Data Structures and Algorithms

(ESO207)

Lecture 29:

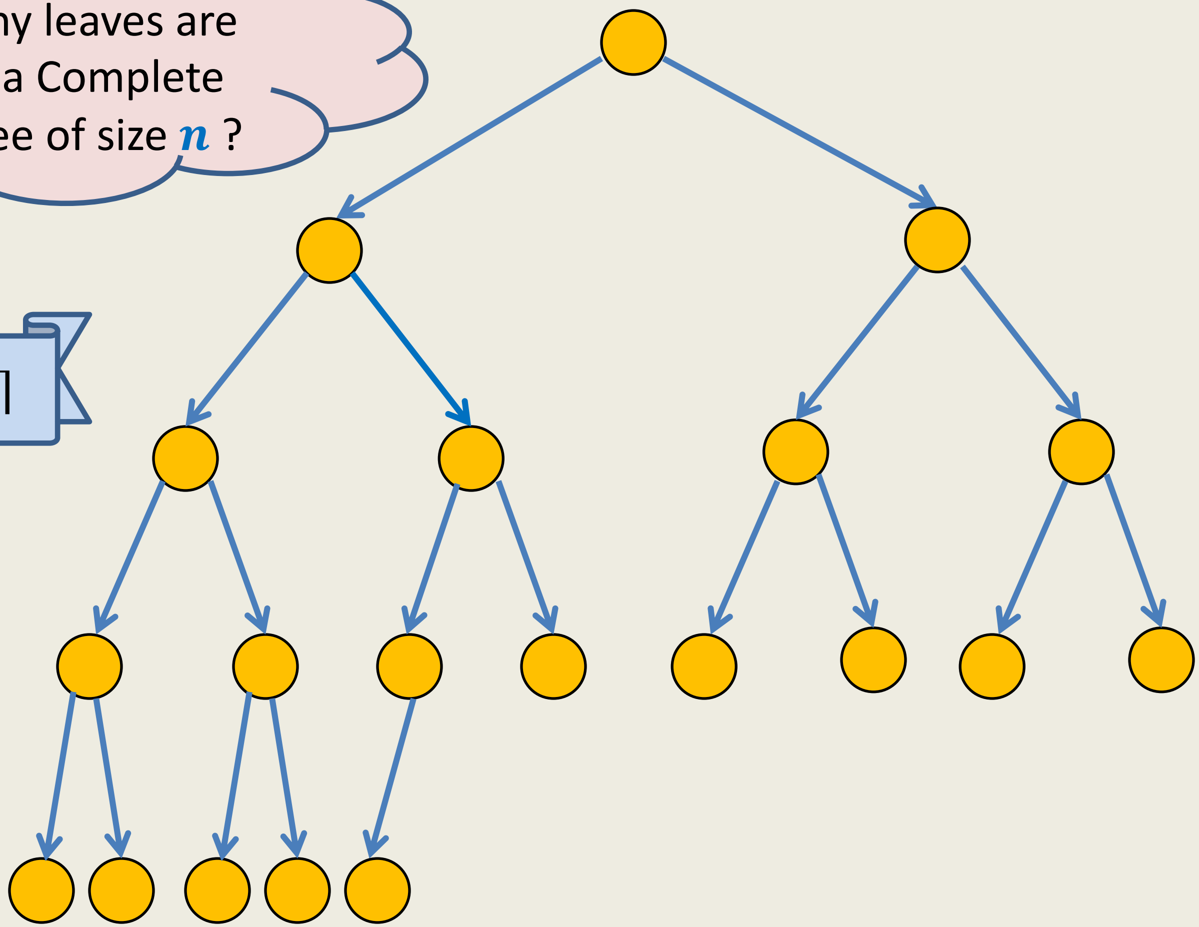
- **Building** a Binary heap on n elements in O(n) time.
- **Applications of Binary heap : sorting**
- **Binary trees: beyond searching and sorting**

Recap from the last lecture

A complete binary tree

How many leaves are there in a Complete Binary tree of size n ?

$\lceil n/2 \rceil$



Building a Binary heap

Problem: Given $\boldsymbol{n}$ elements $\{x_0, \ldots, x_{n-1}\}$, build a binary **heap H** storing them.

Trivial solution:

(Building the Binary heap incrementally)

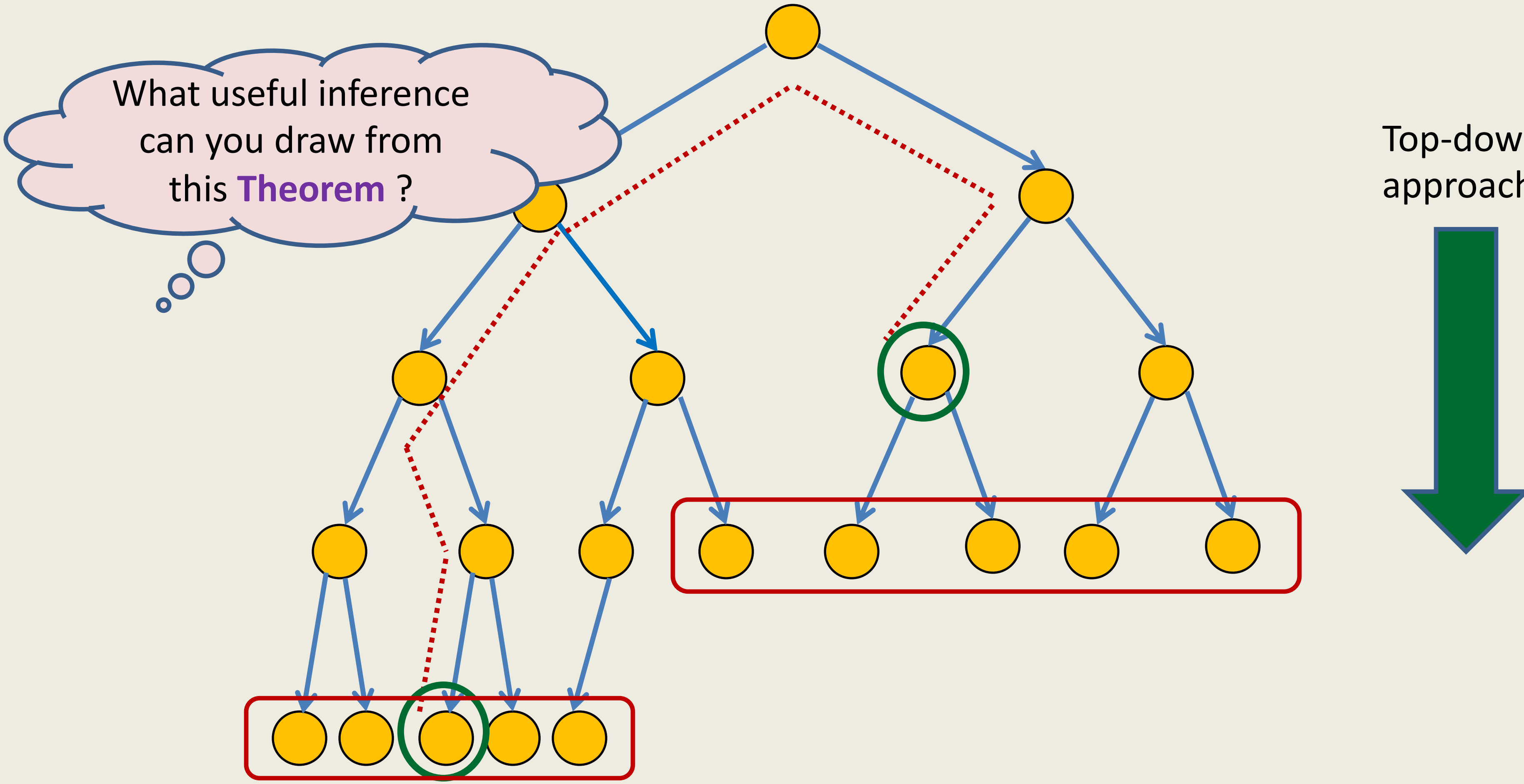
CreateHeap(**H**);

For($\boldsymbol{i}$ = 0 to $\boldsymbol{n-1}$ **)**

Insert(x_i,**H**);

What is the time complexity of this algorithm?

