Data Structures and Algorithms

(ESO207)

Lecture 14:

- Algorithm paradigms
- Algorithm paradigm of Divide and Conquer

Algorithm Paradigms

Algorithm Paradigm

Motivation:

- Many problems whose algorithms are based on a common approach.
- A need of a systematic study of such widely used approaches.

Algorithm Paradigms:

- **Divide and Conquer**
- **Greedy Strategy**
- **Dynamic Programming**
- **Local Search**

Divide and Conquer paradigm for Algorithm Design

Divide and Conquer paradigm
An Overview

1. **Divide** the problem instance into two or more instances of the same problem
2. Solve each smaller instances **recursively** (base case suitably defined).
3. **Combine** the solutions of the smaller instances
   to get the solution of the original instance.

This is usually the main **nontrivial** step in the design of an algorithm using divide and conquer strategy

Example 1

Sorting

A familiar problem

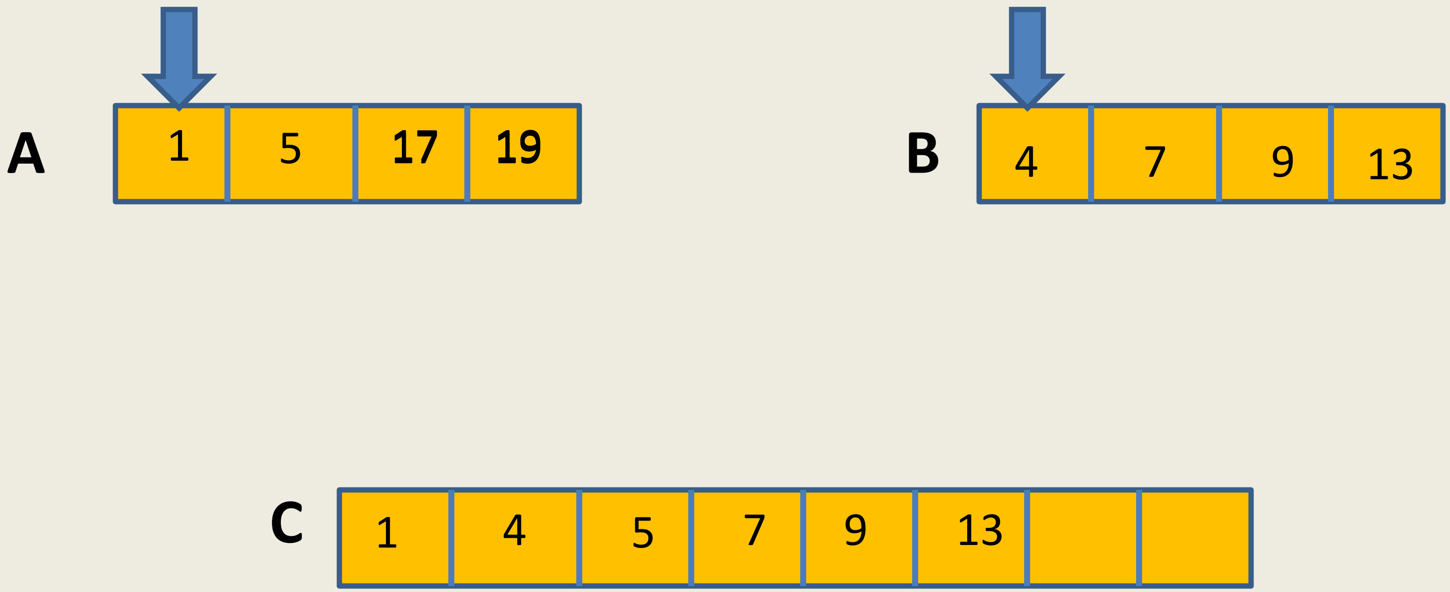
Merging two sorted arrays:

Given two sorted arrays **A** and **B** storing n elements each, Design an **O**(n) time algorithm to output a sorted array **C** containing all elements of **A** and **B**.

Example: If **A**={1,5,17,19} **B=**{4,7,9,13}, then output is

C={1,4,5,7,9,13,17,19}.

