Data Structures and Algorithms

(ESO207)

Lecture 35

- A new algorithm design paradigm: Greedy strategy

part II

Continuing Problem from last class

JOB Scheduling

Largest subset of non-overlapping job

A job scheduling problem

Formal Description

INPUT:

- A set J of n jobs $\{j_1, j_2, \ldots, j_n\}$
- job j_i is specified by two real numbers

  s(i): start time of job j_i

  f(i): finish time of job j_i
- A single server

Constraints:

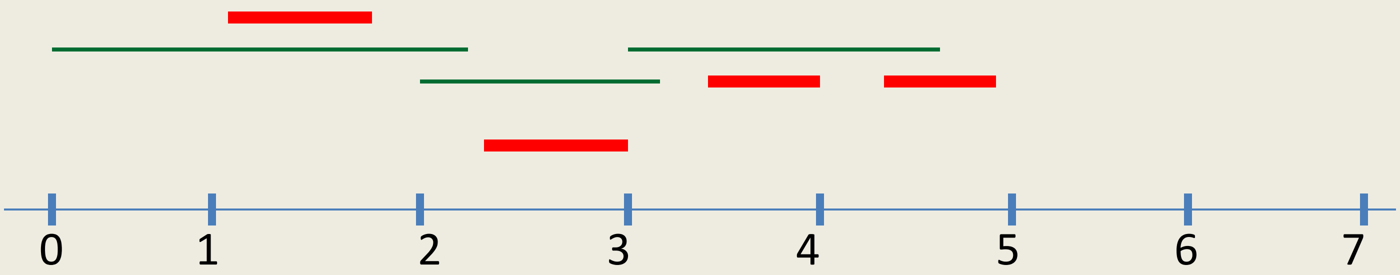
- Server can execute at most one job at any moment of time and a job.
- **Job** j_i, if scheduled, has to be scheduled during[s(i), f(i)] only.

Aim:

To select the **largest** subset

Designing algorithm for the problem

Strategy 4: Select the job with earliest finish time



Intuition:

Selecting such a job will **free** the server **earliest**

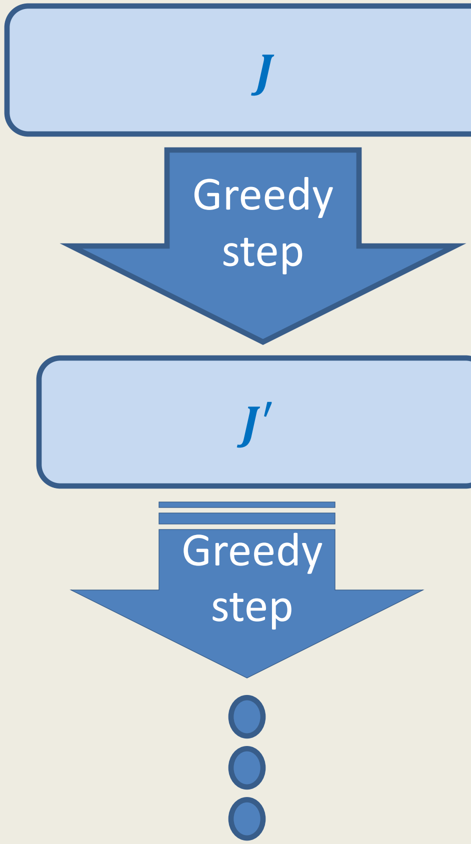
➔ hence more no. of jobs might get scheduled.

Algorithm "earliest finish time"

Proof of correctness ?

Let $x \in J$ be the job with earliest finish time.

Let $J' = J \backslash$ **Overlap**(x)



Algorithm (Input : set J of n jobs.)

1. Define $A \leftarrow \emptyset$;
2. **While** J <>$\emptyset$ **do**

   { Let $x \in J$ has earliest finish time;

   $A \leftarrow A \cup \{x\}$;

   $J \leftarrow J \backslash$ **Overlap**(x);

   }
3. Return A;

Lemma1 (last class):

There exists <u>an</u> optimal solution for J containing the **earliest finish time** job.

Algorithm "earliest finish time"

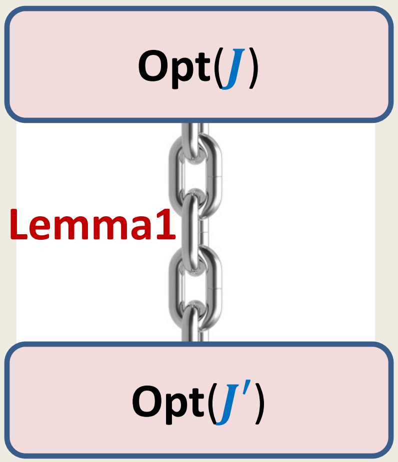
How to prove it ?

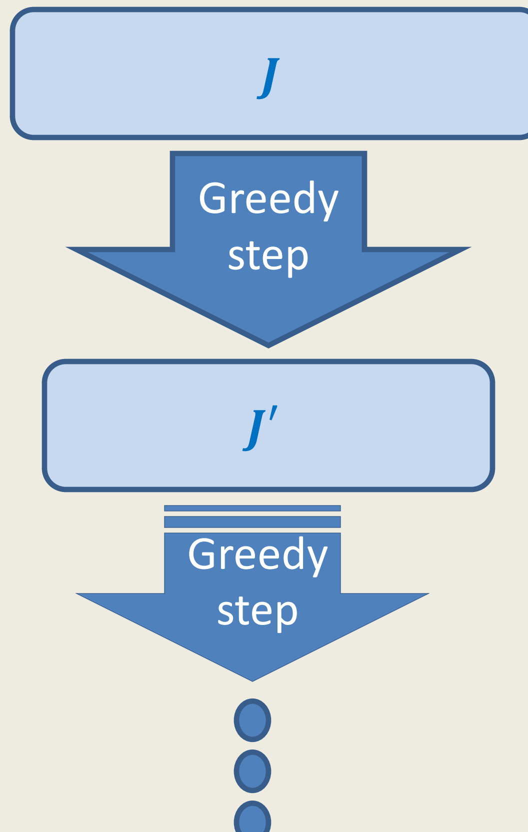
Proof of correctness ?