Building a Binary heap incrementally



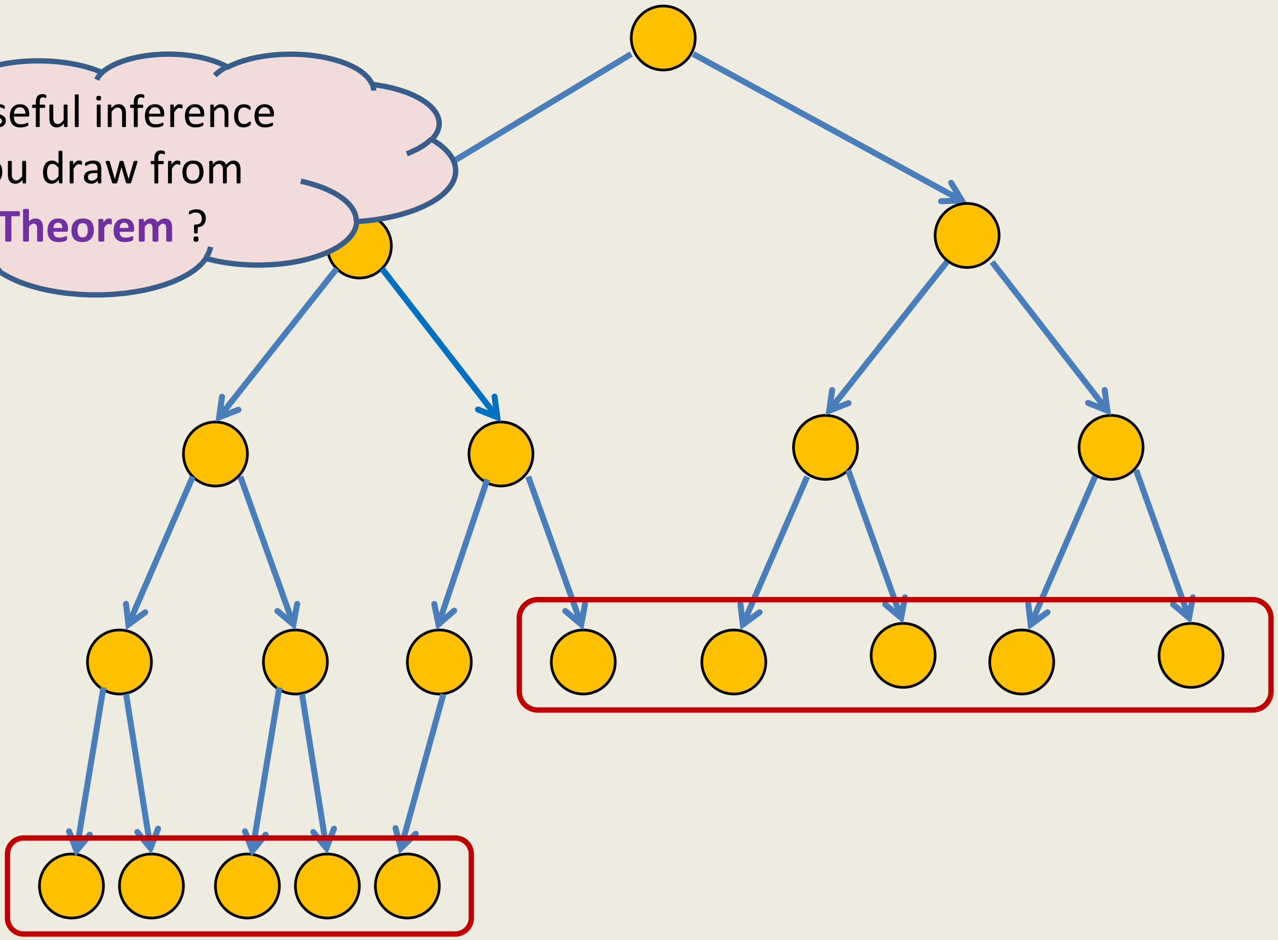
The time complexity for inserting a leaf node = **O**($\log n$)

leaf nodes = $[n/2]$,

➔ **Theorem**: Time complexity of building a binary heap incrementally is **O**($n \log n$).

Building a Binary heap incrementally

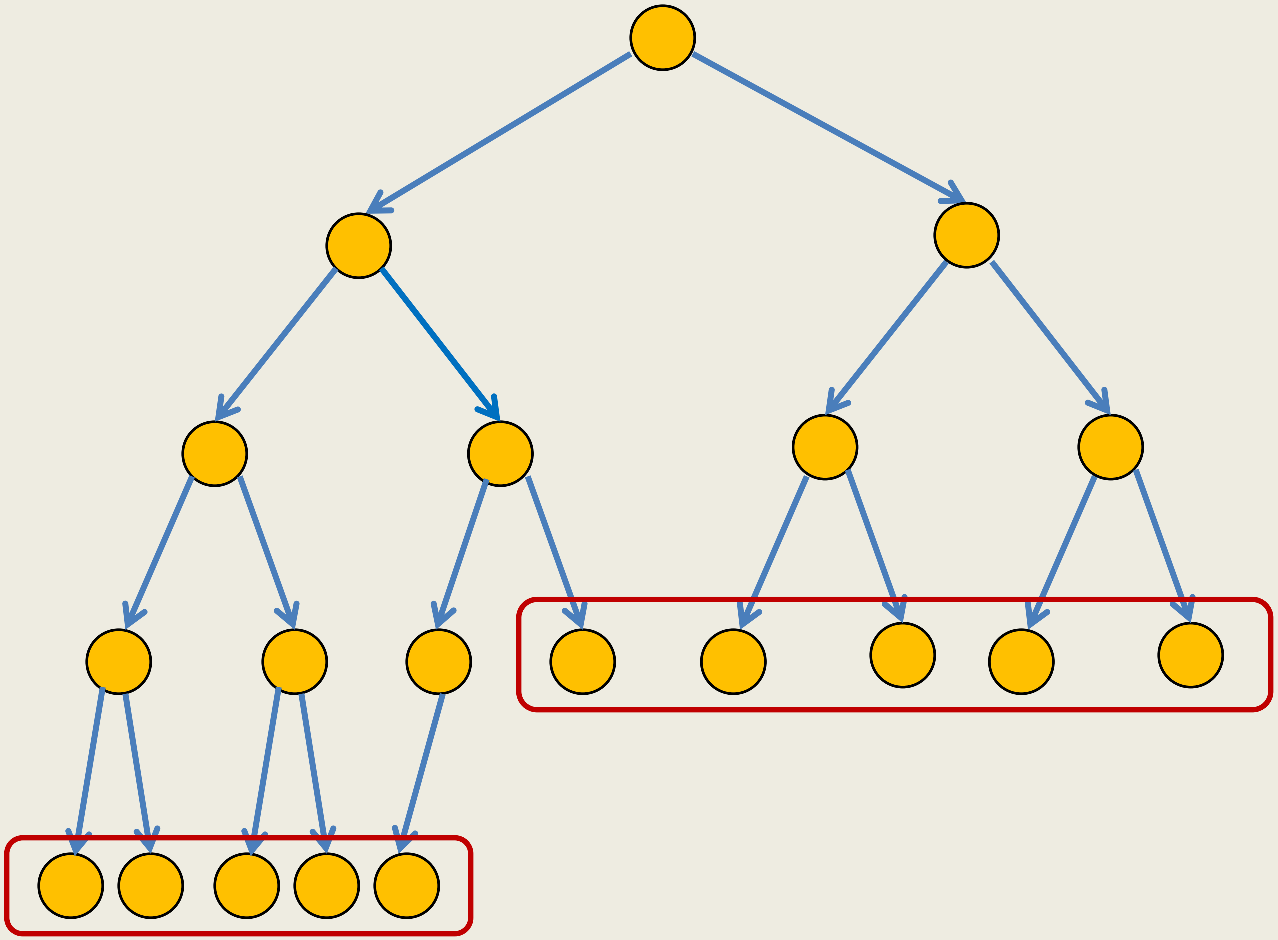
What useful inference can you draw from this **Theorem** ?



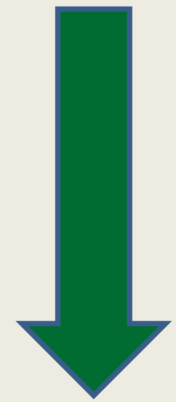
Top-down approach

The **O**(n) time algorithm must take **O**(1) time for each of the $\lceil n/2 \rceil$ leaves.

