Time Value of Money

IME 611

Future Values and Present Values

- Calculating Future Values
  - Future Value
    - Amount to which investment will grow after earning interest
  - Present Value
    - Value today of future cash flow

Future Values and Present Values

- Future Value of $100 =

$$FV = \$100 \times (1+r)^t$$

- Example: FV
  - What is the future value of $100 if interest is compounded annually at a rate of 7% for two years?

$$FV = \$100 \times (1.07) \times (1.07) = \$114.49$$

$$FV = \$100 \times (1+.07)^2 = \$114.49$$

Future Values

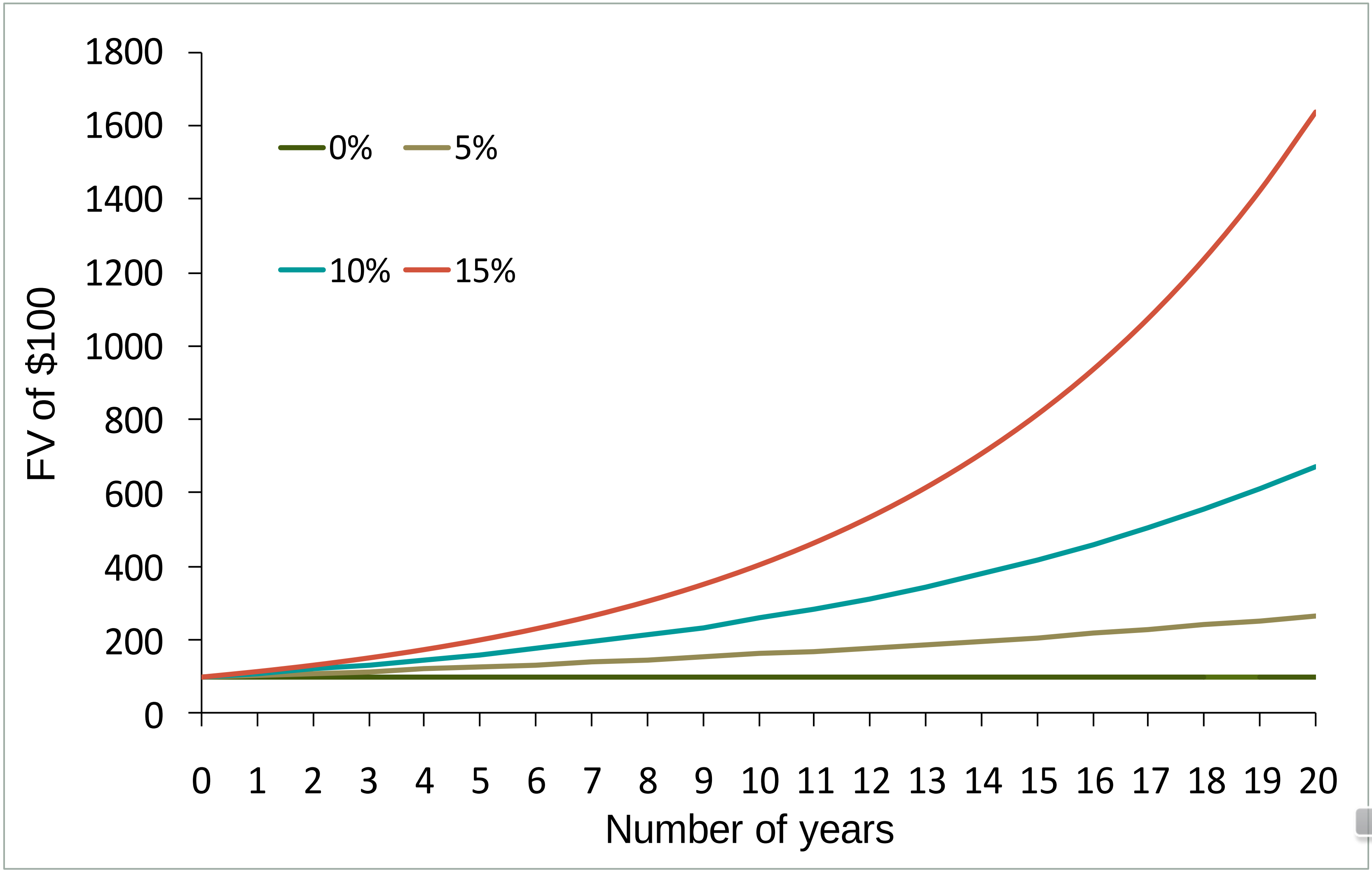
- **The seven-ten rule** *Money invested at 7% per year doubles in approximately 10 years. Also, money invested at 10% per year doubles in approximately 7 years*
  - Doubling time is approx. $\ln 2 / r$