Merging two sorted arrays A and B



Pesudo-code for Merging two sorted arrays

```
Merge(A[0..n-1],B[0..m-1], C)    // Merging two sorted arrays A and B into array C.
{  i← 0;  j← 0;
   k← 0;
  While(i<n  and j<m)
  {      If(A[i]< B[j]) {    C[k] ← A[i]; k++;  i++     }
         Else            {    C[k] ← B[j]; k++;  j++    }
   }
   While(i<n) {  C[k] ← A[i]; k++;  i++   }
   While(j<m) {  C[k] ← B[j]; k++;  j++   }
   return C;
}
```

Correctness : homework exercise

Time Complexity = $O(n+m)$

Divide and Conquer based sorting algorithm

MSort(**A**,i,j) // Sorting the subarray **A**[$i..j$].

{ **If** ($i < j$)

{ **mid** ← (i+j)/2;

MSort(**A**,i,**mid**);

MSort(**A**,**mid**+1,j);

Divide step

Create temporarily **C**[$0..j - i$]

Merge(**A**[i..**mid**], **A**[**mid**+1..j], **C**);

Copy **C**[$0..j - i$] to **A**[$i..j$]

Combine/conquer step

}

This is Merge Sort algorithm

Divide and Conquer based sorting algorithm

MSort(**A**,i,j) // Sorting the subarray **A**[i..j].

{ **If** ($i < j$)

{ **mid** ← (i+j)/2;

MSort(**A**,i,**mid**); ← **T**(n/**2**)

MSort(**A**,**mid**+1,j); ← **T**(n/**2**)

$\left.\begin{array}{l}\text{Create temporarily } \mathbf{C}[0..j-i] \\ \mathbf{Merge}(\mathbf{A}[i..\mathbf{mid}], \mathbf{A}[\mathbf{mid}+1..j], \mathbf{C}); \\ \textit{Copy } \mathbf{C}[0..j-i] \text{ to } \mathbf{A}[i..j]\end{array}\right\} \mathbf{c}\, n$

}

Time complexity:

If n = 1,

T(n) = c for some constant c

If n > 1,

$$
\begin{aligned}
\mathbf{T}(n) &= c\,n + \mathbf{2}\,\mathbf{T}(n/\mathbf{2}) \\
&= c\,n + c\,n + \mathbf{2^2}\,\mathbf{T}(n/\mathbf{2^2}) \\
&= c\,n + c\,n + c\,n + \mathbf{2^3}\,\mathbf{T}(n\,/\mathbf{2^3}) \\
&= c\,n + \ldots(\log n \text{ terms})\ldots + c\,n \\
&= \mathbf{O}(n \,\mathbf{log}\, n)
\end{aligned}
$$

Proof of correctness of Merge-Sort

MSort(**A**,i,j)   // Sorting the subarray **A**[i..j].

```
{  If (  i< j  )
   {  mid←(i+j)/2;
      MSort(A,i,mid);
      MSort(A,mid+1,j);
      Create temporarily  C[0..j − i]
      Merge(A[i..mid], A[mid+1..j], C);
      Copy C[0..j − i]  to A[i..j]
}
```

Question: What is to be proved ?

Answer: **MSort(A,**i**,**j**)** sorts the subarray **A**[i..j]

Question: How to prove ?

Answer:

- By **induction** on the <u>length</u> ($j - i + 1$) of the subarray.
- Use correctness of the algorithm **Merge.**

Example 2

Faster algorithm for

multiplying two integers

Addition is faster than multiplication

Given: any two n-bit numbers **X** and **Y**

Question: how many **bit-operations** are required to compute **X+Y** ?

Answer: O(n)

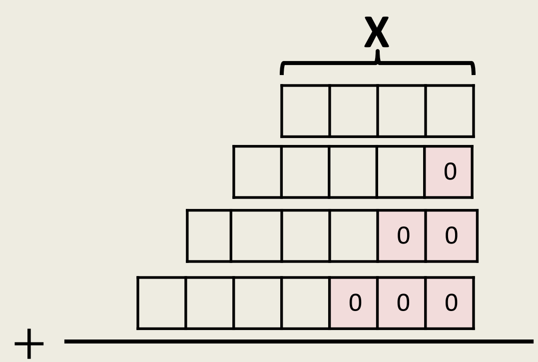
Question: how many **bit-operations** are required to compute **X*** $\mathbf{2^n}$?

