Forwards Futures and Swaps

IME 611

Derivatives

- A security whose payoff is explicitly tied to the value of some other financial security
  - Example: certificate that can be redeemed in six months for an amount equal to the price, then, of a share of IBM stock
    - The certificate is a derivative security since its payoff depends on the future price of IBM

- Forward contract: a forward contract to purchase 2,000 kgs of sugar at 12 cents per kg in 6 weeks

- Option: a contract that gives one the right to purchase 100 shares of Infosys stock for ₹100 per share in exactly 3 months

Derivative Pricing principles

- The security that determines the value of a derivative security is called the underlying security

- Pricing principles apply to well-functioning markets that satisfy a set of perfect market assumptions
  - It is possible to buy, sell, or short-sell any asset
  - There are no transactions costs or taxes
  - No one person's action influences prices
  - Every asset is infinitely devisable
  - No arbitrage opportunity exists in the market

Derivative Pricing principles

1. Use the market
   a. Market to compare and execute strategies

2. Discount cash at the current rate of interest
   a. Discounting cash at the rate of interest – the current value of ₹1000 to be paid in 1 year and the interest rate is 10%

3. Use linear pricing
   a. The total value of 'a' units of A, and 'b' units of B: $aV_A + bV_B$

Forward Contracts

- A forward contract on a security is a contract agreed upon at date $t = 0$ to purchase or sell the security at date T for a price, F, that is specified at $t = 0$

- When a forward contract is designed at $t = 0$, the forward price, F, is set in such a way that the initial value of the forward contract, forward value f_0, satisfies $f_0 = 0$

- At maturity date, T, the forward value is $f_0 = \pm (S_T - F)$
  - Long position in contract: $(S_T - F)$
  - Short position in contract: $(F - S_T)$

Computing Forward Prices

- Forward price – the delivery price of a unit of the underlying asset to be delivered at a specific future date
- Assumptions
  - Security has zero storage cost
  - Short selling is allowed
- The forward price, F, at $t = 0$ for delivery of the security at date T is given by $F = S/d(0,T)$
  - Where $d(0,T)$ is the discount factor and S is the current spot price (at $t = 0$)

Proof: Forward Price

Case 1. $F > S/d(0,T)$

At t = 0	Initial cost	Final receipt
Borrow S	-S	$-S/d(0,T)$
Buy 1 unit of asset	S	0
Short 1 forward	0	F
Total	**0**	$\boldsymbol{F - S/d(0,T)}$

A positive profit of $F - S/d(0,T)$ for zero net investment – arbitrage

Proof: Forward Price

Case 2. $F < S/d(0,T)$

At t = 0	Initial cost	Final receipt
Lend S	S	$S/d(0,T)$
Short 1 unit of asset	-S	0
Long 1 forward	0	-F
Total	**0**	$\boldsymbol{S/d(0,T) - F}$

A positive profit of $S/d(0,T) - F$ for zero net investment – arbitrage

Example: Forward Price

- Consider a forward contract on a non-dividend paying stock that matures in 6 months. The current stock price is ₹50 and the interest rate per annum is 4%. Compute the Forward price, F.
  - $F = S/d(0,T)$
  - $S = 50,\ d(0,0.5) = \frac{1}{1.02}$
  - $F = 50*1.02=51$
- Continuous-time compounding
  - If there is a constant interest rate r compounded continuously, the forward price becomes $F = Se^{rT}$

Cost of Carry (non-zero storage costs)

- Holding a physical asset such as gold, cereals, cement etc. entails storage costs
- Discrete multi-period model with delivery date T is M periods in the future.
  - Storage cost $c(k)$ is paid periodically per unit in the period k to $k+1$ (payable at the beginning of the period)

- Suppose an asset has a holding cost of $c(k)$ per unit in period k, and the asset can be sold short. The initial spot price is S, then the forward price, F is
  - $F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}$

