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ASSIGNMENT - 2

1.) cashflow at time t=0

borrow $\rightarrow$ $S(0)$.
buy one unit and short e^{-qT} units

we invest all the cash borrowed for a net 0 cashflow.

now we pay carrying cost by selling fraction q at each instant from our holdings.
mathematical representation :

$$\frac{dx}{dt} = -qx$$

since, carrying charge per unit time is $\propto$ to the spot price

solving $-\frac{1}{q}\int \frac{dx}{x} = \int dt$

$$\ln x = -qT$$

$$\boxed{x = e^{-qT}}$$

now we clear our account, profit after time T :

$$Fe^{-qT} - e^{rT} S(0) = 0 \cdot \text{(no arbitrage condition)}$$

$$\boxed{F = e^{(r+q)T} S(0)} \quad \underline{HP}$$

r : fixed interest rate

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2.) $u = e^{w} \Rightarrow w = \ln u = N(\mu, \sigma^2)$

$Var(u) = E(u^2) - (E(u))^2$

we will find $E(u)$ first.

$$f_u(u) = f_w(w) \left|\frac{dw}{du}\right| = \frac{f_w(w)}{u}$$

$$f_u(u) = \frac{1}{\sqrt{2\pi}\,\sigma\, u} e^{-1/2\left(\frac{\ln u - \mu}{\sigma}\right)^2} \quad ; x > 0$$

$$E(u) = \int_0^\infty u\, f_u(u)\, du$$

$$E(u) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty e^{-1/2\left(\frac{\ln u - \mu}{\sigma}\right)^2} du$$

let $z = \dfrac{\ln u - \mu}{\sigma}$.

$$dz = \frac{1}{\sigma}\frac{1}{u}\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2} + \mu + \sigma z}\, dz$$

rearranging.

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z-\sigma)^2} \cdot e^{\left(\mu + \frac{\sigma^2}{2}\right)} dz$$

$$\boxed{E(u) = e^{\left(\mu + \frac{\sigma^2}{2}\right)}}$$

now we will find $E(u^2)$

$$E(u^2) = \int_0^\infty u^2\, f_u(u)\, du$$

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$$E(u^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty u \, e^{-\frac{1}{2}\left(\frac{\ln u - \mu}{\sigma}\right)^2} du$$

let $z = \dfrac{\ln u - \mu}{\sigma}$

$$dz = \frac{1}{\sigma} \frac{1}{\mu} du$$

also $u^2 = e^{2(\sigma z + \mu)}$.

$$E(u^2) = \frac{1 \, \cancel{\sigma}}{\sqrt{2\pi}\,\cancel{\sigma}} \int_0^\infty e^{-\frac{1}{2}z^2 + 2\sigma z + 2\mu} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}\left[(z-2\sigma)^2 - 4(\mu+\sigma^2)\right]} dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{2(\mu+\sigma^2)} \int_0^\infty e^{-\frac{1}{2}(z-2\sigma)^2} dz$$

$$\boxed{E(u^2) = e^{2(\mu+\sigma^2)}}$$

now $\mathrm{Var}(u) = E(u^2) - (E(u))^2$

$$\mathrm{Var}(u) = e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$$

$$\mathrm{Var}(u) = e^{2\mu} e^{2\sigma^2} - e^{2\mu} e^{\sigma^2}$$

$$\mathrm{Var}(\mu) = e^{2\mu} e^{\sigma^2} (e^{\sigma^2} - 1).$$

$$\boxed{\mathrm{Var}(u) = e^{2\mu+\sigma^2} (e^{\sigma^2} - 1)}$$ Ans

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3) $dS = aS\,dt + bS\,dz.$

$y(t) = F(S,t) = \sqrt{S(t)}$

$$dy(t) = \left(\frac{dF}{dS}a + \frac{dF}{dt} + \frac{1}{2}\frac{d^2F}{dS^2}\right)dt + \frac{dF}{dS}b\,dz$$

$$\frac{dF}{dS} = \frac{1}{2\sqrt{S(t)}}, \quad \frac{dF}{dt} = 0, \quad \frac{d^2F}{dS^2} = -\frac{1}{4(\sqrt{S(t)})^3}$$

$$dy(t) = \left(\frac{1}{2}(S)^{-1/2}aS - \frac{1}{8}S^{-3/2}b^2S^2\right)dt + \frac{1}{2}S^{-1/2}b\,S\,dz$$

$$\boxed{dy(t) = \left(\frac{1}{2}a - \frac{1}{8}b^2\right)y(t)\,dt + \frac{1}{2}b\,y(t)\,dz}$$