Let $x \in J$ be the job with earliest finish time.

Let $J' = J \setminus$ **Overlap**(x)

Opt(J)= Opt(J') + 1





Notation:

Opt(J):

Theorem: Opt(J) = Opt(J') + 1.

- Proof has two parts

$$\left.\begin{array}{l} \textbf{Opt}(J) \geq \textbf{Opt}(J') + 1 \\ \textbf{Opt}(J') \geq \textbf{Opt}(J) - 1 \end{array}\right\}$$

Try to give a physical interpretation to these inequalities.

- Proof for each part is a proof by **construction**

Algorithm "earliest finish time"

Proving $\boxed{\textbf{Opt}(J) \geq \textbf{Opt}(J') + 1}$.

Observation: start time of every job in J' is greater than finish time of x.

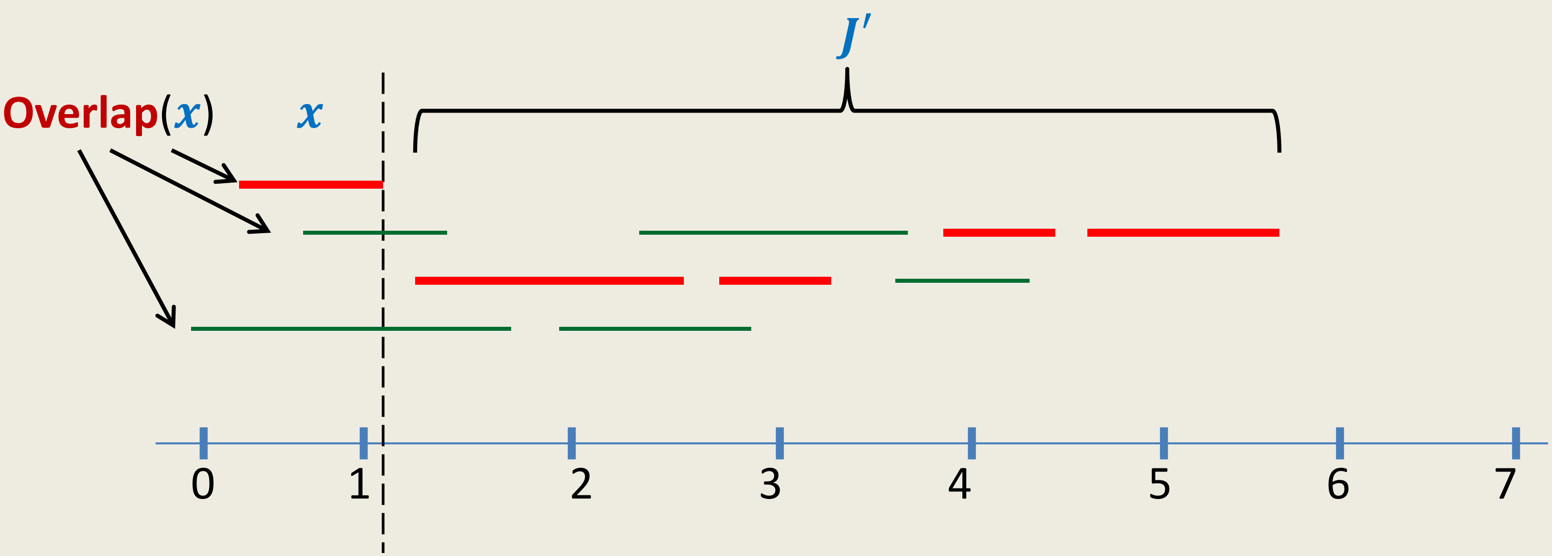
Let O' be any optimal solution for J'.

None of the jobs in

Hence $O' \cup \{x\}$ is a

Therefore **Opt**(J) ≥ **|**O'**|** + 1

From an **optimal solution** of J'
can you derive a **solution** for J with one extra job?



Algorithm "earliest finish time"

Lemma1 (last class): There exists an optimal solution for J in which x is present.

Let O be an optimal solution for J containing x.

