<u>Module 4</u>

Poisson process

Topics: Poisson process, Inter-event and waiting times, Conditional distributions, Superposition and thinning, Generalized Poisson processes

Poisson process

<u>Counting process</u>: Let us count the number of earthquakes that hits India from this moment onwards. The count would increase, as in not decrease, as time progresses. Let $N(t)$ denote the count at time t. Then the stochastic process $\{N(t): t \geq 0\}$ is called a counting process. Formally, $\{N(t): t \geq 0\}$ *is a counting process if* $N(t)$ *is non-negative integer-valued random variable for all* t *and* $N(t)$ *is non-decreasing in* t, *i.e.,* $N(t_1) \leq N(t_2)$ *whenever* $t_1 < t_2$. A counting process is continuous time discrete state stochastic process.

We encounter many counting processes in and around us. For example, number of vehicles passing through a highway location, number of power cuts in a city block, number of failures of a repairable machine, number of customers arriving in a shopping mall, etc. are counting processes. There are counting processes over space, instead of time. For example, number of mango orchard by the side of a highway as we move along it, number of micro cracks in a long shaft along its length, number of tigers in a reserved forest as we move along its length and breadth, number of starts as we 'move' along space in some appropriate manner. Note that counting process over space can be in different dimensions. In this course, we restrict ourselves to counting processes over time or one spatial dimension.

Since the count $N(t)$ is non-decreasing with time t, typically, in all counting processes of interest, $\lim_{t\to\infty} N(t) = \infty$. Thus, long-run behavior is not of interest, unlike the Markov chains. Here, our interest lies in the *waiting time* till n-th event $W_n \coloneqq \min\{t: N(t) = n\}$ for $n = 1,2,3, \dots$, the n-th *inter-event time* $T_n \coloneqq W_n - W_{n-1}$ for $n = 1,2,3, \dots$ with $W_0 = 0$, and the connections between $N(t)$, W_n, and T_n. The way $N(t)$ increases with time differentiates among counting processes and influences W_n and T_n.

Let us obtain the generic relations among $N(t)$, W_n, and T_n. Later, for specific counting processes, we will be more specific. We already mentioned that $T_n = W_n - W_{n-1}$ for $n \geq 1$ with $W_0 = 0$. Then $W_n = T_1 + T_2 + \cdots + T_n$ for $n \geq 1$. *The relation between* W_n *and* $N(t)$ *is:* $\{W_n \leq t\} \equiv \{N(t) \geq n\}$. This is a fundamental relation in counting process. $W_n \leq t$ implies at least n events have taken place by time t; therefore, $N(t) \geq n$. The argument works in the reverse direction as well. Hence, events $\{W_n \leq t\}$ and $\{N(t) \geq n\}$ are equivalent. Then $P(W_n \leq t) = P(N(t) \geq n)$, which will lead to many interesting results about W_n and T_n for specific counting processes. In the general case, you shall verify that $\{W_n > t\} \equiv \{N(t) < n\}$, but $\{W_n < t\} \not\equiv \{N(t) > n\}$ and $\{W_n \geq t\} \not\equiv \{N(t) \leq n\}$.

<u>Poisson process</u>: Before we define Poisson process, let us introduce two important properties that some counting processes have. A counting process is said to have *independent increment*

property if the number of events taking place in two arbitrary disjoint intervals of time are independent, i.e., $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are independent for all $t_1 < t_2 \le t_3 < t_4$. Note that the intervals may have the same end and start points, and in case an event takes place at time point $t_2 = t_3$, it is counted in $N(t_2) - N(t_1)$, not in $N(t_4) - N(t_3)$. A counting process is said to have *stationary increment property* if the number of events taking place in an interval of time depends on its length, but does not depend on its start and end points, i.e., $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are identical random variables whenever $t_2 - t_1 = t_4 - t_3$. Consider the earthquake example; it possesses both independent and stationary increment properties. Most of the naturally occurring counting processes have these two properties. Man-made counting processes, e.g., customer arrival in shopping mall, typically possess the independent increment property, but not the stationary increment property. However, in a shorter span of time, these can be said to have both the properties.

A counting process $\{N(t): t \ge 0\}$ is said to be a Poisson process with intensity or rate λ per unit time if (i) $N(0) = 0$, (ii) it has both independent and stationary increment property, and (iii) $N(t) \sim Pois(\lambda t)$ for all $t > 0$. So, Poisson process is a special type of counting process. Most of the above counting processes are examples of Poisson process. Given a counting process, it's not difficult to check the first two conditions, but it's not clear how to check the third one without empirical study. We have an alternate characterization of Poisson process, which is relatively easier to check.

A counting process $\{N(t): t \ge 0\}$ is said to be a Poisson process with intensity or rate λ per unit time if (i) $N(0) = 0$, (ii) it has both independent and stationary increment property, and (iii) $P(N(h) = 1) = \lambda h + o(h)$ and $P(N(h) \ge 2) = o(h)$ where $o(h)$ is an unspecified function having the property that $\lim_{h\to 0} o(h)/h = 0$. Polynomials of degree 2 or more are $o(h)$ functions, but the linear function is not. The third condition essentially says that *in a sufficiently short interval of time, at most one event can take place and the rate of occurrence of an event, i.e., $\lim_{h\to 0} P(N(h) = 1)/h = \lim_{h\to 0}\{\lambda + o(h)/h\} = \lambda$ is constant*. With this interpretation of the third condition, one can verify if a given counting processes possess this property or not. Most of the above examples have this property.

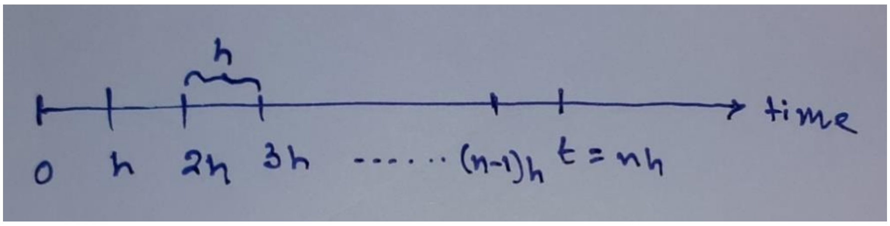
<u>Equivalence</u>: Let us see if the two definitions of Poisson process are equivalent or not. With the first definition, $N(h) \sim Pois(\lambda h)$ for all $h > 0$. Then

$$P(N(h) = 1) = e^{-\lambda h}\lambda h = \left(1 - \lambda h + \frac{(\lambda h)^2}{2} - \frac{(\lambda h)^3}{3!} + \cdots\right)\lambda h = \lambda h + o(h)$$

$$P(N(h) = 0) = e^{-\lambda h} = 1 - \lambda h + \frac{(\lambda h)^2}{2} - \frac{(\lambda h)^3}{3!} + \cdots = 1 - \lambda h + o(h)$$

$$\Rightarrow P(N(h) \ge 2) = 1 - P(N(h) = 0) - P(N(h) = 1) = o(h)$$

Clearly, the first definition implies the second. Note that $o(h) + o(h) = o(h)$, as $o(h)$ is unspecified. For the same reason, $o(h) - o(h) = o(h)$. Now, let us begin with the second definition. Consider an arbitrary $t > 0$ and split it into n equal sub-intervals of length h so that $nh = t$, as depicted below.



