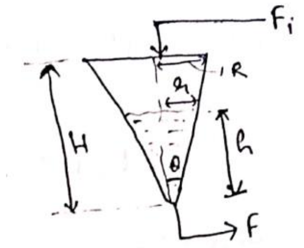
Assignment 2 Solution

1.

Solution:



$$\tan\theta = \frac{r}{h} = \frac{R}{H}$$

$$r = \frac{Rh}{H}$$

Apply material balance over a canonical tank

Accumulation= in-out

$$\frac{dV}{dt} = F_{in} - F \qquad \{\text{at any time, area } A = \pi r^2 = \pi\left(\frac{Rh}{H}\right)^2\}$$

$$\frac{dh}{dt} = \frac{F_{in}}{A} - \frac{F}{A}$$

$$\frac{dh}{dt} = \frac{F_{in}H^2}{\pi R^2 h^2} - \frac{FH^2}{\pi R^2 h^2} = \frac{\alpha F_{in}}{h^2} - \frac{\alpha F}{h^2}$$

$$\frac{dh}{dt} = \frac{\alpha F_{in}}{h^2} - \frac{A_0 \alpha k h}{h^2}$$

$$\frac{dh}{dt} = \frac{\alpha F_{in}}{h^2} - \beta h^{-1}$$

Now linearize the above equation,

$$\frac{dh}{dt} = \frac{\alpha F_i}{h^2} - \frac{\beta}{h} = F(h)$$

$$[\frac{\partial F}{\partial h}]_{h=h_s} = -\frac{2\alpha F_i}{{h_s}^3} + \frac{\beta}{{h_s}^2} = A_1$$

$$[\frac{\partial F}{\partial F_i}]_{h=h_s} = \frac{\alpha}{{h_s}^2} = A_2$$

$$\frac{dh}{dt} = \frac{\partial F}{\partial h})_{h=h_s}(h - h_s) + \frac{\partial F}{\partial h})_{F=F_{is}}(F_i - F_{is})$$

$$h' = h - h_s$$

$$F_i' = F_i - F_{is}$$

$$\frac{dh'}{dt} = A_1 h' + A_2 F_i'$$

$$sH(s) = A_1 H(s) + A_2 F_i(s)$$

$$\frac{H(s)}{F_i(s)} = \frac{A_2}{s - A_1}$$

$$Time\ constant = \frac{1}{A_1}$$

$$Gain = \frac{A_2}{A_1}$$

2. For given differential eqn.

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=f(t)$$

a) For unit step

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=1$$

Solution of this equation will be $y=y_c+y_p$

CF is

$$m^2+5m+6=1$$

$$m=-2,-3$$

$$y_c=c_1e^{-2t}+c_2e^{-3t}$$

Now applying the BC

$$y(0)=0,\left(\frac{dy}{dt}\right)_{(0)}=0$$

We get

$$c_1=\frac{-1}{2}$$

$$c_2=\frac{1}{3}$$

PI is

$$0+5(0)+6y_p=1\Rightarrow y_p=\frac{1}{6}$$

The final solution is

$$y=\frac{-1}{2}e^{-2t}+\frac{1}{3}e^{-3t}+\frac{1}{6}$$

b) For Ramp input

$$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=t$$

Solution of this equation will be $y=y_c+y_p$

CF is

$$m^2+5m+6=1$$

$$m=-2,-3$$

$$y_c=c_1e^{-2t}+c_2e^{-3t}$$

Now applying the BC

$$y(0)=0,\left(\frac{dy}{dt}\right)_{(0)}=0$$

We get

$$c_1 = \frac{-1}{2}$$
$$c_2 = \frac{1}{3}$$

PI is

$$y_p = (D+2)^{-1}(D+3)^{-1}(t)$$
$$y_p = \frac{1}{6}\left(1-\frac{D}{2}\right)\left(1-\frac{D}{3}\right)(t)$$
$$y_p = \frac{1}{6}\left[1-\frac{5D}{6}+\frac{D^2}{6}\right](t)$$
$$y_p = \frac{1}{6}\left[t-\frac{5}{6}\right]$$

The final solution is

$$y = \frac{-1}{2}e^{-2t} + \frac{1}{3}e^{-3t} + \frac{1}{6}\left[t-\frac{5}{6}\right]$$

3. Solve according to question 2(a).

4.

