Plurality Points and Condorcet Points in Euclidean Space

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Outline

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Informal description

We can represent opinion standpoints by points in space. Goal: find a (spacial) equilibrium between these quantified “opinions”.

- Voters: Multiset in Euclidean space ($\mathbb{R}^d$ with ℓ_2 norm)
- Plurality point: Closer to at least as many voters as any other point
- Condorcet point: No other point is closer to an absolute majority of voters

Notation

Definition (Voters)

Voters: $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$ (a multiset, in general). The components are denoted as $v_i = (v_i^{(1)}, \ldots, v_i^{(d)})$.

Definition (Multidimensional medians)

Informally: at least half the voters are on each side, per dimension:

$$\begin{aligned}\mathcal{M}_V = \{\theta : &|\{v \in V : \forall j : v^{(j)} \leq \theta^{(j)}\}| \geq n/2 \\ &\wedge |\{v \in V : \forall j : v^{(j)} \geq \theta^{(j)}\}| \geq n/2\}\end{aligned}$$

Notation

Definition (Median corners)

Let $V^{(i)}$ be the i-th coordinate of points in V, and $x_l^{(i)}$ and $x_h^{(i)}$ denote the $\lceil n/2 \rceil$-th and $\lceil (n+1)/2 \rceil$-th smallest numbers in $V^{(i)}$. Then the set

$$C = \{(x_{t_1}^{(i)}, \ldots, x_{t_d}^{(i)}) : t_1, \ldots, t_d \in \{l, h\}\}$$

collects all corners of $\mathcal{M}_V$. Clearly, $|C| \leq 2^d$.

Definition (Half spaces)

Given a line L, we can partition $\mathbb{R}^d = L^+ \cup L \cup L^-$, where L^+ and L^- are the open half spaces separated by L. We define:

$$V_L = L \cap V, \quad V_L^+ = L^+ \cap V, \quad V_L^- = L^- \cap V$$
$$n_L = |V_L|, \quad n_L^+ = |V_L^+|, \quad n_L^- = |V_L^-|$$

Formal definition

Definition (Preference)

- v_i prefers θ_1 to θ_2 if $d(v_i, \theta_1) < d(v_i, \theta_2)$
- $[\theta_1 \succ \theta_2] := \{v \in V : d(v, \theta_1) < d(v, \theta_2)\}$, and accordingly for other relations

Definition (Plurality point (PP))

$\Delta \in \mathbb{R}^d$ is a plurality point iff

$$\forall \theta \in \mathbb{R}^d : |[\theta \succ \Delta]| \leq |[\Delta \succ \theta]|$$

Definition (Condorcet point (CP))

$\Delta \in \mathbb{R}^d$ is a plurality point iff

$$\forall \theta \in \mathbb{R}^d : |[\theta \succ \Delta]| \leq n/2$$

Equivalence of PPs and CPs

Lemma (Equivalence)

In $\mathbb{R}^d$, a point is a plurality point iff it is a Condorcet point.

Proof.

1. A PP is a CP by definition.
2. - Assume x is a CP, but not a PP, so $\exists y : |[x \succ y]| < |[y \succ x]|$.
   - Let z be the midpoint of x, y. Then for each $c \in [y \succeq x]$, $d(c, z) < d(c, x)$.
   - Thus, $|[z \succ x]| > n/2$, which is a contradiction.
3. Therefore, a CP is also a PP.

From now on, we contstrain ourselves to $\mathbb{R}^2$.

Idea behind algorithm

The presented algorithm works by successively cutting down the space of possible PPs via a series of case distinctions, until only $\mathcal{O}(1)$ candidate points remain, which can be checked exhaustively. For this purpose, the following series of lemmata are employed.

Furthermore:

Definition

Δ_V denotes the set of all PPs of V

Collinear case

Lemma (Collinear points)

If all voters in V are collinear, then

1. $\Delta_V = \mathcal{M}_V$, *and*
2. $|\Delta_v| \geq 1$.

Proof.

1. has been shown by Hansen & Thisse (1981) for CCs in $\mathbb{R}^2$. By the equivalence lemma, this holds also for PPs.
2. By definition, $\mathcal{M}_V \neq \emptyset$. When $|V|$ is even, we may have $|\mathcal{M}_V| = |\Delta_V| > 1$.

Tukey depth condition

This condition is used to check if a candidate point is a valid PP.

Lemma

In $\mathbb{R}^2$, Δ is a plurality point iff for any line L through Δ, $n_L^+ \leq n/2$ and $n_L^- \leq n/2$.

