Module 5

Renewal process

Topics: Renewal process, Inspection paradox, Renewal theorem, Renewal reward process

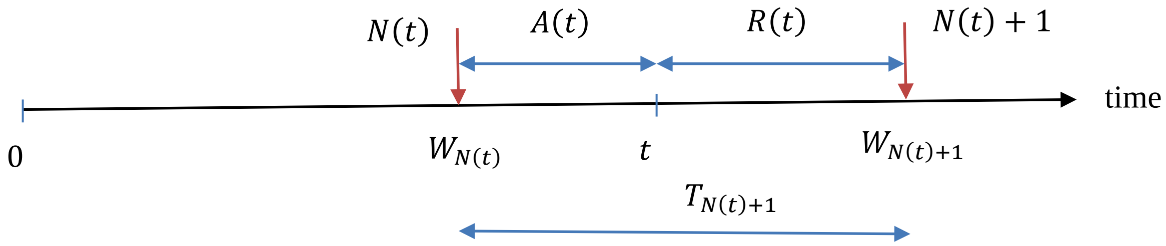
Renewal process

Renewal process: In the previous module, we learned about two different generalizations of Poisson process. Renewal process is another generalization. A counting process $\{N(t): t \geq 0\}$ is a renewal process if its inter-event times are independent and identically distributed positive-valued random variables. It is so named because every time an event occurs, due to independence and identicalness of inter-event times, the system renews itself. In the Poisson process, inter-event times are *iid* exponential random variables. So, it is a renewal process. Note that non-homogeneous Poisson process is not a renewal process.

Renewal processes are specified by the distribution and density/mass functions of the inter-event times. Let $T_1, T_2, \mathrm{T}_3, \dots$ be independent with common distribution $F(\cdot)$ and density $f(\cdot)$ or mass $p(\cdot)$ functions. From these we can obtain distribution and density/mass functions of the waiting times W_n, $n \geq 1$ using the convolution formulas, which in turn gives us the mass and distribution functions of $N(t)$, $t \geq 0$. W_n's and $N(t)$'s generally do not follow common distributions, except for special cases. So, our focus will be primarily on mean, variance, etc., which can be determined without knowing the distribution. $E[W_n] = E[T_1 + T_2 + \cdots + T_n] = nE[T]$, and due to independence, $Var(W_n) = \mathrm{Var}(T_1 + T_2 + \cdots + T_n) = nVar(T)$. Obtaining $E[N(t)]$, however, is not so easy. Its calculation leads to a paradox known as the inspection paradox or the waiting time paradox, which we will study little later.

Continuous renewal: Let us revisit Poisson process once more before proceeding with the renewal process. We know that the Poisson process is a renewal process, as its inter-event times are *iid* (exponential) random variables. Here, we ask about the converse, i.e., whether a renewal process with exponential inter-event times is a Poisson process or not.

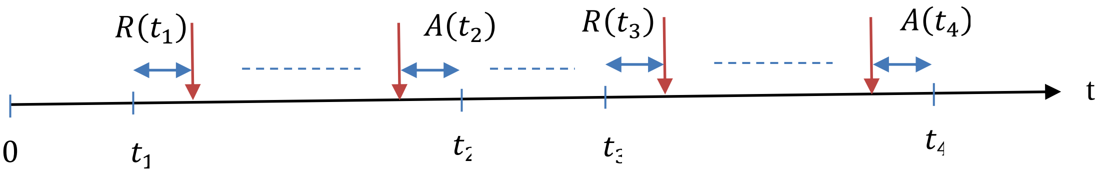
The answer to the above question is yes. To prove this, we need to show that the renewal process with $Exp(\lambda)$ inter-event time possesses the independent and stationary increment properties. First, we show that a renewal process with exponential inter-event times renews itself continuously. From the definition of renewal process, we know that every time an event takes place, the renewal process restarts. By continuous renewal, we mean continuous restart, i.e., we pick any point in time and the process from that point onwards looks stochastically identical to the original process. Consider the following diagram.



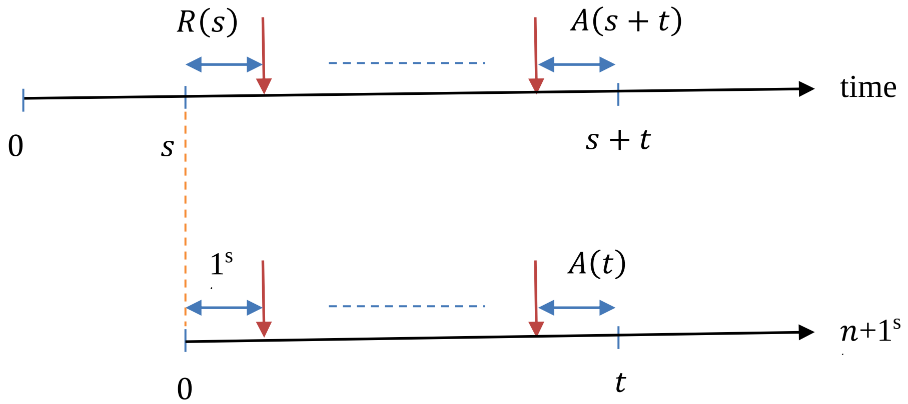
We pick an arbitrary point in time t in the renewal process with $Exp(\lambda)$ interevent times. $R(t)$ denotes the time till the next event, i.e., $N(t)+1^{st}$ event. If we start observing the process from t, then $R(t)$ is the 1^{st} inter-event time, $T_{N(t)+2}$ is the 2^{nd} inter-event time, $T_{N(t)+3}$ is the 3^{rd}, and so on. If we can show that $R(t), T_{N(t)+2}, T_{N(t)+3}, \ldots$ are *iid* $Exp(\lambda)$ random variables, then continuous renewal is established.

With the memoryless property of exponential inter-event times, we can argue that $R(t) \sim Exp(\lambda)$. A formal proof is provided in Appendix A. Thus, $R(t), T_{N(t)+2}, T_{N(t)+3}, \ldots$ all are $Exp(\lambda)$ random variables. For their independence, we need to show that $A(t)$ and $R(t)$ are independent. Since $T_{N(t)+1} = A(t) + R(t)$ and $T_{N(t)+1}, T_{N(t)+2}, T_{N(t)+3}, \ldots$ are independent, then $R(t), T_{N(t)+2}, T_{N(t)+3}, \ldots$ are independent. Common sense may suggest that $A(t), R(t)$ are negatively correlated, but they are independent due to the memoryless property of inter-event times. See Appendix A for a formal proof.