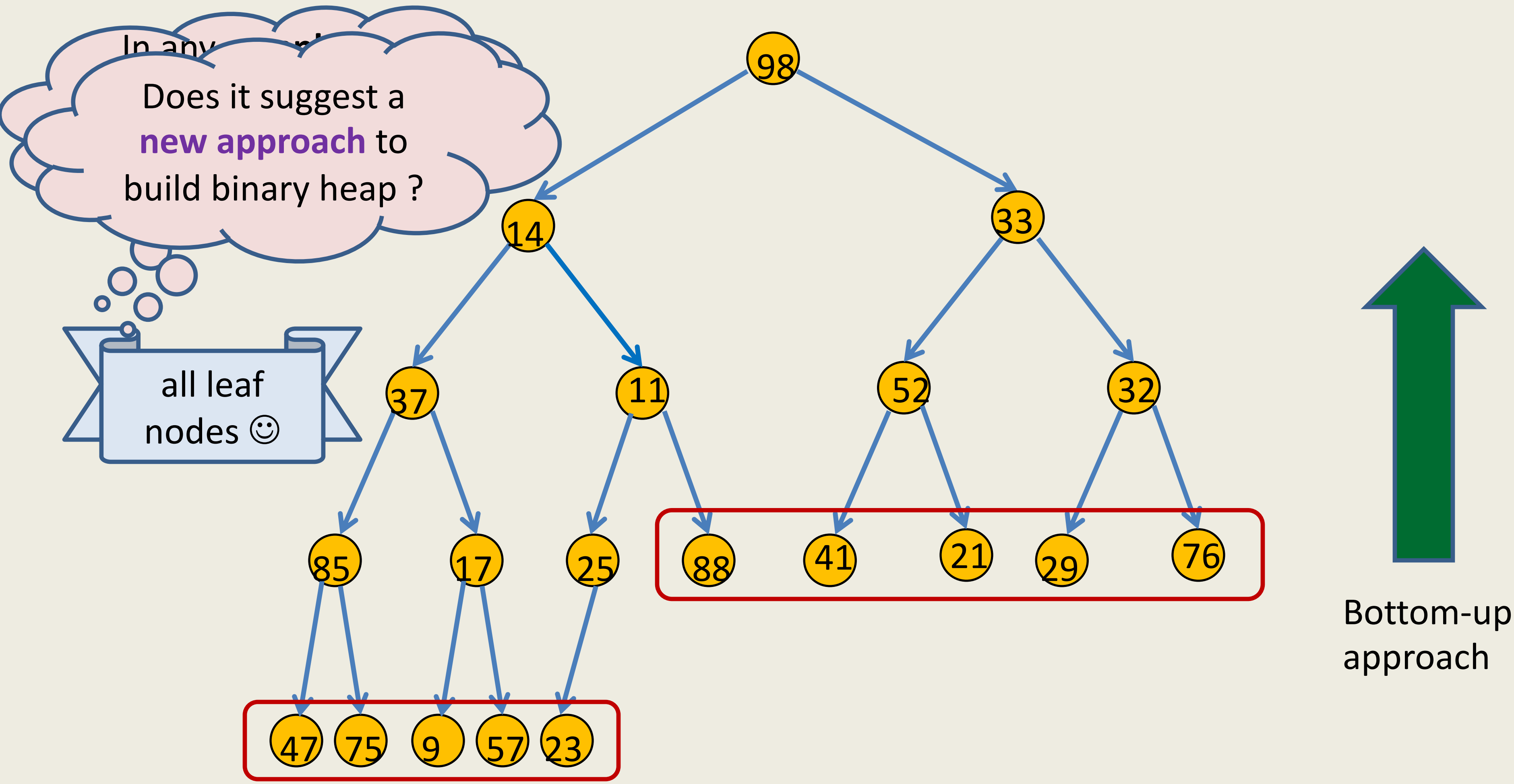
Building a Binary heap incrementally



Top-down approach



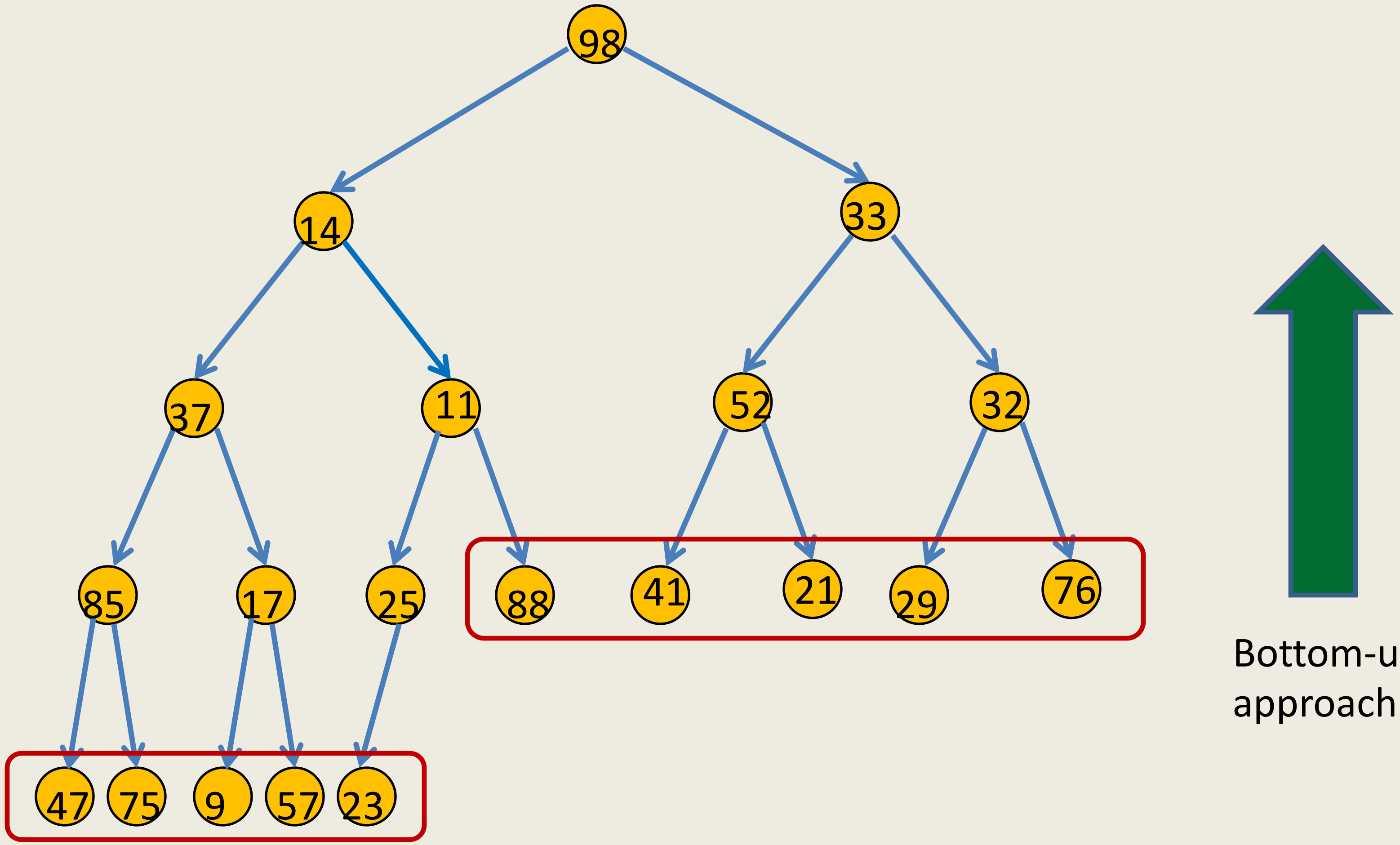
Think of alternate approach for building a binary heap



heap property: *"Every **node** stores value smaller than its **children**"*

We just need to ensure this property at each node.

Think of alternate approach for building a binary heap



heap property: *"Every **node** stores value <u>smaller</u> than its **children**"*

We just need to ensure this property at each node.

