Solutions to Problems in

An Introduction to Stochastic Modeling

3RD EDITION

Howard M. Taylor
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CHAPTER I

2.1 $E[\mathbf{1}\{A_1\}] = Pr\{A_1\} = \frac{1}{13}$. Similarly, $E[\mathbf{1}\{A_1\}] = Pr\{A_k\} = \frac{1}{13}$ for $k = 1, \ldots, 13$. Then, because the expected value of a sum is always the sum of the expected values, $E[N] = E[\mathbf{1}\{A_1\}] + \cdots + E[\mathbf{1}\{A_{13}\}] = \frac{1}{13} + \cdots + \frac{1}{13} = 1$.

2.2 Let X be the first number observed and let Y be the second. We use the identity $(\Sigma x_i)^2 = \Sigma x_i^2 + \Sigma_{i \neq j} x_i x_j$ several times.

$$E[X] = E[Y] = \frac{1}{N}\sum x_i;$$

$$Var[X] = Var[Y] = \frac{1}{N}\sum x_i^2 - \left(\frac{1}{N}\sum x_i\right)^2 = \frac{(N-1)\sum x_i^2 - \sum_{i \neq j} x_i x_j}{N^2};$$

$$E[XY] = \frac{\sum_{i \neq j} x_i x_j}{N(N-1)};$$

$$Cov[X,Y] = E[XY] - E[X]E[Y] = \frac{\sum_{i \neq j} x_i x_j - (N-1)\sum x_i^2}{N^2(N-1)}$$

$$\varrho_{X,Y} = \frac{Cov[X,Y]}{\sigma_X \sigma_Y} = -\frac{1}{N-1}.$$

2.3 Write $S_r = \xi_1 + \ldots + \xi_r$ where ξ_k is the number of additional samples needed to observe k district elements, assuming that k-1 district elements have already been observed. Then, difining $p_k = Pr[\xi_k = 1\} = 1 - \dfrac{k-1}{N}$ we have $Pr\{\xi_k = n\} = p_k(1 - p_k)^{n-1}$ for $n = 1, 2, \ldots$ and $E[\xi_k] = \dfrac{1}{p_k}$. Finally, $E[S_r] = E[\xi_1] + \cdots + E[\xi_r] = \dfrac{1}{p_1} + \cdots + \dfrac{1}{p_r}$ will verify the given formula.

2.4 Using an obvious notation, the event $\{N = n\}$ is equivalent to either $\overbrace{HTH\ldots HT}^{n-1}T$ or $\overbrace{THT\ldots TH}^{n-1}H$ so $P_r\{N = n\} = 2 \times \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}$ for $n = 2, 3, \ldots$; $Pr\{N$ is even$\} = \Sigma_{n=2,4,\ldots}\left(\frac{1}{2}\right)^{n-1} = \frac{2}{3}$ and $P_r\{N \leqslant 6\} = \Sigma_{n=2}^{6}\left(\frac{1}{2}\right)^{n-1} = \frac{31}{32}$. $Pr\{N$ is even and $N \leqslant 6\}$ 5 $\Sigma_{m=2,4,6}\left(\frac{1}{2}\right)^{n-1} = \frac{21}{32}$.

2.5 Using an obvious notation, the probability that A wins on the $2n + 1$ trial is $Pr\left\{\overbrace{A^c B^c \ldots A^c B^c}^{\text{In losses}} A\right\}$

$= [(1 - p)(1 - q)]^n p,\ n = 0, 1, \ldots.$ $Pr\{\text{A wins}\} = \Sigma_{n=0}^{\infty}[(1-p)(1-q)]^n\, p = \dfrac{p}{1-(1-p)(1-q)}.$
$Pr\{\text{A wins on } 2n+1 \text{ play}|\text{A wins}\} = (1-\pi)\pi^n$ where $\pi = (1-p)(1-q)$. $E[{}^{\#}\text{trials}|\text{A wins}] = \Sigma_{n=0}^{\infty}$ $(2n+1)(1-\pi)\pi^n = 1 + \dfrac{2\pi}{1-\pi} = \dfrac{1+(1-p)(1-q)}{1-(1-p)(1-q)} = \dfrac{2}{1-(1-p)(1-q)} - 1.$

2.6 Let N be the number of losses and let S be the sum. Then $Pr\{N = n, S = k\} = \left(\frac{1}{6}\right)^{n-1}\left(\frac{5}{6}p_k\right)$ where $p_3 = p_{11} = p_4 = p_{10} = \frac{1}{15}; p_5 = p_9 = p_6 = p_8 = \frac{2}{15}$ and $p_7 = \frac{3}{15}$. Finally $Pr\{S = k\} = \Sigma_{n=1}^{\infty} Pr\{N = n, S = k\} = p_k$. (It is *not* a correct *argument* to simply say $Pr\{S = k\} = Pr\{\text{Sum of 2 dice} = k|\text{Dice differ}\}$. Compare with Exercise II, 2.1.)

2.7 We are given that (*) $Pr\{U > u, W > w\} = [1 - F_u(u)][1 - F_w(w)]$ for all u, w. According to the difinition for independence we wish to show that $Pr\{U \le u, W \le w\} = F_u(u)F_w(w)$ for all u, w. Taking complements and using the addition law

$$\begin{aligned}
Pr\{U \le u, W \le w\} &= 1 - Pr\{U > u \text{ or } W > w\} \\
&= 1 - \left[Pr\{U > u\} + Pr\{W > w\} - Pr\{U > u, W > w\}\right] \\
&= 1 - \left[\left(1 - F_U(u)\right) + \left(1 - F_W(w)\right) - \left(1 - F_u(u)\right)\left(1 - F_W(w)\right)\right] \\
&= F_U(u)F_W(w) \quad \text{after simplification.}
\end{aligned}$$

2.8 (a) $E[Y] = E[a + bX] = \int(a + bx)dF_X(x) = a\int dF_X(x) + b\int x dF_X(x) = a + bE[X] = a + b\mu$. In words, (a) implies that the expected value of a constant times a random variable is the constant times the expected value of the random variable. So $E[b^2(X-\mu)^2] = b^2E[(X-\mu)^2]$.

(b) $Var[Y] = E[(Y - E\{Y\})^2] = E[(a + bX - a - b\mu)^2] = E[b^2(X-\mu)^2] = b^2E[(X-\mu)^2] = b^2\sigma^2$

2.9 Use the usual sums of numbers formula (See I, 6 if necessary) to establish

$$\sum_{k=1}^{n} k(n-k) = \frac{1}{6}n(n+1)(n-1); \text{ and}$$

$$\sum_{k=1}^{n} k^2(n-k) = n\sum k^2 - \sum k^3 = \frac{1}{12}n^2(n+1)(n-1), \text{ so}$$

$$E[X] = \frac{2}{n(n-1)}\sum k(n-k) = \frac{1}{3}(n+1)$$

$$E[X^2] = \frac{3}{n(n-1)}\sum k^2(n-k) = \frac{1}{6}n(n+1), \text{ and}$$

$$Var[X] = E[X^2] - (E[X])^2 = \frac{1}{18}(n+1)(n-2).$$

2.10 Observe, for example, $Pr\{Z = 4\} = Pr\{X = 3, Y = 1\} = (\frac{1}{2})(\frac{1}{6})$, using independence. Continuing in this manner,

z	1	2	3	4	5	6
$Pr\{Z = z\}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$

2.11 Observe, for example, $Pr\{W = z\} = Pr\{U = 0, V = 2\} + Pr\{U = 1, V = 1\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{3}$. Continuing in this manner, arrive at

w	1	2	3	4
$Pr\{W = w\}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

2.12 Changeing any of the random variables by adding or subtracting a constant will not affect the covariance. Therefore, by replacing U with $U\text{-}E[U]$, if necessary, etc, we may assume, without loss of generality that all of the means are zero. Because the means are zew, $Cov[X, Y] = E[XY] - E[X]E[Y] = E[XY] = E[UV - UW + VW - W^2] = -E[W^2] = -\sigma^2$. ($E[UV] = E[U]E[V] = 0$, etc.)

2.13
$$\begin{aligned} Pr\{v < V,\ U \leq u\} &= Pr\{v < X \leq u,\ v < Y \leq u\} \\ &= Pr\{v < X \leq u\}Pr\{v < Y \leq u\}\ \text{(by independence)} \\ &= (u - v)^2 \\ &= \iint_{(u',v')\, v < v' \leq u' \leq u} f_{u,v}(u', v')du'dv' \\ &= \int_v^u \left\{\int_{v'}^u f_{u,v}(u', v')du'\right\}dv'. \end{aligned}$$

The integrals are removed from the last expression by successive differentiation, first w.r.t. v (changing sign because v is a lower limit) than w.r.t. u. This tells us

$$f_{u,v}(u,v) = -\frac{d}{du}\frac{d}{dv}(u - v)^2 = 2 \quad \text{for} \quad 0 < v \leq u \leq 1.$$

3.1 Z has a discrete uniform distribution on $0, 1, \ldots, 9$.

3.2 In maximizing a continuous function, we often set the desirative equal to zero. In maximizing a function of a discrete variable, we equate the ratio of successive terms to one. More precisely, k^* is the smallest k for which $\frac{p(k+1)}{p(k)} < 1$, or, the smallest k for which $\frac{n-k}{k+1}\left(\frac{p}{1-p}\right) < 1$. Equivently, (b) $k^* = [(n + 1)p]$ where $[x]$ = greatest integer $\leq x$. For (a) let $n \to \infty$, $p \to 0$, $\lambda = np$. Then $k^* = [\lambda]$.

3.3 Recall that $e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$ and $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots$ so that $\sinh\lambda \equiv \frac{1}{2}(e^\lambda - e^{-\lambda}) = \lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots$ Then $Pr\{X \text{ is odd}\} = \sum_{k=1,3,5,\ldots} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\sinh(\lambda) = \frac{1}{2}(1 - e^{-2\lambda})$.

3.4
$$\begin{aligned} E[V] &= \sum_{k=0}^{\infty} \frac{1}{k+1}\frac{\lambda^k e^{-\lambda}}{k!} = \frac{e^{-\lambda}}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!} \\ &= \frac{1}{\lambda}e^{-\lambda}(e^\lambda - 1) = \frac{1}{\lambda}(1 - e^{-\lambda}). \end{aligned}$$

3.5
$$\begin{aligned} E[XY] &= E[X(N - X)] = NE[X] - E[X^2] \\ &= N^2p - [Np(1-p) + N^2p^2] = N^2p(1-p) - Np(1-p) \\ Cov[X,Y] &= E[XY] - E[X]E[Y] = -Np(1-p). \end{aligned}$$

3.6 Your intuition should suggest the correct answers: (a) X_1 is binomially distributed with parameters M and π_1; (b) N is binomial with parameters M and $\pi_1 + \pi_2$; and (c) X_1, given $N = n$, is conditionally binomial with parameters n and $p = \pi_1/(\pi_1 + \pi_2)$. To derive these correct answers formally, begin with

$$Pr\{X_1 = i,\ X_2 = j,\ X_3 = k\} = \frac{M!}{i!\,j!\,k!}\pi_1^i\pi_2^j\pi_3^k;\ i + j + k = M.$$

Since $k = M - (i + j)$

$$Pr\{X_1 = i,\ X_2 = j\} = \frac{M!}{i!\,j!(M - i - j)!}\pi_1^i\pi_2^j\pi_3^{M-i-j};\ 0 \leq i + j \leq M.$$

(a) $Pr\{X_1 = i\} = \sum_j Pr\{X_1 = i, X_2 = j\}$

$$= \frac{M!}{i!(M-i)!}\pi_1^i \sum_{j=0}^{M-i} \frac{(M-i)!}{j!(M-i-j)!}\pi_2^j \pi_3^{M-i-j}$$

$$= \binom{M}{i}\pi_1^i(\pi_2 + \pi_3)^{M-i}, \ i = 0,1,\ldots,M.$$

(b) Observe that $N = n$ if and only if $X_3 = M - n$. Apply the results of (a) to X_3:

$$Pr\{N = n\} = Pr\{X_3 = M - n\} = \frac{M!}{n!(M-n)!}(\pi_1 + \pi_2)^n \pi_3^{M-n}$$

(c) $$Pr\{X_1 = k | N = n\} = \frac{Pr\{X_1 = k, X_2 = n-k\}}{Pr\{N = n\}}$$

$$= \frac{\dfrac{M!}{k!(M-n)!(n-k)!}\pi_1^k \pi_2^{n-k} \pi_3^{M-n}}{\dfrac{M!}{n!(M-n)!}(\pi_1 + \pi_2)^n \pi_3^{M-n}}$$

$$= \frac{n!}{k!(n-k)!}\left(\frac{\pi_1}{\pi_1 + \pi_2}\right)^k \left(\frac{\pi_2}{\pi_1 + \pi_2}\right)^{n-k}, \ k = 0,1,\ldots,n.$$

3.7 $$Pr\{Z = n\} = \sum_{k=0}^{n} Pr\{X = k\} Pr\{Y = n-k\}$$

$$= \sum_{k=0}^{n} \frac{\mu^k e^{-\mu}}{k!} \frac{v^{(n-k)} e^{-v}}{(n-k)!} = e^{-(\mu+v)} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\mu^k v^{n-k}$$

$$= \frac{e^{-(\mu+v)}(\mu + v)^n}{n!} \quad \text{(Using binomial formula.)}$$

Z is Poisson distributed, parameter $\mu + v$.

3.8 (a) X is the sum of N independent Bernoulli random variables, each with parameter p, and Y is the sum of M independent Bernoulli random variables each with the same parameter p. Z is the sum of $M + N$ independent Bernoulli random variables, each with parameter p.

(b) By considering the ways in which a committee of n people may be formed from a group comprised of M men and N women, establish the identity $\binom{M+N}{n} = \sum_{k=0}^{n}\binom{N}{k}\binom{M}{n-k}$.

Then

$$Pr\{Z = n\} = \sum_{k=0}^{n} Pr\{X = k\} Pr\{Y = n-k\}$$

$$= \sum_{k=0}^{n} \binom{N}{k} p^k (1-p)^{N-k} \binom{M}{n-k} p^{n-k} (1-p)^{M-n+k}$$

$$= \binom{M+N}{n} p^n (1-p)^{M+N-n} \quad \text{for} \quad n = 0,1,\ldots,M+N.$$

Note:

$$\binom{\mathrm{N}}{\mathrm{k}} = 0 \quad \text{for} \quad k > N.$$

3.9 $Pr\{X+Y=n\} = \sum_{k=0}^{n} Pr\{X=k,\ Y=n-k\} = \sum_{k=0}^{n}(1-\pi)\pi^k(1-\pi)\pi^{n-k}$

$$= (1-\pi)^2\pi^n\sum_{k=0}^{n}1 = (n+1)(1-\pi)^2\pi^n \text{ for } n \geqq 0.$$

3.10

k	Binomial $n=10$ $p=.1$	Binomial $n=100$ $p=.01$	Poisson $\lambda=1$
0	.349	.366	.368
1	.387	.370	.368
2	.194	.185	.184

3.11 $Pr\{U=u,\ W=0\} = Pr\{X=u,\ Y=u\} = (1-\pi)^2\pi^{2u},\ u \geq 0.$

$Pr\{U=u,\ W=w>0\} = Pr\{X=u,\ Y=u+w\} + Pr\{Y=u,\ X=u+w\} = 2(1-\pi)^2\pi^{2u+w}$

$Pr\{U=u\} = \sum_{w=0}^{\infty} Pr\{U=u,\ W=w\} = \pi^{2u}(1-\pi^2).$

$Pr\{W=0\} = \sum_{u=0}^{\infty} Pr\{U=u,\ W=0\} = (1-\pi)^2/(1-\pi^2).$

$Pr\{W=w>0\} = 2\left[(1-\pi)^2/(1-\pi^2)\right]\pi^w$, and

$Pr\{U=u,\ W=w\} = Pr\{U=u\}Pr\{W=w\}$ for all u, w.

3.12 Let X = number of calls to switch board in a minute. $Pr\{X \geq 7\} = 1 - \sum_{k=0}^{6}\frac{4^k e^{-4}}{k!} = .111.$

3.13 Assume that inspected items are independently defective or good. Let X = # of defects in sample.

$$Pr\{X=0\} = (.95)^{10} = .599$$

$$Pr\{X=1\} = 10(.95)^9(.05) = .315$$

$$Pr\{X \geqq 2\} = 1 - (.599 + .315) = .086.$$

3.14 (a) $E[Z] = \frac{1-p}{p} = 9,\ Var[Z] = \frac{1-p}{p^2} = 90$

(b) $Pr\{Z > 10\} = (.9)^{10} = .349.$

3.15 $Pr\{X \leq 2\} = \left(1 + 2 + \frac{2^2}{2}\right)e^{-2} = 5e^{-2} = .677.$

3.16 (a) $p_0 = 1 - b\sum_{k=1}^{\infty}(1-p)^k = 1 - b\left(\frac{1-p}{p}\right).$

(b) When $b = p$, then p_k is given by (3.4).

When $b = \frac{p}{1-p}$, then p_k is given by (3.5).

(c) $Pr\{N=n>0\} = Pr\{X=0,\ Z=n\} + Pr\{X=1,\ Z=n-1\}$

$$= (1-\alpha)p(1-p)^n + \alpha p(1-p)^{n-1}$$

$$= \left[(1-\alpha)p + \alpha p/(1-p)\right](1-p)^n$$

So $b = (1-\alpha)p + \alpha p/(1-p)$.

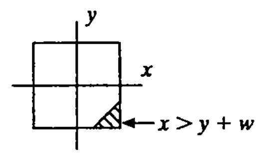
4.1 $E\left[e^{\lambda Z}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2+\lambda z}dz = e^{\frac{1}{2}\lambda^2}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}(z-\lambda)^2}dz\right\} = e^{\frac{1}{2}\lambda^2}.$

4.2 (a) $Pr\left\{W > \frac{1}{\theta}\right\} = e^{-\theta/\theta} = e^{-1} = .368\ldots.$

(b) Mode $= 0$.

4.3 $X - \theta$ and $Y - \theta$ are both uniform over $[-\frac{1}{2}, \frac{1}{2}]$, independent of θ, and $W = X - Y = (X - \theta) - (Y - \theta)$. Therefore the distribution of W is independent of θ and we may determine it assuming $\theta = 0$. Also, the density of W is symmetric since that of both X and Y are.

$$Pr\{W > w\} = Pr\{X > Y + w\} = \frac{1}{2}(1 - w)^2, \quad w > 0$$



So $f_w(w) = 1 - w$ for $0 \le w \le 1$
and $f_w(w) = 1 - |w|$ for $-1 \le w \le +1$

4.4 $\mu_c = .010$; $\sigma_c^2 = (.005)^2$, $Pr\{C < 0\} = Pr\left\{\frac{C - .010}{.005} < \frac{-.010}{.005}\right\} = Pr\{Z < -2\} = .0228.$

4.5 $Pr\{X < Y\} = \int_0^\infty\left\{\int_x^\infty 3e^{-3y}dy\right\}2e^{-2x}dx = \frac{2}{5}.$

5.1 $Pr\{N > k\} = Pr\{X_1 \le \xi, \ldots, X_k \le \xi\} = \left[F(\xi)\right]^k$, $k = 0, 1, \ldots$

$Pr\{N = k\} = Pr\{N > k - 1\} - Pr\{N > k\} = \left[1 - F(\xi)\right]F(\xi)^{k-1}$, $k = 1, 2, \ldots$

5.2 $Pr\{Z > z\} = Pr\{X_1 > z, \ldots, X_n > z\} = Pr\{X_1 > z\}\cdot\ldots\cdot Pr\{X_n > z\}$

$= e^{-\lambda z}\cdot\ldots\cdot e^{-\lambda z} = e^{-n\lambda z}$, $z > 0$.

Z is exponentially distributed, parameter $n\lambda$.

5.3 $Pr\{X > k\} = \sum_{l=k+1}^{\infty} p(1 - p)^l = (1 - p)^{k+1}$, $k = 0, 1, \ldots$

$E[X] = \sum_{k=0}^{\infty} Pr\{X > k\} = \frac{1 - p}{p}.$

5.4 Write $V = V^+ - V^-$ when $V^+ = \max\{V, 0\}$ and $V^- = \max\{-V, 0\}$. Then $Pr\{V^+ > v\} = 1 - F_V(v)$ and $Pr\{V^- > v\} = F_V(-v)$ for $v > 0$. Use (5.3) on V^+ and V^- together with $E[V] = E[V^+] - E[V^-]$. Mean does not exist if $E[V^+] = E[V^-] = \infty$.

5.5 $E\left[W^2\right] = \int_0^\infty Pr\{W^2 > t\}dt = \int_0^\infty\left[1 - F_W\left(\sqrt{t}\right)\right]dt = \int_0^\infty 2y\left[1 - F_W(y)\right]dy$ by letting $y = \sqrt{t}$.

5.6 $Pr\{V > t\} = \int_t^\infty \lambda e^{-\lambda v}dv = e^{-\lambda t}$; $E[V] = \int_0^\infty Pr\{V > t\}dt = \frac{1}{\lambda}\int_0^\infty \lambda e^{-\lambda t}dt = \frac{1}{\lambda}.$

5.7 $Pr\{V > v\} = Pr\{X_1 > v, \ldots, X_n > v\} = Pr\{X_1 > v\}\cdot\ldots\cdot Pr\{X_n > v\}$

$= e^{-\lambda_1 v}\cdot\ldots\cdot e^{-\lambda_n v} = e^{-(\lambda_1 + \ldots + \lambda_n)v}$, $v > 0$.

V is exponentially distributed, parameter $\Sigma\lambda_i$.

5.8

Spares	3	2	1	0
A				
B				
Mean	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$

Expected flash light operating duration $= \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{2}{\lambda} = 2$ Expected battery operating durations!

CHAPTER II

1.1 (a) $Pr\{X=k\} = \sum_{m=k}^{N} \frac{N!}{m!(N-m)!} p^m (1-p)^{N-m} \frac{m!}{k!(m-k)!} \pi^k (1-\pi)^{m-k}$

$= \binom{N}{k} (\pi p)^k (1-\pi p)^{N-k}$ for $k = 0,1,\ldots,N$.

(b) $E[XY] = E[MX] - E[X]^2$; $E[M] = Np$;

$E[M^2] = N^2 p^2 + Np(1-p)$; $E[X|M] = M\pi$

$E[X] = N\pi p$; $E[X^2|M] = M^2\pi^2 + M\pi(1-\pi)$;

$E[X^2] = (N\pi p)^2 + N\pi^2 p(1-p) + N\pi p(1-\pi)$,

$E[MX] = E\{ME[X|M]\} = \pi[N^2 p^2 + Np(1-p)]$

$Cov[X,Y] = E[X(M-X)] - E[X]E[M-X] = -Np^2\pi(1-\pi)$.

1.2 $Pr\{X=x|Y=y\} = \frac{1/x}{[1/y + \ldots + 1/N]}$ for $1 \le y \le x \le N$.

1.3 (a) $Pr\{U=u|V=v\} = \frac{2}{2v-1}$ for $1 \le u < v \le 6$

$= \frac{2}{2v-1}$ for $1 \le u = v \le 6$.

1.3 (b) $Pr\{S=s,\ T=t\}$

t/s	2	3	4	5	6	7	8	9	10	11	12
0	$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$		$\frac{1}{36}$
1		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$	
2			$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$		
3				$\frac{2}{36}$		$\frac{2}{36}$		$\frac{2}{36}$			
4					$\frac{2}{36}$		$\frac{2}{36}$				
5						$\frac{2}{36}$					

1.4 $E[X|N] = \frac{N}{2}; \ E[X] = 20 \times \frac{1}{4} \times \frac{1}{2} = \frac{5}{2}.$

1.5 $Pr\{X = 0\} = \left(\frac{3}{4}\right)^{20} = .00317.$

1.6 $Pr\{N = n\} = \left(\frac{1}{2}\right)^n \quad n = 1, 2, \ldots$

$$Pr\{X = k|N = n\} = \binom{n}{k}\left(\frac{1}{2}\right)^n \quad 0 \le k \le n$$

$$Pr\{X = k\} = \sum_{n=1}^{\infty}\binom{n}{k}\left(\frac{1}{4}\right)^n$$

$$Pr\{X = 0\} = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n = \frac{1}{3}$$

$$Pr\{X = 1\} = \sum_{n=1}^{\infty} n\left(\frac{1}{4}\right)^n = \frac{4}{9}.$$

$$E[X|N] = \frac{N}{2}; \ E[X] = \frac{1}{2}E[N] = 1.$$

1.7 $Pr\{\text{True S.F.}|\text{Diag. S.F.}\} = \frac{.30(.85)}{.30(.85) + .70(.35)} = .51$

1.8 $Pr\{\text{First Red}|\text{Second Red}\} = \frac{Pr\{\text{Both Red}\}}{Pr\{\text{Second Red}\}} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)}{\frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right)} = \frac{2}{3}.$

1.9
$$\begin{aligned} Pr\{X = k\} &= \sum_{n=k-1}^{\infty} Pr\{X = k|N = n\}Pr\{N = n\} \\ &= \sum_{n=k-1}^{\infty}\frac{1}{n+2}\frac{e^{-1}}{n!} = \sum_{n=k-1}^{\infty} e^{-1}\left[\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right] = \frac{e^{-1}}{k!} \end{aligned}$$

for $k = 0, 1, \ldots$ (Poisson, mean = 1).

1.10 A "typical" distribution of 8 families would break down as follows:

Family	#1	#2	#3	#4	#5	#6	#7	#8
Children	$\{G\}$	$\{G\}$	$\{G\}$	$\{G\}$	$\{B, G\}$	$\{B, G\}$	$\{B, B, G\}$	{B, B, B}

(a) Let N = the number of children in a family

$$Pr\{N = 1\} = \frac{1}{2}, \ Pr\{N = 2\} = Pr\{N = 3\} = \frac{1}{4}.$$

(b) Let X = the number of girl children in a family.

$$Pr\{X = 0\} = \frac{1}{8}, \ Pr\{X = 1\} = \frac{7}{8}$$

(c) Family #8 is three times as likely, and family #7 is twice as likely to be chosen as families #5 and #6.

$$Pr\{\text{No sisters}\} = Pr\{\#8\} = \frac{3}{7}.$$

$$Pr\{\text{One sister}\} = \frac{4}{7}.$$

$$Pr\{\text{Two brothers}\} = P_r\{\#8\} = \frac{3}{7}$$

$$Pr\{\text{One brother}\} = P_r\{\#7\} = \frac{2}{7}$$

$$Pr\{\text{No brothers}\} = \frac{2}{7}.$$

2.1 (a) The bid $X_1 = x$ is accepted if $x > A$; Otherwise, if $O \le x \le A$, one is back where one started due to the independent and identically distributed nature of the bids.

$$E[X_N | X_1 = x] = \begin{cases} x & \text{if } x > A; \\ M & \text{if } 0 \le x \le A \end{cases}$$

(b) The given equation results from the law of total probability:

$$E[X_N] = \int_0^\infty E[X_N | X_1 = x]\, dF(x).$$

(c) When X is exponentially distributed the conditional distribution of X, given that $X > A$, is the same as the distribution of $A + X$.

