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Abstract

Modeling and forecasting realized volatility is of paramount importance. Recent econometric developments allow total volatility to be decomposed into its' constituent continuous and jump components. While previous studies have examined the role of both components in forecasting, little analysis has been undertaken into how best to harness the jump component. This paper considers how to get the most out of the jump component for the purposes of forecasting total volatility. In combination with the magnitude of past jumps, the intensity of jump occurrence is examined. Estimated jump intensity from a point process model is used within a forecasting regression framework. Even in the presence of the diffusive part of total volatility, and past jump size, intensity is found to significantly improve forecast accuracy. The improvement is particularly apparent on the days when jumps occur or when markets are turbulent. Overall, the best way to harness the jump component for volatility forecasting is to make use of both the magnitude and probability of jump occurrences.

Keywords
Realized volatility, diffusion, jumps, point process, Hawkes process, forecasting

JEL Classification Numbers
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1 Introduction

Understanding how the volatility of asset returns evolves is of paramount importance. Many financial applications such as risk management, portfolio allocation and derivative pricing utilize such information, and as such, there has been a vast literature relating to the estimation and forecasting of volatility. Much of this literature has stemmed from the development of the univariate GARCH class of models, Engle (1982) and Bollerslev (1986). In recent years, this literature has benefited from the ready availability of high-frequency intraday data which has led to the development of Realized Volatility (RV) by Andersen, Bollerslev, Diebold, and Labys (2001, 2003) and improved measures of volatility at a daily frequency. As opposed to the GARCH approach of treating volatility as latent, RV provides an observable proxy upon which time-series models can be directly used to generate forecasts

Estimates of RV converge to the total quadratic variation in the asset return process, which may be attributable to both continuous diffusion and discrete jump processes. A number of times series approaches have been applied to forecasting total RV. Among these are the Mixed Interval Data Sampling (MIDAS) of Ghysels, Santa-Clara, and Valkanov (2006) and long-memory ARMA models, Oomen (2001), Andersen et al. (2003) and Koopman, Jungbacker, and Hol (2005). Barndorff-Nielsen and Shephard (2006) and Andersen, Bollerslev, and Diebold (2007) develop methods for measuring the contribution to total quadratic variation from both sources. Andersen et al. (2007) propose a formal test for identifying when significant contributions from jump activity occur. Andersen et al. (2007) employ the Heterogeneous AutoRegressive (HAR) model of Muller, Dacorogna, Dave, Olsen, Pictet, and Weizsacker (1997) and Corsi (2009) to forecast volatility using separate jump and diffusion components. They find that most of the predictability in volatility stems from the persistent diffusive component.

This paper reconsiders the information reflected in jump activity for the purposes of forecasting volatility, and in doing so, extends the analysis of Andersen et al. (2007). Andersen et al. (2007) employ simple historical averages of the jump component size in their HAR forecasting regressions. Here, a separate model for the jump intensity (probability) is estimated. The estimated jump intensity is then used within the HAR framework for volatility forecasting along with the smoothed jump size. Our results indicate that the jump intensity significantly contributes to the predicability of volatility. Meanwhile, after including the jump intensity in the HAR model, volatility forecasting performance improves significantly. This improvement is particularly apparent on the days when jumps occur or when markets are turbulent such as the most recent period of financial crisis.

The paper proceeds as follows. Section 2 describes the methods employed to estimate realized volatility and identify jumps. Section 3 describes the S&P500 stock index data used in the empirical analysis. Section 4 presents both the empirical methods employed and results. Section 5 provides concluding comments.

2 Theoretical Framework

In this section, the formation of realized volatility from high frequency financial data is discussed along with how the diffusion and jump components are extracted. Begin by assuming that the logarithm of the asset price within the active part of the trading day evolves in continuous time as a standard jump-diffusion process given by

$$dp(t) = u(t)dt + \sigma(t)dw(t) + \kappa(t)dq(t), \tag{1}$$

where $u(t)$ denotes the drift term that has continuous and locally bounded variation, $\sigma(t)$ is a strictly positive spot volatility process and $w(t)$ is a standard Brownian motion. The $\kappa(t)dq(t)$ term refers to a pure jump component, where $k(t)$ is the size of jump and $dq(t) = 1$ if there is a jump at time t (and 0 otherwise). The corresponding discrete-time within-day geometric returns are

$$r_{t+j\triangle} = p(t + j/M) - p(t + (j-1)/M), \quad j = 1, 2, ...M, \tag{2}$$

where M refers to the number of intraday equally spaced returns over the trading day t, and $\Delta = 1/M$ denotes the sampling interval. As such, the daily return for the active part of the trading day equals $r_{t+1} = \sum_{j=1}^{M} r_{t+j\triangle}$.

As noted in Andersen and Bollerslev (1998), Andersen et al. (2003), and Barndorff-Nielsen and Shephard (2002), the quadratic variation of the process in equation (1) can be estimated by realized volatility (RV), which is defined as the sum of the intraday squared returns, i.e.

$$RV_{t+1}(\Delta) \equiv \sum_{j=1}^{M} r_{t+j\Delta}^2, \tag{3}$$

whereas the integrated variance of the process in equation (1) is typically estimated using realized bi-power variation (BV) defined by

$$BV_{t+1}(\Delta) \equiv \frac{\pi}{2}(\frac{M}{M-1})\sum_{j=2}^{M} |r_{t+j\Delta}||r_{t+(j-1)\Delta}|. \tag{4}$$

Therefore, given appropriate regularity conditions, we have

$$\begin{aligned} \underset{M\to\infty}{plim}\, RV_{t+1}(\Delta) &= \int_t^{t+1} \sigma^2(s)ds + \sum_{t<s<t+1} \kappa^2(s), &(5)\\ \underset{M\to\infty}{plim}\, BV_{t+1}(\Delta) &= \int_t^{t+1} \sigma^2(s)ds, &(6) \end{aligned}$$

Naturally, the difference between the two estimators can be used to estimate the contribution of jump component to the volatility.