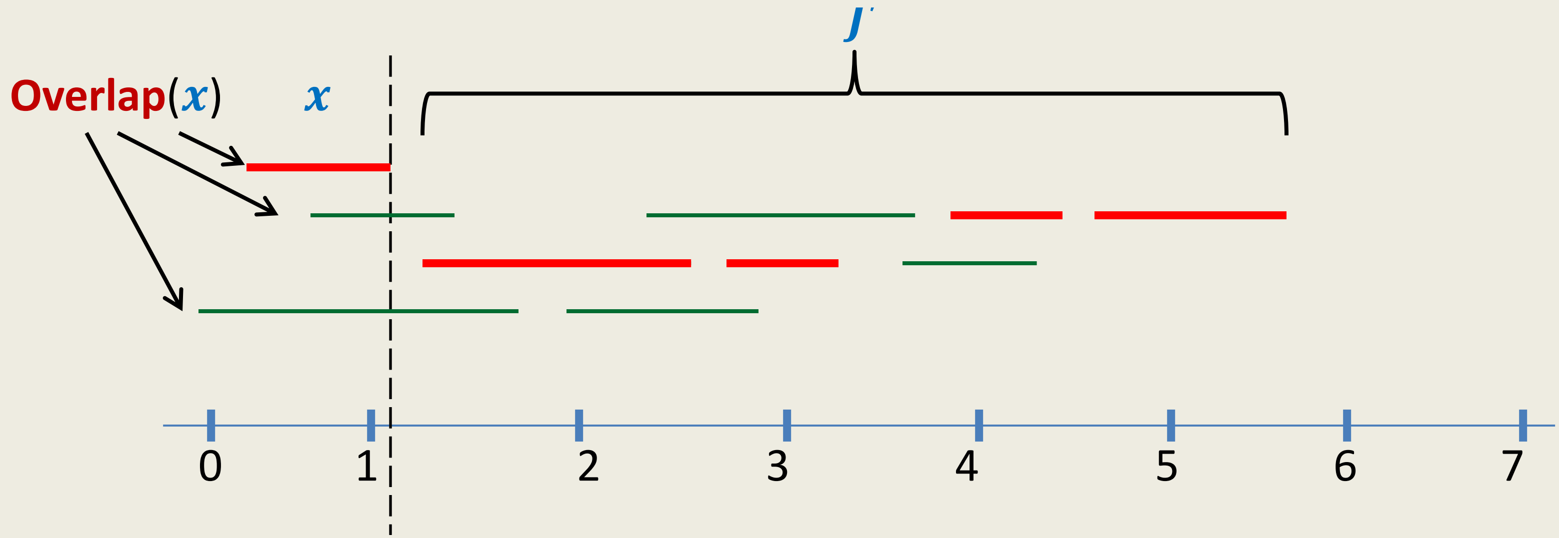
None of the jobs in O overlaps with x.

➔ Every job from O

From an **optimal solution** of J
can you derive a **solution** for J' with one job less?

Hence $O \backslash \{x\}$ is a subset of non-overlapping jobs from J'.

Therefore **Opt**(J') ≥ **|**O**|** **-** 1



Theorem:

Given any set J of n jobs,

the algorithm based on "**earliest finish time**" approach

computes the largest subset of non-overlapping job.

O(n log n) implementation of the Algorithm

This is not the only way to achieve O(n log n) time. It can be done other ways as well.

Algorithm (**Input :** set J of n jobs.)

1. Define $A \leftarrow \emptyset$;
2. **While** J <>$\emptyset$ **do**

   {   Let $x \in J$ have earliest finish time;

   $A \leftarrow A \cup \{x\}$;

   $J \leftarrow J \backslash$**Overlap**($x$);

   }
3. Return A;

Maintain a **binary min-heap** for J based on **finish time** as the key.

Sort J in increasing order of **start time**.

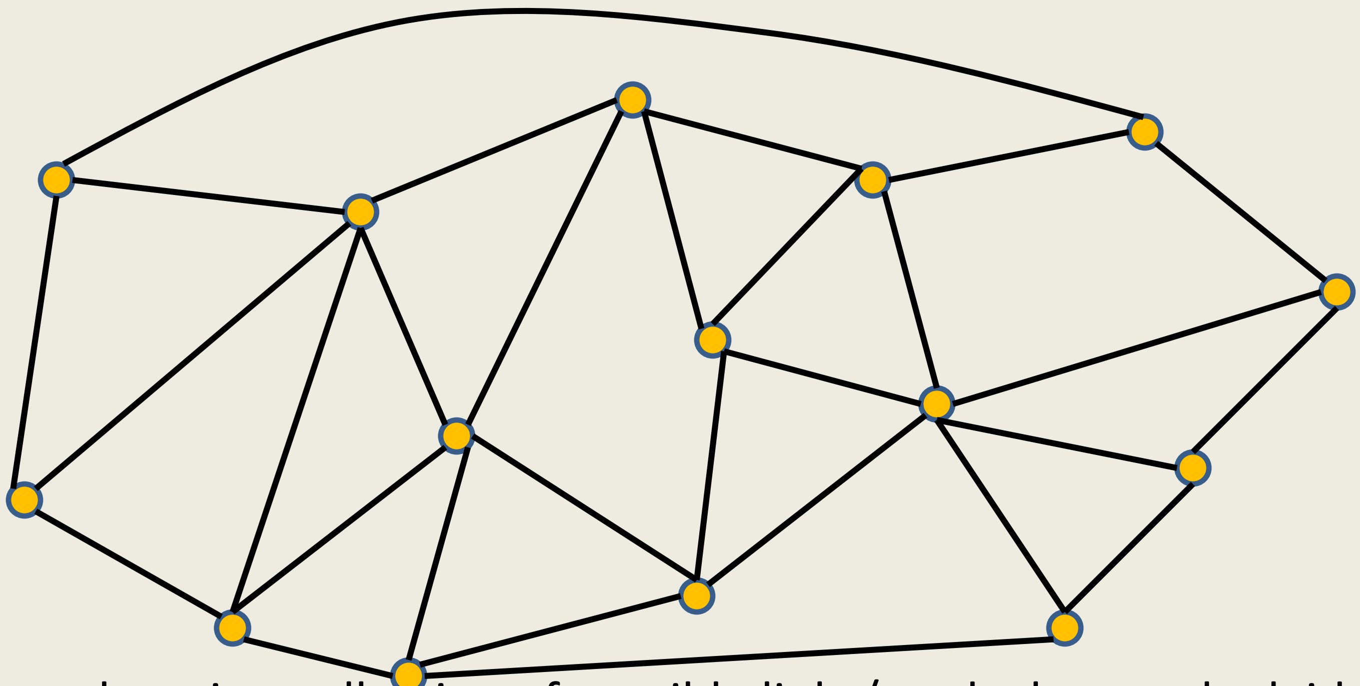
➔ **O(n^2) time complexity is obvious**

Problem 2

First we shall give motivation.