Solution:

$$\begin{array}{c} I + U \xrightarrow{k_1} I \xrightarrow{k_2} R \\ \downarrow k_3 \\ D \end{array}$$

$$\frac{dC_I}{dt} = -k_1 C_I C_U - k_2 C_I - k_3 C_I$$

$$\frac{dC_I}{dT} = -k_1 C_I C_U - (k_2 + k_3) C_I$$

Linearising $C_I C_U$ about its steady state, $\overline{C_I}$ and $\overline{C_U}$

$$C_I C_U = \overline{C_I}\,\overline{C_U} + \overline{C_I}(C_U - \overline{C_U}) + \overline{C_U}(C_I - \overline{C_I})$$

$X - \bar{X} = \hat{X}$, deviation variable

$$C_I C_U = \overline{C_I}\,\overline{C_U} + \overline{C_I}\,\widehat{C_U} + \overline{C_U}\,\widehat{C_I}$$
$$\frac{dC_I}{dt} + (k_2 + k_3) C_I = k_1 \overline{C_I}\,\overline{C_U} + k_1 \overline{C_I}\,\widehat{C_U} + k_1 \overline{C_U}\,\widehat{C_I}$$

Subtract $(k_2 + k_3)\overline{C_I}$ from both sides,

LHS = $\frac{dC_I}{dt} + (k_2 + k_3)(C_I - \overline{C_I})$

RHS = $k_1 \overline{C_I}\,\overline{C_U} - (k_2 + k_3)\overline{C_I} + k_1 \overline{C_I}\,\widehat{C_U} + k_1 \overline{C_U}\,\widehat{C_I}$

We have, $\frac{d\widehat{C_I}}{dt} + (k_2 + k_3)\widehat{C_I} = k_1\overline{C_I}\widehat{C_U} + k_1\overline{C_U}\widehat{C_I}$

$\frac{d\widehat{C_I}}{dt} + (k_2 + k_3 - k_1\overline{C_U})\widehat{C_I} = k_1\overline{C_I}\widehat{C_U}$

$(\frac{1}{k_2 + k_3 - k_1\overline{C_U}})\frac{d\widehat{C_I}}{dt} + \widehat{C_I} = (\frac{k_1\overline{C_I}}{k_2 + k_3 - k_1\overline{C_U}})\widehat{C_U}$

Solution, $\widehat{C_I} = \widehat{C_U}\,(\frac{k_1\overline{C_I}}{k_2+k_3-k_1\overline{C_U}})(1 - e^{(k_2+k_3-k_1C_U)(-t)})$

5.

Solution:

$$A \rightarrow R \rightarrow S$$

Now rate equation

$$r_A = \frac{dC_A}{dt} = -k_1C_A$$

This implies $C_A = C_{A0}e^{-k_1 t}$ (1)

$r_R = \frac{dC_R}{dt} = k_1C_A - k_2C_R$ (2)

From equation 2

$$\frac{dC_R}{dt} = k_1C_A - k_2C_R$$

$$\frac{dC_R}{dt} + k_2C_R = k_1C_A$$

This is of the form of linear equation

$$\frac{dy}{dx} + py = Q$$

So we can solve it by using an integration factor method

$$IF = e^{\int k_2\, dt}$$

This implies $C_Re^{k_2 t} = \int k_1C_A\, e^{k_2 t} + c$

On solving the above equation with BC that at t=0 $C_{R0} = 0$

$C_R = \frac{k_1C_{A0}}{(k_2-k_1)}(e^{-k_1 t} - e^{-k_2 t})$ (3)

For maximum value of R we need to differentiate equation 3 from where we get the for maximum value of C_R

$$\frac{dC_R}{dt} = \frac{k_1C_{A0}}{(k_2 - k_1)}\left(-k_1e^{-k_1 t} + k_2e^{-k_2 t}\right) = 0$$

On solving we get $t = \frac{\ln\frac{k_2}{k_1}}{(k_2-k_1)}$

This implies $t = \frac{1}{k_{logmean}}$

6.