However, the theoretical justification for equations (3) and (4) is based on the notion of increasingly finer sampled returns, or $M \to \infty$. Of course, any practical implementation with a finite fixed sampling frequency, or $M < \infty$, is invariably subject to measurement error, and hence it is desirable to treat small jumps as measurement error and identify significantly large jumps. Barndorff-Nielsen and Shephard (2006) developed such a test, which is modified to account for microstructure noise and improve the finite sample performance following Huang and Tauchen (2005) as

$$Z_{t+1}(\Delta) \equiv \Delta^{-1/2} \times \frac{[RV_{t+1}(\Delta) - BV_{t+1}(\Delta)]RV_{t+!}(\Delta)^{-1}}{[(\mu_1^{-4} + 2\mu_1^{-2} - 5)max\{1, TQ_{t+1}(\Delta)BV_{t+1}(\Delta)^{-2}\}]^{1/2}}, \tag{7}$$

where TQ_{t+1} is the realized tripower quarticity measure, whose general expression is

$$TQ_{t+1}(\Delta) = \Delta^{-1}\mu_{4/3}^{-3} \sum_{j=3}^{1/\Delta} |r_{t+j*\Delta,\Delta}|^{4/3}|r_{t+(j-1)*\Delta,\Delta}|^{4/3} r_{t+(j-2)*\Delta,\Delta}|^{4/3}, \tag{8}$$

and $\mu_{4/3} \equiv E(|Z|^{4/3}) = 2^{2/3} \cdot \Gamma(7/6) \cdot \Gamma(1/2)^{-1}$. The BV and TQ measures are generated based on staggered returns to remove the microstructure noise.

Significant jumps can be identified by the realizations of $Z_{t+1}(\Delta)$ in excess of some critical value Φ_α,

$$J_{t+1,\alpha}(\Delta) \equiv I(Z_{t+1}(\Delta) > \Phi_\alpha) \cdot [RV_{t+1}(\Delta) - BV_{t+1}(\Delta)], \tag{9}$$

where $I(\cdot)$ denotes an indicator function, and α is the significance level. The continuous component is then defined as

$$C_{t+1,\alpha}(\Delta) \equiv I(Z_{t+1}(\Delta) \leq \Phi_\alpha) \cdot RV_{t+1}(\Delta) + I(Z_{t+1}(\Delta) \leq \Phi_\alpha) \cdot BV_{t+1}(\Delta) \tag{10}$$

to ensure that the continuous and jump components add to the realized volatility.

The intuition behind the concept of bipower variation is the following. Suppose $r_{t+j\Delta}$ contains a jump, it multiplies $r_{t+(j-1)\Delta}$ and $r_{t+(j+1)\Delta}$ in the case of bipower variation. Both these returns have to vanish asymptotically and the bipower variation converges to the integrated variance. However, for finite Δ these returns will not vanish, causing a positive bias which will be extreme if consecutive jumps occur. Corsi, Pirino, and Reno (2010) developed the threshold bipower variation (TBV) with the aim of reducing this bias. The TBV is defined as

$$TBV_{t+1}(\Delta) \equiv \frac{\pi}{2}(\frac{M}{M-1}) \sum_{j=2}^{M} |r_{t+j\Delta}||r_{t+(j-1)\Delta}| I_{|r_{t+j\Delta}|^2 <= \vartheta_{t+j}} I_{|r_{t+(j-1)\Delta}| <= \vartheta_{t+(j-1)}}. \tag{11}$$

This estimator sets up a threshold function ϑ to select out the intraday returns containing jumps. For example, if $|r_{t+j\Delta}|$ has a jump larger than ϑ_{t+j}, the corresponding indicator function

	RV_t	BV_t	J_t
Mean	1.1673e-04	1.0023e-04	1.6457e-05
St. dev.	2.5442e-04	2.3167e-04	9.2653e-05
Skewness	9.5872	11.3899	12.3091
Kurtosis	156.9335	217.1928	213.0333
Min	1.8500e-06	6.8700e-07	0
Max	0.0070	0.0070	0.0025
LB_{10}	1.4576e+04	1.3871e+04	107.1182

Table 1: The first six rows in each of the panels report the sample mean, standard deviation, skewness, and kurtosis, along with the sample minimum and maximum for the daily realized volatility, daily bipower variation and the size of jumps . The rows labeled LB_{10} give the Ljung-Box test statistic for up to tenth-order serial correlation for the daily realized volatility, daily bipower variation and the size of jumps.

vanishes, thus the $|r_{t+j\Delta}|$ is not included in the calculation of TBV, and the bias is corrected. Corsi et al. (2010) scales the threshold function ϑ with respect to the local spot variance, and is estimated with the data themselves as

$$\vartheta_{t+j} = c_\vartheta^2 \hat{V_{t+j}}, \tag{12}$$

where $\hat{V_{t+j}}$ is an estimator of the local spot variance σ_t^2 and c_ϑ^2 is a scale-free constant. Appendix B in Corsi et al. (2010) provides the details on the estimation of the local spot variance $\hat{V_{t+j}}$.

As a robustness check, subsequent empirical analysis also employs TBV as a measure of the continuous component, along with the corresponding jump component extracted from RV to explore the role of jumps in volatility forecasting.

3 Data and Summary Statistics

Our empirical data set comprises thirty-minute intra-day prices for S &P500 index for the period 2 Jan 1990 to 1 Jun 2012 (5652 trading days). The realized volatility and bipower variation are constructed from the original thirty-minute returns based on the procedure discussed in Section 2. The jumps are generated by using the Barndorff-Nielsen and Shephard (2006) test with a 99.5% significance level. Jumps are detected in 11.75% of trading days, and are detected on consecutive days, with a maximum duration of 58 days between jumps.[1]

Figure 1 shows the plots of RV, BV, jumps and the binary series of jump occurrence. It is clear that RV, BV and jumps follow a similar pattern and are dominated by the period of 1998-

[1]The jumps reported here are generated based on BV. The ones generating by using TVB are very identical, and thereby are not reported.

