = 1$ with prob. 1.

Note that Wiener process is not differentiable,

Loose explanation, $E\left[\dfrac{Z(t)-Z(s)}{t-s}\right]^2$

$t \to s,\ t > s$

$$= \frac{1}{(t-s)^2} \times (t-s) \to \infty$$

Generalized Wiener Process

$$dx(t) = a\,dt + b\,dz;\quad dz = \epsilon\sqrt{dt},\quad \epsilon \sim N(0,1)$$

a, b are constants.

Ito process: $dx(t) = a(x,t)\,dt + b(x,t)\,dz \qquad (11)$

(a more general process)

Frequently used to describe the behavior of financial assets.

Ito's Lemma: Suppose that the random process, x, is defined by the eq. (11), and $y(t) = F(x,t)$. Then

$$dy(t) = \left(\frac{\partial F}{\partial x}a + \frac{\partial F}{\partial t} + \frac{1}{2}\cdot\frac{\partial^2 F}{\partial x^2}b^2\right)dt + \frac{\partial F}{\partial x}\cdot b\,dz \qquad (11a)$$

Result if ordinary calcus is appied,

$$dy(t) = \frac{\partial F}{\partial x} \cdot dx + \frac{\partial F}{\partial t} \cdot dt$$

$$= \frac{\partial F}{\partial x}(adt + bdz) + \frac{\partial F}{\partial t} \cdot dt = \left(\frac{\partial F}{\partial x} \cdot a + \frac{\partial F}{\partial t}\right) dt + \frac{\partial F}{\partial x} \cdot b \cdot dz$$

We are missing the term, $\frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2} \cdot b^2 \cdot dt$

Rough sketch of proof: (rigorous derivation require measure theory)

Expand y with respect to a change Δy, in expansion keep the 1st order terms in Δx, Δt, and also Δx^2 because Δx has $\sqrt{\Delta t}$ term.

$$y + \Delta y = F(x,t) + \frac{\partial F}{\partial x} \Delta x + \frac{\partial F}{\partial t} \cdot \Delta t + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2} \cdot \Delta x^2$$

Now, $\Delta x^2 = (a\Delta t + b \cdot \epsilon \cdot \sqrt{\Delta t})^2 = \underbrace{a^2 \Delta t^2 + 2ab\epsilon \cdot \Delta t^{3/2}}_{\Delta t \text{ has higher order than } 1.} + b^2 \cdot (\epsilon^2 \sqrt{\Delta t})^2$

Hence, the first two terms are dropped.

Hence,

$$y + \Delta y = F(x,t) + \frac{\partial F}{\partial x} \cdot \Delta x + \frac{\partial F}{\partial t} \cdot \Delta t + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2} \cdot b^2 \cdot \Delta z^2$$

Note that ΔZ has mean $= 0$, variance $= \Delta t$.

It can be shown that $(\Delta Z)^2 \rightarrow \Delta t$, for $\Delta t \rightarrow 0$

Substituting this yields,

$$\Delta y = \left(\frac{\partial F}{\partial x} \cdot a + \frac{\partial F}{\partial t} + \frac{1}{2} \cdot \frac{\partial^2 F}{\partial x^2} \cdot b^2\right) \Delta t + \cancel{\frac{\partial F}{\partial t} \Delta t} \ \frac{\partial F}{\partial x} \cdot b \cdot \Delta Z$$

Stock price process (Geometric Brownian)

From eq. (1) the multiplicative model of stock price is,

$$\ln S(K+1) - \ln S(K) = w(K) \; ; \; w(K) = N(v, \sigma^2)$$

For continuous-time model

$$d \ln S(t) = v\,dt + \sigma\, dz \; ; \quad v, \sigma \geq 0 \text{ (constants)} \quad - (3)$$

dz : Wiener process

Also, note the above expression is consistent with result of eq. (A) for a small 'k'.

Eq. (3) is known as Geometric Brownian motion.