Number of events taking place in these sub-intervals are *iid* random variables, due to the independent and stationary increment properties. If we make h sufficiently small, i.e., $h \to 0$, then at most one event takes place in each of these sub-intervals, as in probability of multiple events taking place in any of the sub-intervals is $o(h)$, which can be ignored as $h \to 0$. Then the number of events taking place in these sub-intervals can be regarded as independent and identical Bernoulli trials with success probability λh. Note that $\lim_{h \to 0} \lambda h = 0$, but we will not apply the limit now, because $h \to 0$ while keeping nh constant $(= t)$ makes $n \to \infty$, and we have n such Bernoulli trials. Then $N(t) \sim Bin(n, \lambda h)$.

$$\begin{aligned}
\Rightarrow P(N(t) = k) = \lim_{h \to 0} \binom{n}{k} (\lambda h)^k (1 - \lambda h)^{n-k} &= \lim_{n \to \infty} \frac{n!}{k!\,(n-k)!} \left(\frac{\lambda t}{n}\right)^k \left(1 - \frac{\lambda t}{n}\right)^{n-k} \\
&= \lim_{n \to \infty} \frac{n(n-1) \cdots (n-k+1)}{n^k} \frac{(\lambda t)^k}{k!} \left(1 - \frac{\lambda t}{n}\right)^n \left(1 - \frac{\lambda t}{n}\right)^{-k} \\
&= \frac{(\lambda t)^k}{k!} \left[\lim_{n \to \infty} \prod_{i=0}^{k-1} \left(1 - \frac{i}{n}\right)\right] \left[\lim_{n \to \infty} \left(1 - \frac{\lambda t}{n}\right)^n\right] \left[\lim_{n \to \infty} \left(1 - \frac{\lambda t}{n}\right)^{-k}\right] \\
&= e^{-\lambda t} \frac{(\lambda t)^k}{k!} \Rightarrow N(t) \sim Pois(\lambda t)\ \forall t > 0
\end{aligned}$$

Inter-event and waiting times

Inter-event times: Let us study nature of the inter-event times, denoted by $T_1, T_2, T_3, \ldots$, in a Poisson process with intensity λ. Note that T_1 is the time until the first event occurs (since the observation starts) and T_n for $n \geq 2$ is the time between the $n - 1^{\text{st}}$ and n-th events.

$$P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}\ \forall t > 0, \text{due to equivalence of events}$$
$$\Rightarrow F_{T_1}(t) = 1 - e^{-\lambda t}\ \forall t > 0 \Rightarrow f_{T_1}(t) = \lambda e^{-\lambda t}\ \forall t > 0$$

Clearly, $T_1 \sim Exp(\lambda)$. Next,

$$\begin{aligned}
P(T_2 > t | T_1 = s) = P(N(s+t) - N(s) = 0 | N(s^-) = 0, N(s) = 1)& \\
= P(N(s+t) - N(s) = 0), &\text{due to independent increment property} \\
= P(N(t) = 0), &\text{due to stationary increment property} \\
= e^{-\lambda t}\ \forall t > 0 \Rightarrow (T_2 | T_1 = s) \sim Exp(\lambda)&
\end{aligned}$$
$$\Rightarrow T_2 \text{ is independent of } T_1 \text{ and } T_2 \sim Exp(\lambda)$$

In a similar manner, we can show that $P(T_2 > t | T_1 = s_1, T_2 = s_2) = e^{-\lambda t}\ \forall t > 0$, implying $T_3 \sim Exp(\lambda)$ and it is independent of T_1, T_2. Continuing in this manner, we can conclude that the inter-event times in a Poisson process with intensity λ are independent and exponentially distributed random variables with parameter λ (rate).

Waiting times: Now, we study the waiting times $W_1, W_2, W_3, \dots$ in a Poisson process with rate λ. We can obtain distribution of W_n for $n \geq 1$ in two different ways. We can use the relation between W_n and $N(t)$, i.e., $P(W_n \leq t) = P(N(t) \geq n)$, and obtain

$$F_{W_n}(t) = P(N(t) \geq n) = 1 - P(N(t) \leq n-1) = 1 - \sum_{k=0}^{n-1} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\ \forall t > 0$$

$$\Rightarrow f_{W_n}(t) = \sum_{k=0}^{n-1} \lambda e^{-\lambda t}\frac{(\lambda t)^k}{k!} - \sum_{k=1}^{n-1} e^{-\lambda t}\frac{(\lambda t)^{k-1}\lambda}{(k-1)!} = \lambda e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}\forall t > 0$$

The above distribution is called the Erlang distribution with parameters λ (rate) and n (shape) and is denoted by $Erlang(\lambda, n)$. Note that $Exp(\lambda) \equiv Erlang(\lambda, 1)$.

The other way of obtaining distribution of W_n is via the relation that $W_n = T_1 + T_2 + \cdots + T_n$. Let us hypothesize that $W_n \sim Erlang(\lambda, n)$ for all $n \geq 1$. It is true for $n = 1$, because $W_1 = T_1 \sim Exp(\lambda) \equiv Erlang(\lambda, 1)$. Assuming its validity for arbitrary $n \geq 1$, we show its validity for $n + 1$. Then the claim is true by induction. Observe that $W_{n+1} = W_n + T_{n+1}$ and $W_n \sim Erlang(\lambda, n)$, by assumption, and $T_{n+1} \sim Exp(\lambda)$ are independent. Then

$$f_{W_{n+1}}(t) = \int_0^t f_{W_n}(s) f_{T_{n+1}}(t-s)ds = \int_0^t \lambda e^{-\lambda s}\frac{(\lambda s)^{n-1}}{(n-1)!}\lambda e^{-\lambda(t-s)}ds$$
$$= e^{-\lambda t}\frac{\lambda^{n+1}}{(n-1)!}\int_0^t s^{n-1}ds = \lambda e^{-\lambda t}\frac{(\lambda t)^n}{n!}\ \ \forall t > 0$$
$$\Rightarrow W_{n+1} \sim Erlang(\lambda, n+1), \text{as required.}$$

Exponential distribution: Here, we study some properties of exponential random variable that can be used to answer questions about Poisson process. First, we show that $P(X_1 < X_2) = \lambda_1/(\lambda_1 + \lambda_2)$ when $X_1 \sim Exp(\lambda_1)$ and $X_2 \sim Exp(\lambda_2)$ are independent.

$$P(X_1 < X_2) = \int_0^\infty P(X_1 < X_2 | X_1 = x) f_{X_1}(x)dx = \int_0^\infty P(X_2 > x) f_{X_1}(x)dx$$
$$= \int_0^\infty e^{-\lambda_2 x}\lambda_1 e^{-\lambda_1 x}dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}\int_0^\infty (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)x}dx = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Next, we show that $\min\{X_1, X_2, \dots, X_n\} \sim Exp(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$ for all $n \geq 2$ when $X_1 \sim Exp(\lambda_1), X_2 \sim Exp(\lambda_2), \dots, X_n \sim Exp(\lambda_n)$ are independent.

$$P(\min\{X_1, X_2, \dots, X_n\} > x) = P(X_i > x \text{ for all } i) = \prod_{i=1}^n P(X_i > x) = \prod_{i=1}^n e^{-\lambda_i x}$$

$= e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)x} \Rightarrow \min\{X_1, X_2, \dots, X_n\} \sim Exp(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

Now, we show that exponential random variables are memoryless. This property arises in the context of positive-valued random variables that describe waiting time for something. Let T denote such a random variable. We say that T is memoryless if 'our wait so far has no influence on our wait hereafter', i.e., $P(T > s + t | T > s) = P(T > t)$ for all $s, t > 0$. In the context of discrete random variables, geometric distribution is memoryless. If we are yet to get the first success, then the number of past failures does not influence how many more trials are necessary. Among continuous random variables, exponential distribution is memoryless.

$$P(T > s + t | T > s) = \frac{P(T > s + t)}{P(T > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t)\ \forall s, t > 0$$

Note that $Erlang(\lambda, n)$ for $n \geq 2$ is not memoryless.