(d) When $dF(x) = \lambda e^{-\lambda x} dx$, then

$$\alpha = e^{-\lambda A} \quad \text{and} \quad \int_A^\infty x\lambda e^{-\lambda x} dx = A\alpha + \alpha \int_0^\infty y \lambda e^{-\lambda y} dy = \alpha\left(A + \frac{1}{\lambda}\right) \quad \text{and} \quad \text{M} = A + \frac{1}{\lambda}.$$

2.2 The distribution for the sum is

$$p(2) = p(12) = \frac{1}{36} \qquad p(5) = p(9) = \frac{4}{36}$$

$$p(3) = p(11) = \frac{2}{36} \qquad p(6) = p(8) = \frac{5}{36} - (.02)^2$$

$$p(4) = p(10) = \frac{3}{36} \qquad p(7) = \frac{6}{36} + 2(.02)^2$$

$$Pr\{\text{Win}\} = .4924065.$$

3.1 (a) $\nu = \tau^2 = \lambda, \quad \mu = p \quad \sigma^2 = p(1-p)$

$$E[Z] = \lambda p; \quad Var[Z] = \lambda p(1-p) + \lambda p^2 = \lambda p.$$

(b) Z has the Poisson distribution.

3.2
$$Pr\{Z = k\} = \sum_{n=k}^{M} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{M!}{n!(M-n)!} q^n (1-q)^{M-n}$$

$$= \binom{M}{k} (qp)^k (1-qp)^{M-k} \quad \text{for} \quad k = 0, 1, \ldots, M.$$

3.3 (a) $\mu = 0, \quad \sigma^2 = 1, \quad \nu = \dfrac{1-\alpha}{\alpha}, \quad \tau^2 = \dfrac{1-\alpha}{\alpha^2}; \quad E[Z] = 0 \quad Var[Z] = \dfrac{1-\alpha}{\alpha}.$

(b) $E[Z^3] = 0 \quad E[Z^4] = E[N^2 + 2N(N-1)]$

$$= 3\left(\frac{(1-\alpha)^2}{\alpha^2} + \frac{1-\alpha}{\alpha^2}\right) - 2\left(\frac{1-\alpha}{\alpha}\right) = \frac{(1-\alpha)(6-5\alpha)}{\alpha^2}$$

3.4 (a) $E[N] = Var[N] = \lambda; \quad E[S_N] = \lambda\mu \quad Var[S_N] = \lambda(\mu^2 + \sigma^2).$

(b) $E[N]=\lambda,\quad Var[N]=\dfrac{\lambda}{p}=\lambda(1+\lambda)$

$E[S_N]=\lambda\mu\quad Var[S_N]=\lambda(\mu^2+\sigma^2)+\lambda^2\mu^2$

(c) For large λ, $Var[S_N]\propto\lambda$ in (a), while $Var[S_N]\propto\lambda^2$ in (b).

3.5 $E[Z]=(1+\nu)\mu;\quad Var[Z]=\nu\sigma^2+\mu^2\tau^2.$

4.1 $E[X_1]=E[X_2]=\int_0^1 p\,dp=\dfrac{1}{2}$

$E[X_1X_2]=\int_0^1 p^2dp=\dfrac{1}{3}$

$Cov[X_1,X_2]=E[X_1X_2]-E[X_1]E[X_2]=\dfrac{1}{12}.$

4.2
$$\begin{aligned}Pr\{X=i,Y=j\}&=Pr\{X=i,N=i+j\}=Pr\{X=i|N=i+j\}Pr\{N=i+j\}\\&=\binom{i+j}{i}p^i(1-p)^j\lambda^{i+j}e^{-\lambda}/(i+j)!\\&=\left\{\frac{(\lambda p)^ie^{-\lambda p}}{i!}\right\}\left\{\frac{[\lambda(1-p)]^je^{-\lambda(1-p)}}{j!}\right\},\quad i,j\geqq 0.\end{aligned}$$

$$\begin{aligned}Pr\{X=i\}&=\sum_{j=0}^{\infty}P_r\{X=i,Y=j\}=\frac{(\lambda p)^ie^{-\lambda p}}{i!},\\Pr\{Y=j\}&=\frac{[\lambda(1-p)]^je^{-\lambda(1-p)}}{j!}\end{aligned}$$

and $Pr\{X=i,Y=j\}=Pr\{X=i\}Pr\{Y=j\}$ for all i,j.

4.3 (a) $Pr\{X=i\}=\int_0^\infty\dfrac{\lambda^ie^{-\lambda}}{i!}\theta e^{-\theta\lambda}d\lambda=\dfrac{\theta}{i!}\int_0^\infty\lambda^ie^{-(1+\theta)\lambda}d\lambda=\left(\dfrac{\theta}{1+\theta}\right)\left(\dfrac{1}{1+\theta}\right)^i,$

$i=0,1,\ldots$ (Geometric distribution).

(b) $f(\lambda|X=k)=\dfrac{(1+\theta)^{k+1}\lambda^ke^{-(1+\theta)\lambda}}{k!}$ (Gamma).

4.4
$$\begin{aligned}Pr\{X=i,Y=j\}&=\frac{\theta}{i!j!}\int_0^\infty\lambda^{i+j}e^{-(z+\theta)\lambda}d\lambda\\&=\binom{i+j}{i}\left(\frac{\theta}{2+\theta}\right)\left(\frac{1}{2+\theta}\right)^{i+j},\quad i,j\geq 0.\end{aligned}$$

$$Pr\{X=i,N=n\}=\binom{n}{i}\left(\frac{\theta}{2+\theta}\right)\left(\frac{1}{2+\theta}\right)^n\quad 0\leq i\leq n.$$

$$Pr\{N=n\}=\left(\frac{\theta}{2+\theta}\right)\left(\frac{1}{2+\theta}\right)^n\sum_{i=0}^n\binom{n}{i}=\left(\frac{\theta}{2+\theta}\right)\left(\frac{2}{2+\theta}\right)^n,\quad n\geq 0.$$

$$Pr\{X=i|N=n\}=\binom{n}{i}\left(\frac{1}{2}\right)^n,\quad 0\leq i\leq n.$$

4.5 $Pr\{X=0, Y=1\} = 0 \neq Pr\{X=0\}Pr\{Y=1\}$ so X and Y are *NOT* independent.

$$E[X] = E[Y] = 0, \quad Cov[X,Y] = E[XY] = \frac{1}{9} + \frac{1}{9} - \frac{2}{9} = 0.$$

4.6 $Pr\{N > n\} = Pr\{X_0$ is the largest among $\{X_0, \ldots, X_n\}\} = \dfrac{1}{n+1}$ since each random variable has the same chance to be the largest. Or, one can integrate $Pr\{N > n\} = \int_0^\infty [1 - F(x)]^n f(x)dx = -\int_0^\infty [1 - F(x)]^n d[1 - F(x)] = \int_0^1 y^n dy = \dfrac{1}{n+1}$

$$Pr\{N = n\} = Pr\{N > n-1\} - Pr\{N > n\} = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.$$

$$E[N] = \sum_{n=0}^\infty Pr\{N > n\} = \sum_{n=0}^\infty \frac{1}{n+1} = \infty.$$

4.7 $f_{X,Z}(x,z) = \alpha^2 e^{-\alpha z}$ for $0 \le x \le z$.

$$f_{X|Z}(x|z) = \frac{f_{x,z}(x,z)}{f_z(z)} = \frac{\alpha^2 e^{-\alpha z}}{\alpha^2 z e^{-\alpha z}} = \frac{1}{z}, \quad 0 < x < z.$$

X is, conditional on $Z = z$, uniformly distributed over the interval $[0, z]$.

4.8 The key algebraic step in simplifying the joint divided by the marginal is

$$\begin{aligned}
Q(x,y) - \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2 &= \frac{1}{1-\varrho^2}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\varrho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \varrho^2\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right\} \\
&= \frac{1}{1-\varrho^2}\left\{\left(\frac{x-\mu_X}{\sigma_X}\right) - \varrho\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right\}^2 \\
&= \frac{1}{(1-\varrho^2)\sigma_X^2}\left\{x - \mu_X - \varrho\left(\frac{\sigma_X}{\sigma_Y}\right)(y - \mu_Y)\right\}^2
\end{aligned}$$

5.1 Following the suggestion

$$\begin{aligned}
E[X_{n+2}|X_0,\ldots,X_n] &= E[E\{X_{n+2}|X_0,\ldots,X_{n+1}\}|X_0,\ldots,X_n] \\
&= E[X_{n+1}|X_0,\ldots,X_n] = X_n
\end{aligned}$$

5.2
$$\begin{aligned}
E[X_{n+1}|X_0,\ldots,X_n] &= E[2U_{n+1}X_n|X_n] \\
&= X_n E[2U_{n+1}] = X_n.
\end{aligned}$$

5.3
$$\begin{aligned}
E[X_{n+1}|X_0,\ldots,X_n] &= E[2e^{-\varepsilon_{n+1}}X_n|X_n] \\
&= X_n E[2e^{-\varepsilon_{n+1}}] = X_n.
\end{aligned}$$

5.4
$$\begin{aligned}
E[X_{n+1}|X_0,\ldots,X_n] &= E\left[X_n \frac{\xi_{n+1}}{p}\middle|X_n\right] \\
&= X_n E\left[\frac{\xi_{n+1}}{p}\right] = X_n \quad \lim_{n\to\infty} X_n = 0.
\end{aligned}$$

5.5 (b) $Pr\{X_n \ge N \text{ for some } n \ge 0 | X_0 = i\} \le \dfrac{E(X_0)}{N} = \dfrac{i}{N}$. (In fact, equality holds. See III, 5.3)

CHAPTER III

1.1 $P_{55} = 1, P_{k,k+1} = \alpha \dfrac{\binom{k}{1}\binom{5-k}{1}}{\binom{5}{2}}, \quad k = 1,2,3,4$

$P_{ij} = 0$, otherwise.

1.2 (a) $Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00}^2 = 1(1-\alpha)^2 = (1-\alpha)^2.$

(b) $Pr\{X_0 = 0, X_2 = 0\} = P_{00}P_{00} + P_{01}P_{10} = (1-\alpha)^2 + \alpha^2.$

1.3 $Pr\{X_2 = G, X_3 = G, X_4 = G, X_5 = D | X_1 = G\} = P_{GG}^3 P_{GD} = \alpha^3(1-\alpha).$

1.4

$$P = \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & .1 & .3 & .2 & .4 \\ 1 & 0 & .4 & .2 & .4 \\ 2 & 0 & 0 & .6 & .4 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

2.1 Observe that the columns of P sum to one. Then $p_i^{(0)}=1/4$ for $i = 0,1,2,3$ and by induction $p_i^{(n+1)} = \sum_{k=0}^{3} p_k^{(n)} P_{ki} = \frac{1}{4}\sum P_{ki} = \frac{1}{4}.$

2.2 $P_{00}^{(5)} = (1-\alpha)^5 + 10(1-\alpha)^3\alpha^2 + 5(1-\alpha)\alpha^4 = \frac{1}{2}\left[1 + (1-2\alpha)^5\right].$

2.3 $P_{11}^{(3)} = .684772$

2.4

$$\begin{array}{c|cccc|} & (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & \alpha & 1-\alpha & 0 & 0 \\ (0,1) & 0 & 0 & 1-\beta & \beta \\ (1,0) & \alpha & 1-\alpha & 0 & 0 \\ (1,1) & 0 & 0 & 1-\beta & \beta \end{array}$$

2.5 $Pr\{X_3 = 0 | X_0 = 0, T > 3\} = P_{00}^{(3)} \Big/ \left(P_{00}^{(3)} + P_{01}^{(3)}\right) = .6652.$

3.1

$$P = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ \frac{1}{15} & \frac{14}{15} & 0 & 0 \\ 0 & \frac{8}{15} & \frac{7}{15} & 0 \\ 0 & 0 & \frac{3}{5} & \frac{2}{5} \end{array} \right\| \end{array}$$

3.2

$$P = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \left\| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array} \right\| \end{array}$$

3.3 (a)

$$P = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} .1 & .4 & .5 \\ .5 & .5 & 0 \\ .1 & .4 & .5 \end{array} \right\| \end{array}$$

(b) Long run lost sales per period = (.1) $\pi_1 = .0444\ldots.$

3.4 $Pr\{\text{Customer completes service} \mid \text{In service}\} = Pr\{Z = k | Z \geq k\} = \alpha.$

$$P_{01} = \beta, \quad P_{00} = 1 - \beta, \quad P_{i,i+1} = \beta(1 - \alpha)$$

$$P_{i,i-1} = (1 - \beta)\alpha, \quad P_{ii} = (1 - \beta)(1 - \alpha) + \alpha\beta, \quad i \geqq 1.$$

3.5

$$P = \begin{array}{cc} & \begin{array}{cccc} 0 & H & HH & HHT \end{array} \\ \begin{array}{c} 0 \\ H \\ HH \\ HHT \end{array} & \left\| \begin{array}{cccc} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array} \right\| \end{array}$$

3.6 $P_{(a,b)(a,b+1)} = 1 - p \qquad a \leq 3, b \leq 2$

$P_{(a,b)(a+1,b)} = p \qquad a \leq 2, b \leq 3$

$P_{(3,b),A\ wins} = p \qquad b \leq 3$

$P_{(a,3),B\ wins} = 1 - p, \qquad a \leq 3$

$P_{A\ wins,A\ wins} = P_{B\ wins,B\ wins} = 1.$

3.7 $P_{0,k} = \alpha_k$ for $k = 1, 2, \ldots$

$P_{0,0} = 0$

$P_{k,k-1} = 1 \quad h \geq 1$

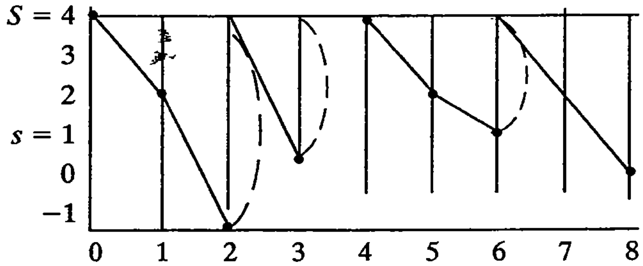
3.8 $P_{k,k-1} = q\left(\frac{k}{N}\right) \quad P_{k,k+1} = p\left(\frac{N-k}{N}\right)$

$$P_{k,k} = p\left(\frac{k}{N}\right) + q\left(\frac{N-k}{N}\right).$$

3.9 $P_{k,k-1} = P_{k,k+1} = \left(\frac{k}{N}\right)\left(\frac{N-k}{N}\right)$

$$P_{k,k} = \left(\frac{k}{N}\right)^2 + \left(\frac{N-k}{N}\right)^2$$

3.10



	ξ	= 2	3	4	0	2	1	2	2
(a)	n	= 1	2	3	4	5	6	7	8
	X_n	= 2	−1	0	4	2	1	2	0

(b) $P_{41} = Pr\{\xi = 3\} = .2$

$P_{04} = Pr\{\xi = 0\} = .1$

4.1 $0 = 0 \quad 1 = H \quad 2 = HH \quad 3 = HHT$

$$\begin{array}{c} \\ \boldsymbol{P} = \end{array} \begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 2 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\left.\begin{aligned} v_0 &= 1 + \tfrac{1}{2}v_0 + \tfrac{1}{2}v_1 \\ v_1 &= 1 + \tfrac{1}{2}v_0 + \tfrac{1}{2}v_2 \\ v_2 &= 1 + \tfrac{1}{2}v_2 \end{aligned}\right\} \qquad \begin{aligned} &\boxed{v_0 = 8} \\ &v_1 = 6 \\ &v_2 = 2 \end{aligned}$$

$0 = 0 \quad 1 = H \quad 2 = HT \quad 3 = HTH$

$$\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 2 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

$$\left.\begin{aligned} v_0 &= 1 + \tfrac{1}{2}v_0 + \tfrac{1}{2}v_1 \\ v_1 &= 1 + \tfrac{1}{2}v_1 + \tfrac{1}{2}v_2 \\ v_2 &= 1 + \tfrac{1}{2}v_0 \end{aligned}\right\} \qquad \begin{aligned} &\boxed{v_0 = 10} \\ &v_1 = 8 \\ &v_2 = 6 \end{aligned}$$

4.2

$$\begin{aligned} v_1 &= 1 & v_1 &= 1 \\ v_2 &= 1 + \frac{1}{2}v_1 & v_2 &= 1 + \frac{1}{2} \\ v_3 &= 1 + \frac{1}{3}v_1 + \frac{1}{3}v_2 & v_3 &= 1 + \frac{1}{3} + \frac{1}{3}\left(\frac{3}{2}\right) = 1 + \frac{1}{2} + \frac{1}{3} \\ &\vdots \\ v_m &= 1 + \frac{1}{m}v_1 + \frac{1}{m}v_2 + \cdots + \frac{1}{m}v_{m-1} \end{aligned}$$

Solve to get $v_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \sim \text{Log m}$

Note: $w_{jj} = 1$ and $w_{ij} = \dfrac{1}{j+1}$ for $i > j$.

4.3 We will verify that $v_m = 2\left(\frac{m+1}{m}\right)\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) - 3$ solves $v_m = 1 + \sum_{j=1}^{m-1} \frac{2j}{m^2} v_j$. Change variables to $V_k = kv_k$. To show: $V_k = 2(k+1)\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right) - 3k$ solves $V_m = m + \frac{2}{m}(V_1 + \cdots + V_{m-1})$. Using the given V_k, use sums of numbers and interchange order as follows:

$$\begin{aligned} \sum_{k=1}^{m-1} V_k &= \sum_{k=1}^{m-1}\sum_{l=1}^{k} 2(k+1)\frac{1}{l} - 3\sum_{k=1}^{m-1} k \\ &= \sum_{l=1}^{m-1}\frac{2}{l}\sum_{k=l}^{m-1}(k+1) - \frac{3m(m-1)}{2} \\ &= \sum_{l=1}^{m-1}\frac{2}{l}\left[\frac{m(m+1)}{2} - \frac{l(l+1)}{2}\right] - \frac{3m(m-1)}{2}. \end{aligned}$$

Then,

$$m + \frac{2}{m}\sum_{k=1}^{m-1} V_k = 2(m+1)\sum_{l=1}^{m-1}\frac{1}{l} - 3m + \frac{2(m+1)}{m} = V_m$$

as was to be shown.

As in Problem 4.2, $v_m \sim \log m$.

4.4 Change the matrix to

$$\begin{array}{c|cccc|} & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & .1 & .2 & .5 & .2 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & .2 & .2 & .3 & .3 \end{array}$$

and calculate the probability that absorption takes place in State O:

$$\left.\begin{aligned} u_{10} &= .1 + .2u_{10} + .2u_{30} \\ u_{30} &= .2 + .2u_{10} + .3u_{30} \end{aligned}\right\} \quad \begin{aligned} &\boxed{u_{10} = .2115} \\ &u_{30} = .3462 \end{aligned}$$

4.5

$$\begin{array}{r||ccccccc||}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
2 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
4 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\
5 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\
6 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$u_{13} = \frac{7}{12}$ $\quad u_{53} = \frac{2}{3}$ $\quad \boxed{u_{43} = \frac{5}{12}}$

$u_{23} = \frac{3}{4}$ $\quad u_{63} = \frac{5}{6}$

4.6 $v_0 = 1 + qv_0 + pv_1$

$v_1 = 1 + qv_0 + pv_2 \qquad q = 1 - p$

$v_2 = 1 + qv_0 + pv_3$

$v_3 = 1 + qv_0$

gives $v_0 = (1 + qv_0)(1 + p + p^2 + p^3) = (1 + p + p^2 + p^3) + v_0(1 - p^4)$
so

$$v_0 = \frac{1 + p + p^2 + p^3}{1 - \left(1 - p^4\right)} = \frac{1 - p^4}{p^4\left(1 - p\right)}.$$

4.7 The stationary Markov transitions imply that $E\left[\sum_{n=1}^{\infty} \beta^n c\left(X_n\right) \middle| X_0 = i, X_1 = j\right] = \beta h_j$ while $E\left[\beta^0 c\left(X_0\right) \middle| X_0 = i\right] = c\left(i\right)$. Now use the law of total probability.

4.8

$$\begin{array}{r||cccccc||}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & 0 \\
2 & 0 & \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 \\
3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
4 & 0 & 0 & 0 & \frac{4}{7} & \frac{3}{7} & 0 \\
5 & 0 & 0 & 0 & 0 & \frac{5}{8} & \frac{3}{8}
\end{array}$$

X_n = # of red balls in use at nth draw.

$$v_5 = 4 + \frac{5}{2} + 2 + \frac{7}{4} + \frac{8}{5} = 11.85.$$

4.9

$$\begin{array}{r||cccccc||}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 & 0 \\
2 & 0 & \frac{2}{8} & \frac{6}{8} & 0 & 0 & 0 \\
3 & 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 \\
4 & 0 & 0 & 0 & \frac{4}{8} & \frac{4}{8} & 0 \\
5 & 0 & 0 & 0 & 0 & \frac{5}{8} & \frac{3}{8}
\end{array}$$

X_n = # of red balls in use.

$$v_5 = \frac{8}{1} + \frac{8}{2} + \frac{8}{3} + \frac{8}{4} + \frac{8}{5} = 18\frac{4}{15} = 18.266\ldots.$$

4.10 Let X_n be the number of coins that fall tails in the nth toss, $X_0 = 5$. Make "1" an absorbing state.

$$P = \begin{array}{c|cccccc|} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 3 & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ 4 & \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} & 0 \\ 5 & \frac{1}{32} & \frac{5}{32} & \frac{10}{32} & \frac{10}{32} & \frac{5}{32} & \frac{1}{32} \end{array}$$

$U_{51} = .7235023041\ldots$

4.11 Label the states (x, y) where x = # of red balls and y = # of green balls, "win" = (1,0), "lose" = {(0,2), (2,0)}.

$$\begin{array}{c|cccccc|} & (2,2) & (1,2) & (2,1) & (1,1) & \textit{win} & \textit{lose} \\ (2,2) & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ (1,2) & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ (2,1) & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ (1,1) & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \textit{win} & 0 & 0 & 0 & 0 & 1 & 0 \\ \textit{lose} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$U_{(2,2),win} = \frac{1}{3}$.

4.12

$$\begin{array}{c|cccc|} & 0 & 1 & \textit{win} & \textit{lose} \\ 0 & .3 & .2 & 0 & .5 \\ 1 & .5 & .1 & .4 & 0 \\ \textit{win} & 0 & 0 & 1 & 0 \\ \textit{lose} & 0 & 0 & 0 & 1 \end{array} \quad z_0 = U_{0,win} = \frac{8}{53}$$

4.13 Use the matrix

$$\begin{array}{c|ccc|ccc|} & 0 & 1 & 2 & 0' & 1' & 2' \\ 0 & 0 & 0 & 0 & .3 & .2 & .5 \\ 1 & 0 & 0 & 0 & .5 & .1 & .4 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0' & .3 & .2 & .5 & 0 & 0 & 0 \\ 1' & .5 & .1 & .4 & 0 & 0 & 0 \\ 2' & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$U_{0,2'} = .67669\ldots$

4.14 The long but sure way to solve the problem is to set up the 36×36 transition matrix for the Markov chain $Z_n = (X_{n-1}, X_n)$, making states (x, y) where $x + y = 5$ or $x + y = 7$ absorbing. However, all that is important prior to ending the game is the last roll, and by aggregating states where possible, we get the Markov transition matrix.

$$
\begin{array}{c|ccccc|}
 & 1 \text{ or } 2 & 3 \text{ or } 4 & 5 \text{ or } 6 & 5 & 7 \\
1 \text{ or } 2 & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
3 \text{ or } 4 & \frac{1}{6} & \frac{1}{6} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\
5 \text{ or } 6 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{6} \\
5 & 0 & 0 & 0 & 1 & 0 \\
7 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$U_{1,5} = \tfrac{22}{51} \quad U_{3\,5} = \tfrac{21}{51} \quad U_{5,5} = \tfrac{16}{51}$$

$$Pr\{5 \text{ before} 7\} = \tfrac{1}{3}\left(U_{1,5} + U_{3\,5} + U_{5,5}\right) = \tfrac{59}{153}.$$

4.15

$$
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
1 & .96 & .04 & 0 & 0 & 0 \\
2 & 0 & .94 & .06 & 0 & 0 \\
3 & 0 & 0 & .94 & .06 & 0 \\
4 & 0 & 0 & 0 & .96 & .04 \\
5 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$v_1 = 133\tfrac{1}{3}$

4.16

$$
\begin{array}{c|cccccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & .2 & 0 & .8 & 0 & 0 & 0 \\
2 & 0 & .4 & 0 & .6 & 0 & 0 \\
3 & 0 & 0 & .6 & 0 & .4 & 0 \\
4 & 0 & 0 & 0 & .8 & 0 & .2 \\
5 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$U_{35} = \tfrac{17}{32}$.

4.17 Let $\varphi_i(s) = E\left[s^T \middle| X_0 = i\right]$ for $i = 0, 1$

Then

$$\varphi_0(s) = s\left[.7\varphi_0(s) + .3\varphi_1(s)\right]$$
$$\varphi_1(s) = s\left[.6\varphi_1(s) + .4\right]$$

which solves to give

$$\varphi_1(s) = \frac{.4s}{1 - .6s}$$
$$\varphi_0(s) = \left(\frac{.4s}{1 - .6s}\right)\left(\frac{.3s}{1 - .7s}\right)$$

4.18 The transition probabilities are: If $X_n = w$, then $X_{n+1} = w \quad w.pr.\dfrac{h}{w+h}$ $\left(\text{and } h \to h-1\right)$ or $X_{n+1} = w-1 \quad w.pr.\dfrac{w}{w+h}$ $\left(\text{and } h \to h+1\right)$ where $h = 2\mathrm{N} - n - 2w$ is the number of half cigars in the box.

The recursion to verify is

$$v_n(w) = \frac{h}{w+h}v_{n+1}(w) + \frac{w}{w+h}v_{n+1}(w-1)$$

This is straight forward, and easiest if one first does the algebra to verify

$$v_{n+1}(w) = v_n(w) + \frac{1}{1+w}$$
$$v_{n+1}(w-1) = v_n(w) + \frac{n+2w-2N}{w(1+w)}$$

4.19 $p_N = \dfrac{N}{2}\sum_{n=0}^{\infty}\left(\dfrac{1}{2}\right)^n\left[1-\left(\dfrac{1}{2}\right)^n\right]^{N-1}.$

As a function of N, this does NOT converge but oscillates very slowly (cycles $\propto \log N$) and very slightly about the (Cesaro limit) $\dfrac{1}{2\log 2}$.