2003 and the Global Financial Crisis period of 2008-2010. They are quite persistent processes and have a slight increase from 1998-2003, followed by a rapid rise during the crisis period of 2008-2010. In general, they remain higher during the latter half of the sample. Jumps occur more frequently and exhibit a great deal of clustering with many jumps occurring during and after the Global Financial Crisis period of 2008-2010. Figure 2 plots the autocorrelations in RV, BV and jumps (along with associated confidence intervals). It is clear RV and BV exhibit the familiar pattern of strong and slowly decaying persistence. The jumps exhibit a relatively small, though at times significant, degree of persistence. It is clear from the plot of the binary series of when jumps occur, there is clustering in the times of their occurrence. It implies that the jump intensity is somewhat persistent. These visual observations are readily confirmed by the Ljung-Box statistics for up to tenth order serial correlation reported in the last rows of Table 1. Meanwhile, the table indicates that the jump component is important, with the mean of the jump size accounting for 0.141 of the mean of RV. The summary statistics of TBV and the corresponding jumps are not reported, as they are quite similar to what are reported here.

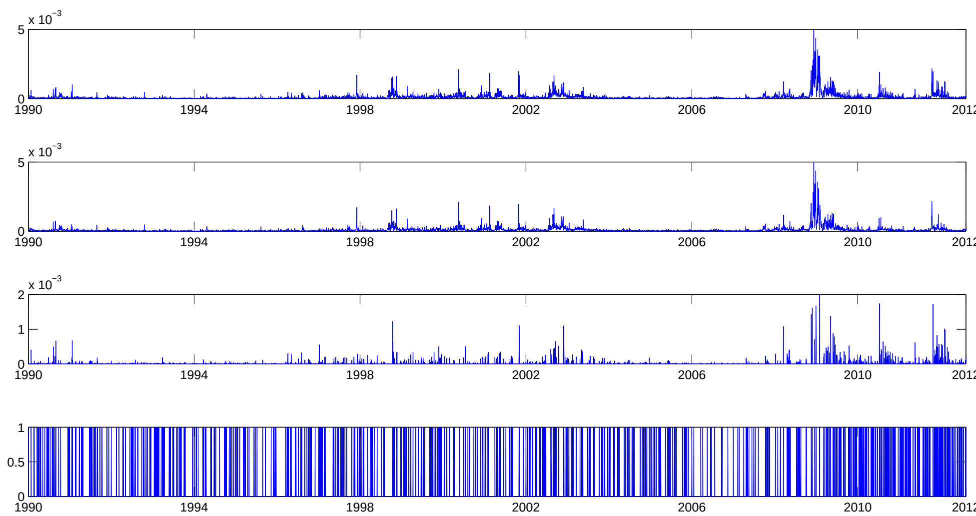

Figure 1: The top panel shows the realized volatility, the second panel graphs the bipower variation, the third and the bottom panels respectively plot the series of jump size and the binary series of jump occurrence.

4 Empirical Methods and Results

This section presents both the empirical methods employed and associated results. Section 4.1 builds up a model to describe the dynamic behavior of jumps intensity, and Section 4.2 examines the impact of jumps intensity on volatility.

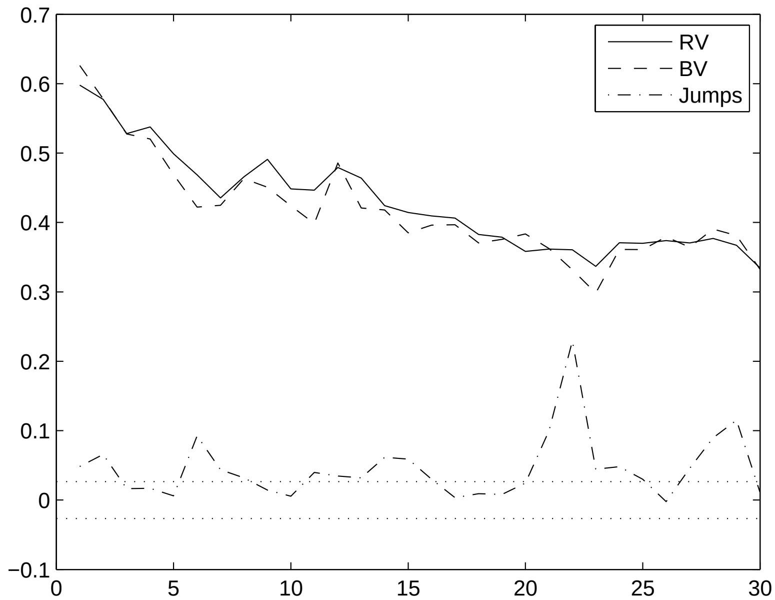

Figure 2: Autocorrelations for the realized volatility, the bipower variation and the jumps size. The dashed lines represent the 95% confidence interval.

4.1 A Hawkes model for jump intensity

To model the intensity of jumps, their occurrence is viewed as point process. Such an approach is common when dealing with events in financial markets such as the arrival of trades or quotes, for an overview of the literature see Bauwens and Hautsch (2009). To begin, a number of definitions are required. Let $\{t_i\}_{i\in 1,\cdots,n}$ be a random sequence of increasing event times $0 \geq t_1 > \cdots > t_n$ which describe a simple point process. Given $N(t) := \sum_{i\geq 1} \mathbf{1}_{t_i \geq t}$ is a counting function, the conditional intensity, $\lambda(t)$ can be viewed as the expected change in $N(t)$ (as a reflection of the probability of an event occurring) over a small time horizon,

$$\lambda(t) = \lim_{s\downarrow t} \frac{1}{s-t} E[N(s) - N(t)] \tag{13}$$

Bauwens and Hautsch (2009) provide a discussion of various specifications for $\lambda(t)$.

A common specification for $\lambda(t)$ is the self-exciting Hawkes process, attributable to Hawkes (1971)

$$\lambda(t) = \mu + \int_0^t w(t-u)dN(u) = \mu + \sum_{t_i<t} w(t-t_i) \tag{14}$$

where μ is a constant and $w()$ is a non-negative weight function. This process is self-exciting in the sense that $\text{Cov}[N(a,b),\, N(b,c)] > 0$ where $0 > a \geq b < c$. The weight function $w()$ is a decreasing function of $t-u$ meaning that subsequent to a spike the intensity decays. Bowsher (2007), Large (2007) and Ait-Sahalia, Cacho-Diaz, and Laeven (2011) model events in financial markets such as trading activity, liquidity supply and contagion across markets using Hawkes processes.