<u>More on memoryless property</u>: Interestingly, exponential distribution is the only continuous random variable that has the memoryless property. Here, we prove this fact. It has profound implications for Poisson process and continuous-time Markov chain. Consider an arbitrary memoryless continuous random variable T. Let F denote its distribution function. Then

$$P(T > s + t | T > s) = \frac{P(T > s + t)}{P(T > s)} = P(T > t) \equiv P(T > s + t) = P(T > s)P(T > t)$$
$$\equiv \bar{F}(s + t) = \bar{F}(s)\bar{F}(t)\ \forall s, t > 0, \text{where } \bar{F}(x) \coloneqq 1 - F(x)$$

Let $\bar{F}(1) = c$. Then for $m = 1,2,3, \dots$, $\bar{F}(m) = \bar{F}(m - 1)\bar{F}(1) = \bar{F}(m - 2)\{\bar{F}(1)\}^2 = \cdots = \{\bar{F}(1)\}^m = c^m$. Similarly, $\bar{F}(1) = \bar{F}((n - 1)/n)\bar{F}(1/n) = \bar{F}((n - 2)/n)\{\bar{F}(1/n)\}^2 = \cdots = \{\bar{F}(1/n)\}^n \Rightarrow \bar{F}(1/n) = \{\bar{F}(1)\}^{1/n} = c^{1/n}$ for $n = 1,2,3, \dots$. With these, for any positive rational number $q = m/n$ where $m, n \in \{1,2,3, \dots\}$, $\bar{F}(q) = \bar{F}(m/n) = \{\bar{F}(1/n)\}^m = c^{m/n} = c^q$. From the number theory, for any positive irrational number r, there exists a sequence of positive rational numbers $\{q_k : k = 1,2,3, \dots\}$ that converges to r, i.e., $\lim_{k\to\infty} q_k = r$. Then $\bar{F}(r) = \bar{F}(\lim_{k\to\infty} q_k) = \lim_{k\to\infty} \bar{F}(q_k) = \lim_{k\to\infty} c^{q_k} = c^{\lim_{k\to\infty} q_k} = c^r$, by changing order of limit and continuous function. So, we have $\bar{F}(x) = c^x$ for all $x > 0$.

Since $c = \bar{F}(1) \in [0,1]$, there exists $\lambda \geq 0$ such that $c = e^{-\lambda}$. Then the arbitrary memoryless distribution satisfies $\bar{F}(x) = e^{-\lambda x}$ for al $x > 0$. Clearly, the distribution is exponential. Thus, exponential distribution is the only continuous memoryless random variable.

<u>A redundancy</u>: With the above result, we can argue that the 3rd condition in the definition of Poisson process, i.e., Poisson distribution of $N(t)$ or its alternate form, is redundant. We can establish that the inter-event times in a counting process having independent and stationary increment properties are *iid* exponential random variables (shown later). Let λ denote the rate of the exponential random variables. Then $W_n \sim Erlang(\lambda, n)$ for all $n \geq 1$, as shown earlier. Next, the equivalence $P(W_n \leq t) = P(N(t) \geq n)$ implies that

$$P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n + 1) = P(W_n \leq t) - P(W_{n+1} \leq t)$$

$$= \left\{1 - \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!}\right\} - \left\{1 - \sum_{k=0}^{n} e^{-\lambda t} \frac{(\lambda t)^k}{k!}\right\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \text{ for } n = 0,1,2, \dots$$

Therefore, $N(t) \sim Pois(\lambda t)$ for all $t > 0$.

It remains to be shown that the inter-event times $T_1, T_2, T_3, \dots$ in a counting process $\{N(t): t \geq 0\}$ with independent and stationary increment properties are *iid* exponential random variables. Let $N_1(t) = N(T_1 + t) - N(T_1)$ for $t \geq 0$. Then the counting process $\{N_1(t): t \geq 0\}$ has the inter-event times $T_2, T_3, T_4, \dots$. Due to independent increment property, T_1 cannot influence $\{N_1(t): t \geq 0\}$, i.e., $T_2, T_3, T_4, \dots$ are independent of T_1. Due to stationary increment property, $P(T_2 > t) = P(N_1(t) = 0) = P(N(T_1 + t) - N(T_1) = 0) = P(N(t) = 0) = P(T_1 > t)$ for all $t > 0$. Hence, T_1 and T_2 are identically distributed. Next, we construct counting process $\{N_2(t): t \geq 0\}$, where $N_2(t) = N_1(T_2 + t) - N_1(T_2)$, and show independence of $T_3, T_4, T_5 \dots$ with T_2 and identicalness of T_2 and T_3. Continuing in this way, we establish that $T_1, T_2, T_3, \dots$ are *iid* random variables. Now, we just need to show that T_1 is exponentially distributed.

$$\begin{aligned}
P(T_1 > s + t | T_1 > s) &= P(N(s + t) = 0 | N(s) = 0) \\
&= \frac{P(N(s) = 0, N(s + t) = 0)}{P(N(s) = 0)} = \frac{P(N(s) = 0, N(s + t) - N(s) = 0)}{P(N(s) = 0)} \\
&= \frac{P(N(s) = 0) P(N(s + t) - N(s) = 0)}{P(N(s) = 0)}, \text{due to independent increment} \\
&= P(N(s + t) - N(s) = 0) = P(N(t) = 0), \text{due to stationary increment} \\
&= P(T_1 > t) \text{ for all } s, t > 0
\end{aligned}$$

So, T_1 is memoryless, and therefore, T_1 is exponentially distributed.

Conditional distributions

Future given current: Let us consider the mass function of $N(t)|N(s)$ for $t > s$ in a Poisson process with intensity λ. For $n \geq m$,

$$\begin{aligned}
P(N(t) = n | N(s) = m) &= \frac{P(N(s) = m, N(t) - N(s) = n - m)}{P(N(s) = m)} \\
&= \frac{P(N(s) = m) P(N(t) - N(s) = n - m)}{P(N(s) = m)}, \text{due to independent increment} \\
&= P(N(t - s) = n - m), \text{due to stationary increment} \\
&= e^{-\lambda(t-s)} \frac{\{\lambda(t - s)\}^{n-m}}{(n - m)!}
\end{aligned}$$

So, $N(t)|N(s)$ for $t > s$ follows Poisson distribution with parameter $\lambda(t - s)$. Now, consider $N_s(t) = N(s + t) - N(s)$. Note that we just showed $N_s(t) \sim Pois(\lambda t)$. Also, $N_s(0) = 0$. If can show that $\{N_s(t): t \geq 0\}$ has independent and stationary increment properties, then we establish that $\{N_s(t): t \geq 0\}$ is a Poisson process with intensity λ. Since we chose arbitrary s, then we can say that one can start observing a Poisson process from any point in time and the resultant counting process is a Poisson process with the same intensity.