Motivation:

A road or telecommunication network



Suppose there is a collection of possible links/roads that can be laid.

But laying down each possible link/road is costly.

Aim: To lay down **least number** of links/roads to ensure **connectivity** between each pair of nodes/cities.

Motivation

Formal description of the problem

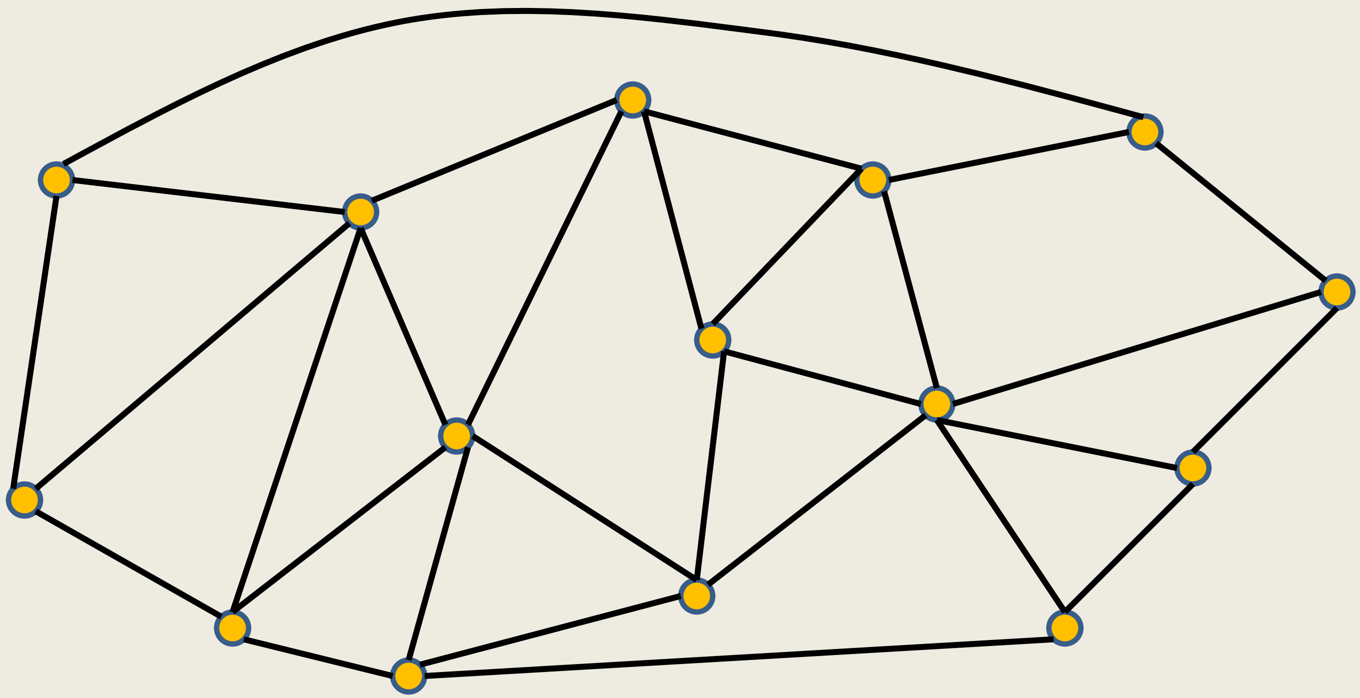
Input: an undirected graph **G**=(**V**,**E**).

Aim: compute a **subgraph** (**V**,**E'**), **E'** ⊆ **E** such that

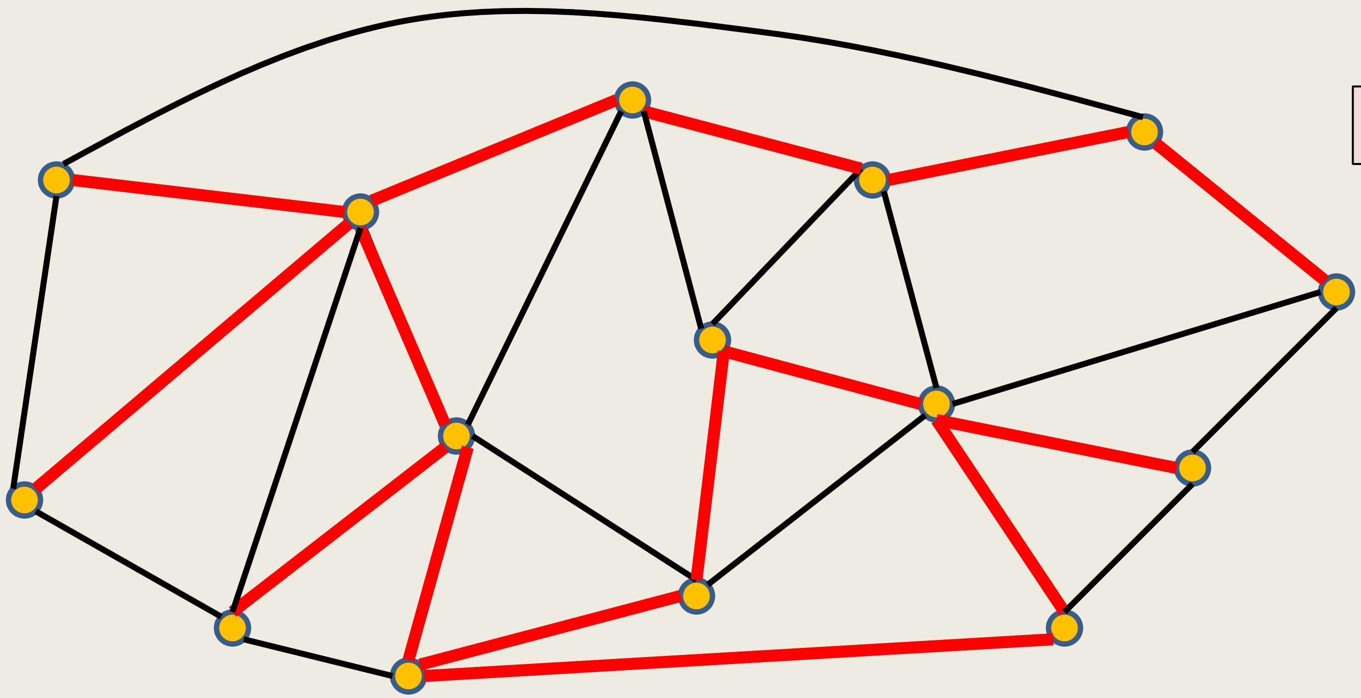
- **Connectivity** among all **V** is guaranteed in the **subgraph**.
- **|E'|** is **minimum.**