From 3,

$$\ln S(t) = \ln S(0) + vt + \sigma \cdot Z(t)$$

$$\ln S(t) \sim N\left(\ln S(0) + vt, \; \sigma^2 \cdot t\right)$$

$$S(t) = \cancel{\exp[\ln(S(t))]} = S(0) \cdot \exp\left[vt + \sigma \cdot Z(t)\right]$$

What is $E(S(t))$, $Var(S(t))$?

Consider this problem, $Y = \ln X = N(\mu, \sigma^2) \; ; \; -\infty < y < \infty$

Then calculate ~~E(Y) and Var(Y)~~, $E(X)$ & $Var(X)$

Note that $\cancel{Y = e^X}$ $X = e^Y$; $\cancel{Y \in (0,\infty)}$ $x \in (0, \infty)$

Using transformation of p.d.f.

$$\cancel{f_Y(y) = f_X(x) \cdot \left|\frac{dx}{dy}\right|} \quad f_X(x) = f_Y(y) \cdot \left|\frac{dy}{dx}\right|$$

$$y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} \cdot \frac{1}{x} ; \quad x > 0$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma\cdot x} \cdot e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} ; \quad (x > 0)$$

$$E(X) = \int_0^\infty x f_X(x)\, dx$$

$$= \int_0^\infty x \cdot \frac{1}{x} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \int_0^\infty e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} \cdot dx$$

Make the substitution, $z = \frac{\ln x - \mu}{\sigma}$

$$dz = \frac{1}{\sigma} \cdot \frac{1}{x} \cdot dx$$

when $x \to 0,\ z \to -\infty;\ x \to \infty,\ z \to \infty$

$=$

$$= \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} \cdot \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \cdot \cancel{\sigma} \cdot e^{(\mu+\sigma z)} \cdot dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} \cdot e^{\mu+\sigma z} \cdot dz$$

Note, $-\frac{z^2}{2} + \sigma z + \mu = -\frac{1}{2}(z^2 - 2\sigma z + \sigma^2) + (\mu + \frac{\sigma^2}{2})$

$$= -\frac{1}{2}(z-\sigma)^2 + (\mu + \frac{\sigma^2}{2})$$

Hence,

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma)^2} \cdot e^{(\mu+\frac{\sigma^2}{2})} \cdot dz$$

$$= e^{(\mu+\frac{\sigma^2}{2})} \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma)^2} \cdot dz$$

$$E(x) = e^{(\mu+\frac{\sigma^2}{2})}$$

Similarly show $Var(x) = e^{2(\mu+\frac{\sigma^2}{2})}\left(e^{\sigma^2} - 1\right)$

Using these results, $E(S(t)) = e^{(\nu t + \frac{\sigma^2}{2}t)}$

$$Var(S(t)) = e^{2(\mu+\frac{\sigma^2}{2})t} \cdot \left(e^{\sigma^2 t} - 1\right)$$

From eq. (3) $\quad d \ln S(t) = \nu dt + \sigma dz$

Defined in terms of $\ln S(t)$ rather than in terms of $S(t)$. How do achieve that?

ordinary calculus results in $\cancel{d(\ln S(t)) = \dfrac{dS(t)}{S(t)}}$

Use Ito lemma, $y = S(t) = e^{\ln S(t)}$, $F = e^{x}$, $x = \ln S(t)$

From (11a),

$$\cancel{dS(t) = \left(\frac{\partial F}{\partial S}\cdot a + \frac{\partial F}{\partial t} + \frac{1}{2}\cdot\frac{\partial^2 F}{\partial S^2} b^2\right)dt + \frac{\partial F}{\partial S}\cdot b\cdot dz}$$

$$\cancel{= \left(e^{S}\cdot a + \frac{\partial F}{\partial t} + \frac{1}{2}\cdot e^{S}\cdot b^2\right)dt + e^{S}\cdot b\cdot dz}$$