Let us show that $\{N_s(t): t \geq 0\}$ possesses independent and stationary increment properties. For arbitrary $t_1 < t_2 \leq t_3 < t_4$, $N_s(t_2) - N_s(t_1) = N(s + t_2) - N(s + t_1)$ and $N_s(t_4) - N_s(t_3) = N(s + t_4) - N(s + t_3)$ are independent, as $s + t_1 < s + t_2 \leq s + t_3 < s + t_4$ and $\{N(t): t \geq 0\}$ has independent increment property. For arbitrary t_1, t_2 and t, $N_s(t_1 + t) - N_s(t_1) = N(s + t_1 + t) - N(s + t_1)$ and $N_s(t_2 + t) - N_s(t_2) = N(s + t_2 + t) - N(s + t_2)$ have the same distribution, because $(s + t_1 + t) - (s + t_1) = (s + t_2 + t) - (s + t_2)$ and $\{N(t): t \geq 0\}$ has stationary increment property. This completes the proof.

Past given current: Now, we consider the mass function of $N(s)|N(t)$ for $s < t$ in a Poisson process with intensity λ. For $m \leq n$,

$$
\begin{aligned}
P(N(s) = m|N(t) = n) &= \frac{P(N(s) = m, N(t) - N(s) = n - m)}{P(N(t) = n)} \\
&= \frac{P(N(s) = m)P(N(t) - N(s) = n - m)}{P(N(t) = n)}, \text{due to independent increment} \\
&= \frac{P(N(s) = m)P(N(t - s) = n - m)}{P(N(t) = n)}, \text{due to stationary increment} \\
&= \frac{e^{-\lambda s}(\lambda s)^m}{m!} \times \frac{e^{-\lambda(t-s)}\big(\lambda(t - s)\big)^{n-m}}{(n - m)!} \times \frac{n!}{e^{-\lambda t}(\lambda t)^n} = \binom{n}{m}\left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m}
\end{aligned}
$$

So, $N(s)|N(t)$ for $s < t$ follows binomial distribution with number of trials $N(t)$ and success probability s/t, as if each of these $N(t)$ events take place independently and uniformly in $(0, t]$. Then by time s, any of these events takes place with probability s/t, and $N(s)|N(t)$ is the number of successes in $N(t)$ number of independent and identical Bernoulli trials with success probability s/t. Therefore, $N(s)|N(t) \sim Bin(N(t), s/t)$.

Uniform distribution: Given $N(t) = n$, we conjectured that each of these n events take place independently and uniformly in $(0, t]$. Here, we show that 'uniform' part. Let W denote the time of occurrence of any of these n events. We do not know if the selected event is 1st or 2nd or … n-th. It can be any of these with equal probabilities. Then for $s \leq t$,

$$
\begin{aligned}
P(W \leq s|N(t) = n) &= \sum_{k=1}^{n} P(W \leq s|W = W_k, N(t) = n)P(W = W_k|N(t) = n) \\
&= \sum_{k=1}^{n} P(W_k \leq s|N(t) = n)\frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} P(N(s) \geq k|N(t) = n) \\
&= \frac{1}{n}\sum_{k=1}^{n} P(Bin(n, s/t) \geq k) = \frac{1}{n}\sum_{k=0}^{\infty} P(Bin(n, s/t) > k) \\
&= \frac{1}{n}E[Bin(n, s/t)] = \frac{1}{n}\left(n\frac{s}{t}\right) = \frac{s}{t}, \text{as expected}
\end{aligned}
$$

For the 'independence' part of the conjecture, we need to learn about order statistics.

Order statistics: Consider n independent and identically distributed random variables $X_1, X_2, \dots, X_n$ with the common distribution function F and density function f. Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ denote the ascending ordering of the random variables. We studied about the minimum $X_{(1)}$ and the maximum $X_{(n)}$ of the random variables. For any $k = 1,2, \dots, n$,

$$F_{X_{(k)}}(x) = P\big(X_{(k)} \le x\big) = P(\text{At least } k \text{ out of } n \text{ random variables are within } x)$$
$$= \sum_{m=k}^{n} P(m \text{ random variables are within } x \text{ and the remaining } n-m \text{ are not})$$
$$= \sum_{m=k}^{n} \binom{n}{m} P^m(\text{a random variable is within } x) P^{n-m}(\text{a random variable is beyond } x)$$
$$= \sum_{m=k}^{n} \binom{n}{m} \{F(x)\}^m \{1-F(x)\}^{n-m}, \text{due to independence and identicalness}$$

Given that $N(t) = n$ in a Poisson process, let $W_1', W_2', \dots, W_n'$ denote the unordered times of occurrences of the n events. The ordered times of occurrences $W_1, W_2, \dots, W_n$ is the ascending ordering of $W_1', W_2', \dots, W_n'$. We already have shown that each of $W_1', W_2', \dots, W_n'$, given that $N(t) = n$, is uniformly distributed in $(0, t]$. Now, if $W_1', W_2', \dots, W_n'$ are independent, then the order statistics formula gives us the following.

$$P(W_k \le s | N(t) = n) = \sum_{m=k}^{n} \binom{n}{m} \{F_{W'}(s)\}^m \{1 - F_{W'}(s)\}^{n-m}$$
$$= \sum_{m=k}^{n} \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m} \text{ for all } s \in (0,t] \text{ and } k = 1,2, \dots, n$$

If we can obtain the above expression without using order statistics, then we can be confident that the unordered times of occurrences are indeed independent.

$$P(W_k \le s | N(t) = n) = P(N(s) \ge k | N(t) = n) = \sum_{m=k}^{n} P(N(s) = m | N(t) = n)$$
$$= \sum_{m=k}^{n} \binom{n}{m} \left(\frac{s}{t}\right)^m \left(1 - \frac{s}{t}\right)^{n-m} \ \forall s \in (0,t] \text{ and } k \le n, \text{as required}$$

Superposition and thinning

Superposition of Poisson processes: Consider two traffic flows, which can be modelled as independent Poisson processes $\{N_1(t): t \ge 0\}$ and $\{N_2(t): t \ge 0\}$ with intensities λ_1 and λ_2, are merging at a point. This phenomenon is known as the superposition of Poisson processes.

The resultant flow, denoted by $\{N(t): t \geq 0\}$ where $N(t) = N_1(t) + N_2(t)\ \forall t$, is a Poisson process with intensity $\lambda_1 + \lambda_2$. Let us prove this fact.

First, observe that $\{N(t): t \geq 0\}$ is a counting process. Also, $N(0) = N_1(0) + N_2(0) = 0$. Consider arbitrary $t_1 < t_2 \leq t_3 < t_4$. Then
$N(t_2) - N(t_1) = \{N_1(t_2) - N_1(t_1)\} + \{N_2(t_2) - N_2(t_1)\} = A + B, \text{say, and}$
$N(t_4) - N(t_3) = \{N_1(t_4) - N_1(t_3)\} + \{N_2(t_4) - N_2(t_3)\} = C + D, \text{say.}$

A and C are independent due to independent increment property of $\{N_1(t): t \geq 0\}$. A and D are independent due to independence of $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$. Similarly, B is independent of both C and D. Therefore, $A + B$ is independent of $C + D$. Thus, $\{N(t): t \geq 0\}$ possesses independent increment property.

