Module 1

Introduction to Markov chain

Topics: Markov chain, Transition probability matrix, Markov chain models

Markov chain

Introduction to stochastic processes: With basic probability theory, we can model and analyze stochastic systems that can be described using one or a few random variables. Sometimes, we need many random variables to describe a system, and often, such random variables evolve over time, for example, daily closing indices of BSE, population size of a country over time, etc. In such cases, we model the system using stochastic processes and study how the system evolves with time. Formally, stochastic process is a sequence of random variables $\{X_t : t \in T\}$, where T is called the index set. Typically, T represents time and X_t represents state of the system at time $t \in T$. Depending on the natures of T and X_t (discrete vs. continuous), we classify stochastic processes into four types.

	X_t is discrete	X_t is continuous
T is discrete	Discrete-time discrete-state stochastic process Example: Markov chain	Discrete-time continuous-state stochastic process Example: Martingale
T is continuous	Continuous-time discrete-state stochastic process Example: Poisson process	Continuous-time continuous-state stochastic process Example: Brownian motion

In this course, we will consider discrete-state stochastic processes such as Markov chain, Branching chain, Poisson process, Renewal process, and Continuous-time Markov chain. The first two are discrete-time process and the last three are continuous-time process.

Markov chain: A Markov chain is a discrete time discrete state stochastic process that poses Markov property, which says that if the current state is known, then the knowledge of past is irrelevant in 'deciding' future. Formally, a discrete time discrete state stochastic process $\{X_n : n = 0,1,2,\dots\}$ is Markov chain if $P(X_{n+1} = j | X_n = i, X_{n-1} = i_1, X_{n-2} = i_2, \dots, X_0 = i_n) = P(X_{n+1} = j | X_n = i)$ for all $j, i, i_1, i_2, \dots, i_n \in \Omega$ (support of the random variables X_0, $X_1, X_2, \dots$) and $n = 01,2, \dots$.

Example 1: Let us consider a simple example of Markov chain. A day is either sunny or cloudy or rainy, denoted by 1, 2, 3 respectively. A sunny day is followed by another sunny day 70% of the times and a cloudy day 30% of the times. A cloudy day is followed by a sunny day 30% of the times, another cloudy day 50% of the times, and a rainy day 20% of the times. A rainy day is followed by a cloudy day 60% of the times and another rainy day 40% of the times. From the description, it's evident that the nature of the next day is decided (in a stochastic sense) by today and the past does not play any role. So, the stochastic process

$\{X_n: n = 0,1,2, \dots\}$, where X_n represents nature of day n (say, 1 for sunny, 2 for cloudy, and 3 for rainy), is a Markov chain. Here, $\Omega = \{1,2,3\}$.

Example 2: Let us consider a slight variation of example 1, as shown in the table below.

Tomorrow →	In rainy season (Jul-Sep)			In other seasons (Oct-Jun)		
Today ↓	Sunny day	Cloudy day	Rainy day	Sunny day	Cloudy day	Rainy day
Sunny day	55% times	45% times	--	75% times	25% times	--
Cloudy day	15% times	50% times	35% times	35% times	50% times	15% times
Rainy day	--	40% times	60% times	--	67⅔% times	33⅓% times

Observe that the weighted average of the numbers in the above table (with weights 0.25 for rainy season and 0.75 for non-rainy seasons) matches with the numbers in example 1. Example 2, too, is a Markov chain, as the nature of tomorrow is decided (in a stochastic sense) by today and the past does not play any role. Only thing that is different now is that the probability of tomorrow being a sunny/cloudy/rainy day is dependent on the time of the year, apart from the state today. Definition of Markov chain allows for both types of dependence. In example 1, time-dependence is absent. State-dependence is common in both the examples. State-dependence is present in all Markov chains; without it, we just have a sequence of independent random variables.

Markov property: In the definition of Markov chain, Markov property is defined as follows: If the current state is known, then the knowledge of past is irrelevant in 'deciding' future; formally, $P(X_{n+1} = j | X_n = i, X_{n-1} = i_1, X_{n-2} = i_2, \dots, X_0 = i_n) = P(X_{n+1} = j | X_n = i)$ for all $n \geq 0$ and $j, i, i_1, i_2, \dots, i_n \in \Omega$. Here, we explore some consequences of this property.

In the definition we considered complete knowledge of the past. Let us consider partial knowledge. For arbitrary $n > n_1 > n_2 > \cdots > n_k \geq 0$,

$$P\left(X_{n+1} = j | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k\right)$$

$$= \sum_{i'_1, i'_2, \dots, i'_l \in \Omega} P\left(X_{n+1} = j, X_{m_1} = i'_1, X_{m_2} = i'_2, \dots, X_{m_l} = i'_l | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k\right),$$

where $m_1, m_2, \dots, m_l$ are the missing time points from the past (by 3rd axiom)

$$= \sum_{i'_1, i'_2, \dots, i'_l \in \Omega} P\left(X_{n+1} = j | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k, X_{m_1} = i'_1, X_{m_2} = i'_2, \dots, X_{m_l} = i'_l\right)$$
$$\cdot P\left(X_{m_1} = i'_1, X_{m_2} = i'_2, \dots, X_{m_l} = i'_l | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k\right)$$