5.1
$$\begin{aligned}
v_0 &= 1 + a_0v_0 + a_1v_1 + a_2v_2 \\
v_1 &= 1 + (a_0 + a_1)v_1 + a_2v_2 \\
v_2 &= 1 + (a_0 + a_1 + a_2)v_2 \\
v_2 &= \frac{1}{1-(a_0+a_1+a_2)} = \frac{1}{a^3} = v_0 = v_1.
\end{aligned}$$

5.2 (a) $p_0 = a_1, \quad q_0 = 1 - a_1 = (a_2 + a_3 + \ldots).$
$p_k = a_{k+1}/(a_{k+1} + a_{k+2} + \ldots) \quad$ for $k \geq 1$.

(b) $p_0 = a_1, \quad q_0 = 1 - a_1$
$p_k = a_{k+1}/(a_{k+1} + a_{k+2} + \ldots), \quad q_k = 1 - p_k \quad$ for $1 \leq k < N-1$.
$p_{N-1} = 1$

The above assumes instantaneous replacement.

5.3

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ P = \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\|\begin{array}{ccc} \alpha & 1-\alpha & 0 \\ 0 & \alpha & 1-\alpha \\ 1-\alpha & 0 & \alpha \end{array}\right\| \end{array}$$

5.4 After reviewing (5.8), set up the matrix

$$
\begin{array}{c|ccccccccccc|cc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 13 & \sim 13 \\
0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 \\
4 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & \frac{2}{6} \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{2}{6} \\
8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{3}{6} \\
9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} \\
10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{5}{6} \\
\hline
13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\sim 13 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$U_{0,13} = .1819$

5.5 If $X_n = 0$, then $X_{n+1} = 0$ and $E[X_{n+1}|X_n = 0] = 0 = X_n$. If $X_n = i > 0$ then $X_{n+1} = i \pm 1$, each *w. pr.* $\frac{1}{2}$ so $E[X_{n+1}|X_n = i] = i = X_n$. So $\{X_n\}$ is a martingale. If $X_0 = i$, then $Pr\{\max X_n \geq N\} \leq \frac{E[X_0]}{N} = \frac{i}{N}$ by the maximal inequality. Of course, (5.13) asserts $Pr\{\max X_n \geq N\} = \frac{i}{N}$.

6.1 $\left.\begin{aligned} v_1 &= 1 + .7v_2 \\ v_2 &= 1 + .1v_1 \end{aligned}\right\} v_1 = \dfrac{170}{93} = 1.827957$

$$\varrho_1 = \frac{3}{7}, \quad \varrho_2 = \frac{1}{21}, \quad \Phi_1 = \frac{10}{7}, \quad \Phi_2 = \frac{80}{63}$$

$$v_1 = \frac{\Phi_1 + \Phi_2}{1 + \varrho_1 + \varrho_2} = 1.827957.$$

6.2 $\left.\begin{aligned} v_0 &= 1 + \alpha v_0 + \beta v_2 \\ v_1 &= 1 + \alpha v_0 \\ v_2 &= 1 + \alpha v_0 + \beta v_1 \end{aligned}\right\} v_0 = \dfrac{1 + \beta + \beta^2}{\beta^3}.$

6.3 For three urns, $v(a, b, c) = E[T] = \dfrac{3abc}{a + b + c}$. The answer is unknown for four urns.

7.1 Observe that each state is visited at most a single time so that $W_{ij} = Pr\{\text{Ever visit } j|X_0 = i\}$. Clearly, then $W_{i,i} = 1$ and $W_{ij} = 0$ for $j > i$.

$$W_{i,i-1} = P_{i,i-1} = \frac{1}{i}$$

$$W_{i,i-2} = P_{i,i-2} + P_{i,i-1}W_{i-1,i-2} = \frac{1}{i} + \frac{1}{i}\left(\frac{1}{i-1}\right) = \frac{1}{i-1.}$$

$$\begin{aligned} W_{i,i-3} &= P_{i,i-3} + P_{i,i-2}W_{i-2,i-3} + P_{i,i-1}W_{i-1,i-3} \\ &= \frac{1}{i}\left(1 + \frac{1}{i-2} + \frac{1}{i-2}\right) = \frac{1}{i-2}. \end{aligned}$$

Continuing in this manner, we deduce that $W_{i,j} = \dfrac{1}{j+1}$ for $1 \le j < i$.

7.2 Observe that each state is visited at most a single time so that $W_{ij} = Pr\{\text{Ever visit } j | X_0 = i\}$. Clearly, then $W_{i,i} = 1$ and $W_{i,j} = 0$ for $j > i$. We claim that

$$W_{k,j} = 2\left(\frac{k+1}{k}\right)\frac{j}{(j+1)(j+2)} \quad \text{for } 1 \le j < k.$$

satisfies the first step equation $W_{k,j} = P_{k,j} + \sum_{l=j+1}^{k-1} P_{k,l} W_{l,j}$.

Evaluating the right side with the given $P_{k,l}$ and $W_{l,j}$ we get

$$\begin{aligned}
R.H.S. &= \frac{2j}{k^2} + \sum_{l=j+1}^{k-1} \frac{2l}{k^2} 2\left(\frac{l+1}{l}\right)\frac{j}{(j+1)(j+2)} \\
&= \frac{2j}{k^2} + \frac{2j}{k^2}\frac{2}{(j+1)(j+2)}\sum_{l=j+1}^{k-1}(l+1) \\
&= \frac{2j}{k^2} + \frac{2j}{k^2}\frac{2}{(j+1)(j+2)}\left[\frac{k(k+1)}{2} - \frac{(j+1)(j+2)}{2}\right] \\
&= 2\left(\frac{k+1}{k}\right)\frac{j}{(j+1)(j+2)} = W_{k,j} \qquad \text{as claimed.}
\end{aligned}$$

7.3 Observe that for j transient and k absorbing, the event $\{X_{n-1} = j, X_n = k\}$ is the same as the event $\{T = n, X_{T-1} = j, X_T = k\}$, whence

$$Pr\{X_{T-1} = j, X_T = k | X_0 = i\} = \sum_{n=1}^{\infty} Pr\{X_{n-1} = j, X_n = k | X_0 = i\} = \sum_{n=1}^{\infty} P_{i,j}^{(n-1)} P_{jk} = W_{i,j} P_{jk}.$$

7.4 (a) $W_{i,j} = \dfrac{1}{j+1}$ for $1 \le j < i$. (See the solution to 7.1). Using 7.3, then,

(b) $Pr\{X_{T-1} = j | X_0 = i\} = W_{i,j} P_{j,0} = \dfrac{1}{(j+1)j}$, $1 \le j < i$, and $Pr\{X_{T-1} = i | X_0 = i\} = P_{i,0} = \dfrac{1}{i}$.

7.5 $v_k = \dfrac{\pi}{2} B_{N,k}\left[(N+1)^2 - (N-2k)^2\right]$

where

$$B_{N,k} = 2^{-2N}\binom{2(N-k)}{N-k}\binom{2k}{k}$$

8.1 No adults survive if a local catastrophy occurs, which has probability $1 - \beta$, and if independently, all N dispersed offspring fail to survive, which has probability $(1 - \alpha)^N$. Another example in which looking at mean values alone is misleading.

8.2 A first step analysis yields $E[Z] = 1 + \mu\, E[Z]$ whence $E[Z] = 1/(1 - \mu)$, $\mu < 1$.

8.3 Possible families:

Possible families:	GG	GB	BG	BBG	BBBG...
Probability:	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}\cdots$

Let N = Total children, X = Male children

(a) $Pr\{N=2\} = \frac{3}{4}, \quad Pr\{N=k\} = \left(\frac{1}{2}\right)^k$ for $k \geqq 3$.

(b) $Pr\{X=1\} = \frac{1}{2}, \quad Pr\{X=0\} = \frac{1}{4}, \quad Pr\{X=k\} = \left(\frac{1}{2}\right)^{k+1}, \quad k \geqq 2.$

8.4 $E\left[Z_{n+1}\middle|Z_0,\ldots,Z_n\right] = \frac{1}{\mu^{n+1}}E\left[\xi_1 + \quad + \xi_{X_n}\middle|X_n = \mu^n Z_n\right] = \frac{1}{\mu^{n+1}}\mu X_n = Z_n$. Suppose $Z_0 = X_0 = 1$. $Pr\{X_n$ is ever greater than $k\mu^n\} \le \frac{1}{k}$.

9.1 ξ = # of male children

$$\begin{array}{rcccc} k = & 0 & 1 & 2 & 3 \\ Pr\{\xi = k\} = & \frac{11}{32} & \frac{9}{32} & \frac{9}{32} & \frac{3}{32} \end{array}$$

$$\varphi(s) = \tfrac{11}{32} + \tfrac{9}{32}s + \tfrac{9}{32}s^2 + \tfrac{3}{32}s^3.$$

$u = \varphi(u)$ has smallest solution $u_\infty = .76887$.

9.2 ξ = # of male children

$$\begin{array}{rcccc} k = & 0 & 1 & 2 & 3 \\ Pr\{\xi = k\} = & \frac{1}{32} & (\frac{3}{32} + \frac{3}{4}) & \frac{3}{32} & \frac{1}{32} \end{array}$$

$$\varphi(s) = \tfrac{1}{32} + \tfrac{27}{32}s + \tfrac{3}{32}s^2 + \tfrac{1}{32}s^3$$

$u = \varphi(u)$ has smallest solution $u_\infty = .23607$.

9.3 Following the hint $Pr\{X=k\} = \int \pi(k|\lambda)f^{(\lambda)}d\lambda = \frac{\Gamma(k+\alpha)}{k!\Gamma(\alpha)}\left(\frac{\theta}{1+\theta}\right)^\alpha\left(\frac{1}{1+\theta}\right)^k, \quad k = 0,1,\ldots.$

9.4 $\varphi(1) = \sum_{k=0}^\infty p_k = 1, \quad p_0 = \varphi(0) = 1 - p.$

Refering to Equations I, (6.18) and (6.20).

$$\varphi(s) = 1 - p\sum_{k=0}^\infty \binom{\beta}{k}(-s)^k,$$

so

$$p_k = -p\binom{\beta}{k}(-1)^k = p\frac{\beta(1-\beta)\ldots(k-1-\beta)}{k!} \ge 0 \text{ because } 0 < \beta < 1.$$

9.5 (a)

$$\begin{array}{rcccc} n = & 1 & 2 & 3 & 4 \\ Pr\{\text{All red}\} = & \frac{1}{4} & \frac{1}{4}(\frac{1}{4})^2 & \frac{1}{4}(\frac{1}{4})^2(\frac{1}{4})^4 & \frac{1}{4}(\frac{1}{4})^2(\frac{1}{4})^4(\frac{1}{4})^8 \end{array}$$

$Pr\{\text{All red at generations}\} = (\frac{1}{4})^{2^n-1}$.

(b) $Pr\{\text{Culture dies out}\} = Pr\{\text{Red cells die out}\}$ $\varphi(s) = \frac{1}{12} + \frac{2}{3}s + \frac{1}{4}s^2$. Smallest solution to $u = \varphi(u)$ is $u_\infty = \frac{1}{3}$.

9.6 $s = \varphi(s)$ is the quadratic $0 = as^2 - (a+c)s + c$ whose smallest solution is $u_\infty = \frac{c}{a} < 1$ if $c < a$.

9.7

Family	GG	GB	BG	BBG	$BBBG$
Probability	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

Let X = # of male children.

$$Pr\{X=0\}=\frac{1}{4}\quad Pr\{X=1\}=\frac{1}{2}$$

$$Pr\{X=k\}=\left(\frac{1}{2}\right)^{k+1}\quad \text{for } k \geqq 2.$$

$$\varphi(s)=\frac{1}{4}+\frac{1}{2}s+\sum_{k=2}^{\infty}\left(\frac{1}{2}\right)^{k+1}s^k=\frac{1}{4}+\frac{1}{2}s+\frac{1}{4}\frac{s^2}{2-s}.$$

$$\varphi'(s)=\frac{1}{2}+\frac{1}{4}\frac{2s(2-s)+s^2}{(2-s)^2}\quad E[X]=\varphi'(1)=\frac{5}{4}.$$

$$Pr\{X>0\}=\frac{3}{4}\quad Pr\{X>k\}=\left(\frac{1}{2}\right)^{k+1}$$

$$E[X]=\sum_{k=0}^{\infty}Pr\{X>h\}=\frac{5}{4}$$

9.8 $\varphi(s)=(1-c)\sum_{k=0}^{\infty}(cs)^k=\dfrac{1-c}{1-cs}.$

$s=\varphi(s)$ is a quadratic whose smallest solution is $u_\infty=\dfrac{1-c}{c}$ provided $c>\frac{1}{2}$.

9.9 (a) Let $X=\#$ of male children.

$$Pr\{X=0\}=\frac{1}{4}+\left(\frac{3}{4}\right)\frac{1}{2}=\frac{5}{8}$$

$$Pr\{X=k\}=\frac{3}{4}\left(\frac{1}{2}\right)^{k+1}\quad \text{for } k\geqq 1.$$

(b) $\varphi(s)=\dfrac{5}{8}+\dfrac{3}{8}\displaystyle\sum_{k=1}^{\infty}\left(\frac{s}{2}\right)^k=\frac{5}{8}+\frac{3}{8}\left(\frac{s}{2-s}\right)$

$u_0=0\quad u_n=\varphi(u_{n-1})\quad u_5=.9414$

9.10 (a) U_∞ is smallest solution to $u=f[g(u)]$.

(b) $[f'(1)g'(1)]^{n/2}$ when $n=0,2,4,\ldots f'(1)^{(n+1)/2}g'(1)^{(n-1)/2}$ when $n=1,3,5,\ldots$.

(c) Yes, both change (unless $f'(1)=g'(1)$ in b).

CHAPTER IV

1.1 Let X_n be the number of balls in Urn A. Then $\{X_n\}$ is a doubly stochastic Markor chain having 6 states, whence $\lim_{n\to\infty} Pr\{X_n = 0 | X_0 = i\} = \frac{1}{6}$.

1.2 Let X_n be the number of balls in Urn A.

$$P = \begin{array}{c|cccccc|} & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 2 & 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 3 & 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 4 & 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{1}{5} \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

$\pi_0 = \pi_5 = \frac{1}{32}$; $\pi_1 = \pi_4 = \frac{5}{32}$; $\pi_2 = \pi_3 = \frac{10}{32}$.

$$\boxed{\pi_0 = \tfrac{1}{32}}$$

1.3 The equations for the stationary distribution simplify to:

$$\begin{aligned}
\pi_0 &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0 \quad \left(\textstyle\sum \alpha_i = 1\right) \\
\pi_1 &= (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0 \\
\pi_2 &= (\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\pi_0 \\
\pi_3 &= (\alpha_4 + \alpha_5 + \alpha_6)\pi_0 \\
\pi_4 &= (\alpha_5 + \alpha_6)\pi_0 \\
\underline{\pi_5} &\underline{=} \alpha_6\pi_0 \\
1 &= \left(\textstyle\sum_{k=1}^{6} k\alpha_k\right)\pi_0 \\
\pi_0 &= \frac{1}{\sum_{k=1}^{6} k\alpha_k} = \frac{1}{\text{Mean of } \alpha \text{ distribution}}
\end{aligned}$$

1.4 Formally, one can look at the Markor chain $Z_n = (X_n, X_{n+1})$ and ask for $\lim_{n\to\infty} Pr\{Z_n = (k,m)\} = \lim Pr\{X_n = k, X_{n+1} = m\} = \pi_k P_{km}$.

1.5

$$P = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ B & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ C & 0 & 1 & 0 & 0 \\ D & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{array}$$

$\pi_A = \frac{2}{8}$; $\pi_B = \frac{3}{8}$; $\pi_C = \frac{1}{8}$ $\boxed{\pi_D = \frac{2}{8}}$

Note: $\pi_{\text{NODE}} \propto$ # arcs at NODE.

1.6 $\lim_{n\to\infty} Pr\{X_{n+1} = j | X_0 = i\} = \pi_j$.

$\lim_{n\to\infty} Pr\{X_n = k, X_{n+1} = j | X_0 = i\} = \pi_k P_{kj}$.

1.7 $\pi_0 = \frac{6}{19}$; $\pi_1 = \frac{3}{19}$; $\pi_2 = \frac{6}{19}$; $\pi_3 = \frac{4}{19}$.

1.8 P^8 has all positive entries.

$$\pi_0 = .2529 \quad \pi_1 = .2299 \quad \pi_2 = .3103 \quad \pi_3 = .1379 \quad \pi_4 = .0690$$

1.9 $\pi_0 = \pi_1 = \frac{1}{10}$ $\pi_2 = \pi_3 = \frac{4}{10}$.

1.10 The matrix is doubly stochastic, whence

$$\pi_k = \frac{1}{N+1} \quad \text{for} \quad k = 0, 1, \ldots, N.$$

1.11 (a) $\pi_0 = \frac{117}{379} = .3087$ $\pi_3 = \frac{62}{379} = .1636$

(b) $\pi_2 + \pi_3 = \frac{143}{379} = .3773$.

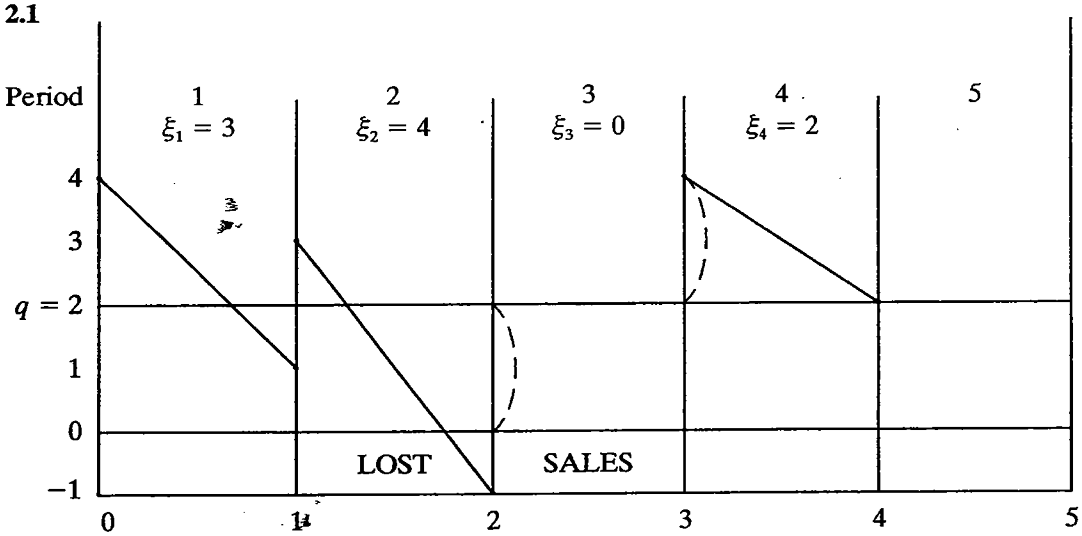
(c) $\pi_0(P_{02} + P_{03}) + \pi_1(P_{12} + P_{13}) = .1559$

1.12 (a) $\mathbf{\Pi P} = \mathbf{P\Pi} = \mathbf{\Pi}^2 = \mathbf{\Pi}$. Then $\mathbf{Q}^2 = (\mathbf{P} - \mathbf{\Pi})(\mathbf{P} - \mathbf{\Pi}) = \mathbf{P}^2 - \mathbf{\Pi P} - \mathbf{P\Pi} + \mathbf{\Pi}^2 = \mathbf{P}^2 - \mathbf{\Pi}$. Similarly, $\mathbf{\Pi P}^n = \mathbf{P}^n\mathbf{\Pi} = \mathbf{\Pi}^{n+1} = \mathbf{\Pi}$ and $\mathbf{Q}^{n+1} = (\mathbf{P} - \mathbf{\Pi})(\mathbf{P}^n - \mathbf{\Pi}) = \mathbf{P}^{n+1} - \mathbf{\Pi}$.

(b) $$Q^n = \frac{1}{2}\begin{Vmatrix} \left(\frac{1}{2}\right)^n & 0 & -\left(\frac{1}{2}\right)^n \\ 0 & 0 & 0 \\ -\left(\frac{1}{2}\right)^n & 0 & \left(\frac{1}{2}\right)^n \end{Vmatrix}$$

1.13 $\lim_{n\to\infty} Pr\{X_{n-1} = 2 | X_n = 1\} = \dfrac{\pi_2 P_{21}}{\pi_1} = \dfrac{6 \times .2}{7} = .1714$ $\pi_0 = \frac{11}{24}$ $\pi_1 = \frac{7}{24}$ $\pi_2 = \frac{6}{24}$.

2.1



(a) $X_0 = 4 \quad X_1 = 1 \quad X_2 = 0 \quad X_3 = 2 \quad X_4 = 2$

(b)
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left\| \begin{array}{ccccc} .6 & .3 & .1 & 0 & 0 \\ .3 & .3 & .3 & .1 & 0 \\ .1 & .2 & .3 & .3 & .1 \\ .3 & .3 & .3 & .1 & 0 \\ .1 & .2 & .3 & .3 & .1 \end{array} \right\| \end{array}$$

2.1 (c) $\pi_0 + \pi_1 + \pi_2 = .3559 + .2746 + .2288 = .8593$

2.2 State is (x, y) where $x = \#$ machines operating and $y = \#$ days repair completed.

(a)
$$P = \begin{array}{c} \\ (2,0) \\ (1,0) \\ (1,1) \\ (0,0) \\ (0,1) \end{array} \begin{array}{c} \begin{array}{ccccc} (2,0) & (1,0) & (1,1) & (0,0) & (0,1) \end{array} \\ \left\| \begin{array}{ccccc} (1-\alpha)^2 & 2\alpha(1-\alpha) & 0 & \alpha^2 & 0 \\ 0 & 0 & 1-\beta & 0 & \beta \\ (1-\beta) & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{array} \right\| \end{array}$$

(b) $\pi_{(2,0)} + \pi_{(1,0)} + \pi_{(1,1)} = .6197 + .1840 + .1472 = .9508.$

2.3 (a) $\pi_0 = .2549; \quad \pi_1 = .2353; \quad \pi_2 = .3529; \quad \pi_3 = .1569$

(b) $\pi_2 + \pi_3 = .5098$

(c) $\pi_0(P_{02} + P_{03}) + \pi_1(P_{12} + P_{13}) = .2235.$

2.4

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\| \begin{array}{cccc} .1 & .3 & .2 & .4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right\| \end{array}$$

$E[\xi] = 2.9$

$$\pi_0 = \tfrac{10}{29} = \frac{1}{E[\xi]}; \quad \pi_1 = \tfrac{9}{29}; \quad \pi_2 = \tfrac{6}{29}; \quad \pi_3 = \tfrac{4}{29}.$$

2.5 (a) $P^{(4)}_{(s,s),(s,s)} + P^{(4)}_{(s,s),(c,s)} = .3421 + .1368 = .4789.$
(b) $\pi_{(s,s)} + \pi_{(c,s)} = .25 + .15 = .40.$

2.6 To establish the Markov property and to get the transition probability matrix, use $Pr\{N = n|N \geqq n\} = \beta$.

$$\pi_1 = \frac{\beta}{p+\beta}.$$

2.7 $P_{00} = p;\quad P_{0,1} = q = 1 - p;\quad P_{i,i+1} = \alpha q$
$P_{ii} = \alpha p + \beta q;\quad P_{i,i-1} = \beta p \quad$ for $i \geqq 1$.

2.8 $\pi_{(1,0)} = \dfrac{1}{1+2p}.$

$p =$	.01	.02	.05	.10
$\pi_{(1,0)} =$	.9804	.9615	.9091	.8333

3.1 (a) $f^{(0)}_{00} = 0 \quad f^{(1)}_{0,0} = 1 - a \quad f^{(k)}_{00} = ab(1-b)^{k-2}, \quad k \geqq 2.$
(b) We are asked to show:

$$P^{(n)}_{00} = \sum_{k=0}^{n} f^{(k)}_{00} P^{(n-k)}_{00} \quad \text{where } n \geqq 1 \quad \text{and}$$
$$P^{(n)}_{00} = \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^n.$$

Some preliminary calculations:

$$(i)\quad \frac{b}{a+b}\sum_{k=2}^{n} f^{(k)}_{00} = \frac{b}{a+b}\sum_{k=2}^{n} ab(1-b)^{k-2} = \frac{ab}{a+b}\left[1-(1-b)^{n-1}\right]$$
$$(ii)\quad \frac{a}{a+b}\sum_{k=2}^{n} f^{(k)}_{00}(1-a-b)^{n-k} = \frac{ab}{a+b}\left[(1-b)^{n-1}-(1-a-b)^{n-1}\right].$$

Then
$$\sum_{k=0}^{n} f^{(k)}_{00}P^{(n-k)}_{00} = f^{(1)}_{00}P^{(n-1)}_{00} + \sum_{k=2}^{n} f^{(k)}_{00}P^{(n-k)}_{00}$$
$$= (1-a)\left[\frac{b}{a+b}+\frac{a}{a+b}(1-a-b)^{n-1}\right] + \frac{ab}{a+b}\left[1-(1-a-b)^{n-1}\right]$$
$$= \frac{b}{a+b} + \left[\frac{(1-a)a}{a+b} - \frac{ab}{a+b}\right](1-a-b)^{n-1}$$
$$= \frac{b}{a+b} + \frac{a}{a+b}(1-a-b)^n = P^n_{00}.$$

as was to be shown.

3.2 For a finite state aperiodic irreducible Markor chain, $P^{(n)}_{ij} \to \pi_j > 0$ as $n \to \infty$ for all i, j. Thus for each i, j, there exists N_{ij} such that $P^{(n)}_{ij} > 0$ for all $n > N_{i,j}$. Because there are only a finite number of states $N = \max_{i,j} N_{i,j}$ is finite, and for $n > N$, we have $P_{i,j} > 0$ for all i,j.

3.3 We first evaluate

$n =$	1	2	3	4	5	6
$P^{(n)}_{00} =$	0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{7}{32}$	$\frac{17}{64}$

Because $P^{(n)}_{00} = 1$, (3.2) may be rewritten

$$f^{(n)}_{00} = P^{(n)}_{00} - \sum_{k=1}^{n-1} f^{(k)}_{00}P^{(n-k)}_{00}.$$

Finally $f^{(1)}_{00} = P^{(1)}_{00} = 0$

$$f_{00}^{(2)} = f_{00}^{(2)} - f_{00}^{(1)}P_{00}^{(1)} = \tfrac{1}{4}$$
$$f_{00}^{(3)} = \tfrac{1}{8}$$
$$f_{00}^{(4)} = \tfrac{3}{8} - (\tfrac{1}{4})(\tfrac{1}{4}) = \tfrac{5}{16}$$
$$f_{00}^{(5)} = \tfrac{7}{32} - (\tfrac{1}{8})(\tfrac{1}{4}) - (\tfrac{1}{4})(\tfrac{1}{8}) = \tfrac{5}{32}$$

4.1 (a) $\pi_0 + \pi_1 = 1$ and $(\beta,\alpha)\,P = (\beta,\alpha)$.

(b) A first return to 0 at time n entails leaving 0 on the first step, staying in 1 for $n-2$ transitions, and then returning to 0, whence $f_{00}^{(n)} = \alpha\beta(1-\beta)^{n-2}$ for $n \geqq 2$.