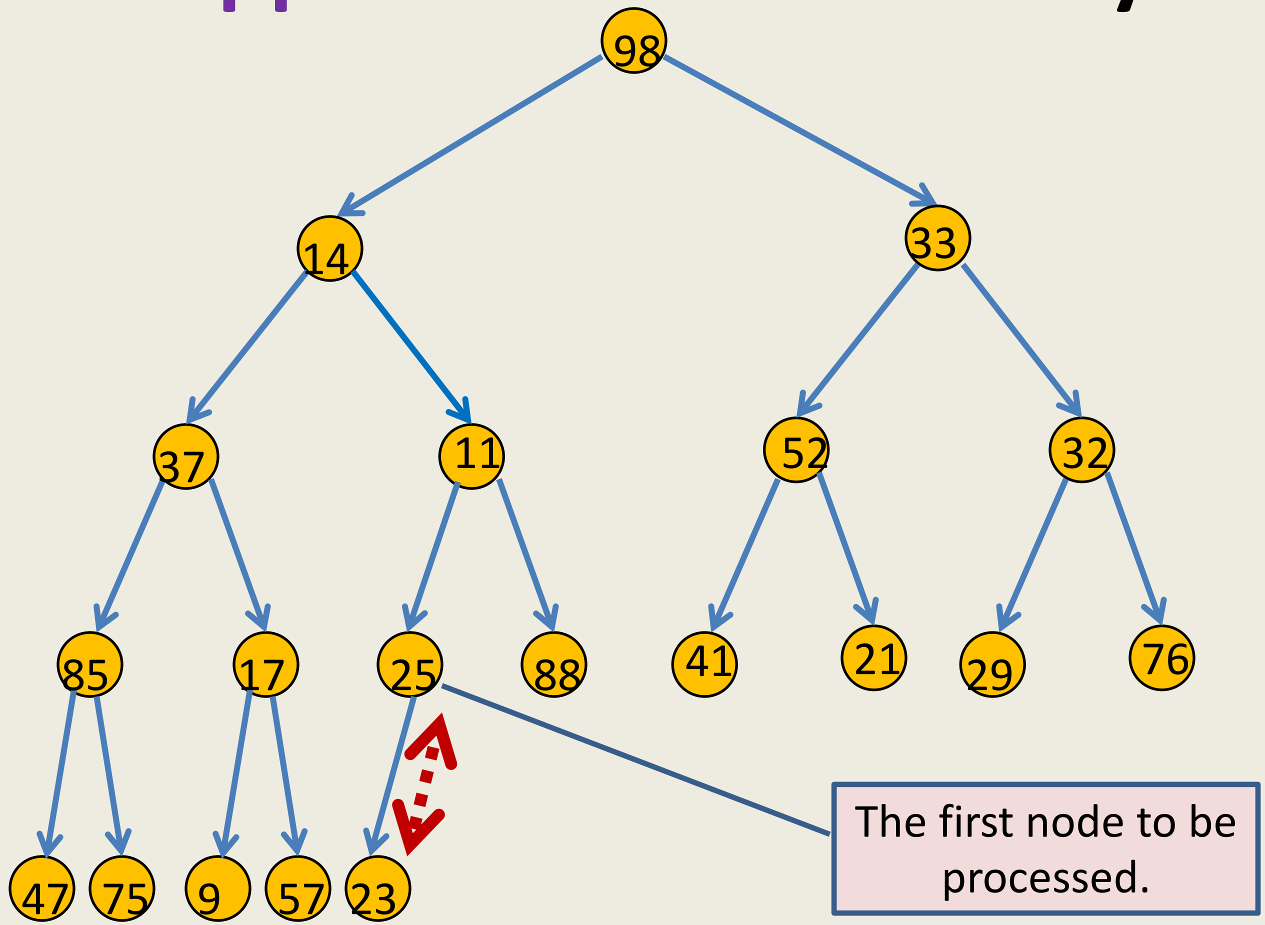
A new approach to build binary heap

1. Just copy the given $\boldsymbol{n}$ elements $\{x_0$, …, $x_{n-1}\}$ into an array **H**.

2. The **heap property** holds for all the leaf nodes in the corresponding complete binary tree.

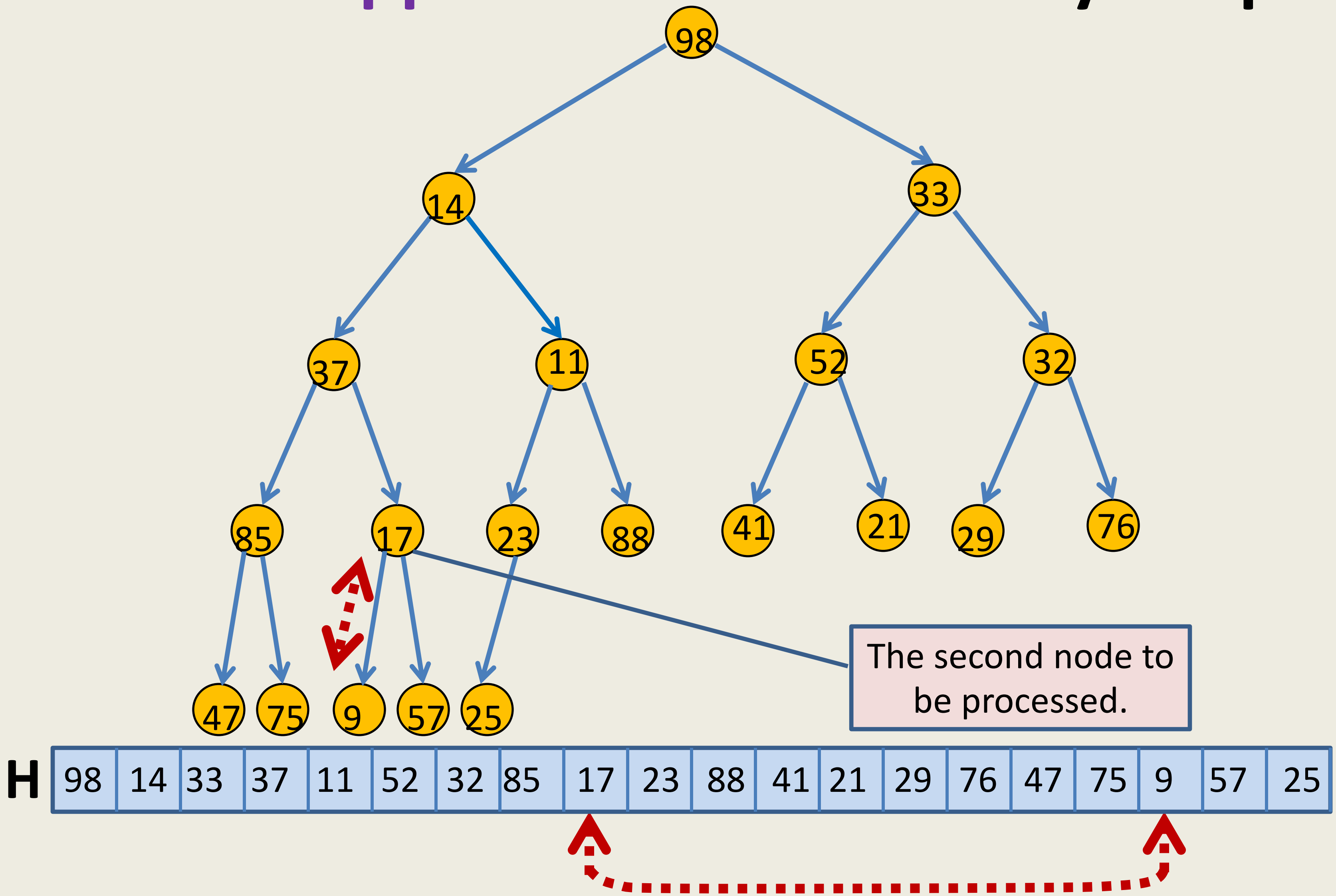
3. Leaving all the leaf nodes,
   process the elements in the decreasing order of their numbering
   and set the heap property for each of them.