$$= \sum_{i'_1, i'_2, \dots, i'_l \in \Omega} P(X_{n+1} = j | X_n = i, X_{n-1} = \cdot, X_{n-2} = \cdot, \dots, X_0 = \cdot)$$
$$\cdot P\left(X_{m_1} = i'_1, X_{m_2} = i'_2, \dots, X_{m_l} = i'_l | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k\right)$$

$$= \sum_{i'_1, i'_2, \dots, i'_l \in \Omega} P(X_{n+1} = j | X_n = i) \quad \text{(by Markov property)}$$
$$\cdot P\left(X_{m_1} = i'_1, X_{m_2} = i'_2, \dots, X_{m_l} = i'_l | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k\right)$$

$$= P(X_{n+1} = j | X_n = i\ ) \cdot \sum_{i'_1, i'_2, \dots, i'_l \in \Omega} P\big(X_{m_1} = i'_1, X_{m_2} = i'_2, \dots, X_{m_l} = i'_l | X_n = i, X_{n_1} = i_1, X_{n_2} = i_2, \dots, X_{n_k} = i_k\big)$$

$$= P(X_{n+1} = j | X_n = i\ ) \text{ (by 3rd axiom)}$$

Thus, we can have partial history in the definition of Markov property.

In the definition of Markov property, we consider the current state (i.e., X_n) to be known. Let us consider absence of this. For arbitrary $m < n$,

$$P(X_{n+1} = j | X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m)$$
$$= \sum_{i'_n \in \Omega} P(X_{n+1} = j, X_n = i'_n | X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m) \quad \text{(by 3rd axiom)}$$
$$= \sum_{i'_n \in \Omega} P(X_{n+1} = j | X_n = i'_n, X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m) \cdot P(X_n = i'_n | X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m)$$
$$= \sum_{i'_n \in \Omega} P(X_{n+1} = j | X_n = i'_n) \quad \text{(by the previous observation)} \cdot P(X_n = i'_n | X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m)$$

We apply the above logic to the 2nd term in the above expression and obtain the following

$$P(X_{n+1} = j | X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m)$$
$$= \sum_{i'_n \in \Omega} P(X_{n+1} = j | X_n = i'_n) \cdot \sum_{i'_{n-1} \in \Omega} P(X_n = i'_n | X_{n-1} = i'_{n-1}) \cdot P(X_{n-1} = i'_{n-1} | X_m = i, X_{m-1} = i_1, \dots, X_0 = i_m)$$
$$= \sum_{i'_n, i'_{n-1} \in \Omega} P(X_{n+1} = j | X_n = i'_n, X_{n-1} = i'_{n-1}) \cdot P(X_n = i'_n | X_{n-1} = i'_{n-1}) \cdot P(X_{n-1} = i'_{n-1} | X_m = i, X_{m-1} = i_1, \dots, X_0 = i_m)$$
$$= \sum_{i'_n, i'_{n-1} \in \Omega} P(X_{n+1} = j, X_n = i'_n | X_{n-1} = i'_{n-1}) \cdot P(X_{n-1} = i'_{n-1} | X_m = i, X_{m-1} = i_1, \dots, X_0 = i_m)$$

Continuing in this manner, we eventually obtain the following

$$P(X_{n+1} = j | X_m = i, X_{m-1} = i_1, X_{m-2} = i_2, \dots, X_0 = i_m)$$
$$= \sum_{i'_n, \dots, i'_{m+2}, i'_{m+1} \in \Omega} P(X_{n+1} = j, X_n = i'_n, \dots, X_{m+2} = i'_{m+2} | X_{m+1} = i'_{m+1}) \cdot P(X_{m+1} = i'_{m+1} | X_m = i, X_{m-1} = i_1, \dots, X_0 = i_m)$$

$$= \sum_{i'_n,\dots,i'_{m+2},i'_{m+1}\in\Omega} P(X_{n+1}=j, X_n=i'_n,\dots,X_{m+2}=i'_{m+2}|X_{m+1}=i'_{m+1},X_m=i)$$
$$\cdot P(X_{m+1}=i'_{m+1}|X_m=i) \quad \text{(by Markov property)}$$
$$= \sum_{i'_n,\dots,i'_{m+2},i'_{m+1}\in\Omega} P(X_{n+1}=j, X_n=i'_n,\dots,X_{m+2}=i'_{m+2},X_{m+1}=i'_{m+1}|X_m=i)$$
$$= P(X_{n+1}=j|X_m=i\,) \quad \text{(by 3rd axiom)}$$

So, if the current state is unknown, the latest known state assumes the role of current state.

Considering the above two observations, we can say that the Markov property implies the latest known state, independent of the past (complete or partial), decides future. If the above arguments look tedious, try X_{n-1} missing in the first part and $m = n-1$ in the second.

Transition probability matrix

Transition probabilities: Since present decides future in a stochastic sense in the Markov chain, we can define transition probabilities from present to (immediate) future as follows: $p_{i,j}^{n,n+1} := P(X_{n+1}=j|X_n=i)$ for all $i,j\in\Omega$ and $n\geq 0$. We can represent these transition probabilities in a matrix form as: $P^{n.n+1} = \left[p_{i,j}^{n,n+1}\right]_{i,j\in\Omega}$. Transition probability matrices for example 1 and 2 are shown below. It is time-invariant for example 1. Observe that the row sums in all three transition probability matrices are 1. This is always true, as $\sum_{j\in\Omega} p_{i,j}^{n,n+1} = \sum_{j\in\Omega} P(X_{n+1}=j|X_n=i) = P(X_{n+1}\in\Omega|X_n=i) = 1$ for all $i\in\Omega$ and $n\geq 0$. Column sums have no restriction, they can be any positive number.