Proof: Non-zero Storage Cost

- Consider the following strategy
  - Buy one unit of the asset
  - Enter a forward to sell one unit at time T
  - Cash flow associated with the strategy
    - $(-S - c(0), -c(1), \dots, -c(j-1), \dots, c(M-1), F)$
    - The present value of this stream must be zero.
    - $Fd(0,M) - S - \sum_{k=0}^{M-1} c(k)\left(\frac{d(0,M)}{d(k,M)}\right) = 0$
- $F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}$

Example: Cost of Carry

- *Consider a Treasury bond with a face value of ₹10,000, a coupon of 8%, and 10 years to maturity. Currently the bond is selling for ₹9,260. The previous coupon has just been paid, further coupons will be paid at 6 months and 1 year. What is the forward price for delivery of this bond in 1 year? The interest rate for 1 year is 9%.*
    - *Coupons are paid at the end of each period*
    - $F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k+1,M)}$
    - S=9260, M=2, $d(0,1) = \frac{1}{1.045}, d(0,2) = \frac{1}{1.045^2}, d(1,2) = \frac{1}{1.045}$
    - $c(1) = 400, c(2) = 400$
    - $F = 9260 * 1.045^2 - 400 * 1.045 - 400 =$ 9294.15

Tight markets

- It is not always possible to short commodities
  - Scarce in supply
  - Holders of the commodity are not willing to lend – the commodity has utility value over and beyond its spot market
- The theoretical relation does hold in one direction if storage is possible
  - Arbitrage exists for $(F > S/d(0,T))$, hence $F \le S/d(0,T)$
  - $F \le \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}$

At t = 0	Initial cost	Final receipt
Borrow S	-S	$-S/d(0,T)$
Buy 1 unit of asset	S	0
Short 1 forward	0	F
Total	**0**	$\boldsymbol{F - S/d(0,T)}$

Tight markets: convenience yield

- The variable used to restore the equality is known as convenience yield
  - $F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)-y}{d(k,M)}$

Value of a Forward Contract $t > 0$

- **Forward Value:** Suppose a forward contract for delivery at time T in the future has a delivery price F_0 and a current forward price F_t . The value of the contract (f_t, t>0) is

$$f_t = (F_t - F_0)d(t, T)$$

- where $d(t, T)$ is the risk-free discount factor over the period from t to T

- Consider the following strategy
  - t=0, long one unit of forward contract (F_0) with maturity T
  - t=t, long one unit of forward contract (F_t) with maturity T, short forward contract (F_0)
  - The above strategy has deterministic cash flow of (F_0- F_t) at T
- Therefore, $f_t + (F_0 - F_t)d(t, T)$=0

Swaps

- A swap is an agreement to exchange one cash flow stream for another
- Plain vanilla swap: one party swaps a series of variable payments for a series of fixed-level payments.
- Plain vanilla interest rate swap: one party swaps a series of variable payments for a series of fixed-level payments
  - A: makes semiannual payments at a fixed rate of interest on a notional principal (There is no loan, this principal simply sets the level of the payments) to B
  - B: makes semiannual payments at a floating rate of interest on the same principal
  - There are M periods with maturity at T

Pricing Interest Rate Swap

- Assuming the payments are made at the end of periods, then the total aggregate cash flow from party A's (long) perspective
  - $C = P \times \left(\underbrace{0}_{t=0}, \underbrace{r_0 - r_f}_{t=1}, \dots, \underbrace{r_{M-1} - r_f}_{t=M} \right)$
  - r_f represents the fixed rate, and r_i floating rate at the beginning of period i
  - The payments are made at the end of each period and the floating rate payment is based on the short rate prevailed at the beginning of the period
  - r_f is chosen such that the initial value of swap is 0
  - Long: A (payments in fixed rate)
  - Short: B (payments in floating rate)