In the current context, a discrete version of equation (14) is estimated, where λ_t will denote the discrete time conditional intensity for the $t-th$ trading day. The intensity of the occurrence of jumps will be driven by past occurrences to allow for self-excitation to capture short term persistence, as well as exogenous variables such as the level of volatility. The model takes the following form

$$\lambda_t = \mu + \gamma X_{t-1} + \begin{cases} \beta\lambda_{t-1} & \text{if } dN_{t-1} = 0 \\ \beta\lambda_{t-1} + \alpha & \text{otherwise,} \end{cases} \tag{15}$$

where μ is a baseline intensity, λ_{t-1} is the conditional intensity from the previous day, β is the decay factor for the intensity from the previous day, and α is the shock to the intensity on day t if a jump occurred on day $t-1$ ($dN_{t-1,d} > 0$). X_{t-1} denotes a set of exogenous variables, in this context, it includes the lags of bipower variation[2]. A base self exciting model without exogenous variables will also be estimated.

The first three columns of Table 2 report the estimation results of Hawkes model for jump intensity of equation (15). Results for Model A in the first column represent the base self exciting model without covariates and are based on jumps extracted from RV using BV. Models B and C include lags of BV, specifically the level and the logarithm of BV. Turning to the results reported in the first column of Table 2, the estimates β close to 1 and α significantly positive confirm that jumps self-excite and exhibit a persistent intensity process.

Interestingly, when we introduce the bipower variation into the jump intensity model, the impact of the occurrence of past jumps although continues to be important, but it decreases. Comparing the estimation results of the first column with the second and the third columns, the estimates for β are very close, but the estimates for α drop from 0.0051 to 0.0037 and 0.0046. Meanwhile, the coefficients of the lags of bipower variation (in either levels or logs) are significant at the 5% significance level, and the values of the log likelihood function improve. In other words, the intensity of jumps appear to be self exciting with the level of intensity influenced by the market condition (the level of the volatility).

To determine whether the results are sensitive to the procedure used to extract the jump component, TBV and corresponding jumps are used to conduct the sample analysis. The results are presented in the last three columns of Table 2 and are consistent with those in the first three columns of the table. Jumps detected based on 1% and 5% significance levels were also used to re-estimate the set of models outlined above. We do not report the results here but they are

[2] We use bipower variation instead of realized volatility as a volatility measure in order to exclude the impact of jumps on itself. Meanwhile, the level of volatility, BV_t, at the same period is not considered here, because from a forecasting perspective the information of BV_t is not available before time t.

$$
\begin{aligned}
A:\ \lambda_t &= \mu + \begin{cases} \beta\lambda_{t-1} & \text{if } dN_{t-1}=0 \\ \beta\lambda_{t-1}+\alpha & otherwise \end{cases} \\
B:\ \lambda_t &= \mu + \gamma X_{t-1} + \begin{cases} \beta\lambda_{t-1} & \text{if } dN_{t-1}=0 \\ \beta\lambda_{t-1}+\alpha & otherwise \end{cases} \\
C:\ \lambda_t &= \mu + \gamma log(X_{t-1}) + \begin{cases} \beta\lambda_{t-1} & \text{if } dN_{t-1}=0 \\ \beta\lambda_{t-1}+\alpha & otherwise \end{cases}
\end{aligned}
$$

	$X = BV$			$X = TBV$		
	A	B	C	A	B	C
μ	0.0388	0.0359	0.0633	0.0527	0.0511	0.0763
	(0.0103)	(0.0096)	(0.0182)	(0.0125)	(0.0133)	(0.0125)
α	0.0051	0.0037	0.0046	0.0052	0.0036	0.0027
	(0.0012)	(0.0015)	(0.0012)	(0.0019)	(0.0020)	(0.0001)
β	0.9927	0.9942	0.9950	0.9926	0.9945	0.9981
	(0.0015)	(0.0018)	(0.0597)	(0.0026)	(0.0024)	(0.0011)
γ		0.6000	0.000025		0.6860	0.000024
		(0.2969)	(0.000012)		(0.3653)	(0.000008)
log likelihood	-2021.4	-2019.6	-2018.9	-2503.1	-2501.5	-2500.1

Note: This table reports the MLE estimates for Hawkes model of jump intensity without covariates or with covariates: the lags of bipower variation. The continuous variation (BV or TBV) and jump component are constructed from the original thirty-minute returns spanning from 2 Jan 1990 to 1 Jun 2012, for a total of 5652 trading days. The standard errors are reported in parentheses. The last row labeled log likelihood reports the negative value of the sum of log likelihood function across time evaluated at the estimates.

Table 2: Estimation results of Hawkes model for jump intensity

very similar to those reported in Table 2.

The estimated jump intensity from the base self exciting model is plotted in Figure 3. The plot shows that the jump intensity fluctuates around 0.12 during period 1990-2006, which is very close to the proportion of trading days with jumps reported in Section 3. After a sudden drop between 2006-2010, the intensity reaches to a very high level. The relatively higher jump intensity after the global financial crisis period (2008-2010) further verifies that the jump intensity is influenced by market conditions. It is the estimated intensity that will incorporated into the subsequent forecasting analysis.

4.2 Forecasting volatility: The role of jump intensity

With the widespread availability of high-frequency financial data, recent literature has focussed on RV to build forecasting models for time-varying financial volatility. A number of empirical