Example 1: For all $n \geq 0$

$p_{i,j}^{n,n+1}$	$j=1$	$j=2$	$j=3$
$i=1$	0.7	0.3	0
$i=2$	0.3	0.5	0.2
$i=3$	0	0.6	0.4

n mod x is remainder of $n \div x$

Example 2: For all n with n mod $365 \in [182,273]$

$p_{i,j}^{n,n+1}$	1	2	3
1	0.55	0.45	0
2	0.15	0.5	0.35
3	0	0.4	0.6

Example 2: For all n with n mod $365 \notin [182,273]$

$p_{i,j}^{n,n+1}$	1	2	3
1	0.75	0.25	0
2	0.35	0.5	0.15
3	0	0.67	0.33

Most of the Markov chains of interest have time-invariant transition probabilities. Thus, *we restrict ourselves to Markov chains with time-invariant or stationary transition probabilities*. For such Markov chains, we write $p_{i,j}$ in place of $p_{i,j}^{n,n+1}$ and P in place of $P^{n.n+1}$. There $p_{i,j} = P(X_{n+1}=j|X_n=i)$ represents probability of transition from state i to state j in one step at any point in time and P represents one-step transition probability matrix.

Sufficiency of transition probabilities: Transition probabilities fully specify a Markov chain, i.e., two different Markov chains must have different transition probabilities, else they would behave in a similar manner. Using the transition probabilities, we can answer any question of interest about a Markov chain. In this section, we answer some such questions.

Path probability: Let us assume that the Markov chain begins in state i_0. Path probability is about the chain taking a path $i_0 \to i_1 \to i_2 \to \cdots \to i_n$ in the first n steps.

$$
\begin{aligned}
&P(\text{path } i_0 \to i_1 \to i_2 \to \cdots \to i_n) = P(X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_1 = i_1 | X_0 = i_0) \\
&= P(X_n = i_n | X_{n-1} = i_{n-1}, \dots, X_0 = i_0) \times P(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}, \dots, X_0 = i_0) \times \cdots \\
&\qquad\qquad \times P(X_1 = i_1 | X_0 = i_0) \\
&= P(X_n = i_n | X_{n-1} = i_{n-1}) \times P(X_{n-1} = i_{n-1} | X_{n-2} = i_{n-2}) \times \cdots \\
&\qquad\qquad \times P(X_1 = i_1 | X_0 = i_0) \text{ (by Markov property)} \\
&= p_{i_0 i_1} \times p_{i_1 i_2} \times \cdots \times p_{i_{n-1} i_n}
\end{aligned}
$$

If $X_m = i_0$ and we are interested in the Markov chain taking the path $i_0 \to i_1 \to i_2 \to \cdots \to i_n$ in the next n steps, then following the above logic, we obtain $P(X_{m+n} = i_{m+n}, X_{m+n-1} = i_{n-1}, \dots, X_{m+1} = i_1 | X_m = i_0) = p_{i_0 i_1} \times p_{i_1 i_2} \times \cdots \times p_{i_{n-1} i_n}$. Clearly, path probabilities are independent of the starting time point, due to stationarity of the transition probabilities. Therefore, we can say that if X_m is known, then the Markov chain 'restarts' from there.

In example 1, the probability of three consecutive rainy days given that today is cloudy would be $p_{23} \times p_{33} \times p_{33} = 0.2 \times 0.4 \times 0.4 = 0.032$. Probability of a sunny weekend given that Friday is rainy would be $p_{31} \times p_{11} = 0 \times 0.7 = 0$.

n-step transition probabilities: In example 1, let us obtain probability of a sunny Sunday given that Friday is rainy, i.e., $P(X_2 = 1 | X_0 = 3)$. In terms of path, we are asking about probability of all 2-step paths that starts at state 3 and ends at state 1. There are three such paths: $3 \to 1 \to 1$, $3 \to 2 \to 1$, $3 \to 3 \to 1$ with path probabilities $p_{31} \times p_{11} = 0$, $p_{32} \times p_{21} = 0.18$, and $p_{33} \times p_{31} = 0$. Adding these, $P(X_2 = 1 | X_0 = 3) = 0.18$. In general,

$$
\begin{aligned}
p_{ij}^{(2)} := P(X_{m+2} = j | X_m = i) &= \sum_{k \in \Omega} P(X_{m+2} = j, X_{m+1} = k | X_m = i) \\
&= \sum_{k \in \Omega} P(X_{m+2} = j | X_{m+1} = k, X_m = i) \times P(X_{m+1} = k | X_m = i) \\
&= \sum_{k \in \Omega} P(X_{m+2} = j | X_{m+1} = k) \times p_{ik} = \sum_{k \in \Omega} p_{ik} \times p_{kj}
\end{aligned}
$$

Using the 2-step transition probabilities, we can obtain 3-step transition probabilities.