Independent and stationary increments: With continuous renewal established for the renewal process with exponential inter-event times, we argue that such a renewal process possesses the independent and stationary increment properties. Consider arbitrary $t_1 < t_2 \le t_3 < t_4$. Following the diagram below, number of events taking place in $(t_1, t_2]$ is decided by $R(t_1)$, $A(t_2)$, and all inter-event times in-between. In a similar manner, number of events taking place in $(t_3, t_4]$ is decided by $R(t_3)$, $A(t_4)$, and all inter-event times in-between. All these times are independent of one another. Hence, the number of events in $(t_1, t_2]$ and $(t_3, t_4]$ are independent, and the independent increment property holds.



Now consider arbitrary s, t. Following the diagram below, number of events taking place in $(s, s+t]$ is decided by $R(s)$, $A(s+t)$, and all inter-event times in-between, and the number of events taking place in $(0, t]$ is decided by T_1, $A(t)$, and all inter-event times in-between. Due to continuous renewal, $R(s) \equiv T_1$, $A(s+t) \equiv A(t)$, and same is the case with all the inter-event times in-between. Therefore, the number of events in these two intervals have the same distribution. Hence, stationary increment property holds.



So, a renewal process with exponentially distributed inter-event times is a Poisson process. This is an alternate characterization of the Poisson process.

Inspection paradox

Renewal equation: Renewal process is defined in terms of inter-event times. We can obtain distribution function of waiting times by the convolution formula, and then the equivalence $\{W_n \le t\} \equiv \{N(t) \ge n\}$ gives us mass function of $N(t)$. These calculations are tedious, and unnecessary if we are interested in expected value alone. Here we calculate $m(t) := E[N(t)]$ using distribution $F(\cdot)$ and density $f(\cdot)$ functions of inter-event times.

$$
\begin{aligned}
m(t) = E_{T_1}\big[E[N(t)|T_1]\big] &= \int_0^\infty E[N(t)|T_1 = s]f(s)ds \\
&= \int_0^t E[N(t)|T_1 = s]f(s)ds + \int_t^\infty E[N(t)|T_1 = s]f(s)ds \\
&= \int_0^t E[1 + N(t-s)|T_1 = s]f(s)ds + 0 = \int_0^t E[1 + N(t-s)]f(s)ds \\
&= \int_0^t f(s)ds + \int_0^t E[N(t-s)]f(s)ds
\end{aligned}
$$

$$
\Rightarrow m(t) = F(t) + \int_0^t m(t-s)f(s)ds \ \forall t > 0
$$

The above equation is known as the renewal equation. Note that the formula is recursive in nature, and therefore, it may not provide closed form expression for $m(t)$ except for some special cases. One such case is the renewal process with *iid* $U(0,1)$ random variables as inter-event times. Here, we can obtain $m(t)$ for $t \le 1$.

$$
m(t) = F(t) + \int_0^t m(t-s)f(s)ds = t + \int_0^t m(t-s)ds = t + \int_0^t m(y)dy
$$

$$
\Rightarrow m'(t) = 1 + m(t) \Rightarrow \frac{d\big(1 + m(t)\big)}{dt} = 1 + m(t) \Rightarrow \frac{d\big(1 + m(t)\big)}{1 + m(t)} = dt \ \forall t \le 1
$$

$$
\Rightarrow \int_0^t \frac{d\big(1 + m(\tau)\big)}{1 + m(\tau)} = \int_0^t d\tau \Rightarrow \ln(|1 + m(t)|) - \ln(|1 + m(0)|) = t
$$

$$
\Rightarrow \ln\big(1 + m(t)\big) - 0 = t \Rightarrow m(t) = e^t - 1 \ \forall t \le 1
$$

Intuitively, we expect $m(t)$ to be same as $\mathrm{t}/E[T] = 2t$, but this is not the case. In the above example, $m(0.5) = e^{0.5} - 1 = 0.65 < 1$, $m(0.75) = e^{0.75} - 1 = 1.12 < 1.5$, $m(1) = e^1 - 1 = 1.72 < 2$. This counter-intuitive observation, i.e., $m(t) \le \mathrm{t}/E[T]$, is quite common and it is a consequence of the inspection paradox. Before we discuss it, let us consider a discrete version of the renewal equation. For integer-valued inter-event times,

$$
m(t) = E_{T_1}\big[E[N(t)|T_1]\big] = \sum_{s=1}^{t} E[N(t)|T_1 = s]p(s) = \sum_{s=1}^{t} E[1 + N(t-s)]p(s)
$$

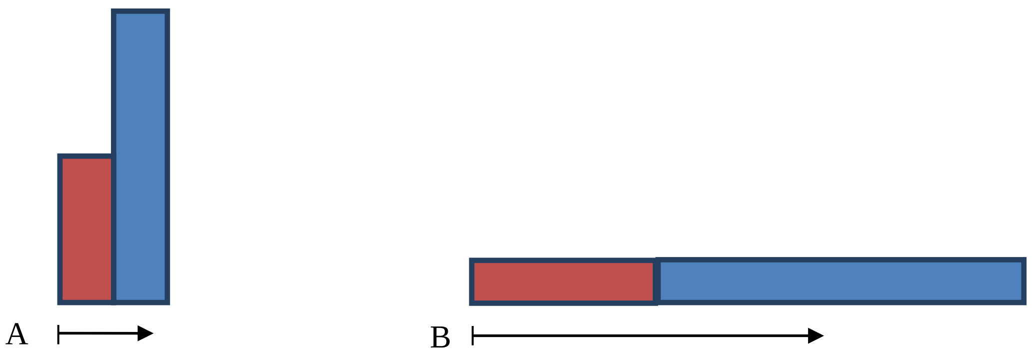
$$= \sum_{s=1}^{t} p(s) + \sum_{s=1}^{t-1} E[N(t-s)]p(s)\ ; \text{ note the range in the 2nd sum}$$

$$\Rightarrow m(t) = F(t) + \sum_{s=1}^{t-1} m(t-s)p(s) \text{ for } t = 1,2,3, \ldots$$

Consider the inter-event time to follow discrete uniform distribution in $\{1,2,3\}$. Let us use the above equation to obtain $m(4)$. Intuitively, the answer is $4/E[T] = 2$, but the correct answer is $148/81 < 2$, which is due to the inspection paradox.