Now, for arbitrary s and t, $N(s+t) - N(s) = \{N_1(s+t) - N_1(s)\} + \{N_2(s+t) - N_2(s)\}$ $\equiv N_1(t) + N_2(t) = N(t)$, by stationary increment property of $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$. Thus, $\{N(t): t \geq 0\}$ possesses stationary increment property. With this, we can conclude that $\{N(t): t \geq 0\}$ is a Poisson process. It remains to be shown that its intensity is $\lambda_1 + \lambda_2$. We do this by establishing the 3rd property of Poisson process.

$$P(N(t) = n) = P(N_1(t) + N_2(t) = n) = \sum_{k=0}^{n} P(N_1(t) = k, N_2(t) = n-k)$$
$$= \sum_{k=0}^{n} P(N_1(t) = k) P(N_2(t) = n-k), \text{due to independence of } N_1(t) \text{ and } N_2(t)$$
$$= \sum_{k=0}^{n} e^{-\lambda_1 t} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_2 t} \frac{(\lambda_2 t)^{n-k}}{(n-k)!} = e^{-(\lambda_1+\lambda_2)t} \frac{(\lambda_1+\lambda_2)^n t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$
$$= e^{-(\lambda_1+\lambda_2)t} \frac{\left((\lambda_1+\lambda_2)t\right)^n}{n!} \text{ for } n = 0,1,2,\dots \text{and } t > 0, \text{as required.}$$

We can also establish the alternate form of the 3rd property.

$$P(N(h) = 1) = P(N_1(h) + N_2(h) = 1)$$
$$= P(N_1(h) = 1, N_2(h) = 0) + P(N_1(h) = 0, N_2(h) = 1)$$
$$= P(N_1(h) = 1)P(N_2(h) = 0) + P(N_1(h) = 0)P(N_2(h) = 1)$$
$$= \big(\lambda_1 h + o(h)\big)\big(1 - \lambda_2 h + o(h)\big) + \big(1 - \lambda_1 h + o(h)\big)\big(\lambda_2 h + o(h)\big)$$
$$= (\lambda_1 + \lambda_2)h + o(h), \text{as required}$$

$$P(N(h) \geq 2) = P(N_1(h) + N_2(h) \geq 2)$$
$$= P(N_1(h) = 1, N_2(h) = 1) + P(N_1(h) \leq 1, N_2(h) \geq 2) + P(N_1(h) \geq 2, N_2(h) \leq 1)$$
$$= P(N_1(h) = 1)P(N_2(h) = 1) + P(N_1(h) \leq 1)P(N_2(h) \geq 2) + P(N_1(h) \geq 2)P(N_2(h) \leq 1)$$
$$= \big(\lambda_1 h + o(h)\big)\big(\lambda_2 h + o(h)\big) + \big(1 + o(h)\big)\big(o(h)\big) + \big(o(h)\big)\big(1 + o(h)\big)$$
$$= o(h), \text{as required}$$

From the above analysis, one can immediately see that if three or more independent Poisson processes merge into one, then the resultant process is a Poisson process with intensity same as the sum of the individual intensities.

Thinning of a Poisson process: Consider a traffic flow, which can be modelled as a Poisson process $\{N(t): t \geq 0\}$ with intensity λ, is splitting into two, say 1 and 2. Let every vehicle in the original flow goes to flow 1 with probability p independent of everything else, and with probability $1-p$ it goes to flow 2. This phenomenon is known as the thinning of a Poisson process. Let $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ denote the flows in 1 and 2, where $N_1(t) + N_2(t) = N(t)\ \forall t$. We show that $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ are independent Poisson processes with intensities $p\lambda$ and $(1-p)\lambda$ respectively.

First, we show that $\{N_1(t): t \geq 0\}$ is a Poisson process with intensity $p\lambda$. Then the other part, i.e., $\{N_2(t): t \geq 0\}$ is a Poisson process with intensity $(1-p)\lambda$, follows automatically. The independence of these two processes is established later. For $0 \leq s < t$, let $Z(s,t)$ denote the fraction of vehicles that goes to flow 1 during $(s,t]$. Then $N_1(t) - N_1(s) = Z(s,t)\big(N(t) - N(s)\big)$. Note that $Z(s,t)$ is a random variable that takes rational values between 0 and 1, and its mass function is determined by p and the mass function of $N(t) - N(s)$.

$\{N_1(t): t \geq 0\}$ is a counting process and $N_1(0) = 0$. For arbitrary $t_1 < t_2 \leq t_3 < t_4$, $N_1(t_2) - N_1(t_1) = Z(t_1,t_2)\big(N(t_2) - N(t_1)\big)$ and $N_1(t_4) - N_1(t_3) = Z(t_3,t_4)\big(N(t_4) - N(t_3)\big)$ are independent, because $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ are independent due to independent increment property of $\{N(t): t \geq 0\}$, and $Z(t_1,t_2)$ and $Z(t_3,t_4)$ depend only on $N(t_2) - N(t_1)$ and $N(t_4) - N(t_3)$ respectively. So, $\{N_1(t): t \geq 0\}$ possesses independent increment property. Next, for arbitrary s and t, $N_1(s+t) - N_1(s) = Z(s,s+t)\big(N(s+t) - N(s)\big) \equiv Z(0,t)N(t) = N_1(t)$, because $N(s+t) - N(s) \equiv N(t)$ due to stationary increment property of $\{N(t): t \geq 0\}$, and $Z(s,s+t) \equiv Z(0,t)$ for the same reason and the nature of dependence of Z on N. So, $\{N_1(t): t \geq 0\}$ possesses stationary increment property. It remains to be shown that its intensity is $p\lambda$. We do this by establishing the 3rd property of Poisson process.

$$
\begin{aligned}
P(N_1(t) = n_1) &= \sum_{n_2=0}^{\infty} P(N_1(t) = n_1, N_2(t) = n_2) = \sum_{n_2=0}^{\infty} P(N_1(t) = n_1, N(t) = n_1 + n_2) \\
&= \sum_{n_2=0}^{\infty} P(N_1(t) = n_1 | N(t) = n_1 + n_2) P(N(t) = n_1 + n_2) \\
&= \sum_{n_2=0}^{\infty} \binom{n_1 + n_2}{n_1} p^{n_1} (1-p)^{n_2} \times e^{-\lambda t} \frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!} \\
&= e^{-p\lambda t} \frac{(p\lambda t)^{n_1}}{n_1!} \sum_{n_2=0}^{\infty} e^{-(1-p)\lambda t} \frac{\big((1-p)\lambda t\big)^{n_2}}{n_2!} \\
&= e^{-p\lambda t} \frac{(p\lambda t)^{n_1}}{n_1!} \text{ for } n_1 = 0,1,2,\ldots \text{ and } t > 0, \text{as required}
\end{aligned}
$$

We can also establish the alternate form of the 3rd property.

$$P(N_1(h)=1) = P(N_1(h)=1|N(h)=1)P(N(h)=1) + P(N_1(h)=1|N(h)\geq 2)P(N(h)\geq 2)$$
$$= p(\lambda h + o(h)) + (\text{a quantity} \leq 1)\times o(h) = p\lambda h + o(h), \text{as required}$$

$$P(N_1(h)\geq 2) = P(N_1(h)\geq 2|N(h)\geq 2)P(N(h)\geq 2) = (\text{a quantity} \leq 1)\times o(h)$$
$$= o(h), \text{as required}$$