Solution:

For P only controller

$u=K_c\, e(t)$ Where, $e(t) = (y_{sp}-y)$

So, equation becomes,

$$\tau \frac{dy}{dt} - y = KK_c\left(y_{sp} - y\right)$$

$$\tau \frac{dy}{dt} + y\left(KK_c - 1\right) = KK_c y_{sp}$$; Closed loop governing equation ODE

Closed loop characteristic equation:

$$\tau\lambda + \left(KK_c - 1\right) = 0$$

$$\lambda = -\frac{\left(KK_c - 1\right)}{\tau}$$ for stable system λ should be negative

So, $KK_c - 1 \geq 0 \Rightarrow K_c \geq 1/K$

Hence if $K_c > 1/K$ **the system with P controller is always stable**.

7.

Solution:

The general ODE for second order system is: $\tau^2 \frac{d^2y}{dt^2} + 2\xi\tau \frac{dy}{dt} + y = Ku\left(t\right)$

The solution of above differential equation is $y=y_p + y_c$

The complementary part y_c is the solution of the equation $\tau^2 \frac{d^2y}{dt^2} + 2\xi\tau \frac{dy}{dt} + y = 0$

Hence the characteristic eqn. of the above ODE is

$$\tau^2 m^2 + 2\xi\tau m + 1 = 0$$

$$m = \frac{-2\xi\tau \pm \sqrt{4\xi^2\tau^2 - 4\tau^2}}{2\tau^2}$$

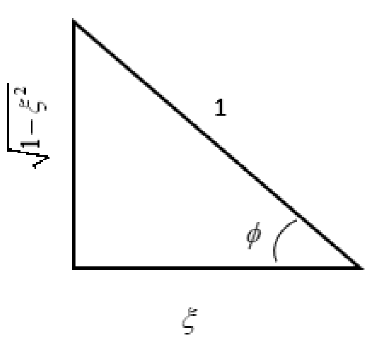
$$m = \frac{-\xi}{\tau} \pm \frac{\sqrt{\xi^2 - 1}}{\tau}$$

For $\xi < 1$; oscillatory response

$$m = \frac{-\xi}{\tau} \pm \frac{\sqrt{\xi^2 - 1}}{\tau} = \frac{-\xi}{\tau} \pm j\frac{\sqrt{1-\xi^2}}{\tau}$$

hence, $\lambda_1 = \dfrac{-\xi}{\tau} + j\dfrac{\sqrt{1-\xi^2}}{\tau} \qquad \lambda_2 = \dfrac{-\xi}{\tau} - j\dfrac{\sqrt{1-\xi^2}}{\tau}$

defining, $\tan\phi = \dfrac{\sqrt{1-\xi^2}}{\xi}$



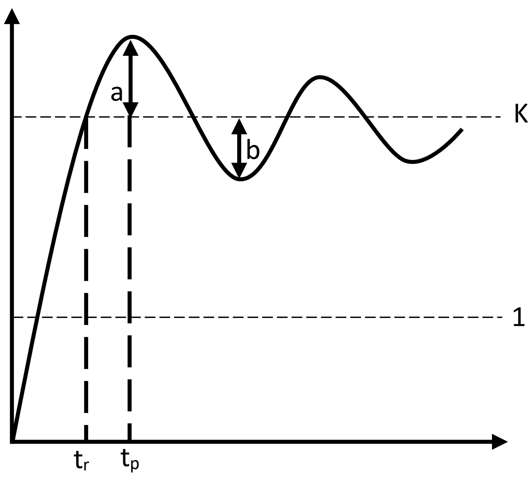
we get on solving,

a) $$y(t) = K\left[1 - \frac{\exp\left(\frac{-\xi}{\tau}t\right)}{\sqrt{1-\xi^2}}\sin\left(\frac{\sqrt{1-\xi^2}}{\tau}t + \phi\right)\right]$$ {sin term led to oscillation}

As $\xi \to 0$ $y(t) = K\left[1 - \sin\left(\frac{t}{\tau} + \frac{\Pi}{2}\right)\right]$, sustained oscillations.

Note:

- Real part of the root decides how quick will the oscillation will die out with time.
- Imaginary part decides the amplitude and time period.