$m(0) = 0$
$m(1) = F(1) = 1/3$
$m(2) = F(2) + m(1)p(1) = 2/3 + 1/3 * 1/3 = 7/9$
$m(3) = F(3) + m(2)p(1) + m(1)p(2) = 1 + 7/9 * 1/3 + 1/3 * 1/3 = 37/27$
$m(4) = F(4) + m(3)p(1) + m(2)p(2) + m(1)p(3) = 1 + 37/27 * 1/3 + 7/9 * 1/3 + 1/3 * 1/3 = 148/81$, as mentioned earlier

<u>Inspection paradox</u>: Consider a random experiment of selecting one bar from among two bars in two different settings A and B. The bars are of equal width, but different length - the blue one is twice compared to the red one, as shown below.



A robotic arm moves in the horizontal direction and randomly selects a point covered by the bars, without any bias to any location. The bar containing the selected point is chosen. In the case of A, both the bars have equal chance of being chosen. In case B, blue bar is twice more likely to be selected as compared to the red bar. This example illustrates the fact that the probability of selection of objects depends on the experimental setup.

In the example with discrete uniform inter-event times, it is reasonable to assume that equal numbers of inter-event times of lengths 5,10,15 minutes cover the time axis $(0, \infty)$, because these values are equally likely. However, the portions of the time axis that they cover are different, as their lengths are different. If we pick a point t in $(0, \infty)$, a longer inter-event time is more likely to 'hold' t than a shorter one. The inter-event time 'containing' t, on an average, is larger than the regular inter-event times, i.e., $E[T_{N(t)+1}] \geq E[T]$. Let us verify if $E[T_{N(4)+1}] \geq E[T] = 2$ in the example. Since $T \in \{1,2,3\}$, $1 \leq N(4) \leq 4$ and we can capture all possibilities by considering different combinations of T_1, T_2, T_3, T_4. Depending upon the scenario, $T_{N(4)+1}$ takes different values, as shown in the table below.

T_1, T_2, T_3, T_4	$N(4)$	$T_{N(4)+1}$	Prob.	T_1, T_2, T_3, T_4	$N(4)$	$T_{N(4)+1}$	Prob.
1, 1, 1, 1	4	1/2/3	1/243	2, 1, 1, any	3	1/2/3	1/81
1, 1, 1, 2/3	3	2/3	1/81	2, 1, 2/3, any	2	2/3	1/27
1, 1, 2, any	3	1/2/3	1/81	2, 2, any, any	2	1/2/3	1/27
1, 1, 3, any	2	3	1/27	2, 3, any, any	1	3	1/9
1, 2, 1, any	3	1/2/3	1/81	3, 1, any, any	2	1/2/3	1/27
1, 2, 2/3, any	2	2/3	1/27	3, 2/3, any, any	1	2/3	1/9
1, 3, any, any	2	1/2/3	1/27	Verify that the probabilities add up to 1.			

From the above table, we obtain the following mass function of $T_{N(4)+1}$.

$$P\left(T_{N(4)+1} = 1\right) = \frac{1}{243} + \frac{1}{81} + \frac{1}{81} + \frac{1}{27} + \frac{1}{81} + \frac{1}{27} + \frac{1}{27} = \frac{37}{243} = 0.15$$
$$P\left(T_{N(4)+1} = 2\right) = \frac{1}{243} + \frac{1}{81} + \frac{1}{81} + \frac{1}{81} + \frac{1}{27} + \frac{1}{27} + \frac{1}{81} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{1}{9} = \frac{85}{243} = 0.35$$
$$P\left(T_{N(4)+1} = 3\right) = \frac{1}{243} + \frac{1}{81} + \frac{1}{81} + \frac{1}{27} + \frac{1}{81} + \frac{1}{27} + \frac{1}{27} + \frac{1}{81} + \frac{1}{27} + \frac{1}{27} + \frac{1}{9} + \frac{1}{27} + \frac{1}{9} = \frac{121}{243} = 0.5$$

We do see that the longer inter-event time is more likely to hold $t = 4$ than a shorter one. As a result, $E\left[T_{N(4)+1}\right] = 1 * \frac{37}{243} + 2 * \frac{85}{243} + 3 * \frac{121}{243} = 2.346 > 2 = E[T]$, as claimed.

If different values of inter-event times are not necessarily equally likely, then the probability of t falling on a particular inter-event time depends on the likelihood of the inter-event time as well as its lengths. Again, there is a positive bias towards longer inter-event times, i.e., the probability of a longer inter-event time holding t is more than its likelihood given by the density/mass function of inter-event times. As a result, $E\left[T_{N(t)+1}\right] \geq E[T]$. This phenomenon is known as the inspection paradox. It is so named because if we inspect the 'current' inter-event time in a renewal process and use it to estimate some quantity of interest, e.g., expected number of events taking place in $(0, t]$, our estimate will be biased.

Refer to the first figure in this document. Part of $T_{N(\mathrm{t})+1}$ lies in $(0, \mathrm{t}]$, as shown above. Since $E\left[T_{N(t)+1}\right] \geq E[T]$, the part of $(0, \mathrm{t}]$ that is covered by $T_{N(\mathrm{t})+1}$, i.e., $A(t)$, is also larger than the usual, and no event takes place in this part. Therefore, the expected number of events in $(0, \mathrm{t}]$ is limited by $\mathrm{t}/E[T]$, i.e., $m(t) \leq t/E[T]$, as noted earlier. With the same logic, $R(t)$, too, is larger than the usual. In the context of waiting for the next event in a renewal process, e.g., waiting for the next bus in a bus stop, $R(t)$ is the waiting time for an observer who starts observing at time t. Since t is equally likely to be anywhere in an inter-event time, we may think that $E[R(t)]$ is $E[T]/2$, but $E[R(t)] = E\left[T_{N(t)+1}\right]/2 \geq E[T]/2$. This consequence of the inspection paradox is known as the waiting time paradox.