How will such a subgraph look like ?

A road or telecommunication network



A road or telecommunication network



No	Yes	No

Is this subgraph meeting our requirement ?

A tree

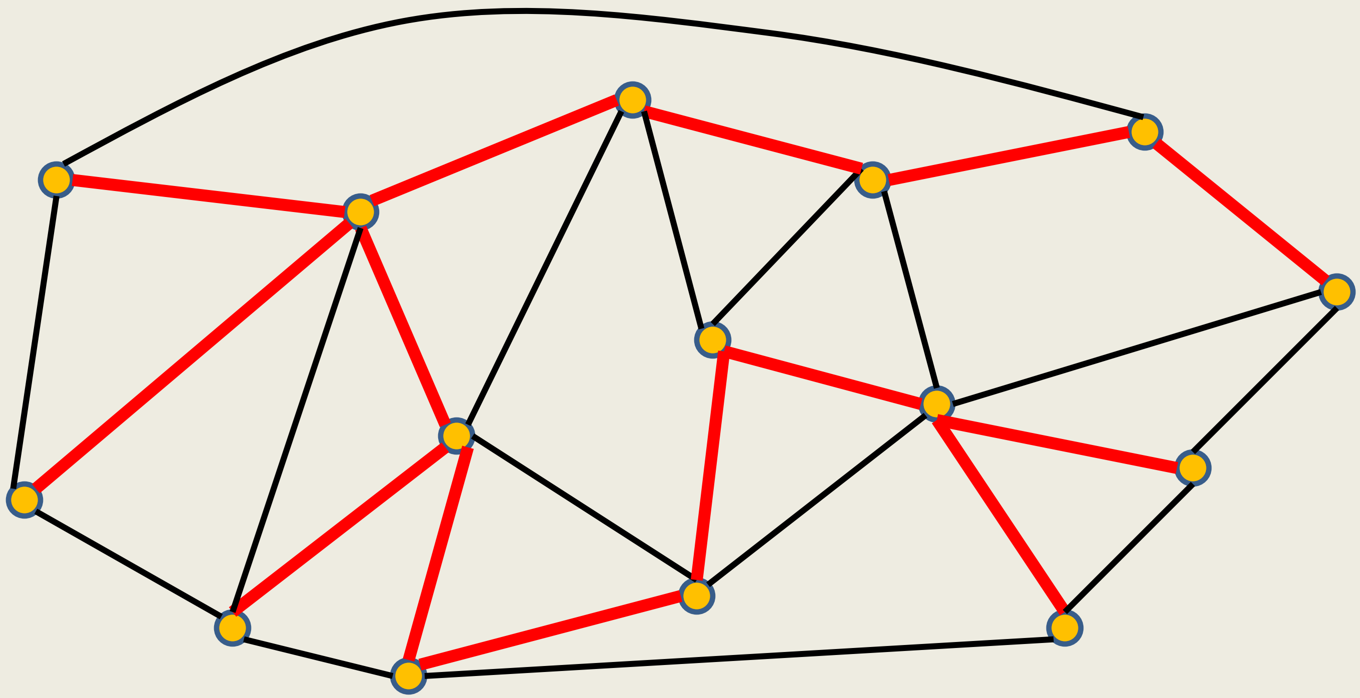
The following definitions are **equivalent**.

- An undirected graph which is **connected**
- An undirected graph where each pair of vertices has
- An undirected **connected** graph on n vertices and
- An undirected graph on n vertices and $n - 1$ **edges** and

A Spanning tree

Definition: For an undirected graph (**V**,**E**),

A spanning tree is a **subgraph** (**V**,**E'**), **E'** ⊆ **E** which is a tree.