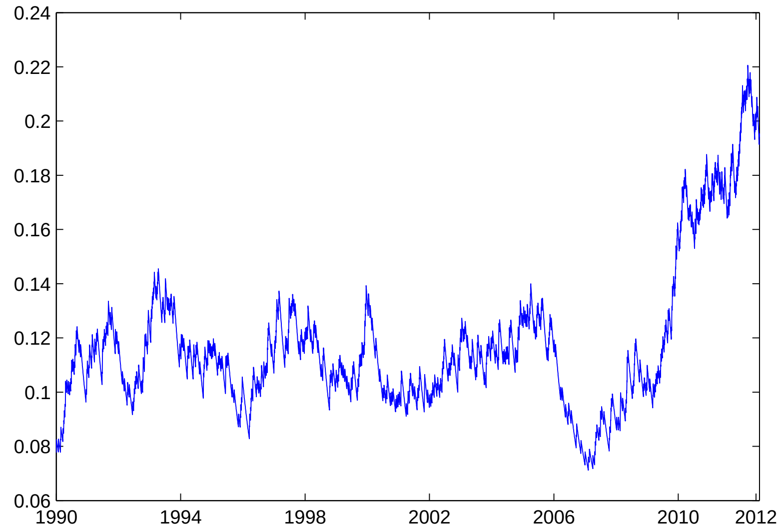

Figure 3: The estimated jump intensity from the Hawkes model as defined in equation (15)

studies have argued for the importance of long-memory dependence in realized volatility. Andersen et al. (2007) eschew such complicated fractionally integrated long-memory formulations and rely instead on the simple-to-estimate Heterogeneous Autoregressive Realized Volatility (HAR-RV) model proposed by Corsi (2009). The HAR-RV formulation is based on a straightforward extension of the so-called Heterogeneous ARCH, or HARCH, class of models analyzed by Muller et al. (1997). In such models the conditional variance of the discretely sampled returns is parameterized as a linear function of the lagged squared returns over the identical return horizon together with the squared returns over longer and/or shorter return horizons. By separately including the continuous sample path and individual jump variability measures in this simple linear volatility forecasting model, Andersen et al. (2007) find that only the continuous part has predictive power, in turn resulting in significant gains relative to the simple realized volatility forecasting models advocated in some of the recent literature.

The current analysis extends the work of Andersen et al. (2007) to further explore the information reflected in the realized jump component. Andersen et al. (2007) only consider the magnitude of jumps in the context of volatility forecasting. However, there is a second dimension to jump activity, the intensity of a jump occurring that has thus far been ignored. Therefore, this section investigates the impact of jump intensity and the joint influence of jump size and jump intensity on future volatility within a HAR framework.

The HAR-RV model specifies RV as a function of a daily, weekly and monthly realized volatility component, and can be expressed as

$$RV_t^d = \alpha_0 + \alpha_1 RV_{t-1}^d + \alpha_2 RV_{t-1}^w + \alpha_3 RV_{t-1}^m + \varepsilon_t, \tag{16}$$

where RV_t^d denotes the daily realized volatility, $RV_{t-1}^w = \frac{1}{5}\sum_{i=1}^{5} RV_{t-i}$ is the weekly realized volatility and $RV_{t-1}^m = \frac{1}{22}\sum_{i=1}^{22} RV_{t-i}$ represents the monthly realized volatility.

Andersen et al. (2007) turned to a simple extension of the HAR-RV, in which they incorporate the time series of magnitude of significant jumps as additional explanatory variables in a straightforward linear fashion, resulting in a new HAR-RV-J model as

$$RV_t^d = \alpha_0 + \alpha_1 RV_{t-1}^d + \alpha_2 RV_{t-1}^w + \alpha_3 RV_{t-1}^m + \alpha_4 J_{t-1} + \varepsilon_t, \tag{17}$$

to explore the predictive power of jump component for the following day's volatility. Meanwhile, Andersen et al. (2007) further extended this model by explicitly decomposing the realized volatilities into the continuous sample path variability and the jump variation utilizing the separate nonparametric measurements based on a statistical jump test to the HAR-RV-CJ model, which is expressed as

$$\begin{aligned} RV_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m \\ +\alpha_4 J_{t-1}^d + \alpha_5 J_{t-1}^w + \alpha_6 J_{t-1}^m + \varepsilon_t, \end{aligned} \tag{18}$$

where C_t and J_t are respectively the continuous sample path variation and the jump variation at time t, and $C_{t-1}^w = \frac{1}{5}\sum_{i=1}^{5} C_{t-i}$, $C_{t-1}^m = \frac{1}{22}\sum_{i=1}^{22} C_{t-i}$, $J_{t-1}^w = \frac{1}{5}\sum_{i=1}^{5} J_{t-i}$, and $J_{t-1}^m = \frac{1}{22}\sum_{i=1}^{22} J_{t-i}$.

To examine the predictive power of the probability of jump occurrences, intensity of jumps λ_t are used to extend the HAR-RV-CJ model in two ways. One is to replace the magnitude of jumps with the intensity of jumps to develop a HAR-RV-CI model, and the other is to include both dimensions of the jump component to produce a HAR-RV-CJI model. From equation 15, it can be seen that λ_t is estimated based on the past jump intensities, which actually incorporates all the historical information of jump occurrences. Hence, this additional information is useful to examine whether future volatility tends to increase or decrease when jumps have been occurring in the near past.

The HAR-RV-CI simply model replaces past jumps with the estimated intensity,

$$RV_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_7 \lambda_t + \varepsilon_t, \tag{19}$$

thus ignoring information about the size of past jumps. Whereas the HAR-RV-CJI model,

$$RV_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_4 J_{t-1}^d + \alpha_5 J_{t-1}^w + \alpha_6 J_{t-1}^m + \alpha_7 \lambda_t + \varepsilon_t, \tag{20}$$

incorporates information relating to past jumps size and occurrence.

Moreover, nonlinear HAR-RV-CI and HAR-RV-CJI models are estimated as a robustness check,

$$log(RV)_t^d = \alpha_0 + \alpha_1 log(C_{t-1}^d) + \alpha_2 log(C_{t-1}^w) + \alpha_3 log(C_{t-1}^m) + \alpha_7 \lambda_t + \varepsilon_t, \tag{21}$$

$$(RV_t^d)^{1/2} = \alpha_0 + \alpha_1 (C_{t-1}^d)^{1/2} + \alpha_2 (C_{t-1}^w)^{1/2} + \alpha_3 (C_{t-1}^m)^{1/2} + \alpha_7 \lambda_t + \varepsilon_t, \tag{22}$$

and

$$\begin{aligned} log(RV)_t^d &= \alpha_0 + \alpha_1 log(C_{t-1}^d) + \alpha_2 log(C_{t-1}^w) + \alpha_3 log(C_{t-1}^m) \\ &+ \alpha_4 log(J_{t-1}^d + 1) + \alpha_5 log(J_{t-1}^w + 1) + \alpha_6 log(J_{t-1}^m + 1) + \alpha_7 \lambda_t + \varepsilon_t, \end{aligned} \tag{23}$$

$$\begin{aligned} (RV_t^d)^{1/2} &= \alpha_0 + \alpha_1 (C_{t-1}^d)^{1/2} + \alpha_2 (C_{t-1}^w)^{1/2} + \alpha_3 (C_{t-1}^m)^{1/2} \\ &+ \alpha_4 (J_{t-1}^d)^{1/2} + \alpha_5 (J_{t-1}^w)^{1/2} + \alpha_6 (J_{t-1}^m)^{1/2} + \alpha_7 \lambda_t + \varepsilon_t. \end{aligned} \tag{24}$$