$T_{N(t)+1}$ is larger than the other inter-event times, i.e., $T_1, T_2, \ldots, T_{N(t)}, T_{N(t)+2}, \ldots$, which follow the given inter-event time distribution. We have illustrated 'largeness' of $T_{N(t)+1}$. Here, we illustrate that $T_1, T_2, \dots, T_{N(t)}, T_{N(t)+2}, \dots$ follow the given inter-event time distribution. We do it for $T_{N(4)}$ in the example with discrete uniform inter-event times.

T_1, T_2, T_3, T_4	$N(4)$	$T_{N(4)}$	Prob.	T_1, T_2, T_3, T_4	$N(4)$	$T_{N(4)}$	Prob.
1, 1, 1, 1	4	1	1/81	2, 1, 1, any	3	1	1/27
1, 1, 1, 2/3	3	1	2/81	2, 1, 2/3, any	2	1	2/27
1, 1, 2, any	3	2	1/27	2, 2, any, any	2	2	1/9
1, 1, 2, any	2	1	1/27	2, 3, any, any	1	2	1/9
1, 2, 1, any	3	1	1/27	3, 1, any, any	2	1	1/9
1, 2, 2/3, any	2	2	2/27	3, 2/3, any, any	1	3	2/9
1, 3, any, any	2	3	1/9	Verify that probabilities add up to 1			

From the above table, we obtain the following mass function of $T_{N(4)}$.

$P(T_{N(4)} = 1) = \frac{1}{81} + \frac{2}{81} + \frac{1}{27} + \frac{1}{27} + \frac{1}{27} + \frac{2}{27} + \frac{1}{9} = \frac{1}{3}$
$P(T_{N(4)} = 2) = \frac{1}{27} + \frac{2}{27} + \frac{1}{9} + \frac{1}{9} = \frac{1}{3}$
$P(T_{N(4)} = 3) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

Clearly, $T_{N(4)}$ has the given inter-event time distribution. *Try some other case.*

Renewal theorem

Elementary renewal theorem: $E[T_{N(t)+1}] \geq E[T] \Rightarrow m(t) \leq t/E[T]$, i.e., the expected rate of events $m(t)/t$ is limited by $1/E[T]$. Since $T_{N(t)+1}$ is the only inter-event time that is larger than usual, we can expect its effect on $m(t)/t$ to diminish as t increases and eventually vanish, i.e., the gap between $m(t)/t$ and $1/E[T]$ decreases with t and $\lim_{t\to\infty} m(t)/t = 1/E[T]$. This limiting behavior is known as the elementary renewal theorem.

For the problem discussed earlier, with 5 minutes equal to unit time, $E[T] = 2$ and

$m(1) = 1/3 \Rightarrow m(1)/1 = 0.333 < 0.5$
$m(2) = 7/9 \Rightarrow m(2)/2 = 0.389 < 0.5$
$m(3) = 37/27 \Rightarrow m(3)/3 = 0.457 < 0.5$
$m(4) = 148/81 \Rightarrow m(4)/4 = 0.457 < 0.5$
$m(10) = 4.83 \Rightarrow m(10)/10 = 0.483 < 0.5$
$m(20) = 9.83 \Rightarrow m(20)/20 = 0.492 < 0.5$
$m(50) = 24.83 \Rightarrow m(50)/50 = 0.497 < 0.5$
$m(100) = 49.83 \Rightarrow m(100)/100 = 0.498 < 0.5$

The above illustration seems to support the claim that $\lim_{t\to\infty} m(t)/t = 1/E[T]$.

$\lim_{t\to\infty} N(t)/t = 1/E[T]$: The elementary renewal theorem says that $\lim_{t\to\infty} E[N(t)]/t = 1/E[T]$. First, we prove that $\lim_{t\to\infty} N(t)/t = 1/E[T]$. Note that $N(\infty) = \infty$, i.e., finite number of events cannot happen in infinite duration. If it is not true, then $N(\infty)$ is finite, say $N(\infty)=k$. Then $T_{k+1}=\infty$ with certainty, which is impossible as inter-event times are proper random variables. Hence, $N(\infty)=\infty$.

Refer to the first figure in this note. $W_{N(t)} \le t \le W_{N(t)+1} = W_{N(t)} + T_{N(t)+1}$ for all $t>0$. Also, $W_{N(t)} = T_1 + T_2 + \cdots + T_{N(t)}$. Then

$$\frac{T_1 + T_2 + \cdots + T_{N(t)}}{N(t)} \le \frac{t}{N(t)} \le \frac{T_1 + T_2 + \cdots + T_{N(t)}}{N(t)} + \frac{T_{N(t)+1}}{N(t)}$$
$$\Rightarrow \lim_{N(t)\to\infty} \frac{T_1 + T_2 + \cdots + T_{N(t)}}{N(t)} \le \lim_{t\to\infty} \frac{t}{N(t)} \le \lim_{N(t)\to\infty} \frac{T_1 + T_2 + \cdots + T_{N(t)}}{N(t)} + \lim_{N(t)\to\infty} \frac{T_{N(t)+1}}{N(t)}$$
$$\Rightarrow E[T] \le \lim_{t\to\infty} \frac{t}{N(t)} \le E[T] + 0, \text{by the law of large numbers}$$
$$\Rightarrow \frac{1}{E[T]} \le \lim_{t\to\infty} \frac{N(t)}{t} \le \frac{1}{E[T]} \Rightarrow \lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{E[T]}$$

From the above, the elementary renewal theorem may seem obvious, but it is not the case because of the problem associated with changing the order of limit and sum. We cannot say that $E[\lim_{t\to\infty} N(t)/t] = \lim_{t\to\infty} E[N(t)/t] = \lim_{t\to\infty} m(t)/t$ is always true. For example, consider $X_1, X_2, X_3, \ldots$ to be independent $U(0,1)$ random variables. Let us define

$$Y_n = \begin{cases} n & \text{if } X_n \le 1/n \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n = 1,2,3,\ldots$$

Y_n for all n takes two values: n with probability $P(X_n \le 1/n) = 1/n$ and 0 with probability $1-1/n$. Then $E[Y_n] = n \times 1/n + 0 \times (1-1/n) = 1$ for all n. As $n\to\infty$, Y_n converges to 0 in a sense, because the other possibility becomes increasingly rare. Thus, $\lim_{n\to\infty} Y_n = 0$; however, $\lim_{n\to\infty} E[Y_n] = 1$.