Pricing Interest Rate Swap

- Consider the cash flows for a bond with face value 1: $r_0 - r_f, \dots, r_{M-1} - r_f$
- Value of the contract with M payments
  - One with M fixed payments: $\left(r_f, \dots, r_f\right) = r_f \sum_{i=1}^{M} d(0, i)$
  - Second has random stream of payments $\sum_{i=1}^{M} r_i d(0, i)$
    - Price of floating rate bond is always at par, $d(0, M)$
    - Value of this stream is 1- $d(0, M)$
    - Value includes the fixed stream 1- $d(0, M)$- $r_f \sum_{i=1}^{M} d(0, i)$
    - Total value of the swap $P(1 - d(0, M) - r_f \sum_{i=1}^{M} d(0, i))$

Commodity Swap

- Part A receives spot price for N units of commodity each period while paying a fixed amount X per unit for N units
- Net cash flow received by A over M periods $N(0, S_1 - X, S_2 - X, \ldots, S_M - X)$, S_i denotes the spot price at the beginning of period i
  - Payments take place at the beginning of each period, with total M payments
- The cash flow, C, by A for the swap is, $C = N(0, S_1 - X, S_2 - X, \ldots, S_M - X)$
  - Random stream, NS_i, and fixed stream, - NX

Pricing a Commodity Swap

- We can see the random stream, NS_i, has the same value of receiving NF_i at period i, where F_i is forward price at $t = 0$, for delivery of one unit of commodity at i
- The forward prices are deterministic and known at $t = 0$, hence the value of commodity swap is

$$\mathrm{V} = N \sum_{i=1}^{M} d(0, i)\,(F_i - X)$$

- X is chosen such that $V = 0$

Currency Swaps

- An agreement between two parties to exchange fixed rate interest payments and the principal on a loan in one currency for fixed rate interest payments and the principal on a loan in another currency
  - Uncertainty in the currency exchange rate
- Example: In a Dollar/Euro swap, a US company may receive the Euro payments of the swap while a German company might receive the dollar payments
  - US company wishes to invest in Europe, while the German company wishes to invest in US
  - Comparative advantage of borrowing in their domestic currency at home as opposed to borrowing in a foreign currency abroad

Futures

- Limitations of forward contracts
  - Forward contracts are not organized through an exchange
  - Forward contracts are not-exchange traded – problems with price transparency and liquidity
  - No standardization of forward prices – contract issued today vs contract issued tomorrow
    - Exchange will have to keep track of all such contracts

- Futures market as an alternative to forward market
  - Multiple delivery prices are eliminated by revising contracts as the price environment changes

Futures

- Contracts are initially written at F_0, and then the next day the price for new contracts is F_1
- At the second day, the exchange revises all the earlier contracts to the new delivery price F_1

- Suppose $F_1 > \ F_0$, the person with one-unit long position with price F_0 receives $F_1 - F_0$
  - Later she must pay F_1 rather than F_0

- The process of adjusting the contract: marking to market
  - An individual is required to open a margin account with a broker - must contain a specified amount of cash for each futures contract (margin requirement)

Mechanics of Futures

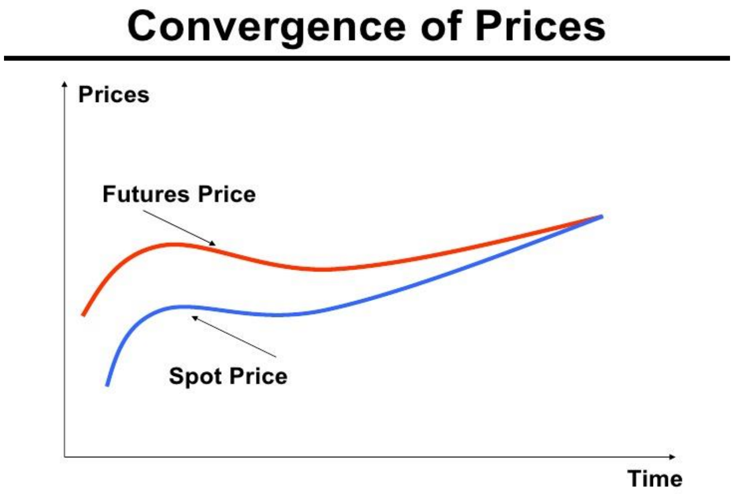
- Margin accounts are marked to market at the end of each trading day
  - If the price of the futures contract (price determined on the exchange) increased that day, then long parties receive a profit equal to the price change times the contract quantity, the profit is deposited in their margin accounts.
  - Vice-versa for short parties
  - With these adjustments, every long futures contract holder has the same contract, as does every short holder
  - At the delivery, delivery is made at the futures contract price at that time