(c)
$$m_0 = \sum_{n=1}^{\infty} n f_{00}^{(n)} = \sum_{n=1}^{\infty}\sum_{k=1}^{n} f_{00}^{(n)} = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} f_{00}^{(n)}$$
$$= 1 + \sum_{k=2}^{\infty} \alpha\beta \sum_{n=k}^{\infty} (1-\beta)^{n-2} = 1 + \alpha \sum_{k=2}^{\infty} (1-\beta)^{k-2}$$
$$= 1 + \frac{\alpha}{\beta} = \frac{\alpha+\beta}{\beta} = \frac{1}{\pi_0}.$$

4.2 $\pi_0 = .1507 \quad \pi_1 = .3493 \quad \pi_2 = .1918 \quad \pi_3 = .3082$

4.3 The period of the Markov chain is $d = 2$. While there is no limiting distribution, there is a stationary distribution. Set $p_0 = q_N = 1$. Solve

$$\begin{array}{ll}
\pi_0 = \pi_0 & \pi_0 = \pi_0 \\
\pi_0 = q_1\pi_1 & \pi_1 = \dfrac{1}{q_1}\pi_0 = \left(\dfrac{p_0}{q_1}\right)\pi_0 \\
\pi_1 = p_0\pi_0 + q_2\pi_2 & \pi_2 = \dfrac{1}{q_2}\left(\pi_1 - p_0\pi_0\right) = \left(\dfrac{p_0 p_1}{q_1 q_2}\right)\pi_0 \\
\pi_2 = p_1\pi_1 + q_3\pi_3 & \pi_3 = \left(\dfrac{p_0 p_1 p_2}{q_1 q_2 q_3}\right)\pi_0 \\
\vdots & \\
\pi_k = p_{k-1}\pi_{k-1} + q_{k+1}\pi_{k+1} & \pi_{k+1} = \rho_{k+1}\pi_0
\end{array}$$

Where

$$\rho_{k+1} = \frac{p_0 p_1 \cdot \ldots \cdot p_k}{q_1 q_2 \cdot \ldots \cdot q_{k+1}}.$$ Upon adding

$$1 = \pi_0 + \ldots + \pi_N = (\rho_0 + \rho_1 + \cdot \ldots \cdot + \rho_N)\pi_0$$
$$\pi_0 = \frac{1}{1 + \rho_1 + \rho_2 + \cdot \ldots \cdot + \rho_N} = \frac{1}{1 + \sum_{k=1}^{N}\prod_{l=1}^{k}\left(\frac{p_{l-1}}{q_l}\right)}$$

and $\pi_k = \rho_k \pi_0$.

4.4

$$\pi_0 = \pi_0 = \left(\sum_{k=1}^{\infty}\alpha_k\right)\pi_0$$

$$\begin{array}{ll}
\pi_0 = \alpha_1\pi_0 + \pi_1 & \pi_1 = \left(1-\alpha_1\right)\pi_0 = \left(\sum_{k=2}^{\infty}\alpha_k\right)\pi_0 \\
\pi_1 = \alpha_2\pi_0 + \pi_2 & \pi_2 = \left(1-\alpha_1-\alpha_2\right)\pi_0 = \left(\sum_{k=3}^{\infty}\alpha_k\right)\pi_0 \\
\pi_2 = \alpha_3\pi_0 + \pi_3 &
\end{array}$$

$$\pi_n = \left(\sum_{k=n+1}^{\infty}\alpha_k\right)\pi_0$$

$$\sum_{n=1}^{\infty}\pi_n = 1 \quad \text{implies} \quad \pi_0 = \frac{1}{\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\alpha_k}$$

Let ξ be a ramdom variable with $Pr\{\xi = k\} = \alpha_k$. Then $\sum_{n=0}^{\infty}\sum_{k=n+1}^{\infty}\alpha_k = \sum_{n=0}^{\infty}Pr(\xi > n) = E[\xi] = \sum_{k=1}^{\infty}k\alpha_k$. In order that $\pi_0 > 0$ we must have $\sum_{k=1}^{\infty}k\alpha_k = E[\xi] < \infty$. Note: The Markov chain here models the remaining life in a renewal process. The result says that, under the conditions $\alpha_1 > 0$, $\alpha_2 > 0$, a renewal occurs during a particular period, in the limit, with probability 1/ mean life of a component. See Chapter VII.

4.5 Recall that a return time is always at least 1.

(a) Straightforward (b) To simplify, use $\Sigma\pi_i = 1$ and

$$\sum_i \pi_i P_{ik} = \pi_k \text{ to get } \sum_i \pi_i m_{ij} = 1 + \sum_{k\neq j}\pi_k m_{kj} = 1 + \sum_{i\neq j}\pi_i m_{ij}$$

and subtract $\sum_{i\neq j}\pi_i m_{ij}$ from both sides to get $\pi_j m_{jj} = 1$.

4.6 $\{n \geqq 1: P_{00}^{(n)} > 0\} = \{4, 5, 8, 9, 10, 12, 13, 14, \ldots\}$

$$d(0) = 1 \qquad P_{-1,-4}^{(19)} = 0, \qquad P_{i,j}^{(20)} > 0 \text{ for all } i, j.$$

4.7 Measure time in *trips*, so there are two trips each day. Let $X_n = 1$ if car and person are at the same location prior to the n*th* trip; $= 0$, if not.

The transition probability matrix is

$$P = \begin{array}{c|cc|} & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1-p & p \end{array}$$

In the long run he is *not* with car for $\pi_0 = \dfrac{1-p}{2-p}$ fraction of trips, and walks in rain $\pi_0 p = \dfrac{p(1-p)}{2-p}$ fraction of trips. The fraction of *days* he/she walks in rain is $\dfrac{2p(1-p)}{2-p}$.

With two case, let X_n be the number of cars at the location of the person.

$$P = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1-p & p \\ 2 & 1-p & p & 0 \end{array}$$

$\pi_0 = \dfrac{1-p}{3-p}$ and fraction of days walk in rain is $2p\pi_0 = \dfrac{2p(1-p)}{3-p}$. Note that the person never gets wet if $p = 0$ *or* $p = 1$.

4.8 The equations are

$$\begin{aligned}
\pi_0 &= \tfrac{1}{2}\pi_0 + \tfrac{1}{3}\pi_1 + \tfrac{1}{4}\pi_2 + \tfrac{1}{5}\pi_3 + \ldots \\
\pi_1 &= \tfrac{1}{2}\pi_0 + \tfrac{1}{3}\pi_1 + \tfrac{1}{4}\pi_2 + \tfrac{1}{5}\pi_3 + \ldots = \pi_0 \\
\pi_2 &= \tfrac{1}{3}\pi_1 + \tfrac{1}{4}\pi_2 + \tfrac{1}{5}\pi_3 + \ldots = \pi_0 - \tfrac{1}{2}\pi_0 \\
\pi_3 &= \tfrac{1}{4}\pi_2 + \tfrac{1}{5}\pi_3 + \ldots = \pi_0 - \tfrac{1}{2}\pi_0 - \tfrac{1}{3}\pi_1 \\
\pi_4 &= \tfrac{1}{5}\pi_3 + \ldots = \pi_0 - \tfrac{1}{2}\pi_0 - \tfrac{1}{3}\pi_1 - \tfrac{1}{4}\pi_2
\end{aligned}$$

Starting with the equation for π_1 and solving recursively we get

$$\pi_k = \pi_{k-1} - \frac{1}{k}\pi_{k-2} = \frac{1}{k!} \quad \text{for} \quad k \geqq 0.$$

$$\Sigma\pi_k = 1 \Rightarrow \pi_0 = e^{-1} \quad \text{and} \quad \pi_k = \frac{e^{-1}}{k!},\ k \geqq 0$$

(Poisson, $\theta = 1$).

5.1 The stationary distributions for P_B and P_C are, respectively, $(\pi_3, \pi_4) = (\frac{1}{2}, \frac{1}{2})$ and (π_5, π_6, π_7) = (.4227 .2371 .3402). The hitting probabilities from the transient states to the recurrent classes are

$$U = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 3\text{–}4 & 5\text{–}7 \end{array} \\ \left\| \begin{array}{cc} .4569 & .5431 \\ .1638 & .8362 \\ .4741 & .5259 \end{array} \right\| \end{array}$$

This gives

$$P^\infty = \begin{array}{c|ccc|cc|ccc|} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & & & & .2284 & .2284 & .2296 & .1288 & .1848 \\ 1 & & & & .0819 & .0819 & .3534 & .1983 & .2845 \\ 2 & & & & .2371 & .2371 & .2223 & .1247 & .1789 \\ \hline 3 & & \mathbf{O} & & \frac{1}{2} & \frac{1}{2} & & \mathbf{O} & \\ 4 & & & & \frac{1}{2} & \frac{1}{2} & & & \\ \hline 5 & & & & & & .4227 & .2371 & .3402 \\ 6 & & & & & \mathbf{O} & .4227 & .2371 & .3402 \\ 7 & & & & & & .4227 & .2371 & .3402 \end{array}$$

5.2 Aggregating states {3, 4} and {5, 6, 7},

$$\begin{array}{c|ccc|cc|} & 0 & 1 & 2 & 3\text{–}4 & 5\text{–}7 \\ \hline 0 & .1 & .2 & .1 & .3 & .3 \\ 1 & 0 & .1 & .2 & .1 & .6 \\ 2 & .5 & 0 & 0 & .3 & .2 \\ \hline 3\text{–}4 & 0 & 0 & 0 & 1 & 0 \\ 5\text{–}7 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$$U = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 3\text{–}4 & 5\text{–}7 \end{array} \\ \left\| \begin{array}{cc} .44 & .56 \\ .23 & .77 \\ .52 & .48 \end{array} \right\| \end{array}$$

$$\left.\begin{array}{r} .3\pi_3 + .6\pi_4 = \pi_3 \\ \pi_3 + \pi_4 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} \pi_3 = \frac{6}{13} = .46 \\ \pi_4 = \frac{7}{13} = .54 \end{array}$$

$$P_{04}^{(\infty)} = U_{0,3\text{–}4} \quad \pi_4 = .44 \times .54 = .24$$

$$
P^{\infty} = \begin{array}{c|ccc:cc:ccc|}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & & & & .20 & .24 & .25 & .15 & .16 \\
1 & & & & .10 & .12 & .35 & .20 & .22 \\
2 & & & & .24 & .28 & .21 & .12 & .14 \\
\hdashline
3 & & & & .46 & .54 & & & \\
4 & & & & .46 & .54 & & & \\
\hdashline
5 & & & & & & .45 & .26 & .29 \\
6 & & & & & & .45 & .26 & .29 \\
7 & & & & & & .45 & .26 & .29
\end{array}
$$

CHAPTER V

1.1 Following the hint:

$$Pr\{X = k\} = \int_0^1 e^{-\lambda(1-x)} \frac{\lambda^k x^{k-1}}{(k-1)!} e^{-\lambda x} dx = \frac{\lambda^k e^{-\lambda}}{(k-1)!} \int_0^1 x^{k-1} dx = \frac{\lambda^k e^{-\lambda}}{k!}$$

1.2 The number of defects in any interval is Poisson distributed, from Theorem 1.1, and the numbers of defects in disjoint intervals are independent random variables because it's true for both major and minor types.

1.3 $g_X(s) = \sum_{k=0}^{\infty} \frac{\mu^k e^{-\mu}}{k!} s^k = e^{-\mu} \sum_{k=0}^{\infty} \frac{(\mu s)^k}{k!} = e^{-\mu} e^{\mu s} = e^{-\mu(1-s)}, \quad |s| < 1.$

1.4 $g_N(s) = E\left[s^{X+Y}\right] = E\left[s^X s^Y\right]$

$= E\left[s^X\right]E\left[s^Y\right]$ (using II, (1.10), (1.12)).

$= g_X(s) g_Y(s).$

In the Poisson case (Problem 1.3)

$$g_N(s) = e^{-\alpha(1-s)} e^{-\beta(1-s)} = e^{-(\alpha+\beta)(1-s)}. \quad \text{(Poisson, } \theta = \alpha + \beta\text{).}$$

1.5 (a) $\frac{1 - p_0(h)}{h} = \frac{1 - e^{-\lambda h}}{h} = \frac{\lambda h - \frac{1}{2}\lambda^2 h^2 + \frac{1}{3!}\lambda^3 h^3}{h}$

$= \lambda - \frac{1}{2}\lambda^2 h + \frac{1}{3!}\lambda^3 h^2 - \cdots \to \lambda \text{ as } h \to 0.$

(b) $\frac{p_1(h)}{h} = \frac{\lambda h e^{-\lambda h}}{h} = \lambda e^{-\lambda h} \to \lambda \text{ as } h \to 0.$

(c) $\frac{p_2(h)}{h} = \frac{\frac{1}{2}\lambda^2 h^2 e^{-\lambda h}}{h} = \frac{1}{2}\lambda^2 h e^{-\lambda h} \to 0 \text{ as } h \to 0.$

1.6 $Pr\{X(t) = k, X(t+s) = n\} = Pr\{X(t) = k, X(t+s) - X(t) = n - k\}$

$$= \frac{(\lambda t)^k e^{-\lambda t}}{k!} \frac{(\lambda s)^{n-k} e^{-\lambda s}}{(n-k)!}$$

$$Pr\{X(t)=k|X(t+s)=n\}=\frac{\dfrac{(\lambda t)^k e^{-\lambda t}}{k!}\dfrac{[\lambda s]^{n-k}e^{-\lambda s}}{(n-k)!}}{\dfrac{[\lambda(t+s)]^n e^{-\lambda(t+s)}}{n!}}$$

$$=\binom{n}{k}\left(\frac{t}{t+s}\right)^k\left(\frac{s}{t+s}\right)^{n-k}\quad\left(\text{Binomial, } p=\frac{t}{t+s}\right)$$

1.7 $Pr\{\text{Survive at time } t\}=\sum_{k=0}^{\infty}Pr\{\text{Survive } k \text{ shocks}\}Pr\{k \text{ shocks}\}=\sum_{k=0}^{\infty}\alpha^k\frac{e^{-\lambda t}(\lambda t)^k}{k!}=e^{-\lambda t(1-\alpha)}.$

1.8 $e^{\lambda t}=1+\lambda t+\frac{1}{2}(\lambda t)^2+\frac{1}{3!}(\lambda t)^3+\frac{1}{4!}(\lambda t)^4+\cdots$

$e^{-\lambda t}=1-\lambda t+\frac{1}{2}(\lambda t)^2-\frac{1}{3!}(\lambda t)^3+\frac{1}{4!}(\lambda t)^4-\cdots$

$\frac{1}{2}\left(e^{\lambda t}-e^{-\lambda t}\right)=\lambda t+\frac{1}{3!}(\lambda t)^3+\frac{1}{5!}(\lambda t)^5+\cdots$

$Pr\{X(t)\text{ is odd}\}=e^{-\lambda t}\left[\frac{1}{2}\left(e^{\lambda t}-e^{-\lambda t}\right)\right]=\frac{1}{2}\left(1-e^{-2\lambda t}\right).$

1.9 (a) $E[X(T)|T=t]=\lambda t \quad E\left[X(T)^2|T=t\right]=\lambda t+\lambda^2t^2.$

(b) $E[X(T)]=\int_0^1\lambda t\,dt=\frac{1}{2}\lambda=1$ when $\lambda=2.$

$E\left[X(T)^2\right]=\int_0^1\left(\lambda t+\lambda^2t^2\right)dt=\frac{1}{2}\lambda+\frac{1}{3}\lambda^2.$

$Var[X(T)]=E\left[X(T)^2\right]-E[X(T)]^2=\frac{1}{2}\lambda+\frac{1}{3}\lambda^2-\frac{1}{4}\lambda^2$

$=\frac{1}{2}\lambda+\frac{1}{12}\lambda^2=\frac{4}{3}$ when $\lambda=2.$

1.10 (a) K

(b) $c\,E\left[\int_0^T X(t)dt\right]=c\int_0^T\lambda t\,dt=\frac{1}{2}\lambda cT^2.$

(c) $A.C.=\frac{1}{T}\left(K+\frac{1}{2}\lambda cT^2\right).$

(d) $\frac{d}{dT}A.C.=O\Rightarrow T^*=\sqrt{\frac{2K}{\lambda c}}.$

Note: In (a), could argue for Dispatch Cost $=K\,\mathbf{1}\{X(T)>0\}$ and $E[\text{Dispatch Cost}]=K\left(1-e^{-\lambda T}\right).$

1.11 The gamma density

$$f_k(x)=\frac{\lambda^k x^{k-1}e^{-\lambda x}}{(k-1)!},\quad x>0$$

1.12 (a) $$Pr\{X'(t)=k\} = \int_0^\infty \frac{(\theta t)^k e^{-\theta t}}{k!} e^{-\theta} d\theta$$
$$= \frac{t^k}{k!} \int_0^\infty \theta^k e^{-(1+t)\theta} d\theta$$
$$= \frac{1}{k!}\left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right) \int_0^\infty x^k e^{-x} dx$$
$$= \left(\frac{t}{1+t}\right)^k \left(\frac{1}{1+t}\right) \quad \text{(See I, (6.4))}.$$

(b) $$Pr\{X'(t)=j, X'(t+s)=j+k\} = \int_0^\infty \frac{(\theta t)^j e^{-\theta t}}{j!} \frac{(\theta s)^k e^{-\theta s}}{k!} e^{-\theta} d\theta$$
$$= \binom{j+k}{j}\left(\frac{t}{1+s+t}\right)^j \left(\frac{s}{1+s+t}\right)^k \left(\frac{1}{1+s+t}\right).$$

2.1 $$Pr\{X(n,p)=0\} = (1-p)^n = \left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}, \quad n\to\infty$$
$$\frac{Pr\{X(n,p)=k+1\}}{Pr\{X(n,p)=k\}} = \frac{(n-k)p}{(k+1)(1-p)} \to \frac{\lambda}{k+1}, \quad n\to\infty, \lambda = np.$$

2.2 In a small sample (sample size $\ll\sqrt{\#\text{tags}}$) there are many potential pairs, and a small probability for each particular pair to be drawn. The probability that pair i is drawn is *approximately* independent of the probability that pair j is drawn.

2.3 $$Pr\{A\} = \frac{2N(2N-2)\cdot\dots\cdot(2N-2n+2)}{2N(2N-1)\cdot\dots\cdot(2N-n+1)} = \frac{2^n\binom{N}{n}}{\binom{2N}{n}}.$$
$$= .7895 \quad \text{when} \quad n=10, N=100.$$
$$\prod_{i=1}^{n-1}\left(1-\frac{i}{2N-i}\right) \cong \prod_{i=1}^{n-1} e^{-\frac{i}{2N-i}} \cong \exp\left\{-\sum_{i=1}^{n-1}\left(\frac{i}{2N}\right)\right\}.$$
$$= \exp\left\{\frac{-n(n-1)}{4N}\right\} = .7985 \quad \text{when} \quad n=10, N=100.$$

2.4 X_N, the number of points in $[0,1)$ when N points are scattered over $[0,N)$ is binomially distributed $N, p=1/N$.
$$Pr\{X_N=k\} = \binom{N}{k}\left(\frac{1}{N}\right)^k\left(1-\frac{1}{N}\right)^{N-k} \to e^{-1}/k!, \quad k \geqq 0.$$

2.5 X_r, the number of points within radius 1 of origin is binomially distributed with parameters N and $p = \frac{\pi}{\pi r^2} = \frac{\lambda}{N}$. $\Pr\{X_r=k\} = \binom{N}{k}\left(\frac{\lambda}{N}\right)^k\left(1-\frac{\lambda}{N}\right)^{N-k} \to \frac{\lambda^k e^{-\lambda}}{k!}, \quad k=0,1,\dots.$

2.6 $Pr\{X_i = k, X_j = l\} = \dfrac{N!}{k!l!(N-k-l)!}\left(\dfrac{1}{M}\right)^k\left(\dfrac{1}{M}\right)^l\left(1-\dfrac{2}{M}\right)^{N-k-l}$

(Multinomial) $\rightarrow \dfrac{1}{k!l!}\lambda^{k+l}e^{-2\lambda} = \left(\dfrac{\lambda^k e^{-\lambda}}{k!}\right)\left(\dfrac{\lambda^l e^{-\lambda}}{l!}\right).$

Fraction of locations having 2 or more accounts $\rightarrow \left(1 - e^{-\lambda} - \lambda e^{-\lambda}\right).$

2.7 $Pr\{k \text{ bacteria in region of area } a\} = \dbinom{N}{k}\left(\dfrac{a}{A}\right)^k\left(1-\dfrac{a}{A}\right)^{N-k} \rightarrow \dfrac{c^k e^{-c}}{k!}$ as $\begin{matrix} N\rightarrow\infty \\ a\rightarrow o \end{matrix}, \ \dfrac{Na}{A}\rightarrow c.$

2.8 $p_1 = .1, p_2 = .2, p_3 = .3, p_4 = .4 \quad \sum p_i^2 = .30$

k	$Pr\{S_n = k\}$	$e^{-1}/k!$	*Diff.*
0	.3024	.3679	−.0655
1	.4404	.3679	.0725
2	.2144	.1840	.0304
3	.0404	.0613	−.0209
4	.00024	.0153	−.0151

2.9 $p_1 = p_2 = p_3 = .1, p_4 = .2 \quad \sum p_i^2 = .07$

k	$Pr\{S_n = k\}$	$\left(\frac{1}{2}\right)^k e^{-\frac{1}{2}}/k!$	*Diff.*
0	.5832	.6065	−.0233
1	.3402	.3033	.0369
2	.0702	.0758	−.0056
3	.0062	.0126	−.0064
4	.0002	.0016	−.0014

2.10 One need only to observe that

$$\begin{aligned}\left|Pr\{S_n \in I\} - Pr\{X(\mu) \in I\}\right| &\le Pr\{S_n \neq X(\mu)\} \\ &\le \sum_{k=1}^{n} Pr\{\varepsilon(p_i) \neq X(p_i)\}.\end{aligned}$$

2.11 From the hint,

$$Pr\{X \text{ in } B\} \le Pr\{Y \text{ in } B\} + Pr\{X \neq Y\}. \text{ Similarly}$$
$$Pr\{Y \text{ in } B\} \le Pr\{X \text{ in } B\} + Pr\{X \neq Y\}.$$

Together

$$\left|Pr\{X \text{ in } B\} - Pr\{Y \text{ in } B\}\right| \le Pr\{X \neq Y\}.$$

2.12 Most older random number generators fail this test, the pairs (U_{2N}, U_{2N+1}) all falling in a region having little or no area.

3.1 To justify the differentiation as the correct means to obtain the density, look at

$$\int_{w_1}^{\infty}\int_{w_2}^{\infty} f_{w_1,w_2}(w_1', w_2')dw_1' dw_2' = [1 + \lambda(w_2 - w_1)]e^{-\lambda w_2}.$$

3.2 $f_{w_2}(w_2) = \int_0^{w_2} f_{w_1,w_2}(w_1, w_2)dw_1 = \lambda^2 w_2 e^{-\lambda w_2}, \quad w_2 > 0$

$$f_{w_1/w_2}(w_1/w_2) = \frac{\lambda^2 e^{-\lambda w_2}}{\lambda^2 w_2 e^{-\lambda w_2}} = \frac{1}{w_2} \quad \text{for } 0 < w_1 < w_2$$

(Uniform on $(0, w_2]$) Theorem 3.3 conditions on an event of positive probability. Here $Pr\{W_2 = w\} = 0$ for all w. $\{W_2 = w_2\}$ is *NOT* the same event as $X(w_2) = 2$.

3.3 $f_{S_0,S_1}(s_0, s_1) = f_{w_1,w_2}(s_0, s_0 + s_1)$ (Jacobean $= 1$)

$$= \lambda^2 \exp\{-\lambda(s_0 + s_1)\} = (\lambda e^{-\lambda s_0})(\lambda e^{-\lambda s_1}).$$

Compare with Theorem 3.2. for another approach.