Wald equation: In order to prove $\lim_{t\to\infty} m(t)/t = 1/E[T]$, first we need to establish Wald's equation, which claims that $E[W_{N(t)+1}] = E[T]E[N(t)+1] = E[T]\{m(t)+1\}$. It may seem an obvious application of random sum of random variables as $W_{N(t)+1} = T_1 + T_2 + \cdots + T_{N(t)+1}$. However, $T_1, T_2, T_3, \ldots$ are not independent of $N(t)+1$ (*why?*), which is necessary for the usual results in random sum of random variables to hold. It shall also be noted that Wald's equation talks about $E[W_{N(t)+1}]$, not $E[W_{N(t)}]$.

$$E[W_{N(t)+1}] = E\left[\sum_{n=1}^{N(t)+1} T_n\right] = E\left[\sum_{n=1}^{\infty} T_n \cdot \mathbb{1}(n \le N(t)+1)\right] = E\left[\sum_{n=1}^{\infty} T_n \cdot \mathbb{1}(N(t) \ge \mathrm{n}-1)\right]$$

Note that $\{N(t) \ge n-1\} \equiv \{W_{n-1} \le t\} \equiv \{T_1 + T_2 + \cdots + T_{n-1} \le t\}$. Then $\mathbb{1}(N(t) \ge \mathrm{n}-1)$ takes value 1 if $T_1 + T_2 + \cdots + T_{n-1} \le t$, otherwise it is zero. Clearly, $\mathbb{1}(N(t) \ge \mathrm{n}-1)$ is a function of $T_1, T_2, \ldots, T_{n-1}$. Since the inter-event times are independent, then $\mathbb{1}(N(t) \ge \mathrm{n}-1)$ is independent of $T_n, T_{n+1}, T_{n+2}, \ldots$. Therefore,

$$E\left[\sum_{n=1}^{\infty} T_n \cdot \mathbb{1}(N(t) \geq \mathrm{n}-1)\right] = \sum_{n=1}^{\infty} E[T_n] \cdot E[\mathbb{1}(N(t) \geq \mathrm{n}-1)]$$

$$= E[T] \sum_{n=1}^{\infty} E[\mathbb{1}(N(t) \geq \mathrm{n}-1)] = E[T] E\left[\sum_{n=1}^{\infty} \mathbb{1}(N(t) \geq \mathrm{n}-1)\right]$$

$$= E[T]\, E\left[\sum_{n=1}^{\infty} \mathbb{1}(n \leq N(t)+1)\right] = E[T]\, E[N(t)+1]$$

$\Rightarrow E\left[W_{N(t)+1}\right] = E[T]\{m(t)+1\}$, as claimed.

Let us consider $E\left[W_{N(t)}\right]$ in a similar manner. Then $E\left[W_{N(t)}\right] = E\left[\sum_{n=1}^{N(t)} T_n\right] = E\left[\sum_{n=1}^{\infty} T_n \cdot \mathbb{1}\big(n \leq N(t)\big)\right] = E[\sum_{n=1}^{\infty} T_n \cdot \mathbb{1}(N(t) \geq n)]$. Now, $\{N(t) \geq n\} \equiv \{W_n \leq t\} \equiv \{T_1 + T_2 + \cdots + T_n \leq t\}$. Clearly, $\mathbb{1}(N(t) \geq n)$ depends on T_n, and we cannot proceed further. Thus, Wald's equation talks about $E\left[W_{N(t)+1}\right]$, not $E\left[W_{N(t)}\right]$.

<u>Proof of the elementary renewal theorem</u>: Now, we can show that $\lim_{t\to\infty} m(t)/t = 1/E[T]$. First, we show that $\lim_{t\to\infty} m(t)/t \geq 1/E[T]$, and then that $\lim_{t\to\infty} m(t)/t \leq 1/E[T]$. Then by the sandwich theorem, $\lim_{t\to\infty} m(t)/t = 1/E[T]$. Since $W_{N(t)+1} = t + R(t)$,

$E\left[W_{N(t)+1}\right] = E[t + R(t)] \Rightarrow E[T]\{m(t)+1\} = t + E[R(t)]$, by Wald′equation

$\Rightarrow \dfrac{\mathrm{m(t)}}{t} + \dfrac{1}{t} = \dfrac{1}{E[T]} + \dfrac{E[R(t)]}{tE[T]}$, by dividing both sides with $tE[T]$

$\Rightarrow \dfrac{m(t)}{t} + \dfrac{1}{t} \geq \dfrac{1}{E[T]}$, as $R(t) \geq 0 \Rightarrow \lim\limits_{t\to\infty} \dfrac{m(t)}{t} \geq \dfrac{1}{E[T]}$

For the upper bound, we consider bounded and unbounded inter-event times separately. For the bounded case, there exists a finite positive number M s.t. $P(T_n \leq M) = 1$ for all n. Then $R(t) \leq T_{N(t)+1} \leq M$ with probability 1, implying $E[R(t) \leq M]$. Then

$$\frac{m(t)}{t} + \frac{1}{t} = \frac{1}{E[T]} + \frac{E[R(t)]}{tE[T]} \leq \frac{1}{E[T]} + \frac{M}{tE[T]} \Rightarrow \lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{E[T]}$$

For the unbounded inter-event times, we consider a new renewal process $\{N'(t): t \geq 0\}$ with $T'_n \coloneqq \min(T_n, M)$ for $n = 1,2,3, \ldots$, where M is a positive number. Clearly, $T'_n \leq T_n$ for all n. Then $N'(t) \geq N(t)$ for all t. The new renewal process has bounded inter-event times. Then by the previous argument, $\lim_{t\to\infty} m'(t)/t \leq 1/E[T']$ for every $M > 0$. With $M \to \infty$, the gap between $E[T']$ and $E[T]$ diminishes. Then $\lim_{t\to\infty} m'(t)/t \leq 1/E[T]$, as $M \to \infty$. Since $N(t) \leq N'(t)$ for all t, then $\lim_{t\to\infty} m(t)/t \leq \lim_{t\to\infty} m'(t)/t \leq 1/E[T]$.