Margins

- Margin accounts
  - Accounts to collect or pay out daily profits
  - Guarantee that contract holders will not default on their obligations
  - If the value of a margin account falls below a defined maintenance margin level (usually 75% of initial requirement), a margin call is issued
  - Margin call demands additional margin, otherwise futures position is closed out by taking an opposite position

Forward-futures equivalent

- Suppose that the interest rates are deterministic and follow expectation dynamics, then the theoretical futures and forward prices of corresponding contracts are identical
- The value of existing futures contract is zero.
  - Because they are marked to market



Forward-futures equivalent

- Initial futures price, F_0, and the corresponding forward price G_0
  - $T + 1$ time points, and T periods
  - $d(j, k)$ is discount rate at time j, for a bond of unit face value at time k $(j < k)$ – remains constant in $(j, j + 1)$

- Strategy A
  - $t = 0$, go long $d(1, T)$ futures
    - Profit at $t = 1$, $(F_1 - F_0)$d(1,T)
    - Invest this profit at $t = 1$ in the interest rate market until T
    - The final amount is, $\frac{(F_1 - F_0) d(1,T)}{d(1,T)} = (F_1 - F_0)$

  - At t=k, increase position to $d(k+1,T)$
    - Final amount $(F_{k+1} - F_k)$
  - Total profit from strategy A: $\sum_{k=0}^{T-1} (F_{k+1} - F_k) = S_T - F_0$

Forward-futures equivalent

- Strategy B: take long position in one forward contract
  - Profit: $S_T - G_0$
- Consider a new strategy, A-B
  - No cash flow until T
  - Generates profit, $(G_0 - F_0)$ - a deterministic amount
  - To avoid arbitrage it must be zero at $t = 0$
- When interest rate is not deterministic, the equivalence may no hold
  - But equivalence is usually accurate for routine analysis

Hedging with Futures

- Primary use of futures contract is to hedge against risk
- **Perfect hedge** – the risk associated with a future commitment to deliver or receive an asset is eliminated by taking an equal and opposite position in the futures market

- It may not be possible to create perfect hedge using futures (forward) contracts
  - Delivery dates of the contracts may not match
  - Lack of liquidity in the futures market
  - Amount of the asset obligated may not be an integral multiple of the contract size
  - The delivery terms may not coincide with those of the obligation
- How to reduce the original risk?

Minimum-Variance Hedge

- Reduce risk to the extent possible
- **Basis**: a measure of the lack of hedging perfection
  - Basis = spot price of asset to be hedged – futures price of contract used
    - Will be zero at the delivery date if asset to be hedged is identical to that of the futures contract

- Suppose at $t = 0$, the asset to be hedged is described by a cash follow x at T
  - Example: for the obligation to purchase W units of an asset at T, $x = -WS$, S is spot price at T
  - Let F is the futures price of the contract used to hedge

Minimum-Variance Hedge

- Let h denote the futures position taken
  - Neglect transaction and interest payments
- Cash flow at T (y)= (original obligation + the profit in the futures account) = $x + (F_T - F_0)h$
  - Var(y) = var($x + (F_T - F_0)h$)

    = var(x) + var($(F_T - F_0)h$)+2cov(x, $(F_T - F_0)h$)

    = var(x) + h^2 var(F_T) + 2$h$cov(x,F_T)
  - The above expression is minimized for $h = \frac{-\text{cov}(x, F_T)}{\text{var}(F_T)} = \frac{-\text{cov}(-WS_T, F_T)}{\text{var}(F_T)} = \frac{\text{cov}(S_T, F_T)}{\text{var}(F_T)} W = \beta W$