We have shown that $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ are Poisson processes with intensities $p\lambda$ and $(1-p)\lambda$. Now, we establish their independent. This requires showing that $N_1(t_1)$ and $N_2(t_2)$ are independent random variables for all t_1 and t_2. For $t_1 = t_2 = t$,

$$P(N_1(t)=n_1, N_2(t)=n_2) = P(N_1(t)=n_1|N(t)=n_1+n_2)P(N(t)=n_1+n_2)$$
$$= \binom{n_1+n_2}{n_1} p^{n_1}(1-p)^{n_2} \times e^{-\lambda t}\frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!}$$
$$= e^{-p\lambda t}\frac{(p\lambda t)^{n_1}}{n_1!} \times e^{-(1-p)\lambda t}\frac{\left((1-p)\lambda t\right)^{n_2}}{n_2!}$$
$$= P(N_1(t)=n_1)\times P(N_2(t)=n_2) \text{ for all } n_1, n_2 = 0,1,2,\ldots.$$

So, $N_1(t_1)$ and $N_2(t_2)$ are independent when $t_1 = t_2$. The proof for $t_1 \neq t_2$ is more involved and is provided in Appendix A.

Generalized Poisson processes

Compound Poisson process: Consider vehicles arrive in a highway toll plaza in accordance with a Poisson process $\{N(t): t \geq 0\}$ with intensity λ. Upon arrival, each vehicle pays a toll amount depending on its type. Let $Z_1, Z_2, Z_3, \ldots$ denote the toll amount paid by the 1st, 2nd, 3rd, … vehicles arriving at the plaza. It's reasonable to assume that $Z_1, Z_2, Z_3, \ldots$ are *iid* random variables and independent of $N(t)$ as well. Then $X(t) \coloneqq \sum_{i=1}^{N(t)} Z_i$ denotes the total collection at the toll plaza till time t. The stochastic process $\{X(t): t \geq 0\}$ is known as the compound Poisson process. It is a continuous-time process, and its states can be continuous as well as discrete depending on the nature of $Z_1, Z_2, Z_3, \ldots$. It's a generalization of the Poisson process. Consider $P(Z=1)=1$; then $X(t) = \sum_{i=1}^{N(t)} 1 = N(t)$ for all t.

It's not easy to obtain distribution function of $X(t)$, except for special cases. However, we can obtain its mean and variance rather easily using the results obtained during the discussion on the random sum of random variables. For $t \geq 0$,

$$E[X(t)] = E\left[\sum_{i=1}^{N(t)} Z_i\right] = E[Z]E[N(t)] = \lambda t E[Z]$$
$$V(X(t)) = V\left(\sum_{i=1}^{N(t)} Z_i\right) = V(Z)E[N(t)] + E^2[Z]V(N(t)) = \lambda t(V(Z)+E^2[Z]) = \lambda t E[Z^2]$$

Non-homogeneous Poisson process: A counting process is a Poisson process if it possesses independent and stationary increment properties. If we look at any counting process around us, typically we see that the increment independent property holds, particularly if there is a large number of potential sources of event. If we look at the customer arrival in a shopping mall, most of the city residents are potential customers, i.e., sources of event. Since there are so many sources of event, occurrences of events in disjoint durations do not influence each other. This is not the case when the sources of events are limited. Consider breakdown of machines in a workshop. Due to the limited number of machines, breakdowns in disjoint intervals can influence one another (with a negative correlation).

Unlike independent increment property, stationary increment property is less universal. If there is a trigger for the sources of event to materialize in a faster or slower manner, then stationary increment property does not hold. Most of the man-made processes have such triggers. Consider trucks passing through a city. The rate increases in the night times. Natural processes typically have no such triggers, and the events occur in a steady manner. If we let go of the stationary increment property, i.e., we allow intensity to be time-dependent, then the counting process is known as the non-homogeneous Poisson process.

A counting process $\{N(t): t \geq 0\}$ *is a non-homogeneous Poisson process with intensity* $\lambda(t)$, $t \geq 0$ *if it satisfies three properties: (i)* $N(0) = 0$, *(ii) it possesses independent increment property, and (iii)* $P(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$ *and* $P(N(t+h) - N(t) \geq 2) = o(h)$ *for all* $t > 0$. The 3rd condition in this definition is not redundant, which was the case with the regular (i.e., homogeneous) Poisson process. If $\lambda(t)$ is a constant function, then that implies stationary increment, and we get the regular Poisson process.

Applicability of Poisson distribution: In the regular Poisson process with rate λ, $N(s+t) - N(s) \sim Pois(\lambda t)$ for all $s \geq 0$ and $t > 0$. For the case of non-homogeneous Poisson process with rate $\lambda(\tau), \tau \geq 0$, $N(s+t) - N(s) \sim Pois\left(\int_s^{s+t} \lambda(\tau) d\tau\right)$ for all $s, t \geq 0$. First, we show this result for $s = 0$, and then we examine the case of $s > 0$.

If two random variables have the same distribution function or the same moment generating function, then the random variables are identical. The same is true with Laplace transform. Let $g(t) = E\left[e^{-uN(t)}\right]$ for $t \geq 0$ denote the Laplace transform of $N(t)$ *w.r.t.* parameter $u \geq 0$. We obtain expression for $g(t)$ from the definition of non-homogeneous Poisson process and show that it is same as the Laplace transform of $Pois\left(\int_0^t \lambda(\tau) d\tau\right)$.

By the independent increment property,

$$g(t+h) = E\left[e^{-uN(t+h)}\right] = E\left[e^{-uN(t)} e^{-u\{N(t+h)-N(t)\}}\right] = g(t) E\left[e^{-u\{N(t+h)-N(t)\}}\right]$$

Now, $N(t+h) - N(t)$ takes values 0, 1, and 2 or more with probabilities $1 - \lambda(t)h + o(h)$, $\lambda(t)h + o(h)$, and $o(h)$. Since $N(t+h) - N(t)$ and u are non-negative, $e^{-u\{N(t+h)-N(t)\}}$ always takes values in $[0,1]$. Then

$$E\left[e^{-u\{N(t+h)-N(t)\}}\right] = e^0\{1-\lambda(t)h+o(h)\} + e^{-u}\{\lambda(t)h+o(h)\} + [\text{a qty} \in (0,1)]\{o(h)\}$$
$$= 1-\lambda(t)h + e^{-u}\lambda(t)h + o(h)$$

$$\Rightarrow g(t+h)-g(t) = g(t)\{\lambda(t)(e^{-u}-1)h+o(h)\}$$
$$\Rightarrow \lim_{h\to 0}\frac{g(t+h)-g(t)}{h} = \lim_{h\to 0} g(t)\left\{\lambda(t)(e^{-u}-1)+\frac{o(h)}{h}\right\}$$
$$\Rightarrow g'(t) = g(t)\lambda(t)(e^{-u}-1)\ \forall t \geq 0$$

Let us solve the above differential equation by integrating it is the range $[0,t]$.