3.4 $f_{w_1}(w_1) = \int_{w_1}^{\infty} \lambda^2 e^{-\lambda w_2} dw_2 = \lambda e^{-\lambda w_1}, \ w_1 > 0.$

$$f_{w_2}(w_2) = \int_0^{w_2} \lambda^2 e^{-\lambda w_2} dw_1 = \lambda^2 w_2 e^{-\lambda w_2}, \ w_2 > 0$$

3.5 One can adapt the solution of Exercise 1.5 for a computational approach. For a different approach,

$$Pr\{X(T) = 0\} = Pr\{T < W_1\} = \frac{\theta}{\lambda + \theta} \text{ (Review I, 5.2).}$$

Using the memory less property and starting afresh at time W_1

$$Pr\{X(T) > 1\} = Pr\{W_2 \le T\} = Pr\{W_1 \le T\}Pr\{W_2 \le T | W_1 \le T\} = \left(\frac{1}{\lambda + \theta}\right)^2.$$

Similarly,

$$Pr\{X(T) > k\} = \left(\frac{\lambda}{\lambda + \theta}\right)^{k+1} \quad \text{and} \quad Pr\{X(T) = k\} = \left(\frac{\theta}{\lambda + \theta}\right)\left(\frac{\lambda}{\lambda + \theta}\right)^k, \quad k \ge 0$$

3.6 $T = W_Q$, the waiting time for the k *th* arrival, whence $E[T] = \frac{Q}{\lambda}$. $\int_0^T N(t)dt$ = Total customer waiting time = Area under $N(t)$ up to $T = W_Q = 1S_1 + 2S_2 + \cdots + (Q-1)S_{Q-1}$ (Draw picture) so

$$E\left[\int_0^T N(t)dt\right] = [1 + 2 + \cdots + Q - 1]\frac{1}{\lambda} = \frac{Q(Q-1)}{2\lambda}.$$

3.7 The failure rate is $\lambda = 2$ per year. Let X = # of failures in a year

If Stock (Spares)	Inoperable unit if	$Pr\{\text{Inoperable}\}$
0	$X \geqq 1$	.8647
1	$X \geqq 2$	.5940
2	$X \geqq 3$	.3233
3	$X \geqq 4$	.1429
4	$X \geqq 5$	.0527
5*	$X \geqq 6$	.0166

3.8 We use the binomical distribution in Theorem 3.3.

$$Pr\{W_r > s|X(t) = n\} = Pr\{X(s) < r|X(t) = n\} = \sum_{k=0}^{r-1}\binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k}.$$

$$f_{W_r|N(t)=n}(s) = -\frac{d}{ds}\sum_{k=0}^{r-1}\binom{n}{k}\left(\frac{s}{t}\right)^k\left(1 - \frac{s}{t}\right)^{n-k} = \frac{n!}{(r-1)!(n-r)!}\left(\frac{s}{t}\right)^{r-1}\frac{1}{t}\left(1 - \frac{s}{t}\right)^{n-r}.$$

3.9 (a) $Pr\left\{W_1^{(1)} < W_1^{(2)}\right\} = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}$

(b) $Pr\left\{W_1^{(1)} < W_2^{(2)}\right\} = Pr\left\{W_2^{(1)} < W_1^{(2)}\right\} + Pr\left\{W_1^{(2)} < W_2^{(1)} < W_2^{(2)}\right\}$

$$= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2 + \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2 + \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)$$

$$= \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^2\left[1 + 2\frac{\lambda_2}{\lambda_1 + \lambda_2}\right].$$

3.10 $X_{n+1} = 2^{n+1}\exp\{-(W_n + S_n)\} = X_n 2e^{-S_n}$.

$$E[X_{n+1}|X_n] = X_n 2E\left[e^{-S_n}\right] = X_n 2\int_0^\infty e^{-2x}dx = X_n.$$

4.1 $f_{W_1,\ldots,W_{k-1},W_{k+1},\ldots,W_n|X(1)=n,\,W_k=w}(w_1,\ldots,w_{k-1},w_{k+1},\ldots,w_n)$

$$= \frac{n!}{\dfrac{n!}{(k-1)!(n-k)!}w^{k-1}(1-w)^{n-k}} = (k-1)!\left(\frac{1}{w}\right)^{k-1}(n-k)!\left(\frac{1}{1-w}\right)^{n-k}.$$

4.2 Appropriately modify the argument leading to (4.7) so as to obtain the joint distribution of $X(t)$ and $Y(t)$. Answer: $X(t)$ and $Y(t)$ are independent Poisson random variables with parameters

$$\lambda\int_0^t[1 - G(v)]dv \quad \text{and} \quad \lambda\int_0^t G(v)dv, \text{ respectively.}$$

4.3 The joint distribution of $U = \dfrac{W_1}{W_2}$

$$V = \left(\frac{1 - W_3}{1 - W_2}\right) \quad \text{and} \quad W = W_2 \text{ is}$$

$$f_{U,V,W}(u,v,w) = 6w(1-w) \quad \text{for } 0 \le u,v,w \le 1.$$

whence

$$f_{U,V}(u,v) = \int_0^1 6w(1-w)dw = 1 \quad \text{for } 0 \le u,v \le 1$$

4.4 Let $Z(t) = \min\left\{W_1 + Z_1,\ldots,W_{X(t)} + Z_{X(t)}\right\}$.

$$\begin{aligned} Pr\{Z(t) > z\} &= \sum_{n=0}^{\infty} Pr\{Z(t) > z | X(t) = n\} Pr\{X(t) = n\} \\ &= \sum_{n=0}^{\infty} Pr\{U + \xi > z\}^n Pr\{X(t) = n\} \\ &= e^{-\lambda t} \exp \lambda t \, Pr\{U + \xi > z\} \quad (U \text{ is unif. } (0, t]) \\ &= \exp\left\{-\lambda t + \lambda t \int_0^t \frac{1}{t}[1 - F(z-u)]du\right\} \\ &= \exp\left\{-\lambda \int_{z-t}^{z} F(v)dv\right\} \end{aligned}$$

Let $t \to \infty$

$$Pr\{Z > z\} = \exp\left\{-\lambda \int_{-\infty}^{z} F(v)dv\right\}$$

4.5 $Pr\{W_1 > w | N(t) = n\} = Pr\{U_1 > w, \ldots, U_n > w\} = \left(1 - \frac{w}{t}\right)^n = \left(1 - \frac{\beta w}{n}\right)^n$

if $n = \beta t \to e^{-\beta w}$ if $t \to \infty$, $n \to \infty$, $n = \beta t$.

Under the specified conditions, W is exponentially distributed, but *not* with rate λ.

4.6 (a) $E[W_1 | X(t) = 2] = \frac{1}{3}t.$

(b) $E[W_3 | X(t) = 5] = \frac{3}{6}t = \frac{1}{2}t.$

(c) $f_{W_2|X(t)=5}(w) = 20w(1-w)^3/t^5$ for $0 < w < t$.

4.7
$$\begin{aligned} E\left[\sum_{i=1}^{X(t)} f(W_i)\right] &= \sum_{n=0}^{\infty} E\left[\sum_{i=1}^{n} f(W_i) \middle| X(t) = n\right] Pr\{X(t) = n\} \\ &= \lambda t \, E[f(U)] \quad U \text{ is unif. } (0, t] \\ &= \lambda \int_0^t f(u)du. \end{aligned}$$

4.8 This is a generalization of the shot noise process in Section 4.1.

$$E[Z(t)] = E[\xi]\lambda \int_0^t e^{-\alpha u} du = E[\xi]\left(\frac{\lambda}{\alpha}\right)\left(1 - e^{-\alpha t}\right).$$

4.9
$$\begin{aligned} E\left[W_1 W_2 \cdots\cdots W_{N(t)}\right] &= \sum_{n=0}^{\infty} E\left[W_1 \cdots\cdots W_n | N(t) = n\right] Pr\{N(t) = n\} \\ &= \sum_{n=0}^{\infty} E\left[U_1, \cdots\cdots U_n\right] Pr\{N(t) = n\} \\ &= \sum_{n=0}^{\infty} \left(\frac{t}{2}\right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-\lambda t\left(1 - \frac{1}{2}t\right)}. \end{aligned}$$

4.10 The generalization allows the impulse response function to also depend on a random variable ξ so that, in the notation of Section 4.1

$$I(t) = \sum_{k=1}^{X(t)} h(\xi_h, t - W_k),$$

the ξ's being independent of the Poisson process.

4.11 The limiting density is

$$\begin{aligned} f(x) &= c && \text{for } 0 < x < 1; \\ &= c(1 - \log_e x) && \text{for } 1 < x < 2; \\ &= c\left[1 - \log x + \int_2^x \frac{\log(t-1)}{t}\,dt\right], && \text{for } 2 < x < 3; \\ &\vdots \end{aligned}$$

where $\quad c = \exp\{-\text{Euler's constant}\} = .561459\ldots$

In principal, one can get the density for any x from the differential equation. In practice,?

5.1 $f(x) = 2\left(\frac{x}{R^2}\right)$ for $0 < x < R$ because $F(x) = \Pr\{X \le x | N = 1\} = \left(\frac{x}{R}\right)^2$.

5.2 It is a homogeneous Poisson process whose intensity is proportional to the mean of the Poisson random variable N. Note: If $X = R\cos\Theta$ and $Y = R\sin\Theta$, then (X, Y) is uniform on the disk.

5.3 For $i = 1, 2, \ldots, n^2$ let $\xi_i = 1$ if two or more points are in box i, and $\xi_i = 0$ otherwise. Then the number of reactions is $\xi_1 + \cdots + \xi_m$ and $p = Pr\{\xi = 1\} = \frac{1}{2}\lambda^2 d^2 + o(d^2)$.

$$n^2 p = \frac{1}{2}\lambda^2 (td)^2 + \cdots \to \frac{1}{2}\lambda^2\mu^2.$$

The number of reactions is asymptotically Poisson with parameter $\frac{1}{2}\lambda^2\mu^2$.

5.4 The probability that the centre of a sphere is located between r and $r + dr$ units from the origin is $2d[\frac{4}{3}\pi r^3] = 4\lambda\pi r^2 dr$. The probability that such a sphere covers the origin is $\int_r^\infty f(x)dx$. The mean number of spheres that cover the origin is $\int_0^\infty 4\lambda\pi r^2 \int_r^\infty f(x)dxdr = \frac{4}{3}\lambda\pi\int_0^\infty r^3 f(r)dr$. The distribution is Poisson with this mean.

5.5
$$\begin{aligned} F_D(x) &= Pr\{D \le x\} = 1 - Pr\{D > x\} \\ &= 1 - Pr\{\text{No particles in circle of radius } x\} \\ &= 1 - \exp\{-\nu\pi x^2\}, \quad x > 0. \end{aligned}$$

5.6
$$\begin{aligned} 1 - F_R(x) &= Pr\{R > x\} = Pr\{\text{No stars in sphere, radius } x\} \\ &= \exp\left\{-\lambda\frac{4}{3}\pi x^3\right\}, \end{aligned}$$

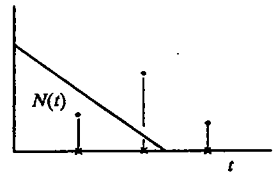
whence

$$f_R(x) = \frac{d}{dx}F_R(x) = 4\lambda\pi x^2 e^{-\frac{4}{3}\lambda\pi x^3}, \quad x > 0$$

5.7 The hint should be sufficient for (a). For (b)

$$\int_0^\infty \lambda(r)dr = 2\pi\lambda\int_0^\infty r\int_r^\infty f(x)dxdr = \int_0^\infty 2\pi\lambda\left\{\int_0^x rdr\right\}f(x)dx = \pi\lambda\int_0^\infty x^2 f(x)dx.$$

6.1 Nonhomogeneous Poisson, intensity $\lambda(t) = \lambda G(t)$. To see this, let $N(t)$ be the number of points $\leq t$ in the relocated process = # points in ◺.



6.2 The key observation is that, if Θ is uniform or $[0, 2\pi)$, and Y is independent with an arbitrary distribution, then $Y + \Theta$ (mod 2π) is uniform.

6.3 From the shock model of Section 6.1, modified for the discrete nature of the damage process, we have

$$\begin{aligned} E[T] &= \frac{1}{\lambda}\sum_{n=0}^{\infty} G^{(n)}(a-1) \\ &= \frac{1}{\lambda}\left[1 + \sum_{n=1}^{\infty}\sum_{k=0}^{a-1}\binom{n+k-1}{k}p^n(1-p)^k\right] \quad \text{(See I, (3.6))} \\ &= \frac{1}{\lambda}\left[1 + \sum_{k=0}^{a-1}p(1-p)^k\left\{\sum_{n=1}^{\infty}\binom{n-1+k}{n-1}p^{n-1}\right\}\right] \\ &= \frac{1}{\lambda}\left[1 + \sum_{k=0}^{a-1}p(1-p)^k(1-p)^{-k-1}\right] \quad \text{(See I, (6.21))} \\ &= \frac{1}{\lambda}\left[1 + \frac{ap}{1-p}\right]. \end{aligned}$$

6.4 One can use the results of I, Section 5.2 or, since T_1 is exponentially distributed, μ

$$\begin{aligned} Pr\{X(T_1) = k\} &= \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!}\mu e^{-\mu t}dt \\ &= \left(\frac{\mu}{\lambda+\mu}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^k, \quad k = 0, 1, \ldots \quad \text{(See I, (6.4))}. \end{aligned}$$

T_a has a gamma density and a similar integration applies. See the last example in II, Section 4.

6.5 $\frac{1}{2}T$ is exponentially distributed with rate parameter 2μ, whence

$$\begin{aligned} Pr\left\{X\left(\frac{1}{2}T\right) = k\right\} &= \int_0^\infty \frac{(\lambda t)^k e^{-\lambda t}}{k!}2\mu e^{-2\mu t}dt \\ &= \left(\frac{2\mu}{\lambda+2\mu}\right)\left(\frac{\lambda}{\lambda+2\mu}\right)^k, \quad k = 0, 1, \ldots \quad \text{(See I, (6.4))} \end{aligned}$$

6.6 Instead of representing $W_k + X_k$, and $W_k + Y_k$ as district points on a single axis, write them as a single point $(W_k + X_k, W_k + Y_k)$ in the plane. The result is a nonhomogeneous Poisson point process in the positive quadrant with intensity

$$\lambda(x, y) = \lambda\int_0^{\min\{x,y\}} f(x-u)f(y-u)du$$

where $f(\cdot)$ is the common density for X, Y.

6.7 Refer to Exercise 6.3.

$$Pr\{Z(t) > z | N(t) > 0\} = \frac{e^{-\lambda z^\alpha t} - e^{-\lambda t}}{1 - e^{-\lambda t}}; \quad 0 < z < 1.$$

Let $Y(t) = t^{1/\alpha}Z(t)$. For large t, we have

$$Pr\{Y(t) > y | N(t) > 0\} = Pr\left\{Z(t) > \frac{y}{t^{1/\alpha}} \middle| N(t) > 0\right\}$$

$$= \frac{e^{-\lambda y^\alpha} - e^{-\lambda t}}{1 - e^{-\lambda t}} \xrightarrow[t\to\infty]{} e^{-\lambda y^\alpha} \quad \text{(Weibull)}.$$

6.8 Following the remarks in Section 1.3 we may write $N(t) = M[\Lambda(t)]$ where $M(t)$ is a Poisson process of unit intensity.

$$Pr\{Z(t) > z\} = Pr\left\{\min\left[Y_1, \ldots, Y_{M[\Lambda(t)]}\right] > z\right\}$$

$$= \exp\{-G(z)\Lambda(t)\} = \exp\{-z^\alpha \Lambda(t)\}.$$

$$Pr\{t^{1/\alpha}Z(t) > z\} = Pr\left\{Z(t) > \frac{z}{t^{1/\alpha}}\right\}$$

$$= \exp\left\{-z^\alpha \frac{\Lambda(t)}{t}\right\} \xrightarrow[t\to\infty]{} e^{-\theta z^\alpha}, \; z > 0 \quad \text{(Weibull)}$$

6.9 To carry the methods of this section a little further, write $N(dt) = N(t + dt) - N(t)$ so $N(dt) = 1$ if and only if $t < W_k \le t + dt$ for some W_k. Then

$$\sum_{k=1}^{N(t)} (W_k)^2 = \int_0^t x^2 N(dx) \quad \text{so}$$

$$E\left[\sum_{k=1}^{N(t)} (W_k)^2\right] = E\left[\int_0^t x^2 N(dx)\right] = \int_0^t x^2 E[N(dx)]$$

$$= \lambda \int_0^t x^2 dx = \frac{\lambda t^3}{3}.$$

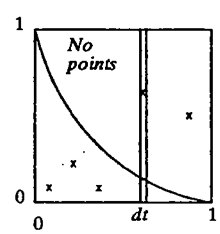
6.10 (a) $Pr\{\text{Sell asset}\} = 1 - e^{-(1-\theta)}$.

(b) Because the offers are uniformly distributed, the expected selling price, given that it is sold, is $\left(\frac{1+\theta}{2}\right)$.

$$E[\text{Return}] = \frac{1+\theta}{2}\left[1 - e^{-(1-\theta)}\right]$$

$$\theta^* = .2079 \quad \max_\theta E[\text{Return}] = .330425$$

(c) $Pr\{\text{Sell in } (t, t+dt]\} = e^{-\int_0^t \{1-\theta(u)\}du}[1 - \theta(t)]dt.$



$$E[\text{Return}] = \int_0^1 \left(\frac{1+\theta(t)}{2}\right) \exp\left\{-\int_0^t [1 - \theta(u)]du\right\}[1 - \theta(t)]dt$$

$$= \frac{2}{9}\int_0^1 (2 - t)dt = \frac{1}{3}, \text{ a miniscule improvement over (b).}$$

CHAPTER VI

1.1 $Pr\{X(U)=k\}=\int_0^1 e^{-\beta u}\left(1-e^{-\beta u}\right)^{k-1}du=\int_0^{1-e^{-\beta}} x^{k-1}\frac{dx}{\beta}=\frac{1}{\beta k}\left(1-e^{-\beta}\right)^k.$

1.2 $\lambda_k=\alpha+\beta k$

$$\lambda_0\lambda_1\cdot\cdots\cdot\lambda_{n-1}=\beta^n\left(\frac{\alpha}{\beta}\right)\left(\frac{\alpha}{\beta}+1\right)\cdot\cdots\cdot\left(\frac{\alpha}{\beta}+n-1\right)$$

$$\left(\lambda_0-\lambda_k\right)\cdot\cdots\cdot\left(\lambda_{k-1}-\lambda_k\right)=\beta^k(-1)^k k!$$

$$\left(\lambda_n-\lambda_k\right)\cdot\cdots\cdot\left(\lambda_{k+1}-\lambda_k\right)=\beta^{n-k}(n-k)!$$

$$P_n(t)=\lambda_0\cdot\cdots\cdot\lambda_{n-1}\sum_{k=0}^{n}B_{k,n}e^{-\lambda_k t}$$

$$=\frac{\left(\frac{\alpha}{\beta}\right)\left(\frac{\alpha}{\beta}+1\right)\cdot\cdots\cdot\left(\frac{\alpha}{\beta}+n-1\right)}{n!}e^{-\alpha t}\sum_{k=0}^{n}\binom{n}{k}\left(-e^{-\beta t}\right)^k$$

$$=\binom{\frac{\alpha}{\beta}+n-1}{n}e^{-\alpha t}\left(1-e^{-\beta t}\right)^n \quad \text{for } n=0,1,\ldots$$

Note: $\sum_{n=0}^{\infty}P_n(t)=1$ using I, (6.18)–(6.20).

1.3 The probabilistic rate of increase in the infected population is jointly proportional to the number who can give the disease, and the number available to catch it.

$$\lambda_k=\alpha k(N-k),\ k=0,1,\ldots,N.$$

1.4 (a) $\lambda_k=\alpha+\theta k=1+2k.$

(b) $P_2(t)=2\left[\frac{1}{8}e^{-t}-\frac{1}{4}e^{-3t}+\frac{1}{8}e^{-5t}\right]$

$P_2(1)=\frac{1}{4}e^{-1}-\frac{1}{2}e^{-3}+\frac{1}{4}e^{-5}=.0688.$

1.5 The two possibilities $X(w_2) = O$ or $X(w_2) = 1$ give us

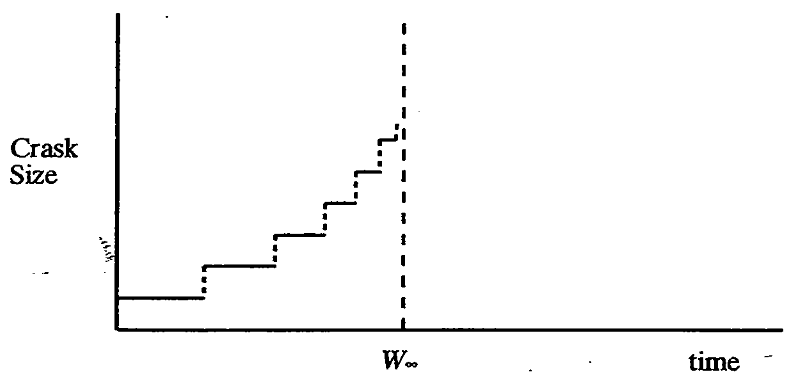
$$\begin{aligned}
Pr\{W_1 > w_1,\ W_2 > w_2\} &= Pr\{X(w_1) = 0,\ X(w_2) = 0\} + Pr\{X(w_1) = 0,\ X(w_2) = 1\} \\
&= P_0(w_1)P_0(w_2 - w_1) + P_0(w_1)P_1(w_2 - w_1) \\
&= e^{-\lambda_0 w_2} + \lambda_0 e^{-\lambda_0 w_1}\left[\frac{1}{\lambda_1 - \lambda_0}e^{-\lambda_0(w_2 - w_1)} + \frac{1}{\lambda_0 - \lambda_1}e^{-\lambda_1(w_2 - w_1)}\right] \\
&= \frac{\lambda_1}{\lambda_1 - \lambda_0}e^{-\lambda_0 w_2} + \frac{\lambda_0}{\lambda_0 - \lambda_1}e^{-\lambda_1 w_2}e^{-(\lambda_0 - \lambda_1)w_1} \\
&= \int_{w_1}^{\infty}\int_{w_2}^{\infty} f_{w_1,w_2}(w_1', w_2')dw_1', dw_2',
\end{aligned}$$

whence

$$f_{w_1,w_2}(w_1, w_2) = \frac{\partial}{\partial w_2}\frac{\partial}{\partial w_1} Pr\{W_1 > w_1, W_2 > w_2\} = \lambda_0\lambda_1 e^{-\lambda_0 w_1}e^{-\lambda_1(w_2 - w_1)}$$

Setting $s_0 = w_1$, $s_1 = w_2 - w_1$ (Jacobean = 1) $f_{s_0,s_1}(s_0, s_1) = (\lambda_0 e^{-\lambda_0 s_0})(\lambda_1 e^{-\lambda_1 s_1})$.

1.6 $E[S_k] = (1 + k)^{-\varrho}$; $E[W_\infty] = \sum_{k=1}^{\infty}(1 + k)^{-\varrho} < \infty$ when $\varrho > 1$.



1.7 (a) $Pr\{S_0 \le t\} = 1 - e^{-\lambda_0 t}$; $Pr\{S_0 > t\} = e^{-\lambda_0 t}$.

$$\begin{aligned}
Pr\{S_0 + S_1 \le t\} &= \int_0^t\left[1 - e^{-\lambda_1(t-x)}\right]\lambda_0 e^{-\lambda_0 x}dx = 1 - e^{-\lambda_0 t} - \left(\frac{\lambda_0}{\lambda_0 - \lambda_1}\right)e^{-\lambda_1 t}\left[1 - e^{-(\lambda_0 - \lambda_1)t}\right] \\
&= 1 - \frac{\lambda_0}{\lambda_0 - \lambda_1}e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_1 - \lambda_0}e^{-\lambda_0 t} = 1 - Pr\{S_0 + S_1 > t\}.
\end{aligned}$$

$$\begin{aligned}
Pr\{S_0 + S_1 + S_2 \le t\} &= \int_0^t\left\{1 - \frac{\lambda_0}{\lambda_0 - \lambda_1}e^{-\lambda_1(t-x)} - \frac{\lambda_0}{\lambda_1 - \lambda_0}e^{-\lambda_0(t-x)}\right\}\lambda_2 e^{-\lambda_2 x}dx \\
&= 1 - \frac{\lambda_0\lambda_2}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)}e^{-\lambda_1 t} - \frac{\lambda_1\lambda_2}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)}e^{-\lambda_0 t} - \frac{\lambda_0\lambda_1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}e^{-\lambda_2 t}.
\end{aligned}$$

(b) $$P_2(t) = \frac{\lambda_0\lambda_2}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)}e^{-\lambda_1 t} + \frac{\lambda_1\lambda_2}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \frac{\lambda_0\lambda_1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t}$$
$$- \frac{\lambda_0}{(\lambda_0-\lambda_1)}e^{-\lambda_1 t} - \frac{\lambda_1}{(\lambda_1-\lambda_0)}e^{-\lambda_0 t}$$
$$= \lambda_0\lambda_1\left[\frac{1}{(\lambda_1-\lambda_0)(\lambda_2-\lambda_0)}e^{-\lambda_0 t} + \frac{1}{(\lambda_0-\lambda_1)(\lambda_2-\lambda_1)}e^{-\lambda_1 t} + \frac{1}{(\lambda_0-\lambda_2)(\lambda_1-\lambda_2)}e^{-\lambda_2 t}\right].$$

1.8 Equations (1.2) become

$$\begin{aligned} P_0'(t) &= -\beta P_0(t) \\ P_n'(t) &= -\beta P_n(t) + \alpha P_{n-1}(t), \quad n = 2,4,6,\ldots \\ P_n'(t) &= -\alpha P_n(t) + \beta P_{n-1}(t), \quad n = 1,3,5,\ldots \end{aligned}$$

Seemming over $n = 0,2,4\ldots$ and $n = 1,3,5,\ldots$ gives

$$\begin{aligned} P_{\text{even}}'(t) &= \alpha P_{\text{odd}}(t) - \beta P_{\text{even}}(t) = \alpha\left[1 - P_{\text{even}}(t)\right] - \beta P_{\text{even}}(t) \\ P_{\text{even}}'(t) &= \alpha - (\alpha+\beta)P_{\text{even}}(t). \end{aligned}$$

Let $Q_{\text{even}}(t) = e^{(\alpha+\beta)t}P_{\text{even}}(t)$, then $Q_{\text{even}}'(t) = \alpha\, e^{(\alpha+\beta)t}$ and since $Q_{\text{even}}(0) = 1$, the solution is $Q_{\text{even}}(t) = \dfrac{\beta}{\alpha+\beta} + \dfrac{\alpha}{\alpha+\beta}e^{(\alpha+\beta)t}$ and $P_{\text{even}}(t) = e^{-(\alpha+\beta)t}Q_{\text{even}}(t) = \dfrac{\alpha}{\alpha+\beta} + \dfrac{\beta}{\alpha+\beta}e^{-(\alpha+\beta)t}$.

1.9 Equations (1.2) become

$$\begin{aligned} P_0'(t) &= -\beta P_0(t) \\ P_n'(t) &= -\beta P_n(t) + \alpha P_{n-1}(t), \quad n = 2,4,6,\ldots \\ P_n'(t) &= -\alpha P_n(t) + \beta P_{n-1}(t), \quad n = 1,3,5,\ldots \end{aligned}$$

Multiply the *n th* equation by *n*, sum, collect terms and simplify to get

$$M'(t) = \alpha P_{\text{odd}}(t) + \beta P_{\text{even}}(t) = \alpha + (\beta-\alpha)P_{\text{even}}(t),$$

and (See Problem 1.8 above)

$$M(t) = \frac{2\alpha\beta}{\alpha+\beta}t + \left(\frac{\beta-\alpha}{\beta+\alpha}\right)\left(\frac{\beta}{\alpha+\beta}\right)\left[1 - e^{-(\alpha+\beta)t}\right].$$

1.10 It is easily checked through (1.5) that $P_n(t) = \binom{N}{n}e^{-(N-n)\lambda t}(1-e^{-\lambda t})^n$, by working with $Q_n(t) = e^{\lambda_n t}P_n(t) = \binom{N}{n}(1-e^{-\lambda t})^n$; $Q_0(t) \equiv 1$; $Q_n(t) = \lambda_{n-1}\int_0^t e^{-\lambda x}Q_{n-1}(x)dx$ (From (1.5)).

