Contents

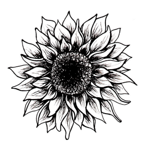
SHAPES & SYMMETRIES

INTRODUCTION

Landscape reflection in water

Symmetry presents itself in nature, with repetitions that can be found in nature's forms and patterns, but often with imperfections.

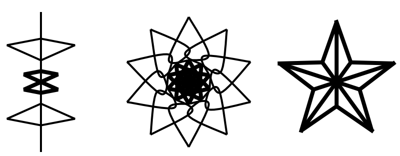
Moth



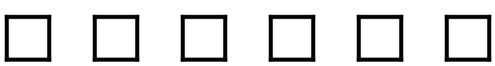
Sunflower

Starfish

Math creates a space where perfect symmetry can be considered.

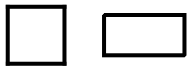


In our real physical world, lines may not be perfectly straight, and squares may not be perfectly square, but mathematics allows us to believe in straight lines and perfect squares.

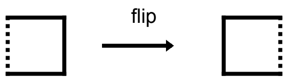


Throughout this book, we will pretend we are in that mathematical space. We will ignore the imperfections in our drawings, and see shapes and patterns as if they are composed of perfect lines and curves. We will play with our shapes and patterns, using color to manipulate their symmetries, and even destroy them at times, all in order to better understand them.

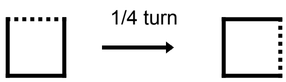
Let's talk about symmetry. Some shapes have more symmetry than others.



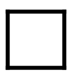
If while you blinked, a square was flipped,



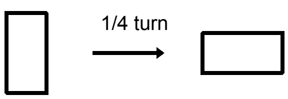
or turned a quarter of the way around,



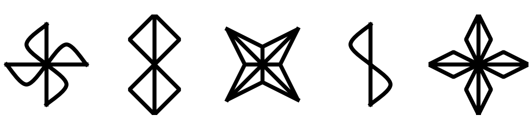
you would then still see the same square and not know.



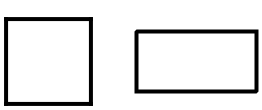
Yet this is not the case for a rectangle...



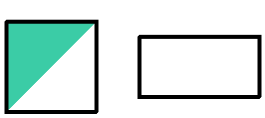
Check in: Which of these shapes can be rotated by a ¼ turn without changing in appearance?



