FINANCIAL ENGINEERING
IME611A

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PORTFOLIO THEORY

Equity pricing and portfolio management

SESSION OBJECTIVES

- The Dividend Discount Model
- Measures of returns
- Short selling
- Portfolio return and variance

REQUIRED PRE-READING: DIVIDEND DISCOUNT MODEL

- A **Stock** which is expected **to pay dividends** $(D_1, D_2, \ldots, D_n)$ at **different time-points** $(t = 1, 2, 3, \ldots, n\ years)$ and can be sold at price P_n at the end of $n^{th}\ year$ can be priced as below, given a discount rate of r.

$$P_0 = \frac{D_1}{(1+r)^1} + \frac{D_2}{(1+r)^2} + ,,, + \frac{D_n}{(1+r)^n} + \frac{P_n}{(1+r)^n}$$

$$P_0 = \sum_{t=1}^{n} \frac{D_t}{(1+r)^t} + \frac{P_n}{(1+r)^n}$$

- Above formula is known as **Dividend Discount Model (DDM)**

UNCERTAIN CASHFLOW

- Investments where **initial cash outlay is known**, but <u>amount to be returned is uncertain</u>
- Uncertainty is handled using
  - Mean –variance analysis
  - Utility function analysis
  - Arbitrage (or comparison analysis)

ASSET RETURN

- Consider buying an asset at time zero (t_0), and selling the same 1 year later (t_1)

$$Total\ return\ (R) = \frac{amount\ received}{amount\ invested} = \frac{X_1}{X_0}$$

$$Rate\ of\ return\ (r) = \frac{amount\ received\ -\ amount\ invested}{amount\ invested} = \frac{(X_1\ - X_0)}{X_0}$$

$$R = 1 + r$$

$$X_1 = \ (1 + r)X_0$$

SHORT SELLING OF AN ASSET

- **Short selling (or shorting):** To sell an asset that you do not own, by borrowing from a broker
- Borrow the stock from a broker
- Sell it at X_0
- Repay the loan later by purchasing the stock at X_1
- Your net payoff = $+X_0 - X_1$
- You earn a **profit if the stock price declines**.
- Short selling is risky, potentially the loss could be unlimited.

RETURN IN SHORT SELLING

- **Short selling results** in receiving a cash inflow of X_0 today at t_0, and experiencing a cash outflow of X_1 at t_1.

$$R = \frac{-X_0}{-X_1} = \frac{X_0}{X_1}$$

$$-X_1 = -X_0 R = -X_0(1+r)$$

Practice Example 6.1, and Exercise 1

PORTFOLIO RETURN (1/2)

- Portfolio is a combination of multiple assets.
- Suppose, there are $\boldsymbol{n}$ **assets**, we can form a **master asset or portfolio**
- $\boldsymbol{X_0}$ amount is invested across $\boldsymbol{n}$ assets.
- $\boldsymbol{X_{0i}}$ is the amount invested in $\boldsymbol{i^{th}}$ asset, where $i = 1, 2, \ldots n$.
- We have $\sum_{i=1}^{n} X_{0i} = X_0$

- Alternatively, if we consider the **weight** or **fraction** of asset i in the portfolio as w_i then
  - $X_{0i} = w_i X_0$
  - $\sum_{i=1}^{n} w_i = 1$

PORTFOLIO RETURN (2/2)

- Let R_i denote the total return of asset i.
- Amount of money generated at the end of period: $R_i X_{0i} = R_i w_i X_0$
- Total amount received at the end of the period $\sum_{i=1}^{n} R_i w_i X_0$
- So, overall **total return on portfolio**

$$R = \frac{\sum_{i=1}^{n} R_i w_i X_0}{X_0} = \sum_{i=1}^{n} w_i R_i$$

- Equivalently,

$$r = \sum_{i=1}^{n} w_i r_i$$

IMPORTANT RESULT

- **Portfolio Return:** Both the total return and rate of return of a portfolio of assets are **equal to the weighted sum of the corresponding individual asset returns**, with the weight of an asset being its relative weight (in purchase cost) in the portfolio, that is,

$$R = \sum_{i=1}^{n} w_i R_i$$

$$r = \sum_{i=1}^{n} w_i r_i$$

An example illustration

SOME PRELIMINARY FROM PROBABILITY AND STATISTICS

- **Random Variable**
- **Expected Value**
  - Properties of expected value
    1. Certainty value
    2. Linearity
    3. Nonnegativity
- **Variance**
  - Several random variables
  - Covariance and correlation
  - Covariance bound, uncorrelated, positively correlated, negatively correlated random variables
  - Properties of variance
    1. Variance of sum of two random variables

RANDOM RETURNS

- An asset, when acquired, typically has an uncertain rate of return
- To summarize the uncertainty
- Expected value: $E(r) \equiv \bar{r}$
- Variance: $E[r - \bar{r}]^2 \equiv \sigma^2$
- Covariance: $E[(r_i - \bar{r})][(r_j - \bar{r})] \equiv Cov(r_i, r_j)$

MEAN RETURN OF A PORTFOLIO

- Suppose, there are n assets with (random) rates of return $r_1, r_2, r_3, \dots, r_n$ having expected values as $E(r_1) = \overline{r_1}, E(r_2) = \overline{r_2}, \dots, E(r_n) = \overline{r_n}$.

- We form a portfolio of these n assets using the weights $w_i, i = 1, 2, \dots, n$.

- The return on portfolio is given by

$$r = w_1 r_1 + w_2 r_2 + \cdots + w_n r_n$$

- Taking expectation and using linearity,

$$E(r) = w_1 E(r_1) + w_2 E(r_2) + \cdots + w_n E(r_n)$$

VARIANCE OF PORTFOLIO RETURN (1/2)

- Suppose,
- σ^2 denote the portfolio variance,
- σ_i^2 denote the variance of i^{th} stock, and
- σ_{ij} denote the covariance of return on asset i and asset j.

VARIANCE OF PORTFOLIO RETURN (2/2)

$$\sigma^2 = E[(r - \bar{r})^2]$$

$$\sigma^2 = E\left[\left(\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \bar{r}_i\right)^2\right]$$

$$\sigma^2 = E\left[\left(\sum_{i=1}^{n} w_i (r_i - \overline{r_i})\right)\left(\sum_{j=1}^{n} w_j (r_j - \overline{r_j})\right)\right]$$

$$\sigma^2 = E\left[\left(\sum_{i,j=1}^{n} w_i w_j (r_i - \overline{r_i})(r_j - \overline{r_j})\right)\right]$$

$$\sigma^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

Practice Example 6.8

DIVERSIFICATION (1/3)

- **Diversification:** The process of achieving **a reduction in the variance** of the portfolio of returns by adding additional assets in the portfolio.

- Consider the case of $\boldsymbol{n}$ **mutually uncorrelated assets**, each having a **mean** $\boldsymbol{m}$ and a **variance of** $\boldsymbol{\sigma^2}$.

- A portfolio with **equal proportion of each of these assets** will have return of

$$r = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$Var\,(r) = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}$$

- The variance decreases rapidly with n.

DIVERSIFICATION (2/3)

- Consider the case of $\boldsymbol{n}$ **correlated assets**,
  - each having a mean m
  - a variance of σ^2
  - $cov\left(r_i, r_j\right) = 0.3\sigma^2$

$$Var\,(r) = ?$$

$$Var\,(r) = E\left[\sum_{i=1}^{n} \frac{1}{n}(r_i - \bar{r})\right]^2$$

$$Var\,(r) = \frac{1}{n^2} E\left\{\left[\sum_{i=1}^{n}(r_i - \bar{r})\right]\left[\sum_{j=1}^{n}(r_j - \bar{r})\right]\right\}$$

DIVERSIFICATION (3/3)

$$Var\ (r) = \frac{1}{n^2}\sum_{i,j=1}^{n}\sigma_{ij}$$

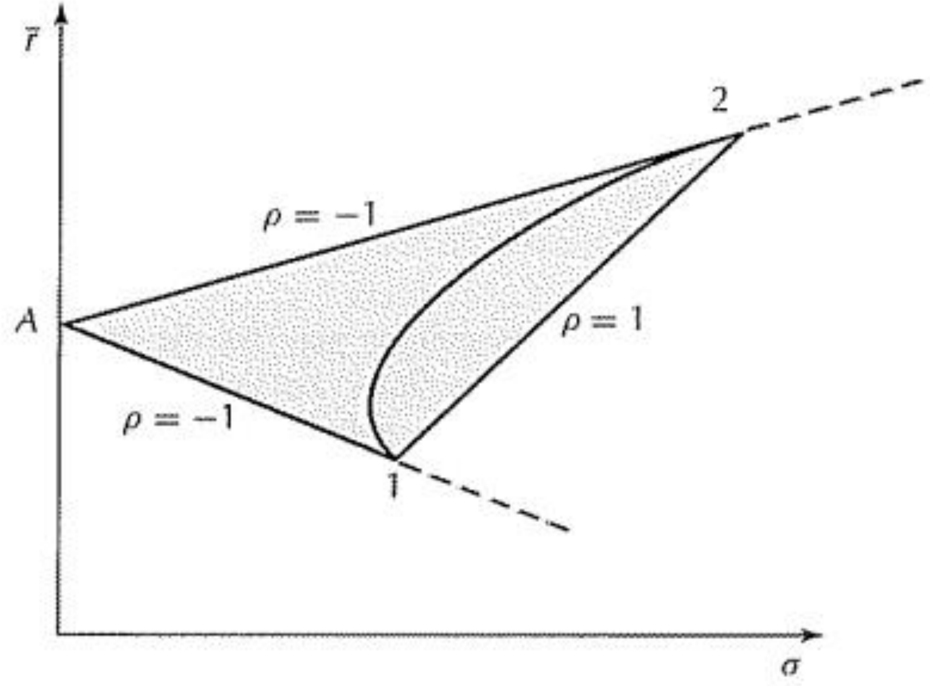
$$Var\ (r) = \frac{1}{n^2}\left\{\sum_{i=j}^{n}\sigma_{ij} + \sum_{i\neq j}^{n}\sigma_{ij}\right\}$$