A new approach to build binary heap



H																				
	98	14	33	37	11	52	32	85	17	25	88	41	21	29	76	47	75	9	57	23

A new approach to build binary heap



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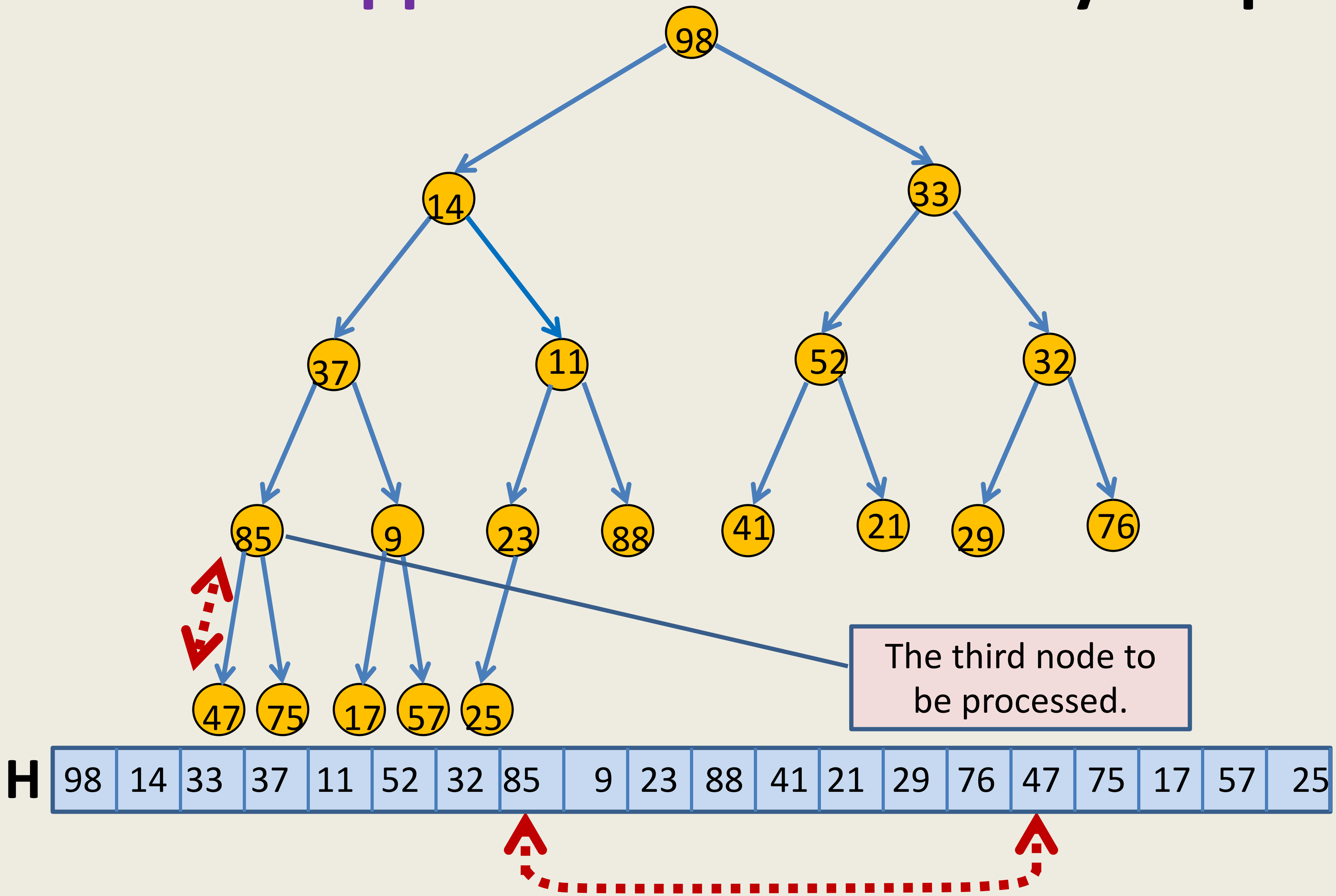

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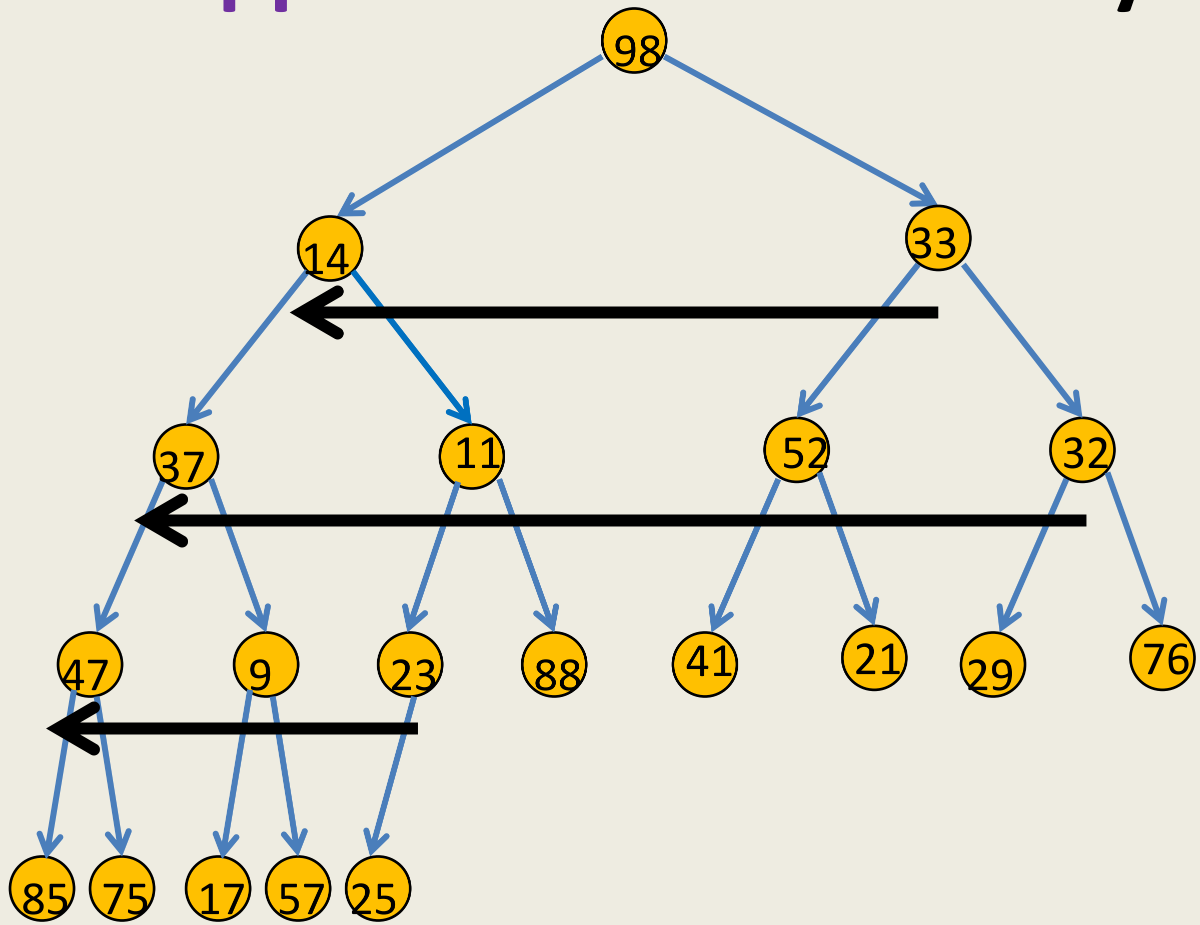

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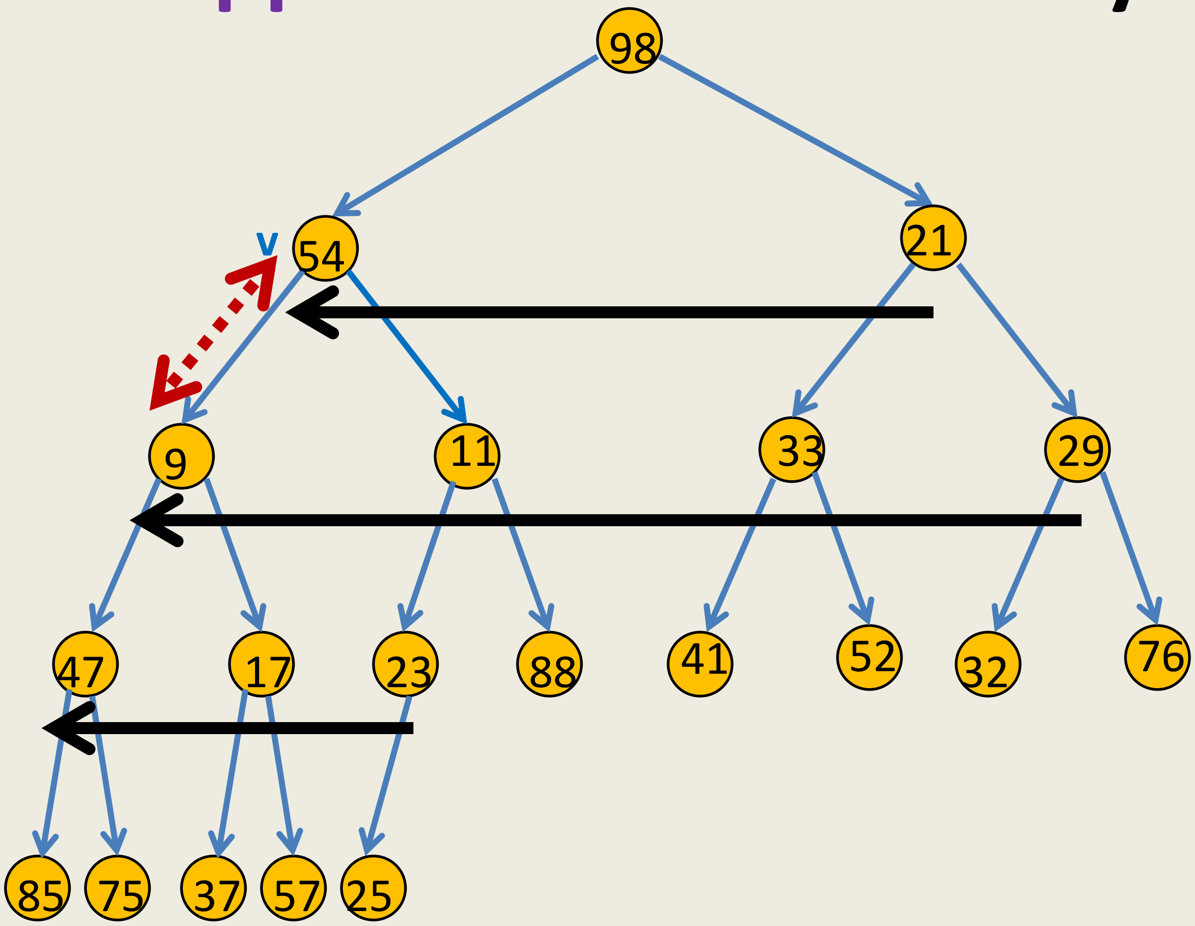
A new approach to build binary heap