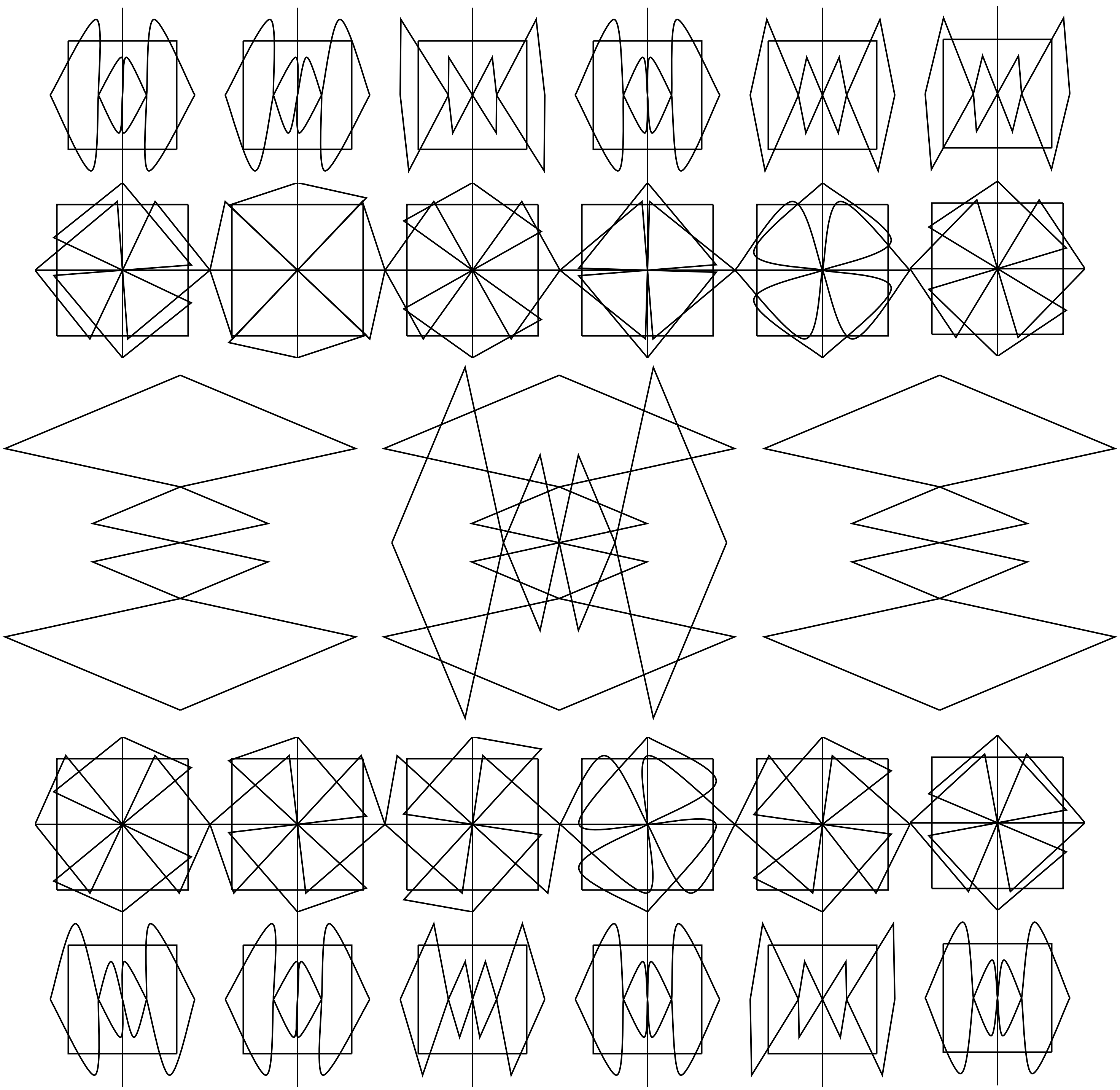
The symmetries of our shapes are the transformations that leave our shapes unchanged. We can see that a quarter turn is a symmetry of a square but not for a rectangle, and we can intuitively see that a square is "more symmetric" than a rectangle because it can be flipped and turned in more ways.



We will also see how this can change once color is added.