$$Var\ (r) = \frac{1}{n^2}\{n\sigma^2 + 0.3(n^2 - n)\sigma^2\}$$

$$Var\ (r) = \frac{\sigma^2}{n} + 0.3\sigma^2\left(1 - \frac{1}{n}\right)$$

$$Var\ (r) = \frac{0.7\sigma^2}{n} + 0.3\sigma^2$$

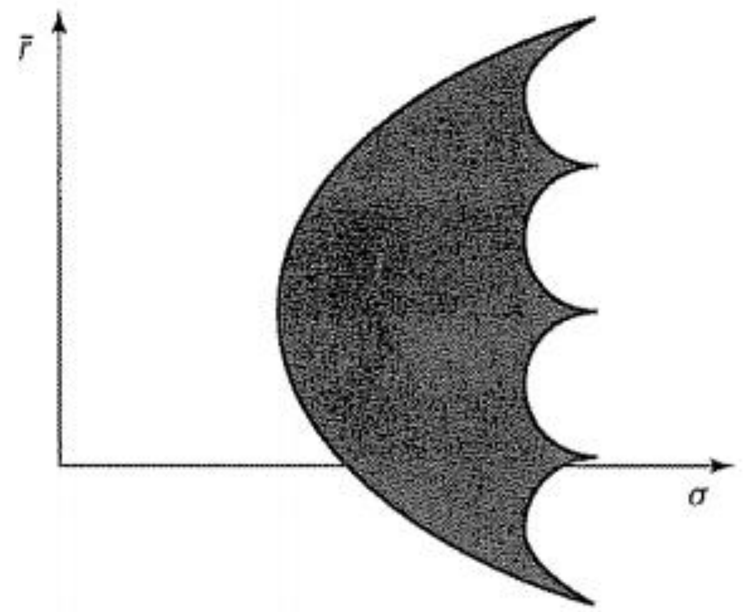
DIAGRAM OF A PORTFOLIO



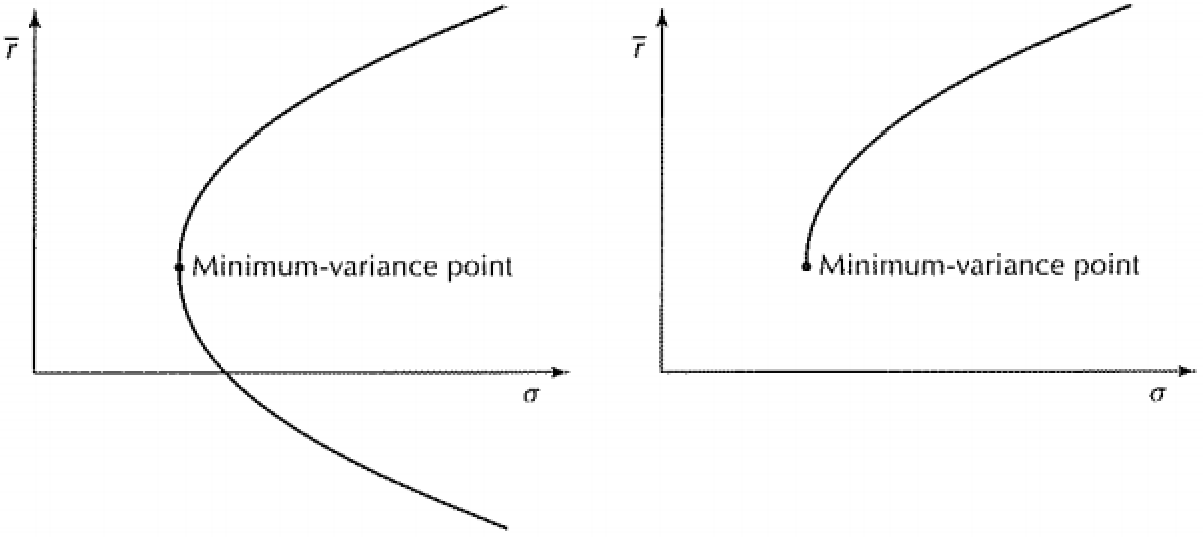
IMPORTANT LEMMA

- **Portfolio diagram lemma:** The curve in $\bar{r} - \sigma$ diagram defined by non-negative mixture of two assets 1 and 2 lies within the triangular region defined by the two original assets and the point on the vertical axis of height $A = (\bar{r}_1\sigma_2 + \bar{r}_2\sigma_1)/(\sigma_1 + \sigma_2)$.
- Proof

THE FEASIBLE SET



MINIMUM VARIANCE SET AND THE EFFICIENT FRONTIER



(*a*) Minimum-variance set (*b*) Efficient frontier

THE MARKOWITZ MODEL (1/3)

- The mathematics of minimum variance portfolios

- Consider n assets with mean (or expected) rates of return as $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n$ and covariances are σ_{ij} for $i, j = 1, 2, \ldots, n$

- A **portfolio** is defined by a set of n weights $w_i, i = 1, 2, \ldots, n$, such that

$$\sum_{i=1}^{n} w_i = 1$$

THE MARKOWITZ MODEL (2/3)

- Portfolio optimization model

Minimize $\frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \, \sigma_{ij}$

subject to $\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$

and $\sum_{i=1}^{n} w_i = 1$

THE MARKOWITZ MODEL (3/3)

- Using Lagrangian multipliers λ and μ

$$L = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j\,\sigma_{ij} \;-\lambda\left(\sum_{i=1}^{n} w_i \bar{r}_i - \bar{r}\right) - \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

- **Solution approach:** Differentiate the lagrangian with respect to each w_i and equate those to zero.

IMPORTANT RESULT

- **Equations for efficient set:** The n portfolio weights w_i for $i = 1, 2, \ldots, n$ and the two Lagrange multipliers λ and μ for an efficient portfolio (with short selling allowed) having mean rate of return $\bar{r}$ satisfy

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0 \text{ for } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1$$

CASE: SHORT SELLING PROHIBITED

- Portfolio optimization model

Minimize $\frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \, \sigma_{ij}$

subject to $\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$

and $\sum_{i=1}^{n} w_i = 1$

and $w_i \geq 0$ for $i = 1, 2, \ldots, n$

- **Optimization: Quadratic program**

THE TWO-FUND THEOREM

- **Two-Fund Theorem:** Two efficient funds (portfolios) can be estimated so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these two funds.

- If the two solution of the Markowitz portfolio optimization are $w^1(w_1^1, w_2^1, \dots, w_n^1)$, λ^1, μ^1 and w^2 $(w_1^2, w_2^2, \dots, w_n^2)$, λ^2, μ^2 with expected returns $\bar{r}^1$ and $\bar{r}^2$, then

- Any portfolio formed by a **combination of the above two will also be an efficient portfolio** (entire minimum variance set).

$$\alpha w^1 + (1-\alpha) w^2$$

IMPLICATION OF TWO FUND THEOREM

- *Two fund theorem:* Two **mutual funds** could provide complete investment service.
- Key assumptions:
  - Everyone **cares only** about <u>mean and variance.</u>
  - Everyone has **same assessment** of the <u>means, variances, and covariances.</u>
  - **Single period** framework is adequate.
- Practice Example 6.11

INCLUSION OF A RISK-FREE ASSET

- A **risk-free asset** has a return that is deterministic and therefore has $\sigma = 0$.

- Inclusion of a risk-free asset allows for lending and borrowing cash at the risk-free rate.

- Consider a risk-free asset with return r_f, and a risky asset with return r and variance σ^2

PORTFOLIO OF RISKY WITH RISK-FREE ASSET

- A portfolio with α in risk free asset and $(1-\alpha)$ in risky asset will have

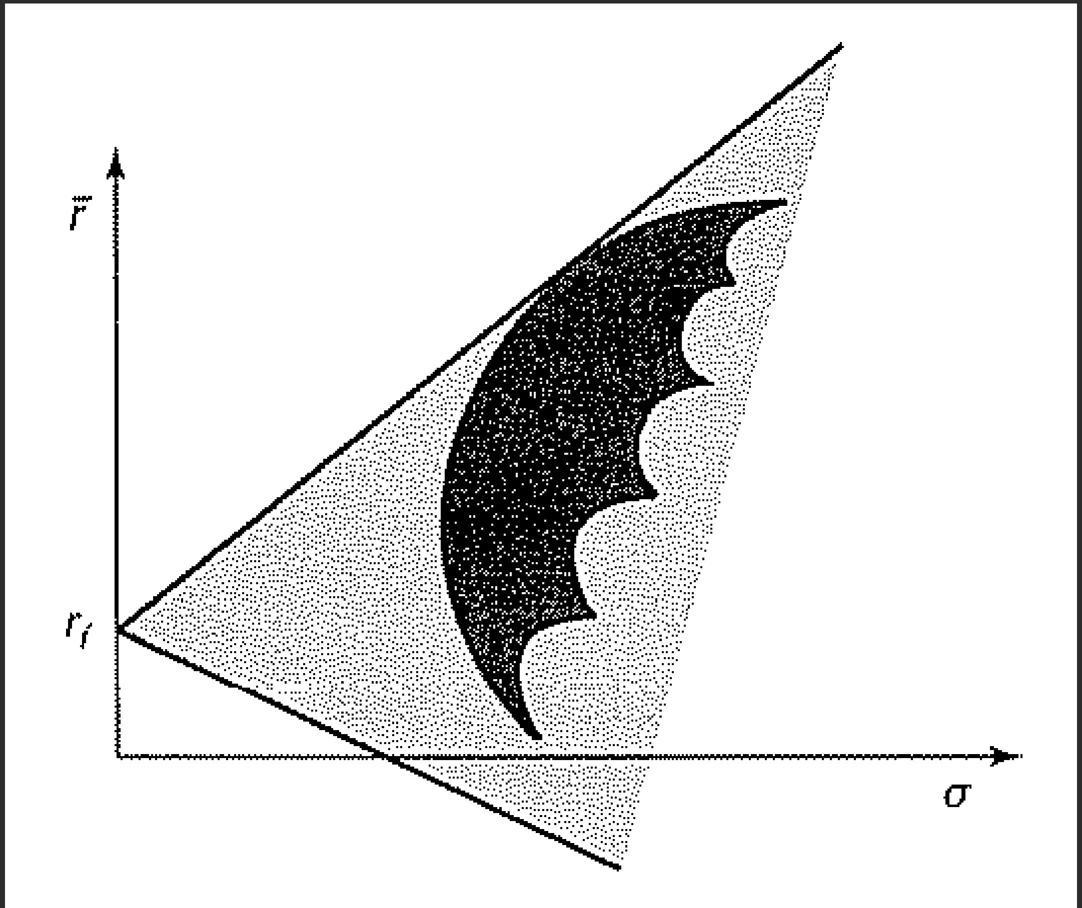
$$mean = \alpha r_f + (1-\alpha)\bar{r}$$

$$standard\ deviation = \ \alpha\sigma_f + (1-\alpha)\sigma$$

- Portfolio represented by a straight line in the $\bar{r}-\sigma$ space

THE ONE-FUND THEOREM

- **The one-fund theorem:** There is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk-free asset (r_f).