$$dS(t) = \left(\frac{\partial F}{\partial x}\cdot a + \frac{\partial F}{\partial t} + \frac{1}{2}\cdot\frac{\partial^2 F}{\partial x^2}\cdot b^2\right)dt + \frac{\partial F}{\partial x}\cdot b\cdot dz$$

$$\frac{\partial F}{\partial x} = e^{x}, \quad \frac{\partial^2 F}{\partial x^2} = e^{x}, \quad \frac{\partial F}{\partial t} = 0, \quad e^{x} = S(t)$$

$$a = v, \quad b = \sigma$$

Hence,

$$dS(t) = \left(S(t)\cdot a + 0 + \frac{1}{2}\cdot S(t)\cdot b^2\right)dt + S(t)\cdot b\cdot dz$$

$$\frac{dS(t)}{S(t)} = \underbrace{\left(v + \frac{1}{2}\sigma^2\right)}_{\mu} dt + \sigma\cdot dz$$

$$\frac{dS(t)}{S(t)} = \mu\cdot dt + \sigma\cdot dz \quad \text{--- (4)}$$

Return of the stock ((4) is eq. for the instantaneous return)

Example: Bond price dyanamics.

$P(t)$ = price of bond with face value ₹1. (no coupons)

r = const, price dynamics, $\dfrac{dP}{P} = r\cdot dt$

deterministic Ito

$$P(t) = e^{r(t-T)}$$

Summary of results on geomteric Brownian motion

~~Geometric~~ $$d(\ln S(t)) = \nu dt + \sigma dz \qquad (5a)$$

$$\frac{dS(t)}{S(t)} = \underbrace{\left(\nu + \frac{\sigma^2}{2}\right)}_{\mu} dt + \sigma dz \qquad (5b)$$

$$dS(t) = \mu S(t) dt + \sigma S(t) dz \qquad (5c)$$

$$E[\ln S(t)] = \nu t \; ; \; S(0) = 1$$

$$Var[\ln S(t)] = \sigma^2 t$$

$$E[S(t)] = e^{\mu t}, \quad Var[S(t)] = e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$$

Simulation

Continuous time prices are simulated using a series of small time steps.

Two widely used techniques are given by 5(a) & 5(c)

From 5(c),

$$S(t_{k+1}) = \left[1 + \mu \Delta t + \sigma \cdot \epsilon(t_k) \cdot \sqrt{\Delta t}\right] \cdot S(t_k)$$

From 5(a) $\quad S(t_{k+1}) = S(t_k) \cdot e^{\nu \Delta t + \sigma \cdot \epsilon(t_k) \cdot \sqrt{\Delta t}}$

The two methods are different; but differences cancel in the long run. & Both methods are used in practice.

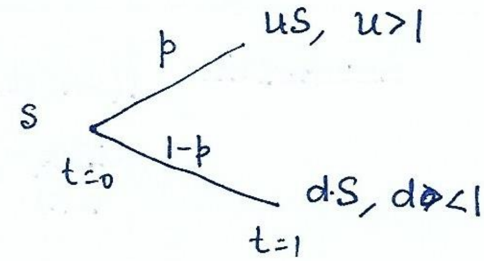
Ex: $\nu = 0.15$, $\sigma = 0.40$, $\Delta t = \frac{1}{52}$, $S(0) =$ ₹10, $\epsilon \sim N(0,1)$

Simulate the stock behaviour for 1 year.