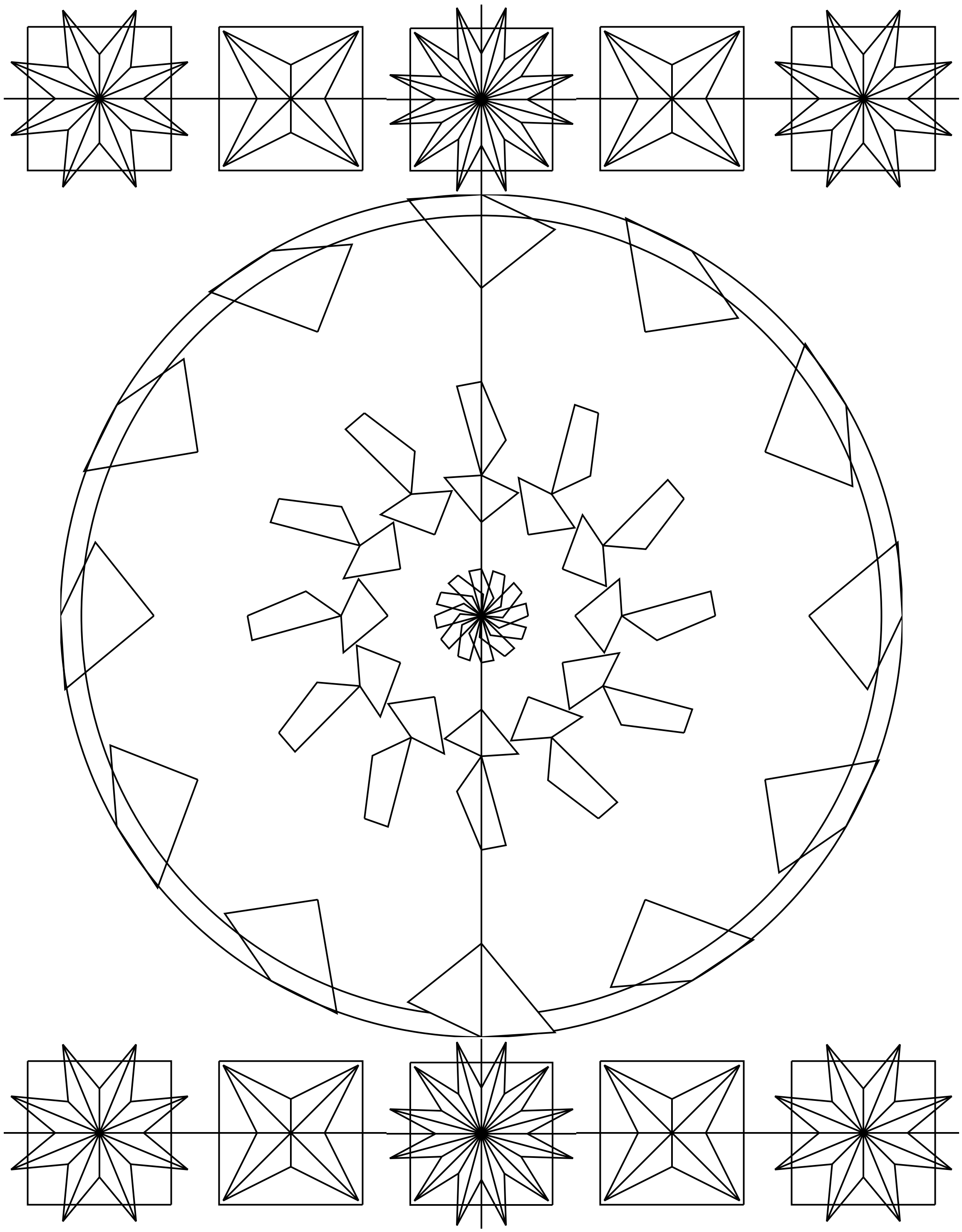


Color the shapes that have ¼ turns with a different set of colors than the shapes that do not have ¼ turns.



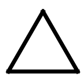
shapes with ½ turns and shapes with ¼ turns

Can you color the shapes to make them "less symmetric"?

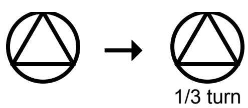


ROTATIONS

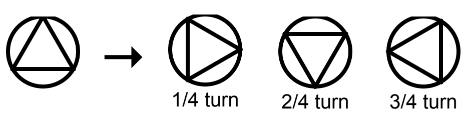
A regular triangle has equal side lengths and equal angles.



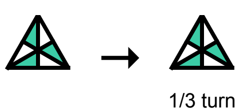
What's more, it can be rotated ⅓ of the way around a circle and appear unchanged. Had our eyes been closed when it rotated, we would not have noticed a difference.



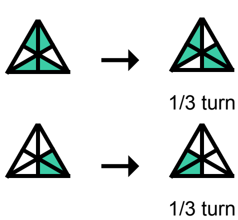
If the triangle is instead rotated by an arbitrary amount, like ¼ of the way around a circle, it will then appear changed, since it is oriented differently.



