MULTIVARIABLE SYSTEMS

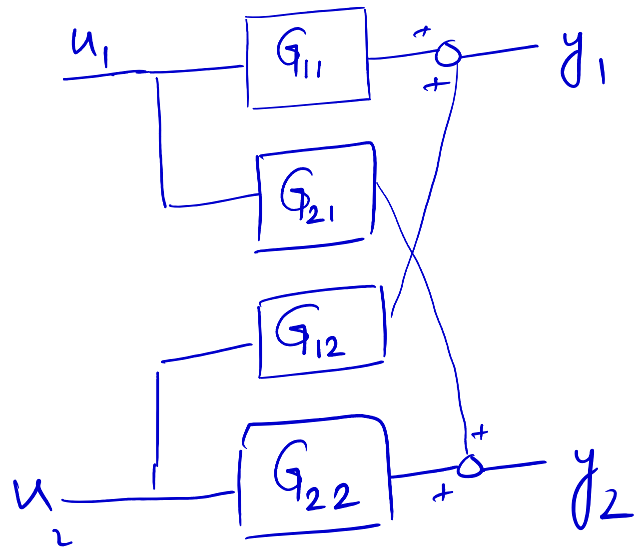
2×2 Process

u_1 y_1

u_2 y_2

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\boxed{\underline{y} = \underline{\underline{G_p}}\ \underline{u}}$$



$$y_1 = G_{11} u_1 + G_{12} u_2$$

$$y_2 = G_{21} u_1 + G_{22} u_2$$

Decentralized

$$u_1 = G_{c_1} e_1 \qquad e_1 = y_1^{sp} - y_1 \qquad e_2 = y_2^{sp} - y_2$$

$$u_2 = G_{c_2} e_2$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} G_{c_1} & 0 \\ 0 & G_{c_2} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \qquad \underline{u} = \underline{\underline{G_c}}\ \underline{e}$$

$$\underline{y} = \underline{\underline{G_p}}\,\underline{u} \qquad \underline{u} = \underline{\underline{G_c}}\,\underline{e}$$

$$\underline{y} = \underline{\underline{G_p}}\,\underline{\underline{G_c}}\,\underline{e}$$

$$\underline{y} = \underline{\underline{G_p}}\,\underline{\underline{G_c}}\left[\underline{y}^{sp} - \underline{y}\right]$$

$$\left[\underline{\underline{I}} + \underline{\underline{G_p}}\,\underline{\underline{G_c}}\right]\underline{y} = \underline{\underline{G_p}}\,\underline{\underline{G_c}}\,\underline{y}^{sp}$$

$$\underline{y} = \underbrace{\left[\underline{\underline{I}} + \underline{\underline{G_p}}\,\underline{\underline{G_c}}\right]^{-1}\underline{\underline{G_p}}\,\underline{\underline{G_c}}}_{\text{matrix of transfer fns}}\,\underline{y}^{sp}$$

$I + G_p G_c = A$

$$\begin{bmatrix} \frac{\bigcirc}{|A|} & \frac{\bigcirc}{|A|} \\ \frac{\bigcirc}{|A|} & \frac{\bigcirc}{|A|} \end{bmatrix}$$

Multivariable CLCE

$$\boxed{\left|\underline{\underline{I}} + \underline{\underline{G_p}}\,\underline{\underline{G_c}}\right| = 0}$$

$$\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} G_{c_1} & 0 \\ 0 & G_{c_2} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I + \underline{\underline{G_p}}\,\underline{\underline{G_c}} = \begin{bmatrix} G_{11}G_{c_1}+1 & G_{12}G_{c_2} \\ \underline{\underline{G_{21}}}G_{c_1} & G_{22}G_{c_2}+1 \end{bmatrix}$$

$$|I + \underline{\underline{G_p}}\,\underline{\underline{G_c}}| = (1 + G_{11}G_{c_1})(1 + G_{22}G_{c_2}) - G_{12}G_{21}G_{c_1}G_{c_2}$$

$$\Rightarrow CLCE^{MV}: \quad 1 + G_{11}G_{c_1} + G_{22}G_{c_2} + G_{11}G_{22}G_{c_1}G_{c_2} - G_{12}G_{21}G_{c_1}G_{c_2} = 0$$

$$\rightarrow 1 + G_{11}G_{c_1} + G_{22}G_{c_2} + G_{c_1}G_{c_2}\,\underline{[G_{11}G_{22} - G_{12}G_{21}]} = 0$$