Answer: O(n) [left shift the number **X** by n places, (do it carefully)]

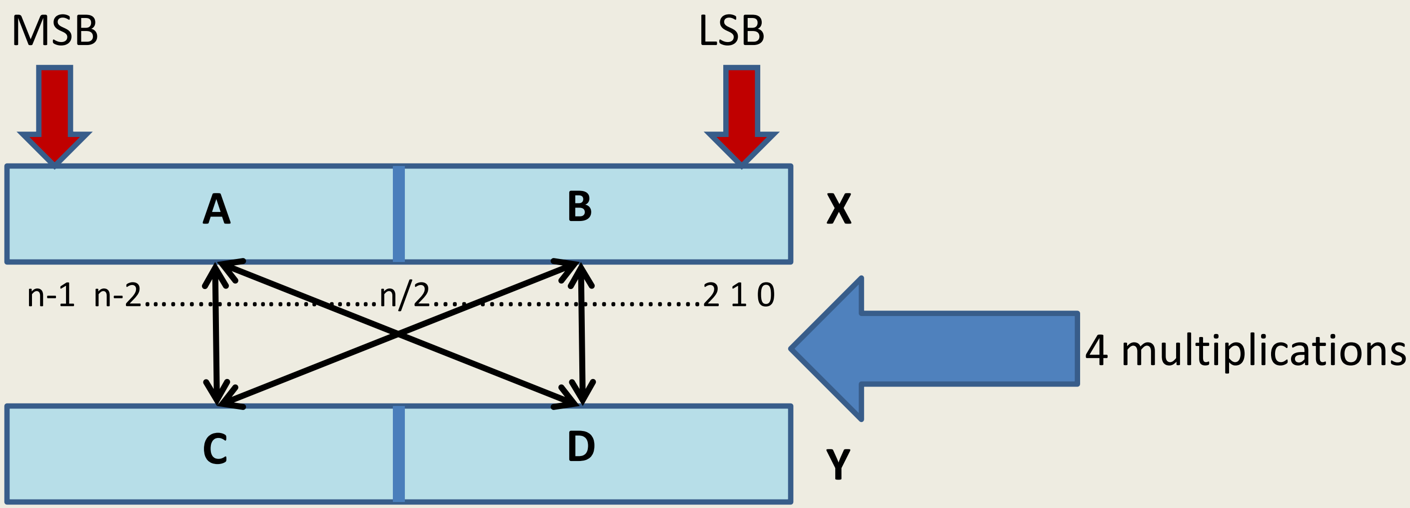
Question: how many **bit-operations** are required to compute **X*Y** ?

Answer: O(n^2)

Can we compute
X*Y faster ??



Pursuing Divide and Conquer approach



Question: how to express **X*Y** in terms of **multiplication/addition** of **{A,B,C,D} ?**

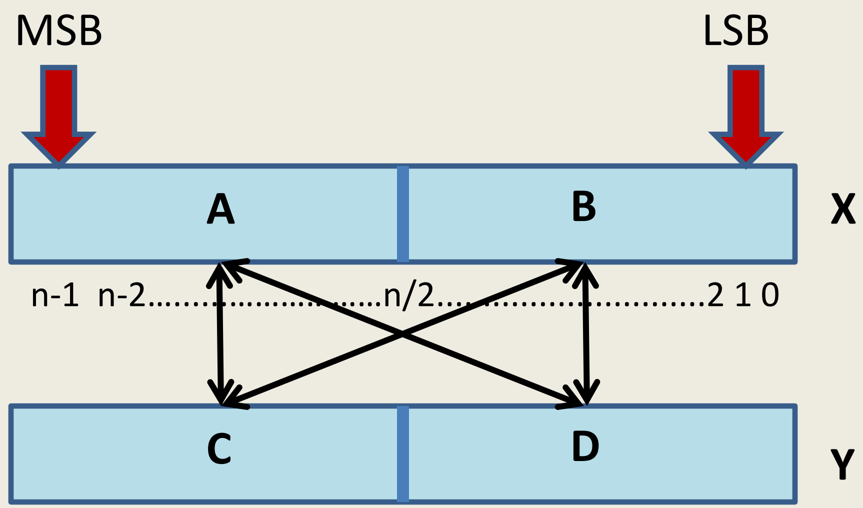
Hint: First Express **X** and **Y** in terms of **{A,B,C,D}.**

$$\mathbf{X} = \mathbf{A} * 2^{n/2} + \mathbf{B} \quad \text{and } \mathbf{Y} = \mathbf{C} * 2^{n/2} + \mathbf{D} \ .$$