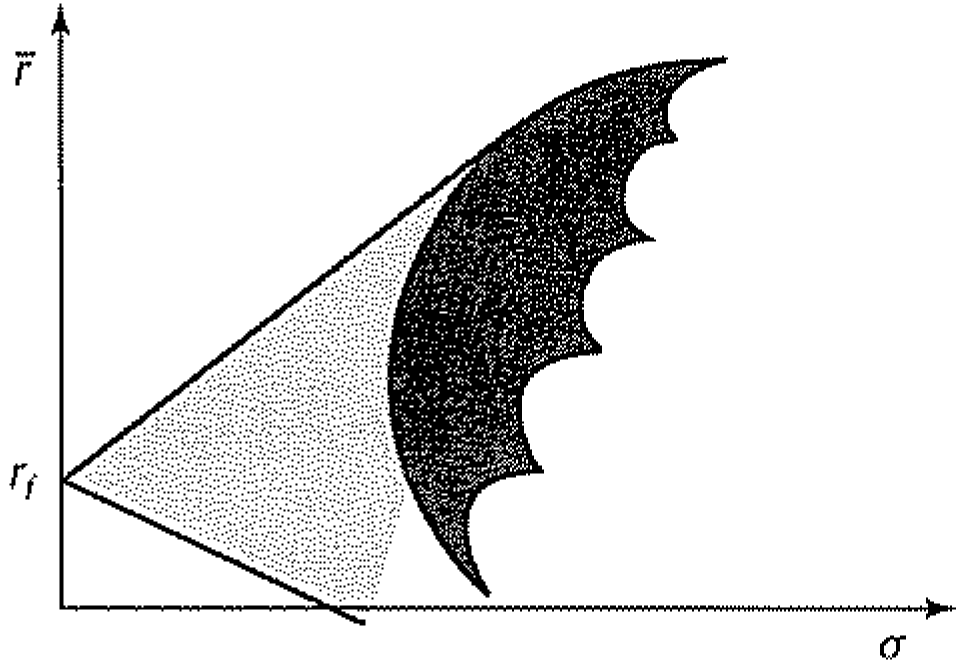
SOLUTION APPROACH

- To obtain the tangent of the line joining the risk-free asset with portfolio on efficient set
- Let θ be the angle between the line joining the risk-free asset with portfolio and the horizontal axis.
- For any feasible (risky) portfolio p, we have

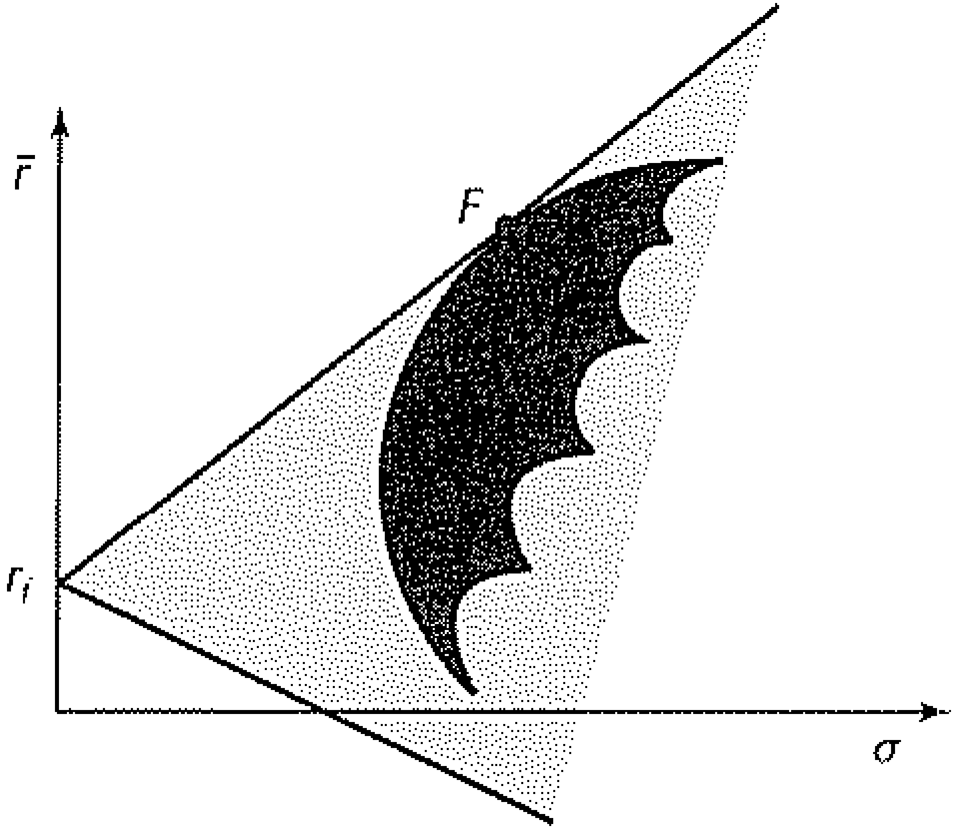
$$tan\theta = \frac{\bar{r}_p - r_f}{\sigma_p}$$

- **Objective:** To maximize θ, (or $tan\theta$) [or maximize Sharpe ratio]
- Practice Example 6.11

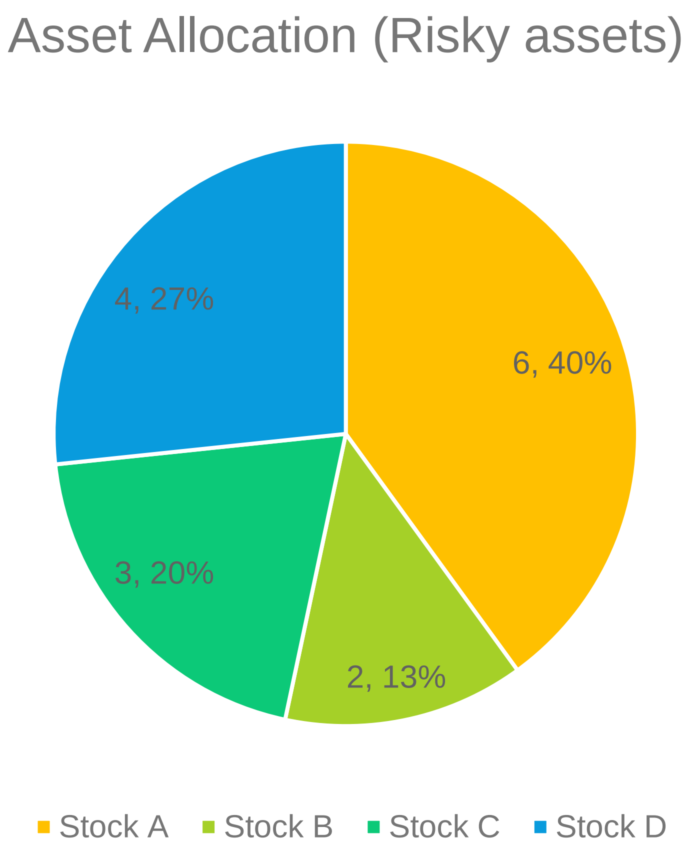
EFFICIENT FRONTIER (ONLY LENDING)



EFFICIENT FRONTIER (LENDING AND BORROWING)



INVESTING: ASSET ALLOCATION DECISION



PRICING MODELS

CAPM

MARKET EQUILIBRIUM

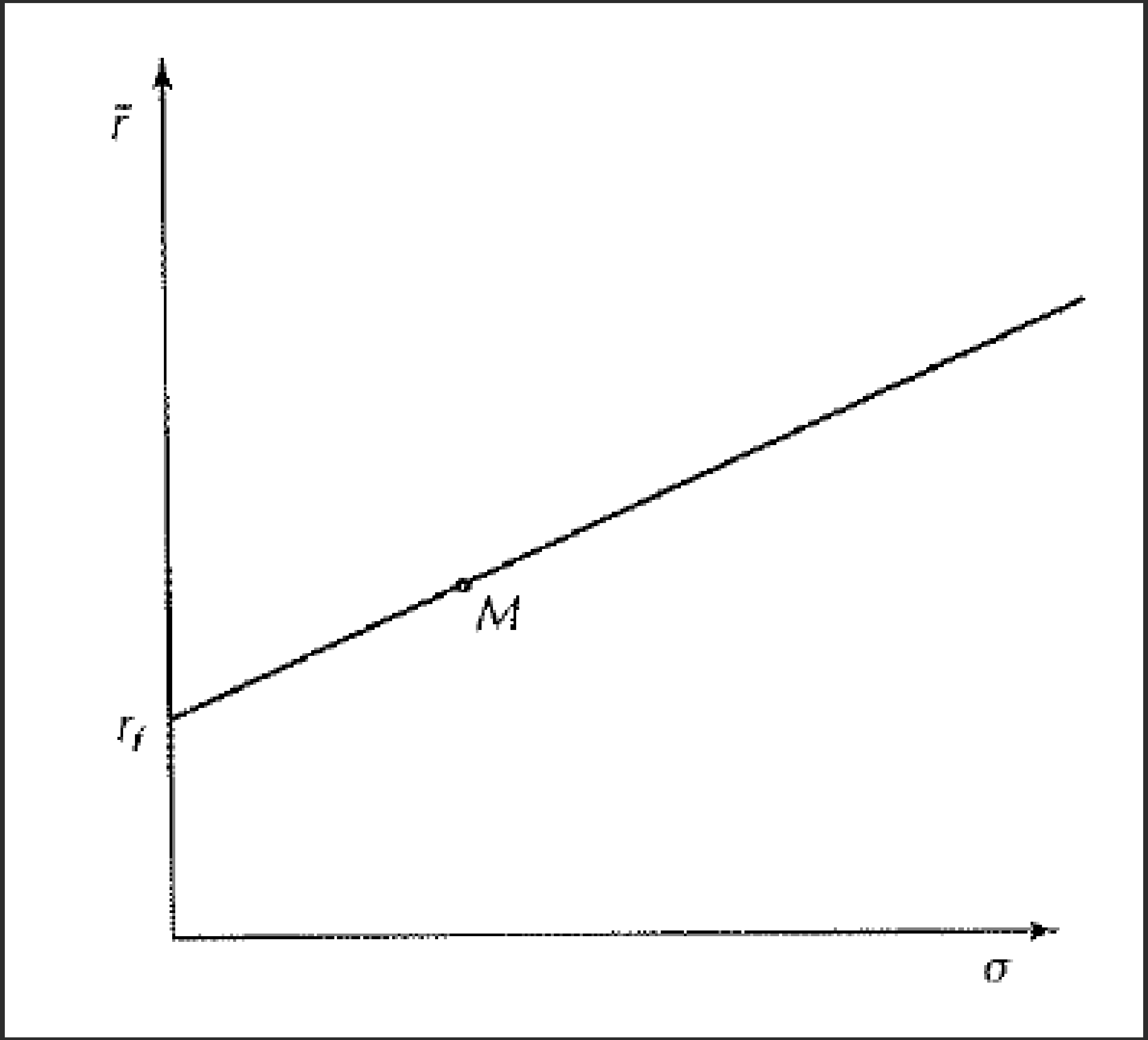
- Suppose, that each investor
  - a mean-variance optimizer
  - agrees on the mean values, variance and covariance values
  - has same risk-free borrowing and lending rate
  - faces no transaction cost

- Using **one-fund theorem**
  - They will invest in risk-free rate (r_f) and one risky fund (r_p)
  - As they have homogeneous expectation, they all will end up holding the same risky fund (r_p)
  - This risky portfolio is the **market portfolio**, summation of all assets.