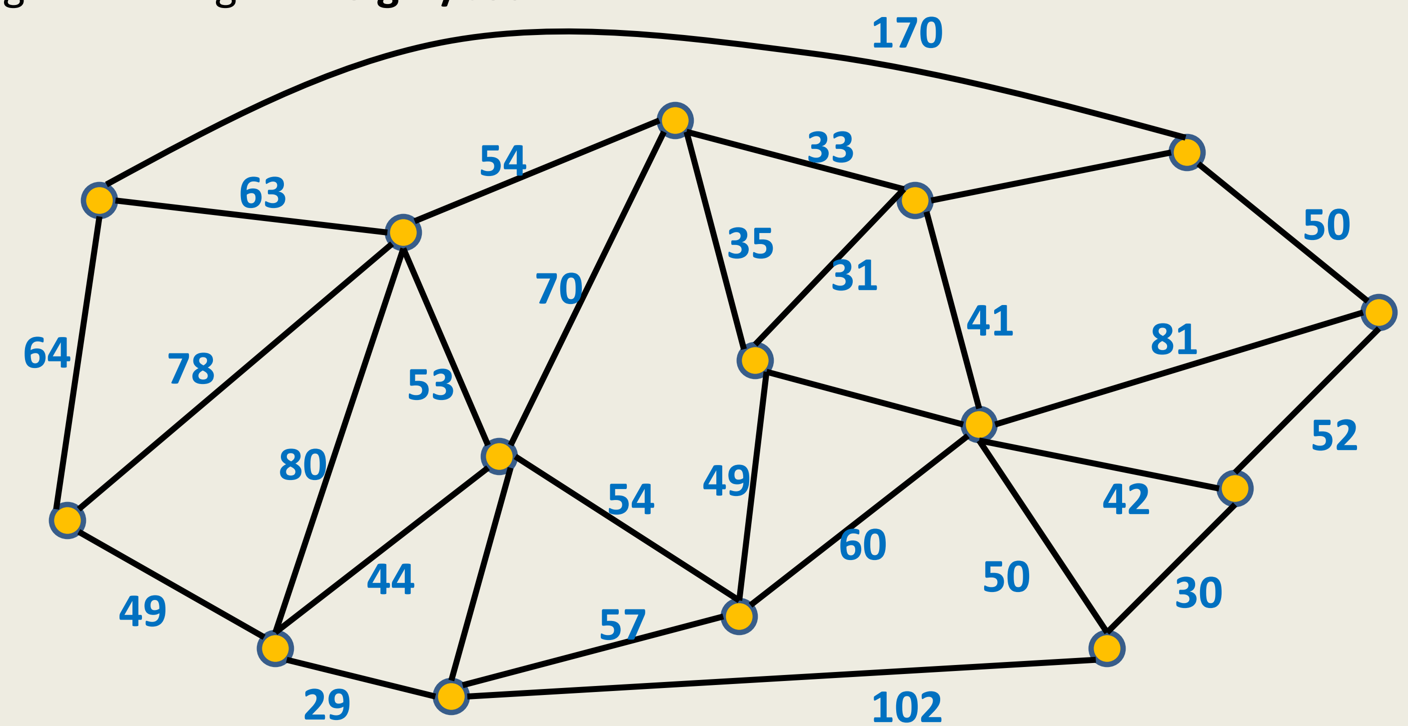


Observation: Given a spanning tree $\boldsymbol{T}$ of a graph $\boldsymbol{G}$, adding a nontree edge $\boldsymbol{e}$ to $\boldsymbol{T}$ creates a unique cycle.

There will be total $\boldsymbol{m-n+1}$ such cycles. These are called **fundamental cycles** in $\boldsymbol{G}$ induced by the spanning tree $\boldsymbol{T}$.

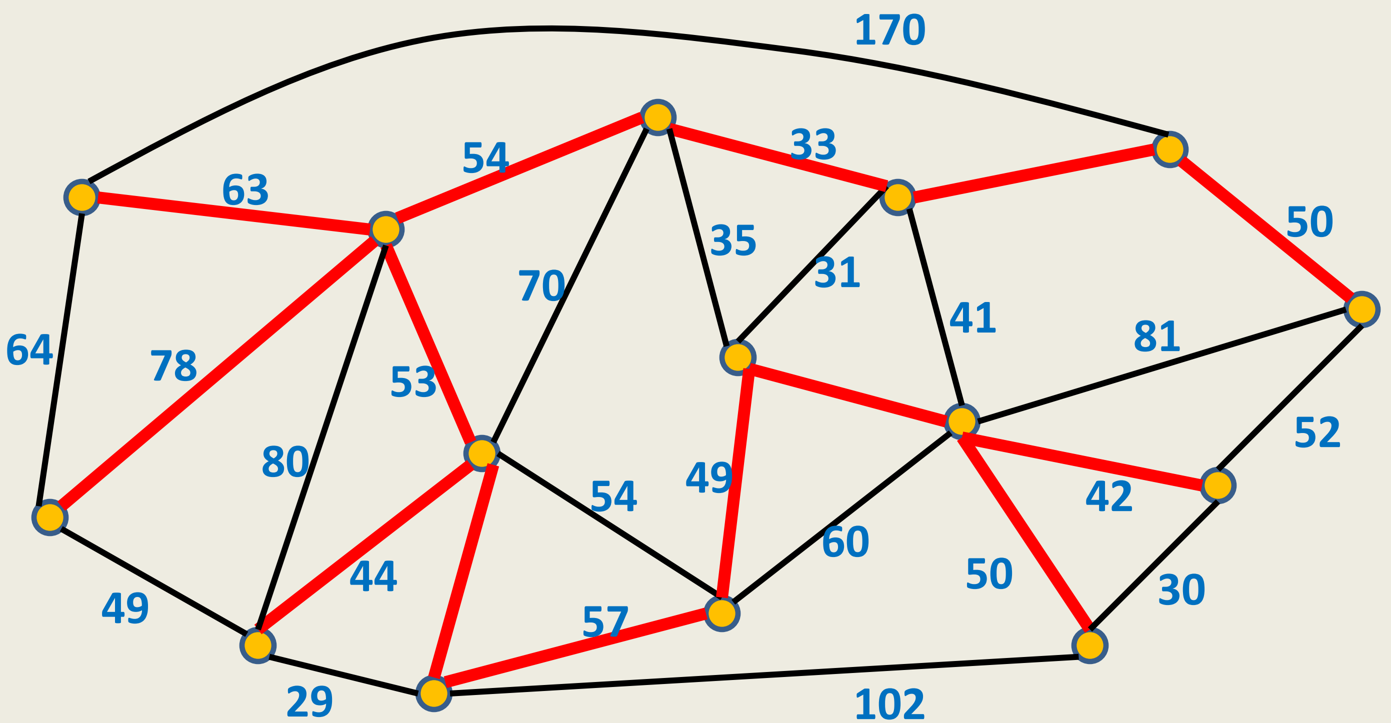
A road or telecommunication network

Assign each edge a **weight/cost**.