Renewal reward process

<u>Renewal reward process</u>: Consider receiving emails to be a renewal process. Different emails

have different sizes that can be regarded as *iid* random variables. Then the inbox size as time progresses constitutes a renewal reward process. Mail sizes are rewards that are accumulated as events occur in the renewal process. In this case, individual rewards are independent of the renewal process itself. Observe the similarity between renewal reward process and compound Poisson process. However, there is difference as illustrated next.

Time between my successive visits to Kanpur City are *iid* positive random variables. Then the city visits with time constitute a renewal process. In each visit, I make purchases and the purchase amount depends on time since the last visit. Purchase amounts in different visits can be regarded as *iid* random variables, even though they depend on the respective inter-event times. Cumulative purchases with time still constitute a renewal reward process. In this case, individual rewards depend on the renewal process, as $N(t)$ influences $T_1, t_2, \ldots, T_{N(t)}$, which influences the associated rewards. In the compound Poisson process, individual rewards are always independent of the Poisson process.

Let us define the renewal reward process formally. Consider a renewal process $\{N(t): t \geq 0\}$, where a reward is earned every time an event occurs. The rewards, denoted by $R_1, R_2, R_3, \ldots$, are *iid* random variables. It is possible that the R_n depends on T_n for all n. Let $X(t) = R_1 + R_2 + \cdots + R_{N(t)}$ denote the total reward earned during $(0, t]$. Then $\{X(t): t \geq 0\}$ is referred to as a renewal reward process. Note that rewards can be negative.

In the compound Poisson process, where $R_1, R_2, R_3, \ldots$ are independent of $N(t)$, apart from being *iid* random variables themselves, computation of mean and variance of $X(t)$ are easy. In the renewal reward process, due to the potential dependence of $R_1, R_2, R_3, \ldots$ on $N(t)$ via $T_1, T_2, T_3, \ldots$, we do not have any simple method for calculating mean and variance of $X(t)$. However, we can study its asymptotic behavior.

<u>Renewal reward theorem</u>: It talks about the long-run reward rate, in real term as well as in expectation: $\lim_{t\to\infty} X(t)/t = \mathrm{E}[R]/E[T]$ and $\lim_{t\to\infty} E[X(t)]/t = \mathrm{E}[R]/E[T]$. The first part follows straightforwardly from the law of large numbers.

$$\lim_{t\to\infty} \frac{X(t)}{t} = \lim_{\substack{t\to\infty \\ N(t)\to\infty}} \frac{R_1 + R_2 + \cdots + R_{N(t)}}{N(t)} \cdot \frac{N(t)}{t} = \frac{E[R]}{E[T]}$$

Proof of the second part is similar to the proof of the elementary renewal theorem. First, we show that $E\left[X\left(W_{N(t)+1}\right)\right] = E[R]\{m(t) + 1\}$, a result similar to the Wald's equation.

$$E\left[X\left(W_{N(t)+1}\right)\right] = E\left[\sum_{n=1}^{N(t)+1} R_n\right] = E\left[\sum_{n=1}^{\infty} R_n \cdot \mathbb{1}(n \leq N(t) + 1)\right] = \sum_{n=1}^{\infty} E[R_n \cdot \mathbb{1}(N(t) \geq n - 1)]$$

Now $\{N(t) \geq n - 1\} \equiv \{W_{n-1} \leq t\}$. Since $W_{n-1} = T_1 + T_2 + \cdots + T_{n-1}$ and $T_1, T_2, T_3, \ldots$ are *iid* random variables, $\mathbb{1}(N(t) \geq n - 1)$ is independent of T_n. Now, R_n is independent of all inter-event times, except possibly T_n. Then R_n and $\mathbb{1}(N(t) \geq n - 1)$ are independent.

$$\Rightarrow E\left[X\left(W_{N(t)+1}\right)\right] = \sum_{n=1}^{\infty} E[R_n] \cdot \mathbb{1}(N(t) \geq n-1) = E[R] \sum_{n=1}^{\infty} E[\mathbb{1}(N(t) \geq n-1)]$$
$$= E[R]E\left[\sum_{n=1}^{\infty} \mathbb{1}(n \leq N(t)+1)\right] = E[R]E[N(t)+1] = E[R]\{m(t)+1\}$$

Now we prove the main result. Since $W_{N(t)+1} = W_{N(t)} + T_{N(t)+1}$, $X\left(W_{N(t)+1}\right) = X\left(W_{N(t)}\right) + R_{N(t)+1}$. Moreover, $X\left(W_{N(t)}\right) = R_1 + R_2 + \cdots + R_{N(t)} = X(t)$. Thus, $X\left(W_{N(t)+1}\right) = X(t) + R_{N(t)+1} \Rightarrow E\left[X\left(W_{N(t)+1}\right)\right] = E[X(t)] + E\left[R_{N(t)+1}\right]$. Then with the above result, $E[X(t)] = E[R]\{m(t)+1\} - E\left[R_{N(t)+1}\right]$. Dividing both sides by t, we get

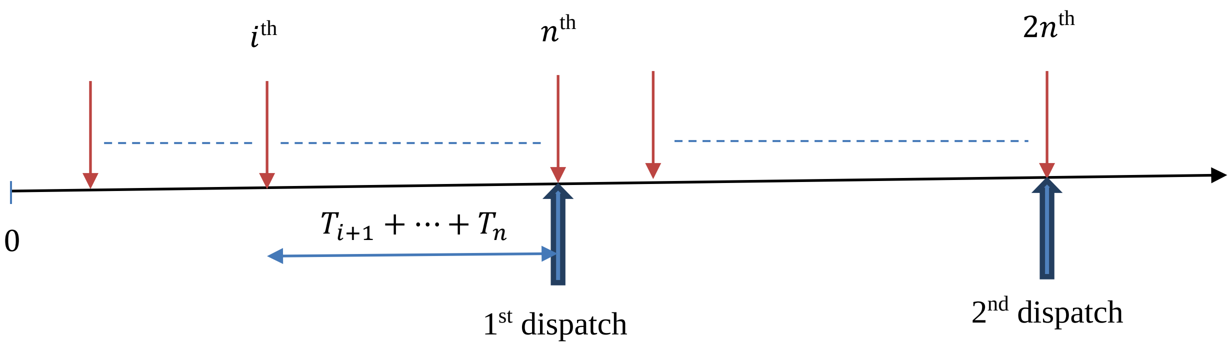
$$\frac{E[X(t)]}{t} = E[R]\frac{m(t)}{t} + \frac{E[R]}{t} - \frac{E\left[R_{N(t)+1}\right]}{t}$$
$$\Rightarrow \lim_{t\to\infty} \frac{E[X(t)]}{t} = E[R] \lim_{t\to\infty} \frac{m(t)}{t} + \lim_{t\to\infty} \frac{E[R] - E\left[R_{N(t)+1}\right]}{t} = \frac{E[R]}{E[T]}$$