CAPITAL MARKET LINE

$$\bar{r} = r_f + \frac{\bar{r}_m - r_f}{\sigma_M}\sigma$$

- The slope of the capital market line (CML):
- $K = \left(\bar{r}_M - r_f\right)/\sigma_M$, is called the **price of risk**.



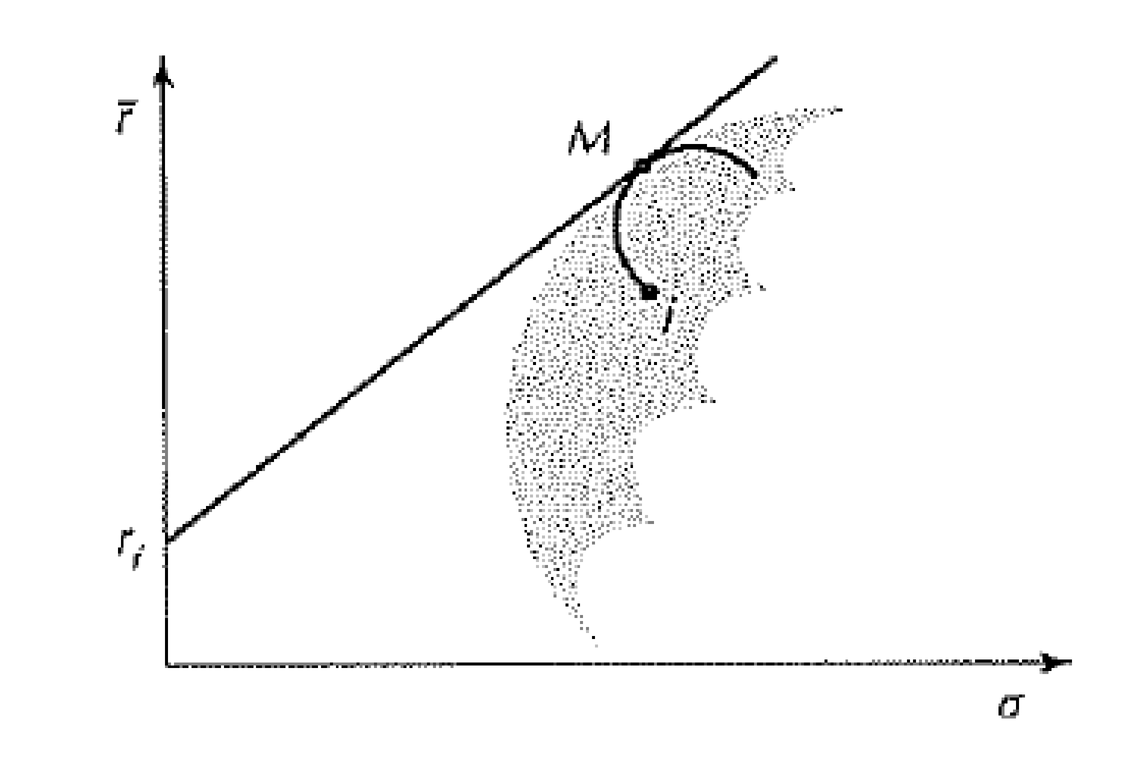
THE PRICING MODEL

- **The Capital Asset Pricing Model (CAPM)**
- If the market portfolio M is efficient, the expected return r_i of any asset i satisfies

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$$

$$\beta_i = \frac{\sigma_{iM}}{\sigma_M^2}$$

- The value β_i is referred to as the beta of an asset.



BETA: A MEASURE OF SYSTEMATIC RISK

- Suppose a portfolio contains n assets with the weights $w_1, w_2, \ldots, w_n$.

- Rate of return on the portfolio is

$$r = \sum_{i=1}^{n} w_i r_i$$

- Then,

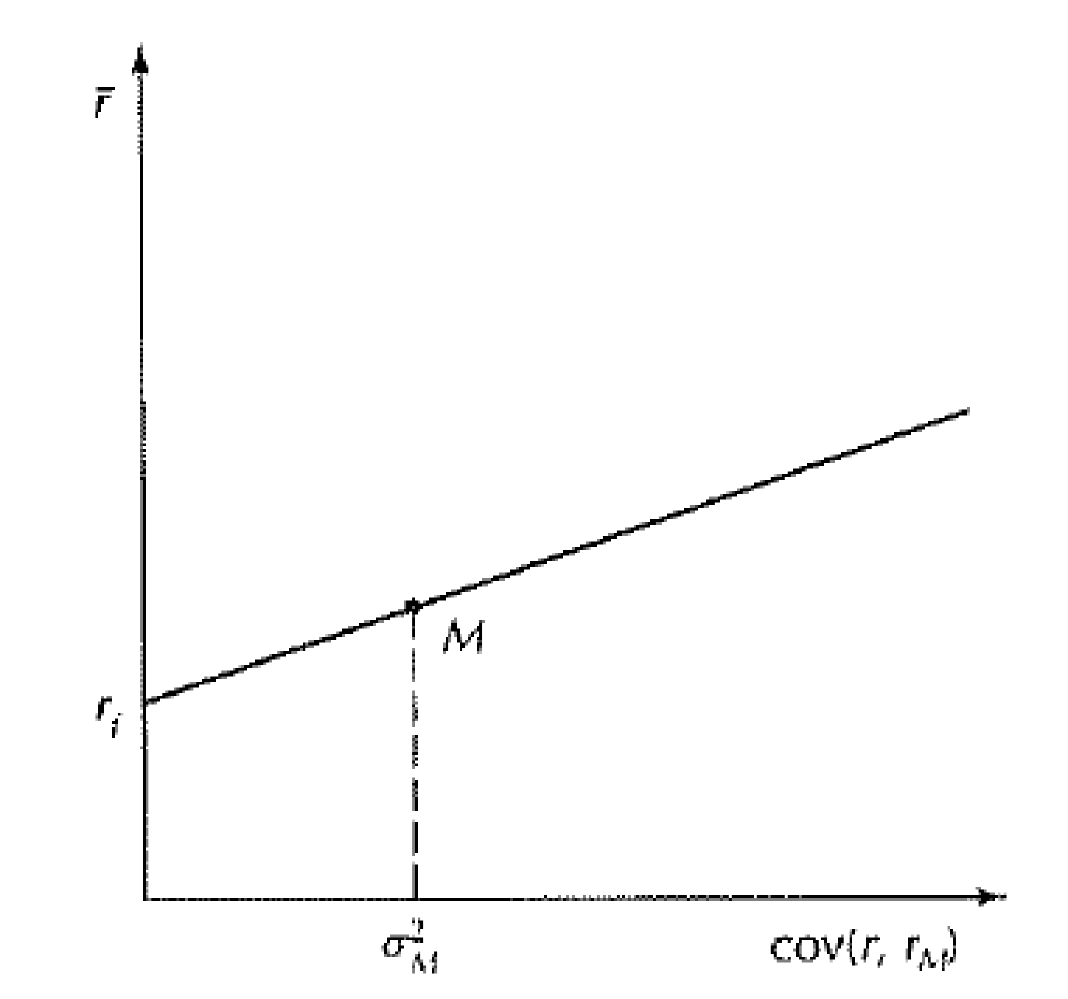
$$Cov(r, r_M) = \sum_{i=1}^{n} w_i \, Cov\,(r_i, r_M)$$