Figure of $\underline{\underline{8}}$

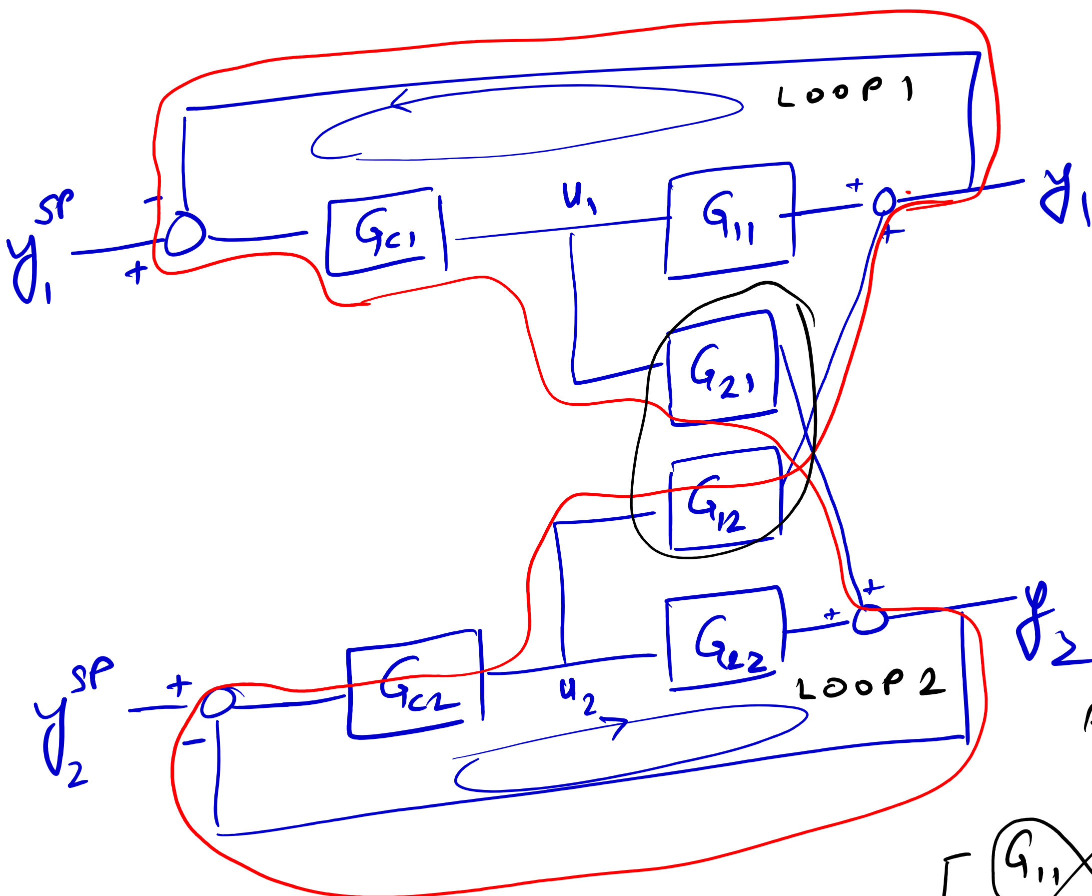
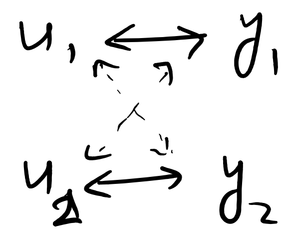
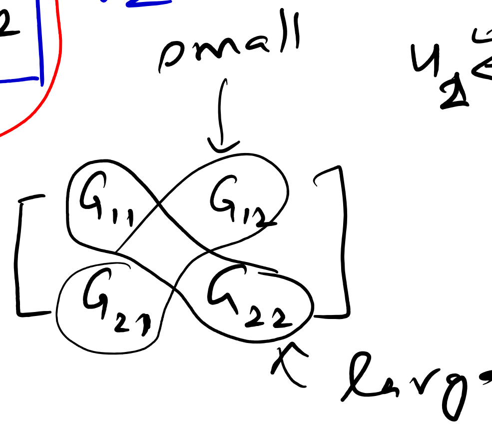


Figure of 8

Extra feedback loop

Loop Interaction





Diagonally Dominant Pairing

$$y_1 = 2u_1 + 4u_2$$

$$y_2 = 3u_1 + 1u_2$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$y_1 - u_1$

$y_2 - u_2$ ✗

small

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$$

large

$y_1 - u_2$ ✓

$y_2 - u_1$

$\because$ diagonally dominant

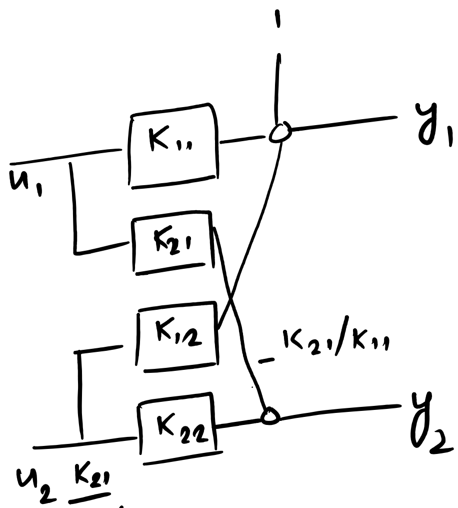
Interaction Metrics

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$NI = \frac{|K|}{\prod_i K_{ii}}$$

- Niederlinski Index (NI)
- Relative Gain Array (RGA)