Table 3 reports estimation results of the linear and nonlinear HAR-RV-CJ, HAR-RV-CI and HAR-RV-CJI models using BV as the proxy for the continuous component of volatility and associated jumps. Coefficient estimates not significant at 5% are omitted. The estimates for α_1, α_2 and α_3 for all models confirm the strong predictive power of the continuous part of volatility. In contrast with Andersen et al. (2007), here the estimates for α_4, α_5 and α_6 in HAR-RV-CJ model and α_4 in HAR-RV-CJI models show that the magnitude of jump component (jump size) contains some predictive power. Most importantly, the estimated coefficient of the jump intensity (α_7) is systematically positive in the HAR-RV-CI models and HAR-RV-CJI models, and overwhelmingly significant. This implies that when jumps are more likely to occur, the higher the volatility and hence beyond the size of jumps, the probability with which they occur is important. While future volatility is significantly related to jump intensity, improvements in-sample model fit are modest. R^2 for any of the linear or nonlinear versions, only increases marginally from the HAR-RV-CJ to either the HAR-RV-CI or HAR-RV-CJI models. The HAR-RV-CJI model, in which both the magnitude of jump component and the probability of jump occurrence are included has the highest R^2, which implies that both dimensions of jump activity are important. Table 4 reports the equivalent results using TBV as the proxy for the continuous component of RV and confirm those based on BV reported in Table 3. Even though improvements in in-sample fit are small, the crucial issue of out-of-sample forecast performance will be considered below.

The magnitude of the estimated coefficients quantify the impact of the occurrences of past jumps on future volatility. For example, when there is a jump occurring at time $t-1$, the probability of jump occurring at time t will increase by $\alpha = 0.0051 \approx 0.51\%$ according to the estimation results of Hawkes model of equation (15), and thereby the volatility at time t will increase by $\alpha_7 * \alpha$. The occurrence of past jumps on the future volatility is not considered in the standard HAR-RV-CJ model.

To consider this impact in further detail, assume that a jump of size $9.64e-05$ occurred at time $t-1$.[3] Figure 4 plots the impact curve of this jump on the volatility over the following five days. The curve from the HAR-RV-CJ model indicates that when a jump occurs, the volatility in the next period will decrease by around 0.00001, and this impact is reduced to 0.000003 in the following four periods. After adding into the influence of the jump intensity, as shown by the curve from the HAR-RV-CJI model, the volatility in the next period increases by 0.000003 after a jump occurs, and the impact on the volatility keeps increasing to 0.00005 until the fifth day. The obvious impact of incorporating jump intensity into the HAR framework has important implications for forecast performance, an issue now considered.

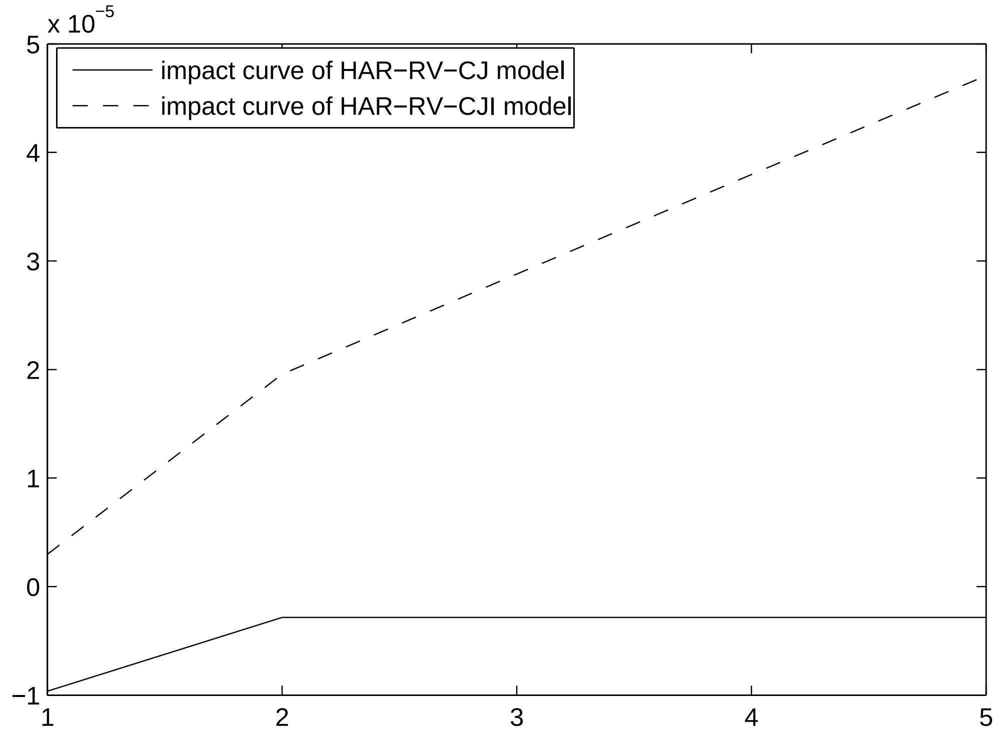

Figure 4: The impact curve of jumps on the volatility

To consider out-of-sample forecast performance, the sample is split into two periods. The first 3652 days of the sample are used for model estimation, with the the last 2000 days used for out-of-sample forecast analysis. After the initial estimation period, a recursive estimation scheme is employed to obtain parameter estimates upon which the sequence of 2000 one-step ahead forecasts of realized volatility are generated.