$$
\begin{aligned}
p_{ij}^{(3)} := P(X_{m+3} = j | X_m = i) &= \sum_{k \in \Omega} P(X_{m+3} = j, X_{m+2} = k | X_m = i) \\
&= \sum_{k \in \Omega} P(X_{m+3} = j | X_{m+2} = k, X_m = i) \times P(X_{m+2} = k | X_m = i) \\
&= \sum_{k \in \Omega} P(X_{m+3} = j | X_{m+2} = k) \times p_{ik}^{(2)} = \sum_{k \in \Omega} p_{ik}^{(2)} \times p_{kj}
\end{aligned}
$$

Continuing this way, we can obtain n-step transition probabilities recursively for $n \geq 2$ as:

$$p_{ij}^{(n)} := P(X_{m+n} = j|X_m = i) = \sum_{k\in\Omega} p_{ik}^{(n-1)} \times p_{kj}$$

The above recursive relation can also be written as: $p_{ij}^{(n)} = \sum_{k\in\Omega} p_{ik}^{(m)} \times p_{kj}^{(n-m)}$ for arbitrary positive $m < n$. This equation is known as the Chapman-Kolmogorov equation.

n-step transition probability matrix: Let us examine $p_{ij}^{(2)} = \sum_{k\in\Omega} p_{ik} \times p_{kj}$ with respect to the transition probability matrix P. The first set of probabilities in the sum-product appears in the i-th row of P and the second set of probabilities appears in the j-th column of P. If we multiply P with P and obtain P^2, then the sum-product appears in the (i,j)-th cell of P^2. This observation is true for all $i, j \in \Omega$. Let us represent the two-step transition probabilities in a matrix form: $P^{(2)} = \left[p_{ij}^{(2)}\right]_{i,j\in\Omega}$; then $P^{(2)} = P \times P = P^2$. Note the distinction between $P^{(2)}$ and P^2. So, with matrix multiplication, we can obtain all 2-step transition probabilities.

In example 1, we calculated $p_{31}^{(2)} = 0.18$. We can see the same in (3,1)-th cell in $P^{(2)}$.

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.6 & 0.4 \end{bmatrix} \Rightarrow P^{(2)} = P^2 = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.6 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0.6 & 0.4 \end{bmatrix}$$
$$= \begin{bmatrix} 0.58 & 0.36 & 0.06 \\ 0.36 & 0.46 & 0.18 \\ 0.18 & 0.54 & 0.28 \end{bmatrix}$$

Let us obtain 3-step transition probability matrix $P^{(3)} = \left[p_{ij}^{(3)}\right]_{i,j\in\Omega}$. Since $p_{ij}^{(3)} = \sum_{k\in\Omega} p_{ik}^{(2)} \times p_{kj}$, then following the above logic, $P^{(3)} = P^{(2)} \times P = P^2 \times P = P^3$. Continuing this way, we obtain n-step transition probability matrix $P^{(n)} = \left[p_{ij}^{(n)}\right]_{i,j\in\Omega} = P^n$ for all $n \geq 2$. Row-sums of all these matrices are always 1.

Mass function of X_n: Now we answer the last question, i.e., mass function of X_n for $n \geq 1$. With $X_0 = i$, $P(X_1 = j|X_0 = i) = p_{ij}$, $P(X_2 = j|X_0 = i) = p_{ij}^{(2)}$, and in general, $P(X_n = j|X_0 = i) = p_{ij}^{(n)}$ for all $j \in \Omega$. Observe that the mass function of X_n appears in the i-th row of $P^{(n)}$. In example 1, if today is sunny, then the day after tomorrow is sunny with probability 0.58, cloudy with probability 0.36, rainy with probability 0.06. If X_0 is unknown, then $P(X_n = j) = \sum_{i\in\Omega} P(X_n = j|X_0 = i) \times P(X_0 = i) = \sum_{i\in\Omega} P(X_0 = i) \times p_{ij}^{(n)}$ for all $j \in \Omega$.

We can obtain joint mass function of (X_m, X_n) for arbitrary $n > m \geq 1$. With $X_0 = i$, $P(X_n = k, X_m = j|X_0 = i) = P(X_n = k|X_m = j, X_0 = i) \times P(X_m = j|X_0 = i) = P(X_n = k| X_m = j) \times p_{ij}^{(m)} = p_{ij}^{(m)} \times p_{jk}^{(n-m)}$ for all $j, k \in \Omega$. With unknown X_0, $P(X_n = k, X_m = j) = \sum_{i\in\Omega} P(X_n = k, X_m = j|X_0 = i) \times P(X_0 = i) = \sum_{i\in\Omega} P(X_0 = i) \times p_{ij}^{(m)} \times p_{jk}^{(n-m)}$ $\forall j, k \in \Omega$. In general, for arbitrary $n_1 < n_2 < \cdots < n_m$, $P\left(X_{n_1} = j_1, X_{n_2} = j_2, \ldots X_{n_m} = j_m|X_0 = i\right) =$

$p_{ij_1}^{(n_1)} \cdot p_{j_1 j_2}^{(n_2-n_1)} \cdot p_{j_2 j_3}^{(n_3-n_2)} \cdots p_{j_{m-1} j_m}^{(n_m-n_{m-1})}$ and $P\left(X_{n_1} = j_1, X_{n_2} = j_2, \ldots X_{n_m} = j_m\right) = \sum_{i \in \Omega} P(X_0 = i) \cdot p_{ij_1}^{(n_1)} \cdot p_{j_1 j_2}^{(n_2-n_1)} \cdot p_{j_2 j_3}^{(n_3-n_2)} \cdots p_{j_{m-1} j_m}^{(n_m-n_{m-1})}$ for all $j_1, j_2, \ldots, j_m \in \Omega$.