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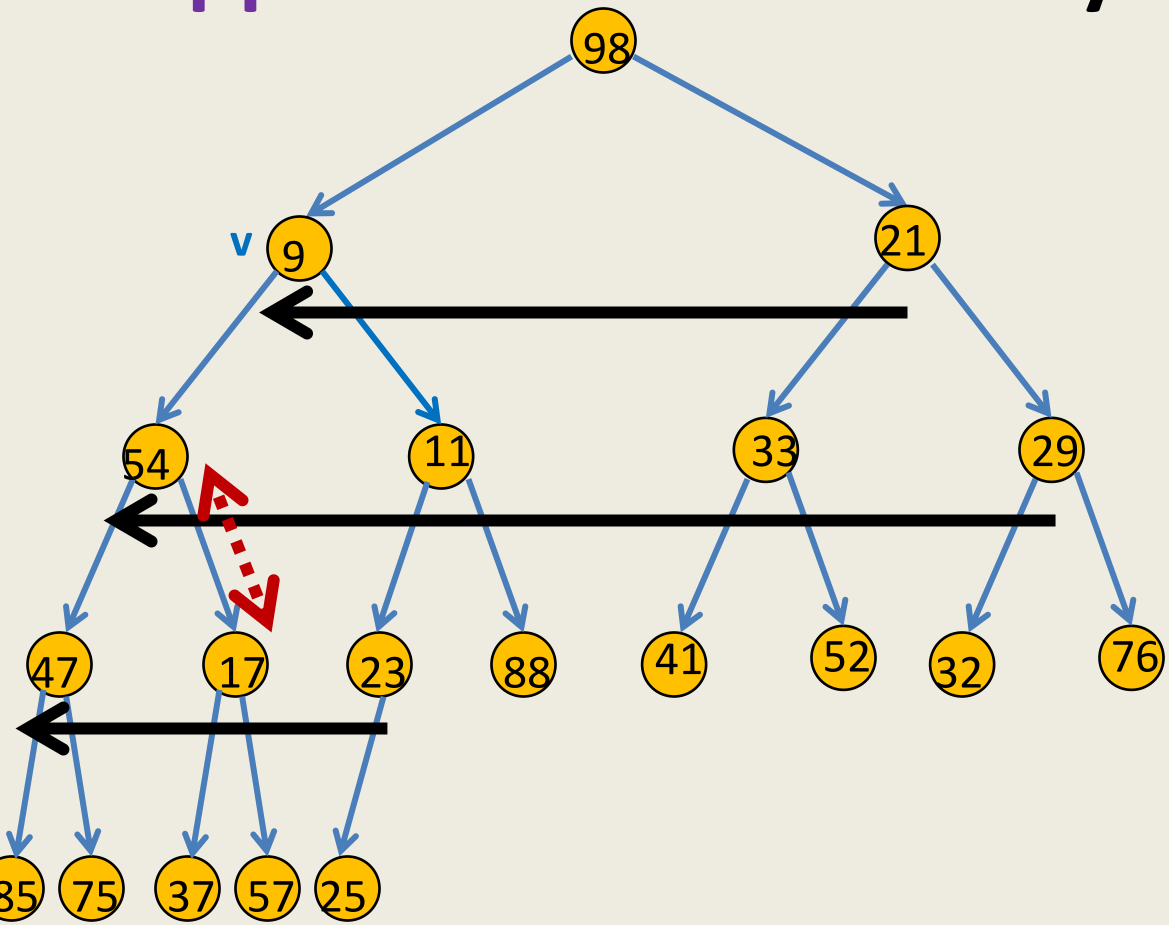

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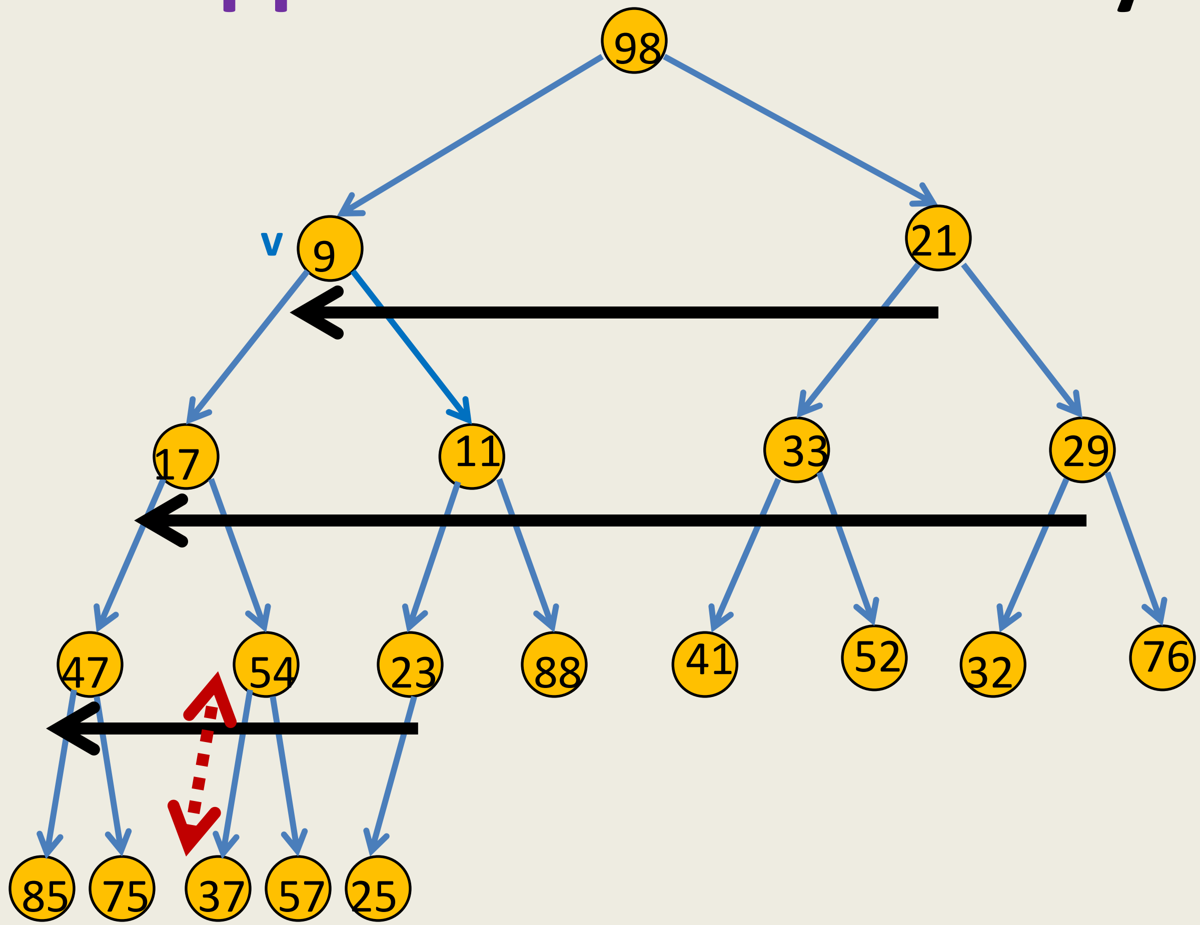