The forecast performance of the HAR-RV-CJ, HAR-RV-CI and HAR-RV-CJI models will be

[3]This magnitude is set up according to an arbitrarily selected jump in our empirical data, and this value is quite close to the mean value of jumps size in the data.

compared using a simple root mean squared one-step ahead forecast errors (RMSE). While doing so indicates relative forecast accuracy, this gives no indication of whether any differences in performance are significant. The Giacomini and White (2006) test (GW test) of conditional predictive accuracy is used to determine whether differences in forecast performance are statistically significant. Given squared forecast errors from competing models A and B, $e^2_{t,A}$ and $e^2_{t,B}$, the null hypothesis is that $E(e^2_{t,A}) = E(e^2_{t,B})$. This test is a t-test on the sample mean of $e^2_{t,A} - e^2_{t,B}$, with a heteroscedasticity and autocorrelation consistent (HAC) standard error. Forecasts from the HAR-RV-CI and HAR-RV-CJI models requires a forecast of the jump intensity, λ_t. In all cases, λ_t is obtained from the Hawkes model in equation (15) using (level of) lags of either BV or TBV as an explanatory variable. The forecast performance of the three models will be considered in three settings, the full sample of 2000 days, days on which jumps occur and the period of high volatility during 2008-2012 associated with (and subsequent to) the Global Financial Crisis. Forecast performance in these subsamples is of great interest, as these are times when volatility is rising rapidly and decision making must adapt quickly to changing conditions.

Table 5 contains the results of the forecasting analysis, with Panels A and B reporting results based on BV and TBV respectively. In all cases, the RMSE of the HAR-RV-CJ provides a benchmark to which the performance of the HAR-RV-CI and HAR-RV-CJI models will be compared. Beginning with the whole sample results, for both linear and nonlinear models, RMSE reduces moving from HAR-RV-CJ to HAR-RV-CI then HAR-RV-CJI. This result is evident for both measures of the continuous component, BV and TBV though the improvements are somewhat more pronounced for TBV. Reported p-values for the GW test (where the benchmark is the HAR-RV-CJ model) show that these differences in RMSEs are statistically significant at the 5% level of significance. The middle section of Table 5 reports equivalent results for days on which jumps occur. Overall, the results are consistent with the whole sample period with improvements over the HAR-RV-CJ model being somewhat more significant. The relative forecast performance during 2008-2012 also follows the same pattern, with the differences in performance once again being more significant than the whole sample period. Hence, using of the probability of jump occurrence in HAR type models leads to the out-of-sample volatility forecasts that are statistically superior to those obtained from the HAR-RV-CJ models. This result is robust to the crucial times when volatility is rising due to jumps or general market uncertainty. Overall, it seems that the best way to harness the jump component for forecasting, is to make use of both the magnitude and probability of jump occurrences.

$$HAR-RV-CJ: \; y_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_4 J_{t-1}^d + \alpha_5 J_{t-1}^w + \alpha_6 J_{t-1}^m + \varepsilon_t$$

$$HAR-CI: \; y_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_7 \lambda_t + \varepsilon_t$$

$$HAR-RV-CJI: \; y_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_4 J_{t-1}^d + \alpha_5 J_{t-1}^w + \alpha_6 J_{t-1}^m + \alpha_7 \lambda_t + \varepsilon_t$$

	HAR-RV-CJ			HAR-RV-CI			HAR-RV-CJI		
	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$
α_0	0.000012	-0.8234	0.0006	-0.000012	-0.7867	-0.0005	-0.000014	-0.7626	-0.0006
	(0.0000029)	(0.1523)	(0.0001)	(0.000011)	(0.1203)	(0.0003)	(0.000011)	(0.1218)	(0.0003)
α_1	0.2091	0.1186	0.1797	0.2193	0.1202	0.1854	0.2119	0.1188	0.1791
	(0.0173)	(0.0157)	(0.0175)	(0.0172)	(0.0157)	(0.0172)	(0.0174)	(0.0157)	(0.0174)
α_2	0.5231	0.4362	0.4887	0.5027	0.4309	0.4737	0.5136	0.4333	0.4801
	(0.0300)	(0.0282)	(0.0283)	(0.0295)	(0.0280)	(0.0278)	(0.0298)	(0.0280)	(0.0280)
α_3	0.2231	0.3672	0.2622	0.2905	0.3941	0.3045	0.2953	0.3958	0.3075
	(0.0317)	(0.0282)	(0.0276)	(0.0275)	(0.0255)	(0.0242)	(0.0275)	(0.0255)	(0.0243)
α_4	-0.0715	-161.7577	-0.0337				-0.0771	-141.0200	-0.0286
	(0.0298)	(123.5974)	(0.0159)				(0.0269)	(111.1573)	(0.0143)
α_5	-0.1674	33.6828	0.0144						
	(0.0715)	(298.0770)	(0.0212)						
α_6	0.7607	1328.8	0.0926						
	(0.1381)	(572.5052)	(0.0330)						
α_7				0.0002	1.7592	0.0091	0.00026	1.8031	0.0098
				(0.00009)	(0.3738)	(0.0020)	(0.00009)	(0.3753)	(0.0020)
R^2	0.4998	0.5574	0.5814	0.4999	0.5586	0.5819	0.5003	0.5587	0.5822
adjusted R^2	0.4989	0.5569	0.5810	0.4992	0.5583	0.5816	0.4998	0.5583	0.5819

Note: This table reports the OLS estimates for linear and nonlinear HAR-RV-CJ models, HAR-RV-CI models and HAR-RV-CJI models, in which the jumps are constructed based on the realized volatility and bipower variation. The table only reports the significant coefficients at the 5% significant level. The standard errors are reported in parentheses. The last two rows labeled R^2 and adjusted R^2 are r-square and adjusted r-square of the regression.