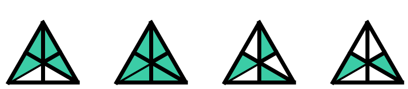
We can even find ways to color the triangle so that a ⅓ turn still does not change it.

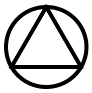


While this will not work for other ways.

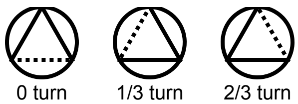


Check in: Which of the following colored triangles can be rotated by a ⅓ turn without changing in appearance?

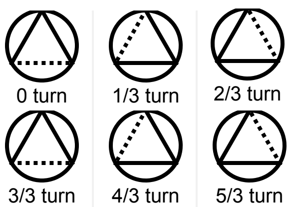




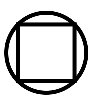
Our triangle can also rotate by more than a $\frac{1}{3}$ turn without changing. It can rotate by twice that much - ⅔ of the way around the circle - or by 3 times that much, which is all the way around the circle.



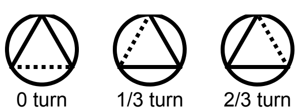
We can keep rotating - by 4 times that much, 5 times that much, 6 times... and keep going. The triangle seems to have an infinite number of rotations, but after 3 they become repetitive.



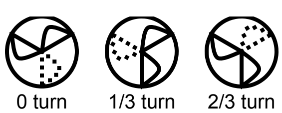
Check in: How many ways can a square rotate without changing before the ways become repetitive?



The triangle has only 3 unique rotations. We'll talk about rotations that are less than a full turn.



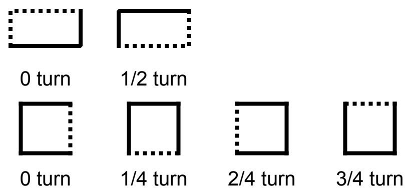
Other shapes have these same 3 rotations. For this reason, we can say they all share the same symmetry group.



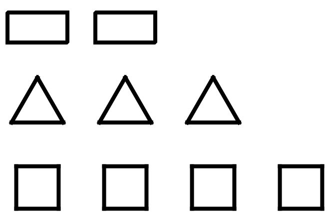
However, their rotational symmetry can be removed by adding color.

Now when our shape is rotated, its color shows it.

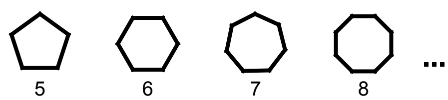
Now that we can count rotations, we can be more precise when we say a square has more symmetry than a rectangle.



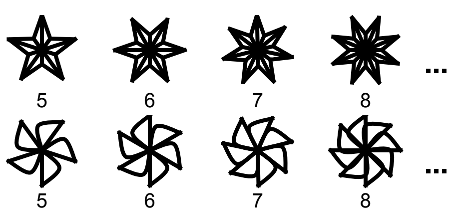
We can also see that a square has more rotational symmetry than a triangle, which in turn has more than a rectangle: A rectangle has only 2 unique rotations, while a triangle has 3, and a square has 4.



We don't need to stop at 4 rotations. We can find shapes with 5 rotations, 6 rotations, 7, 8, ... and keep going towards infinity.