- Compounding at Various Intervals
  - Many banks pay interest more frequently – quarterly, monthly, or even daily
  - Usually, interest is stated on yearly basis, and appropriate proportion of that interest rate is compounded each period
  - Example: quarterly, $\left[1 + \left(\frac{r}{4}\right)\right]^4$

Compound interest

- **Effective interest rate** equivalent yearly interest rate that would produce the same result after 1 year without compounding
  - $1 + r' = \left[1 + \left(\frac{r}{m}\right)\right]^m$ lim $m \longrightarrow \infty$
- **Continuous compounding**
  - $\lim_{m\to\infty} \left[1 + \left(\frac{r}{m}\right)\right]^m \to e^r$
  - If the nominal interest rate is 8%, then with continuous compounding $r' = 8.33\%$
  - For arbitrary length of time t, $t \approx k/m$, that is, k periods coincide with the time t
  - $\lim_{m\to\infty} \left[1 + \left(\frac{r}{m}\right)\right]^{k = mt} \to e^{rt}$

Future Values with Compounding



Present Values

- Present Value = PV

- PV = Discount Factor (DF) x C_1

Future Values and Present Values

- Discount factor = DF = PV of $1

$$DF = \frac{1}{(1+r)^t}$$

- Discount factors can be used to compute present value of any cash flow

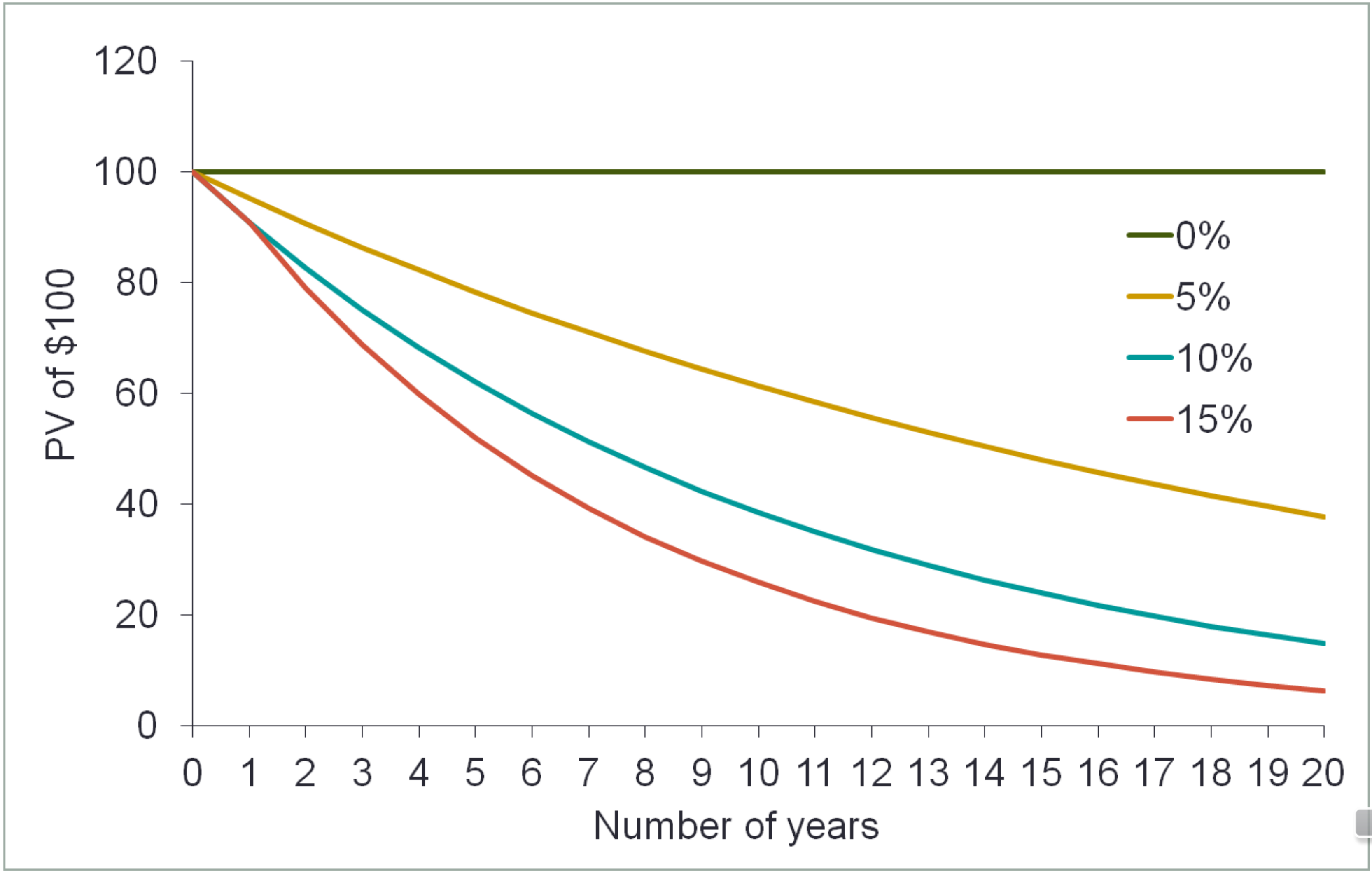
Present Values

- Given any variables in the equation, one can solve for the remaining variable