Table 3: Estimation results of HAR-RV-CJ model, HAR-RV-CI model and HAR-RV-CJI model

$$HAR-RV-CJ: \ y_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_4 J_{t-1}^d + \alpha_5 J_{t-1}^w + \alpha_6 J_{t-1}^m + \varepsilon_t$$

$$HAR-CI: \ y_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_7 \lambda_t + \varepsilon_t$$

$$HAR-RV-CJI: \ y_t^d = \alpha_0 + \alpha_1 C_{t-1}^d + \alpha_2 C_{t-1}^w + \alpha_3 C_{t-1}^m + \alpha_4 J_{t-1}^d + \alpha_5 J_{t-1}^w + \alpha_6 J_{t-1}^m + \alpha_7 \lambda_t + \varepsilon_t$$

	HAR-RV-CJ			HAR-RV-CI			HAR-RV-CJ		
	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$
α_0	0.000012	-0.9155	0.000058	-0.000038	-1.0425	-0.0013	-0.000034	-1.0428	-0.0013
	(0.000003)	(0.1487)	(0.00011)	(0.000013)	(0.1326)	(0.00029)	(0.000012)	(0.1343)	(0.00029)
α_1	0.2766	0.1220	0.2270	0.2776	0.1242	0.2319	0.2844	0.1242	0.2356
	(0.0195)	(0.0152)	(0.0185)	(0.0194)	(0.0151)	(0.0180)	(0.0195)	(0.0157)	(0.0184)
α_2	0.5092	0.4157	0.4637	0.5209	0.4149	0.4481	0.5045	0.4148	0.4444
	(0.0336)	(0.0283)	(0.0298)	(0.0334)	(0.0281)	(0.0294)	(0.0337)	(0.0280)	(0.0296)
α_3	0.1610	0.3685	0.2288	0.2617	0.3919	0.3018	0.2609	0.3919	0.3005
	(0.0353)	(0.0282)	(0.0289)	(0.0314)	(0.0260)	(0.0258)	(0.0314)	(0.0257)	(0.0258)
α_4	0.0261	-133.1003	-0.0193				0.0728	1.5758	0.0113
	(0.0231)	(94.9552)	(0.0135)				(0.0206)	(111.0619)	(0.0121)
α_5	0.0989	705.7103	0.0788						
	(0.0598)	(244.6685)	(0.0194)						
α_6	0.5675	802.4317	0.1304						
	(0.1022)	(389.3177)	(0.0306)						
α_7				0.00034	2.4224	0.0125	0.00031	2.4218	0.0123
				(0.00007)	(0.3163)	(0.0017)	(0.000077)	(0.3951)	(0.0017)
R^2	0.4932	0.5605	0.5839	0.4885	0.5630	0.5826	0.4896	0.5827	0.5826
adjusted R^2	0.4927	0.5600	0.5834	0.4881	0.5627	0.5823	0.4891	0.5626	0.5823

Note: This table reports the OLS estimates for linear and nonlinear HAR-RV-CJ models, HAR-RV-CI models and HAR-RV-CJI models, in which the jumps are constructed based on the realized volatility and threshold bipower variation. The table only reports the significant coefficients at the 5% significance level. The standard errors are reported in parentheses. The last two rows labeled R^2 and adjusted R^2 are r-square and adjusted r-square of the regression.

Table 4: Estimation results of HAR-RV-CJ model, HAR-RV-CI model and HAR-RV-CJI model

	Panel A: $X = BV$								
	The Whole Sample			The Days with Jumps			The Turbulent Volatility Period (2008-2012)		
	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$
HAR-RV-CJ	2.6597e-04	2.7449e-04	2.6157e-04	1.1819e-04	1.2728e-04	1.2149e-04	3.4677e-04	3.5992e-04	3.4296e-04
HAR-RV-CI	2.6446e-04	2.6936e-04	2.5913e-04	1.1725e-04	1.2412e-04	1.1757e-04	3.4536e-04	3.5283e-04	3.3824e-04
	(0.013)	(0.020)	(0.000)	(0.005)	(0.002)	(0.000)	(0.006)	(0.002)	(0.000)
HAR-RV-CJI	2.6356e-04	2.6773e-04	2.5809e-04	1.1593e-04	1.1445e-04	1.1554e-04	3.4483e-04	3.5097e-04	3.3809e-04
	(0.004)	(0.003)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)
	Panel B: $X = TBV$								
	The Whole Sample			The Days with Jumps			The Turbulent Volatility Period(2008-2012)		
	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$	$y = RV$	$y = log(RV)$	$y = (RV)^{1/2}$
HAR-RV-CJ	2.6312e-04	2.7236e-04	2.6035e-04	1.1803e-04	1.2714e-04	1.2140e-04	3.4668e-04	3.5986e-04	3.4276e-04
HAR-RV-CI	2.6165e-04	2.6026e-04	2.5872e-04	1.1714e-04	1.2406e-04	1.1748e-04	3.4521e-04	3.5271e-04	3.3812e-04
	(0.009)	(0.014)	(0.000)	(0.004)	(0.001)	(0.000)	(0.005)	(0.002)	(0.000)
HAR-RV-CJI	2.6056e-04	2.5803e-04	2.5764e-04	1.1591e-04	1.1437e-04	1.1542e-04	3.4469e-04	3.5089e-04	3.3801e-04
	(0.003)	(0.002)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.000)

Note: This table reports the root mean squared one-step-ahead volatility forecast errors for linear and nonlinear HAR-RV-CJ models, HAR-RV-CI models and HAR-RV-CJI models. Panel A and Panel B respectively reports the corresponding results based on bipower variation and threshold bipower variation. The p-value of the Giacomini and White (2006) test is reported in parentheses.

Table 5: Root Mean Squared Forecast Errors of HAR-RV-CJ models, HAR-RV-CI models and HAR-RV-CJI models

5 Conclusion

The modelling and forecasting of financial volatility is a critical issue and hence attracts a great deal of research attention. In recent years, this literature has benefited from the availability of high frequency price data. Such data has led to improvements in the estimation of realized volatility and identification of its diffusion and jump components. While a great deal of work has focused on forecasting realized volatility, less has considered how to use both components of volatility effectively. This paper contributes to this literature and examines how best to use the jump component of volatility for modeling and forecasting total volatility. Specifically, the role of both jump size and probability of jump occurrence were considered. A point process model was developed to capture the intensity, or probability of jump occurrence. It was found that jumps, self-excite, or cluster and are related to the level of diffusive volatility. The estimated intensity was then used in a volatility forecasting framework. It was found that by extending existing forecasting models with jump intensity, superior forecasts were obtained. This improvement in performance was robust to critical periods when jumps occurred or volatility was general high and variable. Thus, the best way to harness the jump component for forecasting, is to make use of both the magnitude and probability of jump occurrences.

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