From this section, it appears that we can answer any question about a Markov chain simply by matrix multiplication. However, the questions involving long-run behavior of Markov chains are exceptions. Also, there are Markov chains with very large (even infinite) number of states, which makes the matrix multiplication computationally challenging.

Markov chain models

An inventory model: Let us consider the stock management problem faced by a university stationary store. It operates from Monday to Friday and remains closed on weekends. Every Friday in the evening, the storekeeper checks the stock levels of the items and places orders with the suppliers, who supply the goods on Monday morning. If the storekeeper orders too much, then the excess stock occupies space and blocks money, and if he orders too little, then some customers return dissatisfied. A good ordering decision balances these two conflicting scenarios. Markov chain can be used to model this situation and make a good decision.

For ease of analysis let us consider only one item. Let the weekly demands of the item, denoted by $D_1, D_2, D_3, \ldots$, be *iid* random variables with mass function: $P(D_1 = 0) = 0.2$, $P(D_1 = 1) = 0.3$, $P(D_1 = 2) = 0.3$, $P(D_1 = 3) = 0.1$, $P(D_1 = 4) = 0.1$. Consider the storekeeper to follow an (s, S) inventory replenishment policy, which requires the following: If the stock level of the item in the end of week-n (i.e., Friday evening) is s or smaller, then an order of size $S - X_n$ is placed, where X_n denotes the closing stock level of week-n, so that the next week begins with a stock level of S. If $X_n > s$, then no order is placed and the next week begins with a stock level of X_n. Note that $0 \le s < S$.

Observe that $X_0, X_1, X_2, \ldots$ describes the above system, where X_0 is the starting inventory level. We can consider $\{X_0, X_1, X_2, \ldots\}$ as a stochastic process. It's a discrete time discrete state stochastic process. A closer look tells that X_{n+1} is determined by X_n and D_{n+1} as:

$$X_{n+1} = \text{Opening stock of week } (n+1) \text{ less weekly demand} = \begin{cases} S - D_{n+1} & \text{if } X_n \le s \\ X_n - D_{n+1} & \text{if } X_n > s \end{cases}$$

Clearly, X_n and D_{n+1} decides X_{n+1}. If X_n is known, then randomness in X_{n+1} is due to D_{n+1}. Since weekly demands are independent, then $X_0, X_1, \ldots, X_{n-1}$ cannot influence D_{n+1}. Thus, Markov property holds and $\{X_0, X_1, X_2, \ldots\}$ is a Markov chain. If weekly demands are not independent, then D_{n-1} influences D_{n+1} through dependence of demands and X_{n-1} through the above expression. Then X_{n-1} can influence X_{n+1} via D_{n-1} and D_{n+1}, even when X_n is known. Then $\{X_0, X_1, X_2, \ldots\}$ is not a Markov chain.

Let us determine the transition probabilities of the Markov chain. The highest possible value of X_n for any n is obviously S, and the lowest possible value is lowest possible opening stock less the highest possible demand, i.e., $s+1-4=s-3$. So, $\Omega=\{s-3, s-2, \ldots, S\}$. Note that Ω may contain negative numbers. A negative closing stock stock means that a customer is waiting for fresh stock to arrive. Then for $i, j \in \Omega$,

$$p_{i,j}^{n,n+1}=P(X_{n+1}=j|X_n=i)=\begin{cases}P(S-D_{n+1}=j|X_n=i)=P(D_{n+1}=S-j) & \text{if } i\leq s\\ P(X_n-D_{n+1}=j|X_n=i)=P(D_{n+1}=i-j) & \text{if } i>s\end{cases}$$

From the above expression, it's evident that transition probabilities are time-invariant, as $D_1, D_2, D_3, \ldots$ are identical random variables. Without this identicalness, the transition probabilities are both state and time dependent. One-step transition probability matrices for different values of s and S are shown below.

$s=1$ and $S=3$

p_{ij}	3	2	1	0	-1	-2
3	0.2	0.3	0.3	0.1	0.1	0
2	0	0.2	0.3	0.3	0.1	0.1
1	0.2	0.3	0.3	0.1	0.1	0
0	0.2	0.3	0.3	0.1	0.1	0
-1	0.2	0.3	0.3	0.1	0.1	0
-2	0.2	0.3	0.3	0.1	0.1	0

$s=1$ and $S=4$

p_{ij}	4	3	2	1	0	-1	-2
4	0.2	0.3	0.3	0.1	0.1	0	0
3	0	0.2	0.3	0.3	0.1	0.1	0
2	0	0	0.2	0.3	0.3	0.1	0.1
1	0.2	0.3	0.3	0.1	0.1	0	0
0	0.2	0.3	0.3	0.1	0.1	0	0
-1	0.2	0.3	0.3	0.1	0.1	0	0
-2	0.2	0.3	0.3	0.1	0.1	0	0