$$\mathrm{PV} = \mathrm{DF}_2 \times C_2$$

$$\mathrm{PV} = \frac{1}{(1+.07)^2} \times 114.49 = 100$$

Present Values with Compounding



Present and Future Values of Streams

- **Ideal Bank** An ideal bank applies the same rate of interest to both deposits and loans – no service charge and transaction fees
  - **Constant Ideal Bank** An Ideal Bank whose interest rate is independent of the length of time for which it applies

Future Value of a Stream

- *Given a cash flow stream* $(x_0, x_1, \dots x_n)$ *and interest rate r each period, the future value of the stream is*
  $$FV = x_0(1+r)^n + x_1(1+r)^{n-1} + \cdots + x_n$$
  - **Example:** cash stream $(-2, 1, 1, 1)$, $r = 10\%$
  - FV $= -2 \times 1.1^3 + 1 \times (1.1)^2 + 1 \times (1.1) + 1 = 0.648$

Present Value of a Stream

- *Given a cash flow stream* $(x_0, x_1, \dots x_n)$ *and interest rate r each period, the PV of this cash flow stream is*

$$PV = x_0 + \frac{x_1}{(1+r)^1} + \frac{x_2}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n}$$

  - **Example:** cash stream $(-2, 1, 1, 1)$, r = 10%
  - PV = 0.487
- PV = $\frac{FV}{(1+r)^n}$

Frequent and continuous compounding

- If r is the nominal annual interest rate and interest is compounded at m equally spaced periods per year

$K = m t_K$

$$PV = \sum_{k=0}^{n} \frac{x_k \ (t_k)}{\underline{\left(1 + \frac{r}{m}\right)}^k}$$