$$\beta_p = \sum_{i=1}^{n} w_i \beta_i$$

- Beta of a **portfolio**

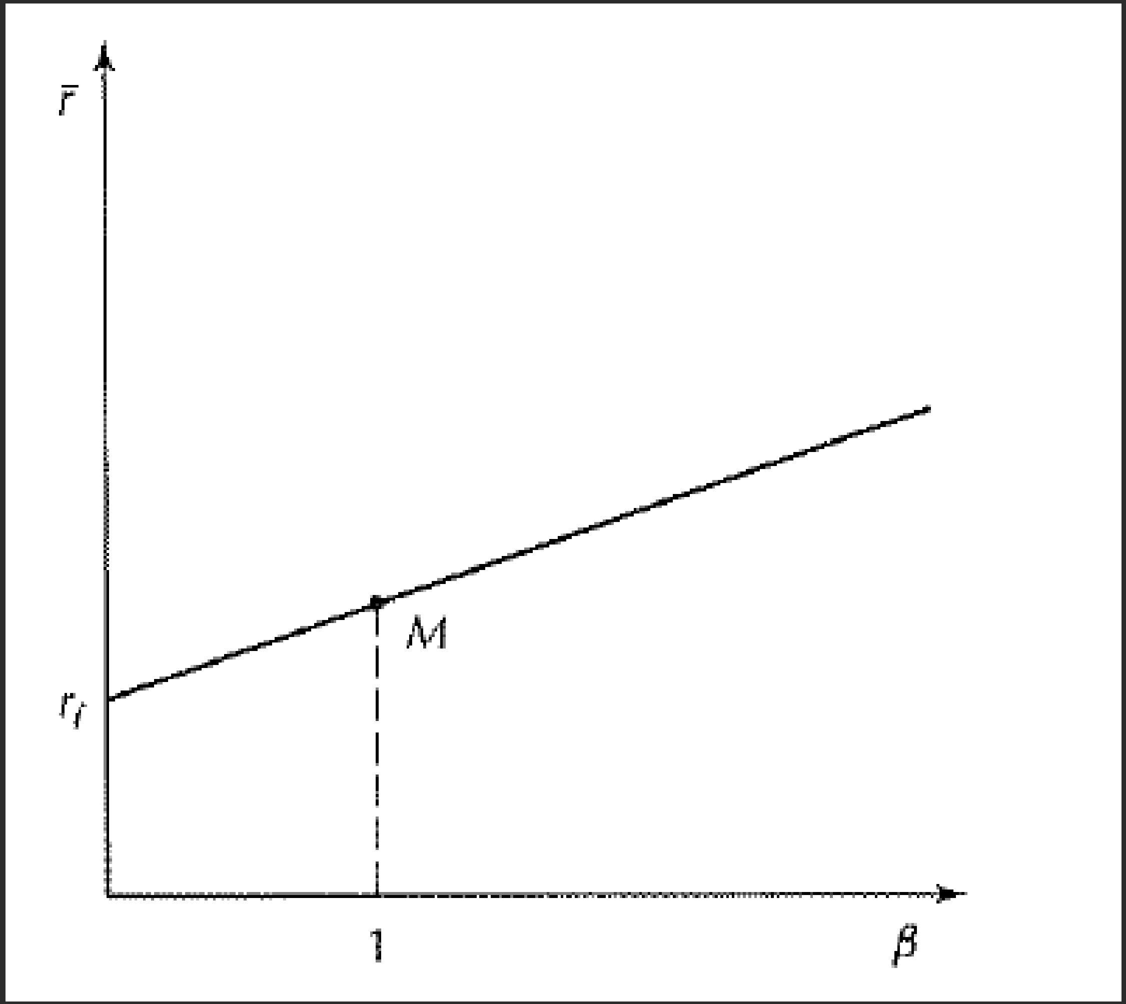
THE SECURITY MARKET LINE

- The CAPM formula when expressed in graphical form is a linear relationship known as **Security Market Line (SML)**.
- SML represents the **risk-reward structure of assets** according to CAPM.
- It emphasizes the **risk of an asset** is a function of its covariance with market, or equivalently a function of its **beta**.



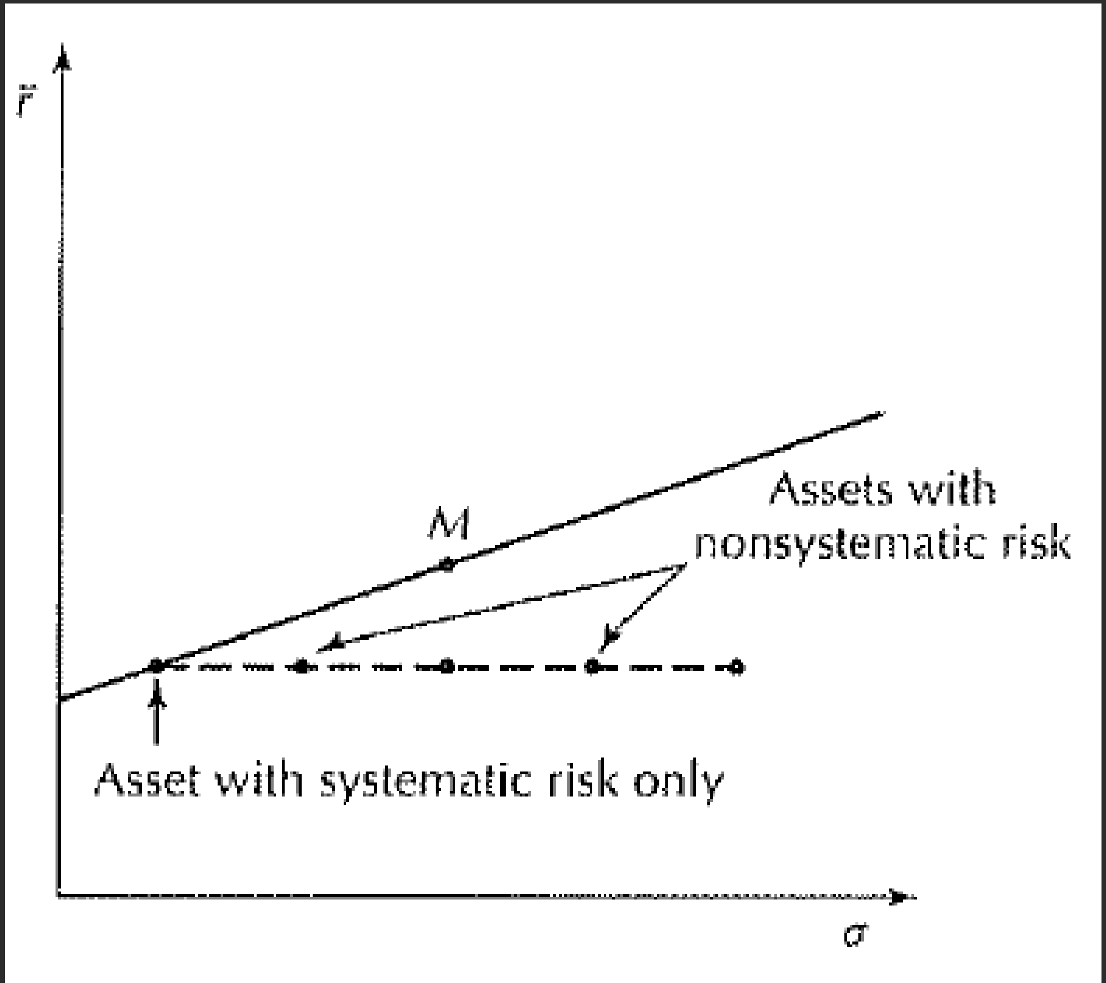
CAPM AND SML

- $\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)$



SYSTEMATIC RISK AND NONSYSTEMATIC RISK

- The CAPM formula
  $r_i = r_f + \beta_i(r_M - r_f) + \epsilon_i$
- The CAPM says that $E(\epsilon_i) = 0$
- $\sigma_i^2 = \beta_i^2 \sigma_M^2 + var(\epsilon_i)$
- Total risk =
  systematic risk
  + non-systematic risk



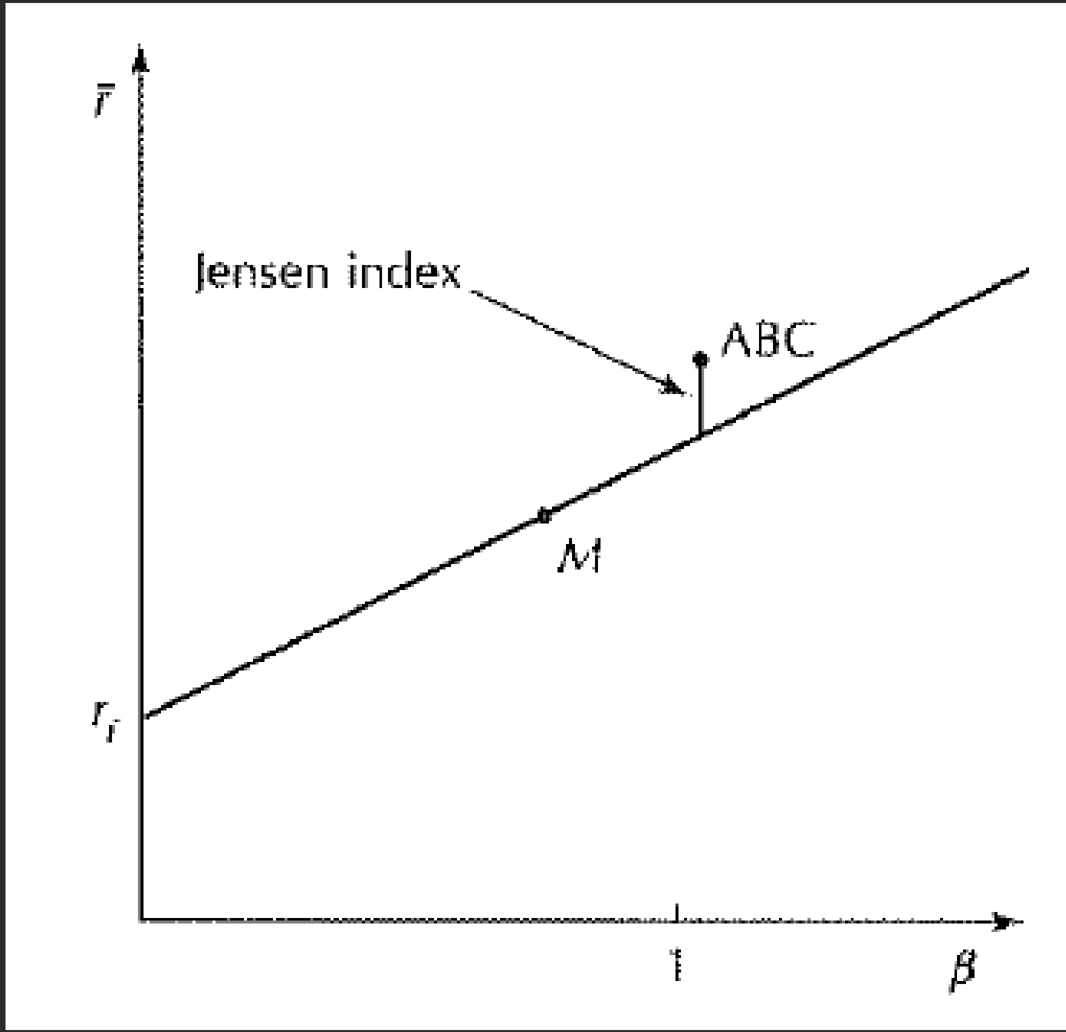
INVESTMENT IMPLICATION

- Can the CAPM help with investment decision?
  - Invest in risk-free assets and one risky fund (market portfolio), instead of the who Markowitz portfolio
  - If all investors chose to invest in some ‘index funds’.
- CAPM provides <u>reasonable prices</u> for those stocks that do not have well established market prices.