1.11 $P_1(t) = \lambda_0 e^{-\lambda_1 t}\int_0^t e^{\lambda_1 x}e^{-\lambda_0 x}dx = \dfrac{\lambda_0}{\lambda_0-\lambda_1}e^{-\lambda_1 t} + \dfrac{\lambda_0}{\lambda_1-\lambda_0}e^{-\lambda_0 t}$

$$P_2(t) = \lambda_1 e^{-\lambda_2 t}\int_0^t \left[\frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 x} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 x}\right] e^{\lambda_2 x}\,dx$$

$$= \lambda_0\lambda_1\left[\frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 t} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 t}\right.$$

$$\left. + \left\{\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)}\right\} e^{-\lambda_2 t}\right]$$

and

$$\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} = \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$$

$$P_3(t) = \lambda_0\lambda_1\lambda_2 e^{-\lambda_2 t}\int_0^t \left[\frac{1}{(\lambda_0 - \lambda_1)(\lambda_2 - \lambda_1)} e^{-\lambda_1 x} + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} e^{-\lambda_0 x}\right.$$

$$\left. + \frac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)} e^{-\lambda_2 x}\right] e^{\lambda_3 x}\,dx$$

$$= \lambda_0\lambda_1\lambda_2\left[\frac{1}{(\lambda_3 - \lambda_0)(\lambda_2 - \lambda_0)(\lambda_1 - \lambda_0)} e^{-\lambda_0 t} + \frac{1}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)(\lambda_0 - \lambda_1)} e^{-\lambda_1 t}\right.$$

$$\left. + \frac{1}{(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)(\lambda_0 - \lambda_2)} e^{-\lambda_2 t} + \frac{1}{(\lambda_0 - \lambda_3)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)} e^{-\lambda_3 t}\right]$$

1.12 $P_2(t) = \lambda_1 e^{-\lambda_2 t}\int_0^t \left[\frac{\lambda_0}{\lambda_0 - \lambda_1} e^{-\lambda_1 x} + \frac{\lambda_0}{\lambda_1 - \lambda_0} e^{-\lambda_0 x}\right] e^{\lambda_2 x}\,dx$

$$= \lambda_0\lambda_1 e^{-\lambda_2 t}\left[\frac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)}\left(1 - e^{-(\lambda_1 - \lambda_2)t}\right) + \frac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)}\left(1 - e^{-(\lambda_0 - \lambda_2)t}\right)\right]$$

$$= \lambda_0\lambda_1\left[\frac{e^{-\lambda_0 t}}{(\lambda_1 - \lambda_0)(\lambda_2 - \lambda_0)} + \frac{e^{-\lambda_1 t}}{(\lambda_2 - \lambda_1)(\lambda_0 - \lambda_1)} + \frac{e^{-\lambda_2 t}}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}\right]$$

Note $\dfrac{1}{(\lambda_0 - \lambda_1)(\lambda_1 - \lambda_2)} + \dfrac{1}{(\lambda_1 - \lambda_0)(\lambda_0 - \lambda_2)} = \dfrac{1}{(\lambda_0 - \lambda_2)(\lambda_1 - \lambda_2)}$.

1.13 Let $Q_n(t) = e^{\lambda t}P_n(t)$. Then $Q_0(t) \equiv 1$ and (1.5) becomes $Q_n'(t) = \lambda\, Q_{n-1}(t)$ which solves to give $Q_1(t) = \lambda t$; $Q_2(t) = \frac{1}{2}(\lambda t)^2 \ldots Q_n(t) = \dfrac{(\lambda t)^n}{n!}$ and $P_n(t) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!}$.

2.1 Using the memoryless property (Section I, 5.2)

$$Pr\{X(T) = 0\} = Pr\{T > W_N\} = \prod_{i=1}^{N} Pr\{T > W_i \mid T > W_{i-1}\} = \prod_{i=1}^{N}\left(\frac{\mu_i}{\mu_i + \theta}\right).$$

2.2 $P_N(t) = e^{-\theta t}$

$P_{N-1}(t) = \theta t\, e^{-\theta t}$

$P_n(t) = (\theta t)^{N-n}\, e^{-\theta t}/(N-n)!$ for $n = 1, 2, \ldots, N$

$P_0(t) = 1 - \sum_{n=1}^{N} P_n(t).$

2.3 Refer to Figure 2.1 to see that Area $= W_N + \ldots + W_1$

Hence $E[Area] = E[W_N] + \ldots + E[W_1] = NE[S_N] + (N-1)E[S_{N-1}] + \ldots + E[S_1] = \sum_{n=1}^{N} \frac{n}{\mu_n}.$

2.4 (a) $\mu_N = NM\theta$

$\mu_{N-1} = (N-1)(M-1)\theta$

$\mu_k = k(M - (N - k))\theta, \quad k = 0, 1, \ldots, N.$

(b)
$$E[W_N] = \sum_{k=1}^{N} E[S_k] = \sum_{k=1}^{N} \frac{1}{k[M-(N-k)]\theta}$$
$$= \frac{1}{\theta} \sum_{k=1}^{N} \left\{ \frac{1}{(M-N)k} - \frac{1}{(M-N)(M-N+k)} \right\}$$
$$= \frac{1}{\theta(M-N)} \left\{ \sum_{k=1}^{N} \frac{1}{k} - \sum_{k=1}^{N} \frac{1}{M-N+k} \right\}$$
$$\sim \frac{1}{\theta(M-N)} \left\{ \log N - \log M + \log(M-N) \right\}.$$

2.5 Breakdown rule is "exponential breakdown".

$$\mu_k = kK\left[\frac{NL}{k}\right] = k \sinh\left[\frac{NL}{k}\right]$$

$$E[W_N] = \sum_{k=1}^{N} \frac{1}{k \sinh\left[\frac{NL}{k}\right]} = \sum_{k=1}^{N} \frac{1}{\left(\frac{k}{N}\right) \sinh\left(\frac{L}{R/N}\right)} \frac{1}{N} \cong \int_0^1 \frac{dx}{x \sinh\left(\frac{L}{x}\right)} \text{ (Riemann approx.)}$$

2.6 (a) $E[T] = E[W_N] = \sum_{k=1}^{N} E[S_k] = \frac{1}{\alpha} \sum_{k=1}^{N} \frac{1}{k}.$

(b)
$$E[T] = \int_0^\infty Pr\{T > t\}dt = \int_0^\infty [1 - P_0(t)]\, dt = \int_0^\infty \left[1 - \left(1 - e^{-\alpha t}\right)^N\right] dt = \frac{1}{\alpha} \int_0^1 \left[1 - y^N\right] \frac{dy}{1-y}$$
$$= \frac{1}{\alpha} \int_0^1 \left[1 + y + y^2 + \ldots + y^{N-1}\right] dy = \frac{1}{\alpha}\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N}\right].$$

3.1 $Pr\{X(t+h)=1|X(t)=0\}=\lambda h+o(h)$ so $\lambda_0=\lambda$.

$Pr\{X(t+h)=0|X(t)=1\}=\lambda(1-\alpha)h+o(h)$ so $\mu_1=\lambda(1-\alpha)$.

The Markov property requires the independent increments of the Poisson process.

3.2 If we assume that the sojourn time on a single plant is exponentially distributed then $X(t)$ is a birth and death process. Reflecting behavior ($\mu_0=0$, $\lambda_0=\frac{1}{m_0}$, $\lambda_N=0$, $\mu_N=\frac{1}{m_N}$) might be assumed. In an actual experiment, and these have been done, one must allow for the escape or other loss of the beetle. We add a state Δ = Escaped (Δ is called the "cemetary") and assume $(0<\alpha<1)$

$$Pr\{X(t+h)=k+1|X(t)=h\}=\frac{\alpha}{2m_k}h+0(h)$$

$$Pr\{X(t+h)=k-1|X(t)=h\}=\frac{\alpha}{2m_k}h+0(h)$$

$$Pr\{X(t+h)=\Delta|X(t)=h\}=\frac{1-\alpha}{m_k}h+0(h)$$

$$Pr\{X(t+h)=\Delta|X(t)=\Delta\}=1.$$

This is a general finite state continuous time Markov process – See Section 6. It is also a birth and death process with "killing".

3.3 $Pr\{V(t)=1\}=\pi$ for all t (See Exercise 3.3)

$$\begin{aligned}E[V(s)V(t)]&=Pr\{V(s)=1,V(t)=1\}=Pr\{V(s)=1\}\times Pr\{V(t)=1|V(s)=1\}\\&=\pi P_{11}(t-s)=\pi[1-P_{10}(t-s)]\end{aligned}$$

$$Cov[V(s)V(t)]=E[V(s)V(t)]-E[V(s)]E[V(t)]=\pi P_{11}(t-s)-\pi^2=\pi(1-\pi)e^{-(\alpha+\beta)(t-s)}.$$

3.4 Because $V(0)=0$, $E[V(t)]=P_{01}(t)$, and

$$E[S(t)]=\int_0^t P_{01}(u)du=\int_0^t(\pi-\pi e^{-\tau u})du=\pi t-\frac{\pi}{\tau}(1-e^{-\tau t}),\quad \tau=\alpha+\beta.$$

4.1 *Single repairman* $R=1$

$\lambda_k=2$ for $k=0,1,2,3,4$, $\mu_k=k$ for $k=0,1,2,3,4,5$.

$\theta_0=1$, $\theta_1=2$, $\theta_2=2$, $\theta_3=\frac{4}{3}$, $\theta_4=\frac{2}{3}$, $\theta_5=\frac{4}{15}$.

$$\sum_{k=0}^5\theta_k=\tfrac{218}{30}=\tfrac{109}{15}$$

Two repairman $R=2$

$\lambda_0=\lambda_1=\lambda_2=\lambda_3=4$; $\lambda_4=2$ $\mu_k=k$

$\theta_0=1$, $\theta_1=4$, $\theta_2=8$, $\theta_3=\frac{32}{3}$, $\theta_4=\frac{32}{3}$, $\theta_5=\frac{64}{15}$,

$$\sum_{k=0}^5\theta_k=\tfrac{579}{15}=\tfrac{193}{5}.$$

	π_0	π_1	π_2	π_3	π_4	π_5
$R=1$	$\frac{15}{109}$	$\frac{30}{109}$	$\frac{30}{109}$	$\frac{20}{109}$	$\frac{10}{109}$	$\frac{4}{109}$
$R=2$	$\frac{15}{579}$	$\frac{60}{579}$	$\frac{120}{579}$	$\frac{160}{579}$	$\frac{160}{579}$	$\frac{64}{579}$

	(a) $\sum k\pi_k$	(b) $\frac{1}{N}\sum k\pi_k$	(c)
$R = 1$	1.93	.39	$\pi_5 = .0367$
$R = 2$	3.01	.60	$2\pi_5 + \pi_4 = .4974$

4.2 $\theta_0 = 1,\ \theta_1 = \frac{\lambda}{\mu},\ \theta_2 = \left(\frac{\lambda}{\mu}\right)^2, \ldots, \theta_k = \left(\frac{\lambda}{\mu}\right)^k$

$$\sum_{k=0}^{\infty}\theta_k = \sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k = \frac{1}{1-\lambda/\mu} \quad \text{if } \lambda < \mu \ (= \infty \text{ if } \lambda \geq \mu)$$

If $\lambda < \mu$, then $\pi_k = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k$ for $k \geqq 0$. This is the geometre distribution. The model corresponds to the M/M/1 queue. If $\lambda \geqslant \mu$, then $\pi_k = 0$ for all k.

4.3 Repairman Model $M = N = 5$, $R = 1$, $\mu = .2$, $\lambda = .5$

k	=	0	1	2	3	4	5
λ_k	=	.5	.5	.5	.5	.5	0
μ_k	=	0	.2	.4	.6	.8	.10
θ_k	=	1	$\frac{5}{2}$	$\frac{25}{8}$	$\frac{125}{48}$	$\frac{625}{384}$	$\frac{625}{768}$
π_k	=	$\frac{768}{8963}$	$\frac{1920}{8963}$	$\frac{2400}{8963}$	$\frac{2000}{8963}$	$\frac{1250}{8963}$	$\frac{625}{8963}$
	=	.086	.214	.268	.223	.139	.070

Fraction of time repairman is idle $= \pi_5 = .07$

4.4 There are $\binom{4}{2} = 6$ possible links, (a) $\lambda_k = \alpha(6-k)$ and $\mu_k = \beta k$ for $0 \leq k \leq 6$, $\theta_0 = 1$, $\theta_1 = 6\left(\frac{\alpha}{\beta}\right)$;

$\theta_2 = \frac{6\cdot 5}{1\cdot 2}\left(\frac{\alpha}{\beta}\right)^2, \ldots \theta_k = \binom{6}{k}\left(\frac{\alpha}{\beta}\right)^k$

$$\sum_{k=0}^{6}\theta_k = \left(1 + \frac{\alpha}{\beta}\right)^6 = \left(\frac{\alpha+\beta}{\beta}\right)^6$$

$$\pi_k = \binom{6}{k}\left(\frac{\alpha}{\beta}\right)^k\left(\frac{\beta}{\alpha+\beta}\right)^6 = \binom{6}{k}\left(\frac{\alpha}{\alpha+\beta}\right)^k\left(\frac{\beta}{\alpha+\beta}\right)^{6-k},\ 0 \leq k \leq 6$$

Binomial distribution. See derivation of (4.8).

4.5 If $X(t) = k$, there are $N - k$ unbonded A molecules and an equal number of unbonded B molecules. $\lambda_k = \alpha(N-k)^2$, $\mu_k = \beta k$.

4.6 This model can be for mulated as a "Repairman model".

k	=	0	1	2	3
λ_k	=	λ	λ	λ	0
μ_k	=	0	μ	2μ	2μ
θ_k	=	1	$\left(\frac{\lambda}{\mu}\right)$	$\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2$	$\frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3$

The computer is fully loaded in states $k = 2$ and 3,

$$\pi_2 + \pi_3 = \frac{\frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3}.$$

4.7 The repairman model with $N = 3$, $R = 2$

$M = 2 \quad \mu = \frac{1}{5} = .2, \quad \lambda = \frac{1}{4} = .25$

k	0	1	2	3
λ_k	$\frac{2}{4}$	$\frac{2}{4}$	$\frac{1}{4}$	0
μ_k	0	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
θ_k	1	$\frac{5}{2}$	$\frac{25}{8}$	$\frac{125}{64}$
π_k	$\frac{64}{549}$	$\frac{160}{549}$	$\frac{200}{549}$	$\frac{125}{549}$

$$\sum \theta_k = 1 + \tfrac{5}{2} + \tfrac{25}{8} + \tfrac{125}{64} + \tfrac{549}{64}.$$

Long run avg. # Machines operating $= 0\pi_0 + 1\pi_1 + 2\pi_2 + 2\pi_3 = \frac{810}{549} = 1.48$; Avg. Output = 148 items per hour.

4.8 $\theta_0 = 1$

$$\theta_1 = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)$$

$$\theta_2 = \frac{1}{3}\left(\frac{\alpha}{\beta}\right)^2$$

$$\theta_k = \frac{1}{k+1}\left(\frac{\alpha}{\beta}\right)^k, \quad k = 0,1,\ldots$$

Because $\log(1-x) = -\left[x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right] = -\sum_{k=1}^{\infty}\frac{1}{k}x^k$ for $|x| < 1$, we can evaluate

$$\sum_{k=0}^{\infty}\theta_k = \frac{\beta}{\alpha}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(\frac{\alpha}{\beta}\right)^{k+1} = \frac{\beta}{\alpha}\log\frac{1}{1-\alpha/\beta}, \quad 0 < \alpha < \beta,$$

and

$$\pi_k = C\left(\frac{1}{k+1}\right)\left(\frac{\alpha}{\beta}\right)^{k+1}, C = \frac{1}{\log\left(\frac{1}{1-\frac{\alpha}{\beta}}\right)}.$$

This is named the "logarithmic distribution".

5.1 Change K into an absorbing state by setting $\lambda_K = \mu_K = 0$. Then $Pr\{\text{Absorption in } 0\} = Pr\{\text{Reach } 0 \text{ before } K\}$. Because $\varrho_i = 0$ for $i \geqq K$, (5.7) becomes $\mu_m = \dfrac{\sum_{i=m}^{K-1}\varrho_i}{1 + \sum_{i=1}^{K-1}\varrho_i}$, as desired.

5.2 $\varrho_0 = 1, \varrho_1 = 4, \varrho_2 = 6, \varrho_3 = 4, \varrho_4 = 1, \varrho_5 = 0$

$$u_2 = \frac{\varrho_2 + \varrho_3 + \varrho_4}{1 + \varrho_1 + \varrho_2 + \varrho_3 + \varrho_4} = \frac{11}{16}$$

(a) Alternatively, solve

$$\left.\begin{aligned} u_1 &= \tfrac{4}{5} + \tfrac{1}{5}u_2 \\ u_2 &= \tfrac{3}{5}u_1 + \tfrac{2}{5}u_3 \\ u_3 &= \tfrac{2}{5}u_2 + \tfrac{3}{5}u_4 \\ u_4 &= \tfrac{1}{5}u_3 \end{aligned}\right\} \quad \begin{aligned} u_1 &= \tfrac{15}{16} \\ u_2 &= \tfrac{11}{16} \\ u_3 &= \tfrac{5}{16} \\ u_4 &= \tfrac{1}{16} \end{aligned}$$

(b) $$\left.\begin{aligned} w_1 &= \tfrac{1}{5} + \tfrac{1}{5}w_2 \\ w_2 &= \tfrac{1}{5} + \tfrac{3}{5}w_1 + \tfrac{2}{5}w_3 \\ w_3 &= \tfrac{1}{5} + \tfrac{2}{5}w_2 + \tfrac{3}{5}w_4 \\ w_4 &= \tfrac{1}{5} + \tfrac{1}{5}w_3 \end{aligned}\right\} \quad \begin{aligned} w_1 &= w_4 = \tfrac{1}{3} \\ w_2 &= w_3 = \tfrac{2}{3}. \end{aligned}$$

6.1 For $i \neq j$, $Pr\{X(t+h) = j | X(t) = i\} = \lambda P_{ij}h + o(h)$ for $h \approx 0$, The independent increments of the Poisson process are needed to establish the Markov property.

6.2 Fraction of time system operates $= \left(\dfrac{\alpha_A}{\alpha_A + \beta_A}\right)\left[1 - \left(\dfrac{\beta_B}{\alpha_B + \beta_B}\right)\left(\dfrac{\beta_C}{\alpha_C + \beta_C}\right)\right]$.

6.3 $Pr\{Z(t+h) = k+1 | Z(t) = k\} = (N-k)\lambda h + o(h)$,

$Pr\{Z(t+h) = k-1 | Z(t) = h\} = k\mu h + o(h)$.

6.4 The infinitesimal matrix is

$$\begin{array}{c} \\ (2,0) \\ (1,0) \\ (1,1) \\ (0,0) \\ (0,1) \\ (0,2) \end{array}
\begin{array}{c} \begin{array}{cccccc} (2,0) & (1,0) & (1,1) & (0,0) & (0,1) & (0,2) \end{array} \\
\left\|\begin{array}{cccccc} -2\mu & 2\mu p & 2\mu q & & & \\ \alpha & -(\alpha+\mu) & & \mu p & \mu q & \\ \beta & & -(\beta+\mu) & & \mu p & \mu q \\ & \alpha & & -\alpha & & \\ & \beta & \alpha & & -(\alpha+\beta) & \\ & & \beta & & & -\beta \end{array}\right\| \end{array}$$

7.1 (a) $(1-\pi)P_{01}(t) + \pi P_{11}(t) = (1-\pi)\pi(1 - e^{-\tau t}) + \pi[\pi + (1-\pi)e^{-\tau t}] = \pi - \pi^2 + \pi^2 = \pi$.

(b) Let $N(dt) = 1$ if there is an event in $(t, t+dt]$, and zero otherwise. Then $N((0,t]) = \int_0^t N(ds)$

and $E[N((0,t])] = E\left[\int_0^t N(ds)\right] = \int_0^t E[N(ds)] = \int_0^t \pi\lambda ds = \pi\lambda t$.

7.2 $\gamma(t) > x$ if and only if $N((t, t+x]) = 0$. (Draw picture)

7.3 $T > t$ if and only if $N((0,t]) = 0$. $Pr\{T > t\} = Pr\{N((0,t]) = 0\} = f(t;\lambda)$ whence $\phi(t) = -\dfrac{d}{dt}f(t;\lambda)$

$= c_+\mu_+e^{-\mu_+t} + c_-\mu_-e^{-\mu_-t}$. When $\alpha = \beta = 1$ and $\lambda = 2$, then $\mu_\pm = 2 \pm \sqrt{2}$, $c_\pm = \dfrac{1}{4}\left(2 \mp \sqrt{2}\right)$ and $\phi(t) =$

$e^{-2t}\dfrac{e^{-\sqrt{2}t} + e^{\sqrt{2}t}}{2} = e^{-2t}\cosh\sqrt{2}\,t$.

7.4 $E[T] = \int_0^\infty Pr\{T > t\}dt = \int_0^\infty f(t;\lambda)dt = \frac{c_+}{\mu_+} + \frac{c_-}{\mu_-}$. When $\alpha = \beta = 1$ and $\lambda = 2$, then

$$E[T] = \frac{1}{4}\left\{\frac{2-\sqrt{2}}{2+\sqrt{2}} + \frac{2+\sqrt{2}}{2-\sqrt{2}}\right\} = \frac{12}{8} = \frac{3}{2}.$$

Since events occur, on average, at a rate of $\pi\lambda$ per unit time, the mean duration between events must be $\frac{1}{\pi\lambda}$ (= 1 when $\alpha = \beta = 1$ and $\lambda = 2$). The discrepancy between the mean time to the first event, and the mean time between events, is another example of length biased sampling.

7.5 $Pr\left\{N\left((t, t+s]\right) = 0 \middle| N\left((0,t]\right) = 0\right\}$

$$= \frac{f(t+s;\lambda)}{f(t;\lambda)} = \frac{c_+e^{-\mu_+(t+s)} + c_-e^{-\mu_-(t+s)}}{c_+e^{-\mu_+t} + c_-e^{-\mu_-t}}$$

$$= \frac{c_+e^{-(\mu_+-\mu_-)t}}{c_+e^{-(\mu_+-\mu_-)t} + c_-}e^{-\mu_+s} + \frac{c_-}{c_+e^{-(\mu_+-\mu_-)t} + c_-}e^{-\mu_-s} \to e^{-\mu_-s} \text{ as } t \to \infty.$$

7.6 $\int_0^\infty e^{-st}ce^{-\mu t}dt = \frac{c}{\mu + s}$ whence

$$\varphi(s;\lambda) = \frac{c_+}{\mu_+ + s} + \frac{c_-}{\mu_- + s} = \frac{c_+\mu_- + c_-\mu_+ + s}{(\mu_+ + s)(\mu_- + s)}$$

$$\mu_\pm = \frac{1}{2}\left\{(\lambda + \tau) \pm \sqrt{(\lambda + \tau)^2 - 4\pi\tau\lambda}\right\}$$

$$c_+\mu_- + c_-\mu_+ = \tau + (1-\pi)\lambda.$$

$$(\mu_+ + s)(\mu_- + s) = s^2 + (\tau + \lambda)s + \pi\tau\lambda.$$

(a) $\lim_{\tau\to\infty}\phi(s;\lambda) = \frac{1}{\pi\lambda + s} = \int_0^\infty e^{-st}e^{-\pi\lambda t}dt$

(b) $\lim_{\tau\to 0}\phi(s;\lambda) = \frac{s + (1-\pi)\lambda}{s^2 + \lambda s} = \pi\left(\frac{1}{\lambda + s}\right) + (1-\pi)\frac{1}{s}.$

7.7 When $\alpha = \beta = 1$ and $\lambda = 2(1-\theta)$ then

$$\mu_\pm = (z - \theta) \pm R, \quad c_\pm = \frac{1}{2} \mp \frac{1}{2R}$$

and

$$g(t;\theta) = e^{-(2-\theta)t}\left\{\left(\frac{1}{2} - \frac{1}{2R}\right)e^{-Rt} + \left(\frac{1}{2} + \frac{1}{2R}\right)e^{Rt}\right\} = e^{-(2-\theta)t}\left\{\cosh Rt + \frac{1}{R}\sinh Rt\right\}.$$

$$\frac{dg}{d\theta} = e^{-(2-\theta)t}t\left\{\cosh Rt + \frac{1}{R}\sinh Rt\right\} + e^{-(2-\theta)t}\left\{t\sinh Rt + \frac{t}{R}\cosh Rt - \frac{1}{R^2}\sinh Rt\right\}\frac{dR}{d\theta}$$

$$R\big|_{\theta=0} = \sqrt{2}, \quad \left.\frac{dR}{d\theta}\right|_{\theta=0} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

$$\left.\frac{dg}{d\theta}\right|_{\theta=0} = e^{-2t}t\left\{\cosh\sqrt{2}t + \frac{1}{2}\sqrt{2}\sinh\sqrt{2}t\right\} - e^{-2t}t\left[\left\{\sinh\sqrt{2}t + \frac{1}{2}\sqrt{2}\cosh\sqrt{2}t\right\} - \frac{1}{2}\sinh\sqrt{2}t\right]\frac{\sqrt{2}}{2}$$

$$= \frac{1}{2}e^{-2t}\left\{t\cosh\sqrt{2}t + \frac{\sqrt{2}}{2}\sinh\sqrt{2}t\right\} = Pr\left\{N\left((0,t]\right) = 1\right\}.$$

Students with symbol manipulating computer skills may test them on $Pr\{N((0,t]) = 2\} = \left.\frac{1}{2}\frac{d^2g}{d\theta^2}\right|_{\theta=0}$.

7.8 $Pr\left\{\lambda(t) = 0 \middle| N\left((0,t]\right) = 0\right\} = \dfrac{Pr\left\{\lambda(t) = 0 \text{ and } N\left((0,t]\right) = 0\right\}}{Pr\left\{N\left((0,t]\right) = 0\right\}} = \dfrac{f_0(t)}{f(t)} \to \dfrac{a_-}{c_-}$ as $t \to \infty$.