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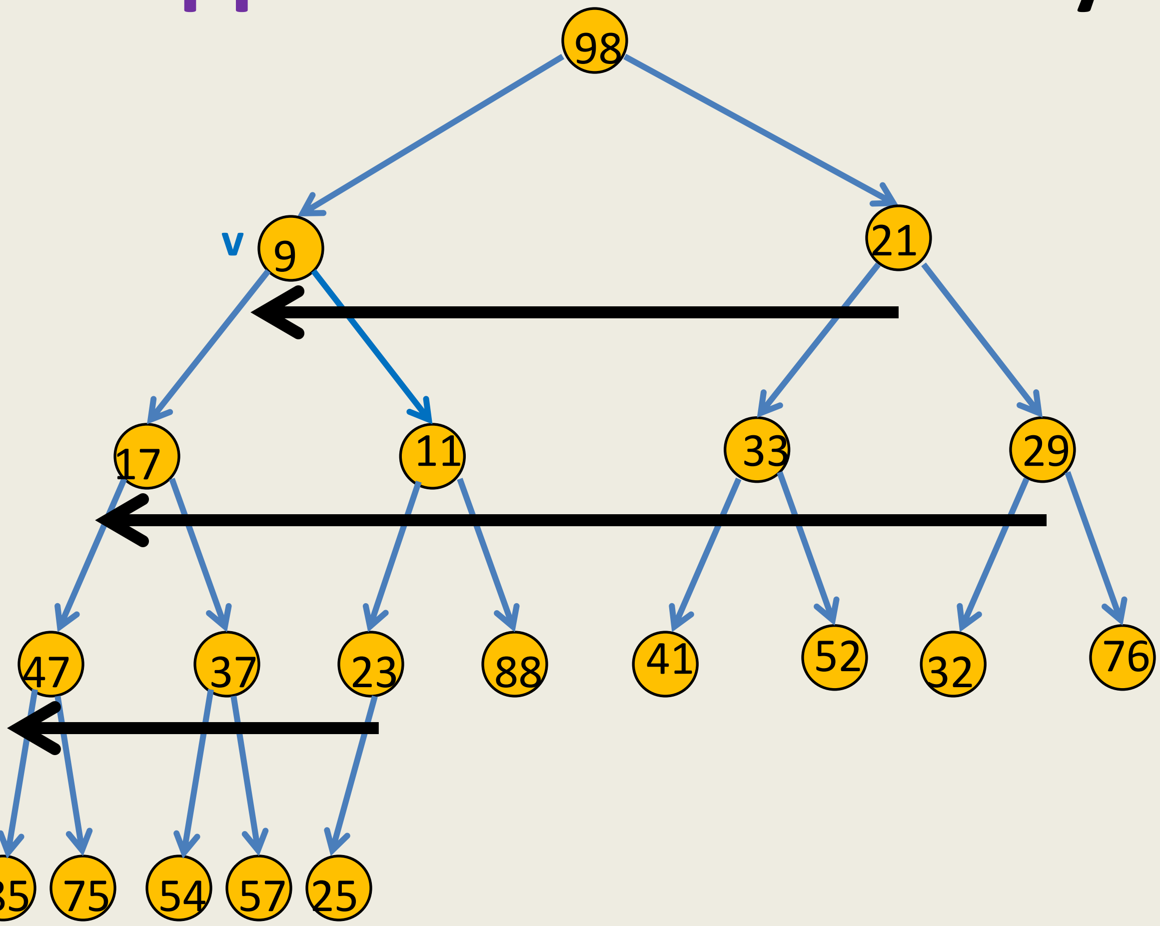

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Let **v** be a node corresponding to index **i** in **H.**
The process of restoring heap property at **i** called **Heapify**(**i**,**H**).

Heapify(i,H)

Heapify(i,H)

{ $n \leftarrow$ size(H) -1 ;

While (? and ?)

{

For node i, compare its value with those of its children

If it is smaller than any of its children

→ Swap it with **smallest** child and move down …

Else stop !

}

}

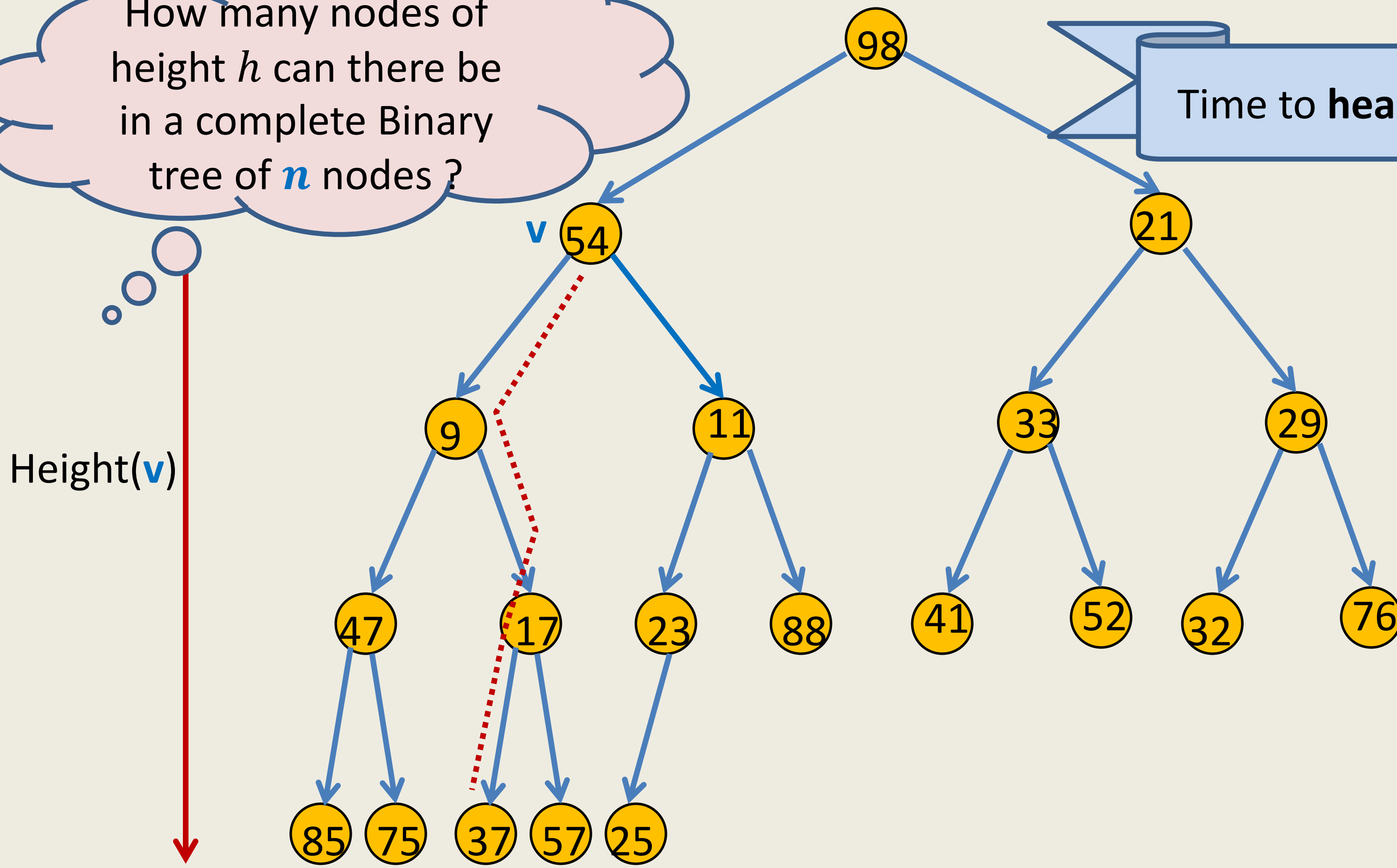
Heapify(i,H)

```
Heapify(i,H)
{     n ← size(H) -1 ;
      Flag ← true;
      While (  i ≤ ⌊(n−1)/2⌋  and  Flag = true  )
      {     min ← i;
            If(      H[i]>H[2i + 1]      )      min ← 2i + 1;
            If(   2i + 2 ≤ n   and H[min]>H[2i + 2]   )     min ← 2i + 2;
            If(min  ≠ i)
               {     H(i) ↔ H(min);
                        i ← min;   }
            else
                     Flag ← false;
       }
}
```

Building Binary heap in O(n) time

How many nodes of height h can there be in a complete Binary tree of n nodes ?

Time to **heapify** node **v** ?