- Decay ratio = b/a
- POR = a/k
- Rise time = t_r
- Peak time = t_p

b) $$\frac{t_r}{\tau} = \frac{\pi - \phi}{\sin\phi}$$

c) $$\frac{t_p}{\tau} = \frac{\phi}{\sin\phi}$$

d) $$\frac{a}{k} = \exp(-\pi\cot\phi)$$

e) $$\frac{b}{a} = \exp(-2\pi\cot\phi)$$

f) $$\frac{T_S}{\tau} = \frac{-\ln(f.\sin\phi)}{\cos\phi}$$

$$y(t)=K\left[1-\frac{\exp\left(\frac{-\xi}{\tau}t\right)}{\sqrt{1-\xi^2}}\sin\left(\frac{\sqrt{1-\xi^2}}{\tau}t+\phi\right)\right]$$

$$y(t)=K\left[1-\frac{\exp(-\xi\omega_n t)}{\sqrt{1-\xi^2}}\sin(\omega t+\phi)\right]$$

$$\Rightarrow y(t)=1-\frac{\exp(-\xi\omega_n t)}{\sqrt{1-\xi^2}}\sin(\omega t+\phi)$$

In order to find the settling time, we assume that it will settle in ±f % of the final value. This definition gives the following value

$$f=\frac{\exp(-\xi\omega_n t)}{\sqrt{1-\xi^2}}\sin(\omega t+\phi)$$

This above equation gives us the conservative estimate of $\cos\left(\omega_n\sqrt{1-\xi^2}t-\phi\right)=1$ at the settling time so the equation comes as follows

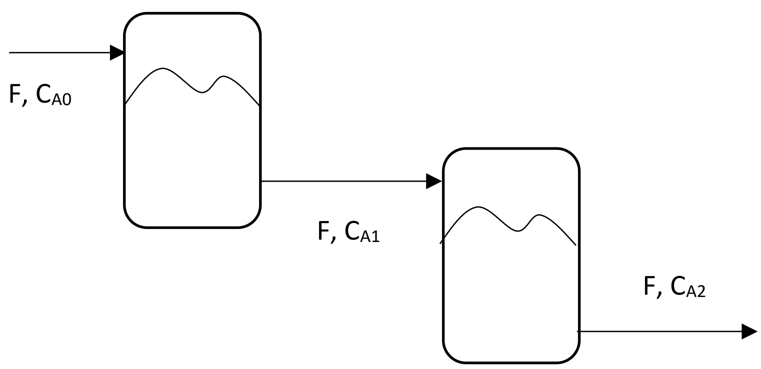
$$T_S=\frac{-\ln(f\sqrt{1-\xi^2})}{\xi\omega_n}$$

$$T_S=\frac{-\ln(f.\sin\phi)}{\omega_n.\cos\phi}$$

8.

Solution:

Dynamics of CSTR in Series, with step change in the Inlet Concentration, Servo Response



$\frac{VdC_{A1}}{dt}=FC_{A0}-FC_{A1}$ $\frac{VdC_{A2}}{dt}=FC_{A1}-FC_{A2}$ $\frac{V}{F}=\tau$

Equations leads to,

$\frac{\tau dC_{A1}}{dt} = C_{A0} - C_{A1}$, Eq (a)
$\frac{\tau dC_{A2}}{dt} = C_{A1} - C_{A2}$, Eq (b)

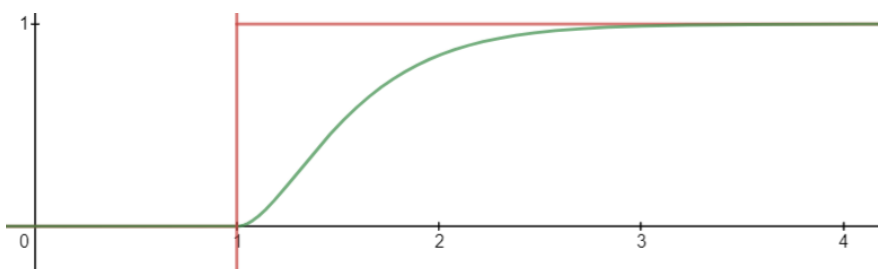
Solving Equation (a) by use of integrating factor
I.F. is $e^{\frac{t}{\tau}}$
Hence solution to the equation (a) is $e^{\frac{t}{\tau}}C_{A1} = e^{\frac{t}{\tau}}C_{A0} + I$
$C_{A0} = \overline{C_{A0}} + \Delta C_{A0}$
From initial condition i.e. at t=0, $C_{A1}=C_{A0}$ we get

$$C_{A1} = \overline{C_{A0}} + \Delta C_{A0}\left(1 - e^{\frac{-t}{\tau}}\right)$$