by the elementary renewal theorem. Note that $\lim_{t\to\infty} E[R]/t = 0$ because $E[R]$ is finite (due to its existence itself), and for the same reason, $\lim_{t\to\infty} E\left[R_{N(t)+1}\right]/t = 0$. However, it shall be noted that $R_{N(t)+1}$ is different from a typical R, because $T_{N(t)+1}$ is larger than a typical T. If R is bounded (from above and below both), then $R_{N(t)+1}$ too is bounded and $E\left[R_{N(t)+1}\right]$ is finite, as required. We skip the proof for the unbounded case.

We implicitly assumed that the rewards are earned at the points of renewals. This is why we wrote $X\left(W_{N(t)}\right) = R_1 + R_2 + \cdots + R_{N(t)} = X(t)$. However, if the rewards are earned over the corresponding inter-event times, then $X(t) = X\left(W_{N(t)}\right) + \text{Reward earned in } \left(W_{N(t)}, t\right]$. Even in this case, the renewal reward theorem holds.

Group dispatching problem: Consider a joyride in an amusement park. Typically, it waits for some number of customers, say n, to gather before it starts. If n is too small, then the ride works with empty seats, which is costly. On the other hand, if n is too large, then the early arriving customers wait for too long, which has implicit cost. Considering these conflicting factors, we need to find an appropriate value for n. This kind of tradeoff arises in many real-life situations. Renewal reward process can help here.

Consider the time between arrival of successive customers, denoted by $T_1, T_2, T_3, \ldots$, to be *iid* positive random variables with mean μ. The ride starts as soon as n customers gather. Then the time for the first dispatch of the ride is given by $T_1' = T_1 + T_2 + \cdots + T_n$. The time between the first and the second dispatches is given by $T_2' = T_{n+1} + T_{n+2} + \cdots + T_{2n}$. In a similar manner, we can obtain the subsequent inter-dispatch times. These are *iid* positive random variables, as $T_1, T_2, T_3, \ldots$, are *iid* positive random variables. Clearly, the dispatches constitute a renewal process with interevent times $T_1', T_2', T_3', \ldots$.



Associated with each renewal, i.e., a dispatch, there is a cost, i.e., the reward. For the first renewal, the cost is: $R_1' = K + \sum_{i=1}^{n-1} c\left(\sum_{j=i+1}^{n} T_j\right)$, where K is the fixed cost of operating the ride and c is the cost of making a customer wait for unit time. For the second renewal, the reward is: $R_2' = K + \sum_{i=n+1}^{2n-1} c\left(\sum_{j=i+1}^{2n} T_j\right)$. In a similar manner, we can obtain the subsequent rewards. These rewards are *iid* random variables, as $T_1, T_2, T_3, \ldots$, are *iid* random variables. Then we have a renewal reward process with rewards $R_1', R_2', R_3', \ldots$.

The long-run (expected) cost rate, by the renewal reward theorem, is:

$$g(n) = \frac{E[R']}{E[T']} = \frac{E\left[K + \sum_{i=1}^{n-1} c\left(\sum_{j=i+1}^{n} T_j\right)\right]}{E[T_1 + T_2 + \cdots + T_n]} = \frac{K + c\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} E[T_j]}{n\mu}$$
$$= \frac{K + c\mu\sum_{i=1}^{n-1}(n-i)}{n\mu} = \frac{K + c\mu\,(n-1)n/2}{n\mu} = \frac{K}{n\mu} + c\frac{n-1}{2}$$

Considering n to be continuous (even though it is integer valued),

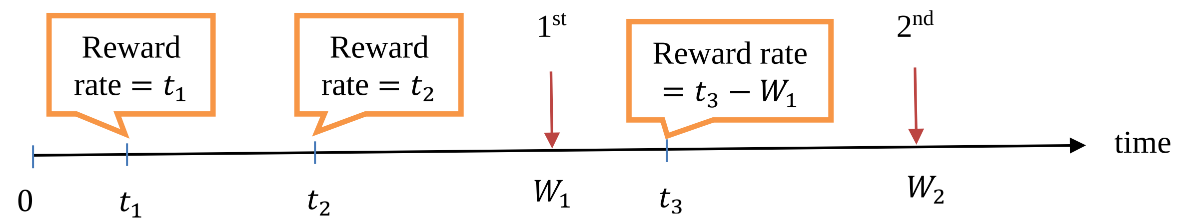
$$g'(n) = -\frac{K}{n^2\mu} + \frac{c}{2}, g''(n) = \frac{2K}{n^3\mu} > 0 \Rightarrow g(n) \text{ is a convex function}$$

Then $g(n)$ has unique minima given by $g'(n) = 0 \Rightarrow n^* = \sqrt{2K/c\mu}$. Since n is integer-valued, one of $\lfloor n^* \rfloor$ or $\lceil n^* \rceil$ is the optimal solution.

Age and residual life: In the inspection paradox, we have seen that $E\left[T_{N(t)+1}\right] = E[A(t)] + E[R(t)] \geq E[T]$, where $T_{N(t)+1} = W_{N(t)+1} - W_{N(t)}$ is the inter-event time that 'contains' t, $A(t) = t - W_{N(t)}$ is the age of the process since the last renewal, and $R(t) = W_{N(t)+1} - t$ is the residual life until the next renewal. Refer to the first figure in this document.