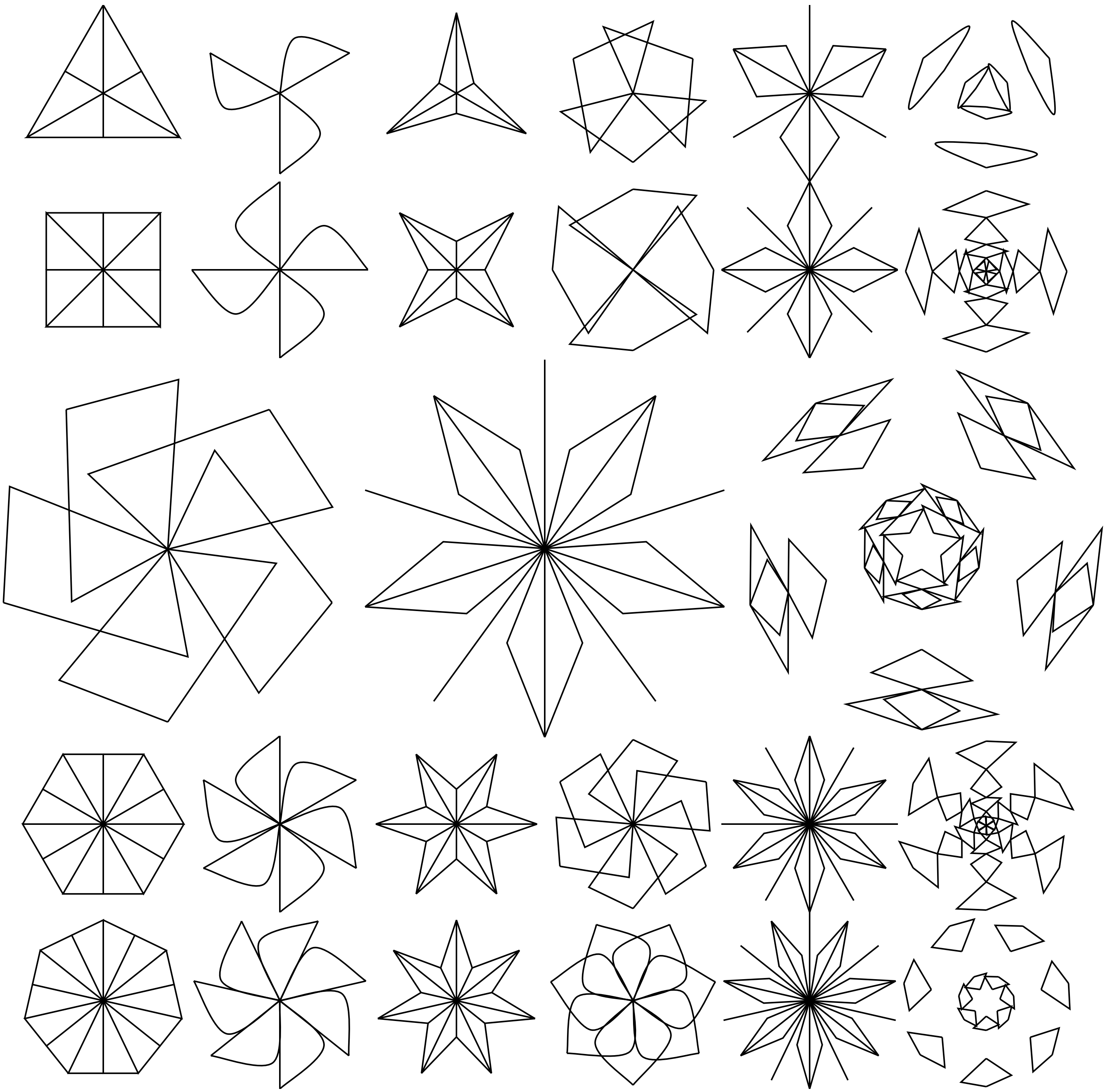


And these shapes don't even need to be so simple.