This can be determined by the number

$$\min_{L \in \mathcal{L}_\Delta} \{|V \cup \gamma| : \gamma \text{ is a closed halfspace separated by } L\},$$

where $\mathcal{L}_\Delta$ is the space of lines through Δ, is called *Tukey depth* of Δ with respect to V.

Non-collinear case & medians

Lemma

If not all voters in V are collinear, then

1. $\Delta_V \subseteq \mathcal{M}_V$, *and*
2. $|\Delta_v| \in \{0, 1\}$.

So, a plurality point must be a median, but not vice versa. We can use this to cut down the searched space to the median; however, this is still infinite.

Δ is in $\mathcal{M}_V$

These lemmata reduce the possibilities in the non-collinear case further to $\mathcal{O}(1)$ candidate points.

Lemma ($\Delta \in V$)

If Δ is a PP of V, and $\Delta \in V$, then $\Delta \in C$, and $V \cup C \leq 2^d$.

Lemma ($\Delta \notin V$)

If $\Delta \notin V$, then Δ is a PP iff $n_L^+ = n_L^-$ for each line L passing through Δ.

Subroutines

- Select(S, n) returns the n-th smallest point in S. This can be calculated in $\mathcal{O}(|S|)$ time by the median-of-medians selection algorithm from Cormen et. al. Used for median calculation.
- VerifyCandidates(C, V) returns all points from C whose Tukey depth in is greater or equal $|V|/2$ (ie., satisfying the condition in the "Tukey depth lemma" above). Tukey depth of one point can be computed in $\mathcal{O}(|V| \log |V|)$ time with an algorithm by Rousseeuw and Struyf (1998).

Subroutines

- InternalTangentIntersection(V, W) returns the intersection of the internal tangents between the convex hulls of V and W. The tangents can be found by solving the linear program

$$\min_{k,d} F(k,d) = k, \quad \text{s.t.}$$

$$\forall\, v \in V : v_y \leq k v_x + d \quad \text{and}$$

$$\forall\, w \in W : w_y \geq k w_x + d,$$

and the opposite formulation. Linear programming with fixed dimension can be done in linear time in the number of constraints, by Megiddo (1984).

Pseudocode

```
procedure PluralityPoint2D($V$)
    if all voters are collinear then
        return medians of $V$
    else                                          ▷ Corners of medians
        $x_h$ = Select($V_x$, $\lceil (n+1)/2 \rceil$)
        $x_l$ = Select($V_x$, $\lceil n/2 \rceil$)
        $y_h$ = Select($V_y$, $\lceil (n+1)/2 \rceil$)
        $y_l$ = Select($V_y$, $\lceil n/2 \rceil$)
        $C = \{(x_h, y_h), (x_h, y_l), (x_l, y_h), (x_l, y_l)\}$

        if $|C| = 1$ then                         ▷ n odd, or C degenerate
            return VerifyCandidates($C$, $V$)
        else
            $P$ = VerifyCandidates($V \cap C$, $V$)
            if $P \neq \emptyset$ then            ▷ $\Delta \in V$
                return $P$
            else                                  ▷ $\Delta \notin V$
                if $x_h = x_l$ then
                    $V_a = \{v \in V : v^{(2)} \leq y_h\}$
                    $V_b = \{v \in V : v^{(2)} \geq y_h\}$
                else
                    $V_a = \{v \in V : v^{(1)} \leq x_h\}$
                    $V_b = \{v \in V : v^{(1)} \geq x_h\}$
                end if

                $p$ = InternalTangentIntersection($V_a$, $V_b$)
                return VerifyCandidates($\{p\}$, $V$)
            end if
        end if
    end if
end procedure
```

Analysis

- Detecting collinearity (line 2): $\mathcal{O}(n)$
- Selecting median corners (lines 5–8): $\mathcal{O}(n)$
- Verifying candidates by calculating Tukey depth (lines 11, 13, 25): $\mathcal{O}(n \log n)$
- Computing internal tangent intersection (line 24): $\mathcal{O}(n)$
- Therefore: $\mathcal{O}(n \log n)$ overall complexity

Conclusion

- In Euclidean space, plurality points and Condorcet points are equivalent notions
- In $\mathbb{R}^2$, they can be calculated in $\mathcal{O}(n \log n)$ time
- This can be generalized to $\mathbb{R}^d$ in $\mathcal{O}(n^{d-1} \log n)$ time