$$\int_0^t \frac{g'(\tau)}{g(\tau)}d\tau = \int_0^t \lambda(\tau)(e^{-u}-1)d\tau \Rightarrow \ln|g(t)| - \ln|g(0)| = (e^{-u}-1)\int_0^t \lambda(\tau)d\tau$$
$$\Rightarrow \ln\big(g(t)\big) - \ln(1) = (e^{-u}-1)\int_0^t \lambda(\tau)d\tau \Rightarrow g(t) = e^{(e^{-u}-1)\int_0^t \lambda(\tau)d\tau}$$

We derived the expression for $g(t)$, the Laplace transform of $N(t)$. Now, consider $Y \sim Pois\left(\int_0^t \lambda(\tau)d\tau\right)$. Laplace transform of Y is

$$E[e^{-uY}] = \sum_{y=0}^{\infty} e^{-uy}e^{-\int_0^t \lambda(\tau)d\tau}\frac{\left(\int_0^t \lambda(\tau)d\tau\right)^y}{y!} = e^{-\int_0^t \lambda(\tau)d\tau}\sum_{y=0}^{\infty}\frac{\left(e^{-u}\int_0^t \lambda(\tau)d\tau\right)^y}{y!}$$
$$= e^{-\int_0^t \lambda(\tau)d\tau}e^{e^{-u}\int_0^t \lambda(\tau)d\tau} = e^{(e^{-u}-1)\int_0^t \lambda(\tau)d\tau}$$

Since the Laplace transforms are identical, $N(t) \sim Pois\left(\int_0^t \lambda(\tau)d\tau\right)$ for all $t \geq 0$. This result is for $s=0$. For $s>0$, consider $\{N_s(t) = N(s+t)-N(s): t \geq 0\}$. It is a counting process with $N_s(0)=0$. It possesses independent increment property, as $\{N(t): t \geq 0\}$ possesses it. Also, $P(N_s(t+h)-N_s(t)=1) = P(N(s+t+h)-N(s+t)=1) = \lambda(s+t)h+o(h)$ and $P(N_s(t+h)-N_s(t)\geq 2) = P(N(s+t+h)-N(s+t)\geq 2) = o(h)$ for all $t \geq 0$. So, $\{N_s(t): t\geq 0\}$ is a non-homogeneous Poisson process with intensity $\lambda_s(t) := \lambda(s+t)$, $t \geq 0$. Then $N_s(t) \sim Pois\left(\int_0^t \lambda_s(\tau)d\tau\right) \Rightarrow N(s+t)-N(s) \sim Pois\left(\int_0^t \lambda(s+\tau)d\tau\right)$. With $s+\tau$ replaced by τ, $N(s+t)-N(s) \sim Pois\left(\int_s^{s+t} \lambda(\tau)d\tau\right)$ for all $s,t \geq 0$.

Impact of non-homogeneity: Non-homogeneity of the intensity does not change the Poisson distribution, but it affects other distributions associated with a regular Poisson process. Let us consider the inter-event times. Let $\Lambda(t) = \int_0^t \lambda(\tau)d\tau$, for ease of notation.

$$P(T_1 > t) = P(N(t)=0) = e^{-\Lambda(t)} \Rightarrow F_{T_1}(t) = 1-e^{-\Lambda(t)}\ \forall t > 0$$
$$\Rightarrow f_{T_1}(t) = e^{-\Lambda(t)}\frac{d}{dt}\left(\int_0^t \lambda(\tau)d\tau\right) = \lambda(t)e^{-\Lambda(t)}\ \forall t > 0$$

This distribution resembles with the exponential distribution, it's not the same. It does not have the memoryless property, because $P(T_1 > s+t) = e^{-\Lambda(s+t)}$ and $P(T_1 > s)P(T_1 > t)$

$= e^{-\Lambda(s)}e^{-\Lambda(t)}$ are generally different. Note that $\Lambda(s+t) = \int_0^{t+s} \lambda(\tau)d\tau$, but $\Lambda(s)\Lambda(t) = \left\{\int_0^s \lambda(\tau)d\tau\right\}\left\{\int_0^t \lambda(\tau)d\tau\right\}$. A constant $\lambda(\tau)$ is an exception.

Now, $P(T_2 > t|T_1 = s) = P(N(s+t) - N(s) = 0|T_1 = s)$
$= P(N(s+t) - N(s) = 0)$, by the independent increament property
$= e^{-\{\Lambda(s+t)-\Lambda(s)\}}$, by Poisson distribution of $N(s+t) - N(s)$

$\Lambda(s+t) - \Lambda(s) = \int_s^{t+s} \lambda(\tau)d\tau$ generally depends on s. Constant $\lambda(\tau)$ is an exception. Thus, T_1 and T_2, in general, are not independent. Also,

$$P(T_2 > t) = \int_0^\infty P(T_2 > t|T_1 = s)f_{T_1}(s)ds = \int_0^\infty e^{-\{\Lambda(s+t)-\Lambda(s)\}}\lambda(s)e^{-\Lambda(s)}ds$$
$$= \int_0^\infty \lambda(s)e^{-\Lambda(s+t)}ds \ \forall t > 0$$

which does not simplify any further. T_1 and T_2, in general, have different distributions. Thus, inter-event times in a non-homogeneous Poisson process generally are non-independent and non-identical, unlike the regular Poisson process.

Waiting times in non-homogeneous Poisson process are no more the Erlang distributions, but the form resembles with the Erlang distribution, as shown below. For $t > 0$,

$$F_{W_n}(t) = P(W_n \le t) = P(N(t) \ge n) = 1 - \sum_{k=0}^{n-1} P(N(t) = k) = 1 - \sum_{k=0}^{n-1} e^{-\Lambda(t)}\frac{\{\Lambda(t)\}^k}{k!}$$
$$\Rightarrow f_{W_n}(t) = \sum_{k=0}^{n-1} \lambda(t)e^{-\Lambda(t)}\frac{\{\Lambda(t)\}^k}{k!} - \sum_{k=1}^{n-1} e^{-\Lambda(t)}\frac{\left(\Lambda(t)\right)^{k-1}\lambda(t)}{(k-1)!} = \lambda(t)e^{-\Lambda(t)}\frac{\left(\Lambda(t)\right)^{n-1}}{(n-1)!}$$

One can explore conditional distributions, in a manner similar to the regular Poisson process, and obtain the following three expressions. *Try these yourself*.

$$P(N(t) = n|N(s) = m) = e^{-\{\Lambda(t)-\Lambda(s)\}}\frac{\{\Lambda(t) - \Lambda(s)\}^{n-m}}{(n-m)!} \quad \text{for } s < t \text{ and } m \le n$$
$$P(N(s) = m|N(t) = n) = \binom{n}{m}\left(\frac{\Lambda(s)}{\Lambda(t)}\right)^m \left(\frac{\Lambda(s)}{\Lambda(t)}\right)^{n-m} \quad \text{for } s < t \text{ and } m \le n$$
$$P(W \le s|N(t) = n) = \frac{\Lambda(s)}{\Lambda(t)} \quad \text{for } s \le t; W \text{ is the time of occurrence of any of the events}$$

One can even go further and show the following results associated with superposition and thinning of non-homogeneous Poisson processes.

Res: Let $\{N_1(t): t \ge 0\}$ and $\{N_2(t): t \ge 0\}$ denote independent non-homogeneous Poisson processes with intensities $\lambda_1(t)$ and $\lambda_2(t)$, $t \ge 0$. Then $\{N(t) = N_1(t) + N_2(t): t \ge 0\}$ is a non-homogeneous Poisson processes with intensity $\lambda(t) = \lambda_1(t) + \lambda_2(t)$, $t \ge 0$.