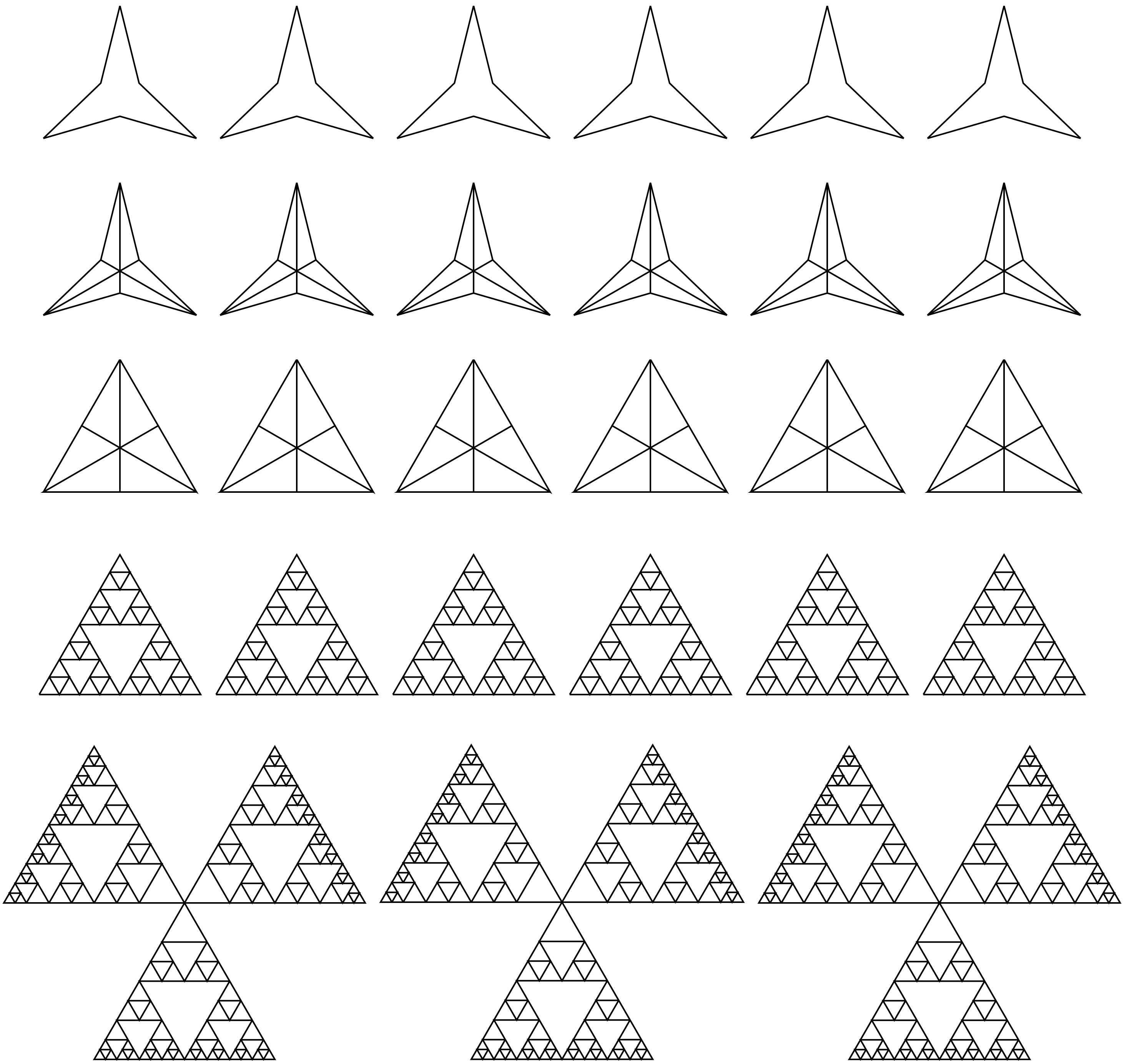
Can you find all of the shapes with 7 rotations?

Color the shapes so that they no longer have any rotations.