Hence …

$$\mathbf{X*Y} = (\mathbf{A*C}) * \boxed{2^{n}} + (\mathbf{A*D} + \mathbf{B*C}) * \boxed{2^{n/2}} + \mathbf{B*D}$$

Pursuing Divide and Conquer approach



$$X*Y = (A*C)*\boxed{2^n} + (A*D + B*C)*\boxed{2^{n/2}} + B*D$$

Let $\mathbf{T}(n)$: time complexity of multiplying **X** and **Y** using the above equation.

$\mathbf{T}(n) = \mathbf{c}\, n + 4\, \mathbf{T}(n/2)$ for some constant **c**

$= \mathbf{c}\, n + 2\mathbf{c}\, n + 4^2\, \mathbf{T}(n/2^2)$

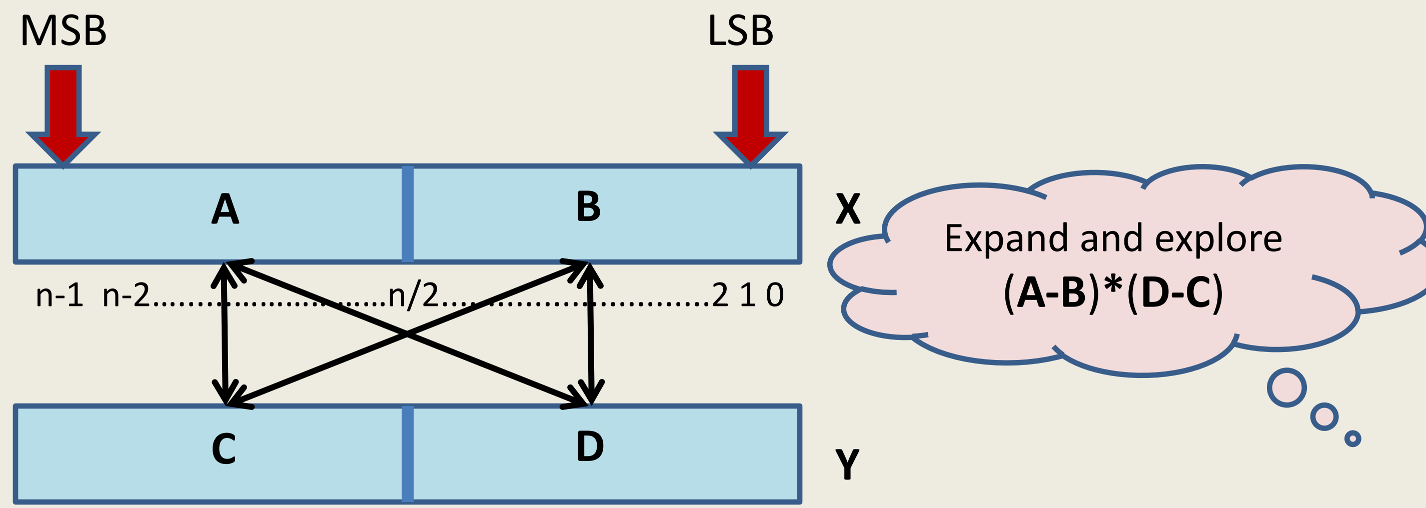
$= \mathbf{c}\, n + 2\mathbf{c}\, n + 4\mathbf{c}\, n + 4^3\, \mathbf{T}(n/2^3)$

$= \mathbf{c}\, n + 2\mathbf{c}\, n + 4\mathbf{c}\, n + 8\mathbf{c}\, n + \ldots + 4^{\log_2 n}\mathbf{T}(1)$