- For continuous compounding $PV = \sum_{k=0}^{n} x\underline{(t}_k)e^{-rt_k};\ t_k = \frac{k}{m}$

Main Theorem on Present Value

- *The cash flow streams* $\boldsymbol{x} = (x_0, x_1, \dots x_n)$ *and* $\boldsymbol{y} = (y_0, y_1, \dots y_n)$ *are equivalent for a constant ideal bank with interest rate r iff PV of the two streams are equal*
  - Only PV is needed to characterize a cash flow stream for an ideal bank
  - Stream can be transformed in variety of ways

Internal Rate of Return

- Pertains to the entire cash flow stream associated with an investment – not a partial
- Given a cash flow stream $(x_0, x_1, \dots x_n)$

$$PV = \sum_{k=0}^{n} \frac{x_k}{(1+r)^k}$$

- Using Main Theorem, the investment corresponding to above stream constructed using deposits and withdrawals from a constant ideal bank at interest rate r should have PV zero

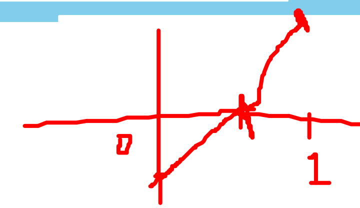
Internal Rate of Return

- Let $\boldsymbol{x} = (x_0, x_1, \dots x_n)$ be a cash flow stream. Then the internal rate of return of this stream is a number r satisfying the equation

$$0 = \sum_{k=0}^{n} \frac{x_k}{(1+r)^k}$$

- Equivalently, $\sum_{k=0}^{n} x_k\, c^k = 0; c = \frac{1}{1+r}$ a polynomial
- **Example:** cash stream $(-2, 1, 1, 1)$
  - $0 = -2 + c + c^2 + c^3$
- Defined internally without reference to the external world

Main Theorem of IRR



- Suppose the cash flow stream $(x_0, x_1, \dots x_n)$ has $x_0 < 0$ and $x_k \geq 0$ for $k = 1,2,..n$, with at least one term being strictly positive. Then there is a unique positive root to the equation

$$0 = \sum_{k=0}^{n} x_k \, c^k$$

Furthermore, if $\sum_{k=0}^{n} x_k > 0$, then the corresponding IRR is positive.

$f(x)$

$f(0) < 0$

$f(\infty) \to +\infty$

$f(1) > 0$

$c = \dfrac{1}{1+R}$

IRR: how to solve the polynomial?

- $\sum_{k=0}^{n} x_k \, c^k = 0$
- Consider $\mathrm{f(c)} = \sum_{k=0}^{n} x_k \, c^k; \mathrm{f}'(\mathrm{c}) = \sum_{k=1}^{n} k.\, x_k \, c^{k-1}$

- Newton's method
  - Start with an estimate close to solution c_0
  - Generate sequences, $c_0, c_1, c_2, \ldots c_k, \ldots$
  - $c_{k+1} = c_k - \frac{\mathrm{f}(c_k)}{\mathrm{f}'(c_k)}$
  - Approximate the function 'f' by a line tangent to its graph at c_k
  - Illustration

Evaluation Criteria

- Selecting the best alternative among cash flow streams
  - PV and IRR

- **NPV** (net present value) is the present value of the benefits minus the present value of the costs
  - Cash flow streams for r = 10%
    - (-1, 2)
    - (-1, 0, 3)
      - 0.82 vs 1.48
    - According to NPV second option is better

Evaluation Criteria

- **IRR** the higher the internal rate of return, the more desirable the investment
  - IRR must be greater than interest rate in money market
  - Previous example:
    - $-1+2c = 0$, and $-1+3c^2 = 0$
    - r = 1, and r = .73
    - According to IRR the first option is better
- Opposite conclusions!