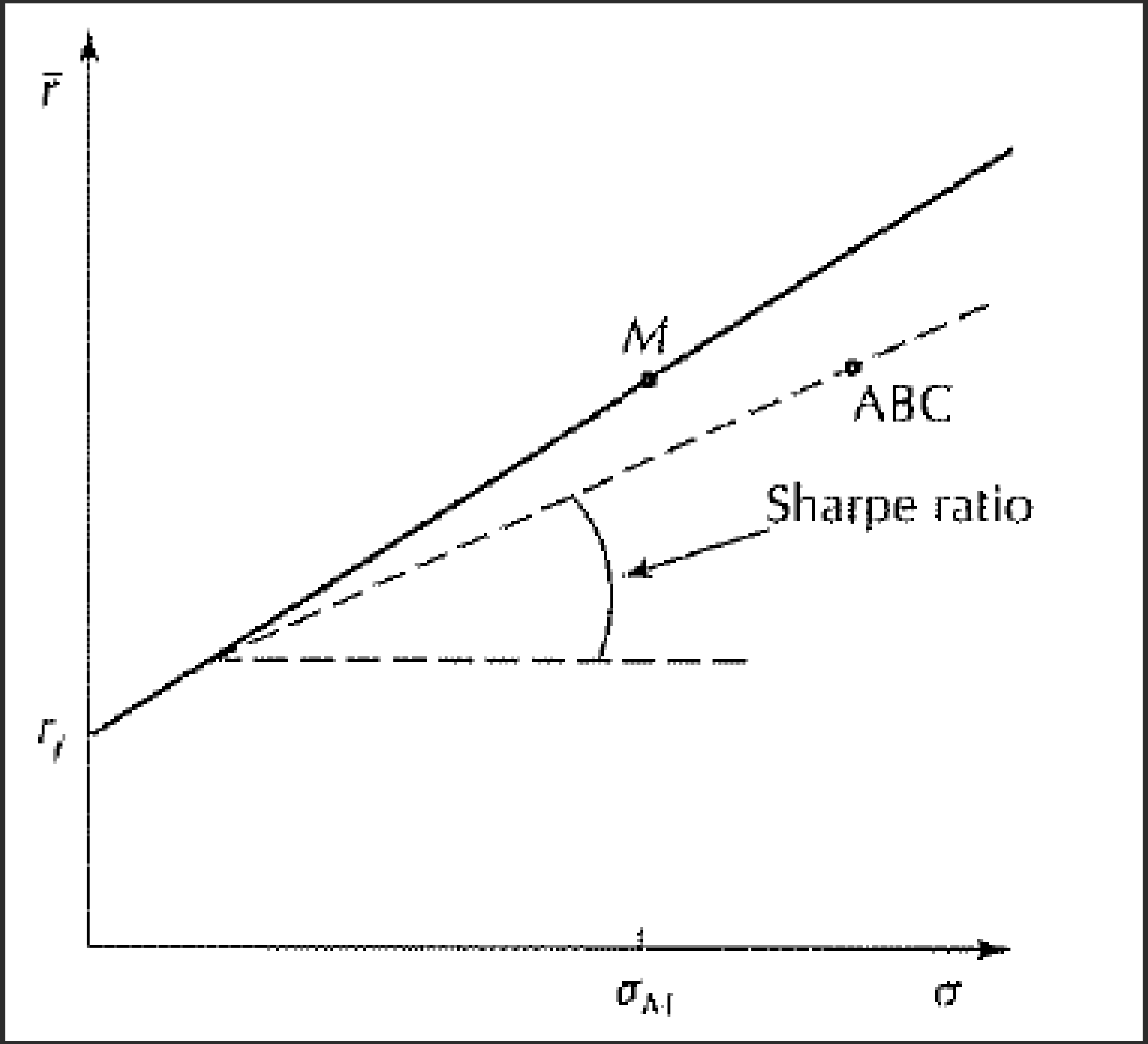
PERFORMANCE EVALUATION

- CAPM theory can be used for performance evaluation of a portfolio.

- An illustration in excel



PERFORMANCE EVALUATION



CAPM AS A PRICING FORMULA

- Suppose an asset is purchased at price P and later sold at a price Q. Then, $r = (Q - P)/P$

$$\frac{\bar{Q} - P}{P} = r_f + \beta\left(\bar{r}_M - r_f\right)$$

- Solving,

$$P = \frac{\bar{Q}}{1 + r_f + \beta\left(\bar{r}_M - r_f\right)}$$

UTILITY THEORY

Capturing individual's preference (risk and return)

UTILITY FUNCTIONS

- Suppose, an individual has different investment opportunities.

- **Objective:** Make the investment grow over the investment horizon
  - For certain cashflows, the choice is simple [Rank them.]
  - For uncertain or random cashflows, the choice **may not be** simple.

- **Utility function**: Useful tool to rank random wealth levels.

UTILITY FUNCTION

- **Utility function**: A function U defined on the real numbers (representing possible wealth levels) and giving a real value.

- Rank the alternative random wealth levels using **Expected Utility** values.

- To compare two random wealth variables x and y,
  - Compare $E[U(x)]$ and $E[U(y)]$
  - Larger value is **preferred.**

UTILITY: DIFFERENT FUNCTIONAL FORMS

- To capture different **individual's risk preferences** different functional form of utility function may be needed.
  - Individual risk tolerance
  - Individual's financial environment

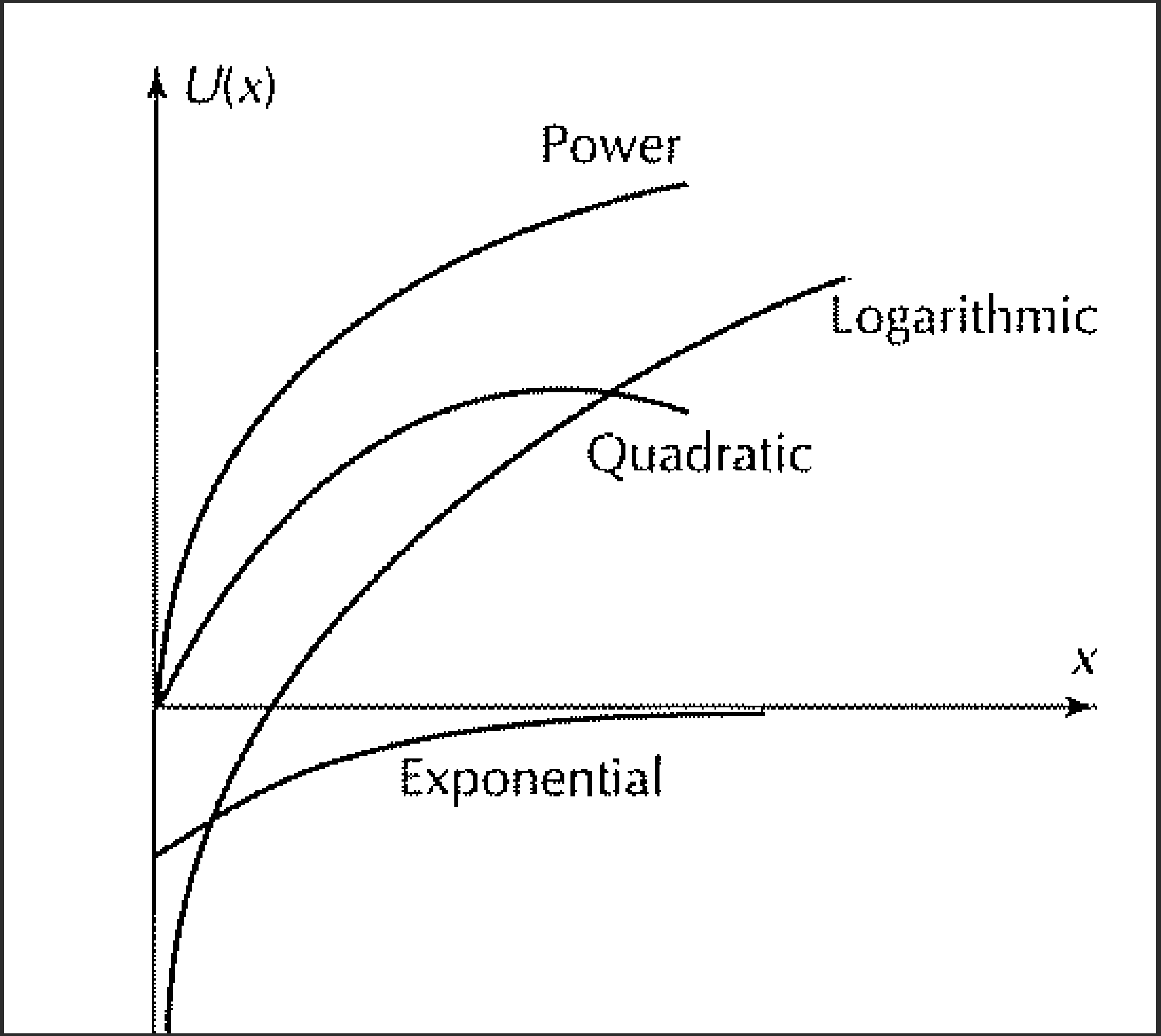
- A simple utility function
  - $U(x) = x$
  - Ranking of random wealth levels based on expected values.
  - Such an individual could be classified as ***risk-neutral.***

- One requirement for any utility function
  - ***Increasing continuous*** function
  - For $x > y$, $U(x) > U(y)$

UTILITY: DIFFERENT FUNCTIONAL FORMS

Utility	Functional Form	Salient feature
Exponential	$U(x) = -e^{-ax}$	• For $a > 0$ • Relative values
Logarithmic	$U(x) = \ln(x)$	• $x > 0$
Power	$U(x) = bx^b$	• For $b \leq 1, b \neq 0$ • For $b = 1$; risk-neutral
Quadratic	$U(x) = x - bx^2$	• For $b > 0$ • Increasing only for $x < 1/(2b)$

UTILITY FUNCTION



UTILITY FUNCTION: AN EXAMPLE

- **Example:** A Venture Capitalist is considering possible alternatives for the coming year.
- Alternative 1: Buy Treasury-bills, which will give $6M for sure.
- Alternative 2: An asset with three possible outcomes with form (p, W):
    - a) (0.2, $10M)
    - b) (0.4, $5M)
    - c) (0.4, $1M)

- Evaluate the alternatives based on power utility function $U(x) = x^{1/2}$

EQUIVALENT UTILITY FUNCTIONS

- Adding a linear constant does not affect the ranking.
  - $V(x) = U(x) + b$
  - Expectation is a linear operation.