7.9 The initial conditions are easy to verify: $f_0(0) = a_+ + a_- = 1 - \pi$, $f_1(0) = b_+ + b_- = \pi$. We wish to check that

$$f_0'(t) = -\alpha f_0(t) + \beta f_1(t)$$
$$f_1'(t) = \alpha f_0(t) - (\beta + \lambda)f_1(t)$$

Now

$$f_0'(t) = -a_+\mu_+e^{-\mu_+t} - a_-\mu_-e^{-\mu_-t}$$

while

$$-\alpha f_0(t) = -\alpha a_+e^{-\mu_+t} - \alpha a_-e^{-\mu_-t}$$
$$\beta f_1(t) = \beta b_+e^{-\mu_+t} + \beta b_-e^{-\mu_-t}$$

Upon equating coefficients, we want to check that $-a_+\mu_+ \stackrel{?}{=} -\alpha a_+ + \beta b_+$ and $-a_-\mu_- \stackrel{?}{=} -\alpha a_- + \beta b_-$. Starting with the first, is $0 \stackrel{?}{=} a_+(\mu_+ - \alpha) + \beta b_+$

$$a_+(\mu_+ - \alpha) = \frac{1}{4}(1 - \pi)\left[(\lambda - \alpha + \beta) + R - \frac{(\lambda - \alpha + \beta)(\lambda + \alpha + \beta)}{R} - (\alpha + \beta + \lambda)\right]$$

$$= \frac{1}{4}(1 - \pi)\left[-2\alpha + R - \frac{(\lambda - \alpha + \beta)(\lambda + \alpha + \beta)}{R}\right]$$

$$\beta b_+ = \frac{1}{2}\pi\beta\left[1 + \frac{\lambda - \alpha - \beta}{R}\right], \quad \pi = \frac{\alpha}{\alpha + \beta}$$

$$2R(\alpha + \beta)\left[a_+(\mu_+ - \alpha) + \beta b_+\right]$$
$$= \beta R^2 - \beta(\lambda - \alpha + \beta)(\lambda + \alpha + \beta) + 2\alpha\beta(\lambda - \alpha - \beta)$$
$$= \beta\left[(\alpha + \beta + \lambda)^2 - 4\alpha\lambda - (\lambda - \alpha + \beta)(\lambda + \alpha + \beta) + 2\alpha(\lambda - \alpha - \beta)\right]$$
$$= \beta\left[(\alpha + \beta + \lambda)\{\alpha + \beta + \lambda - \lambda + \alpha - \beta\} - 4\alpha\lambda + 2\alpha(\lambda - \alpha - \beta)\right]$$
$$= \beta\left[2\alpha(\alpha + \beta + \lambda) - 4\alpha\lambda + 2\alpha(\lambda - \alpha - \beta)\right]$$
$$= \beta\alpha\left[2(\alpha + \beta) + 2(-\alpha - \beta)\right] = 0 \quad \checkmark\checkmark$$

To check: $0 \stackrel{?}{=} a_-(\mu_- - \alpha) + \beta b_-$

$$a_-(\mu_- - \alpha) = \frac{1}{4}(1-\pi)\left[1 + \frac{\alpha+\beta+\lambda}{R}\right]\left[(\lambda-\alpha+\beta) - R\right]$$

$$\stackrel{?}{=} \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[\lambda - \alpha + \beta - R + \frac{(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)}{R} - \alpha - \beta - \lambda\right]$$

$$= \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[-2\alpha - R + \frac{(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)}{R}\right]$$

$$\beta b_- = \frac{1}{2}\frac{\alpha\beta}{\alpha+\beta}\left[1 - \frac{\lambda-\alpha-\beta}{R}\right]$$

$$= \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[2\alpha - \frac{2\alpha(\lambda-\alpha-\beta)}{R}\right]$$

$$a_-(\mu_- - \alpha) + \beta b_- = \frac{1}{4}\frac{\beta}{\alpha+\beta}\left[-R + \frac{(\alpha+\beta+\lambda)(\lambda-\alpha+\beta)}{R} - \frac{2\alpha(\lambda-\alpha-\beta)}{R}\right]$$

$$= \frac{1}{4R}\left(\frac{\beta}{\alpha+\beta}\right)\left[-(\alpha+\beta+\lambda)^2 + 4\alpha\lambda + (\alpha+\beta+\lambda)(\lambda-\alpha+\beta) - 2\alpha(\lambda-\alpha-\beta)\right]$$

$$= \frac{1}{4R}\left(\frac{\beta}{\alpha+\beta}\right)\left[4\alpha\lambda - 2\alpha(\alpha+\beta+\lambda) - 2\alpha(\lambda-\alpha-\beta)\right] = 0 \quad \checkmark$$

Verifying the second differential equation reduces to checking if $0 \stackrel{?}{=} \alpha a_+ + b_+(\mu_+ - \beta - \lambda)$ and $0 \stackrel{?}{=} \alpha a_- + b_-(\mu_- - \beta - \lambda)$. The algebra is similar to that used for the first equation.

7.10 (a) $Pr\{N(0,h] > 0, N(h,h+t] = 0\}$
$= Pr\{N(h,h+t] = 0\} - Pr\{N(0,h] = 0, N(h,h+t] = 0\} = f(t;\lambda) - f(t+h;\lambda).$

(b) $\lim_{h\downarrow 0} Pr\{N(h,h+t] = 0 | N(0,h] > 0\}$

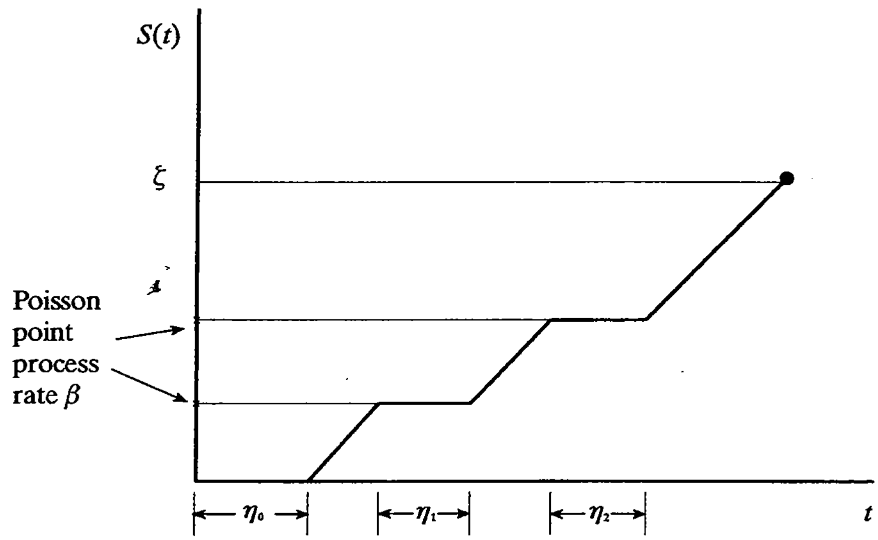
$$= \lim_{h\downarrow 0}\frac{\dfrac{f(t;\lambda) - f(t+h;\lambda)}{h}}{\dfrac{1-f(h;\lambda)}{h}} = \frac{-\dfrac{d}{dt}f(t;\lambda)}{-\dfrac{d}{dt}f(t;\lambda)\Big|_{t=0}} = \frac{f'(t;\lambda)}{f'(0;\lambda)}.$$

(c) $$\frac{f'(t;\lambda)}{f'(0;\lambda)} = \frac{-c_+\mu_+e^{-\mu_+t} - c_-\mu_-e^{-\mu_-t}}{-c_+\mu_+ - c_-\mu_-} = p_+e^{-\mu_+t} + p_-e^{-\mu_-t}.$$

(d) $Pr\{\tau > t \mid \textit{Event at time } 0\} = p_+e^{-\mu_+t} + p_-e^{-\mu_-t}$ whence $E[\tau \mid \text{Event at time } 0] = \dfrac{p_+}{\mu_+} + \dfrac{p_-}{\mu_-}$.
When $\alpha = \beta = 1$ and $\lambda = 2$, we have $\mu_\pm = 2 \pm \sqrt{2}$, $c_\pm = \frac{1}{4}(2 \mp \sqrt{2})$; $p_\pm = \frac{1}{2}$ and $E[\tau \mid \text{Event at time } 0] = 1. =$ Long run mean time between events.

7.11 $V(u)$ plays the role of $\lambda(u)$, and $S(t)$, that of $\Lambda(t)$.

7.12 This model arose in an attempt to describe a scientist observing butterflies. When the butterfly is at rest, it can be observed perfectly. When it is flying, it may fly out-of-range (the "event"). The observation time is random, and the model may guide us in summarizing the data. The stated properties should be clear from the picture.



CHAPTER VII

1.1 $$Pr\{N(t-x)=N(t+y)\}=\sum_{k=0}^{\infty}Pr\{N(t-x)=k=N(t+y)\}$$
$$=\sum_{k=0}^{\infty}Pr\{W_k\le t-x,\ W_{k+1}>t+y\}$$
$$=[1-F(t+y)]+\sum_{k=1}^{\infty}Pr\{W_k\le t-x,\ W_k+X_{k+1}>t+y\}$$
$$=[1-F(t+y)]+\sum_{k=1}^{\infty}\int_0^{t-x}[1-F(t+y-z)]\,dF_k(z).$$

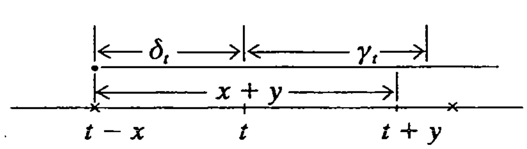
In the exponential case:

$$=e^{-\lambda(t+y)}+\sum_{k=1}^{\infty}\int_0^{t-x}e^{-\lambda(t+y-z)}\frac{\lambda^k z^{k-1}}{(k-1)!}e^{-\lambda z}dz$$
$$=e^{-\lambda(t+y)}+\int_0^{t-x}e^{-\lambda(t+y-z)}\lambda e^{\lambda z}e^{-\lambda z}dz$$
$$=e^{-\lambda(t+y)}+e^{-\lambda(t+y)}\left[e^{\lambda(t-x)}-1\right]=e^{-\lambda(x+y)}.$$

1.2 $$Pr\{N(t)=k\}=Pr\{W_k\le t,\ W_k+X_{k+1}>t\}$$
$=\int_0^t[1-F(t-x)]\,dF_k(\lambda)$, and in the exponential case
$$=\lambda^k e^{-\lambda t}\int_0^t\frac{x^{k-1}}{k!}dx=\frac{(\lambda t)^k e^{-\lambda t}}{k!},\quad k=0,1,\ldots$$

1.3 $$E[\gamma_t]=E\left[W_{N(t)+1}-t\right]=E[X_1][M(t)+1]-t.$$

1.4 $$Pr\{\gamma_t>y\mid\delta_t=x\}=\frac{1-F(x+y)}{1-F(x)}$$



$$E[\gamma_t\mid\delta_t=x]=\int_0^{\infty}\frac{1-F(x+y)}{1-F(x)}dy.$$

2.1 Block Period $\quad \text{Cost} = \dfrac{4+5M(K-1)}{K}$

K	$\Theta(K)$
1	4.00
2	2.25
3	2.183
4	2.1138
5	2.1231

*Replace on failure: $\Theta = 1.923$, is best

2.2 We are asked to verify that $M(n)$, as given, satisfies

$$\begin{aligned}
M(n) &= 1+(1-q)M(n-1)+qM(n-2), \quad n \geqq 2.\\
1 &= 1\\
(1-q)M(n-1) &= \frac{(1-q)(n-1)}{1+q} - \frac{(1-q)q^2}{(1+q)^2} - \frac{(1-q)q^{n+1}}{(1+q)^2}\\
qM(n-2) &= \frac{q(n-2)}{1+q} - \frac{qq^2}{(1+q)^2} + \frac{qq^n}{(1+q)^2}\\
\hline
M(n) &\stackrel{?}{=} 1 - \frac{q^2}{(1+q)^2} + \frac{(1-q)(n-1)+q(n-2)}{1+q} - \frac{(1-q)q^{n+1}-q^{n+1}}{(1+q)^2}\\
&= 1 - \frac{1-q+2q}{1+q} - \frac{q^2}{(1+q)^2} + \frac{n}{1+q} + \frac{qq^{n+1}}{(1+q)^2}\\
&= \frac{n}{1+q} - \frac{q^2}{(1+q)^2} + \frac{q^{n+2}}{(1+q)^2} = M(n)
\end{aligned}$$

2.3 $M(1) = p_1 = \beta$

$M(2) = p_1 + p_2 + p_1M(1) = \beta + \beta(1-\beta) + \beta^2 = 2\beta$

$M(3) = p_1 + p_2 + p_3 + p_1M(2) + p_2M(1)$

$= \beta + \beta(1-\beta) + \beta(1-\beta)^2 + \beta(2\beta) + \beta(1-\beta)\beta$

$= \beta + \beta - \beta^2 + \beta - 2\beta^2 + \beta^3 + 2\beta^2 + \beta^2 - \beta^3 = 3\beta$

In general, $M(n) = n\beta$.

3.1 $E\left[\dfrac{1}{N(t)+1}\right] = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k+1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} = \frac{1}{\lambda t} e^{-\lambda t} \sum_{j=1}^{\infty} \frac{(\lambda t)^j}{j!} = \frac{1}{\lambda t} e^{-\lambda t}\left(e^{\lambda t} - 1\right).$

Using the independence established in Exercise 3.3,

$$E\left[\frac{W_{N(t)+1}}{N(t)+1}\right] = E\left[W_{N(t)+1}\right] E\left[\frac{1}{N(t)+1}\right] = \left(t + \frac{1}{\lambda}\right)\frac{1}{\lambda t}\left(1 - e^{-\lambda t}\right) = \frac{1}{\lambda}\left(1 - \frac{1}{\lambda t}\right)\left(1 - e^{-\lambda t}\right).$$

3.2 Because $W_{N(t)+1} = t + \gamma_t$ and $E[\gamma_t] = \frac{1}{\lambda}$, $E[W_{N(t)+1}] = t + \frac{1}{\lambda}$. On the other hand $E[X_1] = \frac{1}{\lambda}$ and $M(t) = \lambda t$, whence

$$E[X_1]\{M(t)+1\} = \frac{1}{\lambda}(\lambda t + 1) = t + \frac{1}{\lambda} = E[W_{N(t)+1}].$$

3.3 For $t > \tau$, $p(t) = Pr\{\text{No arrivals in } (t-\tau, t]\} = e^{-\lambda\tau}$. For $t \le \tau$, $p(t) = Pr\{\text{No arrivals in } (0, t]\} = e^{-\lambda t}$. Thus $p(t) = e^{-\lambda \min\{\tau, t\}}$.

3.4 Let $l(x, y) = \begin{cases} x & \text{if } 0 \le y \le x \le 1 \\ 1-x & \text{if } 0 \le x < y \le 1. \end{cases}$

$$\begin{aligned} E[L] &= E[l(X, Y)] = \int_0^1\int_0^1 l(xy)\,dx\,dy \\ &= \int_0^1 x\left(\int_0^x dy\right)dx + \int_0^1 (1-x)\left(\int_x^1 dy\right)dx = \int_0^1 x^2\,dx + \int_0^1 (1-x)^2\,dx = \frac{2}{3}. \end{aligned}$$

3.5 $Pr[D(t) > x] = Pr\{\text{No birds in } (t-x, t+x]\} = \begin{cases} e^{-2\lambda x} & \text{for } 0 < x < t \\ e^{-\lambda(x+t)} & \text{for } 0 < t \le x. \end{cases}$

$$E[D(t)] = \int_0^\infty Pr\{D(t) > x\}dx = \int_0^t e^{-2\lambda x}dx + \int_t^\infty e^{-\lambda(x+t)}dx = \frac{1}{2\lambda}\left(1 + e^{-2\lambda t}\right).$$

$$f_T(t) = -\frac{d}{dx}Pr\{D(t) > x\} = \begin{cases} 2\lambda e^{-2\lambda x} & \text{for } 0 < x < t \\ \lambda e^{-\lambda(x+t)} & \text{for } 0 < t \le x. \end{cases}$$

4.1 $\frac{1}{t}M(t) \to \frac{1}{\mu}$ implies $\mu = 1$

$$M(t) - \frac{t}{\mu} \to \frac{\sigma^2 - \mu^2}{2\mu^2} = 1 \quad \text{implies} \quad \sigma^2 = 3$$

4.2 $\mu = \frac{1}{\alpha} + \frac{1}{\beta}$; $\frac{1}{\mu} = \frac{\alpha\beta}{\alpha + \beta}$; $\sigma^2 = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$$M(t) \approx \frac{\alpha\beta t}{\alpha + \beta} + \frac{\sigma^2 - \mu^2}{2\mu^2} = \frac{\alpha\beta t}{\alpha + \beta} + \frac{\alpha\beta}{(\alpha + \beta)^2}.$$

4.3 $1 - F(y) = \exp\left\{-\int_0^y \theta x\,dx\right\} = e^{-\frac{1}{2}\theta y^2}, \quad y \geqq 0$

$$\mu = \int_0^\infty [1 - F(y)]dy = \int_0^\infty e^{-\frac{1}{2}\theta y^2}dy = \sqrt{\frac{\pi}{2\theta}}.$$

$$\sigma^2 + \mu^2 = \int_0^\infty y^2 dF(y) = \frac{2}{\theta}.$$

Limiting mean age $= \frac{\sigma^2 + \mu^2}{2\mu} = \sqrt{\frac{2}{\pi\theta}}$.

4.4 List the children, family after family, to form a renewal process.

Children	MF	MMMF	F	F	MF
Family	#1	#2	#3	#4	#5

$\mu = 2 = E$[#children in a family]. 1 female per family. Long run fraction female $= \dfrac{1}{\mu} = \dfrac{1}{2}$.

4.5 $m_0 = 1 + .3m_0 \quad \Rightarrow \quad m_0 = \dfrac{1}{.7} = \dfrac{10}{7}$

$m_2 = 1 + .5m_2 \quad \Rightarrow \quad m_2 = 2$

Successive visits to state 1 form renewal instants for which $\mu = 1 + .6m_0 + .4m_2 = \frac{93}{35}$. And $\pi_1 = \frac{35}{93}$ $(\pi_0 = \frac{30}{93}, \pi_2 = \frac{28}{93})$.

5.1 (a) $Pr\{\text{A down}\} = \dfrac{\beta_A}{\alpha_A + \beta_A}$; $Pr\{\text{B down}\} = \dfrac{\beta_B}{\alpha_B + \beta_B}$.

$$Pr\{\text{System down}\} = \left(\frac{\beta_A}{\alpha_A + \beta_A}\right)\left(\frac{\beta_B}{\alpha_B + \beta_B}\right).$$

(b) System leaves the failed state upon first component repair. E[Sojourn System down] $= \dfrac{1}{\alpha_A + \alpha_B}$.

(c) $Pr[\text{System down}] = \dfrac{E[\text{Sojourn System Down}]}{E[\text{Cycle}]}$ so

$$E[\text{Cycle}] = \frac{1/(\alpha_A + \alpha_B)}{\left(\frac{\beta_A}{\alpha_A + \beta_A}\right)\left(\frac{\beta_B}{\alpha_B + \beta_B}\right)}.$$

(d) $E[\text{System Sojourn Up}] = E[\text{Cycle}] - E[\text{Sojourn down}]$

$$= \left(\frac{1}{\alpha_A + \alpha_B}\right)\left\{\frac{(\alpha_A + \beta_A)(\alpha_B + \beta_B)}{\beta_A \beta_B} - 1\right\}$$

5.2 $\Pr\{X > y \mid X > x\} = \dfrac{1 - F(y)}{1 - F(x)}, \quad 0 < x < y$

$$E[X \mid X > x] = \int_0^\infty Pr\{X > y \mid X > x\} dy = x + \int_x^\infty \left\{\frac{1 - F(y)}{1 - F(x)}\right\} dy$$

5.3 If the successive fees are $Y_1, Y_2, \ldots$ then $W(t) = \sum_{k=1}^{N(t)+1} Y_k$ when $N(t)$ is the number of customers arriving in $(0, t]$.

$$\lim_{t \to \infty} \frac{E[W(t)]}{t} = \frac{E[Y]}{E[X]} = \frac{\int_0^\infty [1 - G(y)] dy}{\int_0^\infty [1 - F(x)] dx}$$

5.3 The duration of the cycle in the obvious renewal process is the maximum of the two lives. The cycle duration when both are lit is the minimum. Let the bulb lives be ξ_1, ξ_2. The fraction time half lit

$$= 1 - \frac{E[\min\{\xi_1, \xi_2\}]}{E[\max\{\xi_1, \xi_2\}]}$$

(a) $E\left[\min\left\{\xi_1,\xi_2\right\}\right]=\frac{1}{2\lambda}, E\left[\max\right]=\frac{1}{2\lambda}+\frac{1}{\lambda};\quad Pr\left[\text{Half lit}\right]=\frac{2}{3}$

(b) $E\left[\min\right]=\frac{1}{3}, E\left[\max\right]=\frac{2}{3};\quad Pr\left[\text{Half lit}\right]=\frac{1}{2}.$

6.1 (a) $u_0=1$

$$u_1=p_1u_0=\alpha$$
$$u_2=p_2u_0+p_1u_1=\alpha(1-\alpha)+\alpha^2=\alpha$$
$$u_3=p_3u_0+p_2u_1+p_1u_2=\alpha(1-\alpha)^2+\alpha\left[\alpha+\alpha(1-\alpha)\right]=\alpha$$

We guess $u_n=\alpha$ for all n

so
$$\begin{aligned}\alpha &\overset{?}{=} \alpha(1-\alpha)^{n-1}+\alpha\left(p_1+p_2+\ldots+p_{n-1}\right)\\ &=\alpha(1-\alpha)^{n-1}+\alpha^2\left(1+(1-\alpha)+\ldots+(1-\alpha)^{n-2}\right)\\ &=\alpha(1-\alpha)^{n-1}+\alpha\left(1-(1-\alpha)^{n-1}\right)=\alpha\end{aligned}$$

(b) For excess life $b_n=p_{n+m}=\alpha(1-\alpha)^{n+m-1}=p_m(1-\alpha)^n$

$$V_n=\sum_{k=0}^{n}b_{n-k}u_k=p_m(1-\alpha)^n+\alpha\sum_{k=1}^{n}(1-\alpha)^{n-k}p_m=p_m\left[(1-\alpha)^n+1-(1-\alpha)^n\right]=p_m.$$

6.2 We want $Pr\{\gamma_n=3\}$ for $n=10$ $p_1=p_2=\ldots=p_6=\frac{1}{6}$

$$\begin{aligned}
f_0&=\frac{1}{6} &&=.16666\\
f_1&=\frac{1}{6}+\frac{1}{6}f_0 &&=.19444\\
f_2&=\frac{1}{6}+\frac{1}{6}(f_0+f_1) &&=.22685\\
f_3&=\frac{1}{6}+\frac{1}{6}(f_0+f_1+f_2) &&=.26466\\
f_4&=\frac{1}{6}(f_0+f_1+f_2+f_3) &&=.1421\\
f_5&=\frac{1}{6}(f_0+f_1+f_2+f_3+f_4) &&=.16578\\
f_6&=\frac{1}{6}(f_0+f_1+f_2+f_3+f_4+f_5) &&=.1934\\
f_7&=\frac{1}{6}(f_1+f_2+f_3+f_4+f_5+f_6) &&=.197878\\
f_8&=\frac{1}{6}(f_2+f_3+f_4+f_5+f_6+f_7) &&=.198450\\
f_9&=\frac{1}{6}(f_3+f_4+f_5+f_6+f_7+f_8) &&=.1937166\\
f_{10}&=\frac{1}{6}(f_4+f_5+f_6+f_7+f_8+f_9) &&=.181892637\ .
\end{aligned}$$

$$\lim_{n\to\infty} Pr\{\gamma_n = 3\} = \frac{Pr\{X_1 \geqq 3\}}{E[X_1]} = \frac{2/3}{7/2} = \frac{4}{21} = .190476$$

(Compare with problem 5.4 in III. The renewal theory gives us the limit)

6.3 That delaying the age of first birth, even at constant family size, would lower population growth rates, was an important conclusion in early work with this model. $\sum_{\nu=0}^{\infty} m_\nu s^\nu = 2s^2 + 2s^3 = 1$ solves to give $s = .5652$ whence $\lambda = \frac{1}{s} = 1.77$ compared to $\lambda = 2.732$ in the example.

6.4 (a) $\sum_{\nu=0}^{\infty} m_\nu s^\nu = a\sum_{\nu=2}^{\infty} s^\nu = \frac{as^2}{1-s}, \quad |s| < 1.$

$$\sum m_\nu s^\nu = 1 \quad \text{solves to give} \quad s = \frac{-1 \pm \sqrt{1+4a}}{2a}, \ \lambda_1 = \frac{2a}{\sqrt{1+4a}-1}$$

(b) $\sum m_\nu s^\nu = as^2 = 1$ solves to give $s = \sqrt{\frac{1}{a}}$ and $\lambda_2 = \sqrt{a}$. Compare $\lambda_1 = \frac{4}{\sqrt{9}-1} = 2$ when $a = 2$ versus $\lambda_2 = 2$ when $a = 4$. That is 4 offspring at age 2 is equivalent to 2 offspring repeatedly.

CHAPTER VIII

1.1 Let $T_n = \min\{t \geqq 0;\ \ B_n(t) \leqq -a \ \text{ or } \ B_n(t) \geqq b\}$

$$= \min\{t \geqq 0;\ \ S_{[nt]} \leqq -a\sqrt{n} \ \text{ or } \ S_{[nt]} \geqq b\}$$

$$= \frac{1}{n}\min\{k \geqq 0;\ \ S_k \leqq -a\sqrt{n} \ \text{ or } \ S_k \geqq b\sqrt{n}\}$$

$$E[T_n] = \frac{1}{n}[a\sqrt{n}][b\sqrt{n}] \quad \text{(III, Section 5.3)}$$

$$\to ab \quad \text{as} \quad n \to \infty.$$

1.2 Begin with $1 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\lambda\sqrt{t})^2} dx = e^{-\frac{1}{2}\lambda^2 t} \int_{-\infty}^{+\infty} e^{\lambda\sqrt{t}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$$= e^{-\frac{1}{2}\lambda^2 t} \int_{-\infty}^{+\infty} e^{\lambda y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2/t} dy = e^{-\frac{1}{2}\lambda^2 t} E\left[e^{\lambda B(t)}\right] \quad \text{so} \quad E\left[e^{\lambda B(t)}\right] = e^{\frac{1}{2}\lambda^2 t}.$$

1.3 $Pr\left\{\frac{|B(t)|}{t} > \varepsilon\right\} = Pr\{|B(t)| > \varepsilon t\} = 2\{1 - \Phi_t(\varepsilon t)\} = 2\{1 - \Phi(\varepsilon\sqrt{t})\}$ (1.8)

$$Pr\left\{\frac{|B(t)|}{t} > \varepsilon\right\} \to 0 \quad \text{as} \quad t \to \infty \quad \text{because} \quad \Phi(\varepsilon\sqrt{t}) \to 1$$

$$Pr\left\{\frac{|B(t)|}{t} > \varepsilon\right\} \to 1 \quad \text{as} \quad t \to 0 \quad \text{because} \quad \Phi(\varepsilon\sqrt{t}) \to \frac{1}{2}.$$

1.4 Every linear combination of jointly normal random variables is normally distributed.

$$E\left[\sum_{i=1}^n \alpha_i B(t_i)\right] = \sum \alpha_i E[B(t_i)] = 0$$

$$Var\left[\sum_{i=1}^{n} \alpha_i B(t_i)\right] = E\left[\left\{\sum \alpha_i B(t_i)\right\}^2\right] = E\left[\left\{\sum_i \alpha_i B(t_i)\right\}\left\{\sum_j \alpha_j B(t_j)\right\}\right]$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j E\left[B(t_i)B(t_j)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \min\{t_i, t_j\}.$$

1.5 (a) $Pr\{M_\tau = 0\} = Pr\{S_n \text{ drops to } -a < 0 \text{ before rising 1 unit}\} = \dfrac{1}{1+a}$ (Using III, 5.3).

(b) In order that $M_\tau \geqq 2$, we must first have $M_\tau \geqq 1$ followed by moving ahead to 2, starting afresh from $S' = 1$, before dropping a units below 1. The spatial homogeneity gives this second move the same probability as $Pr\{M(\tau) \geqq 1\}$. Thus $Pr\{M_\tau \geqq 2\} = [Pr\{M_\tau \geqq 1\}]^2 = \left(\dfrac{a}{1+a}\right)^2$. The argument repeats to give $Pr\{M_\tau \geqq k\} = \left(\dfrac{a}{1+a}\right)^k$.

(c) Divide the state space into increments of length $\dfrac{1}{n}$ and observe the Brownian motion as it crosses lines $\dfrac{k}{n}$. That is, $\tau_1 = \min\left\{t \geqq 0;\ B(t) = \dfrac{1}{n} \text{ or } B(t) = -\dfrac{1}{n}\right\}$, and if, for example, $B(\tau_1) = \dfrac{1}{n}$, then let $\tau_2 = \min\left\{t \geqq \tau_1;\ B(t) = \dfrac{2}{n} \text{ or } B(t) = \dfrac{0}{n}\right\}$, etc. $B(\tau_1), B(\tau_2), \ldots$ has the same distribution as $\dfrac{1}{n}S_1,\ \dfrac{1}{n}S_2, \ldots$ We apply part (b) to this approximating Brownian motion to see

$$Pr\{M_n(\tau) > x\} = \left(\frac{na}{1+na}\right)^{nx} \to e^{-x/a}$$

As $n \to \infty$, the partially observed Brownian gets closer to the Brownian motion. We conclude that $M(\tau)$ is exponentially distributed with mean a $\left(\text{Rate parameter } \dfrac{1}{a}\right)$.