Adding more reality to the problem

A road or telecommunication network



Any arbitrary spanning tree (like the one shown above) will not serve our goal☹.

We need to select the spanning tree with **least weight/cost**.

Problem 2

Minimum spanning tree

Problem Description

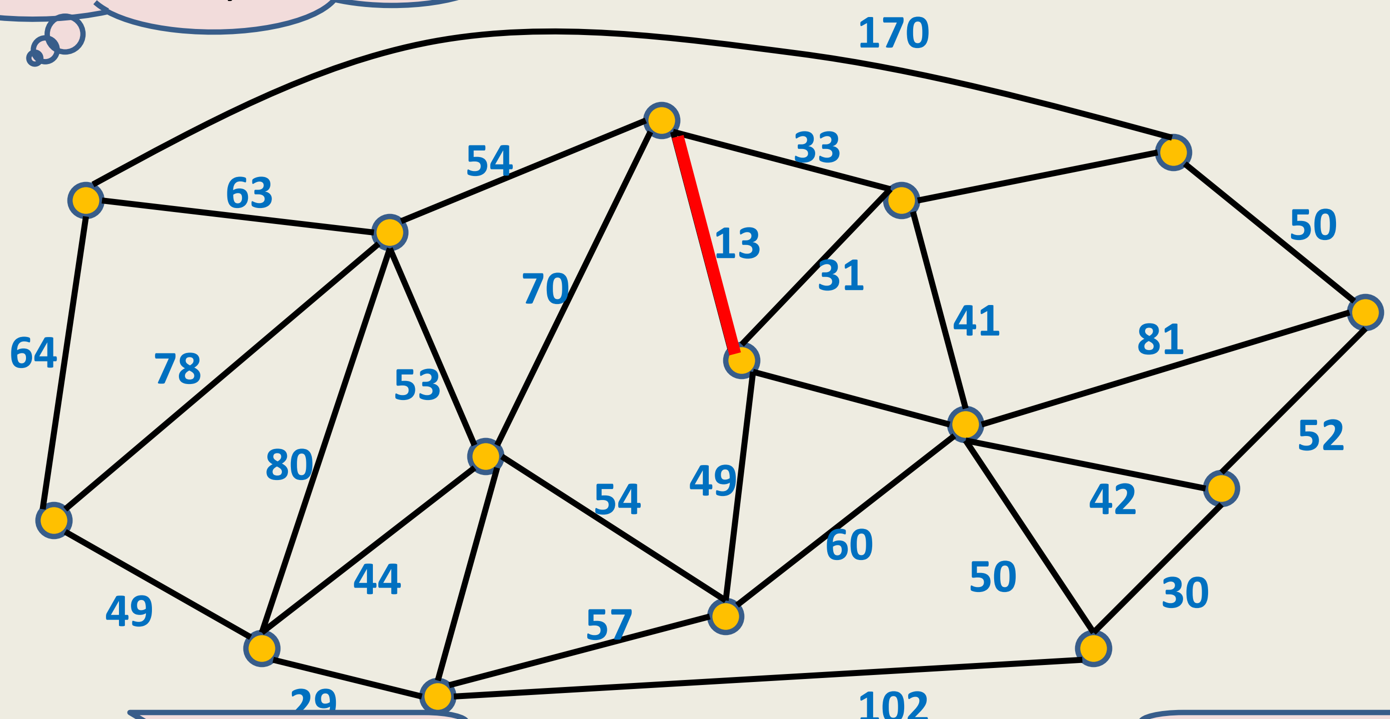
Input: an undirected graph **G**=(**V**,**E**) with **w**: **E** → $\mathbb{R}$,

Aim: compute a **spanning tree** (**V**,**E'**), **E'** ⊆ **E** such that

$\sum_{e \in E'} w(e)$ is **minimum.**

How to compute a MST ?

The least weight edge
should be in MST.
But why ?



Look a … nd.
Ca

Is there any edge for which you feel strongly to be present in MST ?