The choice of s and S shall be such that excess stock and negative stock at the end of weeks are balanced. Let $\pi_k \coloneqq \lim_{m\to\infty}\sum_{n=1}^{m}\mathbb{1}(X_n=k)/m$ for $k\in\Omega$. Note that $\mathbb{1}(X_n=k)$ is the indicator function telling whether $X_n=k$ or not. Then $\sum_{n=1}^{m}\mathbb{1}(X_n=k)/m$ is the fraction of weeks with closing stock k during the first m weeks, and π_k is the long-run fraction of weeks with closing stock k. Then the long-run excess stock can be measured by $I^+=\sum_{k\geq 1}k\pi_k$ and the negative stock can be measured by $I^-=-\sum_{k\leq -1}k\pi_k$. It's not immediately clear how to calculate I^+ and I^-. During the course, we will develop a simple way of obtaining π_k from the transition probabilities, and then we can get I^+ and I^- for different s and S.

A queueing model: Let us consider the congestion problem in a highway toll plaza. It operates round the clock with $c\geq 1$ number of counters, each of which takes around 1 minute to serve a vehicle. If c is too small, then there are many waiting vehicles, and if c is too large, then there are underutilized counters. A good choice for c balances these two conflicting scenarios. Again, Markov chain can help.

For simplicity, let us consider that each counter takes a fixed time of 1 minute to serve a vehicle. Let A_n denote the number of vehicles that arrive at the toll plaza during n-th minute, i.e., during $(n-1, n]$ minute. Let us assume that $A_1, A_2, A_3, \ldots$ are *iid* random variables with mass function: $P(A_1=0)=0.3$, $P(A_1=1)=0.2$, $P(A_1=2)=0.2$, $P(A_1=3)=0.1$, $P(A_1=4)=0.1$, $P(A_1=5)=0.1$. The total number of vehicles in the toll plaza, which changes over time, captures state of the system. It makes sense to consider a continuous time model here. However, we make a simplifying assumption that would allow a discrete time modelling. Let us assume that the vehicles arrive in batches at the end of every minute. With this assumption, total number of vehicles in the toll plaza can change only at the end of 1st, 2nd, 3rd, … minutes. Impact of this assumption on the performance measures is not much; it can be examined later. Let us proceed with the discrete time modelling.

Let X_n for $n = 0,1,2,...$ denote the total number of vehicles in the toll plaza at the end of n-th minute, after arrival and departure of vehicles during n-th minute are complete, with X_0 being the starting position. Then $n + 1^{st}$ minute begins with a total of X_n vehicles, $\min(X_n, c)$ of these vehicles get service during $n + 1^{st}$ minute and departs, and A_{n+1} new vehicles arrive at the end of $n + 1^{st}$ minute, taking the number of vehicles to X_{n+1}. Thus,

$$X_{n+1} = X_n - \min(X_n, c) + A_{n+1} = \max(X_n - c, 0) + A_{n+1} \text{ for } n = 0,1,2,...$$

Clearly, X_n and A_{n+1} decides X_{n+1}. If X_n is known, then randomness in X_{n+1} is due to A_{n+1}. Since $A_1, A_2, A_3, ...$ are independent, then $X_0, X_1, ..., X_{n-1}$ cannot influence A_{n+1}. Therefore, Markov property holds and $\{X_0, X_1, X_2, ...\}$ is a Markov chain. If arrivals are not independent, then the past can influence future via arrivals, even when the present is known (as explained in the inventory model). Then $\{X_0, X_1, X_2, ...\}$ is not a Markov chain.

Let us determine the transition probabilities. The lowest possible value of X_n is obviously 0, and the highest possible value can be unlimited due to accumulation of vehicles, which takes place whenever $X_n > c$. Therefore, $\Omega = \{0,1,2, ...\}$. For $i, j \in \Omega$,

$$p_{i,j}^{n,n+1} = P(X_{n+1} = j | X_n = i) = P(\max(X_n - c, 0) + A_{n+1} = j | X_n = i)$$
$$= \begin{cases} P(0 + A_{n+1} = j | X_n = i) = P(A_{n+1} = j) & \text{if } i \le c \\ P(i - c + A_{n+1} = j | X_n = i) = P(A_{n+1} = j - i + c) & \text{if } i > c \end{cases}$$

From the above expression, it is clear that the transition probabilities are time-invariant, as $A_1, A_2, A_3, ...$ are identical random variables. In absence of this identicalness, the transition probabilities are both state and time dependent. One-step transition probability matrices for different values of c are shown below.