Binomial Lattice

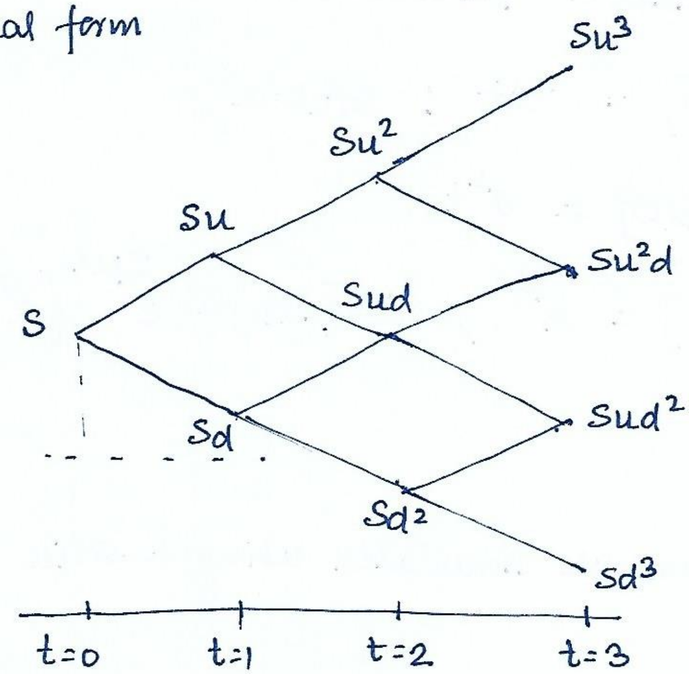
- easy to program

$\Delta t = 1$ week, $S(0)$ is known.



$0 < p < 1$

General form



Up movement followed by down is equivalent to down ——————— up.

To specify the model we must determine, u, d, p. These are chosen such that the true stochastic behavior the stock is captured.

It is similar to the multiplicative model shown earlier.

Note that probability of reaching node $Su^k d^{n-k}$ is

$$\binom{n}{k} p^k (1-p)^{n-k} \xrightarrow[n\to\infty]{} \text{normal dist.}$$ (show it)

Further, it can be shown $Su^k d^{n-k} \xrightarrow[\substack{n\to\infty \\ \Delta t\to 0}]{}$ Log normal (show it)

We match the expected value of the logarithm of a price change, $S(0)=1$, and variance of the log. of the price change.

Note, that, for matching it is only necessary to ensure that the r.v. S_1 has the correct ~~probte~~ properties, the later processes are identical.

$$E[\ln S_1] = \underline{p\cdot\ln u + (1-p)\cdot\ln d}$$

$$\mathrm{Var}[\ln S_1] = (\ln u)^2\cdot p + (\ln d)^2\cdot(1-p) - (p\cdot\ln u + (1-p)\cdot\ln d)^2$$

$$= (p-p^2)\cdot(\ln u)^2 + p(1-p)\cdot(\ln d)^2 - 2p(1-p)(\ln u)\cdot(\ln d)$$

$$= \underline{p(1-p)\left[\ln u - \ln d\right]^2}$$

parameter matching; $\ln u = U$, $\ln d = D$

$$p\cdot U + (1-p)\cdot D = \nu\Delta t$$
$$p(1-p)\left[U-D\right]^2 = \sigma^2\Delta t$$

2 eq, 3 unknown, let's use $U=-D$

$$(2p-1)\cdot U = \nu\Delta t$$
$$4p(1-p)U^2 = \sigma^2\Delta t$$

sq. the first eq. ~~can~~ and add.

$$U^2 = (\nu\Delta t)^2 + \sigma^2\Delta t$$

$$\Rightarrow \quad p = \frac{1}{2} + \frac{1}{2}\cdot\frac{\nu\Delta t}{\sqrt{(\nu\Delta t)^2+\sigma^2\Delta t}} \approx \frac{1}{2} + \cancel{\frac{1}{2\sigma}\cdot\nu\sqrt{\Delta t}} + \frac{1}{2\sigma}\cdot(\nu\sqrt{\Delta t})$$

$$\ln u = \sqrt{\sigma^2\Delta t + (\nu\Delta t)^2}, \quad \ln d = -\sqrt{\sigma^2\Delta t + (\nu\Delta t)^2}$$

$$u \approx e^{\sigma\sqrt{\Delta t}} \qquad\qquad d \approx e^{-\sigma\sqrt{\Delta t}}$$