It is difficult to calculate these quantities for a given t. However, we can obtain long-run average values of these quantities using the renewal reward process. Average age in the interval $(0, u]$ is: $\frac{1}{u}\int_0^u A(t)dt$, and the long-run average age is: $\lim_{u\to\infty}\frac{1}{u}\int_0^u A(t)dt$. Similarly, the long-run average residual life is: $\lim_{u\to\infty}\frac{1}{u}\int_0^u R(t)dt$.

Typically, we consider that the rewards are earned at the point of renewals. However, we can relax this and allow the reward to accumulate over time. The renewal reward theorem is still applicable in this case. Consider a reward scheme that is earned in a continuous manner and the rate at which it is earned is $A(t)$ at time t, as shown below.



The first reward $R_1 = \int_0^{W_1} A(t)dt = \int_0^{T_1} tdt = T_1^2/2$. The second reward $R_2 = \int_{W_1}^{W_2} A(t)dt$ $= \int_{W_1}^{W_2}(t - W_1)dt = \int_0^{W_2-W_1} sds = T_2^2/2$. In this manner, one can obtain n-th reward $R_n = T_n^2/2$ for $n = 1,2,3, \dots$. Clearly, the rewards are *iid* random variables. In this renewal reward process, the long-run reward rate, i.e., $\lim_{u\to\infty} \frac{1}{u}\int_0^u A(t)dt$ is same as $E[R]/E[T]$ by the renewal reward theorem. Therefore, the long-run average age is:

$$\frac{E[T^2/2]}{E[T]} = \frac{E[T^2]}{2E[T]} = \frac{E^2[T] + Var(T)}{2E[T]} = \frac{1}{2}\left(E[T] + \frac{Var(T)}{E[T]}\right) \geq \frac{E[T]}{2}$$

In a similar manner, if we consider the reward rate to be $R(t)$ at time t, then the n-th reward $R_n = \int_{W_{n-1}}^{W_n} R(t)dt = \int_{W_{n-1}}^{W_n}(W_n - t)dt = \int_0^{W_n-W_{n-1}} sds = T_n^2/2$ for $n = 1,2,3, \dots$. Note that $R(t)$ is different from $R_1, R_2, R_3, \dots \equiv R$. Clearly, the rewards are *iid* random variables. Then by the renewal reward theorem, the long-run residual life is:

$$\lim_{u\to\infty} \frac{1}{u}\int_0^u R(t)dt = \frac{E[R]}{E[T]} = \frac{E[T^2/2]}{E[T]} = \frac{1}{2}\left(E[T] + \frac{Var(T)}{E[T]}\right) \geq \frac{E[T]}{2}$$

So, the long-run average age and residual lives are the same, which is at least $E[T]/2$. Also, $\lim_{u\to\infty} \frac{1}{u}\int_0^u T_{N(t)+1}dt = \lim_{u\to\infty} \frac{1}{u}\int_0^u A(t)dt + \lim_{u\to\infty} \frac{1}{u}\int_0^u R(t)dt = E[T] + Var(T)/E[T] \geq E[T]$. In the discussion of inspection paradox, we considered a renewal process with discrete uniform inter-event times and obtained $E\left[T_{N(4)+1}\right] = 2.346 > 2 = E[T]$. With the above results, $\lim_{u\to\infty} \frac{1}{u}\int_0^u T_{N(t)+1}dt = E[T] + Var(T)/E[T] = 2 + (2/3)/2 = 2.333$, which is little less than $E\left[T_{N(4)+1}\right]$. In the Poisson process, $Var(T)/E[T] = \left(1/\lambda^2\right)/(1/\lambda) = 1/\lambda = E[T]$. Then $\lim_{u\to\infty} \frac{1}{u}\int_0^u T_{N(t)+1}dt = 2E[T]$, and $\lim_{u\to\infty} \frac{1}{u}\int_0^u A(t)dt = \lim_{u\to\infty} \frac{1}{u}\int_0^u R(t)dt = E[T]$.

Practice problems

Book-1: Introduction to Probability Models by Sheldon Ross [10th edition]

Renewal process
Book-1, Chapter-7, Exercise No. 5, 10, 19, 31, 51

Renewal reward process
Book-1, Chapter-7, Exercise No. 26, 27, 38, 41

Appendix A

Consider an arbitrary time point t in a Poisson process with intensity λ. With respect to t, let $A(t)$ denote the time since the last event, i.e., the $N(t)$-th event, and $R(t)$ denote the time till the next event, i.e., $N(t)+1^{st}$ event. Refer to the first figure in this document. Here, we show that $R(t) \sim Exp(\lambda)$ and it is independent of $A(t)$. For $x, y > 0$,

$$P(R(t) > x | A(t) = y) = \sum_{n=0}^{\infty} P(R(t) > x | N(t) = n, A(t) = y) P(N(t) = n | A(t) = y)$$

For $n = 0$, $\{N(t) = n, A(t) = y\}$ is possible *iff* $y = t$. Then $P(R(t) > x | N(t) = 0, A(t) = t) = P(T_1 - t > x | T_1 > t) = P(T_1 > x) = e^{-\lambda x}$ by the memoryless property of exponential inter-event times. For $y \neq t$, the above sum starts from $n = 1$.

$$\begin{aligned}
\text{For } n \geq 1, P(R(t) > x | N(t) = n, A(t) = y) &= P(W_{n+1} - t > x | N(t) = n, t - W_n = y) \\
&= P(t - y + T_{n+1} > t + x | N(t) = n, W_n = t - y) \\
&= P(T_{n+1} > y + x | T_{n+1} > y, W_n = t - y) \\
&= P(T_{n+1} > y + x | T_{n+1} > y), \text{as } T_{n+1} \text{ is independent of } W_n \\
&= P(T_{n+1} > x) = e^{-\lambda x}, \text{by memoryless property}
\end{aligned}$$

$$\Rightarrow P(R(t) > x | A(t) = y) = e^{-\lambda x} \sum_{n=0}^{\infty} P(N(t) = n | A(t) = y) = e^{-\lambda x} \; \forall x, y > 0$$

Clearly, $R(t)$ is independent of $A(t)$ and it follows $Exp(\lambda)$ distribution.