$= \mathbf{c}\, n + 2\mathbf{c}\, n + 4\mathbf{c}\, n + 8\mathbf{c}\, n + \ldots + \mathbf{c}\, n^2$

$O(n^2)$ time algo

Pursuing Divide and Conquer approach



$$X*Y = (A*C)*\boxed{2^n} + \boxed{(A*D + B*C)}*\boxed{2^{n/2}} + B*D$$

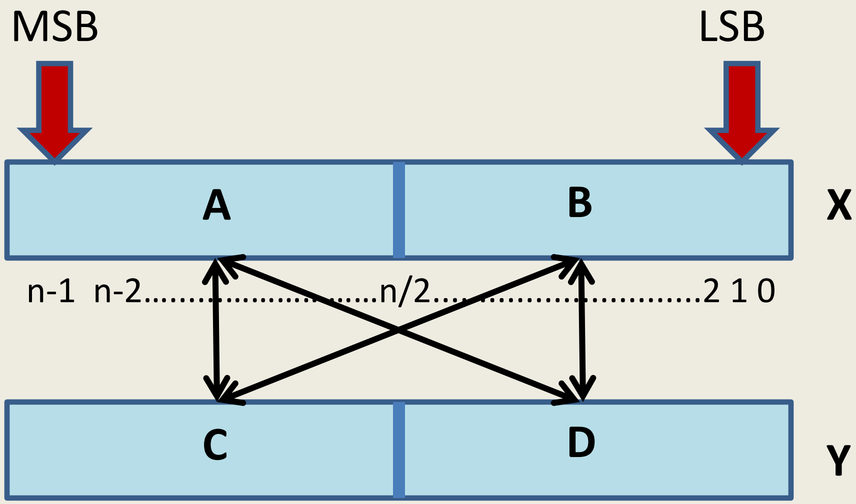
Observation: $A*D + B*C = (A-B)*(D-C) + A*C + B*D$

Question: How many multiplications do we need **now** to compute **X*Y** ?

Answer: **3** multiplications :

- **A*C**
- **B*D**
- **(A-B)*(D-C) .**

Pursuing Divide and Conquer approach



$$X*Y = (A*C)*\boxed{2^n} + \underline{((A-B)*(D-C) + A*C + B*D)}\boxed{2^{n/2}} + B*D$$

Let $T(n)$: time complexity of the **new algo** for multiplying two n-bit numbers

$$T(n) = c\,n + 3\,T(n/2) \text{ for some constant } c$$

$$= c\,n + 3\,c\frac{n}{2} + 3^2\,T(n/2^2)$$

$$= c\,n + 3c\frac{n}{2} + 9c\frac{n}{4} + + \ldots + 3^{\log_2 n}\,T(1)$$

$$= O(n^{\log_2 3}) = O(n^{1.58})$$

Conclusion

Theorem: There is a **divide and conquer** based algorithm for multiplying any two n-bit numbers in **O**($n^{1.58}$) time **(bit operations).**

Note:

The fastest algorithm for this problem runs in almost **O**(n **log** n) time.

Example 3

Counting the number of
“*inversions*” in an array

Counting Inversions in an array

Problem description

Definition (Inversion): Given an array **A** of size n,

a pair (i,j), $0 \le i < j < n$ is called an inversion if **A**$[i]>$**A**$[j]$.

Example:

	0	1	2	3	4	5	6	7
A	3	15	8	19	9	67	11	27

Inversions are :

(1,2), (1,4), (1,6),

(3,4),(3,6),

(5,6), (5,7)

AIM: An efficient algorithm to count the number of inversions in an array **A**.

Counting Inversions in an array

Problem familiarization

```
Trivial-algo(A[0..n-1])
{ count ← 0;
  For(j=1  to n-1) do
  {       For( i=0 to j-1 )
          {       If (A[i]>A[j])  count ← count + 1;
          }
  }
}
```

Ponder over the **divide and conquer algorithm** for this problem. We shall discuss it in the next class.

Time complexity: O(n^2)

Question: What can be the max. no. of inversions in an array **A** ?

Answer: $\binom{n}{2}$, which is O(n^2).

Question: Is the algorithm given above optimal ?

Answer: No, our aim is **not** to report all inversions but to report the count.