Evaluation criteria: NPV vs IRR

- NPV is simple to compute
- IRR only depends on the properties of the cash flow stream, and not on the prevailing interest rate
- Suppose the proceeds of the first investment are used to invest further. Under this plan the investment can be doubled every year
  - Yearly growth by factor of 2 = 1+IRR
- In the second plan investment triples every two years
  - Yearly growth by factor of $\sqrt{3} = 1.73$ = 1+IRR
- When proceeds of the investment can be repeatedly reinvested – use IRR
- For one-time opportunity use NPV- obtained through prevailing interest rate

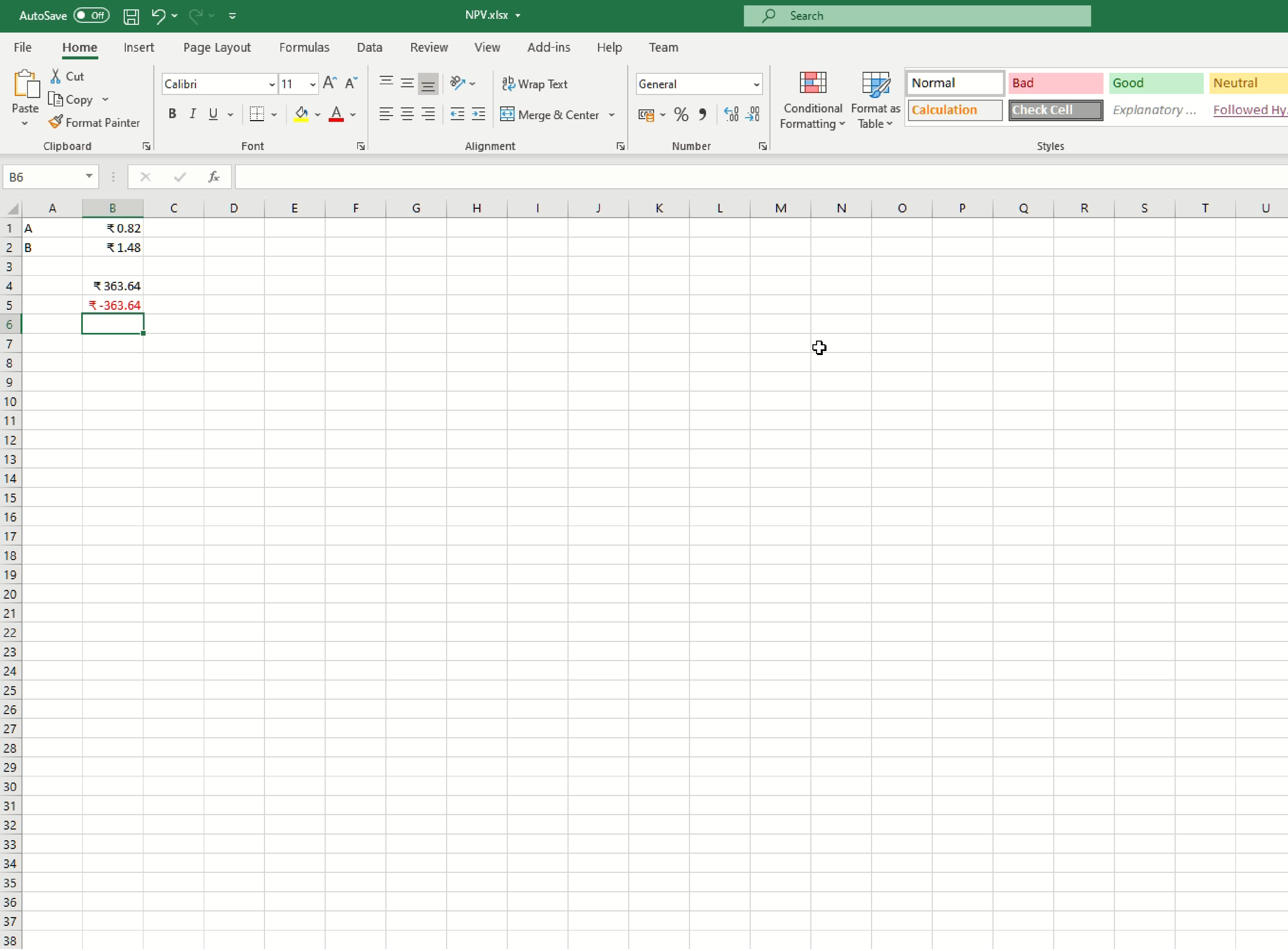
Problems in using IRR

1. Does not distinguish between **lending and borrowing**
2. Investment scenarios yielding **multiple rates of returns**
3. Fails to provide right alternative in case of **mutually exclusive projects**