$$E[V(x)] = E[U(x) + b] = E[U(x)] + \ b$$

- Multiplication with a constant term does not affect the ranking.
  - $V(x) = aU(x)$

$$E[V(x)] = E[aU(x)] = aE[U(x)]$$

IMPORTANT RESULT

- Given a utility function $U(x)$, any function of the form

$$V(x) = aU(x) + b$$

for any $a > 0$ is **equivalent** to $\boldsymbol{U(x)}$

- For example check $V(x) = \ln(cx^a)$, and $U(x) = \ln(x)$

RISK AVERSION

- Utility function: A systematic way to rank alternatives that captures the principle of risk aversion

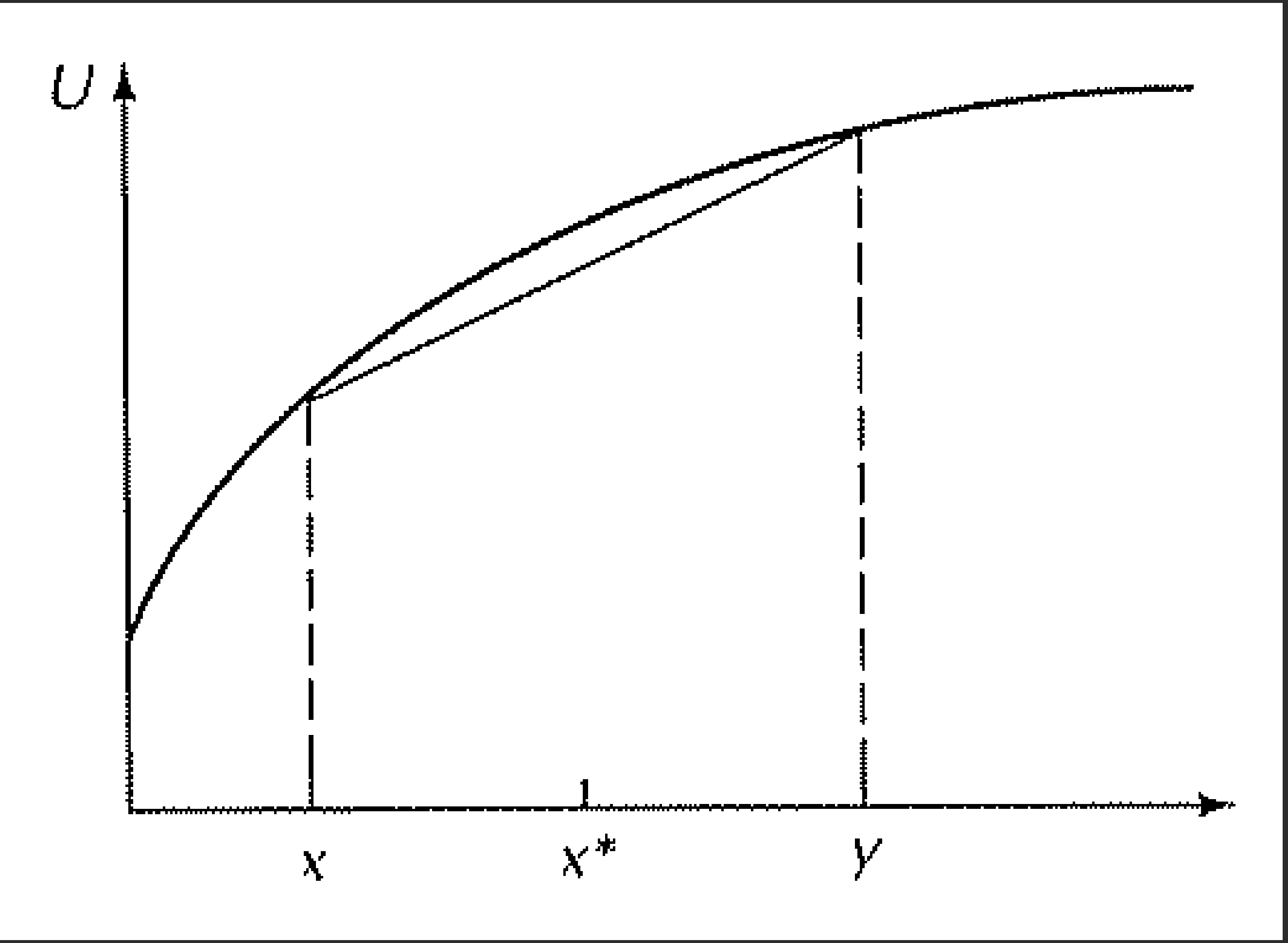
- **Concave utility and risk aversion:** A function U defined on an interval $[a, b]$ of real numbers is said to be **concave** if for any α with $0 \leq \alpha \leq 1$ and any x and y in $[a, b]$ there holds

$$U[\alpha x \ + (1-\alpha)y] \ \geq \alpha U(x) + (1-\alpha)U(y)$$

- A utility function U is said to be **risk averse on** $[a, b]$ if it is **concave** on $[a, b]$**.**

- If U is **concave** everywhere, it is said to be risk averse**.**

CONCAVITY AND RISK AVERSION



AN EXAMPLE: COIN TOSS

- Suppose you are facing following two options.
- Option 1: Based on a coin toss
  - Head: You win \$10
  - Tail: You win 0.
- Option 2: Amount M for certain.

- Your utility function for money: $U(x) = x - 0.04x^2$
- Evaluate the two alternative for $M = \$5$
- Question: What value of M will make you indifferent between these two choices?

SOME PROPERTIES

- Property 1: $U(x)$ is increasing with respect to x if $U'(x) > 0$

- Property 2: $U(x)$ is strictly concave with respect to x if $U''(x) < 0$

- **Example:** Is exponential utility function given by $U(x) = -e^{-ax}$ concave?

RISK AVERSION COEFFICIENTS

- Graphically, the **stronger the bend**, the greater the risk aversion.

- **Arrow-Pratt absolute risk aversion coefficient**

$$a(x) = -\frac{U''(x)}{U'(x)}$$

- Coefficient $a(x)$
  - same for all equivalent utility function.
  - shows how risk aversion changes with the wealth level.

- For many individuals: risk aversion decreases as their wealth increases

RISK AVERSION: DIFFERENT UTILITY FORMS

- **Question:** Calculate the risk aversion and comment on the nature of risk aversion as wealth changes.