NI



y_1	u_1	y_2	u_2
1	$-\frac{1}{K_{11}}$	$-\frac{K_{21}}{K_{11}}$	$+\frac{K_{21}}{K_{11}K_{22}}$
$\frac{K_{12}K_{21}}{K_{11}K_{22}}$	$-\frac{1}{K_{11}}\left(\frac{K_{21}K_{12}}{K_{11}K_{22}}\right)$	$-\frac{K_{21}}{K_{11}}\left(\frac{K_{21}K_{12}}{K_{11}K_{22}}\right)$	$+\frac{K_{21}}{K_{11}K_{22}}\left(\frac{K_{21}K_{12}}{K_{11}K_{22}}\right)$
$\left(\frac{K_{12}K_{21}}{K_{11}K_{22}}\right)^2$			$\frac{K_{21}}{K_{11}K_{22}}\left(\frac{K_{21}K_{12}}{K_{11}K_{22}}\right)^2$

$$u_2 = \frac{K_{21}}{K_{11}K_{22}}\left[1 + \frac{K_{21}K_{12}}{K_{11}K_{22}} + \left(\frac{K_{21}K_{12}}{K_{11}K_{22}}\right)^2 + \ldots\right]$$

Unstable if

$$\frac{K_{21}K_{12}}{K_{11}K_{22}} > 1$$

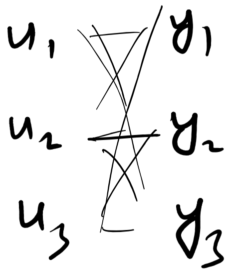
$$\boxed{\frac{K_{11}K_{22} - K_{12}K_{21}}{K_{11}K_{22}} < 0}$$

$$NI = \frac{|K|}{\prod K_{ii}}$$

$NI < 0$ Guaranteed integrally unstable

$NI > 0 \nRightarrow$ Guaranteed integral stability

Necessary but not sufficient condition



1	1	1	2	1	3	1	1
2	2	2	1	2	2	2	3
3	3	3	1	3	1	3	2

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{13} & K_{12} \\ K_{21} & K_{23} & K_{22} \\ K_{31} & K_{33} & K_{32} \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \\ u_2 \end{bmatrix}$$

corr. to that pairing

$$NI = \frac{|K|}{\prod K_{ii}}$$

$NI < 0$

REJECTED

RELATIVE GAIN ARRAY

$i \quad y_1 \; y_2 \; \ldots \; y_N$

$j \quad u_1 \; u_2 \; \ldots \; u_N$

$$\lambda_{ij} \triangleq \frac{\partial y_i / \partial u_j \big|_{u_{k \neq j}}}{\partial y_i / \partial u_j \big|_{y_{k \neq i}}}$$

$\big|_{u_{k \neq j}}$ ← All other inputs constant ie all other controllers off.

$\big|_{y_{k \neq i}}$ ← All other outputs constant. ie all other controllers "on"

$\lambda_{ij} \sim 1$ Prefer such pairings

$\lambda_{ij} < 0 \Rightarrow$ Loop Gain sign flips depending on whether other loops are on or off

Avoid such pairings

$$\lambda_{ij} = \frac{\partial y_i / \partial u_j \, \big|_{u_{k \neq j}} \leftarrow k_{ij}}{\partial y_i / \partial u_j \, \big|_{y_{k \neq i}} \leftarrow 1/k'_{ji}} = \frac{K_{ij} \leftarrow}{\quad}$$

$$y_i = k_{ij} u_j$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_N \end{bmatrix} = \overset{K}{\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1j} & \cdots & k_{1N} \\ k_{21} & k_{22} & \cdots & k_{2j} & \cdots & k_{2N} \\ & & \vdots & & & \\ k_{i1} & k_{i2} & \cdots & k_{ij} & \cdots & k_{iN} \\ & & \vdots & & & \\ k_{N1} & k_{N2} & \cdots & k_{Nj} & \cdots & k_{NN} \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \\ u_N \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_j \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} k'_{11} & k'_{12} & \cdots & k'_{1i} & \cdots & k'_{1N} \\ k'_{21} & k'_{22} & \cdots & k'_{2i} & \cdots & k'_{2N} \\ & & & \vdots & & \\ k'_{j1} & k'_{j2} & \cdots & k'_{ji} & \cdots & k'_{jN} \\ & & & \vdots & & \\ k'_{N1} & k'_{N2} & & k'_{Ni} & \cdots & k'_{NN} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_N \end{bmatrix}$$

$$\lambda_{ij} = k_{ij} \cdot \underline{k'_{ji}}$$

$$\boxed{\Lambda = K \, (.*) \, (K^{-1})^T}$$

$$u_j = k'_{ji} \, y_i$$

$$\frac{u_j}{y_i} = k'_{ji}$$

$$\begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & \bigcirc & \bigcirc \end{bmatrix}$$

$$NI = \frac{|K|}{\prod K_{ii}}$$

$< \underline{\underline{0}} \Rightarrow$ instability

$$RGA = K \; .* \; (K^{-1})^T$$

Prefer ij pairings at $\lambda_{ij} \sim 1$

Avoid $\lambda_{ij} < 0$