4. Does not work under scenarios of **multiple cost of capital**

Problems with IRR

- **Pitfall 1:** Lending or Borrowing?
  - NPV of project B increases as discount rate increases for some cash flows

	Cash Flows ($)			
Project	C_0	C_1	IRR	NPV at 10%
A	−1,000	+1,500	+50%	+364
B	+1,000	−1,500	+50%	−364



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Neutral
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Explanatory ...
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Styles
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A
₹ 0.82
B
₹ 1.48
₹ 363.64
₹ -363.64

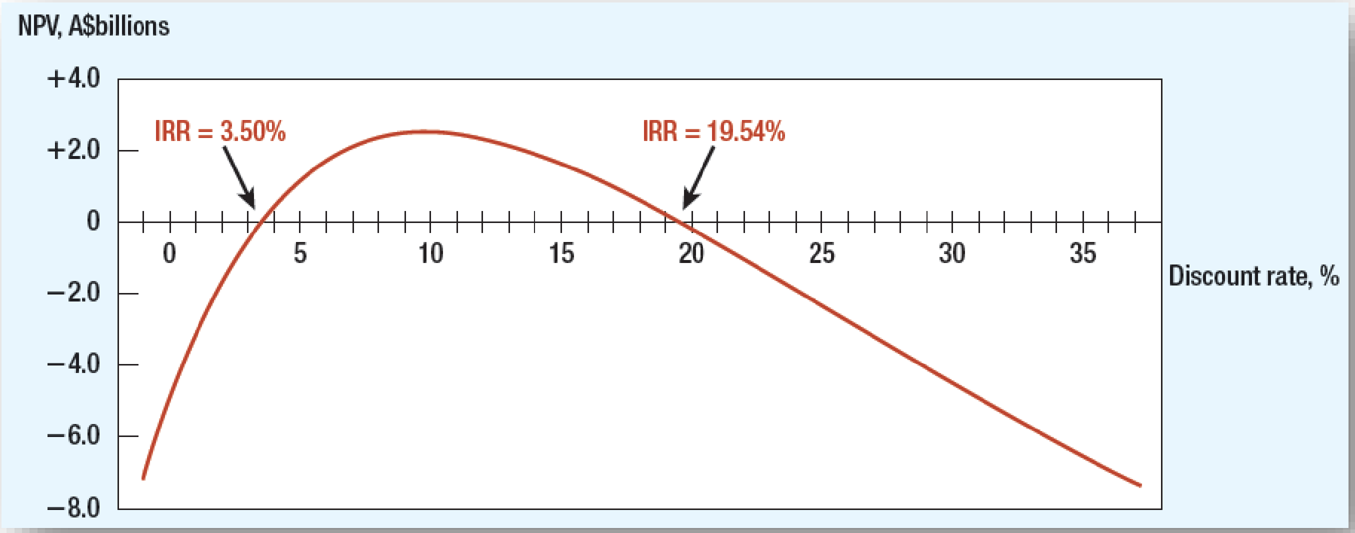
Problems with IRR

- **Pitfall 2:** Multiple Rates of Return
  - Certain cash flows **generate PV = 0** at two different discount rates
    - Following cash flow generates NPV = \$2.53 billion and yields IRR% of 3.5% and 19.54%