A. $U(x) = -e^{-ax}$

B. $U(x) = 1 - be^{-ax}$

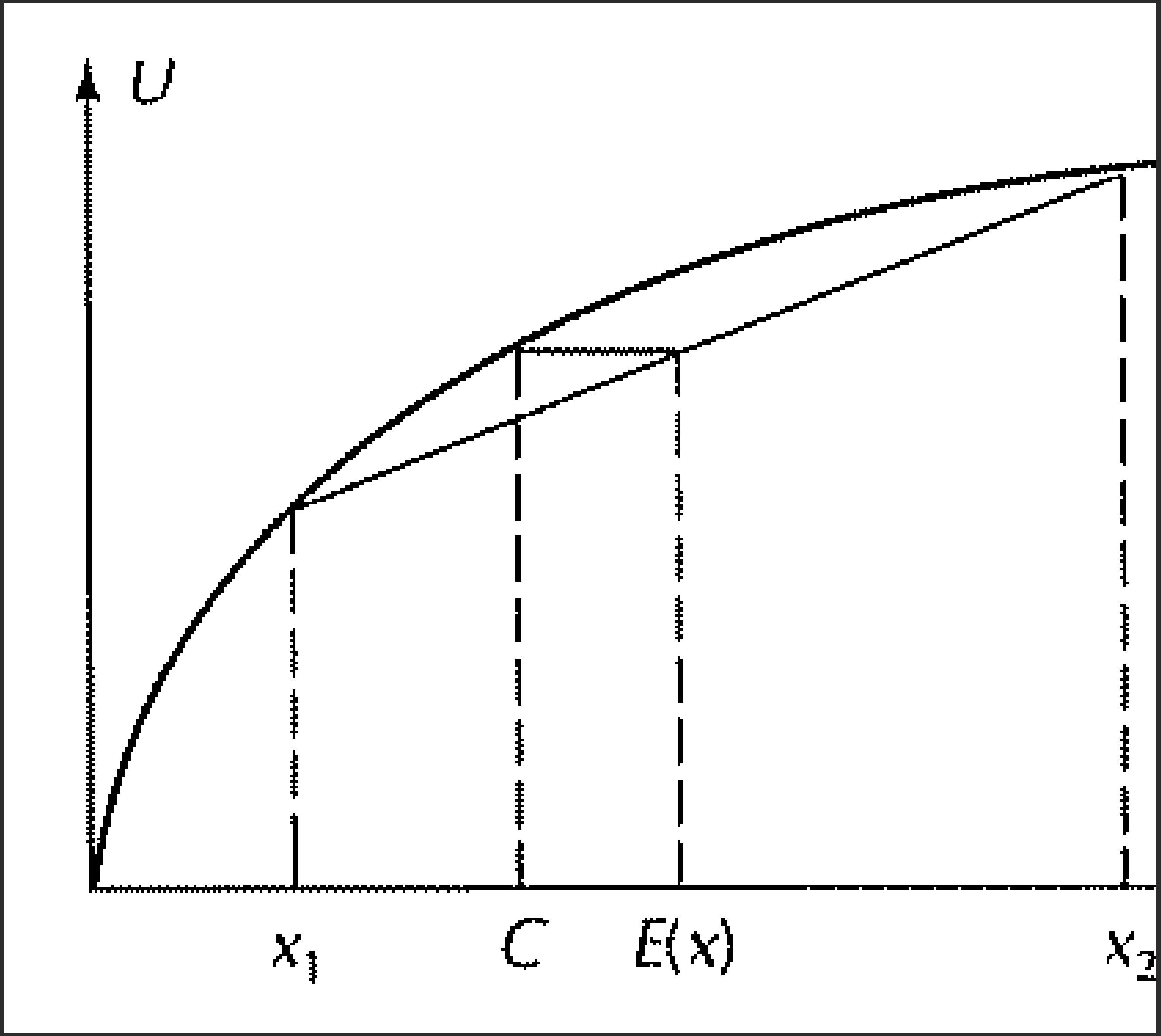
C. $U(x) = \ln(x)$

CERTAINTY EQUIVALENT

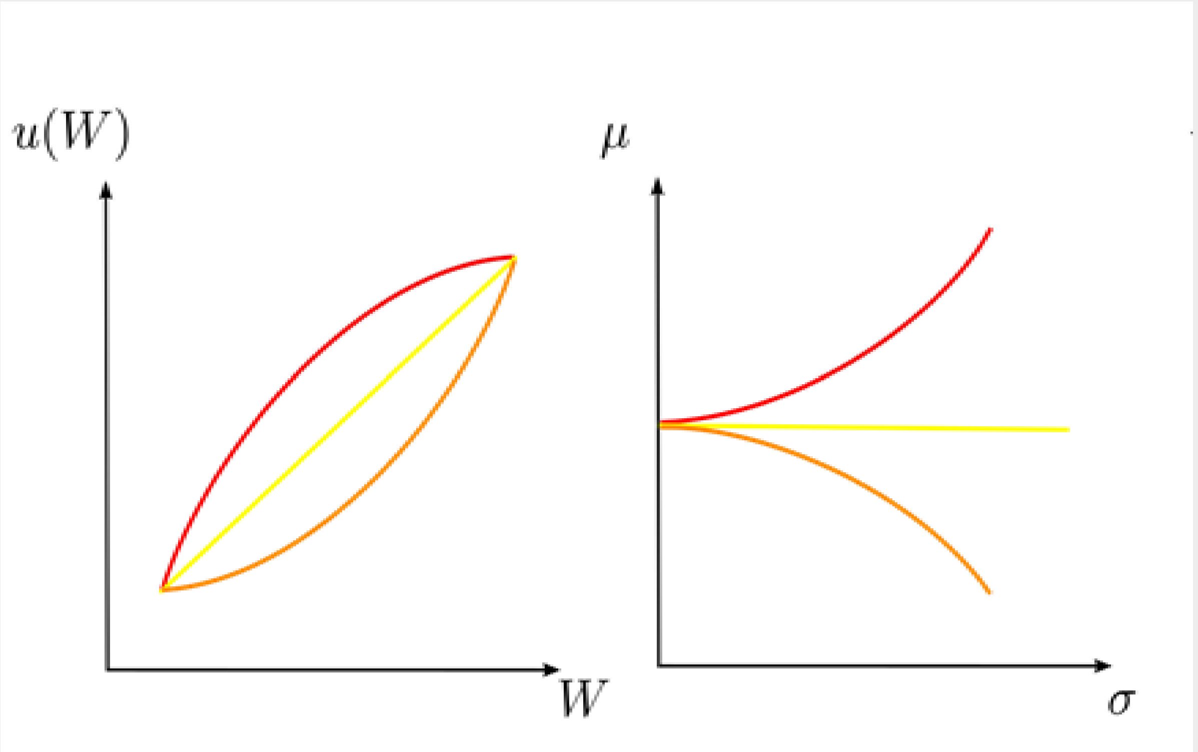
- **Certainty equivalent**

$$U(C) = E[U(x)]$$

- **Risk premium (RP)**

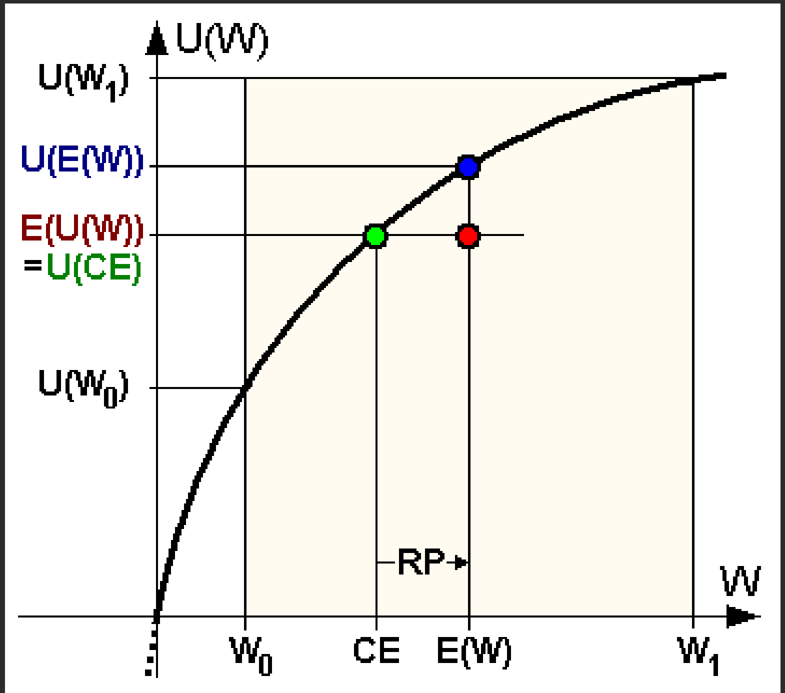


RISK PREFERENCES: AVERSION, NEUTRAL, AND LOVING



- **Legend**
  - Red color: Risk averse
  - Yellow color: Risk-neutral
  - Orange color: Risk-loving
- **Left plot**
  - Utility with changing wealth
  - Risk averse (concave)
- **Right plot**
  - Expected return and risk
  - **Indifference curves**

RISK AVERSE



RELATIVE RISK AVERSION

- **Arrow-Pratt relative risk aversion coefficient**

$$R(x) = -x\frac{U''(x)}{U'(x)}$$

- Relationship between **ARA and RRA**

$$R(x) \;=\; x \;*\; a(x)$$

- Homework:
  - Calculate ARA and RRA for $u(x) = \frac{x^{1-\rho}-1}{1-\rho}$

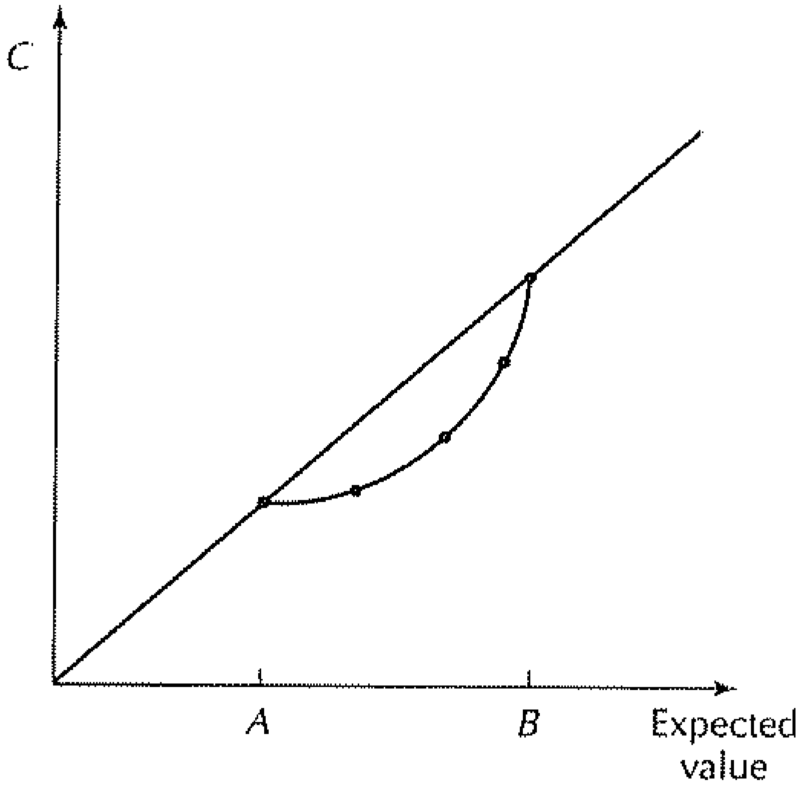
SPECIFICATION OF UTILITY FUNCTION

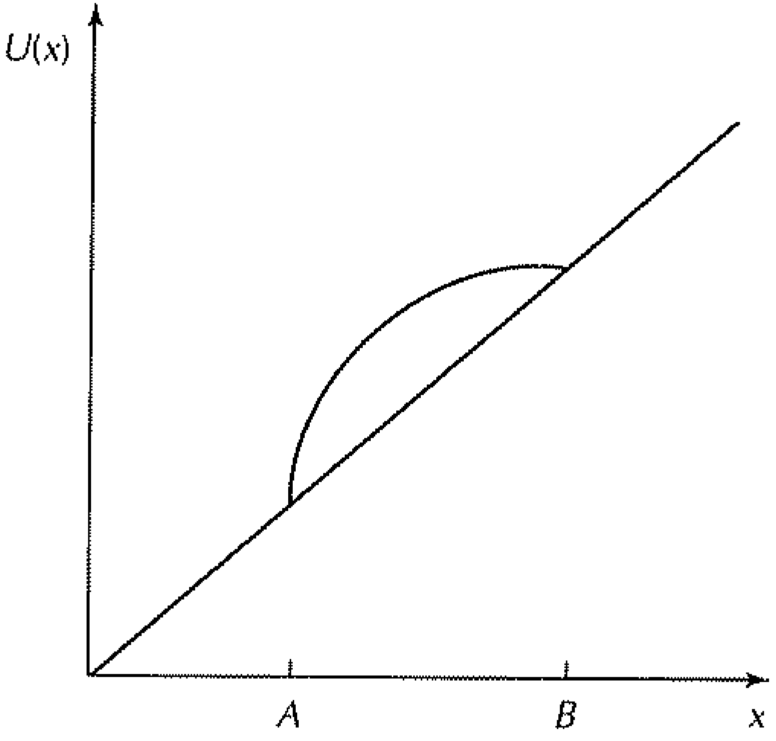
1. Direct measurement of utility
   - Ask the individual to assign **certainty equivalents** to various **risky alternatives**

2. Parameter Families [Exponential, logarithmic, power, etc.]
   - Determine a single lottery in certainty equivalent form
   - An example lottery with p=0.5 [($100,000, $1M)] vs CE($400,000)
   - $U(x) = -e^{-ax}$

3. Questionnaire method [https://www.schwab.com/public/file/P-778947/InvestorProfileQuestionnaire.pdf]

DIRECT METHOD (EXPERIMENTAL)





EUT & MEAN-VARIANCE APPROACH

- Mean-variance criterion reconciles with **Expected Utility Theory** under either of the assumption
  - **Quadratic Utility** $U(x) = ax - \frac{1}{2}bx^2$
  - **Normal returns**

DISCLAIMER

- The information in this presentation has been compiled from the following textbook which has been mentioned as a reference text for this course on **Financial Engineering.**

- Reference Text:
  - **Investment Science**, 2nd Edition, Oxford University Press, David G. Luenberger