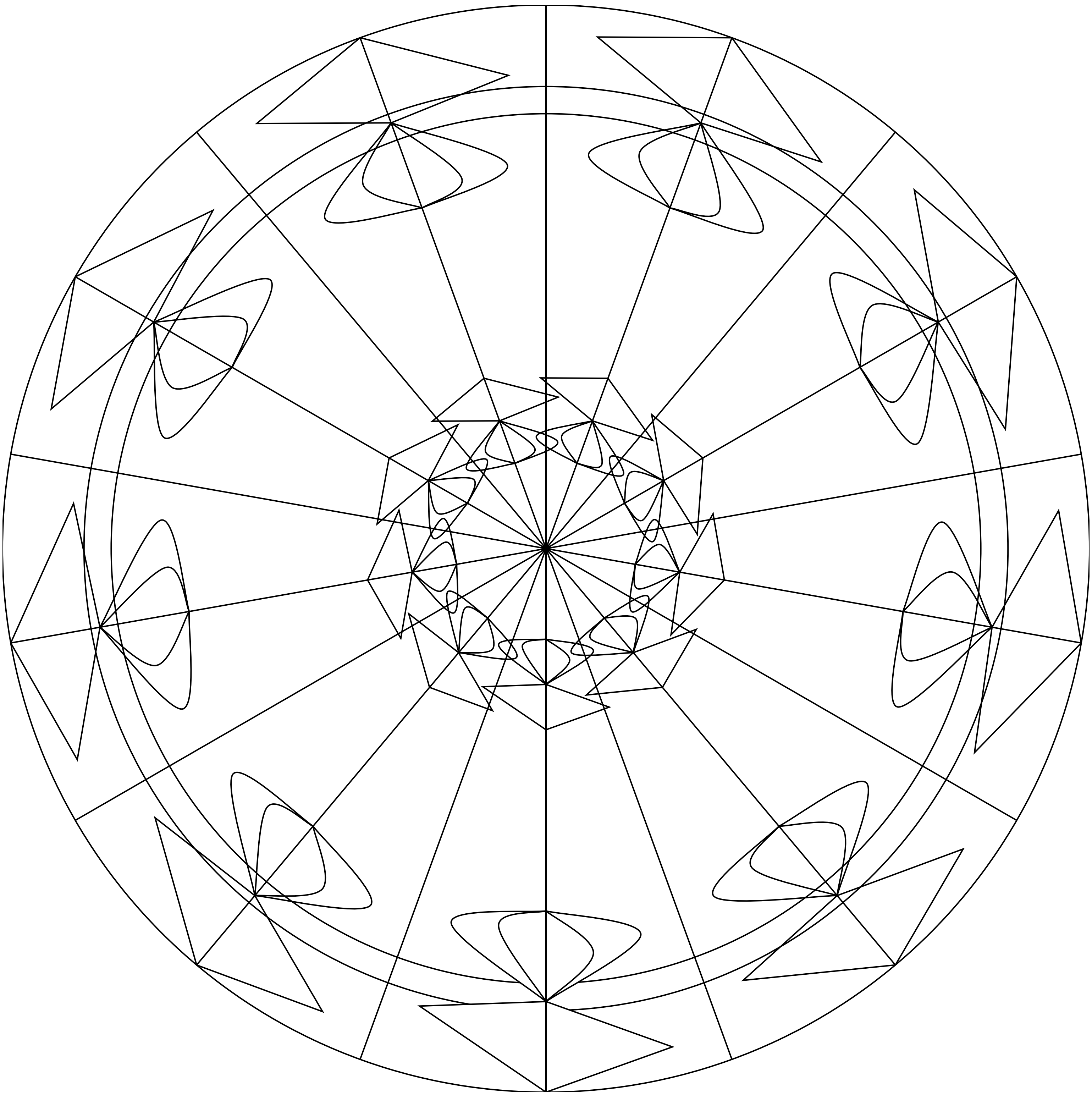
shapes with 3, 4, 5, 6, 7 rotations

Color the shapes so that a ⅓ turn continues to leave their appearance unchanged.