Cash Flows (billions of Australian dollars)				
C_0	C_1	. . .	C_9	C_{10}
−30	10		10	−65

Problems with IRR

- Multiple IRRs



A workaround solution: Use Modified IRR

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F7

	A	B	C	D	E	F
1	amount	cash flow				
2	-30	c0				
3	10	c1	IRR	4%		
4	10	c2		20%		
5	10	c3				
6	10	c4	npv	₹ 2.53		
7	10	c5				
8	10	c6				
9	10	c7				
10	10	c8				
11	10	c9				
12	-65	c10				
13						
14						
15						
16		cf1	cf2			
17		-1000	1000	IRR1	50%	
18		1500	-1500		50%	
19						
20		cf				
21	c0	1000		IRR	#NUM!	
22	c1	-3000		NPV	₹ 338.84	
23	c2	2500				

Problems with IRR

- **Pitfall 2 continued:** Multiple Rates of Return
  - Project can have 0 IRR and positive NPV

Cash Flows, $					
Project	C_0	C_1	C_2	IRR (%)	NPV at 10%
C	+1,000	−3,000	+2,500	None	+339

Problems with IRR

- **Pitfall 3:** Mutually Exclusive Projects
  - IRR sometimes ignores magnitude of project

	Cash Flows, $			
Project	C_0	C_1	IRR (%)	NPV at 10%
D	−10,000	+20,000	100	+ 8,182
E	−20,000	+35,000	75	+11,818

Problem with IRR

- **Pitfall 4:** More than One Opportunity Cost of Capital
  - Term Structure Assumption
    - Assume discount rates stable during term of project
      - May not always hold

Applications of NPV

- **Cycle Problems** When using NPV to evaluate ongoing (repeatable) activities, the alternatives be compared over the same time horizon

- **Example (2.7)** You are contemplating the purchase of a motorbike and have narrowed the choice to two models.
  - Bike A costs Rs. 20,000, is expected to have a low maintenance cost of Rs. 1,000 per year payable at the beginning of each year after the first year. It has useful mileage life that for you translates into 4 years.
  - Bike B costs Rs. 30,000, and has an expected maintenance cost of Rs. 2,000 per year after the first year. It has a useful life of 6 years.
  - Neither bike has a salvage value. The interest rate is 10%. Which bike should you buy?

Applications of NPV

- *Assumption*: similar alternatives will be available in the future (ignore inflation) – so this purchase is one of a sequence of bike purchases. To have same planning horizon, assume a planning period of 12 years, corresponding to three cycles of bike A and two of bike B.
- Bike A:
  - One cycle $PV_A = 20000 + 1000 \sum_{k=1}^{3} \frac{1}{1.1^k} = 22{,}487$
  - Three cycles $PV_{A3} = PV_A \left[1 + \frac{1}{1.1^4} + \frac{1}{1.1^8}\right] = 48{,}336$
- Bike B:
  - One cycle $PV_B = 30000 + 1000 \sum_{k=1}^{5} \frac{1}{1.1^k} = 37{,}582$
  - Two cycles $PV_{B2} = PV_B \left[1 + \frac{1}{1.1^6}\right] = 58{,}795$
- Bike A should be selected

Inflation

- **Inflation** is characterized by an increase in general prices with time. It is described quantitatively in terms of an **inflation rate** f
  - Prices 1 year from now will on average be equal to today's prices multiplied by $(1 + f)$
  - Erodes the purchasing power of money
- Constant (real) dollars: defined relative to a given reference year
- Nominal dollars: used for transactions
  - Real interest rate (r_0): $1 + r_0 = \frac{1+r}{1+f}$; $r_0 = \frac{r-f}{1+f} \approx r - f$

Inflation

Year	Real	PV@5.77	Nominal	PV@10
0	-10000	-10000	-10000	-10000
1	5000	4727	5200	4727
2	5000	4469	5408	4469
3	5000	4226	5624	4226
4	3000	2397	3510	2397
Total		5819		5819