Similary solve equation (b) after use I.F we get
$e^{\frac{t}{\tau}}C_{A2} = \int \frac{1}{\tau} e^{\frac{t}{\tau}} C_{A1} dt$

Integrating above equation by parts as
$e^{\frac{t}{\tau}}C_{A2} = e^{\frac{t}{\tau}}C_{A1} - \int C'_{A1} e^{\frac{t}{\tau}} dt + I$ where $C'_{A1} = \frac{\Delta C_{A0}}{\tau} e^{\frac{-t}{\tau}}$
$e^{\frac{t}{\tau}}C_{A2} = e^{\frac{t}{\tau}}C_{A1} - \int \frac{\Delta C_{A0}}{\tau} e^{\frac{-t}{\tau}} e^{\frac{t}{\tau}} dt + I$
$C_{A2} = C_{A1} - \Delta C_{A0}\frac{t}{\tau}e^{\frac{-t}{\tau}} + Ie^{\frac{-t}{\tau}}$
$C_{A2} = \overline{C_{A0}} + \Delta C_{A0}\left(1 - e^{\frac{-t}{\tau}}\right) - \Delta C_{A0}\frac{t}{\tau}e^{\frac{-t}{\tau}} + Ie^{\frac{-t}{\tau}}$
From initail condition we get I=0
$C_{A2} = \overline{C_{A0}} + \Delta C_{A0}\left(1 - e^{\frac{-t}{\tau}} - \frac{t}{\tau}e^{\frac{-t}{\tau}}\right)$
So, the response of outlet concentration, C_{A2} a unit step in inlet concentration, C_{A0} is given by,

$$\boxed{C_{A2} - \overline{C_{A0}} = \Delta C_{A0}\left(1 - exp\left(-\frac{t}{\tau}\right)\left(1 + \frac{t}{\tau}\right)\right)}$$



9. Solution:

Let V_r= volume of each tank

$A_1=\rho_1 C_{p1}V_r$

$A_2=\rho_2 C_{p2}V_r$

$B_1= w_1 C_{p1}$

$B_2= w_2 C_{p2}$

The energy balance over 4 tanks gives ;

$A_2\frac{dT_6}{dt} = B_2(T_4 - T_6) + K(T_3 - T_6)$

$A_2\frac{dT_4}{dt} = B_2(T_2 - T_4) + K(T_5 - T_4)$

$A_1\frac{dT_5}{dt} = B_1(T_3 - T_5) + K(T_4 - T_5)$

$A_1\frac{dT_3}{dt} = B_1(T_1 - T_3) + K(T_6 - T_3)$

10. For PI controller

$$u = k_C e\{1 + \frac{1}{\tau_i}\int edt \}$$

We know that heated tank follows 1st order kinetics whose ODE is given as

$$\tau\frac{dy}{dt} + y = ku$$

Now,

$$\tau\frac{dy}{dt} + y = kk_C(y_{sp} - y)\{1 + \frac{1}{\tau_i}\int (y_{sp} - y)dt \}$$

$$\tau\frac{d^2y}{dt^2} + \frac{dy}{dt}(1 + kk_C) + \frac{y(kk_C)}{\tau_i} = \{y_{sp} + \tau_i\frac{dy_{sp}}{dt}\}\frac{kk_C}{\tau_i}$$