$c = 1$

p_{ij}	0	1	2	3	4	5	6	7	⋯
0	0.3	0.2	0.2	0.1	0.1	0.1	0	0	⋯
1	0.3	0.2	0.2	0.1	0.1	0.1	0	0	⋯
2	0	0.3	0.2	0.2	0.1	0.1	0.1	0	⋯
3	0	0	0.3	0.2	0.2	0.1	0.1	0.1	⋯
4	0	0	0	0.3	0.2	0.2	0.1	0.1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

$c = 2$

p_{ij}	0	1	2	3	4	5	6	7	⋯
0	0.3	0.2	0.2	0.1	0.1	0.1	0	0	⋯
1	0.3	0.2	0.2	0.1	0.1	0.1	0	0	⋯
2	0.3	0.2	0.2	0.1	0.1	0.1	0	0	⋯
3	0	0.3	0.2	0.2	0.1	0.1	0.1	0	⋯
4	0	0	0.3	0.2	0.2	0.1	0.1	0.1	⋯
5	0	0	0	0.3	0.2	0.2	0.1	0.1	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

The choice of c shall be such that traffic congestion and counter utilization are balanced. The best possible scenario is $X_n = c$ for all n. Then the congestion is at its lowest (i.e., no vehicle waits) and utilization is at its highest (i.e., 100%). This obviously is an impossibility. In order to measure long run congestion and utilization, we need long-run fraction of minutes ending with k vehicles, i.e., $\pi_k = \lim_{m\to\infty} \sum_{n=1}^{m} \mathbb{1}(X_n = k)/m$ for all $k \in \Omega$. Then long-run traffic congestion $v = \sum_{k\geq c+1}(k-c)\pi_k$ and long-run counter utilization $u = \sum_{k\leq c-1}(k/c)\pi_k + \sum_{k\geq c}\pi_k$. With u and v for different c calculated, we can choose the best c.

A population model: Let us consider growth of a viral/bacterial infection in a human body. Consider a virus/bacteria lives for a fixed amount of time in human body, and then it either dies or splits into two new organisms, which then act in a similar manner. Let $q \in (0,1)$ denote the probability that an organism dies without producing any offspring. The value of q can be influenced by medical interventions. We are interested in designing interventions that ensure termination of the infection in quick time.

Let X_0 denote the number of organisms that entered human body in the first place. Let $Z_1, Z_2, \ldots, Z_{X_0}$ denote the number of offspring produced by these entering organisms. $Z_1, Z_2, \ldots, Z_{X_0}$ are *iid* random variables with mass function: $P(Z_1 = 0) = q$ and $P(Z_1 = 2) = p = 1 - q$. We consider splitting of an organism into two as death of the original organism and birth of two new. Then the next generation is of size $X_1 = Z_1 + Z_2 + \cdots + Z_{X_0}$. In a similar manner, $X_2 = Z_1 + Z_2 + \cdots + Z_{X_1}$, and in general, $X_{n+1} = Z_1 + Z_2 + \cdots + Z_{X_n}$.

$\{X_0, X_1, X_2, \ldots\}$ is a discrete time discrete state stochastic process. If X_n is known, say $X_n = i$, then $X_{n+1} = Z_1 + Z_2 + \cdots + Z_i$, sum of i number of *iid* random variables. Since the number of offspring produced by an organism in a particular generation is independent of everything else, then $X_{n+1}|X_n = i$ cannot be influenced by $X_0, X_1, \ldots, X_{n-1}$. Thus, Markov property holds and $\{X_0, X_1, X_2, \ldots\}$ is a Markov chain.

Let us determine the transition probabilities. The lowest possible value of X_n is obviously 0, and the highest possible value can be unlimited due to population growth. Due to the nature of offspring distribution, X_n for $n \geq 1$ assumes only even values. Considering X_0 to be even as well, $\Omega = \{0,2,4,\ldots\}$. For $i, j \in \Omega$,

$$p_{i,j}^{n,n+1} = P(X_{n+1} = j|X_n = i) = P\left(Z_1 + Z_2 + \cdots + Z_{X_n} = j|X_n = i\right)$$
$$= P(Z_1 + Z_2 + \cdots + Z_i = j) = \binom{i}{j/2} p^{j/2} q^{i-j/2}$$

Transition probabilities are time-invariant. The one-step transition probability matrix is shown to the right. Observe that the Markov chain cannot escape state 0 once it reaches there, i.e., if $X_n = 0$, then $X_m = 0 \quad \forall m > n$. This signifies termination of the infection.

p_{ij}	0	2	4	6	8	$\cdots$
0	1	0	0	0	0	$\cdots$
2	q^2	$2pq$	p^2	0	0	$\cdots$
4	q^4	$\binom{4}{1}pq^3$	$\binom{4}{2}p^2q^2$	$\binom{4}{3}p^3q$	p^4	$\cdots$
6	q^6	$\binom{6}{1}pq^5$	$\binom{6}{2}p^2q^4$	$\binom{6}{3}p^3q^3$	$\binom{6}{4}p^4q^2$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Here, we are not interested in the long-run fractions of the previous examples. Instead, we are interested in obtaining $\beta_0 = \lim_{n\to\infty} P(X_n = 0)$, the probability of eventual termination of the infection. In general, $\beta_k = \lim_{n\to\infty} P(X_n = k)$ for $k \in \Omega$ is called the limiting distribution of the Markov chain. Note the distinction between π_k and β_k. Later, we will see that they are the same, but limiting distribution may not exist for some Markov chains, whereas long-run fractions always exist. Here, we want β_0 to be 1 through medical interventions. We are also interested in $T \coloneqq \min\{n: X_n = 0\}$, the time till the termination of infection. We will develop a simple technique called the first-step analysis to obtain $E[T]$.