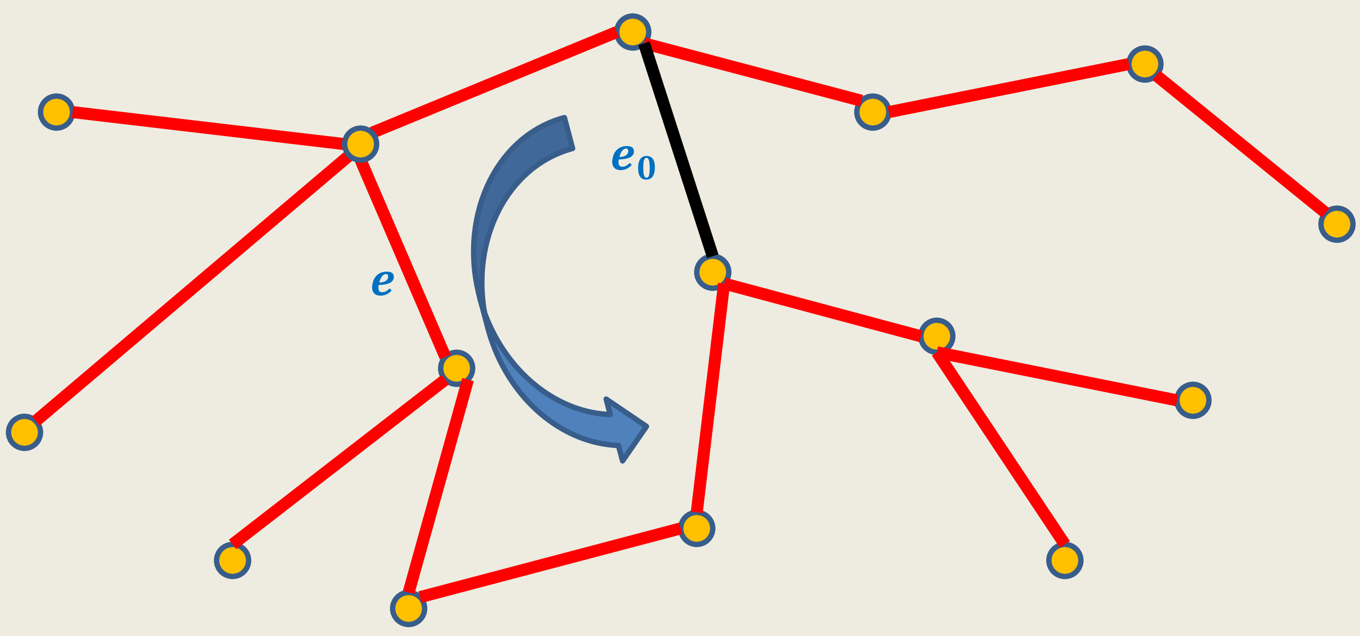
Let $e_0 \epsilon$ **E** be the edge of least weight in the given graph.

Lemma2: There is a **MST** T containing e_0.

Proof: Consider any **MST** T. Let $e_0 \notin T$.

Consider the fundamental cycle C defined by e_0 in T.

Swap e_0 with any edge $e \epsilon\, T$ present in C.



Let $\boldsymbol{e_0} \in$ **E** be the edge of least weight in the given graph.

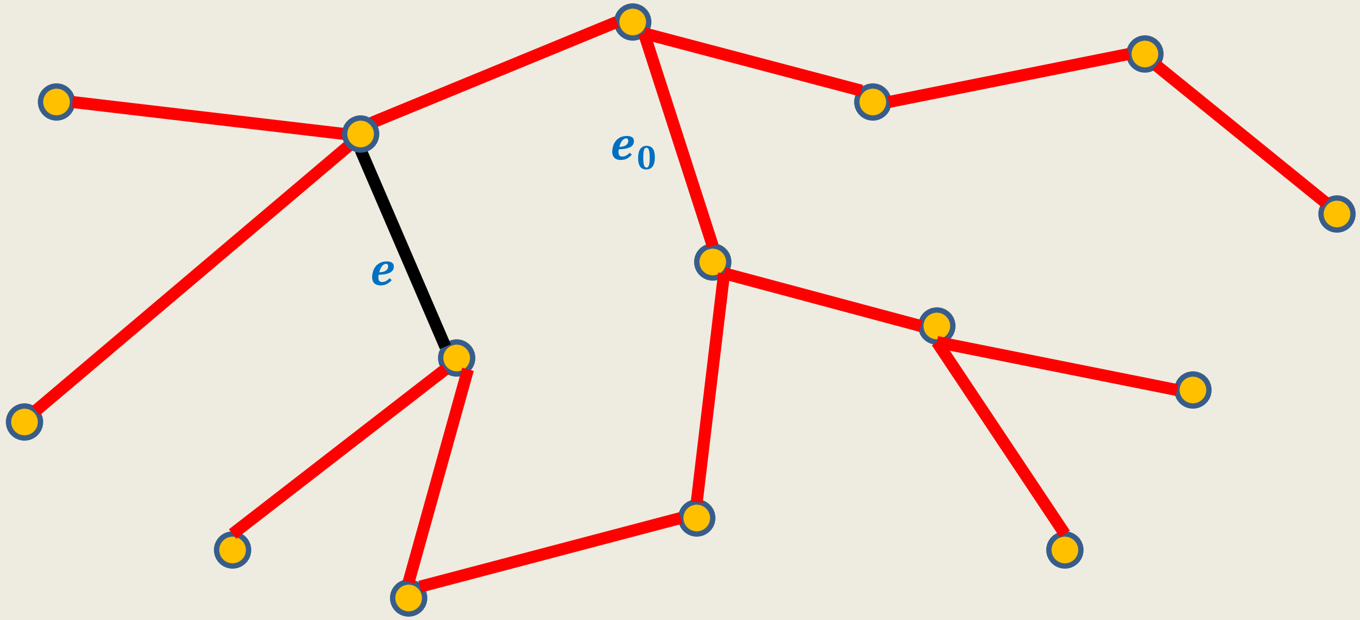
Lemma2: There is a **MST** $\boldsymbol{T}$ containing $\boldsymbol{e_0}$.

Proof: Consider any **MST** $\boldsymbol{T}$. Let $\boldsymbol{e_0} \notin \boldsymbol{T}$.

Consider the fundamental cycle $\boldsymbol{C}$ defined by $\boldsymbol{e_0}$ in $\boldsymbol{T}$.

Swap $\boldsymbol{e_0}$ with any edge $\boldsymbol{e} \in \boldsymbol{T}$ present in $\boldsymbol{C}$.

We get a spanning tree of weight ≤ $\mathbf{w}(\boldsymbol{T})$.



Try to translate Lemma2 to an algorithm for MST ?

with **inspiration** from the job scheduling problem ☺