Characteristic Equation

$$\tau\lambda^2 + \lambda(1 + kk_C) + \frac{(kk_C)}{\tau_i} = 0$$

$$\lambda^2 + \lambda\frac{(1 + kk_C)}{\tau} + \frac{(kk_C)}{\tau\tau_i} = 0$$

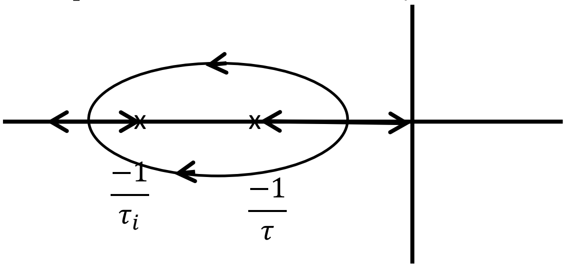
On solving this equation by quadratic formula

$$\lambda = -\frac{(1 + kk_C)}{2\tau} \pm \frac{(1 + kk_C)}{\tau}\sqrt{1 - \frac{4k\tau}{\tau_i}\frac{(k_C)}{(1 + kk_C)^2}}$$

Now for judging the nature of roots

$D = (1 - kk_C)^2 + 4kk_C(1 - r)$ $\qquad\qquad r = \frac{\tau}{\tau_i}$

For complex root r >1 $\qquad \tau > \tau_i$



Hence a PI controller is stable.

11. Solution:

a) For 1st order system

$u = K_c\, e(t)$ Where, $e(t) = (y_{sp} - y)$

So, equation becomes,

$$\tau \frac{dy}{dt} + y = KK_c\left(y_{sp} - y\right)$$

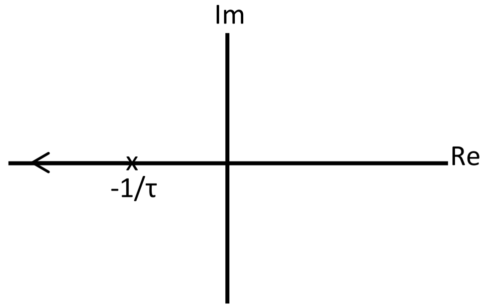
$$\tau \frac{dy}{dt} + y\left(KK_c + 1\right) = KK_c y_{sp}$$; Closed loop governing equation ODE

Closed loop characteristic equation:

$$\tau\lambda + \left(KK_c + 1\right) = 0$$

$$\lambda = -\frac{\left(KK_c + 1\right)}{\tau}$$ for stable system λ should be negative

Now, vary K_c from 0 to infinity ⟶ giving rise to root locus



Hence always stable.

b) For 2nd order system

$$u = k_C e\{t\}$$

We know

$$\tau_1 \tau_2 \frac{d^2y}{dt^2} + (\tau_1 + \tau_2)\frac{dy}{dt} + y = kk_C(y_{sp} - y)$$

$$\tau_1 \tau_2 \frac{d^2y}{dt^2} + \frac{dy}{dt}(\tau_1 + \tau_2) + y(1 + kk_C) = y_{sp} kk_C$$

Characteristic Equation

$$\tau_1 \tau_2 \lambda^2 + \lambda(\tau_1 + \tau_2) + (1 + kk_C) = 0$$

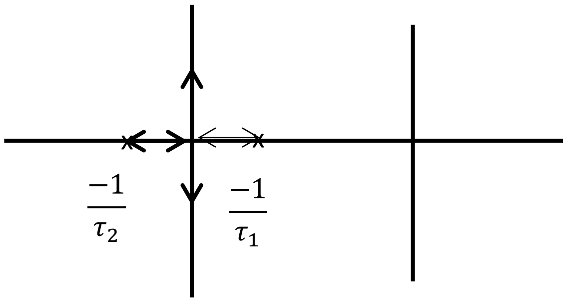
$$\lambda^2 + \lambda\frac{(\tau_1 + \tau_2)}{\tau_1 \tau_2} + \frac{(1 + kk_C)}{\tau_1 \tau_2} = 0$$

On solving this equation by quadratic formula

$$\lambda = -\frac{(\tau_1 + \tau_2)}{2\tau_1 \tau_2} \pm \frac{1}{2\tau_1 \tau_2}\sqrt{(\tau_1 + \tau_2)^2 - 4\tau_1 \tau_2 (1 + kk_C)}$$

Now for judging the nature of roots

$D = (\tau_1 - \tau_2)^2 - 4\tau_1\tau_2 k k_C$



Hence a P controller is stable.