1.6 If $S_n(e)$ is the compressive force on n balloons at a strain of e, we have $E[S_n(e)] = nKe[1 - q(e)] \times [1 - F(e)]$. Because $\log E[S_n(e)] = \log nK + \log e - \log \dfrac{1}{[1-q(e)][1-F(e)]}$, a plot of $\log E[S_n(e)]$ versus $\log e$ would have an intercept $\log nK$ reflecting material volume and modulus. Any departures from linearity would reflect the failure strain distribution plus departures from Hooke's law.

1.7 (a) $E\left[B(n+1) \middle| B(0), \ldots, B(n)\right] = E\left[B(n+1) - B(n) \middle| B(0), \ldots, B(n)\right] + B(n) = B(n).$

(b) $$E\left[B(n+1)^2 - (n+1) \middle| B(n)^2 - n\right] = E\left[\left\{B(n+1)^2 - B(n)^2 - 1\right\} + B(n)^2 - n \middle| B(n)^2 - n\right]$$
$$= B(n)^2 - n + E\left[B(n+1)^2 - B(n)^2\right] - 1 = B(n)^2 - n.$$

1.8 The suggested approximations probably do well in mimicking Brownian motion in some tests, and poorly in others. Because each $B_N(t)$ is differentiable, $\Delta B_N(t)$ will be on the order of Δt, where as $\Delta B(t)$ is on the order of $\sqrt{\Delta t}$.

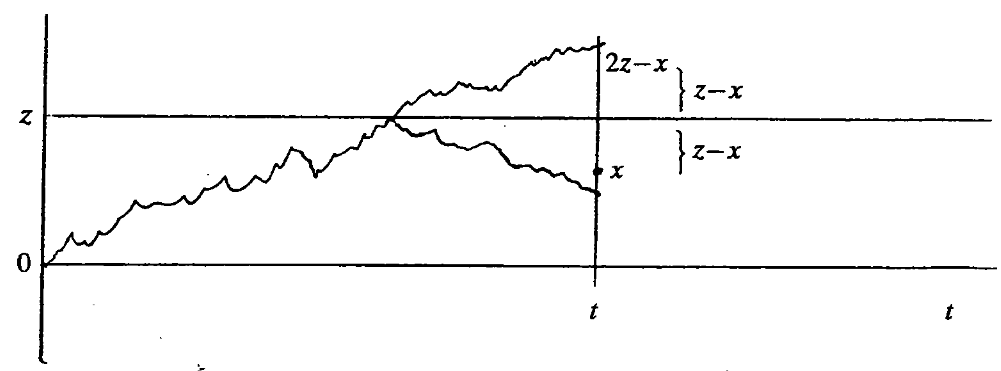
2.1 $$Pr\{B(u)\neq 0, t<u<t+b \mid B(u)\neq 0, t<u<t+a\} = \frac{Pr\{B(u)\neq 0, t<u<t+b\}}{Pr\{B(u)\neq 0, t<u<t+a\}} = \frac{1-\zeta(t,t+b)}{1-\zeta(t,t+a)}$$
$$= \frac{\arcsin\sqrt{t/(t+b)}}{\arcsin\sqrt{t/(t+a)}} \quad 0<a<b.$$

2.2 $$\lim_{t\to 0}\frac{\arcsin\sqrt{t/(t+b)}}{\arcsin\sqrt{t/(t+a)}} = \sqrt{\frac{a}{b}} \quad 0<a<b.$$

2.3 $Pr\{M(t)>a\} = 2\{1-\Phi_t(a)\} = Pr\{B(t)>a\}$.

But the joint distributions clearly differ (For $0<s<t$, it must be $M(t)\geqq M(s)$).

2.4 The appropriate picture is



The two paths are equally likely.

$$\frac{\partial}{\partial x}\left\{1-\Phi\left(\frac{2z-x}{\sqrt{t}}\right)\right\} = \frac{1}{\sqrt{t}}\varphi\left(\frac{2z-x}{\sqrt{t}}\right) - \frac{\partial}{\partial z}\frac{\partial}{\partial x}\left\{1-\Phi\left(\frac{2z-x}{\sqrt{t}}\right)\right\}$$
$$= -\frac{1}{\sqrt{t}}\frac{\partial}{\partial z}\varphi\left(\frac{2z-x}{\sqrt{t}}\right)$$
$$= \frac{2}{t}\left(\frac{2z-x}{\sqrt{t}}\right)\varphi\left(\frac{2z-x}{\sqrt{t}}\right).$$

2.5 The Jacobean is one, whence

$$f_{M(t),Y(t)}(z,y) = f_{M(t),B(t)}(z,z-y) = \frac{2}{t}\left(\frac{z+y}{\sqrt{t}}\right)\varphi\left(\frac{z+y}{\sqrt{t}}\right)$$

2.6 $$f_{Y(t)}(y) = \int_0^\infty f_{M(t),Y(t)}(z,y)\,dz = \frac{2}{\sqrt{t}}\int_{y/\sqrt{t}}^\infty u\varphi(u)\,du = \frac{2}{\sqrt{t}}\varphi\left(\frac{y}{\sqrt{t}}\right) = f_{|B(t)|}(y).$$

3.1 $Pr\left\{\frac{R(t)}{\sqrt{t}} > z\right\} = \iint_{\sqrt{x^2+y^2}>z} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)} dx\, dy = \int_z^\infty \int_0^{2\pi} \frac{1}{2\pi} re^{-\frac{1}{2}r^2} d\theta dr = -\int_z^\infty d\left\{e^{-\frac{1}{2}r^2}\right\} = e^{-\frac{1}{2}z^2}.$

$$E\left[\frac{R(t)}{\sqrt{t}}\right] = \int_0^\infty Pr\left\{\frac{R(t)}{\sqrt{t}} > z\right\} dz = \int_0^\infty e^{-\frac{1}{2}z^2} = \sqrt{\frac{\pi}{2}} \quad \text{and} \quad E[R(t)] = \sqrt{\frac{\pi t}{2}}.$$

3.2 The specified conditional process has the same properties as $bt + B^\circ(t)$ where $B^\circ(t)$ is the Brownian bridge. Whence $E[B(t)|B(1) = b] = bt$ and $Var[B(t)|B(1) = b] = t(1 - t)$ for $0 < t < 1$.

3.3 We need only show that $B(1)$ and $B(u) - uB(1)$ are uncorrelated (Why?)

$$E[B(1)\{B(u) - uB(1)\}] = E[B(1)B(u)] - uE\left[B(1)^2\right] = u - u = 0.$$

(a) $B(t) = \{B(t) - tB(1)\} + tB(1)$
The conditional distribution of $B(t) - tB(1)$, given $B(1)$, is the same as the unconditional distribution, by independence, where as $tB(1)$, given $B(1) = 0$, is zero.

(b)
$$\begin{aligned} E[B^\circ(s)B^\circ(t)] &= E[\{B(s) - sB(1)\}\{B(t) - tB(1)\}] \\ &= E[B(s)B(t)] - sE[B(1)B(t)] - tE[B(s)B(1)] + stE\left[B(1)^2\right] \\ &= s - st - st + st = s(1-t) \quad \text{for } 0 < s < t < 1. \end{aligned}$$

3.4
$$\begin{aligned} E[W^\circ(s)W^\circ(t)] &= E\left[(1-s)B\left(\frac{s}{1-s}\right)(1-t)B\left(\frac{t}{1-t}\right)\right] = (1-s)(1-t)\min\left\{\frac{s}{1-s}, \frac{t}{1-t}\right\} \\ &= s(1-t), \quad 0 < s < t < 1. \end{aligned}$$

3.5 $\int_0^\infty y[\phi_t(y-x) - \varphi_t(y+x)]dy = \int_{-\infty}^{+\infty} y\varphi_t(y-x)dy = x.$

3.6 $M > z$ if and only if a standard Brownian motion reaches z before 0, starting from x.

3.7 The same calculation as in Problem 3.5 showes that $A(t)$ is a martingale. For the (continuous time) martingale $A(t)$, the maximal inequality is an equality.

3.8 Since $\varphi_t(y - x)$ satisfies the diffusion equation, so will $[\varphi_t(y - x) - \varphi_t(y + x)]$ and $[\varphi_t(y - x) + \varphi_t(y + x)]$

3.9 (a) $E[B^\circ(F(s))B^\circ(F(t))] = F(s)[1 - F(t)]$ for $s < t$.

(b) The approximation is $F_N(t) \approx F(t) + \frac{1}{\sqrt{N}} B^\circ(F(t))$.

4.1 $Pr\{\max[B(t) - bt] > a\} = e^{-2ab}.$

4.2 $Pr\left\{\max \frac{b + B(t)}{1+t} > a\right\} = Pr\{\max B(t) - at > a - b\} = e^{-2a(a-b)}.$

4.3
$$\begin{aligned} Pr\{\max_{0\le u\le 1} B^\circ(u) > a\} &= Pr\left\{\max_{t>0} (1+t)B^\circ\left(\frac{t}{1+t}\right) - a(1+t) > 0\right\} \\ &= Pr\left\{\max_{t>0} B(t) - at > a\right\} = e^{-2a^2}. \end{aligned}$$

4.4 If the drift of $X(t)$ is μ_0, then the drift of $X'(t) = X(t) - \frac{1}{2}(\mu_0 + \mu_1)$ is $-\frac{1}{2}\delta$ where $\delta = \mu_1 - \mu_0 > 0$. If the drift of $X(t)$ is μ_1, then the drift of $X'(t) = X(t) - \frac{1}{2}(\mu_0 + \mu_1)$ is $+\frac{1}{2}\delta$. Set $\delta = \mu_1 - \mu_0$ and apply the procedure in the text to $X'(t)$.

4.5 $Pr\{\max X^A(t) > B\} = \dfrac{e^{-2\mu x/\sigma^2} - 1}{e^{-2\mu B/\sigma^2} - 1}$.

4.6 $Pr\{\text{Double your money}\} = \frac{1}{2}$.

4.7
$$Pr\{Z(\tau) > a | Z(0) = z\} = Pr\{\log Z(\tau) > \log a | \log Z(0) = \log z\}$$
$$= 1 - \Phi\left(\frac{\log\frac{a}{z} \left(\alpha - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right).$$

4.8 The Black-Scholes value where $\sigma = .35$ is \$5.21.

4.9 The differential equation is $\dfrac{d^2}{dx^2}w(x) = 2\lambda w(x)$ for $x > 0$. The solution is $w(x) = c_1 e^{\sqrt{2\lambda}x} + c_2 e^{-\sqrt{2\lambda}x}$. The constants are evaluated via the boundary conditions $w(0) = 1$, $\lim_{x\to\infty} w(x) = 0$ whence $w(x) = e^{-1\sqrt{2\lambda}x}$. There are many other ways to evaluate $w(x)$, including martingale methods.

4.10 The relevant computation is $E[Z(t)e^{-rt}|Z(0) = z] = ze^{rt}e^{-rt} = z$.

5.1 (a) The formula follows easily from iteration. It is a discrete analog of (5.22).

(b) Sum $\Delta V_n = V_n - V_{n-1} = -\beta V_{n-1} + \xi_n$ to get the given formula. It is a discrete analog to (5.24).

5.2
$$E[S(t)V(t)] = E\left[\left\{\int_0^t V(s)ds\right\}V(t)\right] = \int_0^t E[V(s)V(t)]ds = \frac{\sigma^2}{2\beta}\int_0^t\left[e^{-\beta(t-s)} - e^{-\beta(t+s)}\right]ds$$
$$= \frac{\sigma^2}{2\beta^2}e^{-\beta t}\int_0^t\left(\beta e^{\beta s} - \beta e^{-\beta s}\right)ds = \frac{\sigma^2}{2\beta^2}\left[e^{\beta t} + e^{-\beta t} - 2\right]$$
$$= \left(\frac{\sigma}{\beta}\right)^2\left[\cosh\beta t - 1\right]$$

5.3 This is merely the result of Exercise 4.6 applied to the position process.

5.4 $\Delta V = V_N\left(t + \frac{1}{N}\right) - V_N(t) = \frac{1}{\sqrt{N}}\left[X_{[Nt]+1} - X_{[Nt]}\right] = \frac{1}{\sqrt{N}}\Delta X, \quad \Delta X = X_{[Nt]+1} - X_{[Nt]}$.

$$Pr\left\{\Delta V = \pm\frac{1}{\sqrt{N}} \middle| V_N(t) = v\right\} = Pr\left\{\Delta X = \pm 1 \middle| X_{[Nt]} = v\sqrt{N}\right\} = \frac{1}{2} \mp \frac{v\sqrt{N}}{2N}.$$

$$E\left[\Delta V \middle| V_N(t) = v\right] = +\frac{1}{\sqrt{N}}\left(\frac{1}{2} - \frac{v\sqrt{N}}{2N}\right) - \frac{1}{\sqrt{N}}\left(\frac{1}{2} + \frac{v\sqrt{N}}{2N}\right) = -v\left(\frac{1}{N}\right) = -v\Delta t.$$

$$E\left[\Delta V^2 \middle| V_N(t) = v\right] = \left(\frac{1}{\sqrt{N}}\right)^2 = \frac{1}{N} = \Delta t.$$

CHAPTER IX

1.1 Let $X(t)$ be the number of trucks at the loader at time t. Then $X(t)$ is a birth and death process for which $\lambda_0 = \lambda_1 = \lambda$ and $\mu_1 = \mu_2 = \mu$. ($\lambda_2 = \mu_0 = 0$). Then $\theta_0 = 1$, $\theta_1 = \varrho = \dfrac{\lambda}{\mu}$, and $\theta_2 = \varrho^2$. $\pi_0 =$ Long run fraction of time no trucks are at the loader $= \dfrac{1}{1 + \varrho + \varrho^2}$. Fraction of time trucks are loading $= 1 - \pi_0 = \dfrac{\varrho + \varrho^2}{1 + \varrho + \varrho^2}$. Since trucks load at a rate of μ per unit time, long run loads per unit time $= \mu(1 - \pi_0) = \mu\left(\dfrac{\varrho + \varrho^2}{1 + \varrho + \varrho^2}\right) = \lambda\left(\dfrac{1 + \varrho}{1 + \varrho + \varrho^2}\right)$

2.1 For the M/M/2 system, $\theta_k = 2\varrho^k$ for $k \geqq 1$.

$$\sum \theta_k = 2\sum \varrho^k - 1 = \frac{2}{1 - \varrho} - 1 = \frac{1 + \varrho}{1 - \varrho}.$$

$$\pi_0 = \frac{1 - \varrho}{1 + \varrho}; \quad \pi_k = \frac{2(1 - \varrho)}{1 + \varrho}\varrho^k, \quad k \geqq 1$$

$$L_0 = \frac{2\varrho^3}{1 - \varrho^2}, \; L = \frac{2\varrho}{1 - \varrho^2}.$$

$$L_0, L \to \infty \text{ as } \varrho \to 1.$$

2.2 $W = \dfrac{1}{\lambda}L = \dfrac{1}{\mu}\left(\dfrac{2}{1 - \varrho^2}\right)$(See solution to 2.1)

When

$$\lambda = 2, \mu = 1.2 \quad \text{then } \varrho = \frac{\lambda}{2\mu} = \frac{1}{1.2}$$

and

$$W = \frac{1}{2}\left(\frac{2}{1 - \left(\frac{1}{1.2}\right)^2}\right) = 3.27$$

For a single server queue $W = \dfrac{1}{\mu - \lambda} = \dfrac{1}{.2} = 5$. This *helps* explain why many banks have a single line feeding several tellers (M/M/s) rather than a separale line for each teller (s parallel M/M/1 systems).

2.3 For the M/M/2 system $\theta_0 = 1$, $\theta_k = 2\varrho^k$ for $k \geqq 1$, $\pi_0 = \dfrac{1-\varrho}{1+\varrho}$, $\pi_k = 2\left(\dfrac{1-\varrho}{1+\varrho}\right)\varrho^k$, $k \geqq 1$.

$$L = \sum k\pi_k = 2\left(\frac{1-\varrho}{1+\varrho}\right)\sum_{k=1}^{\infty} k\varrho^k = \frac{2\varrho}{1-\varrho^2}.$$

$$W = \frac{1}{\mu}\pi_0 + \frac{1}{\mu}\pi_1 + \left(\frac{1}{\mu} + \frac{1}{2\mu}\right)\pi_2 + \left(\frac{1}{\mu} + \frac{2}{2\mu}\right)\pi_2 + \cdots$$

$$= \frac{1}{\mu}\left(\pi_0 + \pi_1 + \pi_2 + \cdots\right) + \frac{1}{2\mu}\sum_{k=2}^{\infty}(k-1)\pi_k$$

$$W = \frac{1}{\mu} + \frac{1}{\mu}\left(\frac{1-\varrho}{1+\varrho}\right)\sum_{k=2}^{\infty}(k-1)\varrho^k = \frac{1}{\mu}\left(\frac{1}{1-\varrho^2}\right)$$

and $L = \lambda W$.

2.4 (a) $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu_2 = \mu_3 = \mu$, $\mu_0 = \lambda_3 = 0$

(b) $\theta_0 = 1$, $\theta_k = \varrho^k$, $k = 1, 2, 3$, $\varrho = \dfrac{\lambda}{\mu}$.

$$\sum \theta_k = 1 + \varrho + \varrho^2 + \varrho^3 = \frac{1-\varrho^4}{1-\varrho}. \quad \pi_0 = \frac{1-\varrho}{1-\varrho^4}.$$

(c) *Fraction* of customers lost $= \pi_3 = \left(\dfrac{1-\varrho}{1-\varrho^4}\right)\varrho^3$.

2.5 Observe that $\mu\pi_k = \lambda\pi_{k-1}$. Following the hint, we get for $j \geqq 1$

$$\pi_0 P_{0j}'(t) = \lambda\pi_0 P_{1j}(t) - \lambda\pi_0 P_{0j}(t) = \mu\pi_1 P_{1j}(t) - \lambda\pi_0 P_{0j}(t)$$

and

$$\pi_k P_{kj}'(t) = \mu\pi_k P_{k-1,j-1}(t) + \lambda\pi_k P_{k+1,j}(t) - (\lambda + \mu)\pi_k P_{k,j}(t)$$
$$= \lambda\pi_{k-1} P_{k-1,j-1}(t) + \mu\pi_{k+1} P_{k+1,j}(t) - (\lambda + \mu)\pi_k P_{k,j}(t).$$

Upon summing the above, the μ terms drop out and we get

$$P_j'(t) = \lambda P_{j-1}(t) - \lambda P_j(t), \quad j \geqq 1$$

Similar analysis yields

$$P_0'(t) = -\lambda P_0(t).$$

Set $Q_n(t) = e^{\lambda t}P_n(t)$. Then $Q_n'(t) = Q_{n-1}(t)$

The solution (using $Q_0(0) = P_0(0) = 1$) is

$$Q_n(t) = \frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, \ldots \quad \text{whence}$$

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Compare with Theorem 5.1.

2.6 We want the smallest c for which $\sum_{k=c+1}^{\infty} \pi_k = \varrho^{c+1} \leq \frac{1}{1000}$. Want smallest c for which

$$c + 1 \geqq \frac{\log 1000}{\log 1/\varrho}$$

$$c^* = \left[\frac{\log 1000}{\log \nu/\varrho}\right] \quad [x] = \text{Integer part of } x.$$

2.7 Since $X(0) = 0$ we set $P_j(t) = P_{0j}(t)$. The forward equations in this case are $P_j'(t) = \lambda P_{j-1}(t) + \mu(j + 1)P_{j+1}(t) - (\lambda + \mu j)P_j(t)$. Multiply by j

$$jP_j'(t) = \lambda(j - 1 + 1)P_{j-1}(t) + \mu\left[(j + 1)^2 - (j + 1)\right]P_{j+1}(t) - \lambda jP_j(t) - \mu j^2 P_j(t), \quad j \geqq 0.$$

Sum:

$$M'(t) = \lambda M(t) + \lambda - \mu M(t) + \mu P_1(t) - \mu P_1(t) - \lambda M(t) = \lambda - \mu M(t), \quad t \geqq 0$$

$$M(0) = 0. \quad \text{Let } Q(t) = e^{\mu t}M(t).$$

$$Q'(t) = e^{\mu t}M'(t) + e^{\mu t}\mu M(t) = e^{\mu t}\left[M'(t) + \mu M(t)\right] = \lambda e^{\mu t}, \quad t \geqq 0$$

$$Q(t) = \frac{\lambda}{\mu}\int_0^t \mu e^{\mu t}dt = \frac{\lambda}{\mu}\left(e^{\mu t} - 1\right)$$

$$M(t) = e^{-\mu t}Q(t) = \frac{\lambda}{\mu}\left(1 - e^{-\mu t}\right).$$

3.1 $X(t)$ and $Y(t)$ are independent random variables, each Poisson distributed

where
$$E[X(t)] = \lambda\int_0^t[1 - G(y)]dy$$
$$E[Y(t)] = \lambda\int_0^t G(y)dy.$$

Observe that $Y(t)$ is the number of points (W_k, V_k) in the triangle $B_t = \{(w, v): 0 \leq w \leq t, 0 \leq v \leq t - w\}$ and that A_t and B_t are disjoint. Apply Theorem V, 6.1.

3.2 *Method A*: $\nu = .5$, $\tau^2 = .2$, $\lambda = 1$, $\varrho = \lambda\nu = .5\lambda$

$$W = \nu + \frac{\lambda\left(\tau^2 + \nu^2\right)}{2(1 - \varrho)} = .5 + \frac{.45\lambda}{2 - \lambda} = .95 \quad \text{when } \lambda = 1.$$

If $\lambda \to 2$ then $W \to \infty$.
Method B: $\nu = .4$, $\tau^2 = .9$, $\lambda = 1$, $\varrho = \lambda\nu = .4\lambda$

$$W = \nu + \frac{\lambda\left(\tau^2 + \nu^2\right)}{2(1 - \varrho)} = .4 + \frac{1.06\lambda}{.8(2.5 - \lambda)} = 1.28 \quad \text{when } \lambda = 1$$

Method A is preferred when $\lambda = 1$, but B is better when $\lambda < .3$ or $\lambda > 1.7\ldots$.

4.1 When the faster server is first

$$D = 1400 \quad \text{and} \quad \pi_{(1,1)} = .3977$$

When the slower is first

$$D = 1600 \quad \text{and} \quad \pi_{(1,1)} = .4082$$

Slightly more customers are lost when the slower server is first.

4.2

	L_p	L_n	L
Slower customers have priority	$\frac{2}{3}$	1.6	2.27
$\sigma = \frac{2}{5}$ $\tau = \frac{4}{15}$			
Faster customers have priority	$\frac{4}{11}$	$\frac{82}{55}$	1.85
$\sigma = \frac{4}{15}$ $\tau = \frac{2}{5}$			

	L_{slow}	L_{fast}	L
Slower have priority	.67	1.60	2.27
Faster have priority	1.49	.36	1.85

Total L reduced with faster customers having priority.

4.3 A birth and death process with $\lambda_n = \lambda$ and $\mu_n = \mu + r_n$

$$\sum \theta_k = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{\mu_1\mu_2 \dots \mu_k}, \quad \pi_0 = \frac{1}{\sum \theta_k} \quad \pi_k = \pi_0\left(\frac{\lambda^k}{\mu_1\mu_2 \dots \mu_k}\right).$$

Rate at which customers depart prior to service is $\Sigma \pi_k r_k$.

4.4 $\lambda_n = \lambda$, $n \geqq 0$; $\mu_0 = 0$, $\mu_1 = \alpha$, $\mu_k = \beta$, $k \geqq 2$.

$$\theta_0 = 1 \quad \theta_1 = \frac{\lambda}{\alpha} = \left(\frac{\beta}{\alpha}\right)\left(\frac{\lambda}{\beta}\right) \quad \theta_k = \left(\frac{\beta}{\alpha}\right)\left(\frac{\lambda}{\beta}\right)^k, \quad k \geqq 1.$$

$$\sum \theta_k = 1 + \frac{\beta}{\alpha}\sum_{k=1}^{\infty}\left(\frac{\lambda}{\beta}\right)^k = 1 + \frac{\beta}{\alpha}\left(\frac{\lambda}{\beta - \lambda}\right); \quad 0 < \lambda < \beta.$$

$$\pi_0 = \frac{\alpha(\beta - \lambda)}{\alpha(\beta - \lambda) + \beta\lambda}, \quad \pi_k = \frac{\beta(\beta - \lambda)}{\alpha(\beta - \lambda) + \beta\lambda}\left(\frac{\lambda}{\beta}\right)^k, \quad k \geqq 1$$

4.5 $\lambda_0 = \lambda_1 = \lambda_2 = \lambda$, $\mu_1 = \mu$, $\mu_2 = \mu_3 = 2\mu$.

$$\theta_0 = 1, \quad \theta_1 = \frac{\lambda}{\mu} \quad \theta_2 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 \quad \theta_3 = \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3.$$

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{4}\left(\frac{\lambda}{\mu}\right)^3} \quad \pi_k = \theta_k \pi_0$$

5.1 $Pr\{X_3 = k\} = \left(1 - \frac{10}{15}\right)\left(\frac{10}{15}\right)^k, \quad k \geqq 0$

$$Pr\{X_3 > k\} = \left(\frac{10}{15}\right)^k$$

Want c such that $\left(\frac{10}{15}\right)^c \leq .01$

such that $c \log \frac{2}{3} \leq \log .01$

such that $c \geqq \dfrac{\log .01}{\log \frac{2}{3}}$

$c \geqq 11.36$

$c^* = 12.$

6.1 Let x be the rate of feedback. Then $x + \lambda$ go in to Server #1, and $.6(x + \lambda)$ go in and out of Server #2. But $x = .2$ of the output of Server #2. Therefore $x = (.2)(.6)(x + \lambda) = .12x + .12\lambda$

$$.88x = .12\lambda \quad x = \frac{12}{88}\lambda = \frac{3}{11}.$$

Server #2: Arrival rate $= .6(x + \lambda) = \frac{6}{10}\left(\frac{3}{11} + 2\right) = \frac{15}{11}.$

$$\pi_{20} = 1 - \frac{15/11}{3} = \frac{6}{11}.$$

Server #3: Arrival rate $= .4(x + \lambda) = \frac{10}{11}$

$$\pi_{30} = 1 - \frac{10/11}{2} = \frac{6}{11}$$

Long run $Pr\{X_2 = 0, X_3 > 0\} = \frac{6}{11} \times \frac{5}{11} = \frac{30}{121}.$

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