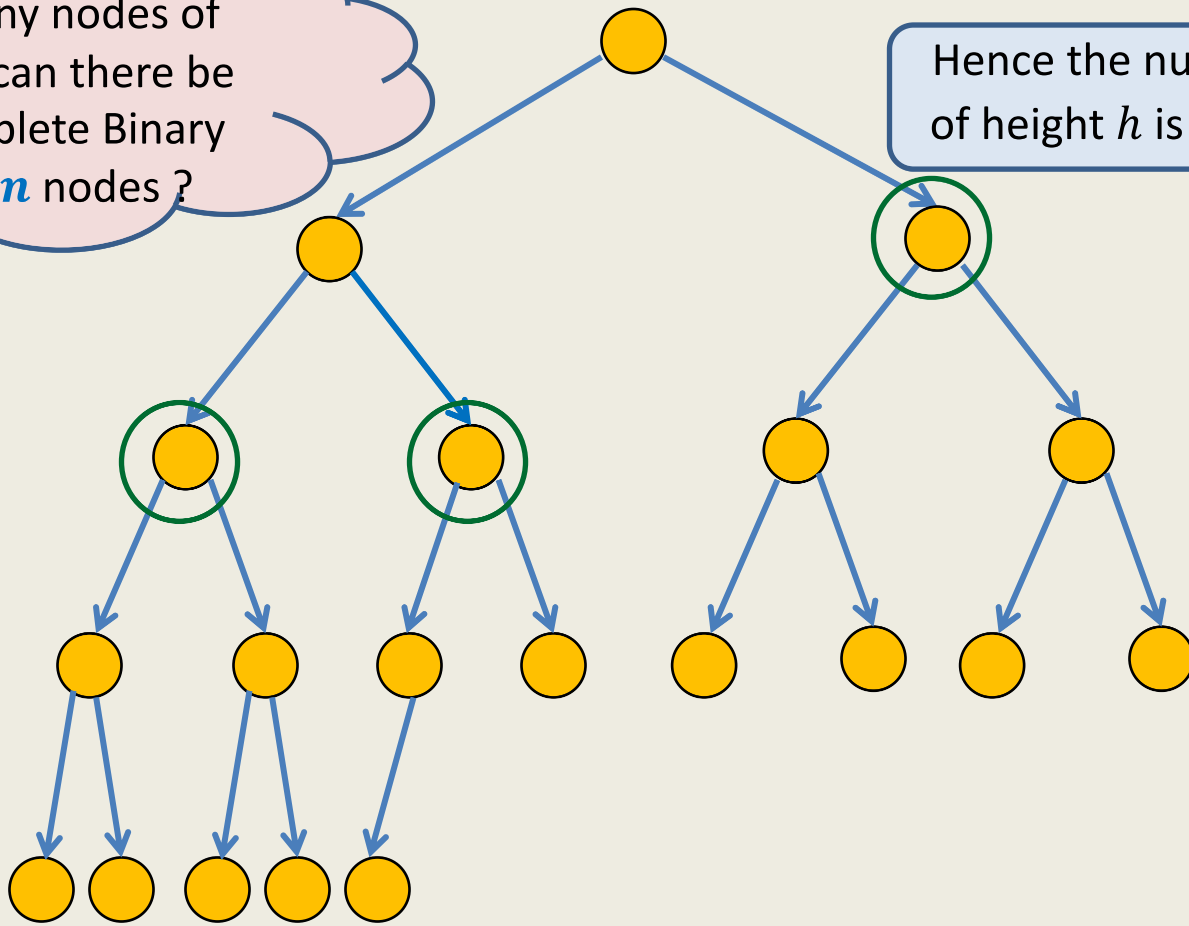
H	98	54	21	9	11	33	29	47	17	23	88	41	52	32	76	85	75	37	57	25

Time complexity of algorithm = $\sum_h \boldsymbol{O}(h) \cdot \underbrace{\boldsymbol{N}(h)}$

No. of nodes of height h

A complete binary tree

How many nodes of height h can there be in a complete Binary tree of n nodes ?



Hence the number of nodes of height h is bounded by $\frac{n}{2^h}$

Each subtree is also a complete binary tree.

➔ A subtree of height h has at least 2^h nodes

Moreover, no two subtrees of height h in the given tree have any element in common

Building Binary heap in O(n) time

Lemma: the number of nodes of height h is bounded by $\frac{n}{2^h}$.

Hence Time complexity to build the heap = $\sum_{h=1}^{\log n} \frac{n}{2^h}\ O(h)$

$= n\ \ c\ \ \sum_{i=1}^{\log n} \frac{h}{2^h}$

$= \mathrm{O}(n)$

As an exercise (using knowledge from your JEE preparation days), show that $\sum_{h=1}^{\log n} \frac{h}{2^h}$ is bounded by 2

Sorting using a Binary heap

Sorting using heap

Build heap **H** on the given n elements;

While (**H** is not empty)

{   **x** ← Extract-min(**H**);

     print **x**;

}

This is **HEAP** SORT algorithm

Time complexity : **O**(n log n)

Question:

Which is the best sorting algorithm : (**Merge** sort, **Heap** sort, **Quick** sort) ?

Answer: Practice programming assignment ☺

Binary trees:
beyond searching and sorting

- **Elegant solution for two interesting problem**

- **An important lesson:**

Lack of **proper understanding** of a problem is a big hurdle to solve the problem

Two interesting problems on sequences

What is a sequence ?

A sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$

- Can be viewed as a mapping from [0, n].
- Order <u>does</u> matter.

Problem 1

Multi-increment

Problem 1

Given an initial sequence **S** = $\prec x_0, \dots, x_{n-1} \succ$ of numbers,
maintain a compact data structure to perform the following operations:

- **ReportElement**($\boldsymbol{i}$):

  Report the current value of x_i.

- **Multi-Increment**($\boldsymbol{i}$, $\boldsymbol{j}$, **Δ**):

  Add **Δ** to each x_k for each $\boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}$

Example:

Let the initial sequence be **S** = $\prec$ 14, 12, 23, 12, 111, 51, 321, -40 $\succ$

After **Multi-Increment**(2,6,10), **S** becomes

$\prec$ 14, 12, **33**, **22**, **121**, **61**, **331**, -40 $\succ$

After **Multi-Increment**(0,4,25), **S** becomes

$\prec$ **39**, **37**, **58**, **47**, **146**, 61, 331, -40 $\succ$

After **Multi-Increment**(2,5,31), **S** becomes

$\prec$ 39, 37, **89**, **78**, **177**, **92**, 331, -40 $\succ$

Problem 1

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  Add **Δ** to each x_k for each $\boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}$

Trivial solution :

Store **S** in an array **A[0.. $\boldsymbol{n}$-1]** such that **A[$\boldsymbol{i}$]** stores **the current value** of x_i.

Multi-Increment($\boldsymbol{i}$, $\boldsymbol{j}$, **Δ**)

{

For ($\boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}$) **A[$\boldsymbol{k}$]← A[$\boldsymbol{k}$]+ Δ;**

}

O($\boldsymbol{j} - \boldsymbol{i}$) = O($\boldsymbol{n}$)

ReportElement($\boldsymbol{i}$)**{** **return** A[$\boldsymbol{i}$] **}**

O(1)

Problem 1

Given an initial sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ of numbers,
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Trivial solution :

Store **S** in an array **A[0.. $\boldsymbol{n}$-1]** such that **A[$\boldsymbol{i}$]** stores **the current value** of x_i.

Question: the source of difficulty in breaking the **O (n)** barrier for **Multi-Increment() ?**

Answer: we need to **explicitly maintain** in **S**.

Question: who asked/inspired us to maintain **S** explicitly.

Answer: 1. **incomplete understanding** of the problem

2. **conditioning** based on incomplete understanding

Towards efficient solution of Problem 1

Assumption: without loss of generality assume n is power of 2.

Explore ways to maintain sequence S **implicitly** such that

- **Multi-Increment**(i, j, Δ) is efficient
- **Report**(i) is efficient too.

Main hurdle: To perform **Multi-Increment**(i, j, Δ) efficiently

Problem 2

Dynamic Range-minima

Problem 2

Given an initial sequence **S** = $\prec x_0, \ldots, x_{n-1} \succ$ of numbers,
maintain a compact data structure to perform the following operations efficiently for any $0 \le \boldsymbol{i} < \boldsymbol{j} < \boldsymbol{n}$.

- **ReportMin**($\boldsymbol{i}$, $\boldsymbol{j}$):

  Report the minimum element from $\{x_k \mid \text{for each } \boldsymbol{i} \le \boldsymbol{k} \le \boldsymbol{j}\}$

- **Update**($\boldsymbol{i}$, **a**):

  a becomes the new value of x_i.

AIM:

- **O**($\boldsymbol{n}$) size data structure.
- **ReportMin**($\boldsymbol{i}$, $\boldsymbol{j}$) in **O**(log $\boldsymbol{n}$) time.
- **Update**($\boldsymbol{i}$, **a**) in **O**(log $\boldsymbol{n}$) time.

All **data structure lovers** must ponder over these two problems ☺.
We shall discuss them in the next lecture.