A stock price model: Let X_n denote closing price of a stock on day n for $n = 0,1,2,...$, with X_0 denoting the issue price. Let $Y_n = X_n/X_{n-1}$ for $n = 1,2,3,....$ Y_n represents fractional change in stock price after trading on day n. $Y_n < 1$ indicates decline in stock price and $Y_n > 1$ indicates appreciation. Let us assume that $Y_1, Y_2, Y_3, ...$ are *iid* random variables. In a steady macro-economic situation, this is a reasonable assumption. For simplicity, let us assume that $P(Y_1 = u) = p$ and $P(Y_1 = d) = q = 1 - p$ for some $0 < d < 1 < u$. One can consider more elaborate mass function for Z_1 or even a density over $(0, \infty)$.

With $Y_1, Y_2, Y_3, ...$ as input, we can represent $X_{n+1} = X_n Y_{n+1}$ for $n \geq 1$. If X_n is known, say $X_n = i$, then $X_{n+1} = iY_{n+1}$. Since $Y_1, Y_2, Y_3, ...$ are independent, then $X_1, X_2, ..., X_{n-1}$ cannot influence X_{n+1} when X_n is known. Thus, $\{X_0, X_1, X_2, ...\}$ is a Markov chain. Sample space for this Markov chain $\Omega = \{X_0 u^a d^b : a, b = 0,1,2, ...\}$. It's not easy to write down the transition probability matrix and work with it. We take help of a related Markov chain, known as the random walk, to model the stock prices.

Imagine a particle moving on integers as follows: It starts at 0, moves to the next integer whenever stock price increases and to the previous integer whenever stock-price decreases. These movements can be captured by $Z_1, Z_2, Z_3, ...$ defined as: $Y_n = u \Rightarrow Z_n = 1$ and $Y_n = d \Rightarrow Z_n = -1$ for all n. Then $Z_1, Z_2, Z_3, ...$ are *iid* random variables with $P(Z_1 = 1) = p$ and $P(Z_1 = -1) = q$. Let S_n denote position of the particle after n such move for $n = 0,1,2,, ...$, with $S_0 = 0$ being the starting position. Then $S_{n+1} = S_n + Z_{n+1}$ for all $n \geq 0$. Since $Z_1, Z_2, Z_3, ...$ are independent, $S_0, S_1, ..., S_{n-1}$ cannot influence S_{n+1} when S_n is known. Therefore, $\{S_0, S_1, S_2, ...\}$ is a Markov chain. It's sample space $\Omega = \mathbb{Z}$, and its transition probabilities are given by: $p_{i,i+1} = p$ and $p_{i,i-1} = q$ for all $i \in \mathbb{Z}$. This Markov chain is known as the simple random walk over integers. There are other types of random walk.
For $n \geq 1$, $X_n = X_{n-1}Y_n = X_{n-2}Y_{n-1}Y_n = \cdots = X_0 \prod_{i=1}^{n} Y_i$. Note that Y_i's takes values u and d. Then the number of Y_i's taking value u is same as the number of Z_i's with value 1, and the number of Y_i's taking value d is same as the number of Z_i's with value -1. Thus, $\prod_{i=1}^{n} Y_i = u^{\sum_{i=1}^{n} \mathbb{1}(Z_i=1)} \cdot d^{\sum_{i=1}^{n} \mathbb{1}(Z_i=-1)}$. Observe that $S_n = S_{n-1} + Z_n = S_{n-1} + Z_{n-1} + Z_n = \cdots = S_0 + \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} Z_i$ (as $S_0 = 0$) $= \sum_{i=1}^{n} \mathbb{1}(Z_i = 1) - \sum_{i=1}^{n} \mathbb{1}(Z_i = -1)$ and $\sum_{i=1}^{n} \mathbb{1}(Z_i = 1) + \sum_{i=1}^{n} \mathbb{1}(Z_i = -1) = n$. Therefore, $\sum_{i=1}^{n} \mathbb{1}(Z_i = 1) = (n + S_n)/2$ and $\sum_{i=1}^{n} \mathbb{1}(Z_i = -1) = (n - S_n)/2$. Then $\prod_{i=1}^{n} Y_i = u^{(n+S_n)/2} \cdot d^{(n-S_n)/2} = (ud)^{n/2} \cdot (u/d)^{S_n/2}$. Hence, $X_n = X_0 (ud)^{n/2} \cdot (u/d)^{S_n/2}$ for all $n \geq 1$.

With the above representation, we can answer any question about $\{X_0, X_1, X_2, \ldots\}$ using $\{S_0, S_1, S_2, \ldots\}$. We may be interested in mean and variance of X_n, or the maximum of X_n during the first N days, or the time for X_n to hit or exceed x for the first time. All these questions have an equivalent question involving S_n. In our discussion of Markov chain, these questions will not be answered, at least explicitly.

We can classify questions about Markov chains into two classes: short-run behavior and long-run behavior. The former can be answered with n-step transition probabilities, as explained in the previous lecture. The later requires in-depth analysis of Markov chains, which is our agenda for the next several classes.

Practice problems

Book-1: Introduction to Probability Models by Sheldon Ross [10th edition]
Book-2: An Introduction to Stochastic Modeling by Taylor and Karlin [3rd edition]

Transition Probabilities
Book-1, Chapter-4, Exercise No. 1, 5
Book-2, Chapter-III, Problem No. 1.1, 1.2, 2.4, 2.5

Markov chain models
Book-2, Chapter-III, Problem No. 3.1, 3.2, 3.5, 3.6, 3.7