Res: Consider a non-homogeneous Poisson process $\{N(t): t \geq 0\}$ with intensity $\lambda(t)$, $t \geq 0$. Every event occurring through this process is classified as a type-1 event with probability $p(t)$, $t \geq 0$, and type-2 with probability $1 - p(t)$. Let $N_1(t)$ and $N_2(t)$ denote the numbers of type-1 and type-2 events occurring during $(0, t]$. Then $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ are independent non-homogeneous Poisson processes with intensities $p(t)\lambda(t)$ and $\{1 - p(t)\}\lambda(t)$, $t \geq 0$. Note that the classification probabilities can change with time.

Both the results, except the independence part in the second result, can be proved following similar proofs for the regular Poisson process. *Try these yourself.*

Practice problems

Book-1: Introduction to Probability Models by Sheldon Ross [10$^{\text{th}}$ edition]

Poisson Process
Book-1, Chapter-5, Exercise No. 32, 34, 36, 39, 44

Exponential Distribution
Book-1, Chapter-5, Exercise No. 20, 21, 23, 24, 28

Conditional Distributions
Book-1, Chapter-5, Exercise No. 60, 64, 72, 73

Superposition and Thinning
Book-1, Chapter-5, Exercise No. 59, 62, 67, 74

Generalized Poisson Processes
Book-1, Chapter-5, Exercise No. 78, 81a, 85, 88

Appendix A

Every event happening through a Poisson process $\{N(t): t \geq 0\}$ with intensity λ is classified either into a type-1 event with probability p or a type-2 event with probability $q = 1 - p$. Let $N_1(t)$ and $N_2(t)$ denote the number of type-1 and type-2 events in $(0, t]$. Then $\{N_1(t): t \geq 0\}$ and $\{N_2(t): t \geq 0\}$ are Poisson processes with intensities $p\lambda$ and $q\lambda$, as shown earlier. Here, we prove independence of these stochastic processes, i.e., we show that $N_1(s)$ and $N_2(t)$ are independent random variables for all $s, t > 0$. Earlier, we proved this for $s = t$.

For $s \leq t$, we show that $P(N_1(s) = m_1, N_2(t) = n_2) = P(N_1(s) = m_1) \cdot P(N_2(t) = n_2)$ for $m_1, n_2 = 0,1,2, \ldots$. If $s > t$, then we can alter the categorization of events, and the same proof works. With $N_1(s) = m_1$ and $N_2(t) = n_2$, we have $m_1 = N_1(s) \leq N(s) \leq N_1(s) + N_2(t) = m_1 + n_2$ and $N(t) \geq N_1(s) + N_2(t) = m_1 + n_2$. Then

$$P(N_1(s)=m_1, N_2(t)=n_2)$$
$$=\sum_{m=m_1}^{m_1+n_2}\sum_{n=m_1+n_2}^{\infty} P(N_1(s)=m_1, N_2(t)=n_2 | N(s)=m, N(t)=n)P(N(s)=m, N(t)=n)$$
$$=\sum_{m=m_1}^{m_1+n_2}\sum_{n=m_1+n_2}^{\infty}\left[\begin{array}{c} P\left(\begin{array}{c} N_1(s)=m_1, \\ N_2(t)-N_2(s)=n_2-(m-m_1)\end{array}\middle| \begin{array}{c} N(s)=m, \\ N(t)-N(s)=n-m\end{array}\right) \\ \times P(N(s)=m, N(t)-N(s)=n-m)\end{array}\right]$$

The conditional probability in the above expression is the probability of "m_1 events out of the m events that occur in $(0,s]$ are of type-1 (and the remaining $m-m_1$ events are of type-2) and $n_2-(m-m_1)$ events out of the $n-m$ events that occur in $(s,t]$ are of type-2 (and the remaining $n-m_1-n_2$ events are of type-1)". Since $N(s)$ and $N(t)-N(s)$ are independent and the categorization of events are independent of "everything else",

$$P(N_1(s)=m_1, N_2(t)=n_2)$$
$$=\sum_{m=m_1}^{m_1+n_2}\sum_{n=m_1+n_2}^{\infty}\left[\begin{array}{c}\left\{\begin{array}{c} P(N_1(s)=m_1|N(s)=m)\cdot \\ P(N_2(t)-N_2(s)=n_2-(m-m_1)|N(t)-N(s)=n-m)\end{array}\right\} \\ \times\{P(N(s)=m)\cdot P(N(t)-N(s)=n-m)\}\end{array}\right]$$
$$=\sum_{m=m_1}^{m_1+n_2}\sum_{n=m_1+n_2}^{\infty}\left[\begin{array}{c}\left\{\binom{m}{m_1}p^{m_1}q^{m-m_1}\cdot\binom{n-m}{n_2-m+m_1}p^{n-m_1-n_2}q^{n_2-m+m_1}\right\} \\ \times\left\{e^{-\lambda s}\frac{(\lambda s)^m}{m!}\cdot e^{-\lambda(t-s)}\frac{\left(\lambda(t-s)\right)^{n-m}}{(n-m)!}\right\}\end{array}\right]$$
$$=\sum_{m=m_1}^{m_1+n_2}\sum_{n=m_1+n_2}^{\infty}\frac{p^{n-n_2}q^{n_2}e^{-\lambda t}\lambda^n s^m(t-s)^{n-m}}{m_1!\,(m-m_1)!\,(n_2-m+m_1)!\,(n-m_1-n_2)!}$$
$$=\sum_{m=m_1}^{m_1+n_2}\sum_{n=m_1+n_2}^{\infty}\left[\begin{array}{c}\left\{e^{-p\lambda s}\frac{(p\lambda s)^{m_1}}{m_1!}\cdot e^{-q\lambda t}\frac{(q\lambda t)^{n_2}}{n_2!}\right\}\times\left\{\binom{n_2}{m-m_1}\frac{s^{m-m_1}(t-s)^{n_2-m+m_1}}{t^{n_2}}\right\} \\ \times\left\{e^{-p\lambda(t-s)}\frac{\left(p\lambda(t-s)\right)^{n-m_1-n_2}}{(n-m_1-n_2)!}\right\}\end{array}\right]$$
$$=\left\{e^{-p\lambda s}\frac{(p\lambda s)^{m_1}}{m_1!}\cdot e^{-q\lambda t}\frac{(q\lambda t)^{n_2}}{n_2!}\right\}\times\left\{\sum_{m-m_1=0}^{n_2}\binom{n_2}{m-m_1}\frac{s^{m-m_1}(t-s)^{n_2-m+m_1}}{t^{n_2}}\right\}$$
$$\times\left\{\sum_{n-m_1-n_2=0}^{\infty}\left(e^{-p\lambda(t-s)}\frac{\left(p\lambda(t-s)\right)^{n-m_1-n_2}}{(n-m_1-n_2)!}\right)\right\}$$
$$=e^{-p\lambda s}\frac{(p\lambda s)^{m_1}}{m_1!}\cdot e^{-q\lambda t}\frac{(q\lambda t)^{n_2}}{n_2!}=P(N_1(s)=m_1)\cdot P(N_2(t)=n_2), \text{as required.}$$