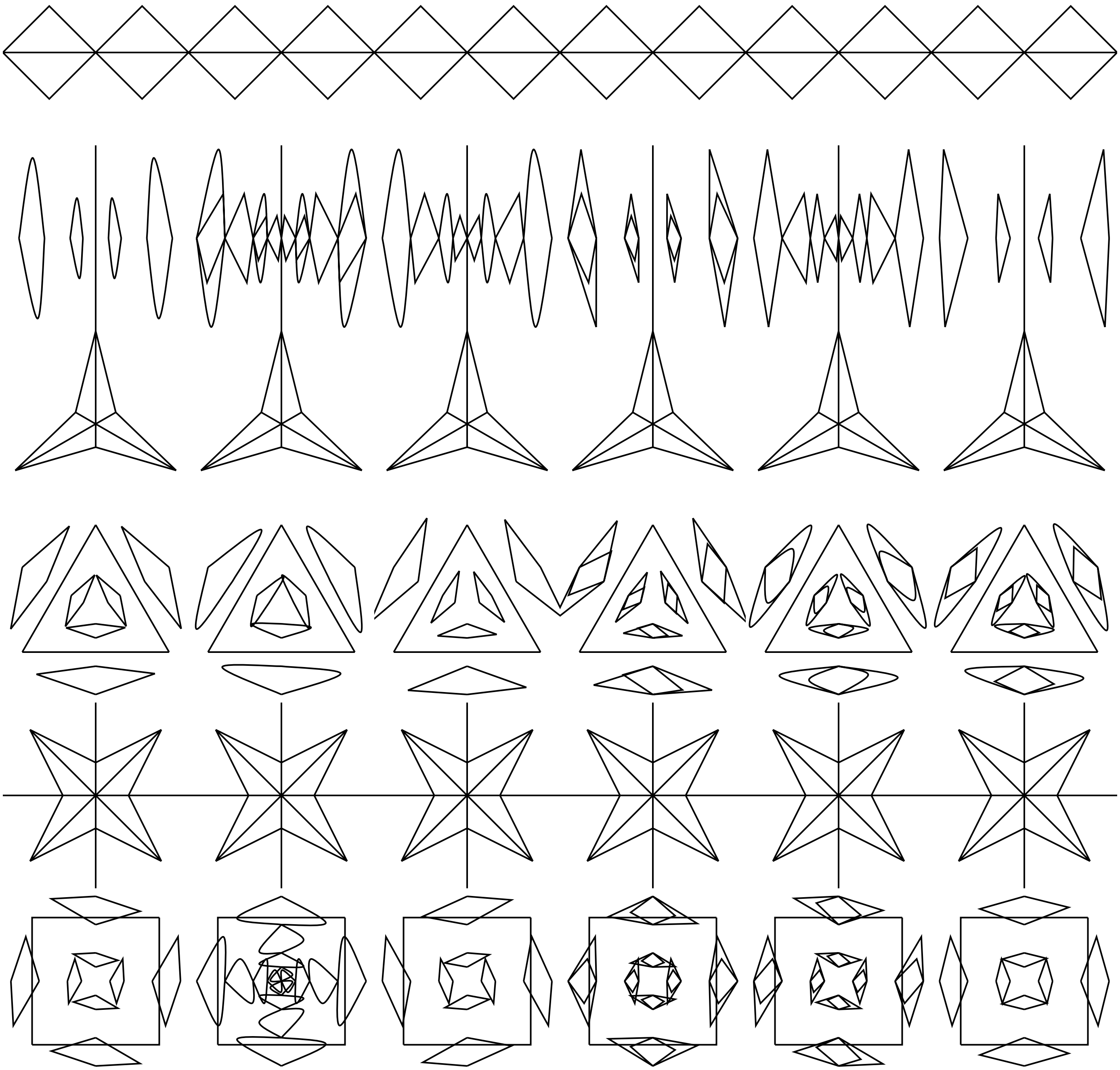
shapes with ⅓ turns and sierpinski triangles

Color the shape so that it has 3 unique rotations.



circular pattern with 9 rotations

Color the shapes with 4 rotations so that they have only 2 rotations.



shapes with 2, 3, 4 rotations